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Critical stationary Kirchhoff equations in $\mathbb{R}^N$ involving nonlocal operators

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Abstract. In this paper we establish existence and multiplicity of nontrivial non-negative entire (weak) solutions of a stationary Kirchhoff eigenvalue problem, involving a general nonlocal integro-differential operator. The model under consideration depends on a real parameter λ and involves two superlinear nonlinearities, one of which could be critical or even supercritical.

1. Introduction

In recent years stationary Kirchhoff problems have been widely studied. We refer to [2], [19], [15], [18], [21] for problems involving the classical Laplace operator, to [16], [4] for the p-Laplacian case, and to [27] for Kirchhoff models with critical exponents. For evolution problems we refer to [5], [8], [3] and the references therein. More recently, following [14], Fiscella and Valdinoci [20] proposed a stationary Kirchhoff variational model, in bounded regular domains of $\mathbb{R}^N$, which takes into account the *nonlocal* aspect of the tension arising from nonlocal measurements of the fractional length of the string. In [2], [18], and [20], the authors use variational methods, as well as a concentration compactness arguments. In [15] and [21], variational methods are still used, but the stationary Kirchhoff problems are set in the whole of $\mathbb{R}^N$. In [16] and [4], the so-called *degenerate* case is covered (see also [5], [8], [27]), that is, the main Kirchhoff non-negative non-decreasing function M could be zero at 0, while in [20] only the *non-degenerate* case is covered. Lately, several papers have been devoted to problems involving critical non-linearities and nonlocal elliptic operators; see [9], [10], [23], [24], [25], [26], [27] in bounded regular domains of $\mathbb{R}^N$, and [7], [21] in all $\mathbb{R}^N$, and the references therein.

In this paper, inspired by the above articles and the fact that several interesting questions arise from the search of nontrivial non-negative (weak) solutions, we

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deal with existence and multiplicity of nontrivial non-negative entire solutions of a Kirchhoff eigenvalue problem, involving critical non-linearities and nonlocal elliptic operators. More precisely, we consider the problem

$$(\mathscr{P}_\lambda) \qquad \begin{aligned} M\left([u]_{s,K}^2\right)\mathscr{L}_K u &= \lambda w(x)|u|^{q-2}u - h(x)|u|^{r-2}u \qquad \text{in } \mathbb{R}^N,\\ [u]_{s,K}^2 &= \iint_{\mathbb{R}^{2N}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy, \end{aligned}$$

where $\lambda \in \mathbb{R}$, $0 < s < 1$, $2s < N$, and $\mathscr{L}_K$ is an integro-differential nonlocal operator, defined pointwise by

$$\mathscr{L}_K\varphi(x) = \frac{1}{2}\int_{\mathbb{R}^N}[2\varphi(x) - \varphi(x+y) - \varphi(x-y)]K(y)\,dy,$$

for any function $\varphi \in C_0^\infty(\mathbb{R}^N)$. The *weight* $K\colon \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^+$ *is a measurable function* satisfying the natural restrictions

(K_1) *there exists a number* $\beta > 0$ *such that* $K(x)|x|^{N+2s} \geq \beta$ *for all* $x \in \mathbb{R}^N \setminus \{0\}$;

(K_2) $mK \in L^1(\mathbb{R}^N)$, *where* $m(x) = \min\{1, |x|^2\}$.

Clearly, when $K(x) = |x|^{-(N+2s)}$, the operator $\mathscr{L}_K$ reduces to the more familiar fractional Laplace operator $(-\Delta)^s$, which, up to a multiplicative constant depending only on N and s, is defined by

$$(-\Delta)^s\varphi(x) = \frac{1}{2}\int_{\mathbb{R}^N}\frac{2\varphi(x) - \varphi(x+y) - \varphi(x-y)}{|y|^{N+2s}}\,dy,$$

for any rapidly decaying function φ of class $C^\infty(\mathbb{R}^N)$, see Lemma 3.5 of [17].

The nonlinear terms in $(\mathscr{P}_\lambda)$ are related to the main elliptic part by the request that

$$2 < q < \min\{r, 2^*\}, \tag{1.1}$$

where $2^* = 2N/(N-2s)$ is the critical Sobolev exponent for $H^s(\mathbb{R}^N)$. The *weight* w *satisfies*

$$w \in L^{\wp}(\mathbb{R}^N) \cap L^{\sigma}_{\mathrm{loc}}(\mathbb{R}^N), \quad \text{with } \wp = 2^*/(2^*-q), \quad \sigma > \wp, \tag{1.2}$$

while h *is a positive weight of class* $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Finally, h *and* w *are related by the condition*

$$\int_{\mathbb{R}^N}\left[\frac{w(x)^r}{h(x)^q}\right]^{1/(r-q)} dx = H \in \mathbb{R}^+. \tag{1.3}$$

The *Kirchhoff function* $M\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *satisfies* the following condition:

($\mathcal{M}$) M *is an increasing and continuous function, with* $M(t) > 0$ *for* $t \geq 0$, *and* $\mathscr{M}(t) = \int_0^t M(\tau)\,d\tau$.

Problem $(\mathscr{P}_\lambda)$ is said to be *degenerate* when $M(0) = 0$, and *non-degenerate* when $M(0) > 0$. In this paper we cover *only* the non-degenerate case, as in [20]. From now on we put $M(0) = m_0$ and recall that $m_0 > 0$ by $(\mathcal{M})$.

A typical prototype of Kirchhoff function is given by

$$M(t) = m_0 + 2bt, \quad \textit{with } m_0 = M(0) \geq 0, \quad b \geq 0 \quad \textit{and} \quad m_0 + b > 0. \tag{1.4}$$

When M is of the type (1.4), problem $(\mathscr{P}_\lambda)$ is *non-degenerate* when $m_0 > 0$ and $b \geq 0$, while it is *degenerate* if $m_0 = 0$ and $b > 0$. We refer to [16], [3], and [4] for further details and references.

In $(\mathscr{P}_\lambda)$ the Kirchhoff function M, which represents the elastic tension term, depends on the Gagliardo fractional norm $[\cdot]_{s,K}$ arising from general kernel K and generating nonlocal operators $\mathscr{L}_K$. It is clear from symmetry properties that if u is a solution of $(\mathscr{P}_\lambda)$ also $-u$ is a solution of $(\mathscr{P}_\lambda)$. The main result of the paper is:

Theorem 1.1. *Under the above assumptions, there exists $\overline{\lambda} > 0$ such that $(\mathscr{P}_\lambda)$ admits at least two nontrivial non-negative entire solutions for all $\lambda > \overline{\lambda}$, one of which is a global minimizer of the underlying functional J_λ of $(\mathscr{P}_\lambda)$, and the second independent solution u_λ is a mountain pass critical point of J_λ. In particular, $\|u_\lambda\| \to 0$ as $\lambda \to \infty$, where $\|\cdot\|$ is the natural solution space norm of $(\mathscr{P}_\lambda)$. Moreover, there exist λ^* and λ^{**}, with $0 < \lambda^* \leq \lambda^{**} \leq \overline{\lambda}$, such that*

(i) $(\mathscr{P}_\lambda)$ *possesses only the trivial solution if $\lambda < \lambda^*$;*

(ii) $(\mathscr{P}_\lambda)$ *admits a nontrivial* non-negative *entire solution if and only if $\lambda \geq \lambda^{**}$.*

The paper is organized as follows. In Section 2 we define the main solution space X and give some preliminary results, from which we derive (i) of Theorem 1.1. In Section 3 we prove the existence of $\overline{\lambda} > 0$ such that, for all $\lambda > \overline{\lambda}$, problem $(\mathscr{P}_\lambda)$ admits a first nontrivial non-negative entire solution and then, thanks to a modified version of the mountain pass theorem established in [6], we construct a second independent nontrivial non-negative entire solution u_λ of $(\mathscr{P}_\lambda)$. We end Section 3 by proving the asymptotic property for u_λ stated in Theorem 1.1. Finally in Section 4 we prove part (ii) of Theorem 1.1.

2. Solution spaces and preliminaries

Throughout the paper $D^s(\mathbb{R}^N)$ denotes the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the Gagliardo norm

$$[u]_s = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right)^{1/2}.$$

The embedding $D^s(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is continuous, that is,

$$\|u\|_{2^*} \leq C_{2^*} [u]_s \quad \text{for all } u \in D^s(\mathbb{R}^N), \tag{2.1}$$

where $C_{2^*}^2 = c(N) \dfrac{s(1-s)}{N-2s}$ by Theorem 1 of [22], see also Theorem 1 of [12].

By (K_2), for all $\varphi \in C_0^\infty(\mathbb{R}^N)$ the function

$$(x,y) \mapsto [\varphi(x) - \varphi(y)] \cdot \sqrt{K(x-y)} \in L^2(\mathbb{R}^{2N}).$$

Let $D_K^s(\mathbb{R}^N)$ be the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the Hilbertian norm $[\cdot]_{s,K}$, defined in $(\mathscr{P}_\lambda)$ and induced by the inner product

$$\langle u, v\rangle_{s,K} = \iint_{\mathbb{R}^{2N}} [u(x) - u(y)] \cdot [v(x) - v(y)] \cdot K(x-y)\, dx\, dy. \tag{2.2}$$

Clearly, by (K_1), the embedding $D_K^s(\mathbb{R}^N) \hookrightarrow D^s(\mathbb{R}^N)$ is continuous, being

$$[u]_s \leq \beta^{-1/2}\, [u]_{s,K} \quad \text{for all } u \in D_K^s(\mathbb{R}^N). \tag{2.3}$$

Hence, by (2.1) we obtain

$$\|u\|_{2^*} \leq C_{2^*}\, \beta^{-1/2}\, [u]_{s,K} \quad \text{for all } u \in D_K^s(\mathbb{R}^N). \tag{2.4}$$

Finally, the space X denotes the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| = \left([u]_{s,K}^2 + \|u\|_{r,h}^2\right)^{1/2}, \quad \text{where } \|u\|_{r,h}^r = \int_{\mathbb{R}^N} h(x)|u|^r dx.$$

The embedding

$$X \hookrightarrow D_K^s(\mathbb{R}^N) \quad \textit{is continuous,} \tag{2.5}$$

with $[u]_{s,K} \leq \|u\|$ for all $u \in X$. In particular, by (2.1) and (2.3),

$$X \hookrightarrow D_K^s(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N).$$

Moreover, for all $R > 0$ and $p \in [1, 2^*)$, the embedding

$$D_K^s(\mathbb{R}^N) \hookrightarrow\!\!\!\!\hookrightarrow L^p(B_R) \tag{2.6}$$

is compact. Indeed, $D_K^s(\mathbb{R}^N) \hookrightarrow D^s(\mathbb{R}^N) \hookrightarrow H^s(B_R)$ by (2.3) and the embedding $H^s(B_R) \hookrightarrow\!\!\!\!\hookrightarrow L^p(B_R)$ is compact for all $p \in [1, 2^*)$ by Corollary 7.2 of [17].

We also have the following main embedding result, whose proof is referred to the Appendix.

Lemma 2.1. *The embedding $D_K^s(\mathbb{R}^N) \hookrightarrow\!\!\!\!\hookrightarrow L^q(\mathbb{R}^N, w)$ is compact, with*

$$\|u\|_{q,w} \leq \mathfrak{C}_w [u]_{s,K} \quad \textit{for all } u \in D_K^s(\mathbb{R}^N), \tag{2.7}$$

and $\mathfrak{C}_w = C_{2^*}\|w\|_{\wp}^{1/q}\beta^{-1/2} > 0$. Furthermore, the embedding

$$X \hookrightarrow\!\!\!\!\hookrightarrow L^q(\mathbb{R}^N, w)$$

is also compact.

An *entire* (*weak*) *solution* u of $(\mathscr{P}_\lambda)$ is a function in X such that

$$M([u]_{s,K}^2)\langle u,\varphi\rangle_{s,K} = \lambda\int_{\mathbb{R}^N} w(x)\,|u|^{q-2}\,u\,\varphi\,dx - \int_{\mathbb{R}^N} h(x)\,|u|^{r-2}\,u\,\varphi\,dx, \tag{2.8}$$

for all $\varphi\in X$, where $\langle\cdot,\cdot\rangle_{s,K}$ is given in (2.2).

Lemma 2.2. *If* $\lambda\in\mathbb{R}$ *and* $u=u_\lambda\in X\setminus\{0\}$ *satisfy*

$$M([u]_{s,K}^2)[u]_{s,K}^2+\|u\|_{r,h}^r=\lambda\|u\|_{q,w}^q\,, \tag{2.9}$$

then $\lambda>0$ *and*

$$\kappa_1\,\lambda^{1/(2-q)}\le\|u_\lambda\|_{q,w}\le\kappa_2\,\lambda^{r/2(r-q)}, \tag{2.10}$$

where κ_1 *and* κ_2 *are positive constants independent of* λ *and* u_λ*.*

Proof. Let $u\in X\setminus\{0\}$ and $\lambda\in\mathbb{R}$ satisfy (2.9). By (2.7), $(\mathcal{M})$ and (2.9),

$$\|u\|_{q,w}^2\le\mathfrak{C}_w^2\,[u]_{s,K}^2\le\frac{\mathfrak{C}_w^2}{m_0}M([u]_{s,K}^2)\,[u]_{s,K}^2\le\lambda\,\frac{\mathfrak{C}_w^2}{m_0}\,\|u\|_{q,w}^q. \tag{2.11}$$

Hence, $\lambda>0$, being $u\not\equiv 0$. Moreover, $\lambda\|u\|_{q,w}^{q-2}\ge m_0/\mathfrak{C}_w^2$, that is $\|u\|_{q,w}\ge\kappa_1\lambda^{1/(2-q)}$, with $\kappa_1=(m_0/\mathfrak{C}_w^2)^{1/(q-2)}$. In other words, the first part of (2.10) holds true. By Young's inequality,

$$ab\le\frac{a^\alpha}{\alpha}+\frac{b^\beta}{\beta},$$

with $a=h(x)^{q/r}|u|^q\ge 0$, $b=\lambda w(x)h(x)^{-q/r}\ge 0$, $\alpha=r/q>1$ and $\beta=r/(r-q)>1$, we find

$$\lambda\,w(x)\,|u|^q\le\frac{q}{r}\,h(x)\,|u|^r+\frac{r-q}{r}\left(\frac{\lambda w(x)}{h(x)^{q/r}}\right)^{r/(r-q)}.$$

Integration over $\mathbb{R}^N$, $(\mathcal{M})$ and (2.9) give

$$m_0[u]_{s,K}^2\le M([u]_{s,K}^2)[u]_{s,K}^2\le\frac{q-r}{r}\|u\|_{r,h}^r+\frac{r-q}{r}\,H\,\lambda^{r/(r-q)}\le\frac{r-q}{r}\,H\,\lambda^{r/(r-q)},$$

being $q<r$. Since $u\not\equiv 0$ by assumption, the last inequality and (2.11) yield the second part of (2.10), with $\kappa_2=[(r-q)\mathfrak{C}_w^2H/m_0r]^{1/2}$. This completes the proof. □

If $(\mathscr{P}_\lambda)$ admits a *nontrivial* entire solution $u\in X$, then $\lambda>0$ by Lemma 2.2, and actually $\lambda\ge\lambda_0$ by (2.10), where $\lambda_0=(\kappa_1/\kappa_2)^{2(r-q)(q-2)/q(r-2)}>0$. Define

$$\lambda^*=\sup\{\lambda>0\,:\,(\mathscr{P}_\mu)\ \textit{admits only the trivial entire solution for all}\ \mu<\lambda\}.$$

Clearly $\lambda^*\ge\lambda_0>0$. Theorem 1.1 (i) follows directly from the definition of λ^*.

For the proof of Theorem 1.1 we use variational arguments since the entire solutions of $(\mathscr{P}_\lambda)$ are exactly the critical points of the natural underlying energy functional J_λ associated to $(\mathscr{P}_\lambda)$, that is

$$J_\lambda(u) = \frac{1}{2}\mathscr{M}([u]_{s,K}^2) - \frac{\lambda}{q}\,\|u\|_{q,w}^q + \frac{1}{r}\,\|u\|_{r,h}^r, \quad u \in X, \tag{2.12}$$

where $\mathscr{M}$ is defined in $(\mathcal{M})$. Clearly, J_λ is Gâteaux-differentiable in X and, for all u, $\varphi \in X$,

$$\begin{aligned}\langle J_\lambda'(u), \varphi\rangle = & M([u]_{s,K}^2)\langle u, \varphi\rangle_{s,K} - \lambda \int_{\mathbb{R}^N} w(x)\,|u|^{q-2}\,u\,\varphi\,dx \\ & + \int_{\mathbb{R}^N} h(x)\,|u|^{r-2}\,u\,\varphi\,dx,\end{aligned} \tag{2.13}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X and its dual space X'.

Thanks to the results of this section, *from now on we assume that* $\lambda > 0$, without loss of generality.

Lemma 2.3. *The functional* $J_\lambda\colon X \to \mathbb{R}$ *is bounded below and coercive in* X*. In particular, any sequence* $(u_n)_n$ *in* X *such that* $(J_\lambda(u_n))_n$ *is bounded admits a weakly convergent subsequence in* X*.*

Proof. Let us consider the following elementary inequality: for every k_1, $k_2 > 0$ and $0 < \alpha < \beta$,

$$k_1\,|t|^\alpha - k_2\,|t|^\beta \le C_{\alpha\beta}\,k_1\left(\frac{k_1}{k_2}\right)^{\alpha/(\beta-\alpha)} \quad \text{for all } t \in \mathbb{R}, \tag{2.14}$$

where $C_{\alpha\beta} > 0$ is a constant depending only on α and β. Taking $k_1 = \lambda w(x)/q$, $k_2 = h(x)/2r$, $\alpha = q$, $\beta = r$ and $t = u(x)$ in (2.14), for all $x \in \mathbb{R}^N$ we have

$$\frac{\lambda}{q}\,w(x)|u(x)|^q - \frac{h(x)}{2r}\,|u(x)|^r \le \mathcal{C}\,\lambda^{r/(r-q)}\left[\frac{w(x)^r}{h(x)^q}\right]^{1/(r-q)},$$

where $\mathcal{C} = C_{qr}\,[2r/q]^{q/(r-q)}\,/q$. Integrating the above inequality over $\mathbb{R}^N$, we get by (1.3),

$$\frac{\lambda}{q}\,\|u\|_{q,w}^q - \frac{1}{2r}\,\|u\|_{r,h}^r \le C_\lambda,$$

where $C_\lambda = \mathcal{C}H\lambda^{r/(r-q)} > 0$ by Lemma 2.2.

Therefore, by $(\mathcal{M})$, for all $u \in X$,

$$
\begin{aligned}
J_\lambda(u) &\geq \frac{1}{2} m_0 [u]_{s,K}^2 - \frac{\lambda}{q}\|u\|_{q,w}^q + \frac{1}{r}\|u\|_{r,h}^r \\
&= \frac{1}{2} m_0 [u]_{s,K}^2 - \left[\frac{\lambda}{q}\|u\|_{q,w}^q - \frac{1}{2r}\|u\|_{r,h}^r\right] - \frac{1}{2r}\|u\|_{r,h}^r + \frac{1}{r}\|u\|_{r,h}^r \\
&\geq \frac{1}{2} m_0 [u]_{s,K}^2 - C_\lambda + \frac{1}{2r}\|u\|_{r,h}^r \\
&\geq \frac{1}{2} m_0 [u]_{s,K}^2 + \frac{1}{2r}\left(\|u\|_{r,h}^2 - 1\right) - C_\lambda \\
&\geq \frac{\min\left\{m_0, r^{-1}\right\}}{2}\|u\|^2 - C_\lambda - \frac{1}{2r}.
\end{aligned}
\tag{2.15}
$$

Hence, J_λ is bounded below and coercive in X. The last part of the lemma follows at once by the coercivity of J_λ and the reflexivity of the space X, proved in Proposition A.1. □

For any $(x, u) \in \mathbb{R}^N \times \mathbb{R}$, put

$$
f(x, u) = \lambda w(x)|u|^{q-2}u - h(x)|u|^{r-2}u, \tag{2.16}
$$

so that

$$
F(x, u) = \int_0^u f(x, v)\, dv = \frac{\lambda}{q}\, w(x)|u|^q - h(x)\frac{|u|^r}{r}. \tag{2.17}
$$

Lemma 2.4. *For any fixed $u \in X$ the functional $\mathcal{F}_u : X \to \mathbb{R}$, defined by*

$$
\mathcal{F}_u(v) = \int_{\mathbb{R}^N} f(x, u(x))\, v(x)\, dx,
$$

is in X'. In particular, if $v_n \rightharpoonup v$ in X then $\mathcal{F}_u(v_n) \to \mathcal{F}_u(v)$ as $n \to \infty$.

Proof. Fix $u \in X$. Clearly $\mathcal{F}_u$ is linear. Moreover, using (2.7), we get for all $v \in X$

$$
\begin{aligned}
|\mathcal{F}_u(v)| &\leq \lambda \int_{\mathbb{R}^N} w(x)|u|^{q-1}|v|\, dx + \int_{\mathbb{R}^N} h(x)|u|^{r-1}|v| dx \\
&\leq \lambda\, \|u\|_{q,w}^{q-1}\|v\|_{q,w} + \|u\|_{r,h}^{r-1}\|v\|_{r,h} \leq \sqrt{2}\left(\lambda\, \mathfrak{C}_w \|u\|_{q,w}^{q-1} + \|u\|_{r,h}^{r-1}\right)\|v\|,
\end{aligned}
$$

and so $\mathcal{F}_u$ is continuous in X. □

Lemma 2.5. *The functional $J_\lambda : X \to \mathbb{R}$ is of class $C^1(X)$ and J_λ is sequentially weakly lower semicontinuous in X, that is, if $u_n \rightharpoonup u$ in X, then*

$$
J_\lambda(u) \leq \liminf_{n\to\infty} J_\lambda(u_n). \tag{2.18}
$$

Moreover, J_λ attains its infimum $e = e_\lambda$ in X, which is an entire solution of $(\mathscr{P}_\lambda)$.

Proof. A simple calculation shows that $\frac{1}{2}\mathscr{M}([u]_{s,K}^2)$ is convex in X, since $\mathscr{M}$ is convex and monotone non-decreasing in $\mathbb{R}_0^+$ by $(\mathcal{M})$ and of class $C^1(X)$. Therefore, $\frac{1}{2}\mathscr{M}([u]_{s,K}^2)$ is sequentially weakly lower semicontinuous in X by Corollary 3.9 of [13], so that

$$\mathscr{M}([u]_{s,K}^2) \le \liminf_{n\to\infty} \mathscr{M}([u_n]_{s,K}^2) \tag{2.19}$$

along any sequence $(u_n)_n$, with $u_n \rightharpoonup u$ in X.

Denote with Φ_w the functional $u \mapsto \|u\|_{q,w}^q/q$. By Lemma 2.1 and Theorem 3.10 of [13] we also have that Φ_w is weakly continuous, so that in particular Φ_w is continuous in X. Furthermore, Φ_w is Gâteaux-differentiable in X and for all u, $\varphi \in X$,

$$\langle \Phi_w'(u), \varphi\rangle = \int_{\mathbb{R}^N} w(x)\,|u|^{q-2}\,u\,\varphi\,dx.$$

Now, let $(u_n)_n$, $u \in X$ be such that $u_n \rightharpoonup u$ in X and fix $\varphi \in X$, with $\|\varphi\| = 1$. By Lemma 2.1 and Proposition A.8 (ii) of [6], it follows that $v_n = |u_n|^{q-2}u_n \to v = |u|^{q-2}u$ in $L^{q'}(\mathbb{R}^N, w)$. Therefore,

$$|\langle \Phi_w'(u_n) - \Phi_w'(u), \varphi\rangle| \le \|v_n - v\|_{q',w}\,\|\varphi\|_{q,w} \le \mathfrak{C}_w\|v_n - v\|_{q',w},$$

by (2.7). Hence, $\|\Phi_w'(u_n) - \Phi_w'(u)\|_{X'} \le \mathfrak{C}_w\|v_n - v\|_{q',w}$, that is $\Phi_w'(u_n) \to \Phi_w'(u)$ in X'. Thus, Φ_w is of class $C^1(X)$ and, as $n \to \infty$,

$$\int_{\mathbb{R}^N} w(x)|u_n|^{q-2}\,u_n\,\varphi\,dx \to \int_{\mathbb{R}^N} w(x)\,|u|^{q-2}\,u\,\varphi\,dx \tag{2.20}$$

for all $\varphi \in X$.

Finally, it remains to show that also the functional $u \mapsto \|u\|_{r,h}^r/r$, denoted by Φ_h, is of class $C^1(X)$. The continuity of Φ_h follows from the continuity of the embedding $X \hookrightarrow L^r(\mathbb{R}^N, h)$. Hence Φ_h is weakly lower semicontinuous in X, again by Corollary 3.9 of [13]. On the other hand, Φ_h is Gâteaux-differentiable in X and for all u, $\varphi \in X$,

$$\langle \Phi_h'(u), \varphi\rangle = \int_{\mathbb{R}^N} h(x)|u|^{r-2}\,u\,\varphi\,dx.$$

Let $(u_n)_n$, $u \in X$ be such that $u_n \to u$ in X. Then, $u_n \to u$ in $L^r(\mathbb{R}^N, h)$, and so $v_n = |u_n|^{r-2}u_n \to v = |u|^{r-2}u$ in $L^{r'}(\mathbb{R}^N, h)$ by Proposition A.8 (ii) of [6]. Therefore,

$$\|\Phi_h'(u_n) - \Phi_h'(u)\|_{X'} \le \sup_{\substack{\varphi\in X\\ \|\varphi\|=1}} \|v_n - v\|_{r',h} \cdot \|\varphi\|_{r,h} \le \|v_n - v\|_{r',h} = o(1)$$

as $n \to \infty$. This gives the C^1 regularity of Φ_h.

Suppose now that $u_n \rightharpoonup u$ in X. Fix a subsequence $(v_{n_k})_k$ of the sequence $n \mapsto v_n = |u_n|^{r-2}u_n$. Of course $u_{n_k} \rightharpoonup u$ in X and by Proposition A.2 there exists a further subsequence $(u_{n_{k_j}})_j$ such that $u_{n_{k_j}} \to u$ a.e. in $\mathbb{R}^N$. Thus $v_{n_{k_j}} \to v = |u|^{r-2}u$ a.e. in $\mathbb{R}^N$. On the other hand, $(v_{n_{k_j}})_j$ is bounded in $L^{r'}(\mathbb{R}^N, h)$, since $\|v_{n_{k_j}}\|_{r',h}^{r'} = \|u_{n_{k_j}}\|_{r,h}^{r}$ and $(u_{n_{k_j}})_j$ is bounded in $L^r(\mathbb{R}^N, h)$. Therefore, $v_{n_{k_j}} \rightharpoonup v$ in $L^{r'}(\mathbb{R}^N, h)$ by Proposition A.8 (i) of [6]. In conclusion, due to the arbitrariness of $(v_{n_k})_k$, the entire sequence $v_n \rightharpoonup v$ in $L^{r'}(\mathbb{R}^N, h)$ as $n \to \infty$. In particular, for all $\varphi \in X$,

$$\int_{\mathbb{R}^N} h(x)\,|u_n|^{r-2}\,u_n\,\varphi\,dx \to \int_{\mathbb{R}^N} h(x)\,|u|^{r-2}\,u\,\varphi\,dx \tag{2.21}$$

as $n \to \infty$.

For the second part of the lemma, let $(u_n)_n$, $u \in X$ be such that $u_n \rightharpoonup u$ in X. The definition of J_λ and (2.17) give

$$J_\lambda(u) - J_\lambda(u_n) = \frac{1}{2}\left[\mathscr{M}([u]_{s,K}^2) - \mathscr{M}([u_n]_{s,K}^2)\right] + \int_{\mathbb{R}^N}[F(x,u_n) - F(x,u)]\,dx.$$

Hence, by (2.19),

$$\limsup_{n\to\infty}\,[J_\lambda(u) - J_\lambda(u_n)] \le \limsup_{n\to\infty} \int_{\mathbb{R}^N}[F(x,u_n) - F(x,u)]\,dx. \tag{2.22}$$

By (2.16) and (2.17), for all $t \in [0,1]$,

$$\begin{aligned} F_u(x, u + t(u_n - u)) &= f(x, u + t(u_n - u)) \\ &= f(x,u) + (u_n - u)\int_0^t f_u(x, u + \tau(u_n - u))\,d\tau, \end{aligned} \tag{2.23}$$

where clearly $f_u(x,z) = \lambda(q-1)w(x)|z|^{q-2} - h(x)(r-1)|z|^{r-2}$. Multiplying (2.23) by $u_n - u$ and integrating over $[0,1]$, we obtain

$$\begin{aligned} F(x,u_n) - F(x,u) &= f(x,u)(u_n - u) \\ &\quad + (u_n - u)^2 \int_0^1 \left(\int_0^t f_u(x, u + \tau(u_n - u))d\tau\right) dt. \end{aligned} \tag{2.24}$$

By (2.14), with $t = z$, $k_1 = \lambda w(x)(q-1)$, $k_2 = h(x)(r-1)$, $\alpha = q - 2 > 0$ and $\beta = r - 2 > 0$, we get

$$f_u(x,z) \le 2\,C_1\,w(x)^{2/q}\left[\frac{w(x)^{r/q}}{h(x)}\right]^{(q-2)/(r-q)}, \tag{2.25}$$

where C_1 is a positive constant, depending only on q, r and λ. Consequently, (2.24) yields

$$\begin{aligned} \int_{\mathbb{R}^N}[F(x,u_n) - F(x,u)]dx &\le \int_{\mathbb{R}^N} f(x,u)(u_n - u)\,dx \\ &\quad + C_1\,H^{(q-2)/q}\|u_n - u\|_{q,w}^2, \end{aligned} \tag{2.26}$$

by Hölder's inequality and (1.3). Now, Lemma 2.4 gives

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f(x,u)(u_n - u)\,dx = 0, \tag{2.27}$$

and Lemma 2.1 implies

$$\lim_{n\to\infty} \|u_n - u\|_{q,w} = 0. \tag{2.28}$$

Combining (2.26)–(2.28) with (2.22) we get the claim (2.18).

Finally, Corollary 3.23 of [13] yields the existence of a global minimizer $e = e_\lambda$ of J_λ in X for each $\lambda > 0$ and e is therefore an entire solution of $(\mathscr{P}_\lambda)$. □

3. Existence of two solutions

The number

$$\overline{\lambda} = \inf_{\substack{u\in X\\ \|u\|_{q,w}=1}} \left\{\frac{q}{2}\mathscr{M}([u]_{s,K}^2) + \frac{q}{r}\,\|u\|_{r,h}^r\right\}$$

is positive. Indeed, for all $u \in X$ with $\|u\|_{q,w} = 1$, by Hölder's inequality and (1.3), we have

$$1 = \|u\|_{q,w}^q = \int_{\mathbb{R}^N} \frac{w(x)}{h(x)^{q/r}}\, h(x)^{q/r}|u|^q dx \le H^{(r-q)/r}\|u\|_{r,h}^q.$$

Consequently, we get

$$\frac{q}{2}\,\mathscr{M}([u]_{s,K}^2) + \frac{q}{r}\,\|u\|_{r,h}^r \ge \frac{m_0 q}{2}\,[u]_{s,K}^2 + \frac{q}{r}\,H^{(q-r)/q} \ge \frac{m_0\, q}{2\,\mathfrak{C}_w^2} + \frac{q}{r}\,H^{(q-r)/q}.$$

In other words, $\overline{\lambda} \ge m_0 q/2\mathfrak{C}_w^2 + qH^{(q-r)/q}/r > 0$, as stated.

Lemma 3.1. *For all $\lambda > \overline{\lambda}$ there exists a global nontrivial non-negative minimizer $e \in X$ of J_λ with negative energy, that is $J_\lambda(e) < 0$. In particular, e is a nontrivial non-negative entire solution of $(\mathscr{P}_\lambda)$.*

Proof. By Lemma 2.5 for each $\lambda > 0$ there exists a global minimizer $e = e_\lambda \in X$ of J_λ, that is

$$J_\lambda(e) = \inf_{v\in X} J_\lambda(v).$$

We prove that $e \not\equiv 0$ whenever $\lambda > \overline{\lambda}$, showing that $J_\lambda(e) < 0$.

Let $\lambda > \overline{\lambda}$. Then there exists a function $v \in X$, with $\|v\|_{q,w} = 1$, such that

$$\lambda\,\|v\|_{q,w}^q = \lambda > \frac{q}{2}\mathscr{M}([v]_{s,K}^2) + \frac{q}{r}\,\|v\|_{r,h}^r,$$

that is

$$J_\lambda(v) = \frac{1}{2}\,\mathscr{M}([v]_{s,K}^2) - \frac{\lambda}{q}\,\|v\|_{q,w}^q + \frac{1}{r}\,\|v\|_{r,h}^r < 0.$$

In particular, $J_\lambda(e) \le J_\lambda(v) < 0$, as required.

Hence, for any $\lambda > \overline{\lambda}$ equation $(\mathscr{P}_\lambda)$ has a nontrivial entire solution $e \in X$ such that $J_\lambda(e) < 0$. Finally, we may assume $e \ge 0$ in $\mathbb{R}^N$. Indeed, $|e| \in X$, being $\big|\,|e(x)| - |e(y)|\,\big| \le |e(x) - e(y)|$ for all x, $y \in \mathbb{R}^N$. Moreover, $J_\lambda(|e|) \le J_\lambda(e)$, since $[|u|]_{s,K} \le [u]_{s,K}$ for all $u \in X$ and so $\mathscr{M}([|e|]_{s,K}^2) \le \mathscr{M}([e]_{s,K}^2)$ by $(\mathcal{M})$. This gives $J_\lambda(e) = J_\lambda(|e|)$, due to the minimality of e. □

Proposition 3.2. *The non-negative entire solutions of* $(\mathscr{P}_\lambda)$ *are exactly the critical points of the* $C^1(X)$ *functional*

$$\mathscr{J}_\lambda(u) = \frac{1}{2}\mathscr{M}([u]_{s,K}^2) - \frac{\lambda}{q}\,\|u^+\|_{q,w}^q + \frac{1}{r}\,\|u\|_{r,h}^r, \quad u \in X.$$

Proof. It is evident from the proof of Lemma 2.5 that also $\mathscr{J}_\lambda$ is of class $C^1(X)$. For any non-negative function $u \in X$, we have $\mathscr{J}_\lambda(u) = J_\lambda(u)$, so that non-negative entire solutions of $(\mathscr{P}_\lambda)$ are critical points of $\mathscr{J}_\lambda$. To see the vice versa, first observe that $|u^+(x)-u^+(y)| \le |u(x)-u(y)|$ and $|u^-(x)-u^-(y)| \le |u(x)-u(y)|$ for all x, $y \in \mathbb{R}^N$, so that both u^+ and $u^- \in X$ for all $u \in X$. Furthermore, for all $u \in X$,

$$\begin{aligned}\langle u, -u^-\rangle_{s,K} &= \iint_{\mathbb{R}^{2N}} \big(u^+(x)u^-(y) + u^-(x)u^+(y) + |u^-(x) - u^-(y)|^2\big) K(x-y)\,dx\,dy \\ &\ge [u^-]_{s,K}^2.\end{aligned}$$

Finally, if $u \in X$ is a critical point of $\mathscr{J}_\lambda$, then, taking the test function $\varphi = -u^- \in X$, we get, by $(\mathcal{M})$,

$$0 = M([u]_{s,K}^2)\langle u, -u^-\rangle_{s,K} + \|u^-\|_{r,h}^r \ge m_0\,[u^-]_{s,K}^2 + \|u^-\|_{r,h}^r \ge 0\,;$$

in other words $u^- = 0$ in X, that is the critical point u of $\mathscr{J}_\lambda$ is non-negative in $\mathbb{R}^N$. □

By Lemma 3.1 and Proposition 3.2, the global nontrivial non-negative minimizer $e \in X$ of J_λ is also a critical point of $\mathscr{J}_\lambda$ and $\mathscr{J}_\lambda(e) = J_\lambda(e) < 0$.

Lemma 3.3. *For any* $v \in X \setminus \{0\}$ *and* $\lambda > 0$ *there exist* ϱ*, depending on* v *and* λ*, with* $\varrho \in (0, [v]_{s,K})$*, and* $\alpha = \alpha(\varrho, \lambda) > 0$ *such that* $\mathscr{J}_\lambda(u) \ge \alpha$ *for all* $u \in X$*, with* $[u]_{s,K} = \varrho$*. Furthermore,* J_λ *also satisfies the mountain pass geometry stated above.*

Proof. Let u be in X, with $[u]_{s,K} = \varrho$. By $(\mathcal{M})$ and (2.7),

$$\mathscr{J}_\lambda(u) \ge J_\lambda(u) \ge \frac{m_0}{2}\,[u]_{s,K}^2 - \frac{\lambda}{q}\,\mathfrak{C}_w^q [u]_{s,K}^q \ge \left(\frac{m_0}{2} - \frac{\lambda}{q}\,\mathfrak{C}_w^q [u]_{s,K}^{q-2}\right)[u]_{s,K}^2.$$

Therefore, it is enough to take ϱ, with $0 < \varrho < \min\big\{(m_0 q/2\lambda\mathfrak{C}_w^q)^{1/(q-2)}, [v]_{s,K}\big\}$ and the number $\alpha = \big(m_0/2 - \lambda\mathfrak{C}_w^q \varrho^{q-2}/q\big)\varrho^2 > 0$ satisfies the assertion. The last part of the lemma follows now at once. □

The proof of Lemma 3.3 shows in particular that, for all $\lambda > 0$, the trivial solution $u = 0$ is a strict local minimum of both J_λ and $\mathscr{J}_\lambda$ in X. Indeed, fix a positive number ϱ, with $\varrho < (m_0 q/2\lambda\mathfrak{C}_w^q)^{1/(q-2)}$. Then for all $u \in X$, with $0 < \|u\| \le \varrho$, by $(\mathcal{M})$ and (2.7),

$$\mathscr{J}_\lambda(u) \ge J_\lambda(u) \ge \left(\frac{m_0}{2} - \frac{\lambda}{q}\,\mathfrak{C}_w^q \varrho^{q-2}\right)[u]_{s,K}^2 > 0,$$

as stated.

Proof of the first part of Theorem 1.1. We recall that $\mathscr{J}_\lambda$ is of class $C^1(X)$ by Proposition 3.2. Moreover, by Lemmas 3.1 and 3.3 and Theorem A.3 of [6], for all $\lambda > \overline{\lambda}$ there exists a sequence $(u_n)_n$ in X such that for all n

$$c_\lambda \le \mathscr{J}_\lambda(u_n) \le c_\lambda + \frac{1}{n^2} \quad \text{and} \quad \|\mathscr{J}_\lambda'(u_n)\|_{X'} \le \frac{2}{n}, \tag{3.1}$$

where

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathscr{J}_\lambda(\gamma(t)) \quad \text{and} \quad \Gamma = \{\gamma \in C([0,1];X) : \gamma(0) = 0,\ \gamma(1) = e\}.$$

By Lemma 2.3 the sequence $(u_n)_n$ is bounded in X. By Propositions A.1, A.2, Lemma 2.1 and the fact that $L^q(\mathbb{R}^N, w)$ and $L^r(\mathbb{R}^N, h)$ are uniformly convex Banach spaces by Proposition A.6 of [6], it is possible to extract a subsequence, still relabeled $(u_n)_n$, satisfying

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } X, \qquad u_n \rightharpoonup u \quad \text{in } L^r(\mathbb{R}^N, h), \qquad [u_n]_{s,K} \to \ell, \\ &u_n \to u \quad \text{in } L^q(\mathbb{R}^N, w), \qquad u_n^+ \to u^+ \quad \text{in } L^q(\mathbb{R}^N, w), \end{aligned} \tag{3.2}$$

for some $u \in X$ and some $\ell \in \mathbb{R}_0^+$, since $|u_n^+(x) - u^+(x)| \le |u_n(x) - u(x)|$ for all $x \in \mathbb{R}^N$. In particular, by $(\mathcal{M})$,

$$M([u_n]_{s,K}^2) \to M(\ell^2) > 0 \qquad \text{as } n \to \infty. \tag{3.3}$$

We claim that $\|u_n - u\| \to 0$ *in* X. Clearly, $\langle \mathscr{J}_\lambda'(u_n) - \mathscr{J}_\lambda'(u), u_n - u\rangle \to 0$ as $n \to \infty$, since $u_n \rightharpoonup u$ in X and $\mathscr{J}_\lambda'(u_n) \to 0$ in X' as $n \to \infty$ by (3.1) and (3.2). Hence, as $n \to \infty$,

$$\begin{aligned} o(1) &= \langle \mathscr{J}_\lambda'(u_n) - \mathscr{J}_\lambda'(u), u_n - u\rangle \\ &= M([u_n]_{s,K}^2)[u_n]_{s,K}^2 + M([u]_{s,K}^2)[u]_{s,K}^2 \\ &\quad - \langle u_n, u\rangle_{s,K} \left[M([u_n]_{s,K}^2) + M([u]_{s,K}^2)\right] - \int_{\mathbb{R}^N} \left[g(x, u_n) - g(x, u)\right](u_n - u)\, dx, \end{aligned} \tag{3.4}$$

where $g(x, z) = \lambda w(x)(z^+)^{q-1} - h(x)|z|^{r-2}z$ for $(x, z) \in \mathbb{R}^{N+1}$. Thus, using the notation (2.16) and putting $\mathscr{I}_n = \langle u_n, u\rangle_{s,K} \left[M([u_n]_{s,K}^2) + M([u]_{s,K}^2)\right]$, we get, by (2.25), Hölder's inequality and (1.3),

$$\begin{aligned} M([u_n]_{s,K}^2)&[u_n]_{s,K}^2 + M([u]_{s,K}^2)[u]_{s,K}^2 \\ &= \mathscr{I}_n + \int_{\mathbb{R}^N} \left[g(x, u_n) - g(x, u)\right](u_n - u)\, dx + o(1) \\ &= \mathscr{I}_n + \int_{\mathbb{R}^N} (u_n - u)^2 dx \int_0^1 g_u(x, u + t(u_n - u))dt + o(1) \\ &\le \mathscr{I}_n + \int_{\mathbb{R}^N} (u_n - u)^2 dx \int_0^1 f_u(x, u + t(u_n - u))dt + o(1) \\ &\le \mathscr{I}_n + 2\, C_1\, H^{(q-2)/q} \|u_n - u\|_{q,w}^2 + o(1). \end{aligned}$$

Passing now to the limit as $n \to \infty$, we have that $M(\ell^2)\ell^2 + M([u]_{s,K}^2)[u]_{s,K}^2 \leq [u]_{s,K}^2 \big[M(\ell^2) + M([u]_{s,K}^2)\big]$, that is $\ell \leq [u]_{s,K}$ by $(\mathcal{M})$ and (3.3). In other words,

$$[u]_{s,K} \leq \lim_{n\to\infty} [u_n]_{s,K} = \ell \leq [u]_{s,K},$$

which implies at once that

$$[u_n - u]_{s,K} \to 0, \tag{3.5}$$

since $u_n \rightharpoonup u$ in $X \hookrightarrow D_K^s(\mathbb{R}^N)$ and $D_K^s(\mathbb{R}^N)$ is a Hilbert space. Thus, (3.4) and (3.5) yield that

$$0 \leq \int_{\mathbb{R}^N} h(x)(|u_n|^{r-2}\, u_n - |u|^{r-2}\, u)\,(u_n - u)\, dx \to 0$$

as $n \to \infty$. Hence,

$$\|u_n - u\|_{r,h}^r \leq k_r \int_{\mathbb{R}^N} h(x)(|u_n|^{r-2}\, u_n - |u|^{r-2}\, u)\,(u_n - u)\, dx \to 0, \tag{3.6}$$

thanks to Simon's inequality $|\xi - \xi_0|^r \leq k_r(|\xi|^{r-2}\xi - |\xi_0|^{r-2}\xi_0) \cdot (\xi - \xi_0)$, valid for all ξ, $\xi_0 \in \mathbb{R}$, being $r > 2$. Therefore, $\|u_n - u\|_{r,h} \to 0$ as $n \to \infty$. Combining this fact with (3.5) we obtain that $\|u_n - u\| \to 0$, that is $u_n \to u$ in X as $n \to \infty$.

We next prove that u is a second independent nontrivial non-negative entire solution of problem $(\mathscr{P}_\lambda)$. Clearly, for any $\varphi \in X$,

$$\langle \mathscr{J}_\lambda'(u_n), \varphi\rangle = M([u_n]_{s,K}^2)\langle u_n, \varphi\rangle_{s,K} - \int_{\mathbb{R}^N} g(x, u_n)\, \varphi\, dx, \tag{3.7}$$

with g defined above. Now, by (2.20) and (3.2), as $n \to \infty$,

$$\int_{\mathbb{R}^N} w(x)\, (u_n^+)^{q-1} \varphi\, dx \to \int_{\mathbb{R}^N} w(x)\, (u^+)^{q-1} \varphi\, dx$$

for all $\varphi \in X$. Hence, since $u_n \to u$ in X, passing to the limit as $n \to \infty$ in (3.7) and using also (2.21) and (3.5), we have for all $\varphi \in X$

$$M([u]_{s,K}^2)\langle u, \varphi\rangle_s = \lambda \int_{\mathbb{R}^N} w(x)\, (u^+)^{q-1} \varphi\, dx - \int_{\mathbb{R}^N} h(x)\, |u|^{r-2}\, u\, \varphi\, dx,$$

since $\langle \mathscr{J}_\lambda'(u_n), \varphi\rangle \to 0$ as $n \to \infty$ for all $\varphi \in X$ by (3.1).

Finally, $\mathscr{J}_\lambda(u) = c_\lambda = \lim_{n\to\infty} \mathscr{J}_\lambda(u_n)$, being $\mathscr{J}_\lambda \in C^1(X)$ by Proposition 3.2. Therefore, u is a second nontrivial independent critical point for $\mathscr{J}_\lambda$, being $\mathscr{J}_\lambda(u) = c_\lambda > 0 > \mathscr{J}_\lambda(e)$, that is u is a second nontrivial non-negative entire solution of $(\mathscr{P}_\lambda)$. □

From Proposition 3.2 it is clear that the second nontrivial non-negative entire solution $u = u_\lambda \in X$, constructed in the proof above, is a critical point of J_λ, with $J_\lambda(u) = \mathscr{J}_\lambda(u) = c_\lambda > 0 > \mathscr{J}_\lambda(e) = J_\lambda(e)$. We next prove an important property of the asymptotic behavior as $\lambda \to \infty$ of c_λ.

Proposition 3.4. *Under the assumptions of Theorem 1.1,*

$$\lim_{\lambda\to\infty} c_\lambda = 0,$$

where c_λ is the level in (3.1) of the mountain pass solution u_λ of $(\mathscr{P}_\lambda)$, constructed in the proof of Theorem 1.1.

Proof. Let $\lambda > \overline{\lambda}$ and let $e \in X$ be the function given in Lemma 3.1. Since J_λ satisfies the mountain pass geometry of Lemma 3.3 and the path $t \mapsto te$, $t \in [0,1]$, is in Γ defined in (3.1), it follows that there exists $t_\lambda \in (0,1)$ such that $J_\lambda(t_\lambda e) = \max_{t\in[0,1]} J_\lambda(te)$, being $c_\lambda > 0$. Hence, $\langle J'_\lambda(t_\lambda e), e\rangle = 0$. Thus, $\langle J'_\lambda(t_\lambda e), t_\lambda e\rangle = 0$ and by (2.13)

$$M([t_\lambda e]^2_{s,K})[t_\lambda e]^2_{s,K} = \lambda\, t_\lambda^q\, \|e\|^q_{q,w} - t_\lambda^r\, \|e\|^r_{r,h}. \tag{3.8}$$

Let $(\lambda_n)_n$ be a sequence, with $\lambda_n \to \infty$. We suppose that $\lambda_n > \overline{\lambda}$ for any $n \in \mathbb{N}$, without loss of generality. Thus, there exists $t \ge 0$ and a subsequence $(t_{n_k})_k$ of $(t_{\lambda_n})_n$ such that $t_{n_k} \to t$ as $k \to \infty$. Clearly $t = 0$. Otherwise, (3.8) implies

$$M([te]^2_{s,K})[te]^2_{s,K} + \|te\|^r_{r,h} = \|te\|^q_{q,w} \lim_{k\to\infty} \lambda_{n_k} = \infty,$$

which gives an obvious contradiction. In particular, the entire sequence $(t_{\lambda_n})_n$ converges to 0. This shows that

$$\lim_{\lambda\to\infty} t_\lambda = 0,$$

being $(\lambda_n)_n$, with $\lambda_n \to \infty$, arbitrary. In conclusion, as $\lambda \to \infty$,

$$\begin{aligned} 0 < c_\lambda \le \max_{t\in[0,1]} J_\lambda(te) = J_\lambda(t_\lambda e) &= \frac{1}{2}\mathscr{M}([t_\lambda e]^2_{s,K}) - \frac{\lambda}{q}\, t_\lambda^q\, \|e\|^q_{q,w} + \frac{1}{r}\, t_\lambda^r\, \|e\|^r_{r,h} \\ &\le \frac{1}{2}\,\mathscr{M}(t_\lambda^2 [e]^2_{s,K}) + \frac{\|e\|^r_{r,h}}{r}\, t_\lambda^r \to 0, \end{aligned}$$

since of course $\mathscr{M}(\tau) \to 0$ as $\tau \to 0^+$. This completes the proof. $\square$

Proposition 3.5. *Under the assumptions of Theorem 1.1,*

$$\lim_{\lambda\to\infty} \|u_\lambda\| = 0,$$

where u_λ is the mountain pass solution of $(\mathscr{P}_\lambda)$, constructed in the proof of Theorem 1.1.

Proof. Using the notation of the statement, it is clear that

$$\limsup_{\lambda\to\infty} [u_\lambda]_{s,K} < \infty \quad \text{and} \quad \limsup_{\lambda\to\infty} \|u_\lambda\|_{r,h} < \infty. \tag{3.9}$$

Otherwise from (2.15) and Proposition 3.4 we would get an easy contradiction. Now, fix a sequence $(\lambda_n)_n$, with $\lambda_n \to \infty$ as $n \to \infty$, and let $n \mapsto u_n = u_{\lambda_n}$ be the corresponding mountain pass sequence of solutions of $(\mathscr{P}_{\lambda_n})$. Hence, there exists a subsequence $(u_{n_k})_k$ of $(u_n)_n$, a function $u \in X$ and a number $\ell \in \mathbb{R}_0^+$ such that $[u_{n_k}]_{s,K} \to \ell$ and

$$u_{n_k} \rightharpoonup u \text{ in } D_K^s(\mathbb{R}^N), \quad u_{n_k} \to u \text{ in } L^q(\mathbb{R}^N, w), \quad u_{n_k} \rightharpoonup u \text{ in } L^r(\mathbb{R}^N, h)$$

as $k \to \infty$, by Proposition A.1 and Lemma 2.1. Assume by contradiction that $u \neq 0$. Then, (2.9) holds along any solution u_{n_k}, so that

$$M(\ell^2)\ell^2 + \limsup_{k\to\infty} \|u_{n_k}\|_{r,h}^r = \|u\|_{q,w}^q \lim_{k\to\infty} \lambda_{n_k},$$

which contradicts (3.9). Therefore, $u = 0$ as stated and the *entire* sequence $(u_n)_n$ satisfies (3.2), with $u = 0$.

By (2.8), for all $n \in \mathbb{N}$ and all $\varphi \in X$,

$$M([u_n]_{s,K}^2)\langle u_n, \varphi\rangle_{s,K} + \int_{\mathbb{R}^N} h(x)\,|u_n|^{r-2}\,u_n\,\varphi\,dx = \lambda_n \int_{\mathbb{R}^N} w(x)\,|u_n|^{q-2}\,u_n\,\varphi\,dx.$$

Thus, by (2.21) the left-hand side approaches zero as $n \to \infty$, since $u_n \rightharpoonup 0$ in X. Hence also the right-hand side should tend to zero as $n \to \infty$. In particular, by (2.20),

$$\lim_{n\to\infty} \lambda_n \|u_n\|_{q,w}^q = 0. \tag{3.10}$$

Therefore, $[u_n]_{s,K} \to \ell = 0$ by (2.11), that is $u_n \to 0$ in $D_K^s(\mathbb{R}^N)$, by (3.2) and the fact that $D_K^s(\mathbb{R}^N)$ is a Hilbert space. Moreover, (2.9) and (3.10) imply at once that $u_n \to 0$ in $L^r(\mathbb{R}^N, h)$. In conclusion, $u_n \to 0$ in X. Since the sequence $(\lambda_n)_n$, with $\lambda_n \to \infty$, is arbitrary, this completes the proof. □

4. Existence of non-negative entire solutions

Put

$$\lambda^{**} = \inf\{\lambda > 0 : (\mathscr{P}_\lambda) \textit{ admits a nontrivial non-negative entire solution}\}.$$

Lemma 3.1 assures that this definition is meaningful and, thanks to Lemma 2.2, we have that $\overline{\lambda} \geq \lambda^{**} \geq \lambda^*$, where λ^* was introduced in Section 2.

Theorem 4.1. *For any $\lambda > \lambda^{**}$, problem $(\mathscr{P}_\lambda)$ admits a nontrivial non-negative entire solution $u_\lambda \in X$.*

Proof. Fix $\lambda > \lambda^{**}$. By definition of λ^{**} there exists $\mu \in (\lambda^{**}, \lambda)$ such that J_μ has a nontrivial critical point $u_\mu \in X$, with $u_\mu \geq 0$ in $\mathbb{R}^N$. Of course, u_μ is a subsolution for $(\mathscr{P}_\lambda)$. Consider the following minimization problem:

$$\inf_{v\in\mathcal{C}} J_\lambda(v), \quad \mathcal{C} = \{v \in X : v \geq u_\mu\}.$$

Clearly $\mathcal{C}$ is closed and convex by Proposition A.2, and in turn also weakly closed in X. Moreover, by Lemmas 2.3 and 2.5, Theorem 6.1.1 of [11] can be applied in X and so in the weakly closed set $\mathcal{C}$. Hence, J_λ attains its infimum in $\mathcal{C}$, i.e., there exists $u_\lambda \geq u_\mu$ such that $J_\lambda(u_\lambda) = \inf_{v\in\mathcal{C}} J_\lambda(v)$.

We claim that u_λ is a solution of $(\mathscr{P}_\lambda)$, which is clearly non-negative. Indeed, take $\varphi \in C_0^\infty(\mathbb{R}^N)$ and $\varepsilon > 0$. Put

$$\varphi_\varepsilon = \max\{0, u_\mu - u_\lambda - \varepsilon\varphi\} \geq 0 \quad \text{and} \quad v_\varepsilon = u_\lambda + \varepsilon\varphi + \varphi_\varepsilon,$$

so that $v_\varepsilon \in \mathcal{C}$. Of course, $0 \leq \langle J'_\lambda(u_\lambda), v_\varepsilon - u_\lambda\rangle = \varepsilon\langle J'_\lambda(u_\lambda), \varphi\rangle + \langle J'_\lambda(u_\lambda), \varphi_\varepsilon\rangle$, and in turn,

$$\langle J'_\lambda(u_\lambda), \varphi\rangle \geq -\frac{1}{\varepsilon}\,\langle J'_\lambda(u_\lambda), \varphi_\varepsilon\rangle. \tag{4.1}$$

Define $\Omega_\varepsilon = \{x \in \mathbb{R}^N : \ u_\lambda(x) + \varepsilon\varphi(x) \leq u_\mu(x) < u_\lambda(x)\}$, so that Ω_ε is a subset of $\operatorname{supp}\varphi$. Since u_μ is a subsolution of $(\mathscr{P}_\lambda)$ and $\varphi_\varepsilon \geq 0$, then $\langle J'_\lambda(u_\mu), \varphi_\varepsilon\rangle \leq 0$. In particular,

$$\langle J'_\lambda(u_\lambda), \varphi_\varepsilon\rangle = \langle J'_\lambda(u_\mu), \varphi_\varepsilon\rangle + \langle J'_\lambda(u_\lambda) - J'_\lambda(u_\mu), \varphi_\varepsilon\rangle \leq \langle J'_\lambda(u_\lambda) - J'_\lambda(u_\mu), \varphi_\varepsilon\rangle.$$

Using the notation of (2.16), we get

$$\Big|\int_{\Omega_\varepsilon}[f(x,u_\lambda) - f(x,u_\mu)]\cdot[-u(x) - \varepsilon\varphi(x)]\,dx\Big| \leq \varepsilon\int_{\Omega_\varepsilon}|f(x,u_\lambda) - f(x,u_\mu)|\cdot|\varphi(x)|\,dx,$$

since $0 \leq -u - \varepsilon\varphi = u_\mu - u_\lambda + \varepsilon|\varphi| < \varepsilon|\varphi|$ in Ω_ε. Therefore,

$$\begin{aligned}\langle J'_\lambda(u_\lambda), \varphi_\varepsilon\rangle \leq \langle M([u_\lambda]^2_{s,K})u_\lambda - M([u_\mu]^2_{s,K})u_\mu, \varphi_\varepsilon\rangle_{s,K}&\\ + \varepsilon\int_{\Omega_\varepsilon}|f(x,u_\lambda) - f(x,u_\mu)|\cdot|\varphi(x)|\,dx\,.&\end{aligned}$$

By the convexity of $\frac{1}{2}\mathscr{M}([u]^2_{s,K})$ in X, we have

$$\begin{aligned}\frac{1}{2}\mathscr{M}([u_\mu]^2_{s,K}) &\geq \frac{1}{2}\mathscr{M}([u_\lambda]^2_{s,K}) + \langle M([u_\lambda]^2_{s,K})u_\lambda, u_\mu - u_\lambda\rangle_{s,K}\\ &\geq \frac{1}{2}\mathscr{M}([u_\mu]^2_{s,K}) + \langle M([u_\mu]^2_{s,K})u_\mu, u_\lambda - u_\mu\rangle_{s,K} + \langle M([u_\lambda]^2_{s,K})u_\lambda, u_\mu - u_\lambda\rangle_{s,K},\end{aligned}$$

so that $\langle M([u_\lambda]^2_{s,K})u_\lambda - M([u_\mu]^2_{s,K})u_\mu, u_\mu - u_\lambda\rangle_{s,K} \leq 0$. Hence,

$$\langle J'_\lambda(u_\lambda), \varphi_\varepsilon\rangle \leq \varepsilon\Big(\int_{\Omega_\varepsilon}\psi(x)\,dx + \iint_{\mathbb{R}^{2N}}\mathscr{U}(x,y)\,dx\,dy\Big),$$

where $\psi(x) = |f(x,u_\lambda) - f(x,u_\mu)|\cdot|\varphi|$ and

$$\begin{aligned}\mathscr{U}(x,y) = \big[M([u_\lambda]^2_{s,K})\big(u_\lambda(x) - u_\lambda(y)\big) - M([u_\mu]^2_{s,K})\big(u_\mu(x) - u_\mu(y)\big)\big]&\\ \times[\varphi(x) - \varphi(y)]\cdot K(x-y).&\end{aligned}$$

Now

$$\begin{aligned}\iint_{\mathbb{R}^{2N}} \mathscr{U}(x,y)dx\,dy &= \iint_{\Omega_\varepsilon\times\Omega_\varepsilon} \mathscr{U}(x,y)\,dx\,dy + \iint_{\Omega_\varepsilon\times(\mathbb{R}^N\setminus\Omega_\varepsilon)} \mathscr{U}(x,y)\,dx\,dy \\ &\quad + \iint_{(\mathbb{R}^N\setminus\Omega_\varepsilon)\times\Omega_\varepsilon} \mathscr{U}(x,y)\,dx\,dy \\ &\le \iint_{\Omega_\varepsilon\times\mathbb{R}^N} |\mathscr{U}(x,y)|\,dx\,dy + \iint_{\mathbb{R}^N\times\Omega_\varepsilon} |\mathscr{U}(x,y)|\,dx\,dy.\end{aligned}$$

Thus,

$$\begin{aligned}\langle J'_\lambda(u_\lambda),\varphi_\varepsilon\rangle \le \varepsilon\Big(\int_{\Omega_\varepsilon}\psi(x)\,dx + \iint_{\Omega_\varepsilon\times\mathbb{R}^N}|\mathscr{U}(x,y)|\,dx\,dy \\ + \iint_{\mathbb{R}^N\times\Omega_\varepsilon}|\mathscr{U}(x,y)|\,dx\,dy\Big) = \varepsilon\mathscr{I}_\varepsilon.\end{aligned} \tag{4.2}$$

We claim that ψ is in $L^1(\operatorname{supp}\varphi)$. Indeed, $|f(x,u_\lambda)-f(x,u_\mu)|$ is in $L^1_{\rm loc}(\mathbb{R}^N)$, being

$$|f(x,u_\lambda)-f(x,u_\mu)| \le \lambda w(x)\left(u_\lambda^{q-1}+u_\mu^{q-1}\right)+h(x)\left(u_\lambda^{r-1}+u_\mu^{r-1}\right).$$

In fact, by Hölder's inequality and (1.2), we obtain

$$\int_{\operatorname{supp}\varphi} w(x)u_\lambda^{q-1}dx \le |\operatorname{supp}\varphi|^{1/2^*}\|w\|_\wp\|u_\lambda\|_{2^*}^{q-1} = C_1, \tag{4.3}$$

and $C_1 = C_1(\operatorname{supp}\varphi)$. Finally, since $h\in L^1_{\rm loc}(\mathbb{R}^N)$ and $u_\lambda\in L^r(\mathbb{R}^N,h)$, then

$$\int_{\operatorname{supp}\varphi} h(x)u_\lambda^{r-1}dx \le \Big(\int_{\operatorname{supp}\varphi} h(x)\,dx\Big)^{1/r}\|u_\lambda\|_{r,h}^{r-1} = C_2, \tag{4.4}$$

with $C_2 = C_2(\operatorname{supp}\varphi)$. The estimates (4.3) and (4.4) hold also for u_μ. The claim is so proved.

We next show that

$$\lim_{\varepsilon\to0^+}\mathscr{I}_\varepsilon = 0. \tag{4.5}$$

Indeed, $\int_{\Omega_\varepsilon}\psi(x)\,dx = o(1)$, since $|\Omega_\varepsilon|\to0$ as $\varepsilon\to0^+$, $\Omega_\varepsilon\subset\operatorname{supp}\varphi$ and by the fact that $\psi\in L^1(\operatorname{supp}\varphi)$.

Now, $\mathscr{U}(x,y)\in L^1(\mathbb{R}^{2N})$, being $X\hookrightarrow D^s_K(\mathbb{R}^N)$ by (2.5). Thus for all $\eta>0$ there exists R_η so large that

$$\iint_{(\operatorname{supp}\varphi)\times(\mathbb{R}^N\setminus B_{R_\eta})}|\mathscr{U}(x,y)|\,dx\,dy < \eta/2, \quad \iint_{(\mathbb{R}^N\setminus B_{R_\eta})\times(\operatorname{supp}\varphi)}|\mathscr{U}(x,y)|\,dx\,dy < \eta/2.$$

Since $|\Omega_\varepsilon\times B_{R_\eta}| = |B_{R_\eta}\times\Omega_\varepsilon|\to0$ as $\varepsilon\to0^+$ and $\mathscr{U}\in L^1(\mathbb{R}^{2N})$, there exist $\delta_\eta>0$ and $\varepsilon_\eta>0$ such that, for all $\varepsilon\in(0,\varepsilon_\eta]$,

$$|\Omega_\varepsilon\times B_{R_\eta}|<\delta_\eta,\quad \iint_{\Omega_\varepsilon\times B_{R_\eta}}|\mathscr{U}(x,y)|\,dx\,dy<\frac{\eta}{2} \quad\text{and}\quad \iint_{B_{R_\eta}\times\Omega_\varepsilon}|\mathscr{U}(x,y)|\,dx\,dy<\frac{\eta}{2}.$$

Therefore, for all $\varepsilon \in (0, \varepsilon_\eta]$,

$$\iint_{\Omega_\varepsilon \times \mathbb{R}^N} |\mathscr{U}(x,y)|\, dx\, dy < \eta \quad \text{and} \quad \iint_{\mathbb{R}^N \times \Omega_\varepsilon} |\mathscr{U}(x,y)|\, dx\, dy < \eta,$$

being $\Omega_\varepsilon \subset \operatorname{supp} \varphi$. Hence (4.5) holds.

In conclusion, by (4.1), (4.2) and (4.5) we finally get $\langle J'_\lambda(u_\lambda), \varphi_\varepsilon \rangle \le o(\varepsilon)$ as $\varepsilon \to 0^+$, so that by (4.1) it follows that $\langle J'_\lambda(u_\lambda), \varphi \rangle \ge o(1)$ as $\varepsilon \to 0^+$. Therefore, $\langle J'_\lambda(u_\lambda), \varphi \rangle \ge 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, that is $\langle J'_\lambda(u_\lambda), \varphi \rangle = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. Since $X = \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|}$, we obtain that u_λ is a nontrivial non-negative solution of $(\mathscr{P}_\lambda)$. $\square$

Theorem 4.2. *Problem $(\mathscr{P}_{\lambda^{**}})$ admits a nontrivial non-negative entire solution in X.*

Proof. Let $(\lambda_n)_n$ be a strictly decreasing sequence converging to λ^{**} and $u_n \in X$ be a nontrivial non-negative entire solution of $(\mathscr{P}_{\lambda_n})$. By (2.8) we get, for all $\varphi \in X$,

$$M([u_n]^2_{s,K})\langle u_n, \varphi\rangle_{s,K} = \int_{\mathbb{R}^N} f_n(x,u_n)\,\varphi\, dx, \tag{4.6}$$

where $n \mapsto f_n(x,u_n) = \lambda_n w(x)|u_n|^{q-2}u_n - h(x)|u_n|^{r-2}u_n$, similarly as defined in (2.16). By (2.8)–(2.10) and the monotonicity of $(\lambda_n)_n$, we obtain

$$\begin{aligned} m_0[u_n]^2_{s,K} + \|u_n\|^r_{r,h} &\le M([u_n]^2_{s,K})[u_n]^2_{s,K} + \|u_n\|^r_{r,h} \\ &= \lambda_n \|u_n\|^q_{q,w} \le \kappa_2^q\, \lambda_n^{1+rq/2(r-q)} \le \kappa_2^q\, \lambda_1^{1+rq/2(r-q)}. \end{aligned}$$

Therefore $([u_n]_{s,K})_n$ and $(\|u_n\|_{r,h})_n$ are bounded, and in turn also $(\|u_n\|)_n$ is bounded. By Propositions A.1, A.2, Lemma 2.1 and the fact that $L^q(\mathbb{R}^N, w)$ and $L^r(\mathbb{R}^N, h)$ are uniformly convex Banach spaces in virtue of Proposition A.6 of [6], it is possible to extract a subsequence, still relabeled $(u_n)_n$, satisfying

$$\begin{gathered} u_n \rightharpoonup u \quad \text{in } X, \qquad u_n \rightharpoonup u \quad \text{in } L^r(\mathbb{R}^N, h), \qquad [u_n]_{s,K} \to \ell, \\ u_n \to u \quad \text{in } L^q(\mathbb{R}^N, w), \qquad u_n \to u \quad \text{a.e. in } \mathbb{R}^N, \end{gathered} \tag{4.7}$$

for some $u \in X$ and some $\ell \in \mathbb{R}^+_0$. In particular, by $(\mathcal{M})$,

$$M([u_n]^2_{s,K}) \to M(\ell^2) > 0 \quad \text{as } n \to \infty.$$

Furthermore, $u \ge 0$ a.e. in $\mathbb{R}^N$. We claim that u is the solution we are looking for.

To this aim, we first show that $[u]_{s,K} = \ell$. Since u_n is a nontrivial non-negative entire solution of $(\mathscr{P}_{\lambda_n})$, it follows that $\langle J'_{\lambda_n}(u_n), \varphi\rangle = 0$ for all $\varphi \in X$ and for all $n \in \mathbb{N}$. In particular, taking $\varphi = u_n - u$, we obtain

$$\begin{aligned} 0 = \langle J'_{\lambda_n}(u_n), u_n - u\rangle &= M([u_n]^2_{s,K})\left([u_n]^2_{s,K} - \langle u_n, u\rangle_{s,K}\right) \\ &\quad - \lambda_n \left[\|u_n\|^q_{q,w} - \int_{\mathbb{R}^N} w(x)\,|u_n|^{q-2}\, u_n\, u\, dx\right] \\ &\quad + \|u_n\|^r_{r,h} - \int_{\mathbb{R}^N} h(x)|u_n|^{r-2}\, u_n\, u\, dx. \end{aligned} \tag{4.8}$$

Clearly $\langle u_n, u\rangle_{s,K} \to [u]_{s,K}^2$ and $\int_{\mathbb{R}^N} w(x)|u_n|^{q-2}u_n u\,dx \to \|u\|_{q,w}^q$ as $n \to \infty$, by (4.7). Thus, passing to the inferior limit in (4.8) and using also (2.21), we get

$$M(\ell^2)\left(\ell^2 - [u]_{s,K}^2\right) + \left(\liminf_{n\to\infty}\|u_n\|_{r,h}^r - \|u\|_{r,h}^r\right) = 0. \tag{4.9}$$

Now, $[u]_{s,K} \le \liminf_{n\to\infty}[u_n]_{s,K} \le \ell$ and $\|u\|_{r,h}^r \le \liminf_{n\to\infty}\|u_n\|_{r,h}^r$, so that the two addends in (4.9) vanish, being both non-negative. In particular, this yields that $[u]_{s,K} = \ell$, since $M(\ell^2) > 0$ by $(\mathcal{M})$. Therefore, passing to the limit in (4.6) as $n \to \infty$, we get, by (2.20) and (2.21),

$$M([u]_{s,K}^2)\langle u, \varphi\rangle_{s,K} = \lambda^{**}\int_{\mathbb{R}^N} w(x)\,|u|^{q-2}\,u\,\varphi\,dx - \int_{\mathbb{R}^N} h(x)\,|u|^{r-2}\,u\,\varphi\,dx$$

for all $\varphi \in X$. Hence u is a non-negative entire solution of $(\mathscr{P}_{\lambda^{**}})$.

We finally claim that $u \not\equiv 0$. Indeed, (2.10) applied to each u_n implies that $\|u_n\|_{q,w} \ge \kappa_1\lambda_n^{1/(2-q)}$, so that, by (4.7),

$$\|u\|_{q,w} = \lim_{n\to\infty}\|u_n\|_{q,w} \ge \kappa_1(\lambda^{**})^{1/(2-q)} > 0,$$

since $\lambda_n \searrow \lambda^{**}$ and $\lambda^{**} > 0$. This shows the claim and completes the proof. □

Proof of part (ii) *of Theorem* 1.1. The existence of $\lambda^{**} \ge \lambda^*$ follows directly from Lemma 3.1, as already noted. The definition of λ^{**}, Theorems 4.1 and 4.2 show at once the validity of (ii) of Theorem 1.1. □

Of course the nontrivial non-negative entire solutions constructed in Theorems 4.1 and 4.2 are also critical points of $\mathscr{J}_\lambda$.

A. Appendix

In this section we present briefly some useful results, which seem not so well known, even if foreseeable. The first can be proved proceeding essentially as in the proof of Proposition A.11 in [6], but we give the argument in order to make the paper self-contained.

Proposition A.1. *The Banach space* $(X, \|\cdot\|)$ *is reflexive.*

Proof. The product space $Y = D_K^s(\mathbb{R}^N) \times L^r(\mathbb{R}^N, h)$, endowed with the norm $\|u\|_Y = [u]_{s,K} + \|u\|_{r,h}$, is a reflexive Banach space by Theorem 1.22 (ii) of [1], since $D_K^s(\mathbb{R}^N)$ is a Hilbert space and $L^r(\mathbb{R}^N, h)$ is a uniformly convex Banach space by Proposition A.6 in [6].

The operator $T\colon (X, \|\cdot\|_Y) \to (Y, \|\cdot\|_Y)$, $T(u) = (u, u)$, is well defined, linear and isometric. Therefore, $T(X)$ is a closed subspace of the reflexive space Y, and so $T(X)$ is reflexive by Theorem 1.21 (ii) of [1]. Consequently, $(X, \|\cdot\|_Y)$ is reflexive, being isomorphic to a reflexive Banach space. Finally, we conclude that also $(X, \|\cdot\|)$ is reflexive, because reflexivity is preserved under equivalent norms, being $\|u\| \le \|u\|_Y \le \sqrt{2}\|u\|$ for all $u \in X$. □

The next proposition is given for functions in X, but of course remains valid also in the main fractional weighted Sobolev space $D^s_K(\mathbb{R}^N)$.

Proposition A.2. *If $(u_n)_n$, $u \in X$ and $u_n \rightharpoonup u$ in X, then, up to a subsequence, $u_n \to u$ a.e. in $\mathbb{R}^N$.*

Proof. Let $(u_n)_n$ and u be as in the statement. Then, $u_n \to u$ as $n \to \infty$ in $L^p(B_R)$ for all $R > 0$ and $p \in [1, 2^*)$ by (2.5) and (2.6). In particular, in correspondence to $R = 1$ there exists a subsequence $(u_{1,n})_n$ of $(u_n)_n$ such that $u_{1,n} \to u$ a.e. in B_1. Clearly $u_{1,n} \rightharpoonup u$ in X and so, in correspondence to $R = 2$, there exists a subsequence $(u_{2,n})_n$ of $(u_{1,n})_n$ such that $u_{2,n} \to u$ a.e. in B_2, and so on. The diagonal subsequence $(u_{n,n})_n$ of $(u_n)_n$, constructed by induction, converges to u a.e. in $\mathbb{R}^N$ as $n \to \infty$. □

For completeness we end by presenting:

Proof of Lemma 2.1. By (1.2), (2.1), (2.3) and Hölder's inequality, for all $u \in D^s_K(\mathbb{R}^N)$,

$$\begin{aligned}\|u\|_{q,w} &\le \Big(\int_{\mathbb{R}^N} w(x)^{\wp} dx\Big)^{1/\wp q} \cdot \Big(\int_{\mathbb{R}^N} |u|^{2^*} dx\Big)^{1/2^*} \le C_{2^*}\|w\|_{\wp}^{1/q}[u]_s \\ &\le C_{2^*}\|w\|_{\wp}^{1/q}\beta^{-1/2}[u]_{s,K},\end{aligned}$$

that is, (2.7) holds.

Let us now show that $\|u_n - u\|_{q,w} \to 0$ as $n \to \infty$ whenever $u_n \rightharpoonup u$ in $D^s_K(\mathbb{R}^N)$. By Hölder's inequality,

$$\int_{\mathbb{R}^N\setminus B_R} w(x)|u_n - u|^q dx \le M\Big(\int_{\mathbb{R}^N\setminus B_R} w(x)^{\wp} dx\Big)^{1/\wp} = o(1)$$

as $R \to \infty$, being $w \in L^{\wp}(\mathbb{R}^N)$ by (1.2) and $M = \sup_n \|u_n - u\|_{2^*}^q < \infty$. For all $\varepsilon > 0$ there exists $R_\varepsilon > 0$ so large that

$$\sup_n \int_{\mathbb{R}^N\setminus B_{R_\varepsilon}} w(x)|u_n - u|^q dx < \varepsilon/2.$$

Moreover, by (1.2), Hölder's inequality and (2.6) we have

$$\int_{B_{R_\varepsilon}} w(x)|u_n - u|^q\, dx \le \|w\|_{L^\sigma(B_{R_\varepsilon})}\|u_n - u\|^q_{L^{\sigma' q}(B_{R_\varepsilon})} = o(1)$$

as $n \to \infty$, since $\sigma' q < 2^*$. Hence, there exists $N_\varepsilon > 0$ such that

$$\int_{B_{R_\varepsilon}} w(x)\,|u_n - u|^q\, dx < \varepsilon/2$$

for all $n \ge N_\varepsilon$. In conclusion, for all $n \ge N_\varepsilon$

$$\|u_n - u\|^q_{q,w} = \int_{\mathbb{R}^N\setminus B_{R_\varepsilon}} w(x)\,|u_n - u|^q\, dx + \int_{B_{R_\varepsilon}} w(x)\,|u_n - u|^q\, dx < \varepsilon,$$

as required.

The last part of the lemma follows at once by (2.5). □

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