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Lower bounds for the truncated Hilbert transform

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Abstract. Given two intervals $I, J \subset \mathbb{R}$, we ask whether it is possible to reconstruct a real-valued function $f \in L^2(I)$ from knowing its Hilbert transform Hf on J. When neither interval is fully contained in the other, this problem has a unique answer (the nullspace is trivial) but is severely ill-posed. We isolate the difficulty and show that by restricting f to functions with controlled total variation, reconstruction becomes stable. In particular, for functions $f \in H^1(I)$, we show that

$$\|Hf\|_{L^2(J)} \geq c_1 \exp\Big(-c_2 \frac{\|f_x\|_{L^2(I)}}{\|f\|_{L^2(I)}}\Big) \|f\|_{L^2(I)},$$

for some constants $c_1, c_2 > 0$ depending only on I, J. This inequality is sharp, but we conjecture that $\|f_x\|_{L^2(I)}$ can be replaced by $\|f_x\|_{L^1(I)}$.

1. Introduction and motivation

1.1. Hilbert transform

The Hilbert transform $H\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a well-studied unitary operator given by

$$(Hf)(x) = \frac{1}{\pi}\,\text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy,$$

where p.v. indicates that the integral is to be understood as a principal value. On $L^2(\mathbb{R})$ it can alternatively be defined via the Fourier multiplier $-i\,\mathrm{sgn}(\xi)$. The Hilbert transform appears naturally in many different settings in pure and applied mathematics. In particular, it plays an important role in the mathematical study of inverse problems arising in medical imaging (see §1.5), which motivates the following fundamental question.

> **Inversion problem.** Given two finite intervals $I, J \subset \mathbb{R}$ and a real-valued function $f \in L^2(I)$, when can f be reconstructed from knowing Hf on J?

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For illustration, let us consider first the particular case in which the intervals I and J are disjoint. The Hilbert transform is an integral operator – thus if a function f is compactly supported and we consider the Hilbert transform Hf only outside of that support, the singularity of the kernel never plays a role and the operator is compact (i.e., smoothing). It is clear from basic principles in functional analysis that the inversion of a compact operator will not yield a bounded operator.

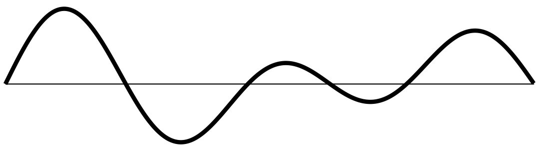


FIGURE 1. A function f on $[0,1]$ with $\|Hf\|_{L^2([2,3])} \sim 10^{-7}\|f\|_{L^2([0,1])}$.

1.2. The phenomenon in practice

Let us understand just how ill-posed the problem actually is. Consider the Hilbert transform applied to functions with support on $I=[0,1]$ and then take its restriction on $J=[2,3]$, leading to the truncated Hilbert transform

$$H_T = \chi_{[2,3]} H(\chi_{[0,1]} f).$$

The operator $H_T\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a compact integral operator. As a consequence of it being compact, for every $\varepsilon > 0$ we can find a function $f \in L^2([0,1])$ with

$$\|H_T f\|_{L^2([2,3])} \leq \varepsilon \, \|f\|_{L^2([0,1])}.$$

Such f are actually very easy to find: let $\phi_1, \phi_2, \dots, \phi_n$ denote any n orthonormal real-valued functions in $L^2([0,1])$ and consider the subspace

$$\mathcal{S} = \operatorname{span}(\phi_1, \phi_2, \dots, \phi_n)$$

spanned by these functions. Then, putting

$$f = \sum_{k=1}^{n} a_k \, \phi_k$$

for real coefficients a_k immediately implies that

$$\begin{aligned}\|H_T f\|^2_{L^2([2,3])} &= \int_2^3 \Big(H_T \sum_{k=1}^{n} a_k \, \phi_k\Big)^2 dx = \int_2^3 \sum_{k,l=1}^{n} a_k a_l (H_T\phi_k)(H_T\phi_l)\, dx \\ &= \sum_{k,l=1}^{n} a_k a_l \int_2^3 (H_T\phi_k)(H_T\phi_l)\, dx.\end{aligned}$$

This yields that the symmetric $n \times n$ matrix

$$A = \left(\int_2^3 (H_T \phi_k)(H_T \phi_l)\, dx \right)_{k,l=1}^{n}$$

will satisfy the relation

$$\inf_{f \in \mathcal{S}} \frac{\|H_T f\|_{L^2([2,3])}}{\|f\|_{L^2([0,1])}} = \sqrt{\lambda_{\min}(A)},$$

where $\lambda_{\min}(A)$ denotes the least eigenvalue of A. Put differently, $H_T \colon \mathcal{S} \to H_T(\mathcal{S})$ may be a bijection but the inverse operator is very sensitive to noise in the measurement because

$$\|H_T^{-1}\|_{H_T(\mathcal{S}) \to \mathcal{S}} = \frac{1}{\sqrt{\lambda_{\min}(A)}} \qquad \text{will be very big.}$$

So far, this discussion could apply to a wide class of operators; focusing on our situation, the key point is that the spectrum of H_T is rapidly decaying and therefore $\lambda_{\min}(A)$ will always be very small, independent of the n orthonormal functions we pick; indeed (see equation (4.3)), there exist real constants $C, \beta > 0$ dependent only on the intervals I, J such that

$$\lambda_{\min}(A) \le C e^{-\beta n}.$$

1.3. An explicit example

We consider a numerical example; let $I = [0,1]$, $J = [2,3]$ and consider the subspace spanned by

$$\phi_i = \sqrt{3}\, \chi_{[\frac{i-1}{3}, \frac{i}{3}]} \quad \text{for } 1 \le i \le 3.$$

The choice of these functions is solely motivated by the fact that $H_T \phi_i$ can be written down in closed form, which simplifies computation. Then the matrix A has the eigenvalues

$$\sim \left\{0.28,\ 0.00013,\ 2.2 \cdot 10^{-8}\right\}, \quad \text{which are very rapidly decaying.}$$

As a consequence, there exists a step function g, which is constant on the three intervals of length $1/3$ in $[0,1]$ (and is therefore certainly quite simple), but nonetheless satisfies

$$\|H_T g\|_{L^2([2,3])} \le 10^{-7}\, \|g\|_{L^2([0,1])}.$$

It is interesting to compare this with a larger subspace. Pick now, for comparison,

$$\phi_i = \sqrt{5} \chi_{[\frac{i-1}{5}, \frac{i}{5}]} \quad \text{for } 1 \le i \le 5.$$

The smallest eigenvector of the arising matrix is $\lambda_{\min}(A) \le 10^{-15}$, allowing for the construction of a step function h with

$$\|H_T h\|_{L^2([2,3])} \le 10^{-15}\, \|h\|_{L^2([0,1])}.$$

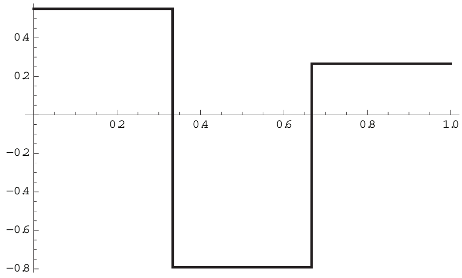


FIGURE 2. The function g ($n = 3$).

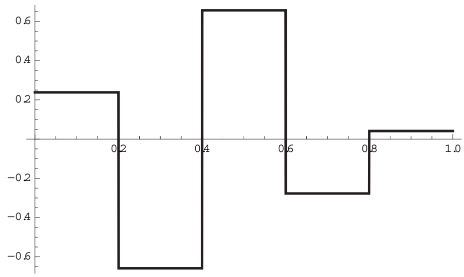


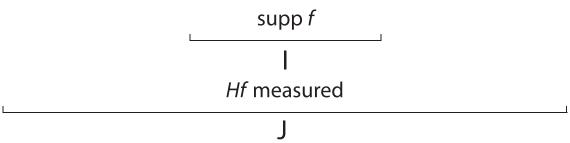
FIGURE 3. The function h ($n = 5$).

The contribution of our paper may now be phrased as follows: the fact that these functions are highly oscillatory is not a coincidence; indeed, it is the purpose of this paper to point out that an inverse inequality is true: *reconstruction becomes stable for functions with controlled total variation.* While there are a variety of techniques for understanding how to bound oscillating quantities from above (e.g. stationary phase), it is usually much harder to control oscillation from below – finding sharp quantitative versions of the above statement falls precisely into this class of problems; as such, we believe it to be very interesting. The same problem could be of great interest for more general integral operators, where a similar phenomenon should be true generically (see Section 8).

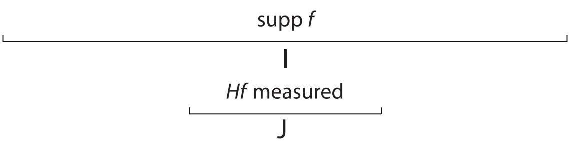
1.4. Configurations of the intervals: four cases

The precise nature of the problem of reconstructing a function supported on I from its Hilbert transform on J will depend on the relation between I and J. To address this question adequately, it is useful to distinguish four cases:

1. The Hilbert transform is known on an interval J that covers the support I of f (that is, $I \subset J$). In this case inversion is stable (the solution operator is bounded) and an explicit inversion formula is known [23].



2. The Hilbert transform is known only on an interval J that is a subset of the support I of f (that is, $I \supseteq J$). In tomographic reconstruction, this case is known as the *interior problem* [6], [11], [13], [14], [24].



3. The Hilbert transform is known only outside of the support of f (that is, $I \cap J = \emptyset$). We will refer to this scenario as the truncated Hilbert transform *with a gap.* The singular value decomposition of the underlying operator has been studied in [10].

4. If none of the above is the case and the Hilbert transform is known on an interval J that overlaps with the support I of f, we call this the truncated Hilbert transform *with overlap.* For this case a pointwise stability estimate has been shown in [7]. The spectral properties of the underlying operator are the subject of [3], [4].

In this paper, we consider Cases 3 and 4. For these, f is supported on I and Hf is known on J, where I and J are non-empty finite intervals on $\mathbb{R}$, such that $I \not\supseteq J$ and $I \not\subseteq J$. Let $\mathcal{P}_\Omega$ stand for the projection operator onto a set $\Omega \subset \mathbb{R}$:

$$(\mathcal{P}_\Omega f)(x) = f(x) \ \text{ if } x \in \Omega, \qquad (\mathcal{P}_\Omega f)(x) = 0 \ \text{ otherwise.}$$

We will use the notation $H_T = \mathcal{P}_J H \mathcal{P}_I$ to denote the truncated Hilbert transform (*with a gap* or *with overlap*), specialized to the intervals I and J.

1.5. Applications in medical imaging

The problem of reconstructing a function from its partially known Hilbert transform arises naturally in computerized tomography: assume a 2D or 3D object is illuminated from various directions by a penetrating beam (usually X-rays) and that the attenuation of the X-ray signals is measured by a set of detectors. Then, one seeks to reconstruct the object from the measured attenuation, which can be modeled as the Radon transform data of the object. If the directions along which the Radon transform is measured are sufficiently dense, the problem and its solution are well-understood (cf. [18]). When the directions are not sufficiently dense the problem is more complicated. One such setting is the case of *truncated projections* and occurs when only a sub-region of the object is illuminated by a sufficiently dense set of directions. Going back to a result by Gelfand and Graev [9], the method of differentiated back-projection allows one to reduce the problem to solving a family of one-dimensional problems which consist of inverting the Hilbert transform data on a finite segment of the line. If one knew Hf on all of $\mathbb{R}$, this would be trivial, since $H^{-1} = -H$.

In practice, Hf is measured on only a finite segment, giving rise to the different configurations 1 through 4 and the resulting reconstruction problems. In this paper, we focus on Case 3 (the truncated Hilbert transform with a gap) and Case 4 (the truncated Hilbert transform with overlap), which are the most unstable from the point of view of functional analysis. In fact, both these cases are *severely* ill-posed, meaning that the singular values of the underlying operator decay to zero at an exponential rate. (For the asymptotic analysis of the singular value decomposition in Case 3 we refer to Katsevich and Tovbis [12]; for Case 4, see [4].) In Case 3, the Hilbert transform is an integral operator with a smooth kernel and is thus compact. In general, one would expect Case 4 to be better behaved with respect to the inversion problem as long as the functions have, say, a fixed proportion of their L^2-mass supported on $I \cap J$. By considering the subproblem arising in Case 4 when we consider functions with compact support bounded away from J, we see that all the difficulties of Case 3 must also be present in Case 4. Inverse estimates specifically tailored to Case 4, which show their strength precisely for functions not supported away from J, are presented in Section 2.4.

1.6. Questions of regularity

In order to situate our results in terms of the role of regularity, it is worth observing that the actual problem of reconstruction is *not* easier for smooth functions. This is easily seen in Case 3: when I and J are disjoint, there is less stability of the inversion problem of the truncated Hilbert transform; in this case the truncated Hilbert transform turns into a highly regular smoothing integral operator (in contrast to the classical Hilbert transform which is the fundamental example of a singular integral operator). Indeed, when I and J are disjoint, the singularity of the Hilbert kernel never comes into play. This smoothing property of the truncated Hilbert transform with a gap allows one to approximate any function $f \in L^2(I)$ by C^∞ functions f_n such that $H_T f_n \to H_T f$ in $L^\infty(J)$. This can be seen from

$$\|H_T f_n - H_T f\|_{L^\infty(J)} \leq \tilde{c}\,\|f_n - f\|_{L^1(I)} \leq c\,\|f_n - f\|_{L^2(I)},$$

where

$$\tilde{c} = \max_{x\in I, y\in J} \frac{1}{|y-x|}$$

and $c = \tilde{c}\cdot|I|^{1/2}$. Yet while the problem of reconstruction is in theory no easier for smooth functions, our current methods will be able to obtain improved estimates for smooth functions (whereas any argument yielding a sharp result should be oblivious to questions of regularity). Another classical property we will make use of is that one can always approximate a function of bounded variation by smooth functions while controlling their total variation (TV). More precisely, we have the following lemma (which we prove in § 9.1):

Lemma 1.1. *Given a function $f \in \mathrm{BV}(I)$ satisfying $f(x_0) = 0$ for at least one $x_0 \in I$, there exists a sequence $f_n \in C_c^\infty(I)$ such that*

$$\|f_n - f\|_{L^2(I)} \to 0 \quad \textit{and} \quad |f_n|_{\mathrm{TV}} \leq 3\cdot|f|_{\mathrm{TV}}\,.$$

We note that the condition that f vanishes at least at one point in the interval will not be a significant restriction in our applications of this lemma (see Lemma 6.2, and subsequent remarks, for example).

Notation. In the following, I and J always denote finite open intervals on $\mathbb{R}$. We write $C_c^N(I)$ for the space of N-times differentiable functions compactly supported on I. As conventional, H^k denotes the Sobolev space $W^{k,2}$, and we recall the following well-known inclusions for a finite interval $\Omega \subset \mathbb{R}$:

$$H^1(\Omega) \subset W^{1,1}(\Omega) \subset BV(\Omega) \subset L^2(\Omega) \subset L^1(\Omega).$$

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2. Statement of results

2.1. Functions of bounded variation

Our first finding establishes a stability result for functions of bounded variation. This seems to be the appropriate notion to exclude strong oscillation while still allowing for rather rough functions with jump discontinuities. The total variation (TV) model has been studied as a regularizing constraint in computerized tomography before, see e.g., [20].

Theorem 2.1. *Let $I, J \subset \mathbb{R}$ be intervals in the configuration of Case 3 or Case 4 and consider functions $f \in \mathrm{BV}(I)$ supported on I. There exists a positive function $h : [0, \infty) \to \mathbb{R}_+$ (depending only on I, J) such that*

$$\|Hf\|_{L^2(J)} \geq h\Big(\frac{|f|_{\mathrm{TV}}}{\|f\|_{L^2(I)}}\Big)\|f\|_{L^2(I)},$$

where $|\cdot|_{\mathrm{TV}}$ denotes the total variation of f.

We conjecture

$$h(\kappa) \geq c_1\, e^{-c_2\kappa}$$

for constants $c_1, c_2 > 0$ depending only on I and J.

The relation between Theorem 2.1 and the reconstruction problem can easily be made explicit. In the application of computerized tomography one needs to solve $H_T f = g$ for f, given a right-hand side g. In practice, g has to be measured and is thus never known exactly, but only up to a certain accuracy. Since the range of the operator H_T is dense but not closed in $L^2(J)$, the inversion of H_T is ill-posed, see [3]. As a consequence, the solution f to $H_T f = g$ does not depend continuously on the right-hand side. In particular, small perturbations in g due to measurement noise might change the solution completely, making the outcome unreliable. Given a function g representing exact data, of which we know only a noisy measurement g^δ and the noise level $\|g - g^\delta\|_{L^2(J)} \leq \delta$, quantitative results taking the form of Theorem 2.1 will enable stable reconstruction, under the assumption that the true solution f_{ex} to $H_T f = g$ has bounded variation (see Corollary 2.1).

2.2. Weakly differentiable functions

We now turn our focus to proving quantitative versions of Theorem 2.1 for more regular functions f. For weakly differentiable functions we can actually write

$$|f|_{\mathrm{TV}} = \int_I |f_x(x)|\, dx$$

and thus identify the total variation with $\|f_x\|_{L^1(I)}$. In light of Theorem 2.1, the total variation seems to be the natural quantity with which to track the behavior of regular functions, and we conjecture that

$$\|Hf\|_{L^2(J)} \geq c_1 \exp\left(-c_2 \frac{\|f_x\|_{L^1(I)}}{\|f\|_{L^2(I)}}\right) \|f\|_{L^2(I)}. \tag{2.1}$$

An inequality of this form would quantify the physically intuitive notion that tomographic reconstruction is more difficult for inhomogeneous objects with high variation in density than it is for relatively uniform objects. Our first result toward this conjecture considers $\|f_x\|_{L^2(I)}$ instead, which provides access to Hilbert space techniques that allow us to prove the following statement:

Theorem 2.2. *Let $I, J \subset \mathbb{R}$ be intervals in the configuration of Case 3 or Case 4. Then, for any $f \in H^1(I)$,*

$$\|Hf\|_{L^2(J)} \geq c_1 \exp\left(-c_2 \frac{\|f_x\|_{L^2(I)}}{\|f\|_{L^2(I)}}\right) \|f\|_{L^2(I)}, \tag{2.2}$$

for some constants $c_1, c_2 > 0$ depending only on I and J.

We note that Theorem 2.2 is weaker than the conjectured inequality (2.1): a step function f, for example, can be approximated by smooth functions f_n in such a way that $\|(f_n)_x\|_{L^1}$ remains controlled by the total variation of f. However, this is no longer true for $\|(f_n)_x\|_{L^2}$, which must necessarily blow up. Yet we may improve on Theorem 2.2 if f is sufficiently smooth and obtain a result which in certain cases is as strong as the conjectured relation (2.1):

Theorem 2.3. *Let $I, J \subset \mathbb{R}$ be intervals in the configuration of Case 3 or Case 4. Then there exists an order 2 differential operator L_I and for any $M \geq 1$ a dense class A_M of L^2-functions (defined in §4) such that, for any $f \in A_M$,*

$$\|Hf\|_{L^2(J)} \geq c_{1,M} \exp\left(- c_{2,M} \left(\frac{\|(L_I^M f)_x\|_{L^2(I)}}{\|f\|_{L^2(I)}}\right)^{\frac{1}{2M+1}}\right) \|f\|_{L^2(I)}, \tag{2.3}$$

for some constants $c_{1,M}, c_{2,M} > 0$ depending only on I, J and M. As $M \to \infty$, $c_{1,M}, c_{2,M}$ tend to finite limits $c_1, c_2 > 0$. Furthermore, $C_c^{2M+1}(I) \subset A_M$ and $C_c^\infty(I) \subset \bigcap_{n=1}^\infty A_n$.

In certain examples, this result approaches the desired conjecture (2.1). Consider, for instance, the interval $I = (0,1)$, use dilation to move the support of the function $f_N(x) = \sin(2\pi Nx)\chi_{[0,1]}$ strictly inside the unit interval and convolve with a compactly supported C^∞ bump function. In this case $(L_I^M f_N)_x$ contains a main term of size $(2\pi N)^{2M+1}\cos(2\pi Nx)$. Then, morally speaking, the theorem implies

$$\begin{aligned}\|Hf_N\|_{L^2(J)} &\geq c_{1,M}\exp\left(-c_{2,M}2\pi N\left(c_J\frac{\|\cos(2\pi Nx)\|_{L^2(I)}}{\|f_N\|_{L^2(I)}}\right)^{\frac{1}{2M+1}}\right)\|f_N\|_{L^2(I)}\\ &\geq c_1\exp\left(-c_2N\right)\|f_N\|_{L^2(I)}\end{aligned}$$

as $M\to\infty$. Here $c_{1,M}, c_{2,M}, c_1, c_2$ are as in Theorem 2.3 and c_J is a constant depending only on J (since we have fixed the interval I). This is of the form (2.1), because in this example

$$\|(f_N)_x\|_{L^1(I)}\,/\,\|f_N\|_{L^2(I)} \approx N. \tag{2.4}$$

2.3. A quantitative result for functions with bounded variation

Our next result gives a different type of result toward the conjecture (2.1), now for functions $f\in W^{1,1}(I)$, and with quadratic scaling within the exponential. We note that the inequality below is superior to the bound given by Theorem 2.2 only for functions with $\|f_x\|_{L^2} \gg \|f_x\|^2_{L^1}/\|f\|_{L^2(I)}$.

Theorem 2.4. *Let $I, J\subset\mathbb{R}$ be intervals in the configuration of Case 3 or Case 4. Then, for any $f\in W^{1,1}(I)$,*

$$\|Hf\|_{L^2(J)} \geq c_1\exp\left(-c_2\frac{|f|^2_{\mathrm{TV}}}{\|f\|^2_{L^2(I)}}\right)\|f\|_{L^2(I)},$$

for some constants $c_1, c_2 > 0$ depending only on I, J.

Theorem 2.4 provides a stability estimate (independent of a specific algorithm) for the reconstruction of a solution f to $H_Tf = g$.

Corollary 2.1 (Stable reconstruction). *Let $g\in\mathrm{Ran}(H_T)$, such that*

$$H_Tf_{\mathrm{ex}} = g,$$

and $|f_{\mathrm{ex}}|_{\mathrm{TV}}\leq\kappa$. Furthermore, let $g^\delta\in L^2(J)$ satisfy

$$\|g-g^\delta\|_{L^2(J)}\leq\delta$$

for some $\delta>0$, and define the set of admissible solutions to be

$$S(\delta, g^\delta) = \{f\in W^{1,1}(I) : \|H_Tf-g^\delta\|_{L^2(J)}\leq\delta, |f|_{\mathrm{TV}}\leq\kappa\}.$$

Then, the diameter of $S(\delta, g^\delta)$ tends to zero as $\delta\to 0$ (at a rate of the order $|\log\delta|^{-1/2}$).

Thus, under the assumption that the true solution f_{ex} to $H_Tf = g$ has bounded variation, any algorithm that, given δ and g^δ, finds a solution in $S(\delta, g^\delta)$, is a regularization method.

As with Theorem 2.2, we are again able to improve on Theorem 2.4 by assuming f is sufficiently smooth, in which case the inequality approaches in the limit an inequality that is in certain cases as strong as the conjecture (2.1).

Theorem 2.5. *Let $I, J \subset \mathbb{R}$ be intervals in the configuration of Case 3 or Case 4. Then there exists an order 2 differential operator L_I (defined in §4) such that for any $M \geq 1$ and any $f \in C_c^{2M+1}(I)$,*

$$\|Hf\|_{L^2(J)} \geq c_{1,M} \exp\left(- c_{2,M} \left(\frac{|L_I^M f|_{\mathrm{TV}}}{\|f\|_{L^2(I)}} \right)^{\frac{2}{4M+1}} \right) \|f\|_{L^2(I)},$$

for some constants $c_{1,M}, c_{2,M} > 0$ depending only on I, J and M, with the property that as $M \to \infty$, $c_{1,M}, c_{2,M}$ tend to finite limits $c_1, c_2 > 0$.

Note that Theorem 2.5 reduces to Theorem 2.4 for $M = 0$. It is again instructive to consider an example. For this purpose we can take $f_N(x) = \sin(2\pi N x)$ with the interval $I = (0,1)$ as before, and again use dilation and convolution with a compactly supported C^∞ bump function to bring f_N into $C_c^{2M+1}(I)$. Then, morally speaking, the theorem implies

$$\begin{aligned} \|Hf_N\|_{L^2(J)} &\geq c_{1,M} \exp\left(- c_{2,M} \left(\frac{(2\pi N)^{2M+1} \|\cos(2\pi N x)\|_{L^1(I)}}{\|f_N\|_{L^2(I)}} \right)^{\frac{2}{4M+1}} \right) \|f_N\|_{L^2(I)} \\ &\geq c_1 \exp\left(-c_2' N\right) \|f_N\|_{L^2(I)} \end{aligned}$$

as $M \to \infty$; the relation (2.4) shows this is as strong as (2.1).

The proofs of both Theorem 2.2 and Theorem 2.4 (see Sections 5 and 6) are in a similar spirit and hinge on TT^* arguments in combination with an eigenfunction decomposition of TT^*. The eigenfunctions are well understood; the difficulty is in putting this information to use in the most effective way. The proof of Theorem 2.2 uses their orthogonality and the fact that an associated differential operator is comparable to $-\Delta$, but does not rely on the asymptotic behavior of the eigenfunctions (merely on asymptotics of the eigenvalues). In contrast, the proof of Theorem 2.4 uses an elementary estimate adapted to the eigenfunctions and inspired from classical Fourier analysis: this estimate is sharp but not sophisticated enough to capture complicated behavior at different scales simultaneously. It is not clear to us whether and how these arguments could be refined.

2.4. An improved estimate for Case 4

Case 3, with disjoint intervals I and J, is the worst case scenario in terms of reconstruction from Hilbert transform data. It seems that reconstruction in Case 4, the truncated Hilbert transform with overlap, is an easier task in the sense that one would expect the inversion problem to be more stable. The singular values decay to zero at a similar exponential rate in both cases, since the Hilbert transform with overlap contains, at this level of generality, the Hilbert transform with a gap as a special case (acting on functions supported away from $I \cap J$). It is this ill-posedness that in practice has led to the concept of *region of interest* reconstruction. Here, the aim is to reconstruct the function f only on the region where the Hilbert transform has been measured. For the truncated Hilbert transform with overlap this means reconstruction of f only on the overlap region $I \cap J$.

The reason this problem of partial reconstruction inside $I \cap J$ may be more stable has an intuitive explanation: one would expect interaction with the singularity of the Hilbert transform to be such that it cannot lead to significant cancellation. More formally, one can consider the singular value decomposition of H_T. In the case where $I \cap J \neq \emptyset$, the singular values accumulate at both 0 and 1. Moreover, the singular functions have the property that they oscillate on $I \cap J$ and are monotonically decaying to zero on $I \backslash J$ as the singular values accumulate at 1. The opposite is true when the singular values decay to zero: the corresponding singular functions oscillate on $I \backslash J$, i.e., outside of the region of interest, and are monotonically decaying to zero on $I \cap J$. (For a proof of these properties we refer to [8].) Figure 4 below illustrates the behavior of the singular functions for a specific choice of overlapping intervals I and J.

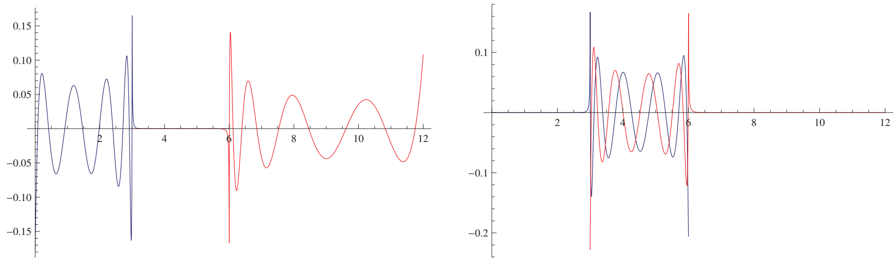

Figure 4. Examples of singular functions u_n (red) and v_n (blue) for the overlap case $I = (0, 6)$, $J = (3, 12)$. *Left:* For σ_n close to 0, the singular functions are exponentially small on $(3, 6)$ and oscillate outside of $(3, 6)$. *Right:* For σ_n close to 1, the functions oscillate on $(3, 6)$ and are exponentially small outside of the overlap region.

A more precise estimate on the decaying part of the singular functions is the subject of joint work by the first author with M. Defrise and A. Katsevich [5]. Let $I = (a_2, a_4)$ and $J = (a_1, a_3)$ for real numbers $a_1 < a_2 < a_3 < a_4$, and let us consider the singular functions u_n on I corresponding to the singular values σ_n decaying to zero. Then, one can show that for any $\mu > 0$ there exist positive constants B_μ and β_μ such that

$$\|u_n\|_{L^2([a_2, a_3 - \mu])} \leq B_\mu \, e^{-\beta_\mu n}$$

for sufficiently large index n. Exploiting this property, we can eliminate the dependence in Theorem 2.4 on the variation of f within the region of interest.

Theorem 2.6. *Let $J = (a_1, a_3)$ and $I = (a_2, a_4) \subset \mathbb{R}$ be open intervals with $a_1 < a_2 < a_3 < a_4$. Fix a closed subinterval $J^* = [a_1^*, a_3^*] \subset J$ with $a_1^* < a_2 < a_3^*$. Then for any function $f \in W^{1,1}(I)$ such that there exists at least one point $x_0 \in I \setminus J^*$*

at which $f(x_0) = 0$, the following holds:

$$\|Hf\|_{L^2(J)} \geq c_1 \exp\Big(- c_2 \frac{|\chi_{I \setminus J^*} f|_{\mathrm{TV}}^2}{\|f\|_{L^2(I)}^2} \Big) \|f\|_{L^2(I)},$$

for some constants $c_1, c_2 > 0$ depending only on I, J and J^*.

Remark 1. Theorem 2.6 can be used in a similar fashion as Theorem 2.4 to obtain a stability estimate analogous to Corollary 2.1. As prior knowledge we assume $|\chi_{I\setminus J^*} f_{\mathrm{ex}}|_{\mathrm{TV}} \leq \kappa$ and $\int_I f_{\mathrm{ex}} = C$. Then, the statement can be formulated similarly as before, with the only change that the set of admissible solutions becomes

$$S(\delta, g^\delta) = \Big\{ f \in W^{1,1}(I) : \|H_T f - g^\delta\|_{L^2(J)} \leq \delta, |\chi_{I\setminus J^*} f|_{\mathrm{TV}} \leq \kappa, \int_I f = C \Big\}.$$

One can then adapt the proof of Corollary 2.1 to obtain that the diameter of $S(\delta, g^\delta)$ tends to zero as $\delta \to 0$ at a similar rate as before of the order $|\log \delta|^{-1/2}$. The only difference to Corollary 2.1 is that now the constants in (6.4) depend not only on I and J, but also on J^*.

Remark 2. Under the assumption that f does not vanish on I, we can improve on Theorem 2.6, giving a lower bound with polynomial decay; see remarks following Lemma 6.2. A stronger version of Theorem 2.6 for smoother functions can also be derived by an iterated argument, analogous to the adaptation of Theorems 2.3 and 2.5 from the proofs of Theorem 2.2 and 2.4; we omit the details.

An interesting question that remains open is whether estimates of the form

$$\|Hf\|_{L^2(J)} \geq h\Big(\frac{|f|_{\mathrm{TV}}}{\|f\|_{L^2(I)}} \Big) \|f\|_{L^2(I\cap J)} \tag{2.5}$$

are possible for a function h that shows a decay that is slower than the quadratically exponential type in Theorems 2.4 and 2.6, yet does not introduce a differential operator such as L_I. Note that (2.5) would give a lower bound on $\|Hf\|_{L^2(J)}$ with respect to $\|f\|_{L^2(I\cap J)}$ instead of $\|f\|_{L^2(I)}$, which is why we could expect such a function h to decay slower than in Theorems 2.4 and 2.6: if f is mainly supported on $I\setminus J$, i.e., away from the overlap, we will most likely not be able to improve on the conjecture (2.1). If, however, f has a significant portion of its L^2-mass inside the overlap $I \cap J$, then $\|Hf\|_{L^2(J)}$ cannot be too small. In terms of a possible stability estimate this implies that a regularization method guarantees good recovery only within the overlap $I \cap J$. Such a stability estimate would be of particular interest, since in practice one only aims at reconstruction within the overlap (i.e., the region of interest).

2.5. A word on the proofs

We note in advance that the results of Theorems 2.2 to 2.5 are such that the statements for the truncated Hilbert transform with overlap follow from the corresponding statement for the truncated Hilbert transform with a gap. Indeed, in Case 4, since $J \not\subset I$, we can always find an interval $J^* \subset J$ such that I and J^*

are disjoint. Trivially, however,

$$\|Hf\|_{L^2(J)} \geq \|Hf\|_{L^2(J^*)}, \tag{2.6}$$

so that a lower bound for $\|Hf\|_{L^2(J^*)}$ suffices. Therefore, in our proofs of Theorems 2.2 to 2.5, we may restrict ourselves to Case 3, i.e., the truncated Hilbert transform with a gap.

3. Proof of Theorem 2.1

Proof. Consider all $g \in \mathrm{BV}(I)$ and define for each such g the corresponding function $\tilde{g} = g/\|g\|_{L^2(I)}$, so that $|\tilde{g}|_{\mathrm{TV}} = |g|_{\mathrm{TV}}/\|g\|_{L^2(I)}$. We will show that for any fixed $\kappa > 0$ if we consider all such normalized $\tilde{g}$ for which $|\tilde{g}|_{\mathrm{TV}} \leq \kappa$, then there exists $c_0(\kappa) > 0$ such that

$$\|H\tilde{g}\|_{L^2(J)} \geq c_0(\kappa). \tag{3.1}$$

From this we may conclude that there exists a positive-valued function h such that

$$\|Hg\|_{L^2(J)} \geq h\left(\frac{|g|_{\mathrm{TV}}}{\|g\|_{L^2(I)}}\right)\|g\|_{L^2(I)}.$$

The proof of (3.1) will proceed by contradiction. We begin by assuming the existence of a sequence $f_n \in \mathrm{BV}(I)$ that has uniformly bounded variation $|f_n|_{\mathrm{TV}} \leq \kappa$, uniform norm $\|f_n\|_{L^2(I)} = 1$, and such that $\|Hf_n\|_{L^2(J)}$ is not bounded below, i.e.,

$$\lim_{n\to\infty} \|Hf_n\|_{L^2(J)} = 0. \tag{3.2}$$

Step 1. The first step of the proof consists of showing that these assumptions imply the uniform boundedness of f_n, more precisely that the following holds:

$$\limsup_{n\to\infty} \|f_n\|_{L^\infty(I)} \leq \kappa + |I|^{-1/2}. \tag{3.3}$$

Suppose that for some index N and some $\varepsilon \in (0, |I|^{-\frac{1}{2}})$, we have $\|f_n\|_{L^\infty(I)} \leq \kappa+\varepsilon$ for all $n \geq N$. Then, we have found a sequence that is uniformly bounded with the above bound in (3.3). If such an index N does not exist, we can find a subsequence f_{n_k} such that

$$\|f_{n_k}\|_{L^\infty(I)} > \kappa + \varepsilon.$$

This together with the assumed bound on $|f_n|_{\mathrm{TV}}$, requires that f_{n_k} does not change sign. Suppose w.l.o.g. that $f_{n_k} > 0$. Then, $|f_{n_k}|_{\mathrm{TV}} \leq \kappa$ implies

$$0 < \|f_{n_k}\|_{L^\infty(I)} - \kappa \leq f_{n_k}(x), \quad x \in I.$$

Hence,

$$\int_I (\|f_{n_k}\|_{L^\infty(I)} - \kappa)^2\,dx \leq \int_I f_{n_k}(x)^2\,dx,$$

which shows that

$$(\|f_{n_k}\|_{L^\infty(I)} - \kappa)^2 \cdot |I| \leq 1,$$

and therefore

$$\|f_{n_k}\|_{L^\infty(I)} \leq \kappa + |I|^{-1/2}.$$

Step 2. This step relies on Helly's selection theorem, which is a compactness theorem for $\mathrm{BV}_{\mathrm{loc}}$. Let $\Omega \subset \mathbb{R}$ be an open set and $f_n : \Omega \to \mathbb{R}$ a sequence of functions with

$$\sup_{n\in\mathbb{N}} \left(\|f_n\|_{L^1(\Omega)} + \left\| \frac{d}{dx} f_n \right\|_{L^1(\Omega)} \right) < \infty,$$

where the derivative is taken in the sense of tempered distributions. Then there exists a subsequence $\{f_{n_k}\}$ and a function $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ such that f_{n_k} converges to f pointwise and in $L^1_{\mathrm{loc}}(\Omega)$. Moreover, $|f|_{\mathrm{TV}} \leq \liminf_{n\to\infty} |f_n|_{\mathrm{TV}}$. Applying Helly's selection theorem to our sequence $\{f_n\}$ implies the existence of a subsequence $\{f_{n_k}\}$, such that their pointwise limit f is in $\mathrm{BV}(I)$. Furthermore, the uniform boundedness established in Step 1 yields that for each n_k,

$$|f_{n_k}(q)| \leq \|f_{n_k}\|_{L^\infty(I)} \leq \kappa + |I|^{-1/2}.$$

Moreover, the dominated convergence theorem implies that the uniform boundedness of f_{n_k} and f, together with their pointwise convergence to f results in convergence in the L^2-sense, i.e.,

$$\|f_{n_k} - f\|_{L^2(I)} \to 0. \tag{3.4}$$

We recall the simple observation that the truncated Hilbert transform remains bounded on L^2, since

$$\|Hf\|_{L^2(J)} = \|\mathcal{P}_J Hf\|_{L^2(\mathbb{R})} \leq \|Hf\|_{L^2(\mathbb{R})} = \|f\|_{L^2(I)}.$$

Consequently, from (3.4) we deduce

$$\|Hf_{n_k} - Hf\|_{L^2(J)} \leq \|f_{n_k} - f\|_{L^2(I)} \to 0.$$

Combining this with (3.2) yields $\|Hf\|_{L^2(J)} = 0$. Lemma 5.1. in [3] states that if $f \in L^2(I)$ and Hf vanishes on an open subset away from I, then $f \equiv 0$. This contradicts the assumption $\|f_n\|_{L^2(I)} = 1$ and hence completes the proof. $\square$

4. A differential operator

Our proofs of the remaining theorems make essential use of the singular value decomposition of H_T. Using an old idea of Landau, Pollak and Slepian [15], [16], [21] (and later of Maass in the context of tomography [17]) in the form of Katsevich [10], [11], we use an explicit differential operator to establish a connection to the singular value expansion. The explicit form of the involved operators will allow us to deduce that if $\|f_x\|_{L^2(I)}$ is small, then there is some explicit part of the L^2-norm of f that is comprised of singular functions associated to the largest singular values.

Let H_T denote the truncated Hilbert transform with a gap, so that we may assume that $J = (a_1, a_2)$ and $I = (a_3, a_4)$ for real numbers $a_1 < a_2 < a_3 < a_4$.

Let $\{\sigma_n; u_n, v_n\}$ be the singular value decomposition of H_T. Note that, by definition, for $\|f\|_{L^2(I)} = 1$,

$$\|H_T f\|^2_{L^2(J)} = \sum_{n=0}^{\infty} |\langle f, u_n\rangle|^2 \, \sigma_n^2.$$

Following Katsevich [10], we define the differential form

$$(L\psi)(x) := (P(x)\psi_x(x))_x + 2(x-\sigma)^2\psi(x),$$

where

$$P(x) = \prod_{i=1}^{4}(x - a_i) \quad \text{and} \quad \sigma = \frac{1}{4}\sum_{i=1}^{4} a_i.$$

For a correct definition of an unbounded operator it is necessary to indicate the domain it is acting on, as unbounded operators cannot be defined on all of L^2 (Hellinger–Toeplitz theorem). Therefore, we let $\mathrm{AC}_{\mathrm{loc}}(I)$ denote the space of locally absolutely continuous functions on I and define the domains

$$D_{\max} := \{\psi : I \to \mathbb{C} : \psi, P\psi_x \in \mathrm{AC}_{\mathrm{loc}}(I); \psi, L\psi \in L^2(I)\}$$

and

$$\mathcal{D} = \{\psi \in D_{\max} : P(x)\psi_x(x) \to 0 \text{ for } x \to a_3^+, x \to a_4^-\}.$$

We let L_I be the restriction of L to the domain $\mathcal{D}$ and note that L_I is a self-adjoint operator [25].

Then, as shown in [10], a commutation property of L_I with H_T proves that the functions $\{u_n\}$ form an orthonormal basis of $L^2(I)$ and that they are the eigenfunctions of L_I, that is $L_I u_n = \lambda_n u_n$ with λ_n being the n-th eigenvalue of L_I. In addition, the asymptotic behavior as $n \to \infty$ of the eigenvalues λ_n of L_I as well as that of the singular values σ_n of H_T is known. Katsevich and Tovbis [12] have given the asymptotics as $n \to \infty$ including error terms, from which we can deduce that for all $n \in \mathbb{N}$

$$\lambda_n \geq k_1 \, n^2, \tag{4.1}$$

$$\sigma_n \geq e^{-k_2 \, n}, \tag{4.2}$$

where $k_1, k_2 > 0$ depend only on the intervals I and J.

We must also consider the M-th iterate L_I^M of L_I. For $M \in \mathbb{N}$, let $D(L_I^M)$ denote the domain of the self-adjoint operator L_I^M. Then, we define the following sets of functions in $L^2(I)$:

$$A_M = \{f \in L^2(I) : f \in D(L_I^{M+1}) \text{ and } L_I^M f \in H^1(I)\}.$$

We note that these classes of functions are dense in $L^2(I)$ and that $C_c^{2M+1}(I)$ is a subset of A_M. Also, $A = \bigcap_{M=1}^{\infty} A_M$ is dense in $L^2(I)$ and $C_c^{\infty}(I) \subset A$.

Remark 3. The asymptotics in the results of Katsevich and Tovbis [12] are actually more precise than stated. In particular, setting $I = (a_3, a_4)$ and $J = (a_1, a_2)$ and

$$K_+ = \frac{\pi}{\sqrt{(a_4 - a_2)(a_3 - a_1)}}\ {}_2F_1\Big(\frac{1}{2}, \frac{1}{2}, 1; \frac{(a_3 - a_2)(a_4 - a_1)}{(a_4 - a_2)(a_3 - a_1)}\Big)$$
$$K_- = \frac{\pi}{\sqrt{(a_4 - a_2)(a_3 - a_1)}}\ {}_2F_1\Big(\frac{1}{2}, \frac{1}{2}, 1; \frac{(a_2 - a_1)(a_4 - a_3)}{(a_3 - a_1)(a_4 - a_2)}\Big),$$

where ${}_2F_1$ is the hypergeometric function, Katsevich and Tovbis derive

$$\lambda_n = \frac{\pi^2}{K_+^2} n^2(1 + o(1)) \quad \text{and} \quad \sigma_n = e^{-\pi \frac{K_+}{K_-} n}(1 + o(1))$$

for sufficiently large n. These asymptotic relations allow one to state (4.1) and (4.2) for all $n \geq N_0$, for some $N_0 \in \mathbb{N}$ depending on I and J. One can then find explicit constants k_1 and k_2 depending on I and J such that relations (4.1) and (4.2) hold for all $n \in \mathbb{N}$. Similarly, exploiting the asymptotics one can derive an upper bound of the form

$$\sigma_n \leq \tilde{K}_2\, e^{-K_2 n} \tag{4.3}$$

with $K_2, \tilde{K}_2$ depending only on I and J.

5. Proof of Theorems 2.2 and 2.3

We now turn to the proof of Theorem 2.2, for which we exploit the following density argument. Since $H^2(I)$ is dense in $H^1(I)$ with respect to the H^1 topology and, as can be easily verified, $H^2(I) \subset \mathcal{D}$, one can conclude that $H^1(I) \cap \mathcal{D}$ is dense in $H^1(I)$ with respect to the H^1 topology. Thus, it suffices to prove the statement of Theorem 2.2 for functions g in $H^1(I) \cap \mathcal{D}$; for each such function we normalize it to $\tilde{g} = g/\|g\|_{L^2(I)}$, so that to prove the theorem it would suffice to show that

$$\|H\tilde{g}\|_{L^2(J)} \geq c_1 \exp(-c_2 \|\tilde{g}_x\|_{L^2(I)}).$$

We now therefore assume we have $f \in H^1(I) \cap \mathcal{D}$ with $\|f\|_{L^2(I)} = 1$. Integration by parts yields that for $f \in H^1(I) \cap \mathcal{D}$,

$$\begin{aligned}\langle L_I f, f\rangle &= -\int_{a_3}^{a_4} P(x) f_x(x)^2\, dx + (P(x) f_x(x)) f(x)\Big|_{a_3}^{a_4} + \int_{a_3}^{a_4} 2(x-\sigma)^2 f(x)^2\, dx \\ &= -\int_{a_3}^{a_4} P(x) f_x(x)^2\, dx + \int_{a_3}^{a_4} 2(x-\sigma)^2 f(x)^2\, dx,\end{aligned}$$

so that

$$|\langle L_I f, f\rangle| \leq \|P\|_{L^\infty(I)} \|f_x\|^2_{L^2(I)} + 2(a_4 - a_1)^2 \|f\|_{L^2(I)} \leq k_3 \|f_x\|^2_{L^2(I)} + k_3,$$

for some constant $k_3 > 0$ depending only on I and J. Altogether, we thus have

$$\sum_{n=0}^{\infty} |\langle f, u_n\rangle|^2 \lambda_n = |\langle L_I f, f\rangle| \le k_3 \, \|f_x\|^2_{L^2(I)} + k_3. \tag{5.1}$$

Hence for any $N \ge 1$, it follows from the asymptotic behavior $\lambda_n \ge k_1 n^2$ that

$$\begin{aligned}
1 = \|f\|^2_{L^2(I)} &= \sum_{n=0}^{N} |\langle f, u_n\rangle|^2 + \sum_{n=N+1}^{\infty} |\langle f, u_n\rangle|^2 \\
&\le \sum_{n=0}^{N} |\langle f, u_n\rangle|^2 + \sum_{n=N+1}^{\infty} |\langle f, u_n\rangle|^2 \, \frac{\lambda_n}{k_1 n^2} \\
&\le \sum_{n=0}^{N} |\langle f, u_n\rangle|^2 + k_1^{-1} N^{-2} \sum_{n=N+1}^{\infty} |\langle f, u_n\rangle|^2 \, \lambda_n, \qquad (5.2)
\end{aligned}$$

so that, by (5.1),

$$\sum_{n=0}^{N} |\langle f, u_n\rangle|^2 \ge 1 - k_3 \, k_1^{-1} \, N^{-2} \left(\|f_x\|^2_{L^2(I)} + 1\right).$$

Hence choosing the least integer N such that

$$N^2 \ge 2 \, k_3 \, k_1^{-1} \, (\|f_x\|_{L^2(I)} + 1)^2 \ge 2 \, k_3 \, k_1^{-1} \, (\|f_x\|^2_{L^2(I)} + 1),$$

and setting $k_4 = (2k_3 k_1^{-1})^{1/2}$ yields

$$\sum_{n \le \lceil k_4 (\|f_x\|_{L^2(I)} + 1)\rceil} |\langle f, u_n\rangle|^2 \ge \frac{1}{2}.$$

Then,

$$\begin{aligned}
\|H_T f\|^2_{L^2(J)} = \sum_{n=0}^{\infty} |\langle f, u_n\rangle|^2 \, \sigma_n^2 &\ge \sum_{n \le \lceil k_4 (\|f_x\|_{L^2(I)} + 1)\rceil} |\langle f, u_n\rangle|^2 \, \sigma_n^2 \\
&\ge \left(\sum_{n \le \lceil k_4 (\|f_x\|_{L^2(I)} + 1)\rceil} |\langle f, u_n\rangle|^2 \right) \sigma^2_{\lceil k_4 (\|f_x\|_{L^2(I)} + 1)\rceil} \\
&\ge \frac{1}{2} e^{-2k_2 k_4 \|f_x\|_{L^2(I)} - 2k_2(k_4+1)} \ge k_5 \, e^{-2k_2 k_4 \|f_x\|_{L^2(I)}}
\end{aligned}$$

for some constant $k_5 > 0$, as desired.

We now modify the above argument to prove the stronger result of Theorem 2.3 when $f \in A_M$ for some arbitrary $M \in \mathbb{N}$. We start with the observation that for $\|f\|_{L^2(I)} = 1$ and any $N \ge 1$,

$$1 = \sum_{n=0}^{N} |\langle f, u_n\rangle|^2 + \sum_{n=N+1}^{\infty} |\langle f, u_n\rangle|^2 \le \sum_{n=0}^{N} |\langle f, u_n\rangle|^2 + \sum_{n=N+1}^{\infty} |\langle f, u_n\rangle|^2 \left(\frac{\lambda_n}{k_1 n^2}\right)^{2M+1}$$

$$
\begin{aligned}
&\leq \sum_{n=0}^{N} |\langle f, u_n\rangle|^2 + (k_1 N^2)^{-(2M+1)} \sum_{n=N+1}^{\infty} |\langle f, \lambda_n^M u_n\rangle|^2 \lambda_n \\
&= \sum_{n=0}^{N} |\langle f, u_n\rangle|^2 + (k_1 N^2)^{-(2M+1)} \sum_{n=N+1}^{\infty} |\langle f, L_I^M u_n\rangle|^2 \lambda_n \\
&= \sum_{n=0}^{N} |\langle f, u_n\rangle|^2 + (k_1 N^2)^{-(2M+1)} \sum_{n=N+1}^{\infty} |\langle L_I^M f, u_n\rangle|^2 \lambda_n. \qquad (5.3)
\end{aligned}
$$

Now we recall that there is a constant $k_3 > 0$ such that for $g \in \mathcal{D} \cap H^1(I)$,

$$\sum_{n=0}^{\infty} |\langle g, u_n\rangle|^2 \lambda_n = |\langle L_I g, g\rangle| \leq k_3 \, \|g_x\|_{L^2(I)}^2 + k_3.$$

We apply this with $g = L_I^M f$, to conclude from (5.3) that

$$\sum_{n=0}^{N} |\langle f, u_n\rangle|^2 \geq 1 - (k_1 N^2)^{-(2M+1)} \, k_3 \, (\|(L_I^M f)_x\|_{L^2(I)}^2 + 1).$$

Hence choosing the least integer N such that

$$
\begin{aligned}
N &\geq \left(2k_3 k_1^{-(2M+1)} (\|(L_I^M f)_x\|_{L^2(I)} + 1)^2\right)^{\frac{1}{2(2M+1)}} \\
&\geq \left(2k_3 k_1^{-(2M+1)} (\|(L_I^M f)_x\|_{L^2(I)}^2 + 1)\right)^{\frac{1}{2(2M+1)}},
\end{aligned}
$$

we see that for $k_4 = (2k_3)^{1/2(2M+1)} k_1^{-1/2}$,

$$\sum_{n \leq \lceil k_4 (\|(L_I^M f)_x\|_{L^2(I)}^{1/(2M+1)} + 1)\rceil} |\langle f, u_n\rangle|^2 \geq \frac{1}{2}.$$

As before, we now obtain a lower bound

$$
\begin{aligned}
\|H_T f\|_{L^2(J)}^2 = \sum_{n=0}^{\infty} |\langle f, u_n\rangle|^2 \sigma_n^2 &\geq \sum_{n \leq \lceil k_4 (\|(L_I^M f)_x\|_{L^2(I)}^{1/(2M+1)} + 1)\rceil} |\langle f, u_n\rangle|^2 \sigma_n^2 \\
&\geq \left(\sum_{n \leq \lceil k_4 (\|(L_I^M f)_x\|_{L^2(I)}^{1/(2M+1)} + 1)\rceil} |\langle f, u_n\rangle|^2 \right) \sigma^2_{\lceil k_4 (\|(L_I^M f)_x\|_{L^2(I)}^{1/(2M+1)} + 1)\rceil} \\
&\geq \frac{1}{2} \exp(-2k_2 k_4 \|(L_I^M f)_x\|_{L^2(I)}^{1/(2M+1)} - 2k_2(k_4 + 1)) \\
&\geq k_5 \, \exp(-2k_2 k_4 \|(L_I^M f)_x\|_{L^2(I)}^{1/(2M+1)})
\end{aligned}
$$

for some constant $k_5 = (1/2)e^{-2k_2(k_4+1)}$. We need only note that as $M \to \infty$, $k_4 \to k_1^{-1/2}$ and $k_5 \to (1/2)e^{-2k_2(k_1^{-1/2}+1)}$, both positive finite limits.

6. Proof of Theorems 2.4 and 2.5

It is well known that smoothness of a function $f : \mathbb{T} \to \mathbb{R}$ translates into decay of the Fourier coefficients $\hat{f}(n)$. This statement is usually proven using integration by parts; in particular, $f \in C^k$ yields $|\hat{f}(n)| \leq C_{f^{(k)},k} n^{-k}$. However, it is easy to see that for $k = 1$ it actually suffices to require f to be of bounded variation: this observation dates back at least to a paper from 1967 (but is possibly quite a bit older) of Taibleson [22], who showed that

$$|\hat{f}(n)| \leq 2\pi \frac{|f|_{\mathrm{TV}}}{n}.$$

We will show the analogous statement with the Fourier system replaced by the singular functions u_n of the operator L_I; the argument exploits an asymptotic expression and, implicitly, Abel's summation formula as a substitute for integration by parts.

Lemma 6.1. *Let I and J be disjoint finite open intervals on $\mathbb{R}$. There exists $c > 0$ depending only on the intervals I, J such that for any f of bounded variation that is supported on I and vanishes at the boundary of the interval I,*

$$|\langle f, u_n \rangle| \leq c \frac{|f|_{\mathrm{TV}}}{n}.$$

Proof. Without loss of generality, we may assume by density that $f \in C^1$ (or, alternatively, replace every integral by summation, and integration by parts by Abel's summation formula). Let $I = (a_3, a_4)$. It suffices to show that

$$\forall\, x \in (a_3, a_4) : \quad \left| \int_{a_3}^{x} u_n(z)\, dz \right| \leq \frac{c}{n}. \tag{6.1}$$

Once this is established (see the appendix in §9.2 for the proof of the above statement), we can write

$$\begin{aligned} \left| \int_{a_3}^{a_4} f(x) u_n(x)\, dx \right| &= \left| \int_{a_3}^{a_4} f(x) \Big(\int_{a_3}^{x} u_n(z) dz \Big)_x dx \right| \\ &= \left| \int_{a_3}^{a_4} f_x(x) \Big(\int_{a_3}^{x} u_n(z) dz \Big) dx \right| \leq \sup_{a_3 \leq x \leq a_4} \left| \int_{a_3}^{x} u_n(z) dz \right| \int_{a_3}^{a_4} |f_x(x)|\, dx, \end{aligned}$$

in which the boundary terms vanish by the assumption on f. □

6.1. Proof of Theorem 2.4

This section is split into two parts: we first assume that there exists a point $x_0 \in I$ such that $f(x_0) = 0$, and argue using that property. The second part of the section is completely independent and establishes a stronger result in the case that f does not change sign.

In the first case, given $g \in W^{1,1}(I)$ we consider the normalization $\tilde{g} = g/\|g\|_{L^2(I)}$, so that it would suffice to show that under the hypotheses of Theorem 2.4,

$$\|H\tilde{g}\|_{L^2(I)} \geq c_1 \exp(-c_2 \, |\tilde{g}|^2_{\mathrm{TV}}). \tag{6.2}$$

Thus we now consider $f \in W^{1,1}(I)$ with $\|f\|_{L^2(I)} = 1$ and such that f vanishes at least at one point in I. If f vanishes at the endpoints of I, we may apply Lemma 6.1 directly; otherwise we use Lemma 1.1 to approximate $f \in \mathrm{BV}(I)$ by a sequence of $f_n \in C_c^\infty(I)$ (in particular, vanishing at the boundary of I) such that $\|f_n - f\|_{L^2(I)} \to 0$ and $|f_n|_{\mathrm{TV}} \leq 3|f|_{\mathrm{TV}}$. Then if we prove (6.2) for each f_n we can conclude it holds for f, since

$$c_1 \exp(-9c_2 \, |f|^2_{\mathrm{TV}}) \leq \|Hf_n\|_{L^2(J)} \leq \|Hf\|_{L^2(J)} + \|H(f - f_n)\|_{L^2(J)}, \tag{6.3}$$

and

$$\|H(f - f_n)\|_{L^2(J)} \leq C\|f - f_n\|_{L^2(I)} \to 0 \quad \text{as} \quad n \to \infty.$$

We may now assume that f vanishes at the boundary of I, and note that by Lemma 6.1,

$$\begin{aligned} 1 = \|f\|^2_{L^2(I)} &= \sum_{n=0}^{N} |\langle f, u_n\rangle|^2 + \sum_{n=N+1}^{\infty} |\langle f, u_n\rangle|^2 \\ &\leq \sum_{n=0}^{N} |\langle f, u_n\rangle|^2 + c^2 \, |f|^2_{\mathrm{TV}} \sum_{n=N+1}^{\infty} \frac{1}{n^2} \leq \sum_{n=0}^{N} |\langle f, u_n\rangle|^2 + \frac{c^2 \, |f|^2_{\mathrm{TV}}}{N}. \end{aligned}$$

This implies that at least half of the L^2-mass is contained within the first $N = \lceil 2c^2 \, |f|^2_{\mathrm{TV}} \rceil$ frequencies. The remainder of the argument can be carried out as in Theorem 2.2.

It remains to show that we can actually restrict ourselves to the case where $f(x_0) = 0$ for some $x_0 \in I$. Assume now that we are in Case 3 (the argument for Case 4 follows completely analogously by reducing it to Case 3 using (2.6)). It is not difficult to see that we get a much stronger inverse inequality (with a polynomial instead of a superexponential decay).

Lemma 6.2. *Let I, J be as in Case 3 and assume that f has no root on I. Then,*

$$\|Hf\|_{L^2(J)} \geq \frac{|J|^{1/2}}{\sup_{x \in I, y \in J} |x - y|} \left(\frac{|f|^2_{\mathrm{TV}}}{\|f\|^2_{L^2(I)}} + \frac{4}{|I|} \right)^{-1/2} \|f\|_{L^2(I)}.$$

Proof. Without loss of generality, we assume $\|f\|_{L^2(I)} = 1$. Since I and J do not overlap, we see that the kernel of the Hilbert transform has constant sign (which sign depends on whether J is to the left or to the right of I). Therefore, since f never changes sign, we have by Hölder and monotonicity,

$$\|Hf\|_{L^2(J)} \geq \frac{1}{|J|^{1/2}} \|Hf\|_{L^1(J)} \geq |J|^{1/2} \frac{1}{\sup_{x \in I, y \in J} |x - y|} \|f\|_{L^1(I)}.$$

Let us now assume that

$$\|f\|_{L^1(I)} \le \varepsilon.$$

Then, there certainly exists a point x_0 with $f(x_0) \le \varepsilon/|I|$ and therefore

$$\|f\|_{L^\infty(I)} \le \frac{\varepsilon}{|I|} + |f|_{\mathrm{TV}}.$$

As a consequence,

$$1 = \int_I f^2\,dx \le \|f\|_{L^\infty(I)} \int_I |f|\,dx,$$

and thus

$$\|f\|_{L^1(I)} \ge \frac{1}{\varepsilon/|I| + |f|_{\mathrm{TV}}},$$

from which we derive that

$$\varepsilon \ge \frac{1}{\varepsilon/|I| + |f|_{\mathrm{TV}}}.$$

This shows that ε cannot be arbitrarily small depending on $|I|$ and $|f|_{\mathrm{TV}}$ and simple algebra implies the stated result. □

Remark 4. When I, J are configured as in Case 4, repeating this argument shows that the result of Lemma 6.2 continues to hold, with the factor $|J|^{1/2}$ replaced by $\frac{1}{2}|J \setminus I|^{1/2}$. This argument may also be suitably adapted to show that if f has no root on I and J^* is a subinterval of J that is disjoint from I, then

$$\|Hf\|_{L^2(J)} \ge \frac{1}{2}|J^* \setminus I|^{1/2} \frac{1}{\sup\limits_{x\in J^*, y\in I} |x-y|} \left(\frac{|\chi_{I\setminus J^*} f|^2_{\mathrm{TV}}}{\|f\|^2_{L^2(I\setminus J^*)}} + \frac{4}{|I \setminus J^*|} \right)^{-1/2} \|f\|_{L^2(I\setminus J^*)}.$$

This result may be seen as a suitable counterpart to Theorem 2.6.

6.2. Proof of Theorem 2.5

Proof. We now modify the argument used in the first part of the proof of Theorem 2.4 to show Theorem 2.5, in which case f is assumed to be in $C_c^{2M+1}(I)$ for some arbitrary $M \ge 1$. We note that under this strong assumption, which ensures that f and all its first $2M+1$ derivatives vanish at the endpoints of I, it follows that $L_I^M f$ also vanishes at the endpoints of I. Thus we may apply Lemma 6.1 directly to $L_I^M f$.

By Lemma 6.1 and the asymptotics for λ_n,

$$
\begin{aligned}
1 = \|f\|^2_{L^2(I)} &= \sum_{n=0}^{N} |\langle f, u_n\rangle|^2 + \sum_{n=N+1}^{\infty} \left|\frac{\lambda_n^M}{\lambda_n^M}\langle f, u_n\rangle\right|^2 \\
&= \sum_{n=0}^{N} |\langle f, u_n\rangle|^2 + \sum_{n=N+1}^{\infty} \left|\frac{1}{\lambda_n^M}\langle f, \lambda_n^M u_n\rangle\right|^2 = \sum_{n=0}^{N} |\langle f, u_n\rangle|^2 + \sum_{n=N+1}^{\infty} \frac{1}{\lambda_n^{2M}} \left|\langle f, L_I^M u_n\rangle\right|^2 \\
&= \sum_{n=0}^{N} |\langle f, u_n\rangle|^2 + \sum_{n=N+1}^{\infty} \frac{1}{\lambda_n^{2M}} \left|\langle L_I^M f, u_n\rangle\right|^2 \\
&\leq \sum_{n=0}^{N} |\langle f, u_n\rangle|^2 + c^2 |L_I^M f|^2_{\mathrm{TV}} \sum_{n=N+1}^{\infty} \frac{1}{(k_1 n^2)^{2M} n^2} \leq \sum_{n=0}^{N} |\langle f, u_n\rangle|^2 + \frac{c^2 k_1^{-2M} |L_I^M f|^2_{\mathrm{TV}}}{N^{4M+1}}.
\end{aligned}
$$

We now choose N to be the least integer such that

$$
N \geq \left(2c^2 k_1^{-2M} \left|L_I^M f\right|^2_{\mathrm{TV}}\right)^{1/(4M+1)},
$$

so that with this choice, we may set $k_4 = (2c^2 k_1^{-2M})^{1/(4M+1)}$ to obtain the lower bound

$$
\begin{aligned}
\|H_T f, \|^2_{L^2(J)} &= \sum_{n=0}^{\infty} |\langle f, u_n\rangle|^2 \sigma_n^2 \geq \sum_{n \leq \lceil k_4 |L_I^M f|^{2/(4M+1)}_{\mathrm{TV}} \rceil} |\langle f, u_n\rangle|^2 \sigma_n^2 \\
&\geq \left(\sum_{n \leq \lceil k_4 |L_I^M f|^{2/(4M+1)}_{\mathrm{TV}} \rceil} |\langle f, u_n\rangle|^2 \right) \sigma^2_{\lceil k_4 |L_I^M f|^{2/(4M+1)}_{\mathrm{TV}} \rceil} \\
&\geq \frac{1}{2} \exp(-2k_2(k_4 |L_I^M f|^{2/(4M+1)}_{\mathrm{TV}} + 1)) \geq k_5 \exp(-2k_2 k_4 |L_I^M f|^{2/(4M+1)}_{\mathrm{TV}})
\end{aligned}
$$

with $k_5 = (1/2)e^{-2k_2}$. We need only note that as $M \to \infty$, $k_4 \to k_1^{-1/2}$. $\square$

6.3. Proof of Corollary 2.1

Proof. Let f_1 and f_2 be elements in $S(\delta, g^\delta)$. From Theorem 2.4 and $|f_1 - f_2|_{\mathrm{TV}} \leq 2\kappa$, we obtain

$$
\|f_1 - f_2\|_{L^2(I)} \leq \frac{1}{c_1} e^{c_2 4\kappa^2 / \|f_1 - f_2\|^2_{L^2(I)}} \|H_T(f_1 - f_2)\|_{L^2(J)}.
$$

Linearity of H_T and the properties of S then yield

$$
\begin{aligned}
\|f_1 - f_2\|_{L^2(I)} &\leq \frac{1}{c_1} e^{c_2 4\kappa^2 / \|f_1 - f_2\|^2_{L^2(I)}} \|H_T f_1 - H_T f_2\|_{L^2(J)} \\
&\leq \frac{1}{c_1} e^{c_2 4\kappa^2 / \|f_1 - f_2\|^2_{L^2(I)}} (\|H_T f_1 - g^\delta\|_{L^2(J)} + \|g^\delta - H_T f_2\|_{L^2(J)}) \\
&\leq \frac{1}{c_1} e^{c_2 4\kappa^2 / \|f_1 - f_2\|^2_{L^2(I)}} 2\delta.
\end{aligned}
$$

This gives

$$\log(\|f_1 - f_2\|_{L^2(I)}) - \frac{c_2 4\kappa^2}{\|f_1 - f_2\|^2_{L^2(I)}} \leq \log\left(\frac{2\delta}{c_1}\right).$$

A lower bound on the left-hand side of the above inequality can be obtained by observing that $x^2 \log|x| \geq -1/(2e)$ for real-valued x. Thus,

$$-\frac{\frac{1}{2e} + 4c_2\kappa^2}{\|f_1 - f_2\|^2_{L^2(I)}} \leq \log\left(\frac{2\delta}{c_1}\right).$$

Hence, if δ is not too large ($\delta \leq c_1/2$), we can conclude that

$$\|f_1 - f_2\|_{L^2(I)} \leq \sqrt{\frac{\frac{1}{2e} + 4c_2\kappa^2}{|\log(2\delta/c_1)|}}. \tag{6.4}$$

□

7. Proof of Theorem 2.6

We recall that Theorem 2.6 considers Case 4, with $I \cap J \neq \emptyset$. Let $I = (a_2, a_4)$ and $J = (a_1, a_3)$ for $a_1 < a_2 < a_3 < a_4$ and let the subinterval J^* of J be defined as $[a_1 + \mu, a_3 - \mu]$ for some $\mu > 0$ sufficiently small so that $a_1 + \mu < a_2 < a_3 - \mu$. We think of J^* as now being fixed for the remainder of the argument. For the two accumulation points of the singular values of H_T (the truncated Hilbert transform with overlap), we use the convention $\sigma_n \to 1$ for $n \to -\infty$ and $\sigma_n \to 0$ for $n \to \infty$. The two main ingredients needed for the statement in Theorem 2.6 are the existence of positive constants B_μ, β_μ and c depending only on I, J and μ such that the following holds for all $n \in \mathbb{N}$:

1. $\|u_n\|_{L^2(I\cap J^*)} \leq B_\mu\, e^{-\beta_\mu n}$,
2. $\sup\limits_{x\in I\setminus J^*} |\int_{a_3-\mu}^{x} u_n(z)\,dz| \leq c/n$.

These properties of the singular functions u_n corresponding to singular values close to zero allow one to estimate the inner products $\langle f, u_n\rangle$. The proof of the first statement can be found in [5] for sufficiently large n, i.e, $n \geq N_0$ for some $N_0 \in \mathbb{N}$. Since $\|u_n\|_{L^2(I)} = 1$ and N_0 depends only on I, J and μ, one can easily deduce the existence of constants B_μ, β_μ depending only on I, J and μ such that (1) holds for all $n \in \mathbb{N}$. Note that we cannot merely apply Lemma 6.1 to prove (2), since in the case where $I \cap J$ is nonempty, the functions u_n behave fundamentally differently at the endpoint a_3 of J, which lies in I. Thus we prove (2) directly in § 9.3.

Given any function $g \in W^{1,1}(I)$, we consider the normalization $\tilde{g} = g/\|g\|_{L^2(I)}$, in which case to prove Theorem 2.6 it would suffice to show

$$\|H\tilde{g}\|_{L^2(J)} \geq c_1 \exp(-c_2\, |\chi_{I\setminus J^*}\tilde{g}|^2_{\mathrm{TV}}), \tag{7.1}$$

as long as $\tilde{g}$ satisfies the remaining hypotheses of Theorem 2.6.

Thus from now on we assume we are working with $f \in W^{1,1}(I)$ and $\|f\|_{L^2(I)} = 1$. If f vanishes at the boundary of $I\setminus J^*$, we may work directly with f. Otherwise, if f merely vanishes at least at one point in $I \setminus J^*$, then we may apply a small modification of Lemma 1.1 to approximate f by functions $f_n \in C_c^\infty(I)$ that vanish at the endpoints of $I \setminus J^*$ and such that $\|f_n - f\|_{L^2(I)} \to 0$ and $|\chi_{I\setminus J^*} f_n|_{\mathrm{TV}} \leq 5|\chi_{I\setminus J^*} f|_{\mathrm{TV}}$. Then having proved (7.1) for each f_n we could conclude it holds for f, since

$$c_1 \exp(-c_2\, 25\, |\chi_{I\setminus J^*} f|_{\mathrm{TV}}^2) \leq \|Hf_n\|_{L^2(J)} \leq \|Hf\|_{L^2(J)} + \|H(f-f_n)\|_{L^2(J)},$$

and $\|H(f-f_n)\|_{L^2(J)} \leq C\|f-f_n\|_{L^2(I)} \to 0$ as $n \to \infty$.

Hence, we can assume $\|f\|_{L^2(I)} = 1$ and f vanishes at the endpoints of $I \setminus J^*$, so that

$$\begin{aligned} \Big| \int_I f(x) u_n(x)\, dx \Big| &\leq \Big| \int_{I\cap J^*} f(x) u_n(x)\, dx \Big| + \Big| \int_{I\setminus J^*} f(x) u_n(x)\, dx \Big| \\ &\leq B_\mu\, e^{-\beta_\mu n} + |\chi_{I\setminus J^*} f|_{\mathrm{TV}} \sup_{x\in I\setminus J^*} \Big| \int_{a_3-\mu}^{x} u_n(z)\, dz \Big| \\ &\leq B_\mu\, e^{-\beta_\mu n} + \frac{c}{n}\, |\chi_{I\setminus J^*} f|_{\mathrm{TV}}. \end{aligned}$$

The remainder of the argument is then similar to the proof of Theorem 2.4. For any $N \geq 1$,

$$\begin{aligned} 1 = \|f\|_{L^2(I)}^2 &\leq \sum_{n=-\infty}^{N} |\langle f, u_n\rangle|^2 + \sum_{n=N+1}^{\infty} \big(B_\mu\, e^{-\beta_\mu n} + \frac{c}{n}\, |\chi_{I\setminus J^*} f|_{\mathrm{TV}}\big)^2 \\ &\leq \sum_{n=-\infty}^{N} |\langle f, u_n\rangle|^2 + 2\, B_\mu^2 \sum_{n=N+1}^{\infty} e^{-2\beta_\mu n} + 2\frac{c^2}{N} |\chi_{I\setminus J^*} f|_{\mathrm{TV}}^2 \\ &\leq \sum_{n=-\infty}^{N} |\langle f, u_n\rangle|^2 + 2\, B_\mu^2\, \frac{e^{-2\beta_\mu N}}{e^{2\beta_\mu} - 1} + 2\, \frac{c^2}{N}\, |\chi_{I\setminus J^*} f|_{\mathrm{TV}}^2. \end{aligned}$$

Let $\tilde{N}$ be the least integer such that, for all $n \geq \tilde{N}$,

$$n\, e^{-2\beta_\mu n} \leq c^2\, B_\mu^{-2} \left(e^{2\beta_\mu} - 1\right)$$

and note that $\tilde{N}$ depends only on I, J and μ. Then, the choice

$$N = \max\{\tilde{N}, \lceil 4c^2(|\chi_{I\setminus J^*} f|_{\mathrm{TV}}^2 + 1)\rceil\}$$

guarantees that the sum $\sum_{n=-\infty}^{N} |\langle f, u_n\rangle|^2$ contains at least half of the energy of f and thus

$$\|H_T f\|_{L^2(J)}^2 = \sum_{n=-\infty}^{\infty} |\langle f, u_n\rangle|^2\, \sigma_n^2 \geq \sum_{n=-\infty}^{N} |\langle f, u_n\rangle|^2\, \sigma_n^2 \geq \frac{1}{2}\sigma_N^2 \geq \tilde{k}_0 e^{-k_0 |\chi_{I\setminus J^*} f|_{\mathrm{TV}}^2},$$

for some constants $k_0, \tilde{k}_0$ depending only on I, J and μ.

8. A remark on generalizations

Let $I, J \subset \mathbb{R}$ be disjoint intervals and let $T : L^2(I) \to L^2(J)$ be an integral operator of convolution type,

$$(Tf)(x) = \int_I K(x-y) f(y)\, dy,$$

for some kernel K. Then we would generically expect an inequality of the type

$$\|Tf\|_{L^2(J)} \geq h\Big(\frac{|f|_{\mathrm{TV}}}{\|f\|_{L^2(I)}} \Big) \|f\|_{L^2(I)} \tag{8.1}$$

to hold true, for some positive function $h : \mathbb{R}_+ \to \mathbb{R}_+$. The purpose of this section is to show how to construct examples where the function h depends very strongly on very fine properties of the kernel K.

8.1. Our example

For reasons of clarity, we set $I = [0, 1]$ and take $K \colon \mathbb{R} \to \mathbb{R}$ to be a 1-periodic smooth function. We define the integral operator $T \colon L^2([0, 1]) \to L^\infty(\mathbb{R})$ by

$$(Tf)(x) = \int_0^1 K(x-y)\, f(y)\, dy.$$

The function Tf is also periodic with period 1. We will not specify the interval J because it will be irrelevant. The main idea is that we can identify

$$Tf = K * f$$

with a function on the torus $\mathbb{T}$ (normalized to have length 1). Expressing everything in terms of Fourier series yields

$$\sum_n \widehat{Tf}(n) = \sum_n \hat{K}(n) \hat{f}(n).$$

We now see that if the Fourier coefficients of K and f are supported on disjoint sets of frequencies, then we immediately get $Tf = 0$. Put differently, the only way to ensure that $Tf \neq 0$ for every $f \neq 0$ is to ensure that K has no vanishing Fourier coefficients.

Lemma 8.1 (Folklore). *Let $K \in L^2(\mathbb{T})$. Then the span of $\{K(x-a) : a \in \mathbb{T}\}$ is dense in $L^2(\mathbb{T})$ if and only if*

$$\forall\, n \in \mathbb{Z}, \qquad \hat{K}(n) \neq 0.$$

Proof. One direction is easy: if $\hat{K}(n) = 0$ for some $n \in \mathbb{Z}$, then e^{inx} serves as a counterexample. As for the other direction, suppose $g \in L^2(\mathbb{T})$ is orthogonal to all translations of K. Then, for any $t \in \mathbb{T}$, by Parseval,

$$0 = \int_{\mathbb{T}} K(x)\, g(x-t)\, dx = \sum_{n \in \mathbb{Z}} \hat{K}(n)\, \overline{e^{int}\, \hat{g}(n)} = \sum_{n \in \mathbb{Z}} \hat{K}(-n)\, \overline{\hat{g}(-n)}\, e^{int}.$$

Since t was arbitrary, this means that the Fourier series

$$\sum_{n\in\mathbb{Z}} \hat{K}(-n)\,\overline{\hat{g}(-n)}\,e^{int}$$

vanishes identically and since for all n, $\hat{K}(-n) \neq 0$, this implies that $g = 0$. $\square$

Having established this lemma, the proof of an estimate of the type

$$\|Tf\|_{L^2(J)} \geq h\Big(\frac{|f|_{\mathrm{TV}}}{\|f\|_{L^2(I)}}\Big)\,\|f\|_{L^2(I)},$$

for some positive-valued function h is easy. If we take a minimizing sequence $f_{n_k} \in \mathrm{BV}(I)$, Helly's compactness theorem implies the existence of a convergent subsequence $f_{n_k} \to f$ with a pointwise limit $f \in \mathrm{BV}(I)$. Assuming that $K \in L^2(\mathbb{T})$ has $\hat{K}(n) \neq 0$ for all n, Lemma 8.1 implies that the translates of K are dense in $L^2(I)$. Then, however, it is impossible for the operator T to map f to 0 and this proves the statement.

8.2. Conclusion

In order for an inequality of the type

$$\|Tf\|_{L^2(J)} \geq h\Big(\frac{|f|_{\mathrm{TV}}}{\|f\|_{L^2(I)}}\Big)\|f\|_{L^2(I)}$$

to hold true at all, fine properties of the Fourier coefficients of the kernel play a crucial role. Furthermore, even assuming such an inequality to be true, the quantitative rate of decay of h will directly depend on the speed with which the Fourier coefficients decay to 0: it is thus possible to construct explicit examples of kernels K for which the associated function h decays faster than any arbitrary given function. These are very serious obstructions for any generalized theory of bounding truncated integral operators from below if one were to hope that such a theory could be stated in 'rough' terms (i.e., smoothness of the function, L^p-norms of the kernel K and its derivatives). In the example above, bounding Fourier coefficients $\hat{K}(n)$ from below seems unavoidable.

9. Appendix

9.1. Proof of Lemma 1.1

Proof. A function $f \in \mathrm{BV}(I)$ can be approximated by smooth functions in the following way (see [2], Section 3.1): There exists a sequence $\{f_n\} \in C^\infty(I) \cap \mathrm{BV}(I)$ such that

$$\|f_n - f\|_{L^1(I)} \to 0, \tag{9.1}$$

$$|f_n|_{\mathrm{TV}} \to |f|_{\mathrm{TV}}. \tag{9.2}$$

We are seeking an approximation by smooth functions that vanish at the boundary of I. Since $C_c^\infty(I) \subset \mathrm{BV}(I)$ is dense in $L^1(I)$, one can find a sequence $f_n \in C_c^\infty(I)$

that satisfies (9.1). Now instead of (9.2), we use the fact that $f(x_0) = 0$ for some $x_0 \in I$ to note that $\|f\|_{L^\infty(I)} \leq |f|_{\mathrm{TV}}$, and so instead of (9.2) we now have

$$|f_n|_{\mathrm{TV}} \leq |f|_{\mathrm{TV}} + 2\,\|f\|_{L^\infty(I)} \leq 3\,|f|_{\mathrm{TV}}. \tag{9.3}$$

Finally, L^2-convergence can be obtained by noting that $\{f_n\}$ is uniformly bounded. Indeed, suppose there exists a subsequence $\{f_{n_k}\}$ such that $\|f_{n_k}\|_{L^\infty(I)} > 3|f|_{\mathrm{TV}}+\varepsilon$ for some small $\varepsilon > 0$. Then, each f_{n_k} does not change sign. For supposing that it did, we would have

$$|f_{n_k}|_{\mathrm{TV}} \geq \|f_{n_k}\|_{L^\infty(I)} > 3\,|f|_{\mathrm{TV}} + \varepsilon,$$

which contradicts (9.3).

Thus we may assume, without loss of generality, $f_{n_k} \geq 0$, in which case we see that for each $x \in I$,

$$\|f_{n_k}\|_{L^\infty(I)} - f_{n_k}(x) \leq |f_{n_k}|_{\mathrm{TV}}.$$

This yields

$$0 < \epsilon < \|f_{n_k}\|_{L^\infty(I)} - 3\,|f|_{\mathrm{TV}} \leq \|f_{n_k}\|_{L^\infty(I)} - |f|_{\mathrm{TV}} \leq f_{n_k}(x), \quad \forall x \in I.$$

Furthermore,

$$\int_I (\|f_{n_k}\|_{L^\infty(I)} - 3\,|f|_{\mathrm{TV}})\,dx \leq \|f_{n_k}\|_{L^1(I)} \leq 2\,\|f\|_{L^1(I)},$$

which results in the uniform bound $\|f_{n_k}\|_{L^\infty(I)} \leq 2\|f\|_{L^1(I)}/|I| + 3\,|f|_{\mathrm{TV}}$. Since L^1-convergence in (9.1) implies the existence of a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \xrightarrow{\mathrm{pw}} f$ almost everywhere, the dominated convergence theorem results in

$$\|f_{n_k} - f\|_{L^2(I)} \to 0.$$

□

9.2. Proof of Lemma 6.1

Proof. Here we will prove the statement

$$\forall\, x \in (a_3, a_4): \quad \left| \int_{a_3}^{x} u_n(z)\,dz \right| \leq \frac{c}{n},$$

where u_n is the n-th eigenfunction of L_I with associated eigenvalue λ_n. We recall we are in the case where I and J are disjoint, with $I = (a_3, a_4)$. We choose $N_0 \in \mathbb{N}$ (depending only on I and J) such that the asymptotic form of u_n in [12] is valid for all $n \geq N_0$. We first show the result for $n \geq N_0$. For this, we note that on (a_3, a_4) and away from the points a_3 and a_4, the function u_n can be approximated by the Wentzel–Kramers–Brillouin (WKB) solution. More precisely, defining $\varepsilon = \varepsilon_n := 1/\sqrt{\lambda_n}$, it is true that for any sufficiently small $\delta > 0$, the representation of u_n in the form

$$\begin{aligned} u_n(z) = \frac{K}{(-P(z))^{1/4}} \Bigg[&\cos\left(\frac{1}{\varepsilon} \int_{a_3}^{z} \frac{dt}{\sqrt{-P(t)}} - \frac{\pi}{4} \right) \cdot (1 + \mathcal{O}(\varepsilon^{1/2-\delta})) \\ &+ \sin\left(\frac{1}{\varepsilon} \int_{a_3}^{z} \frac{dt}{\sqrt{-P(t)}} - \frac{\pi}{4} \right) \cdot \mathcal{O}(\varepsilon^{1/2-\delta}) \Bigg] \end{aligned}$$

is valid for $z \in [a_3 + \mathcal{O}(\varepsilon^{1+2\delta}), a_4 - \mathcal{O}(\varepsilon^{1+2\delta})]$ and some positive constant K depending only on a_1, a_2, a_3, a_4. Having this, we start by estimating

$$\Big| \int_{a_3+\mathcal{O}(\varepsilon^{1+2\delta})}^{x} u_n(z)\, dz \Big|$$

for $x \in [a_3 + \mathcal{O}(\varepsilon^{1+2\delta}), a_4 - \mathcal{O}(\varepsilon^{1+2\delta})]$. We do this by first introducing $\tilde{u}_n(z) = (-P(z))^{-1/4} u_n(z)$, for which

$$\begin{aligned}\int_{a_3+\mathcal{O}(\varepsilon^{1+2\delta})}^{x} \tilde{u}_n(z) dz = K\varepsilon \Big[&\sin\Big(\frac{1}{\varepsilon}\int_{a_3}^{x} \frac{dt}{\sqrt{-P(t)}} - \frac{\pi}{4}\Big) \\ &- \sin\Big(\frac{1}{\varepsilon}\int_{a_3}^{a_3+\mathcal{O}(\varepsilon^{1+2\delta})} \frac{dt}{\sqrt{-P(t)}} - \frac{\pi}{4}\Big)\Big] \cdot (1 + \mathcal{O}(\varepsilon^{1/2-\delta})) \\ &- K\varepsilon\Big[\cos\Big(\frac{1}{\varepsilon}\int_{a_3}^{x} \frac{dt}{\sqrt{-P(t)}} - \frac{\pi}{4}\Big) \\ &- \cos\Big(\frac{1}{\varepsilon}\int_{a_3}^{a_3+\mathcal{O}(\varepsilon^{1+2\delta})} \frac{dt}{\sqrt{-P(t)}} - \frac{\pi}{4}\Big)\Big] \cdot \mathcal{O}(\varepsilon^{1/2-\delta})\end{aligned}$$

and hence

$$\Big| \int_{a_3+\mathcal{O}(\varepsilon^{1+2\delta})}^{x} \tilde{u}_n(z) dz \Big| \le K\, \varepsilon\, (1 + \mathcal{O}(\varepsilon^{1/2-\delta})).$$

It is known from the asymptotics derived in [12] that

$$\varepsilon = \ \varepsilon_n = \frac{2}{K^2\, n\pi} + \mathcal{O}(n^{-1/2+\delta}). \tag{9.4}$$

Thus, there exists a constant $\tilde{c}_1$ depending only on a_1, a_2, a_3, a_4 such that

$$\Big| \int_{a_3+\mathcal{O}(\varepsilon^{1+2\delta})}^{x} \tilde{u}_n(z)\, dz \Big| \le \frac{\tilde{c}_1}{n}.$$

We can use this, together with integration by parts, to find an upper bound on the above expression with $\tilde{u}_n$ replaced by u_n:

$$\begin{aligned}\int_{a_3+\mathcal{O}(\varepsilon^{1+2\delta})}^{x} u_n(z)\, dz &= \int_{a_3+\mathcal{O}(\varepsilon^{1+2\delta})}^{x} (-P(z))^{1/4}\, \tilde{u}_n(z)\, dz \\ &= -\int_{a_3+\mathcal{O}(\varepsilon^{1+2\delta})}^{x} \frac{d}{dz}(-P(z))^{1/4} \int_{a_3+\mathcal{O}(\varepsilon^{1+2\delta})}^{z} \tilde{u}_n(t)\, dt\, dz \\ &\quad + \Big((-P(z))^{1/4} \int_{a_3+\mathcal{O}(\varepsilon^{1+2\delta})}^{z} \tilde{u}_n(t)\, dt\Big)\Big|_{a_3+\mathcal{O}(\varepsilon^{1+2\delta})}^{x}.\end{aligned}$$

This gives

$$\begin{aligned}\Big| \int_{a_3+\mathcal{O}(\varepsilon^{1+2\delta})}^{x} u_n(z)\, dz \Big| &\le \sup_{z \in [a_3+\mathcal{O}(\varepsilon^{1+2\delta}), x]} \Big| \int_{a_3+\mathcal{O}(\varepsilon^{1+2\delta})}^{z} \tilde{u}_n(t)\, dt \Big| \\ &\quad \cdot \int_{a_3+\mathcal{O}(\varepsilon^{1+2\delta})}^{x} \Big| \frac{d}{dz}(-P(z))^{1/4} \Big|\, dz + |P(x)|^{1/4} \cdot \Big| \int_{a_3+\mathcal{O}(\varepsilon^{1+2\delta})}^{x} \tilde{u}_n(t)\, dt \Big| \le \frac{c_1}{n},\end{aligned}$$

for some constant c_1 that depends only on the points a_i. Here we have used that $\frac{d}{dz}(-P(z))^{1/4}$ changes sign exactly once within (a_3, a_4) and hence

$$\int_{a_3+\mathcal{O}(\varepsilon^{1+2\delta})}^{x} \left|\frac{d}{dz}(-P(z))^{1/4}\right| dz \leq \sup_{a_3\leq x\leq a_4} 4\,(-P(x))^{1/4}.$$

What remains to be shown is the estimate for the contributions close to the points a_3 and a_4. Since (by the definition of the operator L_I) the asymptotic behavior of u_n at a_4 is identical to its behavior at a_3, it suffices to find an upper bound on

$$\Big|\int_{a_3}^{x} u_n(z)dz\Big|, \quad x \in (a_3, a_3 + \mathcal{O}(\varepsilon^{1+2\delta})].$$

On this interval, $(a_3, a_3+\mathcal{O}(\varepsilon^{1+2\delta})]$, the eigenfunctions u_n can be approximated by the Bessel function J_0. (This approximation is specific to the case where I and J are disjoint.) For this, we define the variable $t = (a_3 - z)/(\varepsilon^2 P'(a_3))$. Then, the asymptotic behavior of u_n has been found to be

$$u_n(z) = \begin{cases} b_3\,[J_0(2\sqrt{t}) + \mathcal{O}(\varepsilon^{1-2\delta/3})], & \text{for } t \in [0,1) \\ b_3\,[J_0(2\sqrt{t}) + t^{-1/4}\mathcal{O}(\varepsilon^{1-2\delta/3})], & \text{for } t \in [1, \mathcal{O}(\varepsilon^{2\delta-1})] \end{cases}$$

with a constant $b_3 = \mathcal{O}(\varepsilon^{-1/2})$. A change of variables $dx = -\varepsilon^2 P'(a_3)dt$ and $t(x) = \frac{a_3-x}{\varepsilon^2 P'(a_3)} = \mathcal{O}(\varepsilon^{2\delta-1})$ then yield

$$\begin{aligned}
\int_{a_3}^{x} u_n(z)dz = b_3 \cdot \Big\{ &\int_0^1 \big[J_0(2\sqrt{t}) + \mathcal{O}(\varepsilon^{1-2\delta/3})\big]\varepsilon^2(-P'(a_3))\,dt \\
&+ \int_1^{\mathcal{O}(\varepsilon^{2\delta-1})} \big[J_0(2\sqrt{t}) + t^{-1/4}\mathcal{O}(\varepsilon^{1-2\delta/3})\big]\varepsilon^2(-P'(a_3))\,dt\Big\} \\
= \mathcal{O}(\varepsilon^{3/2}) \cdot \Big\{ &\int_0^1 \big[J_0(2\sqrt{t}) + \mathcal{O}(\varepsilon^{1-2\delta/3})\big]\,dt \\
&+ \int_1^{\mathcal{O}(\varepsilon^{2\delta-1})} \big[J_0(2\sqrt{t}) + t^{-1/4}\mathcal{O}(\varepsilon^{1-2\delta/3})\big]\,dt\Big\}.
\end{aligned}$$

The first integral in the above sum is bounded, thus

$$\int_{a_3}^{x} u_n(z)dz = \mathcal{O}(\varepsilon^{3/2}) + \mathcal{O}(\varepsilon^{3/2}) \cdot \int_1^{\mathcal{O}(\varepsilon^{2\delta-1})} \big[J_0(2\sqrt{t}) + t^{-1/4}\mathcal{O}(\varepsilon^{1-2\delta/3})\big]\,dt.$$

To find an upper bound on the remaining integral, we first estimate it by

$$\begin{aligned}
\Big|\int_1^{\mathcal{O}(\varepsilon^{2\delta-1})} &\big[J_0(2\sqrt{t}) + t^{-1/4}\,\mathcal{O}(\varepsilon^{1-2\delta/3})\big]\,dt\Big| \\
&= \Big|\int_1^{\mathcal{O}(\varepsilon^{2\delta-1})} J_0(2\sqrt{t})\,dt + t^{3/4}\,\mathcal{O}(\varepsilon^{1-2\delta/3})\Big|_1^{\mathcal{O}(\varepsilon^{2\delta-1})}\Big| \\
&\leq \Big|\int_1^{\mathcal{O}(\varepsilon^{2\delta-1})} J_0(2\sqrt{t})\,dt\Big| + \mathcal{O}(\varepsilon^{1/4+5\delta/6}). \qquad (9.5)
\end{aligned}$$

Next, we make use of the asymptotic form of J_0 for $t \to \infty$:

$$J_0(2\sqrt{t}) = \frac{1}{\sqrt{\pi}\, t^{1/4}}\Big[\cos\Big(2\sqrt{t} - \frac{\pi}{4}\Big) + \mathcal{O}(t^{-1/2})\Big]. \tag{9.6}$$

For some fixed, sufficiently large T, we can write

$$\begin{aligned}
&\Big|\int_1^{\mathcal{O}(\varepsilon^{2\delta-1})} J_0(2\sqrt{t})\,dt\Big| \\
&\quad \le \Big|\int_1^{T} J_0(2\sqrt{t})\,dt\Big| + \Big|\int_T^{\mathcal{O}(\varepsilon^{2\delta-1})} \Big[\frac{1}{\sqrt{\pi}t^{1/4}}\cos(2\sqrt{t} - \frac{\pi}{4}) + \mathcal{O}(t^{-3/4})\Big]\,dt\Big| \\
&\quad \le \tilde{\tilde{c}}_2 + \Big|\frac{1}{\sqrt{2\pi}} t^{1/4}\big[-\cos(2\sqrt{t}) + \sin(2\sqrt{t})\big]\Big|_T^{\mathcal{O}(\varepsilon^{2\delta-1})}\Big| + \mathcal{O}(\varepsilon^{-1/4+\delta/2}) \\
&\quad \le \tilde{c}_2 + \mathcal{O}(\varepsilon^{-1/4+\delta/2}),
\end{aligned} \tag{9.7}$$

for some constants $\tilde{c}_2$ and $\tilde{\tilde{c}}_2$, where the second inequality is obtained by explicit evaluation in Mathematica.

This yields

$$\Big|\int_{a_3}^{x} u_n(z)\,dz\Big| \le \mathcal{O}(\varepsilon^{3/2}) + \mathcal{O}(\varepsilon^{3/2}\cdot\varepsilon^{-1/4+\delta/2}) + \mathcal{O}(\varepsilon^{3/2}\cdot\varepsilon^{1/4+5\delta/6}) = \mathcal{O}(\varepsilon^{5/4+\delta/2}),$$

where we have recalled from (9.4) that for sufficiently large n, $\epsilon = \epsilon_n < 1$. Consequently, this integral decays at least as fast as $\mathcal{O}(n^{-1})$, and we may conclude that there exists a constant c_2 such that

$$\Big|\int_{a_3}^{x} u_n(z)\,dz\Big| \le \frac{c_2}{n}, \quad x \in (a_3, a_3 + \mathcal{O}(\varepsilon^{1+2\delta})]. \tag{9.8}$$

Altogether, this implies the existence of a constant $\tilde{c}$ depending only on I and J for which

$$\Big|\int_{a_3}^{x} u_n(z)\,dz\Big| \le \frac{\tilde{c}}{n}, \quad x \in (a_3, a_4),$$

given that $n \ge N_0$. Trivially, however, the following upper bound can be derived for $n < N_0$ by noting that $\|u_n\|_{L^2(I)} = 1$: for $C = (a_4 - a_3)^{1/2}$,

$$\Big|\int_{a_3}^{x} u_n(z)\,dz\Big| \le \int_{a_3}^{a_4} |u_n(z)|\,dz \le C \le \frac{C\,N_0}{n}.$$

The assertion holds for all $n \in \mathbb{N}$ choosing $c = \max\{\tilde{c}, CN_0\}$. □

9.3. Proof of Relation (2) in §7

Here, we recall that we are considering Case 4, with $I = (a_2, a_4)$ and $J = (a_1, a_3)$ overlapping intervals with $a_1 < a_2 < a_3 < a_4$, and $\mu > 0$ is fixed so that $a_2 < a_3 - \mu$. We will expand the above argument for bounding integrals of u_n to this case with overlap, As a consequence of the fact that $\sigma_n \to 0$ (or equivalently $\lambda_n \to +\infty$), we will prove that for all $x \in [a_3 - \mu, a_4]$,

$$\Big|\int_{a_3-\mu}^{x} u_n(z)\,dz\Big| \le \frac{c}{n}.$$

As before, we define $\epsilon = \varepsilon_n = 1/\sqrt{\lambda_n}$ and omit the index. For sufficiently large n, the WKB approximation is valid on $[a_3 - \mu, a_3 - \mathcal{O}(\varepsilon^{1+2\delta})]$ and is given by

$$u_n(z) = \frac{K}{(P(z))^{1/4}} \exp\Big(-\frac{1}{\varepsilon}\int_z^{a_3} \frac{dt}{\sqrt{P(t)}}\Big)\cdot\big(1 + \mathcal{O}(\varepsilon^{1/2-\delta})\big),$$

for the same constant K as in §9.2. With this pointwise decay of u_n that is exponential in n, one easily sees that for $x \in [a_3 - \mu, a_3 - \mathcal{O}(\varepsilon^{1+2\delta})]$ the integral $|\int_{a_3-\mu}^{x} u_n(z)dz|$ decays faster than $\mathcal{O}(1/n)$.

Next, we consider $x \in [a_3 - \mathcal{O}(\varepsilon^{1+2\delta}), a_4]$. We distinguish three different cases into which we can split the integrals as follows: for $x \in [a_3 - \mathcal{O}(\varepsilon^{1+2\delta}), a_3]$,

$$\Big|\int_{a_3-\mu}^{x} u_n(z)\,dz\Big| \le \Big|\int_{a_3-\mu}^{a_3-\mathcal{O}(\varepsilon^{1+2\delta})} u_n(z)\,dz\Big| + \Big|\int_{a_3-\mathcal{O}(\varepsilon^{1+2\delta})}^{x} u_n(z)\,dz\Big|;$$

for $x \in [a_3, a_3 + \mathcal{O}(\varepsilon^{1+2\delta})]$,

$$\Big|\int_{a_3-\mu}^{x} u_n(z)dz\Big| \le \Big|\int_{a_3-\mu}^{a_3-\mathcal{O}(\epsilon^{1+2\delta})} u_n(z)dz\Big| + \Big|\int_{a_3-\mathcal{O}(\epsilon^{1+2\delta})}^{a_3} u_n(z)dz\Big| + \Big|\int_{a_3}^{x} u_n(z)dz\Big|;$$

and for $x \in [a_3 + \mathcal{O}(\varepsilon^{1+2\delta}), a_4]$,

$$\Big|\int_{a_3-\mu}^{x} u_n(z)\,dz\Big| \le \Big|\int_{a_3-\mu}^{a_3-\mathcal{O}(\varepsilon^{1+2\delta})} u_n(z)\,dz\Big| + 2\Big|\int_{a_3}^{a_3+\mathcal{O}(\varepsilon^{1+2\delta})} u_n(z)\,dz\Big| + \Big|\int_{a_3+\mathcal{O}(\varepsilon^{1+2\delta})}^{x} u_n(z)\,dz\Big|.$$

The last inequality relies on a property of the singular functions u_n that is referred to as *transmission conditions* (see [3] for details). Roughly, it states that the parts of u_n on regions of size $\mathcal{O}(\varepsilon^{1+2\delta})$ from the left and from the right of the point of singularity a_3 are the same as they approach the limit to a_3.

If we let A represent an integral over an interval at least $\mathcal{O}(\epsilon^{1+2\delta})$ away from the left of a_3, B represent an integral within an $\mathcal{O}(\epsilon^{1+2\delta})$ neighborhood to the left or right of a_3 (the transmission conditions ensure the left-hand and right-hand cases are equivalent), and C represent an integral over an interval at least $\mathcal{O}(\epsilon^{1+2\delta})$ away from the right of a_3, we see that the right hand sides of the above three inequalities take the form $A + B$, $A + B + B$, and $A + 2B + C$, respectively.

Integrals of the form A decay at least to order $\mathcal{O}(1/n)$, as remarked above. Integrals of the form C may be shown to decay to order $\mathcal{O}(1/n)$ by the argument of Section 9.2, since the behavior of u_n away from a_3 is independent of whether I and J intersect.

What remains is to treat the case of integrals of the form B, that is, to show that for $x \in (a_3, a_3 + \mathcal{O}(\varepsilon^{1+2\delta})]$,

$$\Big| \int_{a_3}^{x} u_n(z)\, dz \Big| \le \frac{\tilde{c}}{n},$$

for some $\tilde{c} > 0$. For this, we can proceed in a similar fashion as in §9.2, with the key change that where in §9.2 we used an approximation of u_n by the Bessel function J_0 on this region, now, in the case of overlapping intervals I and J, u_n is no longer a bounded function close to a_3, but can be approximated by a linear combination of the Bessel functions J_0 and Y_0. More precisely, substituting $t = (a_3 - z)/(\varepsilon^2 P'(a_3))$ yields,

$$u_n(z) = \begin{cases} b_3 \left[J_0(2\sqrt{t}) + \mathcal{O}(\varepsilon^{1-2\delta/3})\right] + c_3[Y_0(2\sqrt{t}) + \mathcal{O}(\varepsilon^{3/2-\delta/3})], & t \in [0,1) \\ b_3 \left[J_0(2\sqrt{t}) + t^{-1/4}\mathcal{O}(\varepsilon^{1-2\delta/3})\right] & \\ \quad + c_3 \left[Y_0(2\sqrt{t}) + t^{-1/4}\mathcal{O}(\varepsilon^{1-2\delta/3})\right], & t \in [1, \mathcal{O}(\varepsilon^{2\delta-1})] \end{cases}$$

with constants $b_3 = \mathcal{O}(\varepsilon^{-\delta})$ and $c_3 = \mathcal{O}(\varepsilon^{-1/2})$. As before, with a change of variables $dx = -\varepsilon^2 P'(a_3) dt$, we obtain

$$\begin{aligned} \int_{a_3}^{x} u_n(z) dz = b_3 \varepsilon^2 (-P'(a_3)) \cdot \Big\{ & \int_0^1 \left[J_0(2\sqrt{t}) + \mathcal{O}(\varepsilon^{1-2\delta/3})\right] dt \\ & + \int_1^{\mathcal{O}(\varepsilon^{2\delta-1})} \left[J_0(2\sqrt{t}) + t^{-1/4}\mathcal{O}(\varepsilon^{1-2\delta/3})\right] dt \Big\} \\ & + c_3 \varepsilon^2 (-P'(a_3)) \cdot \Big\{ \int_0^1 \left[Y_0(2\sqrt{t}) + \mathcal{O}(\varepsilon^{3/2-\delta/3})\right] dt \\ & + \int_1^{\mathcal{O}(\varepsilon^{2\delta-1})} \left[Y_0(2\sqrt{t}) + t^{-1/4}\mathcal{O}(\varepsilon^{1-2\delta/3})\right] dt \Big\}. \end{aligned}$$

Using the results from the proof in §9.2 for the terms involving J_0, this simplifies to

$$\begin{aligned} \Big| \int_{a_3}^{a_3 + \mathcal{O}(\varepsilon^{1+2\delta})} u_n(z)\, dz \Big| \le \mathcal{O}(\varepsilon^{2-\delta}) + \mathcal{O}(\varepsilon^{3/2}) \cdot \Big\{ & \Big| \int_0^1 \left[Y_0(2\sqrt{t}) + \mathcal{O}(\varepsilon^{3/2-\delta/3})\right] dt \Big| \\ & + \Big| \int_1^{\mathcal{O}(\varepsilon^{2\delta-1})} \left[Y_0(2\sqrt{t}) + t^{-1/4}\mathcal{O}(\varepsilon^{1-2\delta/3})\right] dt \Big| \Big\}. \end{aligned}$$

The first integral on the right-hand side of the above is bounded, since for small arguments z, $Y_0(z) \sim \frac{2}{\pi} \ln(z)$. For the second integral, the same argument as for J_0 in (9.5)–(9.7) holds, but upon replacing the asymptotic form (9.6) by

$$Y_0(2\sqrt{t}) = \frac{1}{\sqrt{\pi} t^{1/4}} \left[\sin(2\sqrt{t} - \tfrac{\pi}{4}) + \mathcal{O}(t^{-1/2}) \right].$$

This then allows us to state that

$$\Big| \int_{a_3}^{x} u_n(z)\, dz \Big| \le \frac{\tilde{c}}{n}, \quad \forall x \in [a_3, a_3 + \mathcal{O}(\varepsilon^{1+2\delta})],$$

and consequently, that for all $x \in [a_3 - \mu, a_4]$,

$$\Big| \int_{a_3-\mu}^{x} u_n(z)\, dz \Big| \le \frac{c}{n}.$$

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