Leavitt path algebras with at most countably many irreducible representations

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Abstract. Let E be an arbitrary directed graph with no restrictions on the number of vertices and edges and let K be any field. We give necessary and sufficient conditions for the Leavitt path algebra $L_K(E)$ to be of countable irreducible representation type, that is, we determine when $L_K(E)$ has at most countably many distinct isomorphism classes of simple left $L_K(E)$-modules. It is also shown that $L_K(E)$ has finitely many isomorphism classes of simple left modules if and only if $L_K(E)$ is a semi-artinian von Neumann regular ring with finitely many ideals. Equivalent conditions on the graph E are also given. Examples are constructed showing that for each (finite or infinite) cardinal κ there exists a Leavitt path algebra $L_K(E)$ having exactly κ distinct isomorphism classes of simple right modules.

1. Introduction

The notion of Leavitt path algebras was introduced and initially studied in [1] and [7] as algebraic analogues of graph C*-algebras and the study of their various ring-theoretic properties has been the subject of a series of papers in recent years (see, e.g., [1]–[9], [14], and [17]). In [12], Goncalves and Royer indicated a method of constructing various representations of a Leavitt path algebra $L_K(E)$ over a graph E by using the concept of algebraic branching systems. Expanding this, Chen [11] studied special types of irreducible representations of $L_K(E)$ by using the sinks as well as the infinite paths which are not tail-equivalent in the graph E and he noted that these can also be considered as algebraic branching systems. For additional ways of constructing irreducible representations of $L_K(E)$ see [8].

In this paper we investigate the Leavitt path algebras $L_K(E)$ which are of countable irreducible representation type, that is, $L_K(E)$ having at most countably many distinct isomorphism classes of simple left/right $L_K(E)$-modules. For short, we call a Leavitt path algebra with this property as CIRT. In many of the past

investigations describing various algebraic properties of a Leavitt path algebra $L_K(E)$, the individual nature of the field K did not seem to play any role at all. In this context, it may be of of some interest to note that, when $L_K(E)$ is CIRT and the graph E contains cycles, K must necessarily be a countable field. Our structure theorem states that $L_K(E)$ is CIRT if and only if it is the union of a smooth ascending chain of countable length consisting of graded ideals

$$0 < I_1 < \cdots < I_\alpha < I_{\alpha+1} < \cdots \quad (\alpha < \tau),$$

where τ is a countable ordinal and, for each $0 \le \alpha < \tau$, $I_{\alpha+1}/I_\alpha$ is a direct sum of at most countably many matrix rings over K and/or $K[x, x^{-1}]$. Moreover, the field K is countable whenever the graph E contains cycles (equivalently, $K[x, x^{-1}]$ occurs as a direct factor in $I_{\alpha+1}/I_\alpha$ for some α).

We also show that $L_K(E)$ will have at most finitely many non-isomorphic simple left/right $L_K(E)$-modules if and only if $L_K(E)$ is a semi-artinian von Neumann regular ring with at most finitely many ideals. In particular, when E is a finite graph, then $L_K(E)$ has this property exactly when $L_K(E)$ is an artinian semisimple ring (equivalently, the graph E is acyclic). We also construct, for each arbitrary (finite or infinite) cardinal κ, a Leavitt path algebra for which the cardinality of the distinct isomorphism classes of simple right modules is exactly κ.

2. Preliminaries

For the general notation, terminology and results on Leavitt path algebras, we refer the reader to [1], [3] and [17]. We just give a short outline of some of the needed concepts. A (directed) graph $E = (E^0, E^1, r, s)$ consists of two sets E^0 and E^1 together with maps $r, s : E^1 \to E^0$. All the graphs E that we consider here are arbitrary with no restrictions on the number of vertices and the number of edges emitted by a vertex. Also K stands for an arbitrary field. For each $e \in E^1$, we call e^* a *ghost* edge. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. The set of all vertices on the path μ is denoted by μ^0. A vertex v in E is said to be *regular* if $0 < |s^{-1}(v)| < \infty$. A *singular* vertex is a vertex which is not regular.

Given an arbitrary graph E and a field K, the *Leavitt path K-algebra* $L_K(E)$ is defined to be the K-algebra generated by a set $\{v : v \in E^0\}$ of pairwise orthogonal idempotents together with a set of variables $\{e, e^* : e \in E^1\}$ which satisfy the following conditions:

(1) $s(e) e = e = e r(e)$ for all $e \in E^1$;

(2) $r(e) e^* = e^* = e^* s(e)$ for all $e \in E^1$;

(3) (The "CK-1 relations") For all $e, f \in E^1$, $e^* e = r(e)$ and $e^* f = 0$ if $e \ne f$.

(4) (The "CK-2 relations") For every regular vertex $v \in E^0$,

$$v = \sum_{e \in E^1, s(e)=v} e\, e^*.$$

A path $\mu = e_1 \ldots e_n$ in E is *closed* if $r(e_n) = s(e_1)$, in which case μ is said to be based at the vertex $s(e_1)$. The closed path μ is called a *cycle* if $s(e_i) \neq s(e_j)$ for every $i \neq j$. A graph E is said to satisfy *condition* (K) provided no vertex $v \in E^0$ is the base of precisely one simple closed path, i.e., either no simple closed path is based at v, or at least two are based at v. An *exit* for a path $\mu = e_1 \ldots e_n$ is an edge e such that $s(e) = s(e_i)$ for some i and $e \neq e_i$.

If there is a path from vertex u to a vertex v, we write $u \geq v$. A subset D of vertices is said to be *downward directed* if for any $u, v \in D$, there exists a $w \in D$ such that $u \geq w$ and $v \geq w$. A subset H of E^0 is called *hereditary* if, whenever $v \in H$ and $w \in E^0$ satisfy $v \geq w$, then $w \in H$. A hereditary set is *saturated* if, for any regular vertex v, $r(s^{-1}(v)) \subseteq H$ implies $v \in H$.

For any vertex v, the *tree* of v is $T(v) = \{w : v \geq w\}$. We say there is a bifurcation at a vertex v, if v emits more than one edge. In a graph E, a vertex v is called a *line point* if there is no bifurcation or a cycle based at any vertex in $T(v)$. Thus, if v is a line point, there will be a single finite or infinite line segment μ starting at v (μ could just be v) and any other path α with $s(\alpha) = v$ will just be an initial sub-segment of μ. It was shown in [9] that v is a line point in E if and only if $vL_K(E)$ (and likewise $L_K(E)v$) is a simple right (left) ideal. Moreover, the ideal generated by all the line points in E is the socle of $L_K(E)$. If v is a line point, then it is clear that any $w \in T(v)$ is also a line point.

We shall be using the following concepts and results from [17] in our investigation. Although it is assumed in [17] that all the graphs are countable, i.e. have at most countably many vertices and edges, an examination of the proofs shows that this assumption is not needed at all. A *breaking vertex* of a hereditary saturated subset H is an infinite emitter $w \in E^0 \backslash H$ (that is, $s^{-1}(w)$ is an infinite set) with the property that $1 \leq |s^{-1}(w) \cap r^{-1}(E^0 \backslash H)| < \infty$. The set of all breaking vertices of H is denoted by B_H. For any $v \in B_H$, v^H denotes the element $v - \sum_{s(e)=v, r(e) \notin H} e\, e^*$. Given a hereditary saturated subset H and a subset $S \subseteq B_H$, (H, S) is called an *admissible pair*. Given an admissible pair (H, S), $I_{(H,S)}$ denotes the ideal generated by $H \cup \{v^H : v \in S\}$. It was shown in [17] that the graded ideals of $L_K(E)$ are precisely the ideals of the form $I_{(H,S)}$ for some admissible pair (H, S). Moreover, $L_K(E)/I_{(H,S)} \cong L_K(E \backslash (H, S))$. Here $E \backslash (H, S)$ is the *quotient graph of* E in which $(E \backslash (H, S))^0 = (E^0 \backslash H) \cup \{v' : v \in B_H \backslash S\}$ and $(E \backslash (H, S))^1 = \{e \in E^1 : r(e) \notin H\} \cup \{e' : e \in E^1, r(e) \in B_H \backslash S\}$ and r, s are extended to $(E \backslash (H, S))^0$ by setting $s(e') = s(e)$ and $r(e') = r(e)'$.

If $p = e_1 e_2 \cdots e_n \cdots$ is an infinite path where the e_i are edges, then for any positive integer n, let $\tau_{\leq n}(p) = e_1 e_2 \cdots e_n$ and $\tau_{>n}(p) = e_{n+1} e_{n+2} \cdots$. Two infinite paths p and q are said to be *tail equivalent*, in symbols, $p \sim q$, if there exist positive integers m and n such that $\tau_{>n}(p) = \tau_{>m}(q)$. Then $\sim$ is an equivalence relation. Given an equivalence class of infinite paths $[p]$, Chen [11] defines an $L_K(E)$-module

action on the linear span $V_{[p]}$ of the equivalence class $[p]$ of p and shows that $V_{[p]}$ becomes a simple left $L_K(E)$-module. He further shows that, for infinite paths p and q, $V_{[p]} \cong V_{[q]}$ if and only if $p \sim q$.

3. Leavitt path algebras of countable irreducible representation type

Throughout this and the following sections, L will denote the Leavitt path algebra $L_K(E)$. In this section, we give a complete description of the Leavitt path algebra L which has at most countably many distinct isomorphism classes of simple left/right L-modules. For short, we call a Leavitt path algebra with this property as CIRT.

The proof of our main theorem is derived from the following series of Propositions containing the necessary conditions for L to be CIRT.

Proposition 3.1. *If L is CIRT, then distinct cycles in E must be disjoint.*

Proof. We may assume that E contains cycles, since there is nothing to prove if E is acyclic. Suppose that there are two different cycles g, h based at the same vertex v. Consider the infinite path $ggg \cdots$ which, for convenience, we write as $p = g_1 g_2 g_3 \cdots$ indexed by the set **P** of positive integers where $g_i = g$ for all i. Now for every subset S of **P**, define an infinite path p_S by replacing g_i by h if and only if $i \in S$. Observe that this gives an uncountable family of pairwise different infinite paths on E. Denote by Y this family of infinite paths. Now, if $q = \beta_1 \beta_2 \cdots$ is a path in Y, then the paths in Y which are tail-equivalent to q have the form $\gamma_1 \cdots \gamma_j \beta_{i+1} \beta_{i+2} \beta_{i+3} \cdots$, where i, j are non-negative integers and $\gamma_1, \ldots, \gamma_j \in \{g, h\}$. This is a countable family and so each tail-equivalence class contains countably many infinite paths in Y. Consequently, there are uncountably many distinct tail equivalence classes of paths in Y. By Chen [11], L then has uncountably many non-isomorphic simple left/right L-modules. This contradicts the fact that L is a CIRT. Hence no two cycles in E have a common vertex. $\square$

Proposition 3.2. *If the graph E contains cycles and L is CIRT, then the field K must be countable.*

Proof. Consider a cycle c based at a vertex v in E. Then $H = \{u \in E^0 : u \ngeq v\}$ is a hereditary saturated set and $E^0 \backslash H = M(v)$ is downward directed. By Proposition 3.1, distinct cycles have no common vertex. Hence c is a cycle without exits in $M(v)$. By Theorems 3.12 and 4.3 of [14], each irreducible polynomial $g(x) \in K[x, x^{-1}]$ then gives rise to a primitive ideal P_g of L, generated by $I_{(H, B_H)} \cup \{g(c)\}$ (and a corresponding simple module having P_g as its annihilator). Since L is CIRT, $K[x, x^{-1}]$ cannot have uncountably many irreducible polynomials. This implies that K must be a countable field. $\square$

We next establish an important property for E when L is CIRT.

Proposition 3.3. *If L is CIRT, then the graph E contains line points or cycles without exits, or both.*

Proof. Suppose, by way of contradiction, the graph E contains no line points and no cycles without exits.

Let v be a vertex in E, and set $F = T(v)$, seen as a subgraph of E, that is, we consider the tree of v together with all edges in E connecting the vertices of $T(v)$. It is enough to show that F has uncountably many tail-equivalence classes of infinite paths. We wish to build a countable complete subgraph G of F having the same properties.

First, we construct an increasing family of countable complete subgraphs F_n starting with $F_0 = \{v\}$. Indeed, the graphs $F_n \subseteq F$ will satisfy the property that, for $u \in (F_n)^0$, if u is a regular vertex in F_n, then $s_{F_n}^{-1}(u) = s_E^{-1}(u)$. This clearly implies that F_n is a complete subgraph of E, see Definition 1.6.7 in [3]. Now assume that F_n has been built for some $n \geq 0$. For each vertex $u \in (F_n)^0$ which is a sink in F_n and which is a regular vertex in E, add to F_n all the finitely many edges emitted by u in E and the end vertices of these edges. For each vertex $u \in (F_n)^0$ which is a sink in F_n and which is an infinite emitter in E, add a countable number of edges emitted by u in E and the corresponding end vertices of these edges. This way we get the new subgraph F_{n+1}. Let G be the union of all the F_n's. Then G is a complete subgraph of F, and G contains no line points and no cycles without exits.

Clearly $L_K(G)$ is a subalgebra of $L_K(F)$ and the map sending an infinite path in G to the same infinite path in F induces an injective map from the set T_G of the tail-equivalence classes of infinite paths in G to the set T_F of the corresponding tail-equivalence classes of infinite paths in F. Thus if T_G is uncountable so is T_F.

To show that the set T_G is uncountable, let X be the set of all infinite paths in G together with the finite paths in G ending in a singular vertex. For each finite path γ (which could also be a vertex), define

$$\mathcal{Z}(\gamma) = \big\{ p \in X \text{ such that } p = \gamma p', \text{ where } p' \text{ is some path in } G \big\}.$$

Then X can be made a locally compact Hausdorff space, and the collection

$$\big\{ \mathcal{Z}(\gamma) \setminus \textstyle\bigcup_{e \in \mathcal{F}} \mathcal{Z}(\gamma e) \big\},$$

for γ a finite path and $\mathcal{F}$ a finite set of edges starting at $r(\gamma)$, is a basis of compact open sets in X (see Theorem 2.1 in [18]). Since G is a countable graph, these sets form a countable basis of open and closed (compact) sets for X and hence X is second countable. Since G does not have line points or cycles without exits, we see that X has no isolated points.

Now $\mathcal{Z}(v)$ is a compact Hausdorff totally disconnected space with no isolated points and hence is isomorphic to the Cantor space (see Theorem 13 in [16]). This means $\mathcal{Z}(v)$ contains $2^{\aleph_0}$ infinite paths. Since G is countable, an argument similar to the one used in the proof of Proposition 3.1 shows that the tail-equivalence classes of paths in $\mathcal{Z}(v)$ are at most countable. Since $|\mathcal{Z}(v)| = 2^{\aleph_0}$, we conclude that G (and hence E) contains $2^{\aleph_0}$ infinite paths which are not mutually tail-equivalent. By Chen [11], L will then have $2^{\aleph_0}$ non-isomorphic simple left/right L-modules, a contradiction.

Hence E always contains line points and/or cycles without exits. $\square$

For a ring R and any index set Λ, we denote by $M_\Lambda(R)$ the ring of matrices with coefficients in R, indexed on Λ, and having only finitely many nonzero entries.

We are now ready to prove the main result of this section.

Theorem 3.4. *Let E be an arbitrary graph and K be any field. Then the following properties are equivalent:*

(i) $L = L_K(E)$ *is CIRT;*

(ii) L *is the union of a well-ordered smooth ascending chain of countable length consisting of graded ideals*

$$(*) \qquad\qquad 0 < I_1 < \cdots < I_\alpha < I_{\alpha+1} < \cdots \quad (\alpha < \tau)$$

where, τ is a countable ordinal, for each $0 \leq \alpha < \tau$, $I_{\alpha+1}/I_\alpha$ is a direct sum of at most countably many matrix rings $M_\Lambda(R)$, where Λ are arbitrarily-sized index sets and R is either K or $K[x, x^{-1}]$. Moreover, K will be a countable field whenever E contains cycles.

Proof. Assume (i). By Proposition 3.3, the graph E contains line points and/or cycles without exits. Let I_1 be the ideal generated by all the line points and vertices on all the cycles without exits in E. Then $I_1 = I(H, \emptyset)$, where $H = I_1 \cap E^0$. By [3], I_1 is a direct sum of matrices of the form $M_{n_i}(K)$ and/or $M_{l_j}(K[x, x^{-1}])$ where n_i, l_j are cardinal numbers. Suppose we have defined a graded ideal $I_\alpha = I(H_\alpha, B_\alpha)$ for some $\alpha \geq 1$ where $H_\alpha = I_\alpha \cap E^0$ and $B_\alpha = \{v \in B_{H_\alpha} : v^{H_\alpha} \in I_\alpha\}$. If $I_\alpha \neq L$, consider $L/I_\alpha \cong L_K(E \backslash (H_\alpha, B_\alpha))$. Now L/I_α is CIRT and hence, by Proposition 3.3, $E \backslash (H_\alpha, B_\alpha)$ contains line points and/or cycles without exits. Define the ideal $I_{\alpha+1} \supset I_\alpha$ so that $I_{\alpha+1}/I_\alpha$ is the ideal generated by all the line points and the vertices on all the cycles without exits in $E \backslash (H_\alpha, B_\alpha)$. By [3], $I_{\alpha+1}/I_\alpha$ is then a direct sum of matrices of the form $M_{n_i}(K)$ and/or $M_{l_j}(K[x, x^{-1}])$ where n_i, l_j are cardinal numbers. If γ is a limit ordinal and I_α has been defined for all $\alpha < \gamma$, then define $I_\gamma = \bigcup_{\alpha < \gamma} I_\alpha$. By transfinite induction, we then obtain a smooth ascending chain $(*)$ of graded ideals with the desired properties and its union is L. Now each graded ideal is isomorphic to the Leavitt path algebra of a suitable graph (see [15]) and so is a ring with local units. Hence every left/right ideal of $I_{\alpha+1}$ is also a left/right ideal of L. It is then readily seen that every simple $I_{\alpha+1}$-module is isomorphic to a simple L-module. Since L is CIRT, the length of the chain τ is countable and that, for each $\alpha < \tau$, $I_{\alpha+1}/I_\alpha$ is a direct sum of at most countably many matrices of the form $M_{n_i}(K)$ and/or $M_{l_j}(K[x, x^{-1}])$.

Moreover, Proposition 3.2 shows that the field K must be countable if E contains cycles. This proves (ii).

Assume (ii), so that L is the union of a chain of graded ideals satisfying the stated properties. Let $S = L/M$ be a simple module where M is a maximal left ideal of L. Let β be the smallest ordinal such that $I_\beta \nsubseteq M$. Clearly β is a non-limit ordinal. Let $\beta = \alpha + 1$. Then $I_\alpha \subseteq M$. So,

$$S = (M + I_{\alpha+1})/M \cong I_{\alpha+1}/(I_{\alpha+1} \cap M) \cong (I_{\alpha+1}/I_\alpha)/[(I_{\alpha+1} \cap M)/I_\alpha].$$

Thus every simple left L-module is isomorphic to a simple left module over the ring $I_{\alpha+1}/I_\alpha$ for some α with $0 \le \alpha < \tau$. Now for each α, $I_{\alpha+1}/I_\alpha$ is CIRT, due to the fact that $I_{\alpha+1}/I_\alpha$ is a direct sum of at most countably many matrix rings $M_{n_k}(K)$ and matrix rings $M_{r_j}(K[x, x^{-1}])$, where n_k, r_j are arbitrarily-sized cardinal numbers and that K is countable whenever $r_j \neq 0$ for some r_j. Since τ is a countable ordinal, it is readily seen that L is also CIRT. $\square$

Remark 3.5. In Theorem 3.4, if the graph E is acyclic, then for every $\alpha < \tau$, $I_{\alpha+1}/I_\alpha$ is isomorphic to a direct sum of matrix rings over K and hence is a direct sum of simple modules. Thus the chain $(*)$ becomes the socular chain for L. In other words, L becomes a semi-artinian von Neumann regular ring with countable Loewy length. Note that, in this case, there is no restriction on the cardinality of the field K.

4. Leavitt path algebras of finite irreducible representation type

We now wish to specialize to the case when the Leavitt path algebra $L = L_K(E)$ is of finite irreducible representation type, that is, when L has at most finitely many distinct isomorphism classes of simple left L-modules. In this case, L satisfies conditions stronger than those of Theorem 3.4. Specifically, L becomes a semi-artinian von Neumann regular ring with finitely many two-sided ideals. Moreover, the result holds for arbitrary fields K without any restrictions.

We begin with the following lemma, and its corollary, which are used in the proof of Theorem 4.3.

Lemma 4.1. *Suppose R is a von Neumann regular ring and A is a non-zero proper ideal of R. Then every simple left R-module is isomorphic to a simple left module over R/A or A. Conversely, every simple left module over R/A, or A is isomorphic to some simple left R-module.*

Proof. Let R/M be a simple left R-module, where M is a maximal left ideal of R. Then it is easy to see that R/M is isomorphic to a simple left module over R/A or A according as (i) $M \supset A$, or (ii) $M \not\supseteq A$.

To prove the converse, we only consider the case of a simple left A-module S, as the other case is trivial. Write $S = A/N$, where N is a maximal left ideal of A. For $n \in N$ there is an idempotent e in A such that $en = n$. It follows from this that N is indeed a left R-module, and so $S = A/N$ is a simple R-module. $\square$

Corollary 4.2. *Suppose $L_K(E)$ is semi-artinian ring with finitely many ideals. Then $L_K(E)$ has at most finitely many non-isomorphic simple left modules.*

Proof. Suppose the semi-artinian ring $L = L_K(E)$ has only finitely many two-sided ideals. Then we can build a finite ascending chain of two-sided ideals $0 < S_1 < \cdots < S_n = L$ where, for each $i = 1, \dots, n-1$, S_{i+1}/S_i is a simple ring which, being also semi-artinian, is a direct sum of isomorphic simple left ideals. Consequently,

each S_{i+1}/S_i has a single isomorphism class of simple left S_{i+1}/S_i-modules. Also by [6], $L_K(E)$ is von Neumann regular. We then conclude, by Lemma 4.1, that L has exactly n distinct isomorphism classes of simple left L-modules.   □

Theorem 4.3. *Let E be an arbitrary graph and K be any field. Then the following are equivalent for the Leavitt path algebra $L = L_K(E)$:*

   (i) *L has at most finitely many non-isomorphic simple left/right L-modules;*

   (ii) *L is a semi-artinian ring with finitely many two-sided ideals.*

Proof. Assume (i). Since there are only finitely many non-isomorphic simple left modules, $L_K(E)$ has only finitely many distinct primitive ideals. We wish to show that there are no non-graded prime ideals in $L_K(E)$. Now any non-graded prime ideal of $L_K(E)$ is primitive (see Theorem 4.3 in [14]). From Theorem 3.12 of [14], the non-graded prime ideals of $L_K(E)$ are precisely the ideals P generated by $I_{(H,S)} \cup \{f(c)\}$, where $P \cap E^0 = H$, $S = \{v \in B_H : v^H \in P\}$, c is a cycle without exits in $E \backslash H$, and f is an irreducible polynomial in $K[x, x^{-1}]$. So, if $L_K(E)$ has one such non-graded prime ideal P, then corresponding to each of the infinitely many irreducible polynomials g in $K[x, x^{-1}]$, $L_K(E)$ will have a non-graded prime (and hence a primitive) ideal P_g generated by $I_{(H,S)} \cup \{g(c)\}$. This contradicts that $L_K(E)$ has only a finite number of primitive ideals. Thus every prime ideal of $L_K(E)$ must be graded and we appeal to Corollary 3.13 in [14] to conclude that the graph E satisfies condition (K). From Theorem 6.16 in [17], we then conclude that every ideal of $L_K(E)$ is graded.

   Thus if I is an ideal of $L_K(E)$ then, being a graded ideal, $I = I_{(H,S)}$, where $H = I \cap E^0$ and $S = \{v \in B_H : v^H \in I\}$ and, moreover, $L_K(E)/I \cong L_K(E \backslash (H,S))$ by [17]. As the Jacobson radical of the Leavitt path algebra $L_K(E \backslash (H,S))$ is zero (see [6]), we conclude that I is the intersection of all the primitive ideals containing I. As there are only finitely many primitive ideals in $L_K(E)$, we then conclude that $L_K(E)$ contains only finitely many distinct ideals which are all of the form $I_{(H,S)}$ for some admissible pair (H,S).

   We also claim that E contains no cycles. Because if there is a cycle g in E, then by condition (K) there will be another cycle $h \neq g$ sharing a common vertex with g and this contradicts Proposition 3.1. Hence E is acyclic and L is von Neumann regular, by [5]. Also, by Proposition 3.3, E contains line points and so $S_1 = \text{Soc}(L) \neq 0$. Also, since there are only finitely many ideals, S_1 is a direct sum of finitely many homogeneous components: $S_1 = \bigoplus_{i=1}^{n_1} S_{1i}$ with S_{1i} a direct sum of isomorphic simple left ideals of L. Let $H = S_1 \cap E^0$. Now $L/S_1 \cong L_K(E \backslash (H, \emptyset))$ satisfies the same hypothesis as L and hence has a non-zero socle S_2/S_1. Proceeding like this and using the fact that L has at most finitely many ideals, we conclude L is the union of a finite ascending chain of ideals $\{0\} = S_0 \subset S_1 \subset \cdots \subset S_m = L$ where, for each i, $S_{i+1}/S_i = \text{Soc}(L/S_i)$ and is a direct sum of finitely many homogeneous components. This proves (ii).

   (ii)$\Longrightarrow$(i). This follows from Corollary 4.2.   □

Remark 4.4. One might wonder what are the finite lattices appearing as ideal lattices of the algebras $L_K(E)$ characterized in Theorem 4.3. Indeed, observe that

the lattice of ideals of a von Neumann regular ring is a distributive lattice. It follows from Bergman's theorem, [10], that every finite distributive lattice can be represented as the lattice of ideals of a unital ultramatricial algebra. On the other hand, Proposition 2.12 in [13] asserts that this unital ultramatricial algebra is Morita-equivalent to a Leavitt path algebra $L_K(E)$ of an acyclic graph. For this graph E, the conditions in Theorem 4.3 (ii) necessarily hold. In conclusion, we see that the class of lattices appearing as ideal lattices of the algebras described in Theorem 4.3 is exactly the class of all finite distributive lattices.

5. Equivalent graphical conditions

In this section we describe the graphical properties of E under which the Leavitt path algebra $L_K(E)$ is of finite irreducible representation type. We begin with a simple lemma describing when two line points generate isomorphic simple right ideals.

Lemma 5.1. *Given two line points u, v, $uL_K(E) \cong vL_K(E)$ if and only if $T(u) \cap T(v)$ is not empty.*

Proof. Suppose $\theta : u\, L_K(E) \to v\, L_K(E)$ is an isomorphism. Then θ is given by the left multiplication by the non-zero element $\theta(u) = vsu$ for some $s \in L_K(E)$. We can clearly assume that $s = vsu$. Write $s = \sum_{i=1}^{m} k_i\, \alpha_i\, \beta_i^*$ where $k_i \in K$ and α_i, β_i are finite paths in E. If a term $k_i\, \alpha_i\, \beta_i^* = k_i\, v\, \alpha_i\, \beta_i^*\, u \neq 0$, then $v = s(\alpha_i)$, $u = s(\beta_i)$ and $r(\alpha_i) = r(\beta_i) = w$, so $w \in T(u) \cap T(v)$. Conversely, suppose $w \in T(u) \cap T(v)$. Since u is a line point, so is w and there is a unique path μ from u to w. Then $u\, a \longmapsto w\, \mu^* u\, a$ is an isomorphism from the simple module $uL_K(E)$ to $wL_K(E)$ with the map $w\, b \longmapsto u\, \mu\, w\, b$ being the inverse isomorphism. By a similar argument, $vL_K(E) \cong wL_K(E)$. Consequently, $uL_K(E) \cong vL_K(E)$. $\square$

The next theorem describes the graphical conditions on E under which $L_K(E)$ is of finite irreducible representation type.

Theorem 5.2. *Let E be an arbitrary graph and let K be any field. Then the Leavitt path algebra $L_K(E)$ is of finite irreducible representation type if and only if all of the following conditions hold:*

(i) *E is acyclic;*

(ii) *E^0 has only finitely many distinct hereditary saturated subsets H and for each such H the corresponding set B_H of breaking vertices is finite; and*

(iii) *In the poset of admissible pairs in E, $(E^0, \emptyset)$ is the supremum of a finite ascending chain*

$$(H_0 = \emptyset, \emptyset) < (H_1, \emptyset) < (H_1, B_{H_1}) < (H_2, B_{H_1} \cap B_{H_2}) < (H_2, B_{H_2})$$
$$< (H_3, B_{H_2} \cap B_{H_3}) < (H_3, B_{H_3}) < \cdots$$

where, for $j \geq 0$, H_{j+1} is a hereditary saturated subset of E^0 and $H_{j+1} \backslash H_j$ is the hereditary saturated closure of the set of line points in $E \backslash (H_j, B_{H_j})$.

Proof. If $L_K(E)$ has finite irreducible representation type, then in the proof of Theorem 4.3 it was shown that E is acyclic and that $L_K(E)$ has at most finitely many two-sided ideals (which are all graded). The latter property is equivalent to condition (ii) by Theorem 5.7 in [17]. To prove condition (iii), we shall use the fact, established in Theorem 4.3, that $L_K(E)$ (and each of its homomorphic images) is semi-artinian. We wish to construct a chain of ideals of the form $I_{(H,S)}$. Let $J_1 = \mathrm{Soc}(L_K(E))$. By [9], J_1 is the ideal generated by the hereditary saturated closure H_1 of the set T_1 of all the line points in E. Now $J_1 = I_{(H_1,\emptyset)}$ and, by [17], J_1 is the kernel of an epimorphism $f : L_K(E) \longrightarrow L_K(E\backslash(H_1,\emptyset))$ where $(E\backslash(H_1,\emptyset))^0 = E^0\backslash H_1 \cup \{v' : v \in B_{H_1}\}$ and that $f(v^{H_1}) = v'$ for all $v \in B_{H_1}$. Let $J_2 = I_{(H_1,B_{H_1})} = \langle H_1, \{v^{H_1} : v \in B_{H_1}\}\rangle$. so that $J_2/J_1 \cong \langle\{v' : v \in B_{H_1}\}\rangle \subset L_K(E\backslash(H_1,\emptyset))$. By [17], $L_K(E)/J_2 \cong L_K(E\backslash(H_1,B_{H_1})) \cong L_K(E\backslash H_1)$ since $(E\backslash(H_1,B_{H_1}))^0 = E^0\backslash H_1$ and $(E\backslash(H_1,B_{H_1}))^1 = \{e \in E^1 : r(e) \notin H_1\}$. Now $L_K(E)/J_2$ has a non-zero socle (being semi-artinian) and let H_2 be a hereditary saturated subset of E^0 containing H_1 such that $H_2\backslash H_1$ is the hereditary saturated closure of the set T_2 of all the line points in $E\backslash(H_1,B_{H_1})$. Define

$$J_3 = \langle J_2, H_2\rangle = \langle H_2, \{v^{H_1} : v \in B_{H_1}\}\rangle$$

so that

$$J_3/J_2 \cong \langle H_2\backslash H_1\rangle = \mathrm{Soc}(E\backslash(H_1,B_{H_1})) \subset L_K(E\backslash(H_1,B_{H_1})) \cong L_K(E\backslash H_1).$$

If $v \in B_{H_1}\backslash B_{H_2}$, then $r(s^{-1}(v)) \subset H_2 \subset J_3$ and since $v^{H_1} \in J_3$, we conclude that $v \in J_3$. In the isomorphism $L_K(E)/J_2 \cong L_K(E\backslash H_1))$, v gets mapped to an element in $(J_3/J_2) \cap (E^0\backslash H_1) = H_2\backslash H_1$ and so $v \in H_2$. Thus $B_{H_1}\backslash B_{H_2} \subset H_2$ and so $J_3 = \langle H_2, \{v^{H_2} : v \in B_{H_1} \cap B_{H_2}\}\rangle = I_{(H_2,B_{H_1}\cap B_{H_2})}$. Note that, in the isomorphism $L_K(E)/J_1 \cong L_K(E\backslash(H_1,\emptyset))$, J_3/J_1 maps to $\mathrm{Soc}(L_K(E\backslash(H_1,\emptyset)))$. Define $J_4 = I_{(H_2,B_{H_2})}$. Proceeding like this, we obtain a chain of ideals

$$\{0\} = I_{(H_0=\emptyset,\emptyset)} \subset I_{(H_1,\emptyset)} \subset I_{(H_1,B_{H_1})} \subset I_{(H_2,B_{H_1}\cap B_{H_2})}$$
$$(**) \qquad\qquad \subset I_{(H_2,B_{H_2})} \subset I_{(H_3,B_{H_2}\cap B_{H_3})} \subset I_{(H_3,B_{H_3})} \subset \cdots$$

whose union is $L_K(E)$. Here, for each $j \geq 0$, H_{j+1} is a hereditary saturated subset of E^0 and $H_{j+1}\backslash H_j$ is the hereditary saturated closure of the set T_{j+1} of all the line points in $E\backslash(H_j,B_{H_j})$. Note that, for each $j \geq 0$, there are only finitely many equivalence classes of line points in $E\backslash(H_j,B_{H_j})$, due to condition (ii). Our construction shows that, in the poset of admissible pairs, $(E^0,\emptyset)$ is then the supremum of a finite ascending chain

$$(\emptyset,\emptyset) < (H_1,\emptyset) < (H_1,B_{H_1}) < (H_2,B_{H_1} \cap B_{H_2})$$
$$< (H_2,B_{H_2}) < (H_3,B_{H_2} \cap B_{H_3}) < (H_3,B_{H_3}) < \cdots$$

where the sets H_j are as described above.

Conversely, condition (i) implies, by [5], that $L_K(E)$ is von Neumann regular and condition (ii) implies that it has only finitely many two-sided ideals (which are all graded ideals). Consider the chain indicated in condition (iii), and the

corresponding chain of ideals (∗∗) constructed above. Denote $I_{(H_1,\emptyset)}$ by S_1 and, for each $i \geq 2$, denote $I_{(H_i, B_{H_{i-1}} \cap B_{H_i})}$ by S_i. Then we get the ascending finite chain

$$\{0\} \subset S_1 \subset \cdots \subset S_t = L_K(E)$$

where $S_1 = \operatorname{Soc}(L_K(E))$ and for each i, $S_{i+1}/S_i = \operatorname{Soc}(L_{K(E)}/S_i)$. Hence $L_K(E)$ is a semi-artinian ring. By Theorem 4.3, $L_K(E)$ is of finite irreducible representation type.   □

Example 5.3. To illustrate Theorem 5.2, consider the graph P_3 depicted in the next section: it is clearly acyclic. Its hereditary saturated subsets of vertices are: $H_0 = \emptyset$, the empty set; $H_1 = \{v_{11}, v_{12}, v_{13}, \ldots\}$; $H_2 = H_1 \cup \{v_{21}, v_{22}, v_{23}, \ldots\}$; $H_3 = (P_3)^0$. Since the graph is row-finite there are no breaking vertices. Then the chain in condition (iii) is given by

$$(\emptyset, \emptyset) < (H_1, \emptyset) < (H_2, \emptyset) < (H_3 = (P_3)^0, \emptyset).$$

We point out that one can build examples showing that no two of the three conditions in Theorem 5.2 imply the third.

6. Examples

In this section, we wish to illustrate Theorems 3.4 and 4.3 by constructing non-trivial examples of $L_K(E)$ which, being semi-artinian (von Neumann regular) rings, are the union of a finite or infinite socular chain

$$0 < S_1 < S_2 < \cdots < S_n < \cdots$$

where $S_1 = \operatorname{Soc}(L_K(E))$ and, for each $j < n$, $S_{j+1}/S_j = \operatorname{Soc}(L_K(E)/S_j)$. To have a really simple example, one can assume that S_1 and each S_{j+1}/S_j are direct sums of isomorphic simple modules. Since the socle of a Leavitt path algebra is generated by line points in the graph, the assumed property on S_1 and S_{j+1}/S_j imply that the line points belonging to S_1 (and likewise in S_{j+1}/S_j) all must form a single straight line segment.

It is this idea that was used to construct the "pyramid" examples in reference [6]. In some sense these examples are the simplest examples of semi-artinian Leavitt path algebras of infinite graphs with arbitrary Loewy length.

We will show that, for each (finite or infinite) cardinal κ, the Leavitt path algebra $L_K(P_\kappa)$ of the pyramid graph P_κ has exactly κ distinct isomorphism classes of simple right $L_K(E)$-modules.

Let P_1 be the graph

consisting of a single infinite path. Now all the vertices $v_{1,i}$ are line points in P_1 and so $v_{1,i} L_K(P_1)$ is a simple module for all i (see [9]). Also $L_K(P_1) = \oplus_i v_{1,i} L_K(P_1) =$

$Soc(L_K(P_1))$. From Lemma 5.1 it is clear that, for all $i < j$, $v_{1,i}L_K(P_1) \cong$ $v_{1,j}L_K(P_1)$. Since $L_K(P_1)$ is a direct sum of simple modules, every simple right $L_K(P_1)$-module is isomorphic to the simple right ideal $v_{1,i}L_K(P_1)$ and we conclude that all the simple right $L_K(P_1)$-modules are isomorphic.

Let P_2 be the graph
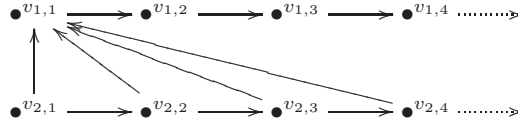
Now the line points in the graph P_2 are the vertices in the first row, namely, $v_{1,1}, v_{1,2}, v_{1,3}, \ldots$ and they generate the socle S of P_2 which, from the explanation in describing $L_K(P_1)$ above, is a direct sum of isomorphic faithful simple right $L_K(P_2)$-modules. Also, by [17], $P_2/S \cong L_K(F)$ where $F \cong P_1$ and so $P_2/S = Soc(P_2/S)$ is a direct sum of isomorphic simple modules annihilated by the ideal S. Thus $L_K(P_2)$ has exactly two distinct isomorphism classes of simple $L_K(P_2)$-modules.

Let P_3 be the graph
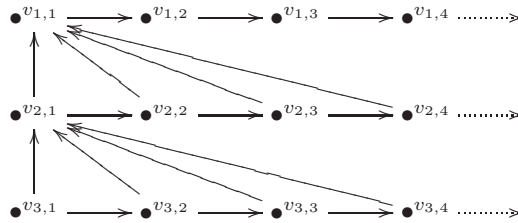
As above, the vertices $v_{1,1}, v_{1,2}, v_{1,3}, \ldots$ generate the socle S of $L_K(P_3)$ which is a direct sum of isomorphic faithful simple $L_K(P_3)$-modules and $L_K(P_3)/S \cong$ $L_K(P_2)$. Clearly, by Lemma 4.1 and the description of $L_K(P_2)$ above, $S_2/S \cong$ $Soc(L_K(P_2))$ is a direct sum of isomorphic simple $L_K(P_3)$-modules annihilated by the ideal S and that

$$L_K(P_3)/S_2 \cong L_K(P_2)/Soc(L_K(P_2)) \cong Soc[L_K(P_2)/Soc(L_K(P_2))]$$

is a direct sum of isomorphic simple $L_K(P_3)$-modules annihilated by the ideal S_2. Thus we conclude that $L_K(P_3)$ has exactly three distinct isomorphism classes of simple right $L_K(P_3)$-modules.

Proceeding like this, we conclude, by simple induction, that for any positive integer n, the Leavitt path algebra $L_K(P_n)$ of the "pyramid" graph P_n with n "layers" has exactly n distinct isomorphism classes of simple right $L_K(P_n)$-modules.

Let $P_\omega = \bigcup_{n \in \mathbb{N}} P_n$ be the "pyramid" graph of length ω constructed inductively and represented pictorially as follows.
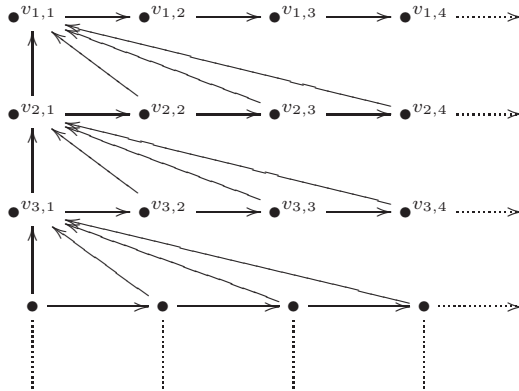
Again, by induction, it follows that $L_K(P_\omega)$ has exactly ω distinct isomorphism classes of simple $L_K(P_\omega)$-modules.

The graph $P_{\omega+1}$ is obtained from the graph P_ω by adding a single vertex $v_{\omega+1}$ and connecting it by an edge to each of the vertices $v_{j,1}$ for $j < \omega$ in the graph P_ω. Specifically, $(P_{\omega+1})^0 = (P_\omega)^0 \cup \{v_{\omega+1}\}$, $(P_{\omega+1})^1 = (P_\omega)^1 \cup \{e_{\omega+1,j}\colon j < \omega\}$ where, for each j, $s(e_{\omega+1,j}) = v_{\omega+1}$ and $r(e_{\omega+1,j}) = v_{j,1}$. If S_ω denotes the ω-socle being the ideal generated by all the vertices in P_ω, then $L_K(P_{\omega+1})/S_\omega$ is a simple $L_K(P_{\omega+1})$-module whose annihilator ideal is S_ω and it is isomorphic to the Leavitt path algebra of a graph $\{\overset{v_{\omega+1}}{\bullet}\}$ consisting of a single vertex and no edges. Clearly $L_K(P_{\omega+1})$ has $\omega + 1$ distinct isomorphism classes of simple $L_K(P_{\omega+1})$-modules.

Proceeding this way, as was shown in [6], we can construct, by transfinite induction, a "pyramid" graph P_λ for each ordinal λ. The Leavitt path algebra $L_K(P_\lambda)$ is a semi-artinian von Neumann ring of Loewy length λ such that, for each $\alpha < \lambda$, the quotient $S_{\alpha+1}/S_\alpha$ of successive socles is isomorphic to the Leavitt path algebra of an infinite line segment (like the graph P_1) or a graph $\{\overset{v}{\bullet}\}$ consisting of a single vertex and no edges, according as α is a successor or a limit ordinal. Thus $S_{\alpha+1}/S_\alpha$ has exactly one isomorphism class of simple modules (annihilated by S_α). By transfinite induction, one can then show that the Leavitt path algebra $L_K(P_\lambda)$ has exactly $|\lambda|$ isomorphism classes of simple $L_K(P_\lambda)$-modules.

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