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The Essential Singularity of the Solution of a Ramified Characteristic Cauchy Problem

By

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§0. Introduction

J. Leray [L] and L. Gårding, T. Kotake and J. Leray [G-K-L] have studied the singularities of the solution of a Cauchy problem with holomorphic data, when the initial surface includes some characteristic points. They have proved that the solution may be ramified around a hypersurface K.

Y. Hamada [H] has studied another class of characteristic Cauchy problem. In his case, the solution may have an essential singularity, although the data are regular.

Let $Pu=v$ be our equation. We already know that we must allow u to be ramified or to have an essential singularity. Now that we understand this necessity, it would be desirable to allow v to be singular without introducing a larger class for u.

[D] and [O-Y] are studies in this direction. They are generalizations of [L] and [G-K-L].

In the present paper, we consider a problem similar to the one in [H]. Although we impose a stronger condition on the operator P than in [H], we assume a weaker condition on v: it is allowed to be singular. Moreover, by employing a symbol calculus like the one in [D], we can explain easily why u has an essential singularity even for a holomorphic v.

§1. Statement of the Results

Let S and K be the hypersurfaces in $\mathbb{C}^n_x$ defined by $x_1=x_2^q$ and $x_1=0$ respec-

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tively, where q is an integer ≥ 2. We introduce a class of the stalk of ramified functions at $x=0$, denoted by $\mathcal{N}_{q,K}$. It is defined by

$$f(x) \in \mathcal{N}_{q,K} \Longleftrightarrow f(x)=\sum_{j=0}^{q-1} f_j(x)\, x_1^{j/q}, \quad f_j \text{ is holomorphic near } x=0.$$

We set

$$\mathcal{N}^{\,l}_{q,K}=\{f(x) \in \mathcal{N}_{q,K};\ \ f \ \text{ vanishes on } S \text{ up to order } l\} \qquad (l\geq 0).$$

Moreover, we set

$$\widetilde{\mathcal{N}}_{q,K}=\sum_{j=0}^{q-1} x_1^{j/q} \varinjlim_{X\ni 0} \mathcal{O}(X\backslash K).$$

A function in $\widetilde{\mathcal{N}}_{q,K}$ may be ramified and have an essential singularity.

To formulate a Cauchy problem, we introduce

$$\widetilde{\mathcal{N}}^{\,l}_{q,K}=\{f \in \widetilde{\mathcal{N}}_{q,K};\ \ f \text{ vanishes on } S \text{ up to order } l\} \qquad (l\geq 0).$$

We have

Theorem 1. *Let $P(x, D)$ be a differential operator near the origin*

$$P(x, D)=D_1^{A_1} D_2^{A_2} - \sum_{|\alpha|<A_1+A_2} D^{\alpha} a_{\alpha}(x), \qquad A_1\geq 0,\ A_2\geq 0$$

where $a_{\alpha}(x)$ is holomorphic near the origin and is a polynomial in x_1 and x_2. Then, for any element $v(x)$ of $D_1^{A_1}\mathcal{N}^{A_1}_{q,K}$, there exists a unique element $u(x)$ of $\widetilde{\mathcal{N}}^{A_1+A_2}_{q,K}$ such that

$$Pu=v$$

holds.

Remark. If $\sum_{|\alpha|<A_1+A_2} D^{\alpha}a_{\alpha}(x)$ is of order less than A_1 with respect to D_1, then P belongs to the class treated in [O-Y] and the solution u is in $\mathcal{N}^{A_1+A_2}_{q,K}$.

Theorem 2. ([O-Y]) *Assume that $A_1\geq 1$. Then*

(A) $x_1^{-\frac{q-1}{q}}\mathcal{N}_{q,K} \subset D_1^{A_1}\mathcal{N}^{A_1}_{q,K}$. *Equality holds if $A_1=1$.*

(B) $x_1^{-\frac{l}{q}} \notin D_1^{A_1}\mathcal{N}_{q,K}^{A_1}$ *if* $l \geq q$.

The proof of Theorem 2 is given in [O-Y]. In the following, we are going to prove Theorem 1.

§2. The Inverse of a Microdifferential Operator

We review the definition of microdifferential operators and formal norms. For details, see [K-K-K].

Definition 1. Let Ω be a conic open set of $T^*\mathbf{C}_x^n$. We denote by ξ the dual variable of x. Let $P(x, \xi)$ be a formal sum of the following form:

$$P(x, \xi) = \sum_{k=0}^{\infty} p_{m-k}(x, \xi),$$

where $p_{m-k}(x, \xi)$ is holomorphic in Ω and is homogeneous of degree $m-k$ with respect to ξ. Then $P(x, \xi)$ is said to be a microdifferential operator of order m in Ω if it satisfies the following growth condition:
For an arbitrary compact subset K in Ω, there exists a positive constant C_K such that

$$\text{(G)} \qquad |p_{m-k}(x, \xi)| \leq C_K^{k+1}\, k!.$$

We sometimes write $P(x, \xi)$ as $P(x, D)$.

The correspondence

$$\Omega \mapsto \{P(x, D);\quad P \text{ is a microdifferential operator of order } m \text{ in } \Omega\}$$

forms a sheaf on $T^*\mathbf{C}^n$, which we denote by $\mathscr{E}(m)$.

In the calculus of microdifferential operators, formal norms defined in [Bou-Kr] are very useful.

Definition 2. In the situation of Definition 1, the formal norm $N_m^K(P; t)$ is a formal sum defined as

$$N_m^K(P;\, t) = \sum_{k,\alpha,\beta} \frac{2(2n)^{-k} k!}{(|\alpha|+k)!(|\beta|+k)!} \sup_K |D_x^\alpha D_\xi^\beta p_{m-k}(x, \xi)|\, t^{2k+|\alpha+\beta|},$$

where the sum is taken with respect to $k \in \mathbb{N}_0=\{0,1,2,\ldots\}$, $\alpha, \beta \in \mathbb{N}_0^n$.

Remark. If $N_m^K(P; \varepsilon)<\infty$ holds for some $\varepsilon>0$, then the growth condition (G) is satisfied. Conversely, if (G) is satisfied, then $N_m^{K'}(P; \varepsilon)<\infty$ for some $K'\subset K$ and $\varepsilon>0$.

We quote two lemmas from [Y].

Lemma 1. (Lemma 10 of [Y]) *Let $R(x, D)$ be a microdifferential operator of order $\leq -j<0$ defined in a neighborhood of a compact set $\omega\subset T^*\mathbb{C}_x^n$, where j is a positive integer. Then we have*

$$N_0^{\omega}(R; t) \ll \frac{(2n)^{-j}}{j!} t^{2j} N_{-j}^{\omega}(R; t).$$

Proof. By definition,

$$N_0^{\omega}(R; t)=\sum_{k,\alpha,\beta} \frac{2(2n)^{-k}k!}{(|\alpha|+k)!(|\beta|+k)!} \sup_{\omega} |D_x^{\alpha} D_{\xi}^{\beta} r_{-k}(x, \xi)| t^{2k+|\alpha+\beta|},$$

where $R=\sum_{k\geq 0} r_{-k}$ and r_{-k} is the homogeneous part of degree $-k$. There is no contribution by the terms corresponding to $k=0, 1, 2, \ldots, j-1$. Hence, if we put $l=k-j$,

$$N_0^{\omega}(R; t)=\sum_{l\geq 0,\alpha,\beta} \frac{2(2n)^{-(l+j)}(l+j)!}{(|\alpha|+l+j)!(|\beta|+l+j)!} \times \sup_{\omega} |D_x^{\alpha} D_{\xi}^{\beta} r_{-(l+j)}(x, \xi)| t^{2(l+j)+|\alpha+\beta|}.$$

We have only to prove that

$$\frac{2(2n)^{-(l+j)}(l+j)!}{(|\alpha|+l+j)!(|\beta|+l+j)!} \leq \frac{(2n)^{-j}}{j!} \frac{2(2n)^{-l}l!}{(|\alpha|+l)!(|\beta|+l)!}.$$

This inequality is obtained by the calculation below.

$$\frac{2(2n)^{-(l+j)}(l+j)!}{(|\alpha|+l+j)!(|\beta|+l+j)!} \times \frac{(|\alpha|+l)!(|\beta|+l)!}{2(2n)^{-l}l!}$$

$$\leq (2n)^{-j} \times \frac{1}{(|\alpha|+l+j)\cdots(|\alpha|+l+1)} \times \frac{(l+j)\cdots(l+1)}{(|\beta|+l+j)\cdots(|\beta|+l+1)}$$

$$\leq (2n)^{-j} \times \frac{1}{j!} \times 1. \qquad \square$$

Lemma 2. (A special case of Lemma 11 of [Y]) *Let Q be a microdifferential operator of order* ≤ -1. *Then we have*

$$N_0^{\omega}(Q^j;\ t) \ll \frac{(2n)^{-j}}{j!} t^{2j} \{N_{-1}^{\omega}Q;\ t)\}^j.$$

Proof. By [B-Kr], we have $N_{-j}^{\omega}(Q^j) \ll \{N_{-1}^{\omega}(Q)\}^j$. Lemma 2 follows from Lemma 1. □

Now let us consider P in Theorem 1. Define a microdifferential operator $\widetilde{P}(x, D)$ by

$$\widetilde{P}(x, D) = D_1^{-A_1} D_2^{-A_2} P(x, D).$$

Obviously we have

$$\widetilde{P} = 1 - \sum_{|\alpha| < A_1 + A_2} D_1^{-A_1} D_2^{-A_2} D^{\alpha} a_{\alpha}(x),$$

and its adjoint $\widetilde{P}^*$ is given by

$$\widetilde{P}^*(x, D) = 1 - \sum_{|\alpha| < A_1 + A_2} a_{\alpha}(x)(-D_1)^{-A_1}(-D_2)^{-A_2}(-D)^{\alpha}.$$

The summation is of order ≤ -1. The inverse of $\widetilde{P}^*$, which we denote by R, is calculated in terms of Neumann series:

$$R = (\widetilde{P}^*)^{-1} = \sum_{j=0}^{\infty} Q(x, D)^j$$

where

$$Q(x, D) = \sum_{|\alpha| < A_1 + A_2} a_{\alpha}(x)(-D_1)^{-A_1}(-D_2)^{-A_2}(-D)^{\alpha} \in \mathscr{E}(-1).$$

Let q_{jk} be the homogeneous term of degree $(-k)$ of Q^j: i.e.

$$Q(x, D)^j = \sum_{k=j}^{\infty} q_{jk}(x, D) \in \mathscr{E}(-j).$$

In fact, this is a finite sum as we will see later). By lemma 2 and the definition of the formal norm, we have

$$\frac{2(2n)^{-k}\ t^{2k}}{k!}\sup|q_{jk}| \leq \frac{(2n)^{-j}}{j!}t^{2j}\{N_{-1}(Q;\ t)\}^{j} \quad \text{if } t>0$$

(For simplicity, we neglect to specify a compact set). Hence

$$(1) \qquad |q_{jk}| \leq \frac{1}{2}(2n)^{-j+k}\frac{k!}{j!}t^{2(j-k)}\{N_{-1}(Q;\ t)\}^{j}.$$

Next, we show the above-mentioned fact that $Q^{j}=\sum_{k} q_{jk}$ is a finite sum. In fact, we have

Lemma 3. *There exists a positive integer m independent of j such that Q^{j} consists of homogeneous terms of degree $-j,\ -(j+1),\cdots,\ -mj$.*

Proof. A term of the form $a(x)D_1^{\gamma_1}D_2^{\gamma_2}\cdots D_n^{\gamma_n}$ is said to be of type $(s,\ -t)$, $s\in \mathbf{N}_0$, $t\in \mathbf{N}_0=\{0,\ 1,\ 2,\ 3,\cdots\}$, where a is a holomorphic function which is a polynomial in x_1 and x_2 of degree $\leq s$ and $\gamma_1+\cdots+\gamma_n\geq -t$, $\gamma_1\in \mathbf{Z}$, $\gamma_2\in \mathbf{Z}$, $\gamma_3\in \mathbf{N}_0,\ldots,\ \gamma_n\in \mathbf{N}_0$. (If $s'\geq s$ and $t'\geq t$, then a term of type $(s,\ -t)$ is of type $(s',-t')$).

Let a_α's be polynomials in x_1 and x_2 of degree $\leq l$. Then Q consists of terms of type $(l,\ -A)$, $A=A_1+A_2$.

It is easy to see that if $r_1(x,\ D)$ (resp. $r_2(x,\ D)$) is of type $(s_1,\ -t_1)$ (resp. $(s_2,\ -t_2)$), then $r_1(x,\ D)\,r_2(x,\ D)$ consists of terms of type $(s_1+s_2,\ -t_1-t_2)$, $(s_1+s_2-1,\ -t_1-t_2-1),\cdots,(0,\ -s_1-s_2-t_1-t_2)$.

By induction, we can prove that Q^{j} consists of terms of type $(jl,-jA),\cdots,(0,-jl-jA)$. Combining this with the fact that $\operatorname{ord} Q^{j}\leq -j$, we obtain the lemma. □

Let $r_k(x,\ D)$ be the homogeneous term of degree $(-k)$ of the operator $R(x,\ D)=\widetilde{P}^{*}(x,\ D)^{-1}=\sum_{j=0}^{\infty} Q(x,\ D)^{j}$. Then $R=\sum_{k=0}^{\infty} r_k(x,\ D)$ and, by the lemma above,

$$r_k=\sum_{j=\lceil \frac{k}{m}\rceil}^{k} q_{jk}, \quad \text{where } \left\lceil \frac{k}{m}\right\rceil=\min\{n\in \mathbf{N}_0;\ n\geq \frac{k}{m}\}.$$

We employ the estimate (1) to obtain

$$|r_k|\leq \sum_{j=\lceil \frac{k}{m}\rceil}^{k}|q_{jk}|\leq \sum_{j=\lceil \frac{k}{m}\rceil}^{k}\frac{1}{2}(2n)^{-j+k}\frac{k!}{j!}t^{2(j-k)}\ \{N_{-1}(Q;\ t)\}^{j}.$$

By using

$$\frac{1}{j!} \le \frac{1}{\lceil \frac{k}{m} \rceil!} \frac{1}{(j-\lceil \frac{k}{m} \rceil)!},$$

we see that

$$\begin{aligned} |r_k| &\le \frac{1}{2}(2n)^k k! \frac{1}{\lceil \frac{k}{m} \rceil!} t^{-2k} (2n)^{-\lceil \frac{k}{m} \rceil} \{t^2 N_{-1}(Q;\, t)\}^{\lceil \frac{k}{m} \rceil} \\ &\quad \times \sum_{j=\lceil \frac{k}{m} \rceil}^{k} (2n)^{-(j-\lceil \frac{k}{m} \rceil)} \frac{1}{(j-\lceil \frac{k}{m} \rceil)!} \{t^2 N_{-1}(Q;\, t)\}^{j-\lceil \frac{k}{m} \rceil} \\ &\le \frac{1}{2}(2n)^k k! \frac{1}{\lceil \frac{k}{m} \rceil!} t^{-2k} (2n)^{-\lceil \frac{k}{m} \rceil} \{t^2 N_{-1}(Q;\, t)\}^{\lceil \frac{k}{m} \rceil} \\ &\quad \cdot \exp\left\{\frac{1}{2n} t^2 N_{-1}(Q;\, t)\right\}. \end{aligned}$$

Therefore, for any compact set ω of $\{x \in \mathbf{C}^n; |x| \ll 1\} \times \{\xi; \xi_1 \neq 0,\ \xi_2 \neq 0\} \subset T^*\mathbf{C}^n_x$, there exists a positive constant C_ω independent of k such that

$$(2) \qquad \sup_\omega |r_k(x,\, \xi)| \le C_\omega^{k+1} \frac{k!}{\lceil \frac{k}{m} \rceil!}.$$

Here $|x| \ll 1$ means that $|x|$ is sufficiently small. Now set

$$r_k(x,\, D) = \sum_{|\beta| = -k} b_\beta(x) D^\beta \in \mathscr{E}(\{|x| \ll 1\} \times \{\xi_1 \neq 0,\ \xi_2 \neq 0\}).$$

Let us obtain an estimate on $b_\beta(x)$ when $\beta_1 > 0\ (\Rightarrow \beta_2 < 0)$. Remark that the partial sum

$$\sum_{k \ge 0} \sum_{\substack{|\beta| = -k \\ \beta_1 \le 0}} b_\beta(x) D^\beta$$

belongs to the class $\mathscr{E}_K$ of [D], and it is already well understood.

Since

$$\begin{aligned} b_\beta(x) = \frac{1}{(2\pi i)^{n-1}} \oint_{|\xi_2| = \delta_2} \oint_{|\xi_3| = \delta'} \cdots \oint_{|\xi_n| = \delta'} & \xi_2^{-\beta_2 - 1} \xi_3^{-\beta_3 - 1} \cdots \xi_n^{-\beta_n - 1} \\ & \times r_k(x;\, 1, \xi_2,\, \xi_3, \ldots, \xi_n)\, d\xi_2 d\xi_3 \cdots d\xi_n, \end{aligned}$$

we obtain, owing to (2)

$$(3) \qquad |b_\beta(x)| \le C_{\delta_2,\delta'}^{k+1} \frac{k!}{\lceil \frac{k}{m} \rceil !} \delta_2^{-\beta_2} \delta'^{-|\beta'|},\ \beta' = (\beta_3, \dots, \beta_n),$$

where $C_{\delta_2,\delta'}$ is a positive constant independent of k.

Before concluding this section, we remark that

$$\widetilde{P}^{-1} = R^* = \sum_{k=0}^{\infty} \{r_k(x, D)\}^* = \sum_{k=0}^{\infty} \sum_{|\beta|=-k} (-D)^\beta b_\beta(x).$$

§3. Some Preparation

Lemma 4.

$$\left(\frac{1}{z^{q-1}} D_z\right)^j = \frac{1}{z^{qj}} \{\theta - q(j-1)\} \cdots \{\theta - q\}\, \theta, \quad j \ge 1$$

where $\theta = zD_z$.

Proof. One has

$$\theta \frac{1}{z^k} = \frac{1}{z^k} \theta - z \frac{k}{z^{k+1}} = \frac{1}{z^k} (\theta - k).$$

The lemma is proved by induction. □

Lemma 5. *Let j be a positive integer. We have for* $0 < y < 1$,

$$\sum_{k=0}^{\infty} \underbrace{\{k+q(j-1)\} \cdots \{k+q\}\, k}_{j \text{ factors}} y^k \le \frac{j!\, y^q}{(1-y)\{y^{q-1}(1-y)\}^j}$$

Proof. In fact,

$$\sum_{k=0}^{\infty} \{\underbrace{k+q(j-1)\} \cdots \{k+q\}\, k}_{j \text{ factors}} y^k$$

$$\le \sum_{k=0}^{\infty} \underbrace{\{k+q(j-1)\}\{k+qj-q-1\} \cdots \{k+(q-1)(j-1)\}}_{j \text{ factors}} y^k$$

$$= \frac{1}{y^{qj-q-j}} \frac{d^j}{dy^j} \sum_{k=0}^{\infty} y^{k+q(j-1)}$$

$$\leq \frac{y^q}{y^{(q-1)j}} \frac{d^j}{dy^j} (1+y+y^2+\cdots)$$
$$= \frac{y^q}{y^{(q-1)j}} \frac{j!}{(1-y)^{j+1}}$$

□

Lemma 6. *Let $f(z)$ be a holomorphic function in $\{z \in \mathbf{C};\ |z|<r+\varepsilon\}$, $r>0$, $\varepsilon>0$. If $|f(z)| \leq M$ holds in $\{z \in \mathbf{C};\ |z| \leqq r\}$ then we have, in $\{z \in \mathbf{C};\ 0<|z|<r\}$,*

$$\left|\left(\frac{1}{z^{q-1}}D_z\right)^j f(z)\right| \leq M \frac{j!\left(\frac{|z|}{r}\right)^q}{\left(1-\frac{|z|}{r}\right)\left\{|z|^q\left(\frac{|z|}{r}\right)^{q-1}\left(1-\frac{|z|}{r}\right)\right\}^j}.$$

Proof. Let the Taylor expansion of f be

$$f(z) = \sum_{k=0}^{\infty} f_k z^k.$$

Then we have

$$f_k = \frac{1}{2\pi i} \oint_{|z|=r} \frac{f(z)}{z^{k+1}}\, dz, \qquad |f_k| \leq Mr^{-k}.$$

By using Lemma 4 we see that

$$\left(\frac{1}{z^{q-1}} D_z\right)^j f(z) = \sum_{k=0}^{\infty} f_k \frac{1}{z^{qj}} \{k-q(j-1)\} \cdots \{k-q\}\, kz^k.$$

The series in the right hand side is estimated by Lemma 5. We obtain

$$\left|\left(\frac{1}{z^{q-1}}D_z\right)^j f(z)\right|$$
$$\leq \sum_{k=0}^{\infty} Mr^{-k} \frac{1}{|z|^{qj}} \{k+q(j-1)\} \cdots \{k+q\}\, k|z|^k$$
$$= \frac{M}{|z|^{qj}} \frac{j!\left(\frac{|z|}{r}\right)^q}{\left(1-\frac{|z|}{r}\right)\left\{\left(\frac{|z|}{r}\right)^{q-1}\left(1-\frac{|z|}{r}\right)\right\}^j}.$$

□

§4. The Action of a Microdifferential Operator on a Ramified Function

For the study of $\mathcal{N}_{q,K}$, we introduce a singular coordinate change $z=x_1^{1/q}$. We denote by $\widetilde{S}$ the hypersurface of $\mathbf{C}^n_{z,x_2,x'}$ defined by $z=x_2$. Here $x'=(x_3,\cdots,x_n)$. The singular coordinate change induces an isomorphism

$$\mathcal{N}_{q,K} \simeq \mathcal{O}_{(z,x_2,x')=0}$$
$$f(x)=\sum_{j=0}^{q-1} f_j(x)\, x_1^{j/q} \mapsto \widetilde{f}(z,\ x_2,\ x')=\sum_{j=0}^{q-1} f_j(z^q,\ x_2,\ x')\, z^j.$$

Moreover $f \in \mathcal{N}^{\,l}_{q,K}$ if and only if $\widetilde{f}$ vanishes on $\widetilde{S}$ up to order l.

Proposition 1. ([D]) Proposition 6) *The characteristic Cauchy problem*

$$D_2 2=f \in \mathcal{N}^{\,l}_{q,K}$$

admits a unique solution $g \in \mathcal{N}^{\,l+1}_{q,K}$. *Moreover, if we have*

$$|\widetilde{f}(z,\ x_2,\ x')| \leq M\{|z|+|x_2-z|\}^m$$

for some positive constant M *and a non-negative integer* m, *then*

$$|\widetilde{g}(z,\ x_2,\ x')| \leq \frac{M}{m+1}\{|z|+|x_2-z|\}^{m+1}.$$

Proof. The equation $D_2 g=f$ is equivalent to $D_2\widetilde{g}=\widetilde{f}$, and the initial surface S is transformed into $\widetilde{S}$. Since $\widetilde{S}$ is *noncharacteristic*, we can find a unique holomorphic solution $\widetilde{g}$. The estimate is obtained by an elementary integral representation. □

This proposition suggests that $\mathcal{N}_{q,K}$ and its variants are more suitable classes for the study of characteristic Cauchy problems than that of holomorphic functions.

Definition 3. We can define

$$D_2^{-1}:\mathcal{N}^{\,l}_{q,K} \longrightarrow \mathcal{N}^{\,l+1}_{q,K}$$

by using the proposition above. It is a right inverse of

$$D_2 : \mathcal{N}^{l+1}_{q,K} \longrightarrow \mathcal{N}^{l}_{q,K},$$

but it is not a left inverse.

Remark. If u is an element of $\mathcal{N}_{q,K}$ and f is holomorphic near $x=0$, then we can define $D_2^{-l}(f(x)u(x))$, $l \in \mathbf{N}_0$. It is the unique solution of the Cauchy problem

$$\begin{cases} D_2^l w(x) = f(x)u(x) \\ w(x) \in \mathcal{N}^{l}_{q,K}. \end{cases}$$

On the other hand, $D_2^{-l} \circ f(x)$ belongs to the symbol class $\mathscr{E}_K$ in [D], and $(D_2^{-l} \circ f(x))u(x) \in \mathcal{N}^{l}_{q,K}$ is defined in [D]. Dunau puts integration on the right:

$$D_2^{-l} \circ f(x) = f(x) D_2^{-l} + \sum_{j=l+1}^{\infty} f_j(x) D_2^{-j}$$

for some $f_j(x)$. He sets

$$(D_2^{-l} \circ f(x))u(x) \underset{\text{def}}{=} f(x) D_2^{-l} u(x) + \sum_{j=l+1}^{\infty} f_j(x) D_2^{-j} u(x).$$

It satisfies the same equation as above and we see that

$$D_2^{-l}(f(x)u(x)) = (D_2^{-l} \circ f(x))u(x).$$

So it makes no difference whether integration comes on the left or on the right.

Now we are ready to define $\widetilde{P}(x, D)^{-1} w(x) \in \widetilde{\mathcal{N}}_{q,K}$, where $\widetilde{P}$ is as in the second section and $w(x) \in \mathcal{N}_{q,K}$.

$\widetilde{P}^{-1}$ has the expression

$$\widetilde{P}^{-1} = \sum_{k=0}^{\infty} \sum_{|\beta|=-k} (-D)^{\beta} b_{\beta}(x) \in \mathscr{E}(\{|x| \ll 1, \xi_1 \neq 0, \xi_2 \neq 0\}), \text{ ord } \widetilde{P}^{-1} \leq 0.$$

The partial sum consisting of the terms corresponding to $\beta_1 \leq 0$ belongs to Dunau's class $\mathscr{E}_K$ and its action on $\mathcal{N}_{q,K}$ is defined in [D]. Therefore, in order to define the action of $\widetilde{P}^{-1}$, we may assume without loss of generality that $b_\beta \equiv 0$ if $\beta_1 \leq 0$. This means that $\beta_2 < 0$ in the sum.

We set

$$\widetilde{P}^{-1}(x, D)\, w(x) = \sum_{k=0}^{\infty} \sum_{|\beta|=-k} (-D)^{\beta} b_{\beta}(x)\, w(x).$$

We are going to prove that it defines an element of $\widetilde{\mathcal{N}}_{q,K}$. Put $x_1^{1/q} = z$, $\widetilde{w}(z, x_2, x') = w(z^q, x_2, x')$, and $\widetilde{b}_{\beta}(z, x_2, x') = b_{\beta}(z^q, x_2, x')$. Then

$$(\widetilde{P}^{-1}w)(x) = \sum_{k=0}^{\infty} \sum_{|\beta|=-k} \left(\frac{1}{qz^{q-1}} D_z\right)^{\beta_1} D_2^{\beta_2} D'^{\beta'} \cdot (-1)^{|\beta|}\, \widetilde{b}_{\beta}(z, x_2, x')\, \widetilde{w}(z, x_2, x').$$

(3) in the second section implies that in a neighborhood X of $(z, x_2, x')=0$, we have

$$\left|(-1)^{|\beta|}\, \widetilde{b}_{\beta}\, \widetilde{w}\right| \leq C_{\delta_2,\delta'}^{k+1} \cdot \frac{k!}{\lceil \frac{k}{m} \rceil !}\, \delta_2^{-\beta_2} \delta'^{-|\beta'|} \sup_X |\widetilde{w}|, \quad |\beta| = -k.$$

In a smaller neighborhood, there exists a positive constant $r'>0$ such that

$$\left|D'^{\beta'} \circ (-1)^{|\beta|}\, \widetilde{b}_{\beta}\, \widetilde{w}\right| \leq \beta'!\, r'^{-|\beta'|} C_{\delta_2,\delta'}^{k+1} \cdot \frac{k!}{\lceil \frac{k}{m} \rceil !}\, \delta_2^{-\beta_2} \delta'^{-|\beta'|} \sup_X |\widetilde{w}|.$$

Then, we employ Proposition 1 repeatedly, first for $m=0$, next for $m=1$ and so on. We obtain

$$\left|D_2^{\beta_2} D'^{\beta'} \circ (-1)^{|\beta|}\, \widetilde{b}_{\beta}\, \widetilde{w}\right| \leq \frac{\lambda^{-\beta_2}}{(-\beta_2)!} \beta'!\, r'^{-|\beta'|} C_{\delta_2,\delta'}^{k+1} \cdot \frac{k!}{\lceil \frac{k}{m} \rceil !}\, \delta_2^{-\beta_2} \delta'^{-|\beta'|} \sup_X |\widetilde{w}|$$

in $\{|z|<\lambda/3,\ |x_2|<\lambda/3,\ \dots,\ |x_n|<\lambda/3\}$.

By using Lemma 6, we see that

$$\text{(4)} \qquad \left|\left(\frac{1}{qz^{q-1}} D_z\right)^{\beta_1} D_2^{\beta_2} D'^{\beta'} \circ (-1)^{|\beta|}\, \widetilde{b}_{\beta}\, \widetilde{w}\right|$$
$$\leq \frac{\beta_1!\left(\frac{|z|}{r}\right)^{q}}{\left(1-\frac{|z|}{r}\right)\left\{q|z|^{q}\left(\frac{|z|}{r}\right)^{q-1}\left(1-\frac{|z|}{r}\right)\right\}^{\beta_1}}$$
$$\times \frac{\lambda^{-\beta_2}}{(-\beta_2)!} \beta'!\, r'^{-|\beta'|} C_{\delta_2,\delta'}^{k+1} \cdot \frac{k!}{\lceil \frac{k}{m} \rceil !} \delta_2^{-\beta_2} \delta'^{-|\beta'|} \sup_X |\widetilde{w}|$$

in $\{0<|z|<r<\lambda/3, |x_2|<\lambda/3, \ldots, |x_n|<\lambda/3\}$.

There exists a constant $C_z>1$ depending continuously on $|z|$, $0<|z|<r$, such that $\left\{q|z|^q(\frac{|z|}{r})^{q-1}(1-\frac{|z|}{r})\right\}^{-1}\leq C_z$. We have

$$\frac{1}{\left\{q|z|^q(\frac{|z|}{r})^{q-1}(1-\frac{|z|}{r})\right\}^{\beta_1}}\leq C_z^{\beta_1}\leq C_z^{\beta_1+|\beta'|+k}=C_z^{-\beta_2}.$$

Moreover, if we take $\delta'>0$ so small that $r'\delta'<1$, then

$$(r'\delta')^{-|\beta'|}\leq(r'\delta')^{-|\beta'|-\beta_1-k}=(r'\delta')^{\beta_2}.$$

In addition, it is easy to see that

$$\frac{\beta_1!\beta'!k!}{(-\beta_2)!}\leq 1$$

because $\beta_1+|\beta'|+k=-\beta_2$. Combining (4) with these three inequalities, we obtain

$$\begin{aligned}&\left|\left(\frac{1}{qz^{q-1}}D_z\right)^{\beta_1}D_2^{\beta_2}D'^{\beta'}\cdot(-1)^{|\beta|}\tilde{b}_\beta\tilde{w}\right|\\&\leq\frac{\left(\frac{|z|}{r}\right)^q}{\left(1-\frac{|z|}{r}\right)\cdot\lceil\frac{k}{m}\rceil!}\left(\frac{C_z\lambda\delta_2}{r'\delta'}\right)^{-\beta_2}C_{\delta_2,\delta'}^{k+1}\sup_X|\tilde{w}|.\end{aligned}$$

For fixed k and β_2, we have

$$\#\{(\beta_1,\beta');\beta_1>0,\beta'\in\mathbf{N}_0^{n-2},\beta_1+\beta_2+|\beta'|=-k\}\leq 2^{n-2-k-\beta_2}.$$

Hence,

$$\begin{aligned}&\left|\sum_{k\geq0}\sum_{|\beta|=-k}\left(\frac{1}{qz^{q-1}}D_z\right)^{\beta_1}D_2^{\beta_2}D'^{\beta'}\cdot(-1)^{|\beta|}\tilde{b}_\beta\tilde{w}\right|\\&\leq\frac{\left(\frac{|z|}{r}\right)^q\sup_X|\tilde{w}|}{1-\frac{|z|}{r}}\sum_{k\geq0}\frac{C_{\delta_2,\delta'}^{k+1}}{\lceil\frac{k}{m}\rceil!}\sum_{\beta_2=-\infty}^{-1}\sum_{\beta_1+|\beta'|=-k-\beta_2}\left(\frac{C_z\lambda\delta_2}{r'\delta'}\right)^{-\beta_2}\\&\leq\frac{2^{n-2}\left(\frac{|z|}{r}\right)^q\sup_X|\tilde{w}|}{1-\frac{|z|}{r}}\sum_{k\geq0}\frac{2^{-k}C_{\delta_2,\delta'}^{k+1}}{\lceil\frac{k}{m}\rceil!}\sum_{\beta_2=-\infty}^{-1}\left(\frac{2C_z\lambda\delta_2}{r'\delta'}\right)^{-\beta_2}\end{aligned}$$

The right hand side converges on every compact set of $\{0<|z|\ll 1, |x_2|\ll 1, \dots, |x_n|\ll 1\}$ if we take a sufficiently small $\delta_2>0$ in accordance with the compact set.

Summing up, we have finally proved that

$$(\widetilde{P}^{-1}w)(x) \in \widetilde{\mathcal{N}}_{q,K}.$$

Moreover, if $w \in \mathcal{N}_{q,K}^{A_1+A_2}$, then it is easy to see that

$$(\widetilde{P}^{-1}w)(x) \in \widetilde{\mathcal{N}}_{q,K}^{A_1+A_2}.$$

§5. Proof of Theorem 1

First, remark that

$$D_2^{A_2}: \mathcal{N}_{q,K}^{A_1+A_2} \xrightarrow{\sim} \mathcal{N}_{q,K}^{A_1}.$$

Hence

$$D_1^{A_1}\mathcal{N}_{q,K}^{A_1} = D_1^{A_1}D_2^{A_2}\mathcal{N}_{q,K}^{A_1+A_2}.$$

Let us solve $Pu = D_1^{A_1}D_2^{A_2}w$, $w \in \mathcal{N}_{q,K}^{A_1+A_2}$. The solution u is given by $u = \widetilde{P}^{-1}w \in \widetilde{\mathcal{N}}_{q,K}^{A_1+A_2}$. In fact,

$$Pu = P(\widetilde{P}^{-1}w) = D_1^{A_1}D_2^{A_2}w$$

holds.

The uniqueness is a consequence of Cauchy-Kowalevski theorem, which we apply at noncharacteristic points.

§6. Hamada's Example

Hamada ([H]) gave the following example.

$$\begin{cases} (D_2^2 - D_1)\, u(x) = 0 \\ u|_S = \gamma_1 x_2^3,\ D_1 u|_S = \gamma_2 x_2 \end{cases}$$

where

$$S=\{x_1=x_2^2\},\quad \gamma_1=\sum_{m=0}^{\infty}(-1)^m\frac{\Gamma(m-\frac{3}{2})}{(2m)!},\quad \gamma_2=\sum_{m=0}^{\infty}(-1)^{m+1}\frac{\Gamma(m-\frac{1}{2})}{(2m)!}.$$

The solution $u(x)$ is given by

$$u(x)=\sum_{m=0}^{\infty}(-1)^m\frac{\Gamma(m-\frac{3}{2})}{(2m)!}x_1^{\frac{3}{2}-m}x_2^{2m}.$$

It is ramified and has an essential singularity. Let us interpret this phenomenon from our viewpoint. First we reduce the problem to the following one.

$$\begin{cases}(D_2^2-D_1)u(x)=v(x),\ v\in\mathcal{O}_{x=0} \text{ is given,}\\ u|_S=0,\qquad D_1u|_S=0.\end{cases}$$

By using

$$\begin{aligned}(D_2^2-D_1)^{-1}&=(1-D_1D_2^{-2})^{-1}D_2^{-2}\\ &=\sum_{j=0}^{\infty}(D_1D_2^{-2})^jD_2^{-2}=\sum_{j=0}^{\infty}D_1^jD_2^{-2j-2},\end{aligned}$$

we can express the solution by

$$u(x)=\sum_{j=0}^{\infty}D_1^jD_2^{-2j-2}v(x).$$

Put $z=x_1^{1/2}$. Then we obtain

$$u(z^2,x_2,x')=\sum_{j=0}^{\infty}\left(\frac{1}{2z}D_z\right)^jD_2^{-2j-2}v(z^2,x_2,x').$$

Ramification is caused by D_2^{-2j-2}. The essential singularity appears because of the factor $(\frac{1}{2z}D_z)^j$.

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