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Symmetries of quasiplatonic Riemann surfaces

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Abstract. We state and prove a corrected version of a theorem of Singerman, which relates the existence of symmetries (anticonformal involutions) of a quasiplatonic Riemann surface $\mathcal{S}$ (one uniformised by a normal subgroup N of finite index in a cocompact triangle group Δ) to the properties of the group $G = \Delta/N$. We give examples to illustrate the revised necessary and sufficient conditions for the existence of symmetries, and we relate them to properties of the associated dessins d'enfants, or hypermaps.

1. Introduction

The category of compact Riemann surfaces is naturally equivalent to that of complex projective algebraic curves. Such a surface or curve $\mathcal{S}$ is real (defined over $\mathbb{R}$) if and only if it possesses an anticonformal involution called a *symmetry*, in which case $\mathcal{S}$ is said to be *symmetric*. (These specialised usages of these terms were introduced by Klein [10]; see [2] for background.)

Among the most important Riemann surfaces $\mathcal{S}$ are the quasiplatonic surfaces, those uniformised by a normal subgroup N of finite index in a cocompact triangle group

$$\Delta = \Delta(l, m, n) = \langle X, Y, Z \mid X^l = Y^m = Z^n = XYZ = 1 \rangle ;$$

this is equivalent to $\mathcal{S}$ carrying a regular dessin $\mathcal{D}$ in Grothendieck's terminology [7], that is, an orientably regular hypermap in combinatorial language. (See [6] or [9] for the background to these connections.) In this case, $G := \Delta/N$ can be identified with the automorphism group $\operatorname{Aut}\mathcal{D}$ of $\mathcal{D}$, the orientation-preserving automorphism group of the hypermap. In Theorem 2 of [16], Singerman stated a result which relates the existence of symmetries of such a surface $\mathcal{S}$ to the properties of G (or equivalently of $\mathcal{D}$). As pointed out by Watson [18], the stated conditions are sufficient for $\mathcal{S}$ to be symmetric, but are not necessary. A correct statement, based on Theorem 3.4 of [18], is as follows (the original [16] omitted conditions (3) and (4)):

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Theorem 1.1. *Let $\mathcal{S}$ be a quasiplatonic Riemann surface, uniformised by a normal subgroup N of finite index in a cocompact triangle group Δ, and let x, y and z be the images in $G := \Delta/N$ of a canonical generating triple X, Y, Z for Δ. Then $\mathcal{S}$ is symmetric if and only if at least one of the following holds:*

(1) *G has an automorphism $\alpha : x \mapsto x^{-1}$, $y \mapsto y^{-1}$;*

(2) *G has an automorphism $\beta : x \mapsto y^{-1}$, $y \mapsto x^{-1}$ (possibly after a cyclic permutation of the canonical generators);*

(3) *Δ has type $(2n, 2n, n)$ for some n (possibly after a cyclic permutation of its generators), G has an automorphism γ transposing x and y, and the extension $\langle G, \gamma\rangle$ of G by $\langle\gamma\rangle$ has an automorphism δ transposing x and $x\gamma$;*

(4) *$\mathcal{S}$ has genus 1.*

The automorphisms $\alpha, \ldots, \delta$ in cases (1), (2) and (3) must have order dividing 2, and in cases (2) and (3), x and y must have the same order.

If we also assume that Δ is *maximal* among all triangle groups normalising N (these exist if $\mathcal{S}$ has genus $g > 1$, but not if $g = 1$), we obtain a version of this theorem which, although a little less general, is simpler to state and to prove, is closer to the original in [16], and for $g > 1$ is equivalent to Theorem 1.5.10 of [2], where G is assumed to be the full group $\mathrm{Aut}^{+}\mathcal{S}$ of conformal automorphisms of $\mathcal{S}$. The authors are grateful to Jürgen Wolfart for suggesting this alternative:

Theorem 1.2. *Let $\mathcal{S}$, N, Δ and G be as in Theorem 1.1, and suppose that Δ is maximal among all triangle groups containing N as a normal subgroup. Then $\mathcal{S}$ is symmetric if and only if either*

(1) *G has an automorphism $\alpha\colon x \mapsto x^{-1}$, $y \mapsto y^{-1}$, or*

(2) *G has an automorphism $\beta\colon x \mapsto y^{-1}$, $y \mapsto x^{-1}$ (possibly after a cyclic permutation of the canonical generators).*

2. Combinatorial interpretation

In view of the importance of quasiplatonic surfaces for the theories of dessins d'enfants and of maps and hypermaps, we will give combinatorial interpretations of the conditions in Theorem 1.1. Each quasiplatonic surface $\mathcal{S}$ inherits from Δ and N a combinatorial structure $\mathcal{D}$ called a *regular dessin*, or *orientably regular hypermap*, of type (l, m, n). For our purposes, one can regard $\mathcal{D}$ as a triangulation of $\mathcal{S}$, the quotient by N of the triangulation of the universal covering space $\hat{\mathcal{S}}$ naturally associated with Δ, together with a preferred orientation of $\mathcal{S}$. The vertices of $\mathcal{D}$ can be coloured black, white or red, and termed *hypervertices*, *hyperedges* or *hyperfaces*, as they are quotients of fixed points of conjugates of X, Y or Z, with valencies $2l, 2m$ or $2n$ respectively. Equivalently, in the language of dessins d'enfants, $\mathcal{D}$ is the inverse image, under a regular Belyĭ function $\beta\colon \mathcal{S} \to \mathbb{P}^1(\mathbb{C})$, of the triangulation of the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ with vertices at the ramification points $0, 1$ and ∞ of β coloured black, white and red, and with edges along $\mathbb{R}$.

The group $G = \Delta/N$ can be identified with the group $\mathrm{Aut}\,\mathcal{D}$ of automorphisms of $\mathcal{D}$; by definition this preserves the orientation and vertex colours of the triangulation, and since $\mathcal{D}$ is regular it acts regularly on incident pairs of vertices of given colours. We say that $\mathcal{D}$ is a *reflexible* dessin (or a *regular* hypermap) if the triangulation has an additional colour-preserving automorphism (induced by the extended triangle group Δ^* corresponding to Δ) which reverses the orientation of $\mathcal{S}$, so that $\mathcal{D}$ is isomorphic to its mirror image $\overline{\mathcal{D}}$; otherwise $\mathcal{D}$ and $\overline{\mathcal{D}}$ form a *chiral pair*. The *black-white dual* $\mathcal{D}^{(01)}$ of $\mathcal{D}$ is the dessin formed by transposing the colours of the black and white vertices, or equivalently by replacing β with $1-\beta$; similarly, there are black-red and white-red duals $\mathcal{D}^{(02)}$ and $\mathcal{D}^{(12)}$ corresponding to $1/\beta$ and $\beta/(\beta-1)$.

For many purposes it is simpler, if less symmetric, to represent $\mathcal{D}$ by its *Walsh map* [17] $\mathcal{W} = W(\mathcal{D})$, a bipartite map on $\mathcal{S}$ formed by deleting the red vertices and their incident edges. This is the inverse image under β of the unit interval $[0,1] \subset \mathbb{R}$. One can recover $\mathcal{D}$ as the stellation of $\mathcal{W}$, formed by placing a red vertex in each face of $\mathcal{W}$, and joining it by an edge to each incident black or white vertex. Many authors take this as the standard model of a dessin.

If one of the elliptic periods l, m or n of Δ is equal to 2 (say $m = 2$ without loss of generality), then the triangulation $\mathcal{D}$ is the barycentric subdivision $B(\mathcal{M})$ of an orientably regular map $\mathcal{M}$ on $\mathcal{S}$, with vertices, edge-centres and face-centres at the black, white and red vertices of $\mathcal{D}$, and with orientation-preserving automorphism group $\mathrm{Aut}^+\mathcal{M} = G$. In the notation of Coxeter and Moser (Chapter 8 of [5]), $\mathcal{M}$ has type $\{n, l\}$, meaning that vertices and faces have valencies l and n. Conversely, any orientably regular map $\mathcal{M}$ on $\mathcal{S}$ determines a dessin $\mathcal{D} = B(\mathcal{M})$ with $m = 2$. The black-red dual $\mathcal{D}^{(02)}$ of $\mathcal{D}$ corresponds to the vertex-face dual map $\mathcal{M}^*$ of $\mathcal{M}$.

In Theorem 1.1, condition (1) corresponds to $\mathcal{D}$ being reflexible, as happens whenever $\mathcal{S}$ has genus 0, for example. Condition (2) corresponds to $\mathcal{D}$ being isomorphic to $\overline{\mathcal{D}^{(01)}}$, the mirror image of its black-white dual, as happens (after a cyclic permutation of generators) for the Edmonds maps $\mathcal{M} \cong \overline{\mathcal{M}^*}$ of genus 7 and type $\{7,7\}$; these are a chiral pair of orientably regular embeddings of the complete graph K_8 in Macbeath's curve [12], described by Coxeter in §21.3 of [4] and denoted by C7.2 in Conder's list of chiral maps [3] (see also p. 29 of [16]). Neither of these conditions applies to the chiral dessins of genus 1, such as the embeddings $\mathcal{M} = \{4,4\}_{1,2}$ and $\overline{\mathcal{M}} = \{4,4\}_{2,1}$ of K_5, or the embeddings $\{3,6\}_{1,2}$ and $\{3,6\}_{2,1}$ of K_7, with $G \cong AGL_1(5)$ or $AGL_1(7)$ (see Chapter 8 of [5]): these groups have only inner automorphisms, and none satisfying (1) or (2), but the underlying tori, uniformised by square and hexagonal lattices respectively, admit symmetries, so condition (4) is required in Theorem 1.1.

Condition (3) is a little harder to explain. The group $G = \mathrm{Aut}\,\mathcal{D}$ acts regularly on edges of the Walsh map $\mathcal{W}$ of $\mathcal{D}$; the existence of an automorphism γ as in (3) is equivalent to $\mathcal{W}$, regarded as an uncoloured map, being orientably regular, with a half-turn γ reversing an edge, so that $\mathrm{Aut}^+\mathcal{W} = \langle G, \gamma\rangle$ acts regularly on arcs (directed edges) of $\mathcal{W}$; the existence of δ is equivalent to $\mathcal{W} \cong \overline{\mathcal{W}^*}$. In §4.5 we will give examples of dessins $\mathcal{D}$ satisfying (3) but not (1), (2) or (4), with underlying Riemann surfaces $\mathcal{S}$ admitting symmetries, so that this condition is required for a correct statement of Theorem 1.1. (Note that G is not normal in $\langle G, \gamma, \delta \rangle$ since x is conjugate to $\gamma x \notin G$.)

Dessins $\mathcal{D}$ satisfying (3) or (4), but not (1) or (2), and hence not covered by Theorem 2 in [16], all correspond to triangle groups with periods $2n, 2n, n$ for $n > 2$, or have genus 1. Thus the original statement is correct when restricted to *maps* of genus $g \neq 1$, and indeed when applied to dessins of all types except permutations of $(2, 3, 6), (3, 3, 3)$ and $(2n, 2n, n)$ for $n \geq 2$.

3. Groups containing triangle groups

In order to prove Theorem 1.1 we need to consider which isometry groups $\tilde{\Delta}$ (of $\hat{\mathcal{S}} = \mathbb{P}^1(\mathbb{C})$, $\mathbb{C}$ or $\mathbb{H}$) can contain a cocompact triangle group $\Delta = \Delta(l, m, n)$ as a subgroup of index 2. Here are some examples:

1. The most obvious example is the extended triangle group $\Delta^* = \Delta^*(l, m, n)$, the extension of Δ by a reflection T in the geodesic through the fixed points of two of its canonical generators X, Y, Z, which it inverts by conjugation; each choice of a pair from the generating triple gives the same group Δ^*.
2. If two of l, m and n are equal, say $l = m$ without loss of generality, Δ has index 2 in a triangle group $\Delta^= = \Delta(l, 2, 2n)$, the extension of Δ by a rotation of order 2 transposing the fixed points of X and Y, and transposing X and Y by conjugation; Δ also has index 2 in a group $\Delta^\times = \Delta^\times(l, m, n)$, the extension of Δ by a reflection transposing the fixed points of X and Y, and sending X to Y^{-1} and Y to X^{-1} by conjugation.
3. If $l = m = n$ then in addition to Δ^* we obtain three triangle groups $\Delta^=$ and three groups $\Delta^\times$ as in case (2), depending in the choice of a pair from the canonical generating triple X, Y, Z.

In the standard notation for NEC groups introduced by Macbeath [13], Δ has signature $(0, +, [l, m, n], \{\,\})$, while the signatures of Δ^*, $\Delta^=$ and $\Delta^\times$ are $(0, +, [\,], \{(l, m, n)\})$, $(0, +, [l, 2, 2n], \{\,\})$ and $(0, +, [l], \{(n)\})$ respectively. The following result shows that these groups are in fact the only possibilities for $\tilde{\Delta}$:

Lemma 3.1. *Let Δ be a cocompact triangle group $\Delta(l, m, n)$, and let $\tilde{\Delta}$ be an isometry group containing Δ as a subgroup of index 2. Then either*

(1) *l, m and n are distinct, and $\tilde{\Delta} = \Delta^*$, or*

(2) *just two of l, m and n are equal, and $\tilde{\Delta} = \Delta^*$, $\Delta^=$ or $\Delta^\times$, or*

(3) *$l = m = n$, and $\tilde{\Delta} = \Delta^*$, one of the three groups $\Delta^=$, or one of the three groups $\Delta^\times$.*

Proof. Any index 2 inclusion must be normal, so the groups $\tilde{\Delta}$ containing Δ as a subgroup of index 2 are contained in the normaliser $N(\Delta)$ of Δ in the full isometry group, and correspond to the involutions in $N(\Delta)/\Delta$. In case (1) we have $N(\Delta) = \Delta^*$, with $N(\Delta)/\Delta \cong C_2$, so $\tilde{\Delta} = \Delta^*$. In case (2) we can assume without loss of generality that $l = m \neq n$, so $N(\Delta) = \Delta^*(l, 2, 2n)$ with $N(\Delta)/\Delta \cong V_4$;

the three involutions in this group yield the possibilities $\tilde{\Delta} = \Delta^*$, $\Delta^=$ and $\Delta^\times$. In case (3) we have $N(\Delta) = \Delta^*(2,3,2n)$ and $N(\Delta)/\Delta \cong \Delta^*(2,3,2) \cong S_3 \times C_2$; there are seven involutions in this group (the central involution, generating the direct factor C_2, and two conjugacy classes of three non-central involutions), giving the seven subgroups $\tilde{\Delta}$ listed in (3). □

Now suppose that $G = \Delta/N$ for some normal subgroup N of Δ. If N is also normal in one of the groups $\tilde{\Delta} = \Delta^*$, $\Delta^=$ or $\Delta^\times$ containing Δ with index 2, let $\tilde{G} = G^*$, $G^=$ or $G^\times$ be the corresponding extension $\tilde{\Delta}/N$ of G by an involution acting as above on the canonical generators x, y, z of G. Now N corresponds to a regular dessin $\mathcal{D}$, with $G \cong \mathrm{Aut}\,\mathcal{D}$, and normality of N in Δ^* corresponds to $\mathcal{D}$ being reflexible, that is, $\mathcal{D} \cong \overline{\mathcal{D}}$, with G^* the full automorphism group of the hypermap; normality in $\Delta^=$ corresponds to $\mathcal{D} \cong \mathcal{D}^{(01)}$, with $G^= = \mathrm{Aut}^+\mathcal{W}$, where $\mathcal{W} = W(\mathcal{D})$ is regarded as an orientably regular uncoloured map; finally, normality of N in $\Delta^\times$ corresponds to $\mathcal{D}$ being isomorphic to the mirror image of its dual $\mathcal{D}^{(01)}$. (When $l = m = n$ there are three duals to consider.)

4. Proof of Theorems 1.1 and 1.2

We will now prove Theorem 1.1, with a short digression in §4.2 to deal with Theorem 1.2.

4.1. The conditions are sufficient

Proof. Let $\mathcal{D}$ have type (l, m, n), so it corresponds to a normal subgroup N of $\Delta = \Delta(l, m, n)$, with $\Delta/N \cong G = \mathrm{Aut}\,\mathcal{D}$. We will show that each of conditions (1) to (4) in Theorem 1.1 is each sufficient for $\mathcal{S}$ to admit a symmetry.

If condition (1) holds, let G^* be the extension of G by $\langle\alpha\rangle \cong C_2$, with α acting naturally by conjugation on G. The epimorphism $\Delta \to G$ with kernel N extends to an epimorphism $\Delta^* = \langle \Delta, T\rangle \to G^*$, $T \mapsto \alpha$ with kernel N, where $\Delta^* = \Delta^*(l, m, n)$, and the reflection T induces a symmetry of $\mathcal{S}$. A similar argument applies to condition (2), with $\Delta^\times$ used instead of Δ^*.

If condition (3) holds, the epimorphism $\Delta \to G$ extends, firstly to an epimorphism $\Delta^= \to \langle G, \gamma\rangle = G^=$, and then to an epimorphism $(\Delta^=)^\times \to \langle G, \gamma, \delta\rangle = (G^=)^\times$, with δ lifting to a reflection inducing a symmetry of $\mathcal{S}$.

If condition (4) holds, $\mathcal{S}$ is a torus $\mathbb{C}/\Lambda$ for some lattice $\Lambda \subset \mathbb{C}$. Having genus 1, $\mathcal{D}$ must have a period $k = l, m$ or n greater than 2, so $\mathcal{S}$ admits an automorphism of order $k > 2$ with a fixed point. This lifts to a rotation of order $k > 2$ of Λ, so Λ is a square or hexagonal lattice. In either case Λ admits reflections of $\mathbb{C}$, inducing symmetries of $\mathcal{S}$. □

4.2. The conditions are necessary

Proof. For the converse, suppose that $\mathcal{S}$ is symmetric. If $\mathcal{S}$ has genus $g = 0$ then condition (1) holds, and condition (4) deals with the case $g = 1$, so we may assume that $g \geq 2$. Any symmetry of $\mathcal{S}$ is induced by an orientation-reversing isometry R of $\mathbb{H}$ normalising N and satisfying $R^2 \in N$. Let $\Gamma = \langle \Delta, R \rangle$. Since Δ and R normalise N, Γ is contained in the normaliser of N in the isometry group $PGL_2(\mathbb{R})$. This is an NEC group, and hence so is Γ. Since it contains R, Γ is a proper NEC group, so let Γ^+ be its orientation-preserving Fuchsian subgroup of index 2.

If Δ is a normal subgroup of Γ then $|\Gamma : \Delta| = 2$, so $\Delta = \Gamma^+$. Then Lemma 3.1 shows that Γ is either Δ^*, giving condition (1), or a group $\Delta^\times$, giving condition (2). We may therefore assume that we have a non-normal inclusion $\Delta < \Gamma$. Since Γ^+ is a Fuchsian group containing the triangle group Δ, it is also a triangle group, as shown by Singerman in [15]; moreover, Δ is a proper subgroup of Γ^+ since it is not normal in Γ. This immediately proves Theorem 1.2, since it contradicts the maximality of Δ assumed there. Continuing with the proof of Theorem 1.1, the inclusion $\Delta < \Gamma^+$ must appear in Singerman's list of triangle group inclusions [15], shown in Table 1. There are seven sporadic examples and seven infinite families: cases (a), (b) and (c) are normal inclusions, while cases (A) to (K) are non-normal. In the fifth column, P denotes the permutation group induced by Γ^+ on the cosets of Δ.

Case	Type of Δ	Type of Γ^+	$\lvert\Gamma^+ : \Delta\rvert$	P	Thm 1.1
a	(s, s, t)	$(2, s, 2t)$	2	S_2	(1), (2)
b	(t, t, t)	$(3, 3, t)$	3	A_3	(2)
c	(t, t, t)	$(2, 3, 2t)$	6	S_3	(1), (2)
A	$(7, 7, 7)$	(2,3,7)	24	$L_2(7)$	(1)
B	$(2, 7, 7)$	(2,3,7)	9	$L_2(8)$	(2)
C	$(3, 3, 7)$	(2,3,7)	8	$L_2(7)$	(2)
D	$(4, 8, 8)$	(2,3,8)	12	$(C_4 \times C_4) \rtimes S_3$	(2)
E	$(3, 8, 8)$	(2,3,8)	10	$PGL_2(9)$	(2)
F	$(9, 9, 9)$	(2,3,9)	12	$L_2(\mathbb{Z}_9)$	(2)
G	$(4, 4, 5)$	(2,4,5)	6	S_5	(2)
H	$(n, 4n, 4n)$	$(2, 3, 4n)$	6	S_4	(2)
I	$(n, 2n, 2n)$	$(2, 4, 2n)$	4	D_4	(2)
J	$(3, n, 3n)$	$(2, 3, 2n)$	4	A_4	(1)
K	$(2, n, 2n)$	$(2, 3, 2n)$	3	S_3	(1)

Table 1. Inclusions between Fuchsian triangle groups.

In all cases except (a), with $s = 2t$, and (b), Γ^+ has no repeated periods, so Lemma 3.1 shows that Γ is the extended triangle group $(\Gamma^+)^*$. As shown by Watson in the Appendix of [18], in each such case the dessin corresponding to the inclusion $\Delta < \Gamma^+$ is reflexible (see §4.3 for an example), so Δ has index 2 in a proper NEC group $\tilde{\Delta} \leq \Gamma$. Lemma 3.1 shows that $\tilde{\Delta} = \Delta^*$ or $\Delta^\times$. Since $\tilde{\Delta}$, as a subgroup of Γ, normalises N, it follows that $\tilde{G} := \tilde{\Delta}/N$ is an extension of $G = \Delta/N$ by an automorphism satisfying condition (1) or (2) of Theorem 1.1 respectively, as indicated in the final column of Table 1.

In the two exceptional cases, a pair of repeated periods allows the additional possibility that $\Gamma = (\Gamma^+)^\times$ (not $(\Gamma^+)^=$, since Γ is a proper NEC group). In case (b), if $\Gamma = (\Gamma^+)^\times$ then Δ is normal in Γ with quotient C_6, and the involution in Γ/Δ corresponds to a proper NEC group $\tilde\Delta < \Gamma$ containing Δ with index 2, as before. Hence there remains only case (a) with $s = 2t$, or equivalently $\Delta = \Delta(2n, 2n, n)$, $\Gamma^+ = \Delta(2n, 2, 2n)$ and $\Gamma = \Delta^\times(2n, 2, 2n)$, considered in §4.4.

4.3. Example of a typical case

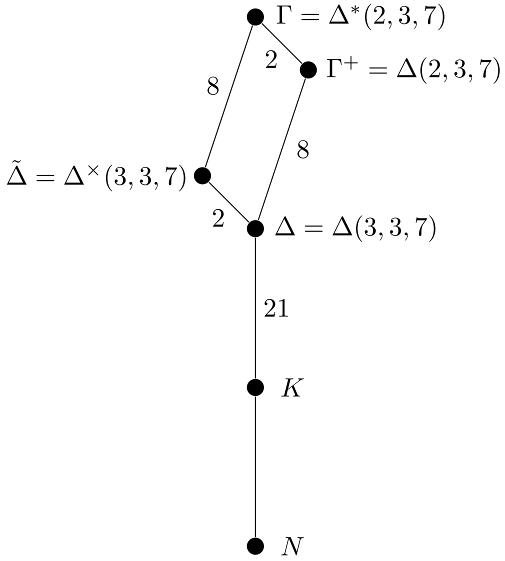

FIGURE 1. Lattice of some subgroups of $\Delta^*(2, 3, 7)$.

Suppose that the inclusion of triangle groups $\Delta < \Gamma^+$ is case C in Table 1. Thus $\Delta = \Delta(3, 3, 7)$ is a subgroup of index 8 in $\Gamma^+ = \Delta(2, 3, 7)$, so that $\Gamma = \Delta^*(2, 3, 7)$ by Lemma 3.1. Since N is a subgroup of Δ and is normal in Γ, it is contained in the core K of Δ in Γ. This is a normal subgroup of Γ, a surface group of genus 3 contained in Γ^+, with $\Gamma^+/K \cong P = L_2(7)$ and $\Gamma/K \cong PGL_2(7)$; the Riemann surface $\mathbb{H}/K$ uniformised by K is Klein's quartic curve $x^3y + y^3z + z^3x = 0$. The image of Δ in $PGL_2(7)$ is a subgroup H of order 21 and index 8 in $L_2(7)$: this is the stabiliser of a point in the natural action on the projective line $\mathbb{P}^1(\mathbb{F}_7)$, isomorphic to the unique subgroup of index 2 in $AGL_1(7)$. The stabiliser in $PGL_2(7)$ of this point is isomorphic to $AGL_1(7)$, an extension of H by an involution which acts on its two canonical generators of order 3 as in condition (2) of Theorem 1.1. This lifts to a proper NEC subgroup $\tilde\Delta$ of Γ which contains Δ with index 2, acting in the same way on its two generators of order 3. By Lemma 3.1, $\tilde\Delta$ must be $\Delta^\times = \Delta^\times(3, 3, 7)$, so it contains a reflection which, since it normalises N, induces on $G := \Delta/N$ an automorphism satisfying condition (2). Figure 1 shows the inclusions between these subgroups of Γ; edges are labelled with indices of inclusions. The dessin corresponding to the inclusion $\Delta < \Gamma^+$ is shown in Figure 2 as a map on the sphere, where we have changed generators to take $\Gamma^+ = \Delta(3, 2, 7)$; the generators x, y and z of G of order $3, 3$ and 7 correspond, as in Theorem 1 of [14], to short cycles of the elliptic generators of Γ^+, at the two vertices and the one face of valency 1; the obvious reflection transposes and inverts x and y.

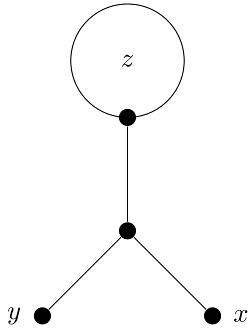

Figure 2. Map for the inclusion $\Delta(3, 3, 7) < \Delta(2, 3, 7)$.

For specific instances of such subgroups N one can use Macbeath's construction in [11] of an infinite sequence of Hurwitz groups Γ^+/N: for any integer $r \geq 1$, the group $N = K'K^r$ generated by the commutators and rth powers of elements of K is a characteristic subgroup of K and hence a normal subgroup of Γ, of index $336r^6$. The surface $\mathcal{S} = \mathbb{H}/N$ carries a regular chiral dessin $\mathcal{D}$ of type $(3, 3, 7)$ and genus $1+2r^6$, a regular covering of the chiral dessin of type $(3, 3, 7)$ and genus 1 corresponding to the inclusion $K < \Delta$. The automorphism group of the Riemann surface $\mathcal{S}$ is an extension Γ^+/N of an abelian group $K/N \cong C_r^6$ by $\Gamma^+/K \cong L_2(7)$; if we include anti-conformal automorphisms we obtain an extension Γ/N of K/N by $\Gamma/K \cong PGL_2(7)$.

4.4. The exceptional case

The exceptional case in the proof of Theorem 1.1 arises in case (a) of Table 1 when $s = 2t$. Putting $t = n$ we have $\Delta = \Delta(2n, 2n, n)$, $\Gamma^+ = \Delta^= = \Delta(2n, 2, 2n)$ and $\Gamma = \Delta^\times(2n, 2, 2n)$. In this case there is no proper NEC group $\tilde{\Delta} \leq \Gamma$ containing Δ with index 2: indeed, the only proper NEC groups containing Δ with index 2 are Δ^* and $\Delta^\times$, both of which are subgroups of $\Delta^*(2n, 2, 2n)$ rather than of $\Delta^\times(2n, 2, 2n)$.

The inclusions between these NEC groups are shown in Figure 3, where Λ is the maximal NEC group $\Delta^*(4, 2, 2n)$ containing Γ. Black and white vertices indicate normal and non-normal subgroups of Λ, and edges indicate inclusions, all of index 2. The normaliser $N(\Delta)$ of Δ (in the isometry group of $\mathbb{H}$) is the extended triangle group $\Delta^*(2n, 2, 2n)$, which has index 2 in Λ. The core of Δ in Λ is the group $K = \Delta \cap \Delta^L$, where L is any element of $\Lambda \setminus N(\Delta)$; this is a quadrilateral group $\Delta(n, n, n, n)$. The quotient Λ/K is isomorphic to $\Delta^*(4, 2, 2) \cong D_4 \times C_2$, the automorphism group of the regular map $\{2, 4\}$ on the sphere; this can be seen by applying the natural epimorphism $\Lambda = \Delta^*(4, 2, 2n) \to \Delta^*(4, 2, 2)$, with kernel K.

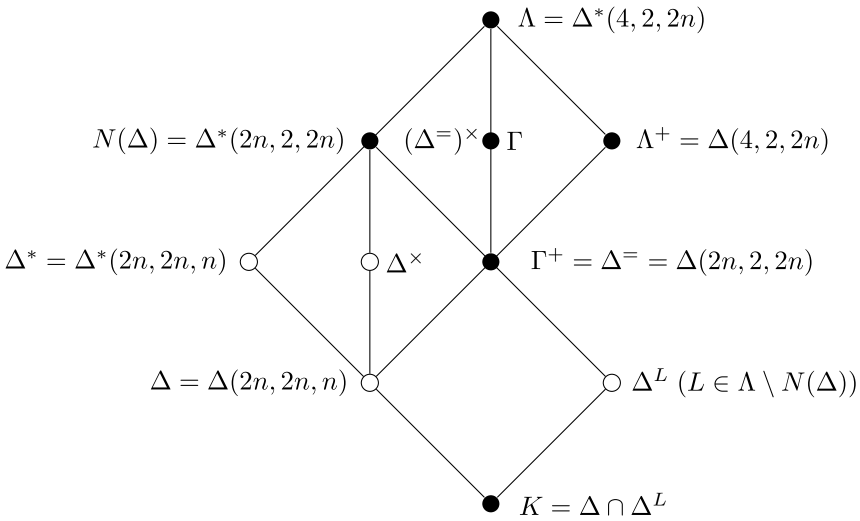

Figure 3. Lattice of groups in the exceptional case.

To complete the proof, since N is normal in Γ with $G \cong \Delta/N$, and since $\Delta \leq \Delta^= \leq \Gamma$, there is an extension $G^= = \langle G, \gamma\rangle \cong \Delta^=/N$ of G, where γ is an automorphism of order 2 of G transposing its canonical generators x and y of order $2n$. The canonical generators of $G^=$, the images of those of $\Delta^= = \Delta(2n,2,2n)$, are x, γ and γx^{-1}. Since $\Delta^= \leq \Gamma = (\Delta^=)^\times$, there is an extension $(G^=)^\times = \langle G^=, \delta\rangle \cong \Gamma/N$ of $G^=$, where δ is an automorphism of order 2 of $G^=$ sending x to $(\gamma x^{-1})^{-1} = x\gamma$. Thus condition (3) is satisfied. □

4.5. Example of the exceptional case

In the exceptional case, N is a torsion-free subgroup of finite index in $\Delta = \Delta(2n,2n,n)$; it is normal $\Gamma = (\Delta^=)^\times = \Delta^\times(2n,2,2n)$, and therefore contained in K. Any torsion-free characteristic subgroup N of finite index in K will correspond to a dessin $\mathcal{D}$ with $G = \operatorname{Aut}\mathcal{D}$ satisfying condition (3) of Theorem 1.1: the normality of N in Γ^+ and Γ provides the required automorphisms γ and δ. However, such a subgroup N is normal in Λ, and hence in Δ^* and $\Delta^\times$, so (1) and (2) are also satisfied. To show that condition (3) is independent of the others, and hence needed for a correct statement of Theorem 1.1, we will give an example where N is normal in Γ but not in Λ, with $n > 2$ so that $\mathcal{S}$ does not satisfy (4). If conditions (1) or (2) were satisfied then N would be normal in Δ^* or $\Delta^\times$ as well as in Γ, and hence normal in Λ, which is false.

In [1], Biggs showed that the complete graph K_q on q vertices has an orientably regular embedding if and only if q is a prime power. When $q = 2^e$ the examples he constructed are maps $\mathcal{M}_1$ of type $\{q-1, q-1\}$ and genus $(q-1)(q-4)/4$ with $\operatorname{Aut}^+\mathcal{M}_1 \cong AGL_1(q)$; for instance, when $q = 8$ they are the Edmonds maps. Each such map corresponds to a normal subgroup M_1 of $\Delta(n, 2, n)$ with $\Delta(n, 2, n)/M_1 \cong AGL_1(q)$, where $n = q - 1$. Using the fact that $\mathrm{Aut}\, AGL_1(q) = A\Gamma L_1(q)$, James and Jones [8] showed that if $q = 2^e \geq 8$ then $\mathcal{M}_1$ is chiral and $\overline{\mathcal{M}}_1 \cong \mathcal{M}_1^*$.

We need these two properties to be satisfied by a bipartite map, which can then be the Walsh map of a dessin $\mathcal{D}$ as in the combinatorial explanation of condition (3) in §2. Since $\mathcal{M}_1$ is not bipartite, we construct a bipartite covering of it with the same properties. Let $\mathcal{M}_2$ be the orientably regular map of type $\{2, 2\}$ on the sphere, with two vertices, joined by two edges, so that $\mathrm{Aut}^+\mathcal{M}_2 \cong V_4$. The join $\mathcal{M}_1 \vee \mathcal{M}_2$ of these two maps is an orientably regular map $\mathcal{M}_3$ of type $\{2n, 2n\}$, corresponding to a torsion-free normal subgroup $N = N_1 \cap N_2$ of finite index in $\Gamma^+ = \Delta(2n, 2, 2n)$, where N_1 and N_2 are the inverse images of M_1 and 1 in Γ^+ under the natural epimorphisms $\Delta(2n, 2, 2n) \to \Delta(n, 2, n)$ and $\Delta(2n, 2, 2n) \to \Delta(2, 2, 2) \cong V_4$. Since the groups $\Gamma^+/N_i \cong AGL_1(q)$ and V_4 for $i = 1, 2$ have no non-trivial common quotients, we have $\Gamma^+ = N_1N_2$; thus $\mathcal{M}_3 = \mathcal{M}_1 \times \mathcal{M}_2$ and

$$\mathrm{Aut}^+\mathcal{M}_3 \cong \Gamma^+/N \cong (N_2/N) \times (N_1/N) \cong AGL_1(q) \times V_4.$$

These subgroups are shown in Figure 4, with black and white vertices indicating normal and non-normal subgroups of Λ.

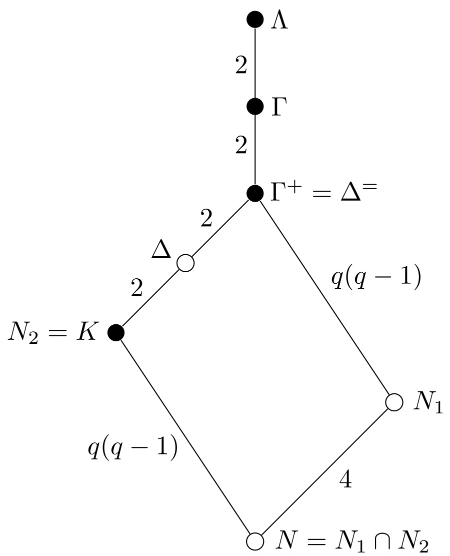

FIGURE 4. Lattice of groups in the example of the exceptional case.

Now K is the unique normal subgroup of Γ^+ with quotient group V_4, so $N_2 = K$ and hence $N \leq \Delta$. Since $\mathcal{M}_i \cong \overline{\mathcal{M}_i^*}$ for $i = 1, 2$, each N_i is normal in Γ, and hence so is N, so $\mathcal{M}_3 \cong \overline{\mathcal{M}_3^*}$. If N were normal in Λ then it would be normal in $\Delta(2n, 2, 2n)^*$; since the direct factor $N_1/N \cong V_4$ is a characteristic subgroup of Γ^+/N (as the centraliser of the subgroup $N_2/N \cong AGL_1(q)$ generated by the elements of odd order) it would follow that N_1 is normal in $\Delta(2n, 2, 2n)^*$, contradicting the chirality of $\mathcal{M}_1$. Thus N is not normal in Λ, as required.

This construction can be interpreted combinatorially as follows. As a covering of the bipartite map $\mathcal{M}_2$, the map $\mathcal{M}_3$ is also bipartite, so it is the Walsh map $\mathcal{W} = W(\mathcal{D})$ of a dessin $\mathcal{D}$: this is a regular dessin of type $(2n, 2n, n)$, corresponding to the normal inclusion of N in $\Delta = \Delta(2n, 2n, n)$. Condition (3) corresponds to the fact that $\mathcal{W}$ is orientably regular and $\mathcal{W} \cong \overline{\mathcal{W}^*}$. The failure of conditions (1) and (2) corresponds to $\mathcal{D}$ not being isomorphic to $\overline{\mathcal{D}}$ or $\overline{\mathcal{D}^{(01)}}$.

In this example, $\mathcal{S}$ has genus $g = n^2 - n - 1$. This is minimised when $e = 3$, so $g = 41$ and the chiral maps $\mathcal{W}$ and $\mathcal{W}^*$ correspond to C41.24 in [3].

Further examples of this type can be found by using the 'Macbeath trick' [11] as in §4.3. Let r be any integer coprime to $2n$. Since $N'N^r$ is a characteristic subgroup of N, it is normal in Γ. If $N'N^r$ were normal in Λ then, since $N/N'N^r$ is a normal subgroup of $\Lambda/N'N^r$ (being generated by its elements of order r), N would be normal in Λ, which is false. Thus $N'N^r$ is not normal in Λ.

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