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Energy estimates for a class of semilinear elliptic equations on half Euclidean balls

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Abstract. For a class of semi-linear elliptic equations with critical Sobolev exponents and boundary conditions, we prove pointwise estimates for blowup solutions and energy estimates. A special case of this class of equations is a locally defined prescribing scalar curvature and mean curvature type equation.

1. Introduction

In this article we consider the problem

$$\begin{cases} -\Delta u = g(u), & \text{in } B_3^+, \\ \dfrac{\partial u}{\partial x_n} = h(u), & \text{on } \partial B_3^+ \cap \partial \mathbb{R}^n_+, \end{cases} \tag{1.1}$$

where $u > 0$ is a positive continuous solution, B_3^+ is the upper half ball centered at the origin with radius 3, g is a continuous function on $(0,\infty)$ and h is locally Hölder continuous on $(0,\infty)$.

If $g(s) = s^{(n+2)/(n-2)}$ and $h(s) = c\, s^{n/(n-2)}$, the equation (1.1) is a typical curvature equation. If we use δ to represent the Euclidean metric, then $u^{4/(n-2)}\,\delta$ is conformal to δ. Equation (1.1) in this special case means that the scalar curvature under the new metric is $4(n-1)/(n-2)$, and that the boundary mean curvature under the new metric is $-\frac{2}{n-2}\,c$. Equation (1.1) is very closely related to the well-known Yamabe problem and to the boundary Yamabe problem. For g and h we assume:

GH_0 : g is a continuous function on $(0,\infty)$, h is Hölder continuous on $(0,\infty)$, and

$$GH_1 : \begin{cases} g(s)\, s^{-(n+2)/(n-2)} \ \text{ is non-increasing}, \ \ \lim_{s\to\infty} g(s)\, s^{-(n+2)/(n-2)} \in (0,\infty), \\ s^{-n/(n-2)}\, h(s) \ \text{ is non-decreasing and } \lim_{s\to\infty} s^{-n/(n-2)}\, h(s) < \infty. \end{cases}$$

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Let

$$c_h := \lim_{s\to\infty} s^{-n/(n-2)}\, h(s). \tag{1.2}$$

Then if $c_h > 0$ we assume

$$GH_2: \quad \sup_{0<s\le 1} \frac{g(s)}{s} < \infty, \quad \text{and} \quad \sup_{0<s\le 1} \frac{|h(s)|}{s} < \infty.$$

If $c_h \le 0$ our assumption on g, h is

$$GH_3: \quad \sup_{0<s\le 1} g(s) < \infty, \quad \text{and} \quad \sup_{0<s\le 1} |h(s)| < \infty.$$

The main result of this article is concerned with the case $c_h > 0$:

Theorem 1.1. *Let $u > 0$ be a solution of (1.1) where g and h satisfy GH_0 and GH_1. Suppose $c_h > 0$ and GH_2 also holds. Then,*

$$\int_{B_1^+} |\nabla u|^2 + u^{2n/(n-2)} \le C, \tag{1.3}$$

for some $C > 0$ that depends only on g, h and n.

Obviously, if

$$g(s) = c_1\, s^{(n+2)/(n-2)}, \ \ c_1 > 0 \quad \text{and} \quad h(s) = c_h\, s^{n/(n-2)}, \ \ c_h > 0, \tag{1.4}$$

then g and h satisfy the assumptions in Theorem 1.1. The energy estimate (1.3) for this special case has been proved by Li–Zhang [9]. It is easy to see that the assumptions on g and h in Theorem 1.1 include a much larger class of functions. For example, for any non-increasing function $c_1(s)$ satisfying $\lim_{s\to\infty} c_1(s) > 0$ and $\lim_{s\to 0+} c_1(s)\, s^{4/(n-2)} < \infty$, $g(s) = c_1(s)\, s^{(n+2)/(n-2)}$ satisfies the assumptions of g. Similarly $h(s) = c_2(s)\, s^{n/(n-2)}$ for a nondecreasing function $c_2(s)$ with $\lim_{s\to\infty} c_2(s) = c_h$ and $\lim_{s\to 0+} |c_2(s)|\, s^{2/(n-2)} < \infty$, satisfies the requirement of h in Theorem 1.1.

For the case $c_h \le 0$ we have:

Theorem 1.2. *Let $u > 0$ be a solution of (1.1) where g and h satisfy GH_0 and GH_1. Suppose $c_h \le 0$ and g and h satisfy GH_3. Then the energy estimate (1.3) holds for C depending only on g, h and n.*

If we allow $\lim_{s\to\infty} s^{-(n+2)/(n-2)}\, g(s) = 0$, then the energy estimate (1.3) may not hold. For example, let $g(s) = \frac{1}{4}(s+1)^{-3}$; then g satisfies the assumption in Theorem 1.2 except that $\lim_{s\to\infty} s^{-(n+2)/(n-2)}\, g(s) = 0$. Let $u_j(x) = \sqrt{x_1 + j} - 1$. It is easy to verify that u_j satisfies

$$\begin{cases} -\Delta u_j = g(u_j) & \text{in } B_3^+, \\ \dfrac{\partial u_j}{\partial x_n} = 0, & \text{on } \partial B_3^+ \cap \partial \mathbb{R}^n_+. \end{cases}$$

Note that $h = 0$ in this case. Then clearly (1.3) does not hold for u_j.

The energy estimate (1.3) is closely related to the following Harnack type inequality:

$$\left(\max_{B_1^+} u\right)\left(\min_{B_2^+} u\right) \le C, \tag{1.5}$$

which was proved by Li–Zhang [9] for the special case (1.4). Li–Zhang [9] also proved (1.3) for equation (1.4) using (1.5) in their argument in a nontrivial way.

In the past two decades Harnack type inequalities similar to (1.5) have played an important role in blowup analysis for semilinear elliptic equations with critical Sobolev exponents. Pioneer works in this respect can be found in Schoen [13], Schoen–Zhang [14], Chen–Lin [3] and Li [8], etc. Further results can be found in [2], [4], [5], [7], [9], [10], [12], [14], [16] and the references therein. Usually for a semi-linear equation without boundary condition, for example the conformal scalar curvature equation

$$\Delta u + K(x)\, u^{(n+2)/(n-2)} = 0, \quad \text{in } B_3,$$

a Harnack inequality of the type

$$\left(\max_{B_1} u\right)\left(\min_{B_2} u\right) \le C$$

immediately leads to the energy estimate

$$\int_{B_1} |\nabla u|^2 + u^{2n/(n-2)} \le C$$

by Green's representation theorem and integration by parts (see [3] for a proof). However, when a boundary condition as in (1.1) intervenes, using the Harnack inequality (1.5) to derive (1.3) is much more involved. In order to derive energy estimate (1.3) and pointwise estimates for blow up solutions, Li and Zhang prove the following results in [9]:

Theorem A (Li–Zhang, [9]). *Let $u > 0$ be a solution of (1.1), where g and h satisfy GH_0, GH_1 and GH_3. Then*

$$\left(\max_{\overline{B_1^+}} u\right)\left(\min_{\overline{B_2^+}} u\right) \le C.$$

Here we note that in Theorem A no sign of c_h is specified. One would expect the energy estimate (1.3) to follow directly from Li–Zhang's theorem. This is indeed the case if $c_h \le 0$. However for $c_h > 0$ substantially more estimates are needed in order to establish a precise pointwise estimate for blowup solutions. As a matter of fact we need to assume (GH_2) instead of (GH_3) in order to obtain (1.3).

The organization of this article is as follows. In Section 2 we prove Theorem 1.1. The idea of the proof is as follows. First we use a selection process to locate regions in which the bubbling solutions look like global solutions. Then we consider the interaction of the bubbling regions. Using delicate blowup analysis and Pohozaev identity we prove that bubbling regions must be a positive distance apart. In Section 3 we prove Theorem 1.2 using Theorem A and integration by parts.

The following notations will be used throughout the paper:

$B(x,\sigma)$ is the ball centered at x with radius σ,

$B^+(x,\sigma) := B(x,\sigma)\cap\mathbb{R}^n_+$, $\quad B_\sigma = B(0,\sigma)$, $\quad B^+_\sigma = B_\sigma\cap\mathbb{R}^n_+$

$B^T(x,\sigma) := B(x,\sigma)\cap\{y_n > T\}$, $\quad B^T_R = B^T(0,R)$,

$\partial' B^T(x,\sigma) := \partial B^T(x,\sigma)\cap\{y_n = T\}$, $\quad \partial'' B^T(x,\sigma) = \partial B^T(x,\sigma)\cap\{y_n > T\}$.

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2. Proof of Theorem 1.1

First we recall that Theorem 1.1 deals with the $c_h > 0$ case, which is substantially harder than the other case. The proof of Theorem 1.1 is by contradiction. Suppose there is no energy bound; then there exists a sequence u_k such that

$$\int_{B_1} |\nabla u_k|^2 + u_k^{2n/(n-2)} \longrightarrow \infty. \tag{2.1}$$

We claim that $\max_{B^+_{3/2}} u_k \to \infty$. Indeed, if this is not the case, which means that there is a uniform bound for u_k on $B^+_{3/2}$, we just take a cut-off function $\eta\in C^\infty$ such that $\eta\equiv 1$ on B_1^+ and $\eta\equiv 0$ on $B^+_{3/2}\setminus B^+_{5/4}$ and $|\nabla\eta|\le C$. Multiplying the equation (1.1) by $u_k\eta^2$, using integration by parts and Cauchy's inequality we obtain a uniform bound of $\int_{B_1}|\nabla u_k|^2$, a contradiction to (2.1).

Since the remaining part of the proof is technical in nature, it may be helpful to explain the outline of the approach. First we use a selection process to determine a bubbling area which consists of disjoint balls. Each ball is shrinking to a point as k tends to infinity and the profile of bubbling solutions in each ball is very similar to that of a globally defined solution. In the second step we focus on the interaction of the bubbling balls, which is the most essential part of the proof. We shall employ the standard moving sphere method to obtain an upper bound of solutions not only within each bubbling ball, but also on the region outside the bubbling balls. Then by comparing the lower and upper bound of solutions we prove that all the bubbles are of comparable magnitude. Then we use the Pohozaev identity, a balancing condition, around a bubbling ball to prove that all bubbling balls have to be a positive distance apart and the distance is independent of k. Finally in step three the conclusion of Theorem 1.1 follows from standard elliptic estimates.

2.1. Step one: locating bubbling balls

Proposition 2.1. *Let u be a solution of (1.1). Then, for any $\epsilon\in(0,1)$ and $R>1$, there exist positive constants $C_0(n,\epsilon,R,g,h)$ and $C_1(n,\epsilon,R,g,h)$ such that if $\max_{B_1^+} u > C_0$, there exists a set $Z=\{q_1,\dots,q_L\}\subset B_2^+$ of local maximum points of u such that*

$$\left\| u(q_j)^{-1} u\big(u(q_j)^{-2/(n-2)}\, y + q_j\big) - \Big(1+\frac{A}{n(n-2)}\,|y|^2\Big)^{-(n-2)/2} \right\|_{C^2(B_R^{-T_j})} < \epsilon,$$

where $T_j = u(q_j)^{2/(n-2)}\, q_{jn}$, q_{jn} *is the last component of* q_j, *and*

$$A = \lim_{s\to\infty} g(s)\, s^{-(n+2)/(n-2)}. \tag{2.2}$$

Moreover, letting $r_j = u(q_j)^{-2/(n-2)}\, R$, *we have*

$$\begin{aligned} &\overline{B(r_i,q_i)} \cap \overline{B(r_j,q_j)} = \emptyset, && \textit{for } i \neq j,\\ &|q_i - q_j|^{(n-2)/2} u(q_j) < C_0, && \textit{for } j > i,\\ &u(q) \leq C_1 \operatorname{dist}(q, Z)^{-(n-2)/2}, && \textit{for all } q \in \overline{B_2^+}. \end{aligned}$$

The proof of Proposition 2.1 requires the following lemma.

Lemma 2.2. *Given any* $R > 1$, $\epsilon \in (0,1)$, *there exists* $C_2(n, R, \epsilon, g, h) > 1$ *such that for any compact* $K \subset \bar{B}_1^+$ *and any* u *of* (1.1) *with*

$$\max_{q \in \overline{B_2^+} \setminus K} \operatorname{dist}(q, K)^{2/(n-2)}\, u(q) \geq C_2,$$

there exists $q_0 \in B_{5/2}^+ \setminus K$, *which is a local maximum of* u, *and*

$$\left\| u^{-1}(q_0)\, u\big(u(q_0)^{-2/(n-2)}\, y + q_0\big) - \Big(1 + \frac{A}{n(n-2)}\, |y|^2\Big)^{-(n-2)/2} \right\|_{C^2(B_R^{-T})} < \epsilon, \tag{2.3}$$

where $T = u(q_0)^{2/(n-2)}\, q_{0n}$.

Proof of Lemma 2.2. We prove it by contradiction. Suppose no such C_2 exists for some ϵ and R. Then there exist compact subsets $K_k \subset \bar{B}_1^+$ and a sequence of solutions u_k such that $\max_{\overline{B_2^+}\setminus K_k} \operatorname{dist}(x, K_k)^{2/(n-2)}\, u_k(x) > k$ and no q_0 as in (2.3) exists. Let x_k satisfy

$$u_k(x_k) \operatorname{dist}(x_k, K_k)^{(n-2)/2} > k, \qquad d_k = \operatorname{dist}(x_k, K_k),$$

and

$$S_k(y) = u_k(y)\, \big(d_k - |y - x_k|\big)^{(n-2)/2}, \qquad \forall y \in B_{5/2}^+.$$

Suppose that S_k reaches its maximum in $\overline{B^+(x_k, d_k)}$ at $\hat{x}_k$. Then

$$S_k(\widehat{x}_k) \geq S_k(x_k) = u_k(x_k)\, d_k^{(n-2)/2} > k. \tag{2.4}$$

Let $\sigma_k = \frac{1}{2}\,(d_k - |x_k - \widehat{x}_k|)$. Then clearly (2.4) can be written as

$$u_k(\widehat{x}_k)\, 2^{(n-2)/2}\, \sigma_k^{(n-2)/2} \geq u_k(x_k)\, d_k^{(n-2)/2} \longrightarrow \infty \quad \text{as} \quad k \longrightarrow \infty. \tag{2.5}$$

For all $x \in B^+(\hat{x}_k, \sigma_k)$, since

$$u_k(x)\, \big(d_k - |x - x_k|\big)^{(n-2)/2} \leq u_k(\widehat{x}_k)\, \big(d_k - |x_k - \widehat{x}_k|\big)^{(n-2)/2},$$

we have

$$u_k(x) \le u_k(\widehat{x}_k)\Big(\frac{d_k - |x_k - \widehat{x}_k|}{d_k - |x - x_k|}\Big)^{(n-2)/2}.$$

Using $|x - \hat{x}_k| \le \sigma_k$, and

$$d_k - |x - x_k| \ge d_k - |x_k - \hat{x}_k| - |x - \hat{x}_k| \ge \sigma_k,$$

we obtain

$$u_k(x) \le 2^{(n-2)/2} u_k(\widehat{x}_k), \quad \text{for all } x \in B^+(\widehat{x}_k, \sigma_k). \tag{2.6}$$

Let $M_k = u_k(\widehat{x}_k)$ and

$$v_k(y) = {M_k}^{-1} u_k({M_k}^{-2/(n-2)}\, y + \widehat{x}_k), \quad M_k^{-2/(n-2)}\, y + \hat{x}_k \in B_3^+.$$

Direct computation shows

$$\Delta v_k(y) + (M_k v_k(y))^{-(n+2)/(n-2)}\, g\big(M_k v_k(y)\big) \cdot v_k(y)^{(n+2)/(n-2)} = 0. \tag{2.7}$$

By (2.6) we have

$$0 \le v_k(y) \le 2^{(n-2)/2} \quad \forall y \in B\big(0, {M_k}^{2/(n-2)}\, \sigma_k\big) \cap \big\{y_n \ge -{M_k}^{2/(n-2)}\, \widehat{x}_{kn}\big\}. \tag{2.8}$$

We consider two cases.

Case one in Lemma 2.2: along a subsequence $\lim_{k\to\infty} {M_k}^{2/(n-2)}\, \widehat{x}_{kn} = \infty$.

Throughout this article we do not distinguish sequences and their subsequences. Since we always consider subsequence instead of the whole sequence with no difference on their notation, this process will not be repeatedly stated in the remaining part of the article.

Since $M_k^{2/(n-2)}\sigma_k$ and $M_k^{2/(n-2)}\hat{x}_{nk}$ both tend to infinity, (2.7) is defined on $|y| \le l_k$ for some $l_k \to \infty$. By (2.8) v_k is bounded above in B_{l_k}. We claim that along a subsequence $v_k \to V$ uniformly over all compact subsets of $\mathbb{R}^n$, where V satisfies

$$\Delta V + AV^{(n+2)/(n-2)} = 0, \quad \mathbb{R}^n, \quad V > 0 \text{ in } \mathbb{R}^n. \tag{2.9}$$

with $A = \lim_{s\to\infty} s^{-(n+2)/(n-2)} g(s)$. To prove the claim we shall show that for any $R > 1$,

$$v_k(y) \ge C(R) > 0, \quad |y| \le R. \tag{2.10}$$

Once (2.10) is established, we have $M_k v_k \to \infty$ over all B_R, thus

$$\begin{aligned} M_k^{-(n+2)/(n-2)}\, g(M_k v_k) &= (M_k v_k)^{-(n+2)/(n-2)}\, g(M_k v_k)\, v_k^{(n+2)/(n-2)} \\ &\longrightarrow A\, V^{(n+2)/(n-2)} \end{aligned}$$

over all compact subsets of $\mathbb{R}^n$. Then it is easy to see that V solves (2.9).

Therefore we only need to establish (2.10) for fixed $R > 1$. Let

$$\Omega_{R,k} := \{y \in B_R : v_k(y) \leq 3M_k^{-1}\}$$

and

$$a_k(y) = M_k^{-(n+2)/(n-2)} g(M_k v_k)/v_k.$$

It follows from (GH_1) that in $B_R \setminus \Omega_{R,k}$

$$a_k(y) \leq g(3)\, v_k^{4/(n-2)} \leq 4\, g(3).$$

For $y \in \Omega_{R,k}$ we use (GH_2) to obtain

$$a_k(y) \leq CM_k^{-4/(n-2)}, \quad y \in \Omega_{R,k}.$$

In either case $a_k(y)$ is a bounded function. From

$$\Delta v_k(y) + a_k(y)\, v_k(y) = 0 \quad \text{in } B_R$$

and the standard Harnack inequality we have

$$1 = v_k(0) \leq \max_{B_{R/2}} v_k \leq C(R) \min_{B_{R/2}} v_k.$$

Thus (2.10) is established.

By the classification theorem of Caffarelli–Gidas–Spruck [1], V is comparable to $O(|y|^{2-n})$ at infinity and there is only one maximum point, $\bar{y}$, in $\mathbb{R}^n$. Correspondingly there exists a sequence of local maximum points of u_k, denoted $\bar{x}_k$, that tends to $\bar{y}$ after scaling. Thus if the scaling is centered at $\bar{x}_k$ in the first place, the limit function would be a solution to (2.9) with $V(0) = 1 = \max_{\mathbb{R}^n} V$. By the classification theorem of Caffarelli–Gidas–Spruck,

$$V(y) = \left(1 + \frac{A}{n(n-2)}|y|^2\right)^{-(n-2)/2}. \tag{2.11}$$

Thus (2.3) holds for all large k. Consequently this case is ruled out and we only need to consider:

Case two in Lemma 2.2: $\lim_{k\to\infty} M_k^{2/(n-2)} \hat{x}_{kn} < \infty$.

It is easy to verify that v_k satisfies

$$\begin{cases} \Delta v_k + (M_k v_k)^{-\frac{n+2}{n-2}} g(M_k v_k)\, v_k^{\frac{n+2}{n-2}} = 0, & \text{in } \{y;\ M_k^{-2/(n-2)} y + \hat{x}_k \in B_3^+\}, \\ \dfrac{\partial v_k}{\partial y_n} = (M_k v_k)^{-n/(n-2)}\, h(M_k v_k)\, v_k^{2/(n-2)}\, v_k, & \text{on } \{y_n = -M_k^{2/(n-2)} \hat{x}_{kn}\}. \end{cases}$$

We claim that for any $R > 1$, there exists $C(R) > 0$ such that

$$v_k(y) \geq C(R) \quad \text{in } B_R \cap \{y_n \geq -M_k^{2/(n-2)}\, \hat{x}_{kn}\}. \tag{2.12}$$

The proof of (2.12) is similar to the interior case. Let $T_k = M_k^{2/(n-2)} x_{kn}$ and $p_k = (0', -T_k)$. On $B(p_k, R) \cap \{y_n \geq -T_k\}$ we write the equation for v_k as

$$\begin{cases} \Delta v_k + a_k v_k = 0, & \text{in } \; B(p_k, R) \cap \{y_n > -T_k\}, \\ \partial_n v_k + b_k v_k = 0, & \text{on } \; B(p_k, R) \cap \{y_n = -T_k\}. \end{cases} \tag{2.13}$$

where it is easy to use GH_2 to prove that $|a_k| + |b_k| \leq C$ for some C independent of k and R. By a classical Harnack inequality with boundary terms (see, for example, Lemma 6.2 of [15], or Han–Li [6]), we have

$$1 = v_k(0) \leq \max_{B(p_k, R/2) \cap \{y_n \geq -T_k\}} v_k \leq C(R) \min_{B(p_k, R/2) \cap \{y_n \geq -T_k\}} v_k.$$

Therefore v_k is bounded below by positive constants over all compact subsets. Thus the limit function V_1 solves

$$\begin{cases} \Delta V_1 + A V_1^{(n+2)/(n-2)} = 0, & \text{in } \; \mathbb{R}^n \cap \{y_n > -T\}, \quad V_1 > 0, \\ \dfrac{\partial V_1}{\partial y_n} = c_h V_1^{n/(n-2)}, & \text{on } \; \{y_n = -T\}, \end{cases} \tag{2.14}$$

where $T = \lim_{k \to \infty} T_k$. Note that $T > 0$ because $c_h > 0$. By Li–Zhu's classification theorem, V_1 is just the restriction of a solution to (2.9) to $\{y_n > -T\}$. Thus there is a global maximum of V_1 in the interior of $\{y_n > -T\}$. Correspondingly there is a sequence of local maximum points x_k of u_k tending to that point after scaling. If the scaling is centered at x_k in the first place, the limit function V_1 is just V as in (2.11), and (2.3) holds for all large k in this case as well. Lemma 2.2 is established. □

Proof of Proposition 2.1. First we apply Lemma 2.2 by letting $K = \emptyset$ (which implies $d(q, K) = 1$). From Lemma 2.2 we obtain q_1. Then we let $K = \overline{B^+(q_1, r_1)}$, where $r_1 = R u^{-2/(n-2)}(q_1)$. If

$$\max_{q \in B_2^+ \setminus K} \operatorname{dist}(q, K)^{2/(n-2)} u(q) \leq C_0,$$

we stop. Otherwise, there is q_2 that satisfies $B(q_2, r_2) \cap \overline{B(q_1, r_1)} = \emptyset$, where $r_2 = u(q_2)^{-2/(n-2)} R$. We continue this process by adding $\overline{B^+(q_2, r_2)}$ to K. This process stops in a finite number of steps, since each selection process implies $\int_{B^+(q_i, r_i)} |\nabla u|^2 \geq C(n)$ because of the profile of the standard bubbles. Then it is easy to conclude that Proposition 2.1 holds. □

2.2. Step two: all bubbles are far apart

The following theorem plays an important role in the proof of our main theorem, Theorem 1.1.

Theorem 2.3. *Let u be a solution to (1.1) and Z be the set of maximum points determined in Proposition 2.1. Then for suitably large R (that only depends on n, g, h) and $\epsilon \in (0, e^{-R})$, there exists $d_0(R, \epsilon) > 0$ such that if $\max_{B_1^+} u \geq C_0(n, R, \epsilon, g, h)$,*

$$\min \left\{ \operatorname{dist}(q_i, q_j), \; \forall q_i, q_j \in Z \cap B_2^+, \, q_i \neq q_j \right\} \geq d_0.$$

Proof of Theorem 2.3. By the way of contradiction, we assume that there exists a sequence of solutions $\{u_k\}$ such that $\max_{B_1^+} u_k \to \infty$ and

$$\min\left\{\text{dist}\,(q_a^k, q_b^k);\ 1 \le a, b \le N_k,\ a \neq b\right\} \longrightarrow 0 \quad \text{as } k \to \infty,$$

where $q_1^k, \ldots, q_{N_k}^k$ are the points determined by Proposition 2.1 for $u = u_k$.

Let Z_k be the set of local maximum points of u_k determined in Proposition 2.1. Let $q_k \in Z_k$ and suppose $\sigma_k = \text{dist}\,(q_k, Z_k \setminus \{q_k\})$ and we let

$$\tilde{u}_k(y) = \sigma_k^{(n-2)/2} u_k(q_k + \sigma_k y), \qquad \text{in } \Omega_k$$

where $\Omega_k := \{y : q_k + \sigma_k y \in B_3^+\}$. By the selection process we have

$$\tilde{u}_k(y) \le C|y|^{-(n-2)/2}, \quad |y| \le 3/4, y \in \Omega_k \tag{2.15}$$

and

$$\tilde{u}_k(0) \longrightarrow \infty. \tag{2.16}$$

We prove in the following proposition that $\tilde{u}_k$ decays like a harmonic function.

Proposition 2.4. *There exists $C > 0$ independent of k such that along a subsequence*

$$\tilde{u}_k(0)\, \tilde{u}_k(y)\, |y|^{n-2} \le C, \qquad \textit{for } y \in B_{2/3} \cap \Omega_k. \tag{2.17}$$

Proof of Proposition 2.4. Direct computation shows that $\tilde{u}_k$ satisfies

$$\begin{cases} \Delta \tilde{u}_k(y) + \sigma_k^{(n+2)/2} g\big(\sigma_k^{-(n-2)/2} \tilde{u}_k\big) = 0, & \text{in } \ \Omega_k, \\ \partial_n \tilde{u}_k(y) = \sigma_k^{n/2} h\big(\sigma_k^{-(n-2)/2} \tilde{u}_k\big), & \text{on } \ \partial\Omega_k \cap \{y_n = -\sigma_k^{-1} q_{kn}\}, \end{cases} \tag{2.18}$$

Let $\tilde{M}_k = \tilde{u}_k(0)$. By (2.16) $\tilde{M}_k \to \infty$. Set

$$v_k(z) = \tilde{M}_k^{-1} \tilde{u}_k(\tilde{M}_k^{-2/(n-2)} z), \qquad \text{for } z \in \tilde{\Omega}_k,$$

where

$$\tilde{\Omega}_k := \left\{z : \ |z| \le \tilde{M}_k^{2/(n-2)},\ \tilde{M}_k^{-2/(n-2)} z \in \Omega_k\right\}$$

Note that v_k is defined on a bigger set, but for the proof of Proposition 2.4 we only need to consider the part in $\tilde{\Omega}_k$.

Direct computation gives

$$\begin{cases} \Delta v_k(z) + l_k^{-(n+2)/(n-2)} g(l_k v_k) = 0, & z \in \tilde{\Omega}_k, \\ \dfrac{\partial v_k}{\partial z_n} = l_k^{-n/(n-2)} h(l_k v_k), & \{z_n = -T_k\} \cap \partial\tilde{\Omega}_k. \end{cases} \tag{2.19}$$

where $l_k = \sigma_k^{-(n-2)/2} \tilde{M}_k$ and $T_k = l_k^{2/(n-2)} q_{kn}$. We consider two cases.

Case one in Proposition 2.4: $T_k \longrightarrow \infty$.

As in the proof of Proposition 2.1, there exist $R_k \to \infty$ such that

$$\Big\| v_k(z) - \Big(1 + \frac{A}{n(n-2)}|z|^2\Big)^{-(n-2)/2} \Big\|_{C^{1,\alpha}(B_{R_k})} \le C R_k^{-1}.$$

Clearly (2.17) holds for $|z| \le \tilde{M}_k^{-2/(n-2)} R_k$, so we just need to prove (2.17) for the case $|z| > \tilde{M}_k^{-2/(n-2)} R_k$.

Lemma 2.5. *There exists $k_0 > 1$ such that for all $k \ge k_0$ and $r \in \big(R_k, \tilde{M}_k^{2/(n-2)}\big)$,*

$$\min_{\partial B_r \cap \tilde{\Omega}_k} v_k \le 2\left(\frac{n(n-2)}{A}\right)^{(n-2)/2} r^{2-n}. \tag{2.20}$$

Proof of Lemma 2.5. Suppose (2.20) does not hold. Then there exist r_k such that

$$v_k(z) \ge 2\left(\frac{n(n-2)}{A}\right)^{(n-2)/2} r_k^{2-n}, \quad |z| = r_k,\ z \in \tilde{\Omega}_k. \tag{2.21}$$

Clearly $r_k \ge R_k$. Let

$$v_k^\lambda(z) = \left(\frac{\lambda}{|z|}\right)^{n-2} v_k(z^\lambda), \quad z^\lambda = \frac{\lambda^2 z}{|z|^2}.$$

One checks that v_k^λ satisfies

$$\Delta v_k^\lambda(z) + \left(\frac{\lambda}{|z|}\right)^{n+2} l_k^{-(n+2)/(n-2)} g\left(l_k \left(\frac{|z|}{\lambda}\right)^{n-2} v_k^\lambda(z)\right) = 0, \quad \text{in } \Sigma_\lambda \tag{2.22}$$

where

$$\Sigma_\lambda := \big\{ z \in \tilde{\Omega}_k; \quad |\lambda| < |z| < r_k \big\}.$$

Clearly $v_k^\lambda \to V^\lambda$ in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n)$ for fixed $\lambda > 0$. By direct computation,

$$\begin{aligned} &V(z) > V^\lambda(z), \text{ for } \lambda \in \left(0, \left(\frac{n(n-2)}{A}\right)^{1/2}\right), \quad |z| > \lambda \\ &V(z) < V^\lambda(z), \text{ for } \lambda > \left(\frac{n(n-2)}{A}\right)^{1/2}, \quad |z| > \lambda. \end{aligned} \tag{2.23}$$

We shall apply the method of moving spheres for $\lambda \in (\frac{1}{2}(\frac{n(n-2)}{A})^{1/2}, 2(\frac{n(n-2)}{A})^{1/2})$. First we prove that, for

$$\lambda_0 = \frac{1}{2}\left(\frac{n(n-2)}{A}\right)^{1/2}, \tag{2.24}$$

we have

$$v_k(z) > v_k^{\lambda_0}(z), \quad z \in \Sigma_{\lambda_0}. \tag{2.25}$$

To prove (2.25) we first observe that $v_k > v_k^{\lambda_0}$ in $B_R \setminus B_\lambda$ for any fixed R large. Indeed, $v_k = v_k^{\lambda_0}$ on ∂B_{λ_0}. On ∂B_{λ_0} we have $\partial_\nu V > \partial_\nu V^{\lambda_0}$. Thus the $C^{1,\alpha}$ convergence of v_k to V gives that $v_k > v_k^{\lambda_0}$ near ∂B_{λ_0}. Then by the uniform convergence we further know that (2.25) holds on $B_R \setminus B_{\lambda_0}$. On ∂B_R, we have

$$v_k(z) \geq \Big(\Big(\frac{n(n-2)}{A}\Big)^{(n-2)/2} - \epsilon\Big)\,|z|^{2-n}, \quad |z| = R \tag{2.26}$$

and

$$v_k^{\lambda_0}(z) \leq \Big(\Big(\frac{n(n-2)}{A}\Big)^{(n-2)/2} - 2\epsilon\Big)\,|z|^{2-n}, \quad |z| \geq R \tag{2.27}$$

for some $\epsilon > 0$ independent of k. Next we shall use the maximum principle to prove that

$$v_k(z) > \Big(\Big(\frac{n(n-2)}{A}\Big)^{(n-2)/2} - 2\epsilon\Big)\,|z|^{2-n} > v_k^{\lambda_0}(z), \quad z \in \Sigma_{\lambda_0} \setminus B_R. \tag{2.28}$$

The proof of (2.28) is by contradiction. We shall compare v_k and

$$f_k := \Big(\Big(\frac{n(n-2)}{A}\Big)^{(n-2)/2} - 2\epsilon\Big)\,|z|^{2-n}.$$

Clearly $v_k - f_k$ is super harmonic in $\Sigma_{\lambda_0} - B_R$ and, by (2.26), (2.27) and (2.21), $v_k - f_k > 0$ on ∂B_R and $\partial\Sigma_{\lambda_0} \cap (\mathbb{R}^n_+ \setminus B_R)$. If there exists $z_0 \in \partial\Sigma_{\lambda_0} \cap \{z_n = -T_k\}$ and

$$0 > v_k(z_0) - f_k(z_0) = \min_{\Sigma_{\lambda_0} \setminus B_R} v_k - f_k$$

we would have

$$0 < \partial_n(v_k - f_k)(z_0) = l_k^{-n/(n-2)}\,h(l_k v_k(z_0)) - \partial_n f_k(z_0). \tag{2.29}$$

It is easy to verify that $\partial_n f_k(z_0) > N_k f_k(z_0)^{n/(n-2)}$ for some $N_k \to \infty$. However, by GH_1,

$$l_k^{-n/(n-2)}\,h(l_k v_k(z_0)) \leq C\,v_k(z_0)^{n/(n-2)}.$$

Thus it is impossible to have $v_k(z_0) < f_k(z_0)$ and (2.29). Therefore, (2.28) is established.

Before we employ the method of moving spheres, we set

$$O_\lambda := \Big\{z \in \Sigma_\lambda \,:\, v_k(z) < \min\Big(\Big(\frac{|z|}{\lambda}\Big)^{n-2}, 2\Big)\,v_k^\lambda(z)\Big\}.$$

Clearly O_λ contains a neighborhood of ∂B_λ in Σ_λ. Later we shall consider the equation of $v_k - v_k^\lambda$ in O_λ only, since outside this region v_k is already much greater than v_k^λ, there is no need to apply maximum principles.

In order to apply the maximum principle in O_λ we first estimate the second term in (2.22): by GH_1,

$$\begin{aligned}\Big(\frac{\lambda}{|z|}\Big)^{n+2} l_k^{-(n+2)/(n-2)}\, g\Big(l_k\Big(\frac{|z|}{\lambda}\Big)^{n-2} v_k^\lambda\Big) \\ = \Big(\Big(\frac{|z|}{\lambda}\Big)^{n-2} l_k v_k^\lambda\Big)^{-(n+2)/(n-2)} g\Big(\Big(\frac{|z|}{\lambda}\Big)^{n-2} l_k v_k^\lambda\Big)\,(v_k^\lambda)^{(n+2)/(n-2)} \\ \le (l_k v_k)^{-(n+2)/(n-2)}\, g(l_k v_k)\,(v_k^\lambda)^{(n+2)/(n-2)}, \quad \text{in } O_\lambda.\end{aligned}$$

Therefore, we have

$$\Delta v_k^\lambda + (l_k v_k)^{-(n+2)/(n-2)}\, g(l_k v_k)\,(v_k^\lambda)^{(n+2)/(n-2)} \ge 0, \quad \text{in } O_\lambda. \tag{2.30}$$

Then we write (2.19) as

$$\Delta v_k + (l_k v_k)^{-(n+2)/(n-2)}\, g(l_k v_k)\, v_k^{(n+2)/(n-2)} = 0. \tag{2.31}$$

Let $w_{\lambda,k} = v_k - v_k^\lambda$. We have, from (2.30) and (2.31),

$$\Delta w_{\lambda,k} + n(n-2)\,(l_k v_k)^{-(n+2)/(n-2)}\, g(l_k v_k)\,\xi_k^{4/(n-2)} w_{\lambda,k} \le 0, \quad \text{in } O_\lambda, \tag{2.32}$$

where ξ_k is obtained from the mean value theorem.

Now we apply the method of moving spheres to $w_{\lambda,k}$. Let

$$\bar\lambda_k = \sup\big\{\lambda \in [\lambda_0,\lambda_1];\ v_k > v_k^\mu \text{ in } \Sigma_\mu,\ \forall \mu \in (0,\lambda)\big\}$$

where λ_0 is defined in (2.24), λ_1 is given by

$$\lambda_1 = \Big(\frac{n(n-2)}{A}\Big)^{1/2} + \epsilon_0,$$

$\epsilon_0 > 0$ is chosen to be independent of k, and

$$v_k(z) > v^\lambda(z), \quad \text{on } \partial B(0,r_k) \cap \tilde\Omega_k, \quad \forall \lambda \in [\lambda_0,\lambda_1]. \tag{2.33}$$

Here we recall that r_k is defined in (2.21). From (2.21) we see that ϵ_0 can be chosen easily. By (2.25), $\bar\lambda_k > \lambda_0$. We claim that $\bar\lambda_k = \lambda_1$. Suppose that this is not the case and that we have $\bar\lambda_k < \lambda_1$. By continuity, $w_{\bar\lambda_k,k} \ge 0$, and by (2.21), $w_{\bar\lambda_k,k} > 0$ on the outside boundary: $\partial\Sigma_{\bar\lambda_k} \setminus (\partial B_{\bar\lambda_k} \cup \{z_n = -T_k\})$. By (2.32), if $\min_{\bar\Sigma_{\bar\lambda_k}} w_{\bar\lambda_k,k} = 0$, the minimum has to appear on $\partial\Sigma_{\bar\lambda_k}$. From (2.33) we see that the minimum does not appear on $\partial\Sigma_{\bar\lambda_k} \setminus (\partial B_{\bar\lambda_k} \cup \{z_n = -T_k\})$. If there exists a minimum $x_0 \in \partial\Sigma_{\bar\lambda_k} \cap \{z_n = -T_k\}$, we have

$$\partial_{z_n} w_{\bar\lambda_k,k}(x_0) > 0.$$

Note that we have strict inequality because of Hopf's lemma. On the other hand, using $T_k \to \infty$ we have

$$\begin{aligned}
\frac{\partial v_k^{\bar\lambda_k}}{\partial z_n} &= \frac{\partial}{\partial z_n}\Big(\frac{\bar\lambda_k}{|z|}\Big)^{n-2} v_k(z^{\bar\lambda_k}), \qquad \Big(\text{where } \ z^{\bar\lambda_k} := \frac{\bar\lambda_k^2 z}{|z|^2}\Big),\\
&= (n-2)\bar\lambda_k^{n-2}\frac{T_k}{|z|^n}\, v_k(z^{\bar\lambda_k}) - 2\left(\frac{\bar\lambda_k}{|z|}\right)^{n-2}\sum_{j=1}^{n-1}\partial_j\, v_k(z^{\bar\lambda_k})\frac{\bar\lambda_k^2 z_j z_n}{|z|^4}\\
&\quad + \left(\frac{\bar\lambda_k}{|z|}\right)^n \partial_n v_k(z^{\bar\lambda_k})\frac{\bar\lambda_k^2|z|^2 - 2\bar\lambda_k^2 z_n^2}{|z|^4}\\
&> \frac{n-2}{2}\ \bar\lambda_k^{n-2}\frac{T_k}{|z|^n}\, v_k(z^{\bar\lambda_k}) > N_k\,(v_k^{\bar\lambda_k})^{n/(n-2)}, \qquad \text{in } O_{\bar\lambda_k}\cap\{\, z_n = -T_k\},
\end{aligned}$$

for some $N_k \to \infty$.

For v_k, GH_1 implies

$$\partial_{z_n} v_k \le c_h\, v_k^{n/(n-2)}, \qquad \text{in } O_{\bar\lambda_k}\cap\{\, z_n = -T_k\}$$

where $c_h = \lim_{s\to\infty} s^{-n/(n-2)}h(s)$. It is easy to see that $w_{\bar\lambda_k} > 0$ on $\{z_n = -T_k\}$. Finally, an application of Hopf's lemma on $\partial B_{\bar\lambda_k}$ gives that $\partial_\nu w_{\bar\lambda_k} > 0$ on $\partial B_{\bar\lambda_k}$. Then it is easy to see that one can move the spheres a little further than $\bar\lambda_k$, a contradiction of the definition of $\bar\lambda_k$. Thus we have proved $\bar\lambda_k = \lambda_1$. However by (2.23) it is impossible to have $\lim_{k\to\infty}\bar\lambda_k > (n(n-2)/A)^{1/2}$. This contradiction proves (2.20) under **Case one**. Lemma 2.5 is established. □

From Lemma 2.5 we further prove the spherical Harnack inequality for v_k. For fixed k, consider $2R_k \le r \le \frac{1}{2}\tilde M_k^{2/(n-2)}$ and let

$$\tilde v_k(z) = r^{(n-2)/2} v_k(rz).$$

By (2.15), $\tilde v_k(z) \le C$. Direct computation yields

$$\begin{cases}
\Delta \tilde v_k(z) + r^{(n+2)/2}\, l_k^{-(n+2)/(n-2)} g\big(l_k\, r^{-(n-2)/2}\,\tilde v_k\big) = 0, & \dfrac12 < |z| < 2,\ \ r\,z \in \tilde\Omega_k,\\
\partial_n \tilde v_k = r^{n/2}\, l_k^{-n/(n-2)}\, h\big(r^{-(n-2)/2}\, l_k\, \tilde v_k\big), & \partial'\tilde\Omega_k.
\end{cases}$$

Let

$$\begin{aligned}
a_k &= r^{(n+2)/2}\, l_k^{-(n+2)/(n-2)}\, g\big(l_k\, r^{-(n-2)/2}\,\tilde v_k\big)/\tilde v_k\\
b_k &= r^{n/2}\, l_k^{-n/(n-2)}\, h\big(r^{-(n-2)/2}\, l_k\, \tilde v_k\big)/\tilde v_k
\end{aligned}$$

By the definition of l_k and r, we see that $r = o(1)\, l_k^{2/(n-2)}$ (recall that $l_k = \sigma_k^{-(n-2)/2}\tilde M_k$). Using the assumptions on g, h we have

$$a_k(z) \le \begin{cases}
g(1)\,\tilde v_k^{4/(n-2)} \le C, & \text{if } \ l_k\, r^{-(n-2)/2}\,\tilde v_k(z) \ge 1,\\
C\, r^2\, l_k^{-4/(n-2)} = o(1), & \text{if } \ l_k\, r^{-(n-2)/2}\,\tilde v_k(z) \le 1,
\end{cases}$$

and

$$|b_k(z)| \le \begin{cases} c_h\, \tilde{v}_k^{2/(n-2)} \le C, & \text{if } \ l_k\, r^{-(n-2)/2}\, \tilde{v}_k(z) \ge 1, \\ C\, r\, l_k^{-2/(n-2)} = o(1), & \text{if } \ l_k\, r^{-(n-2)/2}\, \tilde{v}_k(z) \le 1. \end{cases}$$

Hence a_k and b_k are both bounded functions.

Consequently, the equation for $\tilde{v}_k$ can be written as

$$\begin{cases} \Delta \tilde{v}_k(z) + a_k\, \tilde{v}_k = 0, & \frac{1}{2} < |z| < 2, rz \in \tilde{\Omega}_k, \\ \partial_n\, \tilde{v}_k = b_k\, \tilde{v}_k, & \partial\tilde{\Omega}_k \cap \{\, z_n = -T_k/r\}. \end{cases}$$

We apply the classical Harnack inequality for two cases: either $T_k/r > 1$ or $T_k/r \le 1$. In the first case we have

$$\max_{|z|=3/4} \tilde{v}_k(z) \le C \min_{|z|=3/4} \tilde{v}_k.$$

In the second case we have

$$\max_{|z|=1, z_n \ge -T_k/r} \tilde{v}_k(z) \le C \min_{|z|=1, z_n \ge -T_k/r} \tilde{v}_k.$$

Now (2.17) follows from (2.20) and the spherical Harnack inequality above. Proposition 2.4 is established for **Case one**.

Case two in Proposition 2.4: $\lim_{k\to\infty} T_k = T$.

Recall that v_k satisfies (2.19). As in **Case one** there exists $R_k \to \infty$ such that

$$\left\| v_k(y) - \left(1 + \frac{A}{n(n-2)}\, |y|^2\right)^{-(n-2)/2} \right\|_{C^{1,\alpha}(B_{R_k}^{-T_k})} \le C R_k^{-1}.$$

Clearly (2.17) holds for $|y| \le \tilde{M}_k^{-2/(n-2)} R_k \cap \{y_n \ge -T_k\}$, so we just need to prove (2.17) for $\{\, |y| > \tilde{M}_k^{-2/(n-2)} R_k\} \cap \{\, y_n \ge -T_k\}$.

Lemma 2.6. *There exists $k_0 > 1$ such that, for all $k \ge k_0$ and $r \in (R_k, \tilde{M}_k^{2/(n-2)})$, the estimate (2.20) still holds.*

Remark 2.7. Even though (2.20) also holds for case two, the domain for case two is different.

Proof of Lemma 2.6. Just like in the interior case, suppose there exist $r_k \ge R_k$ such that

$$\min_{\partial B_{r_k} \cap \tilde{\Omega}_k} v_k > 2 \left(\frac{n(n-2)}{A}\right)^{(n-2)/2} r_k^{2-n}. \tag{2.34}$$

Let

$$\tilde{v}_k(z) = v_k(z - T_k e_n), \quad \tilde{v}_k^{\lambda}(z) = \left(\frac{\lambda}{|z|}\right)^{n-2} \tilde{v}_k\left(\frac{\lambda^2 z}{|z|^2}\right)$$

and

$$D_k := \{\, z;\ \tilde{M}_k^{-2/(n-2)}(z - T_k e_n) \in \Omega_k \cap B_{r_k} \}$$

be the domain of $\tilde{v}_k$. Then $D_k \subset \mathbb{R}^n_+$. Set

$$\Sigma_\lambda := \{\, z \in D_k;\ |z| > \lambda \}.$$

Let $\tilde{V}$ be the limit of $\tilde{v}_k$ in $C^2_{\text{loc}}(\mathbb{R}^n_+)$:

$$\tilde{V}(z) = \Big(1 + \frac{A}{n(n-2)}\,|z - Te_n|^2\Big)^{-(n-2)/2}.$$

Then there exist λ_2 and λ_3 ($\lambda_2 < \lambda_3$), depending only on n, A and T, such that

$$\tilde{V} > \tilde{V}^{\lambda_2} \quad \text{in } \mathbb{R}^n_+ \setminus B_{\lambda_2}$$

and

$$\tilde{V} < \tilde{V}^{\lambda_3} \quad \text{in } \mathbb{R}^n_+ \setminus B_{\lambda_3}. \tag{2.35}$$

We shall employ the method of moving spheres to compare $\tilde{v}_k$ and $\tilde{v}_k^\lambda$ on Σ_λ for $\lambda \in [\lambda_2, \lambda_3]$.

We use the uniform convergence of $\tilde{v}_k$ to $\tilde{V}$ to assert that, for any fixed $R > 1$,

$$\tilde{v}_k(y) > \tilde{v}_k^{\lambda_2}(y), \quad y \in \Sigma_{\lambda_2} \cap B_R. \tag{2.36}$$

For R large we have, with $a_1 = (n(n-2)/A)^{(n-2)/2}$,

$$\tilde{v}_k(y) \geq (a_1 - \epsilon/5)\,|y|^{2-n} \quad \text{on } \partial B_R \cap \mathbb{R}^n_+$$

and

$$\tilde{v}_k^{\lambda_0}(y) \leq (a_1 - 2\epsilon/5)\,|y|^{2-n}, \quad |y| > \lambda_2.$$

To prove $\tilde{v}_k > \tilde{v}_k^{\lambda_2}$ in $\Sigma_{\lambda_2} \setminus B_R$, we compare $\tilde{v}_k$ with

$$w = (a_1 - 3\epsilon/10)\,|y - A_1 e_n|^{2-n}$$

where $A_1 = \frac{1}{n-2}\, c_h\, a_1^{2/(n-2)}$. For R chosen sufficiently large we have

$$w \geq \tilde{v}_k^{\lambda_2} \quad \text{in } \Sigma_{\lambda_2} \setminus B_R, \quad \text{and} \quad \tilde{v}_k > w \quad \text{on } \partial B_R \cap \Sigma_{\lambda_2}.$$

To compare $\tilde{v}_k$ and w over $\Sigma_{\lambda_2} \setminus B_R$, it is easy to see that $\tilde{v}_k > w$ on $\partial B_R \cap \Sigma_{\lambda_2}$ and $\partial\Sigma_{\lambda_2} \setminus (B_R \cup \{z_n \geq 0\})$. Since $\tilde{v}_k - w$ is super-harmonic, the only thing we need to prove is, on $\partial\mathbb{R}^n_+ \setminus B_{\lambda_2}$,

$$\partial_n(\tilde{v}_k - w) < \xi_k(\tilde{v}_k - w), \quad \text{on } \{z_n = 0\} \setminus B_R. \tag{2.37}$$

for some positive function ξ_k. Then standard maximum principle can be used to conclude that $\tilde{v}_k > w_k$ on $\Sigma_{\lambda_0} \setminus B_R$.

To obtain (2.37) first for $\tilde{v}_k$ we use GH_2 to have

$$\partial_n \tilde{v}_k \le c_h\, \tilde{v}_k^{n/(n-2)}, \quad \{z_n = 0\}.$$

On the other hand, by the choice of A_1 we verify easily that

$$\partial_n w > c_h\, w^{n/(n-2)}, \quad \text{on} \ \{z_n = 0\}.$$

Thus (2.37) holds from mean value theorem. We have proved that the moving sphere process can start at $\lambda = \lambda_2$:

$$\tilde{v}_k > \tilde{v}_k^{\lambda_2} \quad \text{in } \Sigma_{\lambda_2}.$$

Let $\bar{\lambda}$ be the critical moving sphere position:

$$\bar{\lambda} := \sup\big\{\lambda \in [\lambda_2, \lambda_3] \,:\, \tilde{v}_k > \tilde{v}_k^{\mu} \ \text{ in } \ \Sigma_\mu, \ \forall \mu \in (0, \lambda)\big\}.$$

As in **Case one** we shall prove that $\bar{\lambda} = \lambda_3$, thus getting a contradiction to (2.35). To this end we let

$$w_{\lambda,k} = \tilde{v}_k - \tilde{v}_k^{\lambda}.$$

To derive the equation for $w_{\lambda,k}$, we first recall from (2.19) and the definition of $\tilde{v}_k$ that

$$\left\{\begin{array}{ll} \Delta \tilde{v}_k(z) + l_k^{-(n+2)/(n-2)}\, g(l_k \tilde{v}_k) = 0, & z \in \tilde{\Omega}_k, \\ \dfrac{\partial \tilde{v}_k}{\partial z_n} = l_k^{-n/(n-2)}\, h(l_k \tilde{v}_k), & \{z_n = 0\} \cap \partial \tilde{\Omega}_k. \end{array}\right. \tag{2.38}$$

where $l_k = \sigma_k^{-(n-2)/2} \tilde{M}_k$. Correspondingly $\tilde{v}_k^{\lambda}$ satisfies

$$\left\{\begin{array}{ll} \Delta \tilde{v}_k^{\lambda} + \Big(\dfrac{\lambda}{|z|}\Big)^{n+2} l_k^{-(n+2)/(n-2)}\, g\Big(l_k \Big(\dfrac{|z|}{\lambda}\Big)^{n-2} \tilde{v}_k^{\lambda}(z)\Big) = 0, & \text{in } \tilde{\Sigma}_\lambda, \\ \dfrac{\partial \tilde{v}_k^{\lambda}}{\partial z_n} = \Big(\dfrac{\lambda}{|z|}\Big)^{n} l_k^{-n/(n-2)}\, h\Big(l_k \Big(\dfrac{|z|}{\lambda}\Big)^{n-2} \tilde{v}_k^{\lambda}(z)\Big) & \text{on } \partial\Sigma_\lambda \cap \{z_n = 0\}. \end{array}\right. \tag{2.39}$$

Let O_λ be defined as before. Then in O_λ we have, by GH_1,

$$\begin{aligned} &\Big(\frac{\lambda}{|z|}\Big)^{n+2} l_k^{-(n+2)/(n-2)}\, g\Big(l_k \Big(\frac{|z|}{\lambda}\Big)^{n-2} \tilde{v}_k^{\lambda}(z)\Big) \\ &\qquad\qquad \le (v_k l_k)^{-(n+2)/(n-2)}\, g(l_k v_k)\, (v_k^{\lambda})^{(n+2)/(n-2)}, \end{aligned}$$

and on $\partial O_\lambda \cap \{z_n = 0\}$,

$$\Big(\frac{\lambda}{|z|}\Big)^{n} l_k^{-n/(n-2)}\, h\Big(l_k \Big(\frac{|z|}{\lambda}\Big)^{n-2} \tilde{v}_k^{\lambda}(z)\Big) \ge (l_k v_k)^{-\frac{n}{n-2}}\, h(l_k v_k)\, (v_k^{\lambda})^{\frac{n}{n-2}},$$

The inequalities above yield

$$\begin{aligned} &\Delta w_{\lambda,k} + \xi_{1,k}\, w_{\lambda,k} \le 0, && \text{in } \ O_\lambda, \\ &\partial_n w_{\lambda,k} \le \xi_{2,k}\, w_{\lambda,k}, && \text{on } \ \partial O_\lambda \cap \{z_n = 0\}, \end{aligned}$$

where $\xi_{1,k} > 0$ and $\xi_{2,k}$ are continuous functions obtained from mean value theorem. It is easy to see that the moving sphere argument can be employed to prove that $\bar{\lambda} = \lambda_3$, which leads to a contradiction from the limiting function $\tilde{V}$. Thus Lemma 2.6 is established. □

Lemma 2.6 guarantees that on each radius $R_k \le r \le \frac{1}{2}\tilde{M}_k$ the minimum of v_k is always comparable to $|z|^{2-n}$. Re-scaling v_k as $r^{(n-2)/2}v_k(rz)$ we see the spherical Harnack inequality holds by the GH_2 and GH_3. Thus Proposition 2.4 is established in **Case Two** as well. □

Lemma 2.8. *Let $\{u_k\}$ be a sequence of solutions of (1.1) and $q_k \to q \in \overline{B_1^+}$ be a sequence of points in Z_k. Then there exist $C > 0$, $r_2 > 0$ independent of k and $R_k \to \infty$ such that*

$$u_k(q_k)u_k(x) \ge C|x-q_k|^{2-n} \quad \text{in} \quad u_k(q_k)^{-2/(n-2)}R_k \le |x-q_k| \le r_2, \quad x \in B_3^+.$$

Proof. We consider two cases:

Case one: $u_k(q_k)^{2/(n-2)}q_{kn} \to \infty$**.**

Let $M_k = u_k(q_k)$ and

$$v_k(y) = {M_k}^{-1}u_k({M_k}^{-2/(n-2)}y + q_k), \tag{2.40}$$

for $y \in \Omega_k := \{y : M_k^{-2/(n-2)}y + q_k \in B_3^+\}$. In this case, v_k converges uniformly to

$$V(y) = \Big(1 + \frac{A}{n(n-2)}\,|y|\Big)^{-(n-2)/2} \tag{2.41}$$

over all compact subsets of $\mathbb{R}^n$. For $\epsilon > 0$ small we let

$$\phi = \Big(\frac{n(n-2)}{A} - \epsilon\Big)^{(n-2)/2}\big(|y|^{2-n} - M_k^{-2}\big), \quad |y| \le M_k^{2/(n-2)}$$

on $|y| \ge R$, where $R > 1$ is chosen so that $v_k > \phi$ on ∂B_R. By direct computation we have

$$\frac{\partial \phi}{\partial y_n} > N_k\,\phi^{n/(n-2)}, \quad \text{on } \{y_n = -q_{kn}M_k\}$$

for some $N_k \to \infty$. It is easy to see that $v_k \ge \phi$ on $\partial\Omega_k \setminus \{y_n = -M_k^{2/(n-2)}q_{kn}\}$. On $\{y_n = -M_k^{2/(n-2)}q_{kn}\}$ we have

$$\partial_{y_n}(v_k - \phi) \le c_h(v_k - \phi).$$

Thus standard maximum principle implies $v_k \ge \phi$ on Ω_k. Lemma 2.8 is established in this case.

Case two: $M_k^{2/(n-2)}q_{kn} \le C$.

Let v_k be defined as in (2.40). In this case the boundary condition is written as

$$\partial_{y_n}v_k = \big(M_k^{-2/(n-2)}v_k\big)^{-n/(n-2)}\,h\big(M_k^{-2/(n-2)}v_k\big)v_k^{n/(n-2)},$$

on $\{y_n = -M_k^{2/(n-2)}q_{kn}\}$.

The function v_k converges to V of (2.41) over all compact subsets of $\{y_n \geq -T\}$, where

$$T = \lim_{k\to\infty} M_k^{2/(n-2)} q_{kn}.$$

For R large and $\epsilon > 0$ small, both independent of k, we have

$$v_k(y) \geq \Big(\frac{n(n-2)}{A} - \epsilon\Big)^{(n-2)/2} |y|^{2-n}, \quad |y| = R.$$

In B_R^{-T} we have the uniform convergence of v_k to V_1. Our goal is to prove that v_k is bounded below by $O(1)\,|y|^{2-n}$ outside B_R. To this end let

$$w(y) = \Big(\frac{n(n-2)}{A} - 2\epsilon\Big)^{(n-2)/2} |y - A_1 e_n|^{2-n}$$

where

$$A_1 = c_h\Big(\frac{n(n-2)}{A}\Big) - T.$$

Then it is easy to check that

$$\frac{\partial w}{\partial y_n} > c_h w(y)^{n/(n-2)}, \quad \text{on } \big\{y_n = -M_k^{2/(n-2)} q_{kn}\big\}.$$

By choosing R larger if needed we have

$$v_k(y) > \Big(\frac{n(n-2)}{A} - \epsilon\Big)^{(n-2)/2} |y|^{2-n} > w(y), \quad |y| = R, \quad y \in \mathbb{R}^n_+.$$

Then it is easy to apply maximum principle to prove $v_k > w$ in $\Omega_k \backslash B_R$. Lemma 2.8 is established. □

Let $q_1^k \in Z_k$ and q_2^k be its nearest or almost nearest sequence in Z_k:

$$|q_2^k - q_1^k| = \big(1 + o(1)\big)\, d\big(q_1^k, Z_k \setminus \{q_1^k\}\big).$$

Lemma 2.9. *There exists $C > 0$ independent of k such that*

$$\frac{1}{C}\, u_k(q_1^k) \leq u_k(q_2^k) \leq C\, u_k(q_1^k).$$

Proof. Let $\sigma_k = d(q_1^k, Z_k \setminus \{q_1^k\})$ and

$$\tilde{u}_k(y) = \sigma_k^{(n-2)/2} u_k(q_1^k + \sigma_k y).$$

We use e_k to denote the image of q_2^k after scaling (so $|e_k| \to 1$). Then in B_1, $\tilde{u}_k(x) \sim \tilde{u}_k(0)^{-1}|x|^{2-n}$ for $|x| \sim 1/2$. On one hand, for $|x| = 1/2$ we have, by Lemma 2.8 applied to e_k,

$$\tilde{u}_k(0)^{-1}\Big(\frac{1}{2}\Big)^{2-n} \geq C\, \tilde{u}_k(e_k)^{-1}$$

which is just $u_k(q_1^k) \leq C u_k(q_2^k)$.

On the other hand, the same moving sphere argument can be applied to u_k near q_2^k with no difference. The Harnack type inequality gives

$$\max_{B(q_2^k,1/4)\cap B_3^+} u_k \min_{B(q_2^k,1/2)\cap B_3^+} u_k \le C.$$

Using

$$\max_{B(q_2^k,1/4)\cap B_3^+} u_k \ge u_k(q_2^k), \quad \text{and} \quad \min_{B(q_2^k,1/2)\cap B_3^+} u_k \ge \min_{B(q_1^k,\sigma_k)\cap B_3^+} u_k,$$

we have

$$\tilde{u}_k(e_k)\,\tilde{u}_k(0)^{-1} \le C. \tag{2.42}$$

Thus (2.42) gives $u_k(q_2^k) \le C u_k(q_1^k)$. Lemma 2.9 is established. □

Remark 2.10. Proposition 2.4 is not needed in the proof of Lemma 2.9.

The following lemma is concerned with Pohozaev identity that can be verified by direct computation.

Lemma 2.11. *Let u solve*

$$\begin{cases} \Delta u + g(u) = 0, & \text{in } B_\sigma^+, \\ \dfrac{\partial u}{\partial x_n} = h(u) & \text{on } \partial' B_\sigma^+. \end{cases}$$

Then

$$\int_{\partial' B_\sigma^+} h(u)\Big(\sum_{i=1}^{n-1} x_i\partial_i u + \frac{n-2}{2}u\Big) + \int_{B_\sigma^+}\Big(n\,G(u) - \frac{n-2}{2}g(u)u\Big) \tag{2.43}$$
$$= \int_{\partial'' B_\sigma^+}\Big(\sigma\big(G(u) - \frac{1}{2}|\nabla u|^2 + (\partial_\nu u)^2\big) + \frac{n-2}{2}u\,\partial_\nu u\Big)$$

where $G(s) = \int_0^s g(t)dt$, ν stands for the outer normal vector of the domain.

Now we finish the proof of Theorem 2.3.

Recall that $\sigma_k = (1+o(1))|q_1^k - q_2^k|$. We prove by contradiction. Suppose $\sigma_k \to 0$. Let $\tilde{M}_k = \tilde{u}_k(0)$. We claim that

$$\tilde{M}_k\tilde{u}_k(y) \to a|y|^{2-n} + b(y) \ \text{ in } \ C^2_{\text{loc}}(B_{3/4}\cap\tilde{\Omega}_k\setminus\{0\}), \text{ with } a>0,\ b(0)>0 \tag{2.44}$$

where $\tilde{\Omega}_k = \{y;\ \sigma_k y + q_1^k \in B_3^+\}$.

Proof of (2.44). As usual we consider the following two cases:

Case one in (2.44): $\lim_{k\to\infty} q_{1n}^k\,\tilde{M}_k^{2/(n-2)} \to \infty$.

Let

$$T_k = \tilde{M}_k^{2/(n-2)} q_{1n}^k.$$

Recall the equation for $\tilde{u}_k$ is (2.18). Multiplying $\tilde{M}_k$ on both sides and letting $k\to\infty$ we see from the assumptions of g and h that $\tilde{M}_k\tilde{u}_k \to h$ in $C^2_{\text{loc}}(B_1\setminus\{0\})$ where h is a harmonic function defined in $B_1\setminus\{0\}$.

Thus,

$$h(y) = a|y|^{2-n} + b(y)$$

for some harmonic function $b(y)$ in B_1. From the pointwise estimate in Lemma 2.8 we see that $a > 0$. Given any $\epsilon > 0$, we compare $\tilde{u}_k$ and

$$w_k := (a - \epsilon)\left(|y|^{2-n} - R_k^{2-n}\right)$$

on $|y| \le R_k$. Here $R_k \to \infty$ is less than T_k. Observe that $\tilde{u}_k > w_k$ on ∂B_{R_k} and $|y| = \epsilon_1$ for ϵ_1 sufficiently small. Thus $\tilde{u}_k > w_k$ by the maximum principle. Letting $k \to \infty$, we have, in B_1,

$$a|y|^{2-n} + b(y) \ge (a - \epsilon)|y|^{2-n}, \quad B_1 \setminus B_{\epsilon_1}.$$

Then let $\epsilon \to 0$, which implies $\epsilon_1 \to 0$ we have $b(y) \ge 0$ in B_1. Next we claim that $b(0) > 0$ because by Lemma 2.8 and Lemma 2.9 we have

$$a|y|^{2-n} + b(y) \ge a_1|y - e|^{2-n} \quad \text{in } B_1$$

for some $a_1 > 0$, where $e = \lim_{k\to\infty} e_k$. Thus $b(y) > 0$ when y is close to e, which leads to $b(0) > 0$. (2.44) is established in **Case one**.

Case two in (2.44): $\lim_{k\to\infty} q_{1n}^k\, \tilde{M}_k^{2/(n-2)} \to T < \infty$.

Again we first have $\tilde{M}_k\tilde{u}_k \to h$ in $C^2_{\mathrm{loc}}(B_1^{-T} \setminus \{0\})$ and h is of the form

$$h(y) = a|y|^{2-n} + b(y), \quad y_n \ge -T.$$

To prove $b(y) \ge 0$ we compare, for fixed $\epsilon > 0$, $\tilde{M}_k\tilde{u}_k$ with

$$w_k(y) = (a - \epsilon)\left(|y - b_k e_n|^{2-n} - (R_k - 1)^{2-n}\right)$$

where $b_k \to 0$ and $R_k \to \infty$ are chosen to satisfy

$$(n-2)\, b_k\, R_k^{-2} > c_0\sigma_k, \quad c_0 = \sup_{0<s\le 1} s|h(s)|$$

and

$$(n-2)b_k > c_h\, \tilde{M}_k^{-2/(n-2)} a^{2/(n-2)}.$$

It is easy to see that such b_k and R_k can be found easily. Let $h_k = \tilde{M}_k\tilde{u}_k$, and $\partial'\Omega_k = \partial\Omega_k \cap \{y_n = -T_k\}$. We divide $\partial'\Omega_k$ into two parts:

$$E_1 = \left\{z \in \partial'\Omega_k;\ \tilde{u}_k(z)\, \sigma_k^{-(n-2)/2} \ge 1\right\}, \quad E_2 = \partial'\Omega_k \setminus E_1.$$

Then, by the assumptions on h,

$$\partial_n \tilde{h}_k \le \begin{cases} c_0\, \sigma_k\, h_k, & x \in E_2, \\ c_h\, \tilde{M}_k^{-2/(n-2)}\, h_k^{n/(n-2)}, & x \in E_1, \end{cases}$$

With the choice of b_k and R_k it is easy to verify that

$$\partial_n w_k \geq \max\left\{c_0\, \sigma_k\, w_k,\ c_h \tilde{M}_k^{-2/(n-2)} w_k^{n/(n-2)}\right\} \text{ on } \partial'\Omega_k \cap B_{R_k}.$$

Thus standard maximum principle can be applied to prove $h_k \geq w_k$ on $\Omega_k \cap B_{R_k}$. Letting $k \to \infty$ first and $\epsilon \to 0$ next we have $b(y) \geq 0$ in $B_1 \cap \{y_n \geq -T\}$. Then by Proposition 2.9 we see that $b(y) > 0$ when y is close to e, the limit of e_k. The fact $\partial_n b = 0$ at 0 implies $b(0) > 0$. Claim (2.44) is proved in both cases. □

Finally to finish the proof of Theorem 2.3 we derive a contradiction from each of the following two cases:

Case one: $\lim_{k\to\infty} \tilde{M}_k^{2/(n-2)} q_{1n}^k > 0$**.**

We use the following Pohozaev identity on B_σ for $\sigma < \lim_{k\to\infty} \tilde{M}_k^{2/(n-2)} q_{1n}^k$:

$$\int_{B_\sigma} \Big(n\, G_k(\tilde{u}_k) - \frac{n-2}{2}\, \tilde{u}_k\, g_k(\tilde{u}_k)\Big) = \int_{\partial B_\sigma} \Big(\sigma\,\big(G_k(\tilde{u}_k) - \frac{1}{2}\,|\nabla \tilde{u}_k|^2 + |\partial_\nu \tilde{u}_k|^2\big) + \frac{n-2}{2}\, \tilde{u}_k\, \partial_\nu \tilde{u}_k\Big), \tag{2.45}$$

where

$$g_k(s) = \sigma_k^{(n+2)/2} g\big(\sigma_k^{-(n-2)/2} s\big), \quad G(t) = \int_0^t g(s)\, ds, \quad G_k(s) = \sigma_k^n G\big(\sigma_k^{-(n-2)/2} s\big).$$

First we claim that for $s > 0$,

$$G_k(s) \geq \frac{n-2}{2n}\, s\, g_k(s). \tag{2.46}$$

Indeed, writing $g(t) = c(t)\, t^{(n+2)/(n-2)}$, we see from GH_1 that $c(t)$ is a non-increasing function, thus

$$\begin{aligned} G_k(s) = \sigma_k^n\, G\big(\sigma_k^{-(n-2)/2} s\big) &= \sigma_k^n \int_0^{\sigma_k^{-(n-2)/2} s} c(t)\, t^{(n+2)/(n-2)}\, dt \\ &\geq \sigma_k^n\, c\big(\sigma_k^{-(n-2)/2} s\big) \int_0^{\sigma_k^{-(n-2)/2} s} t^{(n+2)/(n-2)}\, dt \\ &= \frac{n-2}{2n}\, c\big(\sigma_k^{-(n-2)/2} s\big)\, s^{2n/(n-2)} = \frac{n-2}{2n}\, s\, g_k(s). \end{aligned}$$

Replacing s by $\tilde{u}_k$ we see that the left hand side of (2.45) is non-negative. Next we prove that

$$\lim_{k\to\infty} \tilde{M}_k^2 \int_{\partial B_\sigma} \Big(\sigma\big(G_k(\tilde{u}_k) - \frac{1}{2}\,|\nabla \tilde{u}_k|^2 + |\partial_\nu \tilde{u}_k|^2\big) + \frac{n-2}{2}\, \tilde{u}_k\, \partial_\nu \tilde{u}_k\Big) < 0 \tag{2.47}$$

for $\sigma > 0$ small. Clearly after (2.47) is established we obtain a contradiction to (2.45). To this end first we prove that

$$\tilde{M}_k^2\, G_k(\tilde{u}_k) = o(1). \tag{2.48}$$

Indeed, by GH_1 and GH_3,

$$G_k(\tilde u_k) = \sigma_k^n \int_0^{\sigma_k^{-(n-2)/2}\tilde u_k} g(t)\,dt$$
$$\le \begin{cases} \sigma_k^n \displaystyle\int_0^{\sigma_k^{-(n-2)/2}\tilde u_k} c\,t\,dt, & \text{if } \sigma_k^{-(n-2)/2}\tilde M_k^{-1} \le 1,\\ \sigma_k^n\Big(\displaystyle\int_0^1 c\,t\,dt + \int_1^{\sigma_k^{-(n-2)/2}\tilde u_k} c\,t^{(n+2)/(n-2)}\,dt\Big), & \text{if } \sigma_k^{-(n-2)/2}\tilde M_k^{-1} > 1.\end{cases}$$

Therefore,

$$G_k(\tilde u_k) \le \begin{cases} C\,\sigma_k^2\,\tilde M_k^{-2}, & \text{if } \sigma_k^{-(n-2)/2}\tilde M_k^{-1} \le 1,\\ C\,\sigma_k^n + C\,\tilde M_k^{-2n/(n-2)}, & \text{if } \sigma_k^{-(n-2)/2}\tilde M_k^{-1} > 1.\end{cases}$$

Clearly (2.48) holds in either case. Consequently we write the left hand side of (2.47) as

$$\int_{\partial B_\sigma}\Big(-\frac{1}{2}\sigma|\nabla h|^2 + \sigma|\partial_\nu h|^2 + \frac{n-2}{2}h\partial_\nu h\Big) + o(1),$$

where

$$h(y) = a|y|^{2-n} + b(y), \quad b(0) > 0,\ a > 0.$$

By direct computation we have

$$\int_{\partial B_\sigma}\Big(-\frac{1}{2}\sigma|\nabla h|^2 + \sigma|\partial_\nu h|^2 + \frac{n-2}{2}h\,\partial_\nu h\Big)$$
$$= \int_{\partial B_\sigma}\Big(-\frac{(n-2)^2}{a}\cdot b(0)\cdot\sigma^{1-n} + O(\sigma^{2-n})\Big)dS.$$

Thus (2.47) is verified when $\sigma > 0$ is small.

Case two: $\lim_{k\to\infty}\tilde M_k^{2/(n-2)}q_{1n}^k = 0$.

In this case we use the following Pohozaev identity on B_σ^+: let

$$h_k(s) = \sigma_k^{n/2}\,h\big(\sigma_k^{-(n-2)/2}s\big).$$

Then we have

$$\int_{\partial B_\sigma^+\cap\partial\mathbb{R}_+^n} h_k(\tilde u_k)\Big(\sum_{i=1}^{n-1}x_i\partial_i\tilde u_k + \frac{n-2}{2}\tilde u_k\Big) + \int_{B_\sigma^+}\Big(n\,G_k(\tilde u_k) - \frac{n-2}{2}g_k(\tilde u_k)\,\tilde u_k\Big)$$
$$= \int_{\partial B_\sigma^+\cap\mathbb{R}_+^n}\Big(\sigma\,\big(G_k(\tilde u_k) - \frac{1}{2}|\nabla\tilde u_k|^2 + (\partial_\nu\tilde u_k)^2\big) + \frac{n-2}{2}\tilde u_k\,\partial_\nu\tilde u_k\Big). \tag{2.49}$$

Multiplying $\tilde M_k^2$ on both sides and letting $k\to\infty$ we see by the same estimate as in **Case one** that the second term on the left hand side is non-negative, the right hand side is strictly negative.

The only term we need to consider is

$$\lim_{k\to\infty} \tilde{M}_k^2 \int_{\partial B_\sigma^+\cap\partial\mathbb{R}_+^n} h_k(\tilde{u}_k)\Big(\sum_{i=1}^{n-1} x_i\,\partial_i\tilde{u}_k + \frac{n-2}{2}\,\tilde{u}_k\Big).$$

Let $H(s)=\int_0^s h(t)\,dt$, then from integration by parts we have

$$\begin{aligned}(2.50)\quad &\int_{\partial B_\sigma^+\cap\partial\mathbb{R}_+^n} h_k(\tilde{u}_k)\Big(\sum_{i=1}^{n-1} x_i\,\partial_i\tilde{u}_k + \frac{n-2}{2}\,\tilde{u}_k\Big)\\ &= \int_{\partial B_\sigma\cap\partial\mathbb{R}_+^n} \sigma_k^{n-1} H\big(\sigma_k^{-(n-2)/2}\tilde{u}_k\big)\,\sigma\\ &\quad + \int_{\partial B_\sigma^+\cap\partial\mathbb{R}_+^n} \Big(-(n-1)\,\sigma_k^{n-1} H\big(\sigma_k^{-(n-2)/2}\tilde{u}_k\big) + \frac{n-2}{2}\,\tilde{u}_k\,h_k(\tilde{u}_k)\Big)dx'.\end{aligned}$$

For the first term on the right hand side of (2.50) we claim that

$$(2.51)\qquad \tilde{M}_k^2\,\sigma_k^{n-1} H\big(\sigma_k^{-(n-2)/2}\tilde{u}_k\big) = o(1) \quad \text{on } \partial B_\sigma.$$

Indeed, by GH_1 and GH_2,

$$\big|H\big(\sigma_k^{-(n-2)/2}\tilde{u}_k\big)\big| \le \begin{cases} \displaystyle\int_0^{\sigma_k^{-(n-2)/2}\tilde{u}_k} c\,t\,dt, & \text{if } \sigma_k^{-(n-2)/2}\tilde{u}_k \le 1,\\ \displaystyle\int_0^1 c\,t\,dt + \int_1^{\sigma_k^{-(n-2)/2}\tilde{u}_k} c\,t^{\frac{n}{n-2}}\,dt, & \text{if } \sigma_k^{-(n-2)/2}\tilde{u}_k > 1,\end{cases}$$

Using $\tilde{u}_k = O(1/\tilde{M}_k)$ on ∂B_σ we then have

$$\tilde{M}_k^2\,\sigma_k^{n-1}\big|H\big(\sigma_k^{-(n-2)/2}\tilde{u}_k\big)\big| \le \begin{cases} O(\sigma_k), & \text{if } \sigma_k^{-(n-2)/2}\tilde{u}_k \le 1,\\ O(\sigma_k) + O\big(\tilde{M}_k^{-2/(n-2)}\big), & \text{if } \sigma_k^{-(n-2)/2}\tilde{u}_k > 1.\end{cases}$$

Thus (2.51) is verified and the first term on the right hand side of (2.50) is $o(1)$.

Therefore we only need to estimate the last term of (2.50), which we claim is non-negative. Indeed, for $t>0$, we write $h(t)=b(t)\,t^{n/(n-2)}$ for some non-decreasing function b. Then we have

$$\begin{aligned}\sigma_k^{n-1} H\big(\sigma_k^{-(n-2)/2}s\big) &= \sigma_k^{n-1}\int_0^{\sigma_k^{-(n-2)/2}s} h(t)\,dt = \sigma_k^{n-1}\int_0^{\sigma_k^{-(n-2)/2}s} b(t)\,t^{n/(n-2)}\,dt\\ &\le \sigma_k^{n-1} b\big(\sigma_k^{-(n-2)/2}s\big)\int_0^{\sigma_k^{-(n-2)/2}s} t^{n/(n-2)}\,dt\\ &= \frac{n-2}{2n-2}\,b\big(\sigma_k^{-(n-2)/2}s\big)\,s^{(2n-2)/(n-2)} = \frac{n-2}{2n-2}\,h_k(s)\,s.\end{aligned}$$

Replacing s by $\tilde{u}_k$ in the above we see that the last term of (2.50) is non-negative. Thus there is a contradiction in (2.49) in **Case two** as well. Theorem 2.3 is established. □

2.3. Step three: the completion of the proof of Theorem 1.1

Let u_k be a sequence of blowup solutions of (1.1) that satisfies $\max_{B_2^+} u_k \to \infty$ and (2.1) (See the beginning part of the proof before Proposition 2.1). By Proposition 2.3 there are finite local maximum points $q_i^k \in \overline{B_1^+}$ for $i = 1, \ldots, L$, where L is independent of k and $\min\{q_i^k, q_j^k\} \geq 2\delta_0$ for all $q_i^k \neq q_j^k$ and some $\delta_0 > 0$ independent of k. In each $B^+(q_i^k, \delta_0)$ if we define

$$v_i^k(y) = u_k(q_i^k)^{-1} u_k\big(u_k^{-2/(n-2)}(q_i^k)\, y + q_i^k\big)$$

by the proof of Theorem 2.3 we have

$$v_i^k(y) \leq C(1+|y|)^{2-n}, \qquad \text{for} \quad |y| \leq \delta_0\, u_k(q_i^k)^{2/(n-2)}.$$

The corresponding estimate for u_k is

$$u_k(x) \leq C\, u_k(q_i^k)\big(1 + u_k(q_i^k)^{2/(n-2)}\, |x - q_i^k|\big)^{2-n}, \quad \text{in} \quad B^+(q_i^k, \delta_0). \tag{2.52}$$

Then direct computation shows that

$$\int_{B^+(q_i^k, \delta_0)} u_k^{2n/(n-2)} \leq C$$

for some $C > 0$ independent of k. By Proposition 2.1 we also see that $u_k \leq C$ in $B_2^+ \setminus (\cup_i B(q_i^k, \delta_0))$. Thus we have

$$\int_{B_{3/2}^+} u_k^{2n/(n-2)} \leq C. \tag{2.53}$$

Let ϕ be a radial and smooth function such that $\phi \equiv 1$ in B_1 and $\phi \equiv 0$ near $\partial B_{3/2}$. Moreover, $\phi \geq 0$. Multiplying $u_k \phi^2$ to both sides of (1.1), we have

$$\int_{\partial' B_{3/2}^+} h(u_k)\, u_k\, \phi^2 + \int_{B_{3/2}^+} \nabla u_k \nabla(u_k\, \phi^2) = \int_{B_{3/2}^+} g(u_k)\, u_k\, \phi^2.$$

Using Cauchy's inequality we obtain

$$\frac{1}{2}\int_{B_{3/2}^+} |\nabla u_k|^2 \phi^2 \leq 2\int_{B_{3/2}^+} u_k^2 |\nabla \phi|^2 + \int_{B_{3/2}^+} g(u_k)\, u_k\, \phi^2 - \int_{\partial' B_{3/2}^+} h(u_k)\, u_k\, \phi^2.$$

By (2.52) and the assumptions on g, h it is easy to see that the right hand side of the above is bounded by C independent of k. Therefore $\int_{B_1^+} |\nabla u_k|^2 \leq C$ for some $C > 0$ independent of k. A contradiction to (2.1). Theorem 1.1 is established. □

Remark 2.12. Finally we summarize some technical points in the proof of Theorem 1.1. In the proof of Proposition 2.1 the assumptions on g, h play a central role. The proof of Proposition 2.4 relies on the classification theorems of Caffarelli–Gidas–Spruck [1] and Li–Zhu [11] in an essential way. The main reason is equation (1.1) is not scaling invariant so we have to prove the decay rate of bubbling solutions in each blowup disk (Lemma 2.5 and Lemma 2.6). At the first glance the reader may feel that the proof of Theorem 2.3 is similar to corresponding theorems in [8] or [6]. Actually this is not the case. The main difference lies on the fact that (1.1) is a locally defined equation, while the equations in [8] or [6] are globally defined. This difference is particularly subtle when the interaction of bubbles is concerned. In [8] and [6] it is possible to find two bubbles closest to each other because the equations are globally defined. This is not possible for the local equation (1.1) and the approach in this article is significantly different. We mainly follow the line of proof in [9] for this part.

3. Proof of Theorem 1.2

If h is non-positive, the energy estimate follows from the Harnack inequality in a straight forward way. Indeed, let $G(x,y)$ be a Green's function on B_3^+ such that $G(x,y)=0$ if $x\in B_3^+, y\in\partial B_3^+\cap\mathbb{R}^n_+$ and $\partial_{y_n}G(x,y)=0$ for $x\in B_3^+, y\in \partial B_3^+\cap\partial\mathbb{R}^n_+$. It is easy to see that G can be constructed by adding the standard Green's function on B_3 its reflection over $\partial\mathbb{R}^n_+$. It is also immediate to observe that

$$G(x,y)\ge C_n|x-y|^{2-n},\quad x\in B_3^+,\quad y\in B_2^+.$$

Multiplying G on both sides of (1.1) and integrating by parts, we have

$$u(x)+\int_{\partial B_3^+\cap\partial\mathbb{R}^n_+} h(u(y))\,G(x,y)\,dS_y+\int_{\partial B_3^+\cap\mathbb{R}^n_+} u(y)\frac{\partial G(x,y)}{\partial\nu}\,dS_y = \int_{B_3^+} g(u(y))\,G(x,y)\,dy.$$

Here ν represents the outer normal vector of the domain. Using $h\le 0$ and $\partial_\nu G\le 0$, we have

$$u(x)\ge\int_{B_3^+} g(u(y))\,G(x,y)\,dy,\quad x\in B_3^+.$$

In particular, let $u(x_0)=\min_{\partial B_2^+}u$. Then $|x_0|=2$, thus

$$C\ge\max_{B_1^+}u\cdot\min_{B_2^+}u\ge\int_{B_{3/2}^+} g(u(y))\,u(y)\,G(x_0,y)\,dy\ge C\int_{B_{3/2}^+} g(u)u\,dy.$$

Therefore we have obtained the bound on $\int_{B_{3/2}^+} g(u)u\,dy$. To obtain the bound on $\int_{B_1^+}|\nabla u|^2$, we use a cut-off function η which is 1 on B_1^+ and is 0 on $B_2^+\setminus B_{3/2}^+$ and $|\nabla\eta|\le C$. Multiplying $u\eta^2$ to both sides of (1.1) and using integration by parts and Cauchy inequality we obtain the desired bound on $\int_{B_1^+}|\nabla u|^2$. Theorem 1.2 is established. □

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