Rev. Mat. Iberoam. **32** (2016), no. 1, 57–77
DOI 10.4171/RMI/881

Isotropic p-harmonic systems in 2D Jacobian estimates and univalent solutions

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Abstract. The core result of this paper is an inequality (rather tricky) for the Jacobian determinant of solutions of nonlinear elliptic systems in the plane.

The model case is the isotropic (rotationally invariant) p-harmonic system

$$\operatorname{div} |Dh|^{p-2} Dh \;=\; 0\,, \quad h = (u,v) \in \mathscr{W}^{1,p}(\Omega, \mathbb{R}^2)\,,\; 1 < p < \infty\,,$$

as opposed to a pair of scalar p-harmonic equations:

$$\operatorname{div} |\nabla u|^{p-2} \nabla u \;=\; 0 \quad \text{and} \quad \operatorname{div} |\nabla v|^{p-2} \nabla v \;=\; 0\,.$$

Rotational invariance of the systems in question makes them meaningful, both physically and geometrically. An issue is to overcome the nonlinear coupling between ∇u and ∇v. In the extensive literature dealing with coupled systems various differential expressions of the form $\Phi(\nabla u\,, \nabla v)$ were subjected to thorough analysis. But the Jacobian determinant $\det Dh = u_x v_y - u_y v_x$ was never successfully incorporated into such analysis. We present here new nonlinear differential expressions of the form $\Phi(|Dh|, \det Dh)$ and show they are superharmonic, which yields much needed lower bounds for $\det Dh$. To illustrate the utility of such bounds we extend the celebrated univalence theorem of Radó–Kneser–Choquet on harmonic mappings ($p = 2$) to the solutions of the coupled p-harmonic system.

1. Introduction

Suppose a domain $\Omega \subset \mathbb{R}^2 \simeq \mathbb{C}$ (thin plate or film), subject to a deformation $h : \Omega \xrightarrow{\text{onto}} \Omega'$, is occupied by an isotropic elastic material. This amounts to saying that a response of the material to the energy-minimal deformations is the same in all directions. The precise mathematical statement is that the stored energy functional for h is invariant under rotations.

Mathematics Subject Classification (2010): Primary 35J60; Secondary 30C70.
Keywords: Nonlinear systems of PDEs, variational integrals, p-harmonic mappings, Jacobian determinants, energy-minimal deformations.

Such are the functionals of the form:

$$\mathscr{E}[h] = \int_\Omega \mathbf{E}\big(\,|Dh(z)|\,\big)\,\mathrm{d}x\,\mathrm{d}y\,, \quad z = x+iy\,. \tag{1.1}$$

where $\mathbf{E}\colon [0,\infty) \to \mathbb{R}$ depends only on the norm (Hilbert–Schmidt norm) of the deformation gradient $Dh : \Omega \to \mathbb{R}^{2\times 2}$. The Lagrange–Euler equation for the energy-minimal map takes the form of a coupled system of nonlinear PDEs:

$$\operatorname{Div}\Big[\frac{\mathbf{E}'(|Dh|)}{|Dh|}\,Dh\Big] = 0\,. \tag{1.2}$$

In geometric function theory (GFT) [4], [26] and nonlinear elasticity [6], [7], [9], [39] we want to control the Jacobian determinant $J(z,h) := \det Dh(z)$ from below, hopefully by a positive constant. Let us invoke the simplest case; that is, the uncoupled system of two Laplace equations $\Delta u = 0$ and $\Delta v = 0$.

Proposition 1.1 (Minimum principle). *Suppose $\mathcal{H} = u+iv : \Omega \to \mathbb{C}$ is a harmonic map whose Jacobian determinant is positive in Ω. Then*

$$\min_{z\in\mathbb{G}} J(z,\mathcal{H}) \geqslant \min_{z\in\partial\mathbb{G}} J(z,\mathcal{H}) \quad \textit{for every compact } \mathbb{G}\subset\Omega\,. \tag{1.3}$$

It may be worth pointing out that the Dirichlet energy-minimal deformations are invertible exactly where they are harmonic, thus having positive Jacobian [27]. Proposition 1.1 is straightforward from the following superharmonicity result [35].

Proposition 1.2 (Superharmonicity). *The function $z \rightarrowtail \log J(z,\mathcal{H})$ is superharmonic in Ω.*

Returning to more general energy integrals in (1.1) we assume, for ease of presentation rather than a desire for more generality, that $\mathbf{E} = \mathbf{E}(s)$ is continuous in $[0,\infty)$ and $\mathscr{C}^\infty$-smooth in $(0,\infty)$. The assumptions essential for the results are:

ı) $\mathbf{E}'(s) > 0$, for $0 < s < \infty$;

ıı) $0 < \inf\kappa(s) \leqslant \sup\kappa(s) < \infty$, where $\kappa(s) = s\,\mathbf{E}''(s)/\,\mathbf{E}'(s)$.

This furnishes a number

$$0 \leqslant \tau = \sup\Big|\,\frac{1-\kappa(s)}{1+\kappa(s)}\,\Big| < 1\,. \tag{1.4}$$

Needless to say, for general energy integrals, even in the model p-harmonic case, it is not at all obvious which nonlinear differential expressions are suited to the Jacobian estimates. The answer to this puzzle is contained in the following theorem (our prime result).

Theorem 1.3 (Minimum principle). *Let $h : \Omega \to \mathbb{C}$ satisfy the Lagrange–Euler equation for the energy integral (1.1), and suppose that its Jacobian determinant $\det Dh = |h_z|^2 - |h_{\overline{z}}|^2$ is positive. Then for every compact subset $\mathbb{G}\Subset\Omega$, we have*

$$\min_{\mathbb{G}} \mathbf{T}(Dh) \geqslant \min_{\partial\mathbb{G}} \mathbf{T}(Dh)\,, \quad \textit{where} \quad \mathbf{T}(Dh) \stackrel{\mathrm{def}}{=} \frac{\mathbf{E}'(|Dh|)}{|Dh|}\det Dh. \tag{1.5}$$

This is a straightforward corollary from the following:

Theorem 1.4 (Superharmonicity). *The nonlinear differential expression*

$$\Phi\Big(\frac{\mathbf{E}'(|Dh|)}{|Dh|}\det Dh\Big) \quad \textit{is a superharmonic function,} \tag{1.6}$$

where we retrieve Φ *from* Φ', *uniquely up to a constant, by the rule:*

$$\Phi\,'(\mathbf{T}) = \mathbf{T}^{-N}\,, \quad N = \frac{(2-\tau)^2}{4(1-\tau)}\,. \tag{1.7}$$

The Dirichlet energy, which corresponds to $\tau = 0$, gives $\Phi(\mathbf{T}) = \log \mathbf{T}$. We save the reader from an uphill analysis leading to the peculiar nonlinear differential expression $\mathbf{T}(Dh)$ in (1.5) and only say that it is unique for our purpose. Moreover, the exponent N in (1.7) seems to be smallest possible. The proof of Theorem 1.4 goes through rather subtle (though elementary) algebra of quadratic forms with respect to carefully selected linear combinations of the second derivatives of h. We know of no way of simplifying the forthcoming computation; try it!

Remark 1.5. The utility of the minimum principle for the Jacobian of a harmonic map is best demonstrated in [29] where we obtained a proof (not the shortest one) of the celebrated theorem of Radó–Kneser–Choquet (RKC-theorem), [15], [13], [30], and [36]. The novelty of this approach lies in the construction of a homotopy between the given harmonic mapping and a conformal one. The minimum principle in Proposition 1.1 guarantees that the mappings remain injective all the way through the homotopy. The advantage of the homotopy method is that it works for fairly general coupled systems of PDEs as well. An observant reader may also find this method in Section 12 where, by way of illustration, we prove the following isotropic p-harmonic version of the RKC-theorem.

Theorem 1.6 (Univalence criterion). *Let* $\mathbb{X}$ *and* $\mathbb{Y}$ *be bounded* $\mathscr{C}^{1,\alpha}$*-smooth simply connected domains in* $\mathbb{R}^2$, $\mathbb{Y}$ *being convex. Consider an isotropic* p*-harmonic map* $h \in \mathscr{C}^{1,\alpha}(\overline{\mathbb{X}}, \mathbb{R}^2)$ *whose boundary data* $f = h : \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ *is an immersion. Then* h *is a* $\mathscr{C}^{1,\alpha}$*-diffeomorphism of* $\overline{\mathbb{X}}$ *onto* $\overline{\mathbb{Y}}$ *and it is* $\mathscr{C}^{\infty}$*-smooth in* $\mathbb{X}$.

Here the term immersion refers to a $\mathscr{C}^1$-map $f : \partial\mathbb{X} \xrightarrow{\text{into}} \mathbb{R}^2$ whose tangential derivative along $\partial\mathbb{X}$ is nowhere vanishing.

Most recent developments in the approximation of Sobolev homeomorphisms with diffeomorphisms [25], [24], [23], [28] rely on local p-harmonic replacements, encouraging enough to merit Theorem 1.6.

2. Remarks and a historical account

On the regularity. It should be noted that solutions to the (isotropic) p-harmonic systems belong to $\mathscr{C}^{1,\alpha}_{\mathrm{loc}}(\mathbb{X}, \mathbb{R}^2)$, with the Hölder exponent depending only on p ([12], [20], [32], [38]). In Section 11 we discuss the very much needed explicit local $\mathscr{C}^{1,\alpha}$-bounds. We work out a fairly detailed arguments, maybe too much elementary and routine for some readers or specialists in PDEs; they may skip this section. Unfortunately, in the vast literature on nonlinear coupled systems ([2], [10], [33]) there are no explicit statements concerning $\mathscr{C}^{1,\alpha}$-bounds *up to the boundary*. Thus we appeal only to the interior $\mathscr{C}^{1,\alpha}$-regularity results, as in the paper by C. Hamburger [20]. However, if those estimates indeed extend up to the boundary, as claimed by some authors, then the restriction to $h \in \mathscr{C}^{1,\alpha}(\overline{\mathbb{X}}, \mathbb{R}^2)$ in Theorem 1.6 is redundant.

Historical account. Search for univalent solutions to a given system of PDEs has a long and fascinating history. In his famous work, Beltrami (1867) introduced a complex equation (first order) whose homeomorphic solutions have come to be known in GFT as quasiconformal mappings. On the other hand the injective energy-minimal deformations (solutions of second order variational PDEs) have come to the core of nonlinear *hyperelasticity* [1], [5], [14] to act in accordance with the *principle of no interpenetration of matter*. From the very beginning of both theories, the Jacobian determinant has been subjected to a great deal of investigation. It tells us something about invertibility of the mapping. For the uncoupled pair of Laplace equations, the theorem of Radó–Kneser–Choquet [15], conjectured in 1926 by Radó [36], was first proved by Kneser [30] and, independently in 1945, by Choquet [13], see [22] (p. 78–80) for the original Kneser's proof. Multiply connected planar domains are treated in [16], [34]. The interested reader is also referred to the survey paper by Bshouty and Hengartner [11] for further reading about planar harmonic mappings. A generalization of RKC-theorem for a pair of planar elliptic equations (uncoupled system) has been given by Bauman, Marini and Nesi [8]. Finally, an extension of RKC-theorem to nonlinear (*still uncoupled*) systems of p-harmonic type has been offered by G. Alessandrini and M. Sigalotti [3]. At this point we strongly emphasize that Theorem 1.6 deals with the isotropic p-harmonic deformations. An issue is not a greater generality, but to overcome the difficulties with the coupling that occurs between the coordinate components of h. Finally, we refer to [31] where many interesting energy functionals (including Neohookean models) are sorted out for the minimum principle.

3. Words on harmonic mappings

In this context the complex variables $z = x + iy$ and $h = u + iv$ are particularly convenient, so we work with the Cauchy–Riemann partial derivatives:

$$h_z = \tfrac{1}{2}(h_x - ih_y) \ , \ h_{\bar z} = \tfrac{1}{2}(h_x + ih_y)\,,$$
$$|Dh|^2 = 2\left(|h_z|^2 + |h_{\bar z}|^2\right) \quad \text{and} \quad \det Dh(z) \,=\, |\,h_z\,|^2 \,-\, |\,h_{\bar z}\,|^2\,.$$

Proposition 1.2 was shown by V. Manojlović [35] with the aid of a local representation $\mathcal{H}(z) = g(z) + \overline{f(z)}$ in terms of analytic functions g and f. In fact such representation gives further information about the Jacobian:

Proposition 3.1. *Let $\mathcal{H}$ be a harmonic map in Ω, whose Jacobian is strictly subharmonic, meaning that*

$$\Delta J = \Delta(|\mathcal{H}_z|^2 - |\mathcal{H}_{\bar z}|^2) \,=\, |\mathcal{H}_{zz}|^2 - |\mathcal{H}_{\bar z \bar z}|^2 \ > 0$$

in Ω. Then $\log(\Delta J)$ is superharmonic in Ω.

Unfortunately, these stylish arguments are unavailable for nonlinear systems of PDEs. Before proceeding to nonlinear systems let us reveal a direct computation for Proposition 1.2. It follows from the identity [29]:

$$\Delta \log J = 4 (\log J)_{z\bar{z}} = -4 \frac{|\mathcal{H}_{zz} \overline{\mathcal{H}_{\bar{z}}} - \overline{\mathcal{H}_{\bar{z}\bar{z}}} \mathcal{H}_z|^2}{|\mathcal{H}_z|^2 - |\mathcal{H}_{\bar{z}}|^2} \leqslant 0 . \tag{3.1}$$

Proof. This identity is very easy relative to the laborious computation yet to come for the nonlinear p-harmonic type systems. Below, the rules of complex differentiation are self-evident. Indeed, we have:

$$J = \mathcal{H}_z \overline{\mathcal{H}_z} - \mathcal{H}_{\bar{z}} \overline{\mathcal{H}_{\bar{z}}} , \quad J_z = \mathcal{H}_{zz} \overline{\mathcal{H}_z} - \mathcal{H}_{\bar{z}} \overline{\mathcal{H}_{\bar{z}\bar{z}}} , \quad J_{z\bar{z}} = |\mathcal{H}_{zz}|^2 - |\mathcal{H}_{\bar{z}\bar{z}}|^2 ,$$

$$(\log J)_{z\bar{z}} = \left(\frac{J_z}{J}\right)_{\bar{z}} = \frac{J_{z\bar{z}} J - J_z J_{\bar{z}}}{J^2} .$$

The terms in the numerator are:

$$\begin{aligned} J_z J_{\bar{z}} &= |J_z|^2 = |\mathcal{H}_{zz} \overline{\mathcal{H}_z} - \mathcal{H}_{\bar{z}} \overline{\mathcal{H}_{\bar{z}\bar{z}}}|^2 \\ &= |\mathcal{H}_{zz}|^2 |\mathcal{H}_z|^2 + |\mathcal{H}_{\bar{z}\bar{z}}|^2 |\mathcal{H}_{\bar{z}}|^2 - 2 \operatorname{Re}(\mathcal{H}_{zz} \mathcal{H}_{\bar{z}\bar{z}} \overline{\mathcal{H}_{\bar{z}}} \, \overline{\mathcal{H}_z}) \end{aligned}$$

and

$$\begin{aligned} J_{z\bar{z}} J &= (|\mathcal{H}_{zz}|^2 - |\mathcal{H}_{\bar{z}\bar{z}}|^2) \cdot (|\mathcal{H}_z|^2 - |\mathcal{H}_{\bar{z}}|^2) \\ &= |\mathcal{H}_{zz}|^2 |\mathcal{H}_z|^2 - |\mathcal{H}_{\bar{z}\bar{z}}|^2 |\mathcal{H}_z|^2 - |\mathcal{H}_{zz}|^2 |\mathcal{H}_{\bar{z}}|^2 + |\mathcal{H}_{\bar{z}\bar{z}}|^2 |\mathcal{H}_{\bar{z}}|^2 . \end{aligned}$$

Hence the identity (3.1) follows;

$$\begin{aligned} J_z J_{\bar{z}} - J_{z\bar{z}} J &= |\mathcal{H}_{\bar{z}\bar{z}}|^2 |\mathcal{H}_z|^2 + |\mathcal{H}_{zz}|^2 |\mathcal{H}_{\bar{z}}|^2 - 2 \operatorname{Re}(\mathcal{H}_{zz} \mathcal{H}_{\bar{z}\bar{z}} \overline{\mathcal{H}_{\bar{z}} \mathcal{H}_z}) \\ &= |\mathcal{H}_{zz} \overline{\mathcal{H}_{\bar{z}}} - \overline{\mathcal{H}_{\bar{z}\bar{z}}} \mathcal{H}_z|^2 , \end{aligned}$$

completing the calculation. □

Analogous computational attempts for harmonic mappings fail in higher dimensions, though they can be reconstructed for the Hessian of a real-valued harmonic function in dimension $n = 3$. We refer to [19] for this later result and for further generalizations to dimensions $n \geqslant 4$.

4. The equation in complex notation

Let us write the Lagrange–Euler system (1.2) in complex variables:

$$[\lambda(\boldsymbol{D}) h_z]_{\bar{z}} + [\lambda(\boldsymbol{D}) h_{\bar{z}}]_z = 0 , \quad \text{where } \boldsymbol{D} = |h_z|^2 + |h_{\bar{z}}|^2 \tag{4.1}$$

and

$$\lambda(t) = \frac{\mathbf{E}'(\sqrt{2\,t})}{\sqrt{2\,t}} . \tag{4.2}$$

Our hypotheses on $\mathbf{E}$ is now equivalently reformulated as:

$$-\frac{1}{2} < \inf \alpha(t) \leqslant \sup \alpha(t) < \infty , \quad \text{where } \alpha(t) = \frac{t \lambda'(t)}{\lambda(t)} = \frac{\kappa(\sqrt{2t}) - 1}{2} .$$

Regarding parameter τ, we have

$$0 \leqslant \tau = \sup \left| \frac{\alpha(t)}{1+\alpha(t)} \right| < 1 . \tag{4.3}$$

It should be noted that the solutions of the system (4.1) are at least $\mathscr{C}^1$-smooth. They are actually $\mathscr{C}^\infty$-smooth whenever the Jacobian is positive, which is the case. It is therefore legitimate to differentiate the system (4.1) as many times as needed.

5. A linear system

One might view (4.1) as an uncoupled system of two linear equations, simply by assuming that $\lambda(z) = \lambda(\mathbf{D}(z))$ is a given function, positive and $\mathscr{C}^\infty$-smooth:

$$[\,\lambda(z)\, h_z\,]_{\bar z} \;+\; [\,\lambda(z)\, h_{\bar z}\,]_z = 0 . \tag{5.1}$$

Let us introduce more notation; $\mathbf{T}(z) = \mathbf{T}(Dh) = \lambda(z)\mathbf{J}(z)$, where

$$\mathbf{J} = \mathbf{J}(z) = \det Dh = |h_z|^2 - |h_{\bar z}|^2 = h_z\, \overline{h_z} \;-\; h_{\bar z}\, \overline{h_{\bar z}} . \tag{5.2}$$

The objective is to show that the function $z \to \Phi(\mathbf{T}(z))$ is superharmonic, so we must compute its Laplacian:

$$[\Phi(\mathbf{T})]_{z\bar z} = [\Phi'(\mathbf{T})\, \mathbf{T}_z]_{\bar z} = [\mathbf{T}^{-N}\, \mathbf{T}_z]_{\bar z} = \left(\mathbf{T}\, \mathbf{T}_{z\bar z} - N\, |\mathbf{T}_z|^2\right)\, \mathbf{T}^{-N-1} . \tag{5.3}$$

Therefore, we are reduced to proving the following:

Lemma 5.1. *Under the above notation and the conditions of Theorem 1.4, we have*

$$N\, |\mathbf{T}_z|^2 \geqslant \mathbf{T}\, \mathbf{T}_{z\bar z} . \tag{5.4}$$

6. The third order derivatives cancel out

Clearly, the left hand side $N\,|\mathbf{T}_z|^2$ in (5.4) (once written explicitly in terms of h) involves only first and second order derivatives of h. A priori, the right hand side might depend (linearly) on the third derivatives. But this is not the case. And that is why one would expect that Inequality (5.4) holds with N being a suitably large constant.

Lemma 6.1. *We have*

$$\mathbf{T}_{z\bar z} \;=\; \lambda\left(|h_{zz}|^2 \;-\; |h_{\bar z\bar z}|^2\right) \;+\; \operatorname{Re}\left[\lambda_z\left(h_z\overline{h_{zz}} \;-\; \overline{h_{\bar z}}h_{\bar z\bar z} \;-\; \overline{h_z}h_{z\bar z} \;+\; h_{\bar z}\overline{h_{z\bar z}}\right)\right] . \tag{6.1}$$

Proof. Let us begin with the product rule

$$\mathbf{T}_z = (\lambda\, \mathbf{J})_z = \lambda_z \mathbf{J} \;+\; \lambda\, \mathbf{J}_z . \tag{6.2}$$

Hence

$$\mathbf{T}_{z\bar z} = [\,\lambda_{z\bar z}\, \mathbf{J} \;+\; \lambda\, \mathbf{J}_{z\bar z}\,] + 2 \operatorname{Re}\left(\lambda_z\, \mathbf{J}_{\bar z}\right) . \tag{6.3}$$

The last term contains no third derivatives of h. The cancellation of third derivatives will take place within the rectangular bracket. To see this, we compute

$$\mathbf{J}_{\bar z} = (h_z\overline{h_z}) - h_{\bar z}\overline{h_{\bar z}})_{\bar z} \;=\; h_z\,\overline{h_{zz}} + \overline{h_z}\,h_{z\bar z} \;-\; \overline{h_{\bar z}}\,h_{\bar z\bar z} \;-\; h_{\bar z}\,\overline{h_{z\bar z}}\,. \tag{6.4}$$

One more differentiation reveals that

$$\lambda\,\mathbf{J}_{z\bar z} = \lambda\left(|h_{zz}|^2 - |h_{\bar z\bar z}|^2\right) \;+\; 2\lambda\,\mathrm{Re}\left(h_z\,\overline{h_{zz\bar z}} \;-\; \overline{h_{\bar z}}\,h_{z\bar z\bar z}\right). \tag{6.5}$$

At this stage we appeal to the Euler–Lagrange equation (5.1), which upon differentiation takes the form:

$$2\,\lambda\,h_{z\bar z} \;+\; \lambda_{\bar z}\,h_z \;+\; \lambda_z\,h_{\bar z} \;=\; 0\,. \tag{6.6}$$

Note that $\lambda_{\bar z} = \overline{\lambda_z}$ since λ is real. Thus (6.6) can be written in two ways, the first one

$$2\,\lambda\,h_{z\bar z} \;+\; \overline{\lambda_z}\,h_z \;+\; \lambda_z\,h_{\bar z} \;=\; 0\,,$$

and the second one, via conjugation,

$$2\,\lambda\,\overline{h_{z\bar z}} \;+\; \lambda_z\,\overline{h_z} \;+\; \overline{\lambda_z}\;\overline{h_{\bar z}} \;=\; 0\,.$$

This linear system of equations, with λ_z and $\overline{\lambda_z}$ as unknowns, can be solved for λ_z. The formula for λ_z involves only the first and second order derivatives of h:

$$\mathbf{J}\,\lambda_z \;=\; 2\,\lambda\left(\overline{h_{\bar z}}\,h_{z\bar z} \;-\; h_z\overline{h_{z\bar z}}\right). \tag{6.7}$$

Next, we apply the Cauchy–Riemann operator $\partial/\partial\bar z$ to both sides of (6.7). The term $|h_{z\bar z}|^2$ will cancel out and we obtain

$$\mathbf{J}\,\lambda_{z\bar z} \;=\; -\mathbf{J}_{\bar z}\,\lambda_z \;+\; 2\,\lambda_{\bar z}\left(\overline{h_{\bar z}}\,h_{z\bar z} \;-\; h_z\,\overline{h_{z\bar z}}\right) \;+\; 2\,\lambda\left(\overline{h_{\bar z}}\,h_{z\bar z\bar z} \;-\; h_z\,\overline{h_{zz\bar z}}\right).$$

Since $\mathbf{J}\,\lambda_{z\bar z}$ is real we can write it as

$$\begin{aligned}\mathbf{J}\,\lambda_{z\bar z} = &-\mathrm{Re}\,(\mathbf{J}_{\bar z}\,\lambda_z) + 2\;\mathrm{Re}\left[\lambda_{\bar z}\left(\overline{h_{\bar z}}\,h_{z\bar z} \;-\; h_z\overline{h_{z\bar z}}\right)\right] \\ &+\; 2\lambda\,\mathrm{Re}\left[\overline{h_{\bar z}}\,h_{z\bar z\bar z} \;-\; h_z\,\overline{h_{zz\bar z}}\right].\end{aligned} \tag{6.8}$$

Substituting (6.5) and (6.8) into (6.3) results in cancellation of the third order derivatives of h. Specifically,

$$\mathbf{T}_{z\bar z} = \lambda\,\left(|h_{zz}|^2 - |h_{\bar z\bar z}|^2\right) + 2\;\mathrm{Re}\left[\lambda_z\left(h_{\bar z}\overline{h_{z\bar z}} \;-\; \overline{h_z}\,h_{z\bar z}\right)\right] \;+\; \mathrm{Re}\,(\lambda_z\,\mathbf{J}_{\bar z})\,.$$

Further cancelation will occurs upon substitution of $\mathbf{J}_{\bar z}$ by (6.4), completing the proof of Formula (6.1). □

At this stage we introduce the following linear forms of the second derivatives:

$$\begin{cases}\mathcal{A} = \lambda\left(h_z\,\overline{h_{\bar z\bar z}} \;-\; \overline{h_{\bar z}}\,h_{zz}\right), \\ \mathcal{B} = \lambda\left(\overline{h_z}\,h_{zz} \;-\; h_{\bar z}\,\overline{h_{\bar z\bar z}}\right), \\ \mathcal{L} = \lambda\left(h_z\,\overline{h_{z\bar z}} \;-\; \overline{h_{\bar z}}\,h_{z\bar z}\right).\end{cases} \tag{6.9}$$

Note that the coefficients of these linear forms contain no gradient of λ. In fact, by (6.7), the complex gradient of λ can be computed through the formula

$$\mathbf{J}\,\lambda_z \;=\; -2\,\mathcal{L}\,. \tag{6.10}$$

Then, in view of (6.4) , we write the second term in (6.2) as

$$\lambda\,\mathbf{J}_z = \lambda\,\overline{\mathbf{J}_{\bar z}} = \lambda\left(\overline{h_z}h_{zz} + h_z\overline{h_{z\bar z}} - h_{\bar z}\overline{h_{\bar z\bar z}} - \overline{h_{\bar z}}h_{z\bar z}\right) = \mathcal{L} + \mathcal{B}\,. \tag{6.11}$$

Adding the last two formulas, we obtain

$$\mathbf{T}_z = (\lambda\,\mathbf{J})_z = \lambda_z\,\mathbf{J} + \lambda\,\mathbf{J}_z = \mathcal{B} - \mathcal{L}\,. \tag{6.12}$$

Now (6.1) reduces to $\lambda\mathbf{T}_{z\bar z} = \lambda^2\left(|h_{zz}|^2 - |h_{\bar z\bar z}|^2\right) \,+\, \mathrm{Re}\left[\lambda_z(\overline{\mathcal{B}} - \overline{\mathcal{L}})\right]$. We multiply it by $\mathbf{J}$. In view of the definition of $\mathbf{T}$ ($\mathbf{T} = \lambda\,\mathbf{J}$) and by (6.7), we conclude with an identity:

$$\mathbf{T}\cdot\mathbf{T}_{z\bar z} \;=\; \lambda^2\,\mathbf{J}\,\left(\,|h_{zz}|^2 - |h_{\bar z\bar z}|^2\right) \,+\, 2\,|\mathcal{L}|^2 \,-\, 2\;\mathrm{Re}\,(\mathcal{L}\,\overline{\mathcal{B}})\,. \tag{6.13}$$

7. Subtracting the term $|\mathrm{T}_z|^2 = |\mathcal{B} - \mathcal{L}|^2$

The expression $\mathbf{T}\cdot\mathbf{T}_{z\bar z} - |\mathbf{T}_z|^2$, that corresponds to $N = 1$ in (5.4), naturally arises for the Dirichlet integral. In any case it simplifies to the following:

Lemma 7.1. *We have*

$$\mathbf{T}\cdot\mathbf{T}_{z\bar z} - |\mathbf{T}_z|^2 = |\mathcal{L}|^2 - |\mathcal{A}|^2\,. \tag{7.1}$$

Proof. Subtract $|\mathbf{T}_z|^2 = |\mathcal{B} - \mathcal{L}|^2$ from $\mathbf{T}\cdot\mathbf{T}_{z\bar z}$ in (6.13):

$$\begin{aligned}\mathbf{T}\cdot\mathbf{T}_{z\bar z} - |I_z|^2 &= \lambda^2\mathbf{J}\left(|h_{zz}|^2 - |h_{\bar z\bar z}|^2\right) - 2\;\mathrm{Re}\left[\mathcal{L}\left(\overline{\mathcal{B}} - \overline{\mathcal{L}}\right)\right] - |\mathcal{B} - \mathcal{L}|^2\\ &= |\mathcal{L}|^2 + \lambda^2\mathbf{J}\left(|h_{zz}|^2 - |h_{\bar z\bar z}|^2\right) - |\mathcal{B}|^2\,.\end{aligned} \tag{7.2}$$

The last two terms add up to $-|\mathcal{A}|^2$. This is because

$$\begin{aligned}\lambda^2\left(|h_z|^2 - |h_{\bar z}|^2\right)\left(|h_{zz}|^2 - |h_{\bar z\bar z}|^2\right) - \lambda^2|\overline{h_z}h_{zz} - h_{\bar z}\overline{h_{\bar z\bar z}}|^2 &= -\lambda^2|h_z\overline{h_{\bar z\bar z}} - \overline{h_{\bar z}}h_{zz}|^2\\ &= -|\mathcal{A}|^2\,,\end{aligned}$$

as is easy to verify. We then conclude with (7.1). □

8. An upper bound of $\mathcal{L}$

Up to this point we did not use any particular dependance of λ on $|Dh|$. We now recall the equation (4.1) and the associated notation;

$$\alpha = \alpha(\boldsymbol{D}) = \frac{\boldsymbol{D}\,\lambda'(\boldsymbol{D})}{\lambda(\boldsymbol{D})}\,, \quad \boldsymbol{D} = |h_z|^2 + |h_{\bar z}|^2\,.$$

Lemma 8.1. *We have*

$$|\mathcal{L}| \leqslant \frac{|\alpha|}{2+2\alpha-|\alpha|}\Big(|\mathbf{T}_z|+|\mathcal{A}|\Big), \quad \text{where} \quad 0 \leqslant \frac{|\alpha|}{2+2\alpha-|\alpha|} < 1 . \tag{8.1}$$

The factor in front of $|\mathbf{T}_z|+|\mathcal{A}|$ *is smaller than* 1, *because* $\alpha=\alpha(\boldsymbol{D})>-1/2$.

Proof. Let us first express $h_{z\bar z}$ in terms of $\mathcal{L}$ by using (6.9),

$$\lambda\,\mathbf{J}\,h_{z\bar z} = h_{\bar z}\mathcal{L} + h_z\overline{\mathcal{L}} . \tag{8.2}$$

The proof of (8.1) relies on another formula for $\mathbf{T}_z$. This time we exploit the structure of $\lambda=\lambda(\boldsymbol{D})=\lambda(|h_z|^2+|h_{\bar z}|^2)$,

$$\mathbf{T}_z = \lambda\mathbf{J}_z + \lambda_z\mathbf{J} = \mathcal{L}+\mathcal{B}+\mathbf{J}\lambda'\boldsymbol{D}_z = \mathcal{L}+\mathcal{B}+\frac{\alpha}{\boldsymbol{D}}\lambda\,\mathbf{J}\boldsymbol{D}_z . \tag{8.3}$$

By an elementary algebra we find that

$$\begin{aligned}
\mathbf{J}\,\boldsymbol{D}_z &= (h_z\overline{h_z}-h_{\bar z}\overline{h_{\bar z}})(\overline{h_z}h_{zz}+h_z\overline{h_{z\bar z}}+\overline{h_{\bar z}}h_{z\bar z}+h_{\bar z}\overline{h_{\bar z\bar z}})\\
&= (h_z\overline{h_z}+h_{\bar z}\overline{h_{\bar z}})(\overline{h_z}h_{zz}-h_{\bar z}\overline{h_{\bar z\bar z}})+2\overline{h_z}h_{\bar z}(h_z\overline{h_{\bar z\bar z}}-\overline{h_{\bar z}}h_{zz})\\
&\quad + (h_z\overline{h_z}-h_{\bar z}\overline{h_{\bar z}})(h_z\overline{h_{z\bar z}}+\overline{h_{\bar z}}h_{z\bar z}) .
\end{aligned}$$

Hence,

$$\lambda\,J\,\boldsymbol{D}_z \;=\; \boldsymbol{D}\,\mathcal{B} \;+\; 2\,\overline{h_z}\,h_{\bar z}\,\mathcal{A} \;+\; \lambda(h_z\overline{h_z}-h_{\bar z}\overline{h_{\bar z}})(h_z\overline{h_{z\bar z}}+\overline{h_{\bar z}}h_{z\bar z}) .$$

We eliminate $h_{z\bar z}$ by using (8.2),

$$\lambda\,J\,\boldsymbol{D}_z \;=\; \boldsymbol{D}\,\mathcal{B} \;+\; 2\,\overline{h_z}\,h_{\bar z}\,\mathcal{A} \;+\; \boldsymbol{D}\,\mathcal{L} \;+\; 2\,h_z\,\overline{h_{\bar z}}\,\overline{\mathcal{L}} .$$

Therefore,

$$\mathbf{T}_z = (1+\alpha)(\mathcal{B}+\mathcal{L}) + \frac{2\overline{h_z}h_{\bar z}}{\boldsymbol{D}}\alpha\,\mathcal{A} + \frac{2h_z\overline{h_{\bar z}}}{\boldsymbol{D}}\alpha\,\overline{\mathcal{L}} .$$

It follows from (6.12) that $\mathcal{B}+\mathcal{L}=2\mathcal{L}+\mathcal{B}-\mathcal{L}=2\mathcal{L}+\mathbf{T}_z$. Hence the last equation takes the form

$$-\alpha\mathbf{T}_z = 2(1+\alpha)\mathcal{L} + \frac{2\,\overline{h_z}\,h_{\bar z}}{\boldsymbol{D}}\,\alpha\,\mathcal{A} + \frac{2\,h_z\,\overline{h_{\bar z}}}{\boldsymbol{D}}\,\alpha\,\overline{\mathcal{L}} .$$

By the triangle inequality, in view of $|2h_zh_{\bar z}|\leqslant\boldsymbol{D}$, we obtain

$$|\alpha|\,|\mathbf{T}_z| \geqslant \big(\,2+2\alpha-|\alpha|\,\big)\,|\mathcal{L}| \;-\; |\alpha|\,|\mathcal{A}| ,$$

which gives the desired estimate of $|\mathcal{L}|$, completing the proof of Lemma 8.1. □

9. Proof of Theorem 1.4

It is natural to look first at the case of the Dirichlet energy. This case corresponds to $\alpha \equiv 0$ and $\mathcal{L} \equiv 0$. The identity (7.1) yields

$$|\mathbf{T}_z|^2 - \mathbf{T}_z \mathbf{T}_{z\bar{z}} = |\mathcal{A}|^2 \geqslant 0\,, \quad \text{hence} \quad [\Phi(\mathbf{J})]_{z\bar{z}} = (\log \mathbf{J})_{z\bar{z}} \leqslant 0\,.$$

For such inequality to hold in a general case, even for the genuine p-harmonic system with $p \neq 2$, one needs a large factor in front of $|\mathbf{T}_z|^2$. The actual estimate is straightforward from Lemma 8.1,

$$\begin{aligned} N|\mathbf{T}_z|^2 - \mathbf{T}_z \mathbf{T}_{z\bar{z}} &= (N-1)|\mathbf{T}_z|^2 + |\mathcal{A}|^2 - |\mathcal{L}|^2 \\ &\geqslant (N-1)|\mathbf{T}_z|^2 + |\mathcal{A}|^2 - \Big(\frac{|\alpha|}{2+2\alpha-|\alpha|}\Big)^2 (|\mathbf{T}_z| + |\mathcal{A}|)^2 \\ &\geqslant \frac{\tau^2}{4(1-\tau)}|\mathbf{T}_z|^2 + |\mathcal{A}|^2 - \frac{\tau^2}{(2-\tau)^2}(|\mathbf{T}_z| + |\mathcal{A}|)^2 \\ &= \frac{1}{4(1-\tau)(2-\tau)^2}\big[\tau^2|\mathbf{T}_z| - 4(1-\tau)|\mathcal{A}|\big]^2 \geqslant 0\,. \end{aligned}$$

This is because $N - 1 = \frac{\tau^2}{4(1-\tau)}$ and, in view of (4.3), we have

$$\frac{|\alpha|}{2+2\alpha-|\alpha|} = \frac{\left|\frac{\alpha}{1+\alpha}\right|}{2 - \frac{|\alpha|}{1+\alpha}} \leqslant \frac{\tau}{2-\tau}$$

This yields the estimate (1.6).

10. Geometric properties of the p-harmonic mappings

Before proceeding to the proof of Theorem 1.6, let us collect some geometric properties of the isotropic p-harmonic mappings. We shall appeal to familiar technique of elliptic PDEs. First note, since $h : \overline{\mathbb{X}} \to \mathbb{R}^2$ is continuous and its boundary map $f = h : \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ is a homeomorphism, we have the inclusion

$$h(\mathbb{X}) \supset \mathbb{Y}\,. \tag{10.1}$$

On the other hand, since $\partial\mathbb{Y}$ is convex, with the aid of a *weak maximum principle* we obtain another inclusion:

Lemma 10.1.

$$h(\mathbb{X}) \subset \overline{\mathbb{Y}}\,. \tag{10.2}$$

Proof. Suppose that, on the contrary, there is $z_\circ \in \mathbb{X}$ whose image $w_\circ = h(z_\circ)$ lies outside $\overline{\mathbb{Y}}$. Since $\overline{\mathbb{Y}}$ is convex there is a straight line that separates $\overline{\mathbb{Y}}$ from $w_\circ$. Choose an orthogonal coordinate system in the target space $\mathbb{C} = \{u + iv,\ u, v \in \mathbb{R}\}$

so that this line becomes the horizontal axis $\{u+iv\,, v=0\}$. We assume that $\partial\mathbb{Y}$ lies in the upper half plane and $w_\circ$ lies in the lower half plane. Thus $v(z)>0$ for every $z\in\partial\mathbb{X}$ and $v(z_\circ)<0$. This leads to a contradiction by the following arguments. Consider a test mapping $\eta=(0,v^-)\in \mathscr{W}^{1,p}_\circ(\mathbb{X},\mathbb{R}^2)$, where $v^-(z)=\min\{v(z)\,,\,0\}$. The integral form of the p-harmonic system with η as a test function reads as:

$$0=\int |Dh|^{p-2}\,\langle Dh\,|\,D\eta\rangle=\int |Dh|^{p-2}\,|\nabla v^-|^2\,.$$

This implies that $\nabla v^-\equiv 0$ in $\mathbb{X}$, because $|Dh|\geqslant|\nabla v^-|$. Since $v^-\in\mathscr{W}^{1,p}_\circ(\mathbb{X},\mathbb{R})$, it follows that $v^-\equiv 0$ in $\mathbb{X}$. In particular, $v(z_\circ)\geqslant 0$, which is a contradiction. □

We shall need the following technical term:

Definition 10.2. A one-sided neighborhood of $\partial\mathbb{X}$ ($\partial\mathbb{Y}$, respectively) is any topological annulus whose outer boundary coincides with $\partial\mathbb{X}$ ($\partial\mathbb{Y}$, respectively). Since $\mathbb{X}$ and $\mathbb{Y}$ are simply connected such neighborhoods lay in $\mathbb{X}$ ($\mathbb{Y}$, respectively).

We now choose and fix a one-sided neighborhood of $\partial\mathbb{X}$, say

$$\mathscr{U}\overset{\text{def}}{=}\{z\in\mathbb{X}\colon \operatorname{dist}\{z,\partial\mathbb{X}\}<\rho\}\quad\text{with } \rho>0 \text{ small enough to satisfy}$$

$$0<m\overset{\text{def}}{=}\inf_{z\in\mathscr{U}}|Dh(z)|\leqslant\sup_{z\in\mathscr{U}}|Dh(z)|\overset{\text{def}}{=}M<\infty\,. \tag{10.3}$$

For the inequality $0<m$ one must appeal to the tangential and normal derivatives at $\partial\mathbb{X}$. Indeed, we see that $|Dh|^2=|h_{\mathbb{N}}|^2+|h_T|^2\geqslant|f_T|^2>0$, because the boundary data $f\colon\partial\mathbb{X}\to\mathbb{R}^2$ is an immersion. Since h is $\mathscr{C}^{1,\alpha}$-smooth in $\overline{\mathbb{X}}$, the inequalities (10.3) follow.

Lemma 10.3. *Under the map h no point in $\mathscr{U}$ goes into $\partial\mathbb{Y}$. In other words,*

$$h(\mathscr{U})\subset\mathbb{Y}\,. \tag{10.4}$$

Proof. Suppose otherwise, some point $z_\circ\in\mathscr{U}$ goes to $w_\circ=h(z_\circ)\in\partial\mathbb{Y}$. As noted before, we may assume that $\overline{\mathbb{Y}}$ lies in the closed upper half plane $\{u+iv\,;\,v\geqslant0\}$ and $w_\circ=0+i\,0\in\partial\mathbb{Y}$. Since $h(\mathscr{U})\subset h(\mathbb{X})\subset\overline{\mathbb{Y}}$, it follows that

$$v(z)\geqslant0\,,\quad\text{for every } z\in\overline{\mathscr{U}}\,,\quad\text{whereas } v(z_\circ)=0\,. \tag{10.5}$$

In other words, the function $v:\overline{\mathscr{U}}\to\mathbb{R}$ reaches its minimum in $\mathscr{U}$. This time we arrive at a contradiction by viewing the p-harmonic system for $h=u+i\,v$ as a pair of linear uniformly elliptic scalar equations:

$$\begin{cases}\operatorname{div}\lambda(z)\nabla u=0,\\ \operatorname{div}\lambda(z)\nabla v=0,\end{cases}\qquad\text{where } \lambda(z)=|Dh(z)|^{p-2}\,,\quad z\in\mathscr{U}\,. \tag{10.6}$$

Recall from (10.3) that $\lambda(z)$ lies between m^{p-2} and M^{p-2}. By virtue of strong maximum principle ([21], § 6.5), any solution that reaches its infimum inside the domain must be constant. Thus $v\equiv0$ in $\overline{\mathscr{U}}$. But this would mean that $h(\partial\mathbb{X})=\partial\mathbb{Y}$ is a straight horizontal segment, a clear contradiction. □

Lemma 10.4. *The Jacobian determinant* $\mathbf{J} = \det Dh$ *does not vanish on* $\partial\mathbb{X}$ *.*

Proof. Regarding orientation of h , we may assume that the boundaries $\partial\mathbb{X}$ and $\partial\mathbb{Y}$ are positively oriented (counterclockwise); when traveling in such direction the domains are in the left side. We may as well assume that the boundary homeomorphism $f : \partial\mathbb{X} \xrightarrow{\text{onto}} \partial\mathbb{Y}$ is orientation preserving.

Now suppose, to the contrary, that the Jacobian determinant $\mathbf{J} = u_x v_y - u_y v_x = u_{_T} v_{_N} - u_{_N} v_{_T}$ vanishes at some point $z_\circ \in \partial\mathbb{X}$. Hereafter, the subscript T refers to the tangential differentiation along $\partial\mathbb{X}$ (in positive direction) and N to inward drawn normal derivative. We can certainly assume (upon suitable rotation and translation) that the domain $\mathbb{X}$ lies in the upper half plane $\{z = x + i\,y\,;\; y > 0\}$ and $z_\circ = 0 + i\,0$ is the lowest point in $\partial\mathbb{X}$, whereas the target $\mathbb{Y}$ lies in the upper half plane $\{w = u + i\,v\,;\; v > 0\}$ and $w_\circ = 0 + i\,0$ is the lowest point in $\partial\mathbb{Y}$. Since v assumes its minimum value along $\partial\mathbb{X}$ at $z_\circ$, we have $v_{_T} = 0$ at $z_\circ$. However, by assumption, $h_{_T} = u_{_T} + iv_{_T} \neq 0$, everywhere in $\partial\mathbb{X}$. In particular, $u_{_T} \neq 0$ at $z_\circ$. Actually, $u_{_T}(z_\circ) > 0$, because of the orientation. Therefore $\mathbf{J} = |h_T|\, v_{_N}$ at $z_\circ$. It remains to observe that $v_{_N} > 0$. For this we refer to the seminal paper by R. Finn and D. Gilbarg [17] , Lemma 7 . Recall our assumption $h \in \mathscr{C}^{1,\alpha}(\overline{\mathbb{X}}, \mathbb{R}^2)$. In view of (10.3) we then see that the coefficient λ in (10.6) belongs to $\mathscr{C}^\alpha(\overline{\mathscr{U}})$. We then have a uniformly elliptic single equation for v:

$$\operatorname{div} \lambda(z) \nabla v = 0 \quad \text{in } \mathscr{U}\,, \quad \text{where } v \in \mathscr{C}^{1,\alpha}(\overline{\mathscr{U}}) \text{ and } \lambda \in \mathscr{C}^{\alpha}(\overline{\mathscr{U}})\,.$$

Let us express this equation in the form of a Beltrami type elliptic system,

$$\begin{cases} \lambda\, v_x = \varphi_y, \\ \lambda\, v_y = -\varphi_x, \end{cases} \qquad \text{for some } \varphi \in \mathscr{C}^{1,\alpha}(\overline{\mathscr{U}})\,. \tag{10.7}$$

Recall that $v > 0$ in $\mathscr{U}$, by Lemma 10.3, and that $v(z_\circ) = 0$. We now appeal to Lemma 7 in [17]. Accordingly, $v_N > 0$ and hence $J(z_\circ) > 0$. Since $\mathbf{J}$ is continuous along $\partial\mathbb{X}$, we conclude with

$$\inf_{\partial\mathbb{X}} \mathbf{J}(z) > 0\,. \qquad \square$$

We may, and do, further assume that for some positive constant d

$$\det Dh(z) \geqslant d\,, \quad \text{for all } z \in \mathscr{U}\,; \tag{10.8}$$

for if not, we replace $\mathscr{U}$ by a slightly thinner one-sided neighborhood of $\partial\mathbb{X}$. One more thinning operation on $\mathscr{U}$ will be in order. Before proceeding we observe that the map $h : \overline{\mathbb{X}} \to \mathbb{R}^2$ admits a $\mathscr{C}^1$ - extension to a neighborhood of $\overline{\mathbb{X}}$. Thus $\det D\widetilde{h} > 0$ in a neighborhood of $\partial\mathbb{X}$. By the implicit function theorem, the extended map $\widetilde{h}$ is a local $\mathscr{C}^1$-diffeomorphism, say in a domain $\widetilde{\mathscr{U}} \supset \partial\mathbb{X}$. The following fact is an exercise in the first course of topology:

Lemma 10.5. *Every local homeomorphism* $\widetilde{h} \colon \widetilde{\mathscr{U}} \xrightarrow{\text{into}} \mathbb{R}^2$ *in a domain* $\widetilde{\mathscr{U}} \subset \mathbb{R}^2$*, that is injective on a compact subset, say* $\partial\mathbb{X} \Subset \widetilde{\mathscr{U}}$ *, is injective* (*thus a homeomorphism*) *in a small neighborhood of* $\partial\mathbb{X}$ *.*

Corollary 10.6. *In a sufficiently thin one-sided neighborhood of $\partial\mathbb{X}$, still denoted by $\mathscr{U} \subset \mathbb{X}$, the map $h\colon \mathscr{U} \to \mathbb{R}^2$ is injective. Its image $\mathscr{V} \overset{\text{def}}{=\!=} h(\mathscr{U}) \subset \mathbb{Y}$ is a one-sided neighborhood of $\partial\mathbb{Y}$.*

No further thinning of $\mathscr{U}$ or $\mathscr{V}$ will be performed. From (10.3) it follows that

$$|Dh(z)|^{p-2} \det Dh(z) \geqslant c > 0 \quad \text{for all } z \in \overline{\mathscr{U}}\,, \tag{10.9}$$

where $c = \min\{m^{p-2}d\,,\, M^{p-2}d\}$. We aim to prove the following.

Proposition 10.7. *With the same constant c as in (10.9), we have*

$$|Dh(z)|^{p-2} \det Dh(z) \geqslant c > 0 \quad \text{for all } z \in \overline{\mathbb{X}}\,. \tag{10.10}$$

This can be done, as one may have expected, with the aid of the minimum principle at (1.5). Let us postpone the proof until we reminisce about p-harmonic mappings.

11. The coupled p-harmonic systems

The arguments follow very closely the paper of C. Hamburger [20], except for more explicit (very much needed) statements.

We shall work with a one-parameter family of p-harmonic type systems, non-degenerate when $\varepsilon \neq 0$.

$$\begin{cases} \operatorname{div}\big[\, \big(\varepsilon^2 + |Dh^\varepsilon|^2\big)^{(p-2)/2}\, Dh^\varepsilon \big] \;=\; 0\,, \quad -\infty < \varepsilon < \infty\,, \\ h^\varepsilon \in f + \mathscr{W}_\circ^{1,p}(\mathbb{X}, \mathbb{R}^2)\,. \end{cases} \tag{11.1}$$

Lemma 11.1. *The system (11.1) has unique solution. The $\mathscr{L}^p$-norm of its gradient matrix Dh^ε is uniformly controlled by that of Df. Explicitly, we have the inequality*

$$\int_{\mathbb{X}} \big(\varepsilon^2 + |Dh^\varepsilon|^2\big)^{(p-2)/2} |Dh^\varepsilon|^2 \;\leqslant\; 2^{p+1} \int_{\mathbb{X}} \big(\varepsilon^2 + |Df|^2\big)^{(p-2)/2} |Df|^2\,. \tag{11.2}$$

We also include to this set of estimates the following one for the map $h = h^0$:

$$\int_{\mathbb{X}} \big(\varepsilon^2 + |Dh^\varepsilon|^2 + |Dh|^2\big)^{p/2} \leqslant 4^p \int_{\mathbb{X}} \big(1 + |Df|^2\big)^{p/2}\,, \quad \text{when } 0 \leqslant \varepsilon \leqslant 1. \tag{11.3}$$

Proof. We begin with the weak form of the system:

$$\int_{\mathbb{X}} \big(\varepsilon^2 + |Dh^\varepsilon|^2\big)^{(p-2)/2} \langle Dh^\varepsilon \mid D\eta\rangle = 0\,, \;\; \text{with } \eta = h^\varepsilon - f \in \mathscr{W}_\circ^{1,p}(\mathbb{X}, \mathbb{R}^2)\,. \tag{11.4}$$

This identity gives

$$\int_{\mathbb{X}} \big(\varepsilon^2 + |Dh^\varepsilon|^2\big)^{(p-2)/2} |Dh^\varepsilon|^2 \;\leqslant\; \int_{\mathbb{X}} \big(\varepsilon^2 + |Dh^\varepsilon|^2\big)^{(p-2)/2} |Dh^\varepsilon|\,|Df|\,.$$

The rest of the derivation relies on an elementary inequality:

$$(\varepsilon^2 + x^2)^{(p-2)/2}\, xy \leqslant \frac{1}{2}\,(\varepsilon^2 + x^2)^{(p-2)/2}\, x^2 \;+\; 2^p\,(\varepsilon^2 + y^2)^{(p-2)/2}\, y^2\,. \tag{11.5}$$

For verification, consider two cases. The obvious one is $|y| \leqslant \frac{1}{2}|x|$; the other, $|x| \leqslant 2\,|y|$, is also easy. □

Lemma 11.2. *As $\varepsilon \to 0$, the gradients Dh^ε converge to Dh in $\mathscr{L}^p(\mathbb{X}, \mathbb{R}^{2\times 2})$.*

Proof. The proof begins with the weak form of the system

$$\int_{\mathbb{X}} \left\langle\, (\varepsilon^2 + |Dh^\varepsilon|^2)^{(p-2)/2} Dh^\varepsilon \;-\; |Dh|^{p-2} Dh \;\middle|\; Dh^\varepsilon \;-\; Dh \,\right\rangle \;= 0\,. \tag{11.6}$$

We are going to use a uniform (independent of ε) bound for the monotone operator $X \to (\varepsilon^2 + |X|^2)^{(p-2)/2}\, X$, which holds in any inner product space,

$$\begin{aligned} &\big\langle (\varepsilon^2 + |X|^2)^{(p-2)/2}\, X - \,(\varepsilon^2 + |Y|^2)^{(p-2)/2} Y \;\big|\; X - Y \big\rangle \\ &\qquad \succcurlyeq\; \big(\varepsilon^2 + |X|^2 + |Y|^2\big)^{(p-2)/2}\, |X - Y|^2\,. \end{aligned} \tag{11.7}$$

The symbol $\succcurlyeq$ (and $\preccurlyeq$) indicates that in the inequalities there is a missing constant, called *implied constant*, of no importance. It depends on p, but not on ε. Thus, in particular, (11.7 holds for matrices $X, Y \in \mathbb{R}^{2\times 2}$. The reader will have no trouble verifying (11.7) when $\varepsilon = 1$, with implied constant obviously depending only on $p \in (1, \infty)$. The general case for all ε follows by re-scaling.

Now we exploit Hölder's inequality. Using (11.6) and (11.7), we proceed as follows:

$$\begin{aligned} \int_{\mathbb{X}} |Dh^\varepsilon \,-\, Dh|^p &\leqslant \int_{\mathbb{X}} \big(\, |Dh^\varepsilon| + |Dh|\,\big)^{p-1} |Dh^\varepsilon \,-\, Dh| \\ &\preccurlyeq \int_{\mathbb{X}} \big(\, \varepsilon^2 + |Dh^\varepsilon|^2 + |Dh|^2\,\big)^{(p-1)/2} |Dh^\varepsilon \,-\, Dh| \\ &= \int_{\mathbb{X}} \big(\, \varepsilon^2 + |Dh^\varepsilon|^2 + |Dh|^2\,\big)^{p/4} \big(\, \varepsilon^2 + |Dh^\varepsilon|^2 + |Dh|^2\,\big)^{(p-2)/4} |Dh^\varepsilon \,-\, Dh| \\ &\leqslant \left[\int_{\mathbb{X}} \Big(\, \varepsilon^2 + |Dh^\varepsilon|^2 + |Dh|^2 \Big)^{p/2} \right]^{1/2} \\ &\qquad\qquad \times \left[\int_{\mathbb{X}} \big(\, \varepsilon^2 + |Dh^\varepsilon|^2 + |Dh|^2\,\big)^{(p-2)/2}\, |Dh^\varepsilon \,-\, Dh|^2 \right]^{1/2} \\ &\preccurlyeq \left[\int_{\mathbb{X}} \big(\, 1{+}|Df|^2 \big)^{p/2} \right]^{1/2} \left[\int_{\mathbb{X}} \big(\, \varepsilon^2 + |Dh^\varepsilon|^2 + |Dh|^2\,\big)^{(p-2)/2} |Dh^\varepsilon {-} Dh|^2 \right]^{1/2}. \end{aligned} \tag{11.8}$$

by (11.3). The computation will henceforth be valid only for $0 \leqslant \varepsilon \leqslant 1$. The last integral can be estimated using (11.7) and the identity (11.6):

$$\begin{aligned}
&\int_{\mathbb{X}} \left(\varepsilon^2 + |Dh^\varepsilon|^2 + |Dh|^2\right)^{(p-2)/2} |Dh^\varepsilon - Dh|^2 \\
&\preccurlyeq \int_{\mathbb{X}} \left\langle \left(\varepsilon^2 + |Dh^\varepsilon|^2\right)^{(p-2)/2} Dh^\varepsilon - \left(\varepsilon^2 + |Dh|^2\right)^{(p-2)/2} Dh \;\middle|\; Dh^\varepsilon - Dh \right\rangle \\
&= \int_{\mathbb{X}} \left\langle |Dh|^{p-2} Dh - \left(\varepsilon^2 + |Dh|^2\right)^{(p-2)/2} Dh \;\middle|\; Dh^\varepsilon - Dh \right\rangle \\
&= \left\{ \int_{\mathbb{X}} \left| |Dh|^{p-2} Dh - \left(\varepsilon^2 + |Dh|^2\right)^{(p-2)/2} Dh \right|^{p/(p-1)} \right\}^{(p-1)/p} \left\{ \int_{\mathbb{X}} |Dh^\varepsilon - Dh|^p \right\}^{1/p}.
\end{aligned}$$

Inserting this estimate into (11.8) the norm $\|Dh^\varepsilon - Dh\|_p$ will be absorbed by the left hand side. We arrive at the desired estimate:

$$\begin{aligned}
\left\{ \int_{\mathbb{X}} |Dh^\varepsilon - Dh|^p \right\}^{(2p-1)/p} &\preccurlyeq \left\{ \int_{\mathbb{X}} \left(1 + |Df|^2\right)^{p/2} \right\} \\
&\quad \cdot \left\{ \int_{\mathbb{X}} \left| |Dh|^{p-2} Dh - \left(\varepsilon^2 + |Dh|^2\right)^{(p-2)/2} Dh \right|^{p/(p-1)} \right\}^{(p-1)/p}.
\end{aligned}$$

By Lebesgue's convergence theorem, $\|Dh^\varepsilon - Dh\|_p \to 0$, as $\varepsilon \to 0$, completing the proof of the lemma. □

11.1. Local $\mathscr{C}^{1,\beta}$-estimates

It will be necessary to control the dependence of Dh^ε on the parameter ε. For this, we ought to settle for local estimates (no satisfactory boundary estimates are accessible in the literature).

Lemma 11.3. *Let $g \in \mathscr{W}^{1,p}(\mathbb{X}, \mathbb{R}^2)$ satisfy (in the weak Sobolev sense) the non-degenerate p-harmonic system:*

$$\operatorname{div} \left(1 + |Dg|^2\right)^{(p-2)/2} Dg = 0, \quad 1 < p < \infty. \tag{11.9}$$

Define

$$V \overset{\text{def}}{=} \left(1 + |Dg|^2\right)^{(p-2)/4} Dg. \tag{11.10}$$

Then $V \in \mathscr{W}^{1,2}_{\mathrm{loc}}(\mathbb{X}, \mathbb{R}^{2\times 2})$. It satisfies the following "reverse Poincáre" inequalities:

$$\int_B |DV|^2 \leqslant C_p \fint_{2B} \left| V - V_{2B} \right|^2 \leqslant C_p \fint_{2B} \left| V \right|^2 \tag{11.11}$$

for every pair of concentric balls $B \subset 2B \subset \mathbb{X}$. The constant C_p depends only on the exponent p.

We have used the notation $\fint_{2B}$ (or equivalently, $(\ \ )_{2B}$) for integral averages. The above explicit estimate, under the name Caccioppoli inequality, can be found in the paper by C. Hamburger [20], page 28. Then, by the Poincáre–Sobolev inequality, we arrive at the "reverse Hölder inequalities"

$$\fint_B |DV|^2 \preccurlyeq C_p \left\{ \fint_{2B} |DV| \right\}^2 . \tag{11.12}$$

Next we invoke the celebrated Lemma of F. W. Gehring [18]. Accordingly, there exist an exponent $s > 2$ and a constant A_p (both depend only on the constant C_p in the reverse Hölder inequalities (11.12), thus only on p) such that

$$\fint_B |DV|^s \leqslant A_p \left\{ \fint_{2B} |DV|^2 \right\}^{s/2} \preccurlyeq \left\{ \fint_{4B} |V|^2 \right\}^{s/2} \tag{11.13}$$

provided $4B \subset \mathbb{X}$. These local estimates may be added up to yield the following.

Corollary 11.4. *To every compact subset $\mathbb{K} \Subset \mathbb{X}$ there corresponds a constant $C_p(\mathbb{K}, \mathbb{X})$ such that*

$$\left\{ \int_{\mathbb{K}} |DV|^s \right\}^{1/s} \leqslant C_p(\mathbb{K}, \mathbb{X}) \left\{ \int_{\mathbb{X}} |V|^2 \right\}^{1/2}, \quad \textit{where } s = s(p) > 2. \tag{11.14}$$

Returning to the system $\operatorname{div}(\varepsilon^2 + |Dh^\varepsilon|^2)^{(p-2)/2} Dh^\varepsilon = 0$, we set $g = \varepsilon^{-1} h^\varepsilon$ to obtain (11.9). Inequality (11.14) reads as:

$$\left\{ \int_{\mathbb{K}} |DV_\varepsilon|^s \right\}^{1/s} \leqslant C_p(\mathbb{K}, \mathbb{X}) \left\{ \int_{\mathbb{X}} |V_\varepsilon|^2 \right\}^{1/2}, \quad V_\varepsilon = \left(\varepsilon^2 + |Dh^\varepsilon|^2\right)^{\frac{p-2}{4}} Dh^\varepsilon, \tag{11.15}$$

where we recall from Lemma 11.1 the following uniform (independent of ε) bound of the right-hand side: when $0 \leqslant \varepsilon \leqslant 1$,

$$\int_{\mathbb{X}} |V_\varepsilon|^2 = \int_{\mathbb{X}} \left(\varepsilon^2 + |Dh^\varepsilon|^2\right)^{(p-2)/2} |Dh^\varepsilon|^2 \leqslant 4^p \int_{\mathbb{X}} \left(1 + |Df|^2\right)^{p/2} .$$

Implications are immediate. The family $\{V_\varepsilon\}_{0 \leqslant \varepsilon \leqslant 1}$, restricted to any compactly contained subdomain $Q \Subset \mathbb{X}$, is bounded with respect to the Sobolev norm $\|V_\varepsilon\| \overset{\text{def}}{=} \|V_\varepsilon\|_{\mathscr{L}^2(Q)} + \|DV_\varepsilon\|_{\mathscr{L}^s(Q)}$. By the Sobolev imbedding theorem it is also bounded in the space $\mathscr{C}^\beta(Q)$ of Hölder continuous functions with $\beta = 1 - 2/s$. In particular, the family $\{V_\varepsilon\}_{0 \leqslant \varepsilon \leqslant 1}$ is locally equicontinuous. Recall that, by Lemma 11.2, $\lim_{\varepsilon \to 0} V_\varepsilon \to |Dh|^{(p-2)/2} Dh$, almost everywhere. The Arzelá–Ascoli theorem ensures that this convergence is also c-uniform. Hence:

Corollary 11.5. *As ε approaches 0, the gradient matrices Dh^ε converge c-uniformly to Dh. In particular,*

$$\det V_\varepsilon = \left(\varepsilon^2 + |Dh^\varepsilon|^2\right)^{(p-2)/2} \det Dh^\varepsilon \to |Dh|^{p-2} \det Dh ,$$

uniformly on compact subsets.

12. Proof of Proposition 10.7

We fix a point $z_\circ \in \mathbb{X}$ to show that

$$|Dh(z_\circ)|^{p-2} \det Dh(z_\circ) \geqslant c > 0 \, . \tag{12.1}$$

This inequality already appeared in (10.9) for $z_\circ \in \mathscr{U}$. Thus assume that $z_\circ \notin \mathscr{U}$. Recall that h takes $\mathscr{U}$ diffeomorphically onto a one-sided neighborhood $\mathscr{V}$ of $\partial\mathbb{Y}$. Recall that $\mathscr{V}$ is a topological annulus whose outer boundary agrees with $\partial\mathbb{Y}$. We choose and fix a smooth *convex* Jordan curve $\Upsilon \subset \mathscr{V}$ that separates the boundary components of $\mathscr{V}$. Define $\mathfrak{X} = h^{-1}(\Upsilon) \subset \mathscr{U}$. This is a smooth closed Jordan curve in a topological annulus $\mathscr{U}$ which separates its boundary components. Thus $\mathfrak{X} = \partial\Omega$, for some simply connected domain $\Omega \Subset \mathbb{X}$ containing $z_\circ$. Fix a neighborhood of $\mathfrak{X}$ that is compactly contained in $\mathscr{U}$; say,

$$\mathfrak{X} \Subset \mathscr{O} \Subset \mathscr{U} \, .$$

We then look at the mappings $h^\varepsilon : \mathscr{O} \xrightarrow{\text{into}} \mathbb{R}^2$ with ε approaching zero. They converge on $\mathscr{O}$ to h, uniformly together with the first derivatives. Recall from (10.8) that on $\mathscr{U}$ we already have $\det Dh \geqslant d > 0$. Thus, for sufficiently small ε, say $0 \leqslant \varepsilon \leqslant \kappa'$, all the mappings $h^\varepsilon : \mathscr{O} \xrightarrow{\text{into}} \mathbb{R}^2$ have positive Jacobian,

$$\det Dh^\varepsilon \; > 0 \, , \quad \text{in } \mathscr{O} \, , \quad \text{for } 0 \leqslant \epsilon \leqslant \kappa' \, . \tag{12.2}$$

Thus $h^\varepsilon : \mathscr{O} \xrightarrow{\text{into}} \mathbb{R}^2$ are local diffeomorphisms, and we know that the limit map $h : \mathscr{O} \xrightarrow{\text{into}} \mathscr{V}$ is a global homeomorphism. Now the following topological argument comes into play.

12.1. A topological analogue of Hurwitz's theorem

Nonconstant holomorphic functions in a planar domain $\mathscr{O}$ are discrete and open. If a sequence of such functions converges to a conformal homeomorphism, then all but finite number of them are also conformal when restricted to a compact subdomain $\mathscr{O}' \Subset \mathscr{O}$. We ascribe this fact to Adolf Hurwitz. A topological variant of Hurwitz's theorem reads as:

Theorem 12.1. *Let $\{h^\varepsilon\}_{0 \leqslant \varepsilon < \sigma}$ be a continuous family of sense-preserving discrete open mappings $h^\varepsilon : \mathscr{O} \to \mathbb{R}^n$ in a domain $\mathscr{O} \subset \mathbb{R}^n$, with $h^0 \overset{\text{def}}{=} h : \mathscr{O} \xrightarrow{\text{into}} \mathbb{R}^n$ being a homeomorphism. Fix a continuum $\mathfrak{X} \subset \mathscr{O}$ and its neighborhood $\mathscr{O}' \Subset \mathscr{O}$. Then for sufficiently small $0 < \sigma' \leqslant \sigma$ the following holds:*

- *the mappings $h^\varepsilon : \mathscr{O}' \to \mathbb{R}^n$ are homeomorphisms whenever $0 \leqslant \varepsilon \leqslant \sigma'$,*
- *$h^\varepsilon(\mathscr{O}') \supset h(\mathfrak{X})$, whenever $0 \leqslant \varepsilon \leqslant \sigma'$.*

Proof. We appeal to topological degree. One can certainly assume that h is a homeomorphism on the closure of $\mathscr{O}$; for if not, replace $\mathscr{O}$ by a slightly smaller neighborhood of $\mathfrak{X}$.

The sets $h(\overline{\mathscr{O}'})$ and $h(\partial\mathscr{O})$ are disjoint and $h(\mathscr{O})$ is a domain. Thus there exists a domain $\mathbb{G}$ such that $h(\overline{\mathscr{O}'}) \subset \mathbb{G} \Subset h(\mathscr{O})$. In particular, $\mathbb{G} \cap h(\partial\mathscr{O}) = \emptyset$. Since the mappings h^ε are uniformly close to h, we see that:

$$h^\varepsilon(\overline{\mathscr{O}'}) \subset \mathbb{G} \text{ and } \mathbb{G} \cap h^\varepsilon(\partial\mathscr{O}) = \emptyset, \quad \text{whenever } 0 \leqslant \varepsilon \leqslant \sigma' \quad - \text{sufficiently small.}$$

This is precisely the condition on σ' that guarantees the first statement in Theorem 12.1. Indeed, since $\mathbb{G}$ is connected and disjoint with $h^\varepsilon(\partial\mathscr{O})$ it lies in one and only one component of $\mathbb{R}^n \setminus h^\varepsilon(\partial\mathscr{O})$. It is therefore legitimate to speak of the topological degree of h^ε at the points of $\mathbb{G}$. The degree is independent of the choice of a point in $\mathbb{G}$, so we denote it by

$$\deg_{\mathscr{O}}[\mathbb{G}\,;\, h^\varepsilon]\,.$$

Moreover, the function $\varepsilon \to \deg_{\mathscr{O}}[\mathbb{G}\,;\, h^\varepsilon]$ is integer valued and continuous, thus constant. This constant equals to 1, because $\deg_{\mathscr{O}}[\mathbb{G}\,;\, h^\varepsilon] = \deg_{\mathscr{O}}[\mathbb{G}\,;\, h] = 1$. The latter equation holds because $\mathbb{G} \Subset h(\mathscr{O})$ and h is a sense preserving homeomorphism. At this point the assumption of discreteness and openness of h^ε becomes essential (such are local homeomorphisms). For such mappings the cardinality of the preimage of any admissible point does not exceed its degree, see Proposition 4.10 in [37]. Precisely, we have

$$0 \leqslant \operatorname{Card}\{z \in \mathscr{O} : h^\varepsilon(z) = y\} \leqslant \deg_{\mathscr{O}}[y\,;\mathscr{O}]\,, \quad \text{whenever } y \notin h^\varepsilon(\partial\mathscr{O})\,.$$

This applies to all points $y \in h^\varepsilon(\overline{\mathscr{O}'})$, where we obviously have $\deg_{\mathscr{O}}[y\,; h^\varepsilon] = 1$. A geometric meaning of this fact is that given any $z_\circ \in \overline{\mathscr{O}'}$, the equation

$$h^\varepsilon(z) = h^\varepsilon(z_\circ)\,, \quad \text{for } z \in \mathscr{O} \tag{12.3}$$

admits exactly one solution; $z = z_\circ$. In particular, h^ε is one-to-one in $\overline{\mathscr{O}'}$, thus a homeomorphism.

The second statement of Theorem 12.1 also follows by a degree argument. The set $\Upsilon = h(\mathfrak{X})$ is a continuum. Since $\mathfrak{X} \subset \mathscr{O}'$ and $h\colon \mathscr{O}' \to \mathbb{R}^n$ is a homeomorphism, it follows that $h(\partial\mathscr{O}')\cap\Upsilon = \emptyset$. Thus we can adjust $\sigma' > 0$ small enough so that $h^\varepsilon(\partial\mathscr{O}')\cap\Upsilon = \emptyset$, for all $0 \leqslant \varepsilon \leqslant \sigma'$. This allows us to speak of $\deg_{\mathscr{O}'}[\Upsilon;\, h^\varepsilon]$. It equals 1, again because in the limit case $\deg_{\mathscr{O}'}[\Upsilon;\, h] = 1$. Since Υ is a continuum, it lies in one and only one component of $\mathbb{R}^n \setminus h^\varepsilon(\partial\mathscr{O}')$. Therefore, for every $y \in \Upsilon$ we have $\deg_{\mathscr{O}'}[y;\, h^\varepsilon] = 1 \neq 0$, yielding $y \in h^\varepsilon(\mathscr{O}')$. □

Define $\mathfrak{X}_\varepsilon = (h^\varepsilon)^{-1}(\Upsilon) \Subset \mathscr{O}' \Subset \mathscr{O}$ and let Ω_ε denote the bounded component of $\mathbb{R}^2 \setminus \mathfrak{X}_\varepsilon$. This is a smooth simply connected domain. Let us look at $h^\varepsilon\colon \Omega_\varepsilon \to \mathbb{R}^2$ as a solution to a linear elliptic system in Ω_ε:

$$\begin{cases} \operatorname{div} \lambda_\varepsilon(z) Dh^\varepsilon \;=\; 0\,, & \text{where } \lambda_\varepsilon(z) = \left(\varepsilon^2 + |Dh^\varepsilon|^2\right)^{(p-2)/2}, \\ h^\varepsilon : \partial\Omega_\varepsilon \xrightarrow{\text{onto}} \Upsilon & - \text{smooth convex curve.} \end{cases} \tag{12.4}$$

For fixed $\varepsilon > 0$ this system is uniformly elliptic because $0 < \inf_{\Omega_\varepsilon} \lambda_\varepsilon(z) \leqslant \sup_{\Omega_\varepsilon} \lambda_\varepsilon(z) < \infty$. Moreover, λ_ϵ is Hölder continuous up to the boundary of Ω_ϵ.

Now we may appeal to an extension of Radó–Kneser–Choquet theorem to divergence type linear elliptic systems with Hölder continuous coefficients, a result of P. Bauman, A. Marini and V. Nesi [8]. Accordingly, by Theorem 3.1 in [8], the solution h^ϵ is univalent in Ω_ϵ and $\det Dh^\epsilon > 0$. No uniform lower bounds for $\det Dh^\epsilon$ will be needed. But we notice that $\det Dh^\epsilon > 0$ also in Ω, because $\Omega \setminus \Omega_\varepsilon \subset \mathscr{O}$. Let us record it as

$$\det Dh^\epsilon > 0 \ \text{ in } \Omega\,, \quad \text{for } 0 < \varepsilon \leqslant \sigma'.$$

We now proceed to the final stage of the proof of Theorem 1.6.

Step 1. Since $\det Dh^\epsilon > 0$ in Ω, for every $0 < \epsilon \leqslant \sigma'$, it is legitimate to apply the minimum principle in Theorem 1.3. We have $z_\circ \in \Omega$ and $\partial\Omega = \mathfrak{X}$, so

$$\left(\epsilon^2 + |Dh^\epsilon(z_\circ)|^2\right)^{(p-2)/2} \det Dh^\epsilon(z_\circ) \ \geqslant \ \inf_{z \in \mathfrak{X}} \left(\epsilon^2 + |Dh^\epsilon(z)|^2\right)^{(p-2)/2} \det Dh^\epsilon(z)\,.$$

Step 2. We pass to the limit as $\varepsilon \to 0$,

$$|Dh(z_\circ)|^{p-2} \det\, Dh(z_\circ) \ \geqslant \ \inf_{z\in\mathfrak{X}} |Dh(z)|^{p-2} \det\, Dh(z) \ \geqslant c > 0\,,$$

by (10.9). We then summarize,

$$|Dh(z)|^{p-2} \det\, Dh(z) \ \geqslant c > 0\,, \quad \text{for every } z \in \overline{\mathbb{X}}\,.$$

Step 3. Here $h\colon \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ is a local $\mathscr{C}^{1,\alpha}$-diffeomorphism on $\overline{\mathbb{X}}$, and its boundary map $h\colon \partial\mathbb{X} \xrightarrow{\text{onto}} \partial Y$ is a homeomorphism. By topology we conclude that $h\colon \overline{\mathbb{X}} \xrightarrow{\text{onto}} \overline{\mathbb{Y}}$ is a homeomorphism as well. This completes the proof of Theorem 1.6.

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Received March 15, 2014.

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T. Iwaniec and A. Koski were supported by the NSF grant DMS-1301558 and the Academy of Finland project 1128331. J. Onninen was supported by the NSF grant DMS-1301570.