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A two weight theorem for α-fractional singular integrals with an energy side condition

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Abstract. Let σ and ω be locally finite positive Borel measures on $\mathbb{R}^n$ with no common point masses, and let T^α be a standard α-fractional Calderón–Zygmund operator on $\mathbb{R}^n$ with $0 \le \alpha < n$. Furthermore, assume as side conditions the $\mathcal{A}_2^\alpha$ conditions and certain α-*energy conditions*. Then we show that T^α is bounded from $L^2(\sigma)$ to $L^2(\omega)$ if the cube testing conditions hold for T^α and its dual, and if the weak boundedness property holds for T^α.

Conversely, if T^α is bounded from $L^2(\sigma)$ to $L^2(\omega)$, then the testing conditions hold, and the weak boundedness condition holds. If the vector of α-fractional Riesz transforms $\mathbf{R}_\sigma^\alpha$ (or more generally a strongly elliptic vector of transforms) is bounded from $L^2(\sigma)$ to $L^2(\omega)$, then the $\mathcal{A}_2^\alpha$ conditions hold. We do not know if our energy conditions are necessary when $n \ge 2$.

The innovations in this higher dimensional setting are the control of functional energy by energy modulo $\mathcal{A}_2^\alpha$, the necessity of the $\mathcal{A}_2^\alpha$ conditions for elliptic vectors, the extension of certain one-dimensional arguments to higher dimensions in light of the differing Poisson integrals used in $\mathcal{A}_2$ and energy conditions, and the treatment of certain complications arising from the Lacey–Wick monotonicity lemma. The main obstacle in higher dimensions is thus identified as the pair of energy conditions.

1. Introduction

In this paper we prove a two weight inequality for standard α-fractional Calderón–Zygmund operators T^α in Euclidean space $\mathbb{R}^n$, where we assume the n-dimensional $\mathcal{A}_2^\alpha$ conditions and certain α-*energy conditions* as side conditions (in higher dimensions the Poisson kernels used in these two conditions differ). In particular, we show that for locally finite Borel measures σ and ω with no common point masses, and *assuming* the energy conditions in the Theorem below, a strongly elliptic collection of standard α-fractional Calderón–Zygmund operators $\mathbf{T}^\alpha$ is bounded

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from $L^2(\sigma)$ to $L^2(\omega)$,

$$\|\mathbf{T}^\alpha(f\sigma)\|_{L^2(\omega)} \lesssim \|f\|_{L^2(\sigma)}, \tag{1.1}$$

(with $0 \le \alpha < n$) if and only if the $\mathcal{A}_2^\alpha$ conditions hold, the cube testing conditions for $\mathbf{T}^\alpha$ hold, and the weak boundedness property for $\mathbf{T}^\alpha$ holds. This identifies the culprit in higher dimensions as the pair of energy conditions. We point out that these energy conditions are implied by higher dimensional analogues of essentially all the other side conditions used previously in two weight theory, in particular doubling conditions and the Energy Hypothesis (1.16) in [3].

The final argument by M. Lacey ([1]) in the proof of the Nazarov–Treil–Volberg conjecture for the Hilbert transform is the culmination of a large body of work on two-weighted inequalities beginning with the work of Nazarov, Treil and Volberg ([10], [11], [12], [13] and [23]) and continuing with that of Lacey and the authors ([2], [3], [4] and [5]), just to mention a few. See the references for further work. We consider standard singular integrals T, as well as their α-fractional counterparts T^α, and include

1. the control of the functional energy condition by the energy condition modulo $\mathcal{A}_2^\alpha$,
2. a proof of the necessity of the $\mathcal{A}_2^\alpha$ condition for the boundedness of the vector of α-fractional Riesz transforms $\mathbf{R}^{\alpha,n}$,
3. the extensions of certain one-dimensional arguments to higher dimension in light of the differing Poisson integrals used in the $\mathcal{A}_2^\alpha$ and energy conditions,
4. and the treatment of certain complications arising from the Lacey–Wick monotonicity lemma.

These are the main innovations in this paper. The final point is to adapt the clever stopping time and recursion arguments of M. Lacey [1] to complete the proof of our theorem, but only after splitting the stopping form into two sublinear stopping forms dictated by the right-hand side of the Lacey–Wick monotonicity lemma. The basic idea of the generalization is that all of the decompositions of functions are carried out independently of α, while the estimates of the resulting nonlinear forms depend on the α-Poisson integral and the α-energy conditions.

It turns out that in higher dimensions, there are two natural 'Poisson integrals' P and $\mathcal{P}$ that arise, the usual Poisson integral P that emerges in connection with energy considerations, and a different Poisson integral $\mathcal{P}$ that emerges in connection with size considerations (in dimension $n = 1$ these two coincide). The standard Poisson integral P appears in the energy conditions, and the reproducing Poisson integral $\mathcal{P}$ appears in the $\mathcal{A}_2$ condition. These two kernels coincide in dimension $n = 1$ for the case $\alpha = 0$ corresponding to the Hilbert transform.

Acknowledgement. We are grateful to Michael Lacey for pointing out a number of problems with our arguments and various oversights in the versions of [16], [17] (now withdrawn), [18] on the `arXiv`, including the mistake in our monotonicity lemma, which has been corrected by M. Lacey and B. Wick in [8], and in our consequent adaptation of the stopping time and recursion argument in [1]. See these preprints for some of the details.

Remark 1.1. Version 5 of this paper [16] appeared in the *arXiv* in May 9, 2013, with essentially the same results as appear here, except for a serious error in the monotonicity lemma. We thank both M. Lacey and B. Wick for pointing us to this error, as well as various others occurring in versions 1 through 4 of [16]. Seven months after version 5 appeared, M. Lacey and B. Wick posted version 1 of their paper [8] claiming to prove the same $T1$ theorem we had claimed in our earlier versions, and which had significant overlap with version 5 of [16], but they did not acknowledge this overlap there, and only referred to our work in version 2 of [8].

The monotonicity Lemma 6.1 here is due to Lacey and Wick in Lemma 4.2 of [8]; Lemma 7.1 here is proved in [8], but with the larger bound $\mathcal{A}_2^\alpha$ there in place of our bound A_2^α; and an argument treating the additional term in the Lacey–Wick monotonicity lemma as it arises in functional energy is essentially in [8]. We note that the side condition in [8] – uniformly full dimension – permits a reversal of energy, something not assumed in this paper, that implies our energy conditions.

Finally we point to more recent results to be found in our papers [20] and [21], and with M. Lacey and B. Wick in [7].

2. Statements of results

Now we turn to a precise description of our two weight theorem. We will prove a two weight inequality for standard α-fractional Calderón–Zygmund operators T^α in Euclidean space $\mathbb{R}^n$, where we assume the n-dimensional $\mathcal{A}_2^\alpha$ and certain α*-energy conditions* as side conditions. In higher dimensions the Poisson kernels $\mathcal{P}^\alpha$ and P^α used in defining these two conditions differ. In particular, we show that for locally finite Borel measures σ and ω in $\mathbb{R}^n$ with no common point masses, and assuming that both the *energy condition* and its dual hold, a strongly elliptic vector of standard α-fractional Calderón–Zygmund operators $\mathbf{T}^\alpha$ is bounded from $L^2(\sigma)$ to $L^2(\omega)$ *if and only if* the $\mathcal{A}_2^\alpha$ conditions hold, along with the cube testing conditions and the weak boundedness property. In order to state our theorem precisely, we need to define standard fractional singular integrals, the two different Poisson kernels, and an energy condition sufficient for use in the proof of the two weight theorem. These are introduced in the following subsections.

2.1. Standard fractional singular integrals

Let $0 \leq \alpha < n$. Consider a kernel function $K^\alpha(x,y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying the following fractional size and smoothness conditions of order $1+\delta$ for some $\delta > 0$,

$$\begin{aligned}
|K^\alpha(x,y)| &\leq C_{CZ}\,|x-y|^{\alpha-n}, \\
|\nabla K^\alpha(x,y)| &\leq C_{CZ}\,|x-y|^{\alpha-n-1}, \\
|\nabla K^\alpha(x,y)-\nabla K^\alpha(x',y)| &\leq C_{CZ}\left(\frac{|x-x'|}{|x-y|}\right)^\delta |x-y|^{\alpha-n-1}, \quad \frac{|x-x'|}{|x-y|}\leq\frac{1}{2}, \\
|\nabla K^\alpha(x,y)-\nabla K^\alpha(x,y')| &\leq C_{CZ}\left(\frac{|y-y'|}{|x-y|}\right)^\delta |x-y|^{\alpha-n-1}, \quad \frac{|y-y'|}{|x-y|}\leq\frac{1}{2}.
\end{aligned} \tag{2.1}$$

Then we define a standard α-fractional Calderón–Zygmund operator associated with such a kernel as follows.

Definition 2.1. We say that T^α is a *standard α-fractional integral operator with kernel K^α* if T^α is a bounded linear operator from some $L^p(\mathbb{R}^n)$ to some $L^q(\mathbb{R}^n)$ for some fixed $1 < p \le q < \infty$, that is

$$\|T^\alpha f\|_{L^q(\mathbb{R}^n)} \le C\,\|f\|_{L^p(\mathbb{R}^n)}, \quad f \in L^p(\mathbb{R}^n),$$

if $K^\alpha(x,y)$ is defined on $\mathbb{R}^n \times \mathbb{R}^n$ and satisfies (2.1), and if T^α and K^α are related by

$$T^\alpha f(x) = \int K^\alpha(x,y)\, f(y)\, dy, \qquad \text{a.e.- } x \notin \operatorname{supp} f, \tag{2.2}$$

whenever $f \in L^p(\mathbb{R}^n)$ has compact support in $\mathbb{R}^n$. We say $K^\alpha(x,y)$ is a *standard α-fractional kernel* if it satisfies (2.1).

We note that a more general definition of kernel has only order of smoothness $\delta > 0$, rather than $1+\delta$, but the use of the monotonicity and energy lemmas below requires order of smoothness more than 1. A *smooth truncation* of T^α has kernel $\eta_{\delta,R}(|x-y|)K^\alpha(x,y)$ for a smooth function $\eta_{\delta,R}$ compactly supported in (δ, R), $0 < \delta < R < \infty$, and satisfying standard CZ estimates. A typical example of an α-fractional transform is the α-fractional *Riesz* vector of operators

$$\mathbf{R}^{\alpha,n} = \{R_\ell^{\alpha,n} : 1 \le \ell \le n\}, \quad 0 \le \alpha < n.$$

The Riesz transforms $R_\ell^{n,\alpha}$ are convolution fractional singular integrals $R_\ell^{n,\alpha} f \equiv K_\ell^{n,\alpha} * f$ with odd kernel defined by

$$K_\ell^{\alpha,n}(w) \equiv \frac{w^\ell}{|w|^{n+1-\alpha}} \equiv \frac{\Omega_\ell(w)}{|w|^{n-\alpha}}, \quad w = (w^1, \dots, w^n).$$

The *tangent line truncation* of the Riesz transform $R_\ell^{\alpha,n}$ has kernel $\Omega_\ell(w)\psi_{\delta,R}^\alpha(|w|)$ where $\psi_{\delta,R}^\alpha$ is continuously differentiable on an interval $(0,S)$ with $0 < \delta < R < S$, and where $\psi_{\delta,R}^\alpha(r) = r^{\alpha-n}$ if $\delta \le r \le R$, and has constant derivative on both $(0,\delta)$ and (R,S) where $\psi_{\delta,R}^\alpha(S) = 0$. As shown in the one dimensional case in [6], boundedness of $R_\ell^{n,\alpha}$ with one set of appropriate truncations together with the $\mathcal{A}_2^\alpha$ condition below, is equivalent to boundedness of $R_\ell^{n,\alpha}$ with all truncations.

2.2. Cube testing conditions

The following 'dual' cube testing conditions are necessary for the boundedness of T^α from $L^2(\sigma)$ to $L^2(\omega)$:

$$\mathfrak{T}_{T^\alpha}^2 \equiv \sup_{Q \in \mathcal{Q}^n} \frac{1}{|Q|_\sigma} \int_Q |T^\alpha(\mathbf{1}_Q \sigma)|^2 \omega < \infty,$$
$$(\mathfrak{T}_{T^\alpha}^*)^2 \equiv \sup_{Q \in \mathcal{Q}^n} \frac{1}{|Q|_\omega} \int_Q |(T^\alpha)^*(\mathbf{1}_Q \omega)|^2 \sigma < \infty.$$

2.3. Weak boundedness property

The weak boundedness property for T^α with constant C is given by

$$\left|\int_Q T^\alpha(1_{Q'}\sigma)d\omega\right| \leq \mathcal{WBP}_{T^\alpha}\sqrt{|Q|_\omega |Q'|_\sigma},$$

$$\text{for all cubes } Q, Q' \text{ with } \frac{1}{C} \leq \frac{|Q|^{1/n}}{|Q'|^{1/n}} \leq C,$$
$$\text{and either } Q \subset 3Q' \setminus Q' \text{ or } Q' \subset 3Q \setminus Q.$$

Note that the weak boundedness property is implied by either the *tripled* cube testing condition,

$$\|\mathbf{1}_{3Q}\,\mathbf{T}^\alpha(\mathbf{1}_Q\sigma)\|_{L^2(\omega)} \lesssim \|\mathbf{1}_Q\|_{L^2(\sigma)}, \quad \text{for all cubes } Q \text{ in } \mathbb{R}^n,$$

or the tripled dual cube testing condition. In turn, the tripled cube testing condition can be obtained from the cube testing condition for the truncated weight pairs $(\omega, \mathbf{1}_Q\sigma)$. See also Remark 2.9 below.

2.4. Poisson integrals and $\mathcal{A}_2^\alpha$

Now let μ be a locally finite positive Borel measure on $\mathbb{R}^n$, and suppose Q is a cube in $\mathbb{R}^n$. The two α-fractional Poisson integrals of μ on a cube Q are given by:

$$\mathrm{P}^\alpha(Q,\mu) \equiv \int_{\mathbb{R}^n} \frac{|Q|^{1/n}}{(|Q|^{1/n}+|x-x_Q|)^{n+1-\alpha}}d\mu(x),$$
$$\mathcal{P}^\alpha(Q,\mu) \equiv \int_{\mathbb{R}^n} \left(\frac{|Q|^{1/n}}{(|Q|^{1/n}+|x-x_Q|)^2}\right)^{n-\alpha} d\mu(x).$$

We refer to P^α as the *standard* Poisson integral, and to $\mathcal{P}^\alpha$ as the *reproducing* Poisson integral. Let σ and ω be locally finite positive Borel measures on $\mathbb{R}^n$ with no common point masses, and suppose $0 \leq \alpha < n$. Recall that the classical A_2^α constant is defined by

$$A_2^\alpha \equiv \sup_{Q\in\mathcal{Q}^n} \frac{|Q|_\sigma}{|Q|^{1-\alpha/n}}\,\frac{|Q|_\omega}{|Q|^{1-\alpha/n}}.$$

We now define the *one-tailed* $\mathcal{A}_2^\alpha$ constant using $\mathcal{P}^\alpha$. The energy constants $\mathcal{E}_\alpha$ introduced in the next subsection will use the standard Poisson integral P^α. Let $\mathcal{Q}^n$ denote the collection of all cubes in $\mathbb{R}^n$, and denote by $\mathcal{D}^n$ or simply $\mathcal{D}$ a dyadic grid in $\mathbb{R}^n$.

Definition 2.2. The one-sided constants $\mathcal{A}_2^\alpha$ and $\mathcal{A}_2^{\alpha,*}$ for the weight pair (σ,ω) are given by

$$\mathcal{A}_2^\alpha \equiv \sup_{Q\in\mathcal{Q}^n} \mathcal{P}^\alpha(Q,\sigma)\,\frac{|Q|_\omega}{|Q|^{1-\alpha/n}} < \infty,$$
$$\mathcal{A}_2^{\alpha,*} \equiv \sup_{Q\in\mathcal{Q}^n} \mathcal{P}^\alpha(Q,\omega)\,\frac{|Q|_\sigma}{|Q|^{1-\alpha/n}} < \infty.$$

Convention. We will use the expressions $|Q|^{1/n}$ and $\ell(Q)$ interchangeably to denote the side length of a cube Q in $\mathbb{R}^n$.

2.5. Good grids and energy conditions

Given a dyadic cube $K \in \mathcal{D}$ and a positive measure μ we define the Haar projection $\mathsf{P}_K^\mu \equiv \sum_{J\in\mathcal{D}:\, J\subset K} \triangle_J^\mu$ on K by

$$\mathsf{P}_K^\mu f = \sum_{J\in\mathcal{D}:J\subset K} \sum_{a\in\Gamma_n} \langle f, h_J^{\mu,a}\rangle_\mu h_J^{\mu,a} \quad \text{and} \quad \|\mathsf{P}_K^\mu f\|_{L^2(\mu)}^2 = \sum_{J\in\mathcal{D}:J\subset K} \sum_{a\in\Gamma_n} |\langle f, h_J^{\mu,a}\rangle_\mu|^2,$$

and where a Haar basis $\{h_J^{\mu,a}\}_{a\in\Gamma_n \text{ and } J\in\mathcal{D}}$ adapted to the measure μ is defined in the section on a weighted Haar basis below. Now we recall the definition of a *good* dyadic cube −see [13] and [3] for more detail.

Definition 2.3. Let $\mathbf{r} \in \mathbb{N}$ and $0 < \varepsilon < 1$. A dyadic cube J is $(\mathbf{r},\varepsilon)$-*good*, or simply *good*, if for *every* dyadic supercube I, it is the case that **either** J has side length at least $2^{-\mathbf{r}}$ times that of I, **or** $J \Subset_{\mathbf{r}} I$ is $(\mathbf{r},\varepsilon)$-deeply embedded in I.

Here we say that a dyadic cube J is $(\mathbf{r},\varepsilon)$-*deeply embedded* in a dyadic cube K, or simply $\mathbf{r}$-*deeply embedded* in K, which we write as $J \Subset_{\mathbf{r}} K$, when $J \subset K$ and both

$$|J|^{1/n} \le 2^{-\mathbf{r}}\,|K|^{1/n}, \quad \text{and} \quad \operatorname{dist}(J,\partial K) \ge \frac{1}{2}\,|J|^{\varepsilon/n}|\,K|^{(1-\varepsilon)/n}. \tag{2.3}$$

We say that J is $\mathbf{r}$-*nearby* in K when $J \subset K$ and

$$|J|^{1/n} > 2^{-\mathbf{r}}\,|K|^{1/n}.$$

The parameters $\mathbf{r},\varepsilon$ will be fixed sufficiently large and small respectively later in the proof, and we denote the set of such good dyadic cubes by $\mathcal{D}_{\text{good}}$. Throughout the proof, it will be convenient to also consider pairs of cubes J,K where J is $\boldsymbol{\rho}$-*deeply embedded* in K, written $J \Subset_{\boldsymbol{\rho}} K$ and meaning (2.3) holds with the same $\varepsilon > 0$ but with $\boldsymbol{\rho}$ in place of $\mathbf{r}$; as well as pairs of cubes J,K where J is $\boldsymbol{\rho}$-*nearby* in K, $|J|^{1/n} > 2^{-\boldsymbol{\rho}}|K|^{1/n}$, for a parameter $\boldsymbol{\rho} \gg \mathbf{r}$ that will be fixed later.

Then we define the smaller 'good' Haar projection $\mathsf{P}_K^{\text{good},\omega}$ by

$$\mathsf{P}_K^{\text{good},\mu} f \equiv \sum_{J\in\mathcal{G}(K)} \triangle_J^\mu f = \sum_{J\in\mathcal{G}(K)} \sum_{a\in\Gamma_n} \langle f, h_J^{\mu,a}\rangle_\mu h_J^{\mu,a},$$

where $\mathcal{G}(K)$ consists of the good subcubes of K:

$$\mathcal{G}(K) \equiv \{J \in \mathcal{D}_{\text{good}} : J \subset K\},$$

and also the larger 'subgood' Haar projection $\mathsf{P}_K^{\text{subgood},\mu}$ by

$$\mathsf{P}_K^{\text{subgood},\mu} f \equiv \sum_{J\in\mathcal{M}_{\text{good}}(K)} \sum_{J'\subset J} \triangle_{J'}^\mu f = \sum_{J\in\mathcal{M}_{\text{good}}(K)} \sum_{J'\subset J} \sum_{a\in\Gamma_n} \langle f, h_{J'}^{\mu,a}\rangle_\mu h_{J'}^{\mu,a},$$

where $\mathcal{M}_{\text{good}}(K)$ consists of the *maximal* good subcubes of K. We thus have

$$\begin{aligned}\|\mathsf{P}_K^{\text{good},\,\mu}\mathbf{x}\|_{L^2(\mu)}^2 &\leq \|\mathsf{P}_K^{\text{subgood},\,\mu}\mathbf{x}\|_{L^2(\mu)}^2\\ &\leq \|\mathsf{P}_I^{\mu}\mathbf{x}\|_{L^2(\mu)}^2 = \int_I \left|\mathbf{x} - \left(\frac{1}{|I|_\mu}\int_I \mathbf{x}dx\right)\right|^2 d\mu(x), \quad \mathbf{x} = (x_1,\dots,x_n),\end{aligned}$$

where $\mathsf{P}_I^{\mu}\mathbf{x}$ is the orthogonal projection of the identity function $\mathbf{x}:\mathbb{R}^n\to\mathbb{R}^n$ onto the vector-valued subspace of $\oplus_{k=1}^n L^2(\mu)$ consisting of functions supported in I with μ-mean value zero.

Recall that in dimension $n=1$, and for $\alpha=0$, the energy condition constant was defined by

$$\mathcal{E}^2 \equiv \sup_{I=\dot\cup I_r} \frac{1}{|I|_\sigma}\sum_{r=1}^\infty \left(\frac{\mathrm{P}^\alpha(I_r,\mathbf{1}_I\sigma)}{|I_r|}\right)^2 \|\mathsf{P}_{I_r}^{\omega}\mathbf{x}\|_{L^2(\omega)}^2 .$$

Our extension of the energy conditions to higher dimensions in this paper will use the collection $\mathcal{M}_{\mathbf{r}\text{-deep}}(K)$ of *maximal* $\mathbf{r}$-deeply embedded dyadic subcubes of a cube K (a subcube J of K is a *dyadic* subcube of K if $J\in\mathcal{D}$ when $\mathcal{D}$ is a dyadic grid containing K). We let $J^*=\gamma J$ where $\gamma\geq 2$. Then the goodness parameter $\mathbf{r}$ is chosen sufficiently large, depending on ε and γ, that the bounded overlap property

$$\sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(K)} \mathbf{1}_{J^*} \leq \beta\,\mathbf{1}_K\,, \tag{2.4}$$

holds for some positive constant β depending only on $n,\gamma,\mathbf{r}$ and ε. Indeed, the maximal $\mathbf{r}$-deeply embedded subcubes J of K satisfy the condition

$$c_n\,|J|^{\varepsilon/n}\,|K|^{(1-\varepsilon)/n} \leq \operatorname{dist}(J,K^c) \leq C_n\,|J|^{\varepsilon/n}\,|K|^{(1-\varepsilon)/n}.$$

Now with $0<\varepsilon<1$ and $\gamma\geq 2$ fixed, choose $\mathbf{r}$ so large that $2^{-(1-\varepsilon)\mathbf{r}} < c_n/(2\gamma)$. Let $y\in K$. Then if $y\in\gamma J$, we have

$$c_n\,|J|^{\varepsilon/n}\,|K|^{(1-\varepsilon)/n} \leq \operatorname{dist}(J,K^c) \leq \gamma\,|J|^{1/n} + \operatorname{dist}(\gamma J,K^c) \leq \gamma\,|J|^{1/n} + \operatorname{dist}(y,K^c),$$

which implies

$$\frac{c_n}{2}\,|J|^{\varepsilon/n}\,|K|^{(1-\varepsilon)/n} \leq \operatorname{dist}(y,K^c).$$

But we also have

$$\begin{aligned}\operatorname{dist}(y,K^c) &\leq |J|^{1/n} + \operatorname{dist}(J,K^c) \leq |J|^{1/n} + C_n\,|J|^{\varepsilon/n}\,|K|^{(1-\varepsilon)/n}\\ &\leq \left(\frac{c_n}{2\gamma}+C_n\right)|J|^{\varepsilon/n}\,|K|^{(1-\varepsilon)/n},\end{aligned}$$

and so altogether,

$$\frac{1}{\frac{c_n}{2\gamma}+C_n}\operatorname{dist}(y,K^c) \leq |J|^{\varepsilon/n}\,|K|^{(1-\varepsilon)/n} \leq \frac{2}{c_n}\operatorname{dist}(y,K^c),$$

which proves (2.4) since the number β of dyadic numbers $2^j=|J|^{1/n}$ that satisfy this last inequality is bounded independent of K and y.

A cube K is said to be a *shifted $\mathcal{D}$-dyadic cube* if K is a union of 2^n $\mathcal{D}$-dyadic cubes K', each with sidelength half that of K.

We will also need the following refinement of $\mathcal{M}_{\mathbf{r}\text{-deep}}(K)$ for each $\ell \geq 0$ and each shifted $\mathcal{D}$-dyadic cube K:

$$\mathcal{M}^{\ell}_{\mathbf{r}\text{-deep}}(K) \equiv \{J \in \mathcal{M}_{\mathbf{r}\text{-deep}}(\pi^{\ell} K') \text{ for some } K' \text{ a child of } K : \\ J \subset L \text{ for some } L \in \mathcal{M}_{\text{deep}}(K)\}.$$

Since $J \in \mathcal{M}^{\ell}_{\mathbf{r}\text{-deep}}(K)$ implies $\gamma J \subset K$, we also have from (2.4) that

$$\sum_{J \in \mathcal{M}^{(\ell)}_{\mathbf{r}\text{-deep}}(K)} \mathbf{1}_{J^*} \leq \beta\, \mathbf{1}_K\,, \quad \text{for each } \ell \geq 0\,. \tag{2.5}$$

Of course $\mathcal{M}^{0}_{\mathbf{r}\text{-deep}}(K) = \mathcal{M}_{\mathbf{r}\text{-deep}}(K)$, but $\mathcal{M}^{\ell}_{\mathbf{r}\text{-deep}}(K)$ is in general a finer subdecomposition of K the larger ℓ is, and may in fact be empty. We suppress in the notation $\mathcal{M}^{\ell}_{\mathbf{r}\text{-deep}}(K)$ the dependence on the dyadic grid $\mathcal{D}$.

Definition 2.4. Suppose σ and ω are positive Borel measures on $\mathbb{R}^n$ without common point masses, and fix $\gamma \geq 2$. Then the deep energy condition constant $\mathcal{E}^{\text{deep}}_{\alpha}$, the refined energy condition constant $\mathcal{E}^{\text{refined}}_{\alpha}$, and finally the energy condition constant $\mathcal{E}_{\alpha}$ itself, are given by

$$(\mathcal{E}^{\text{deep}}_{\alpha})^2 \equiv \sup_{I=\dot\cup I_r} \frac{1}{|I|_\sigma} \sum_{r=1}^{\infty} \sum_{J \in \mathcal{M}_{\mathbf{r}\text{-deep}}(I_r)} \left(\frac{\mathrm{P}^{\alpha}(J, \mathbf{1}_{I\setminus\gamma J}\sigma)}{|J|^{1/n}}\right)^2 \left\|\mathsf{P}^{\text{subgood},\,\omega}_J \mathbf{x}\right\|^2_{L^2(\omega)},$$

$$(\mathcal{E}^{\text{refined}}_{\alpha})^2 \equiv \sup_{\mathcal{D}} \sup_{\ell\geq 0} \sup_{I} \frac{1}{|I|_\sigma} \sum_{J \in \mathcal{M}^{\ell}_{\mathbf{r}\text{-deep}}(I)} \left(\frac{\mathrm{P}^{\alpha}(J, \mathbf{1}_{I\setminus\gamma J}\sigma)}{|J|^{1/n}}\right)^2 \left\|\mathsf{P}^{\text{subgood},\,\omega}_J \mathbf{x}\right\|^2_{L^2(\omega)},$$

$$(\mathcal{E}_{\alpha})^2 \equiv (\mathcal{E}^{\text{deep}}_{\alpha})^2 + (\mathcal{E}^{\text{refined}}_{\alpha})^2\,.$$

where $\sup_I$ in the second line is taken over all shifted $\mathcal{D}$-dyadic cubes I, and $\sup_{I=\dot\cup I_r}$ in the first line is taken over

1. all dyadic grids $\mathcal{D}$,
2. all $\mathcal{D}$-dyadic cubes I,
3. and all subpartitions $\{I_r\}_{r=1}^{\infty}$ of the cube I into $\mathcal{D}$-dyadic subcubes I_r.

Note that in the refined energy condition there is no outer decomposition $I = \dot\cup I_r$. There are similar definitions for the dual (backward) energy conditions that simply interchange σ and ω everywhere. These definitions of the energy conditions depend on the choice of γ and the goodness parameters $\mathbf{r}$ and ε. Note that we can 'plug the γ-hole' in the Poisson integral $\mathrm{P}^{\alpha}(J, \mathbf{1}_{I\setminus\gamma J}\sigma)$ for both $\mathcal{E}^{\text{deep}}_{\alpha}$ and $\mathcal{E}^{\text{refined}}_{\alpha}$ using the A^{α}_2 condition and the bounded overlap property (2.5).

Indeed, define

(2.6)

$$(\mathcal{E}_\alpha^{\mathrm{deepplug}})^2 \equiv \sup_{I=\dot{\cup}I_r} \frac{1}{|I|_\sigma} \sum_{r=1}^{\infty} \sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(I_r)} \left(\frac{\mathrm{P}^\alpha(J,\mathbf{1}_I\sigma)}{|J|^{1/n}}\right)^2 \left\|\mathsf{P}_J^{\mathrm{subgood},\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 ,$$

$$(\mathcal{E}_\alpha^{\mathrm{refinedplug}})^2 \equiv \sup_{\mathcal{D}}\sup_{\ell\geq 0}\sup_{I} \frac{1}{|I|_\sigma} \sum_{J\in\mathcal{M}^\ell_{\mathbf{r}\text{-deep}}(I)} \left(\frac{\mathrm{P}^\alpha(J,\mathbf{1}_I\sigma)}{|J|^{1/n}}\right)^2 \left\|\mathsf{P}_J^{\mathrm{subgood},\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 .$$

Then we have both

$$\begin{aligned}(\mathcal{E}_\alpha^{\mathrm{deepplug}})^2 &\lesssim \sup_{I=\dot{\cup}I_r} \frac{1}{|I|_\sigma} \sum_{r=1}^{\infty} \sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(I_r)} \left(\frac{\mathrm{P}^\alpha(J,\mathbf{1}_{I\setminus\gamma J}\sigma)}{|J|^{1/n}}\right)^2 \left\|\mathsf{P}_J^{\mathrm{subgood},\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 \\ &\quad + \sup_{I=\dot{\cup}I_r} \frac{1}{|I|_\sigma} \sum_{r=1}^{\infty} \sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(I_r)} \left(\frac{\mathrm{P}^\alpha(J,\mathbf{1}_{\gamma J}\sigma)}{|J|^{1/n}}\right)^2 \left\|\mathsf{P}_J^{\mathrm{subgood},\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 \\ &\lesssim (\mathcal{E}_\alpha)^2 + \sup_{I=\dot{\cup}I_r} \frac{1}{|I|_\sigma} \sum_{r=1}^{\infty} \sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(I_r)} \left(\frac{|\gamma J|_\sigma}{|J|^{1/n}}\right)^2 |J|^{2/n}|J|_\omega \\ &\lesssim (\mathcal{E}_\alpha)^2 + A_2^\alpha \sup_{I=\dot{\cup}I_r} \frac{1}{|I|_\sigma} \sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(I_r)} |\gamma J|_\sigma \lesssim (\mathcal{E}_\alpha)^2 + \beta\, A_2^\alpha , \qquad (2.7)\end{aligned}$$

and similarly

$$(\mathcal{E}_\alpha^{\mathrm{refinedplug}})^2 \lesssim (\mathcal{E}_\alpha^{\mathrm{refined}})^2 + \beta\, A_2^\alpha, \qquad (2.8)$$

by (2.4) and (2.5) respectively.

In the next remark we give a brief description of how and where these energy conditions will be implemented in the proof.

Remark 2.5. There are two layers of dyadic decomposition in the energy condition; the outer layer $I = \dot{\cup} I_r$ which is essentially arbitrary, and an inner layer $I_r = \dot{\cup}_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(I_r)} J$ in which the cubes J are 'nicely arranged' within I_r. Relative to this doubly layered decomposition we sum the products

$$\left(\frac{\mathrm{P}^\alpha(J,\mathbf{1}_{I\setminus\gamma J}\sigma)}{|J|^{1/n}}\right)^2 \left\|\mathsf{P}_J^{\mathrm{subgood},\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 ,$$

which resemble a type of A_2^α expression as defined above. The point of the outer decomposition is to capture 'stopping time cubes', which are essentially arbitrary in this proof, although sometimes restricted to certain collections of good cubes. The point of the inner decomposition is that with $J^* = \gamma J$ for $J \in \mathcal{M}_{\mathbf{r}\text{-deep}}(I_r)$, we have $J^* \subset I_r$ and we can then write

$$\mathrm{P}^\alpha(J,\mathbf{1}_I\sigma) = \mathrm{P}^\alpha(J,\mathbf{1}_{J^*}\sigma) + \mathrm{P}^\alpha(J,\mathbf{1}_{I\setminus J^*}\sigma),$$

and use that

$$\left\|\mathsf{P}_J^{\mathrm{subgood},\omega}\mathbf{x}\right\|_{L^2(\omega)}^2 = \left\|\mathsf{P}_J^{\mathrm{subgood},\omega}(\mathbf{x}-\mathbf{c}_J)\right\|_{L^2(\omega)}^2 \leq |J|^{2/n}\,|J|_\omega$$

to estimate the product involving $\mathbf{1}_{J^*}\sigma$ by

$$\Big(\frac{\mathrm{P}^\alpha(J,\mathbf{1}_{J^*}\sigma)}{|J|^{1/n}}\Big)^2\|\mathsf{P}_J^{\mathrm{subgood},\omega}\mathbf{x}\|^2_{L^2(\omega)} \lesssim \Big(\frac{|J^*|^{\alpha/n-1}\,|J^*|_\sigma}{|J|^{1/n}}\Big)^2|J|^{2/n}\,|J|_\omega \lesssim A_2^\alpha|J^*|_\sigma\,,$$

to which we apply the bounded overlap property (2.4), while the remaining product involving $\mathbf{1}_{I\setminus J^*}\sigma$,

$$\Big(\frac{\mathrm{P}^\alpha(J,\mathbf{1}_{I\setminus J^*}\sigma)}{|J|^{1/n}}\Big)^2\,\|\mathsf{P}_J^{\mathrm{subgood},\omega}\mathbf{x}\|^2_{L^2(\omega)}\,,$$

has a 'hole' in the support of $\mathbf{1}_{I\setminus J^*}\sigma$ that contains the support of ω in the cube J well inside the hole, and moreover these holes are 'nicely arranged' within I_r. Of particular importance is that for pairwise disjoint subcubes $J'\subset J$, the projections $\|\mathsf{P}_{J'}^{\mathrm{subgood},\omega}\mathbf{x}\|^2_{L^2(\omega)}$ are additive, and the Poisson ratios are essentially constant $\mathrm{P}^\alpha(J',\mathbf{1}_{I\setminus J^*}\sigma)/|J'|^{1/n}\approx\mathrm{P}^\alpha(J,\mathbf{1}_{I\setminus J^*}\sigma)/|J|^{1/n}$. The *deep* energy condition suffices for all arguments in the proof except for bounding the two testing conditions for the Poisson operator $\mathbb{P}$, in which case we also use the *refined* energy condition – see Lemma 10.5 below.

2.6. Statement of the theorem

We can now state our main two weight theorem. Let $\mathcal{Q}^n$ denote the collection of all cubes in $\mathbb{R}^n$, and denote by $\mathcal{D}^n$ a dyadic grid in $\mathbb{R}^n$.

Theorem 2.6. *Suppose that T^α is a standard α-fractional Calderón–Zygmund operator on $\mathbb{R}^n$, and that ω and σ are positive Borel measures on $\mathbb{R}^n$ without common point masses. Set $T^\alpha_\sigma f=T^\alpha(f\sigma)$ for any smooth truncation of T^α_σ.*

(1) *Suppose $0\le\alpha<n$ and that $\gamma\ge 2$ is given. Then the operator T^α_σ is bounded from $L^2(\sigma)$ to $L^2(\omega)$, i.e.,*

$$\|T^\alpha_\sigma f\|_{L^2(\omega)}\le\mathfrak{N}_{T^\alpha_\sigma}\|f\|_{L^2(\sigma)}, \tag{2.9}$$

uniformly in smooth truncations of T^α, and moreover

$$\mathfrak{N}_{T^\alpha_\sigma}\le C_\alpha\Big(\sqrt{\mathcal{A}_2^\alpha+\mathcal{A}_2^{\alpha,*}}+\mathfrak{T}_{T^\alpha}+\mathfrak{T}^*_{T^\alpha}+\mathcal{E}_\alpha+\mathcal{E}^*_\alpha+\mathcal{WBP}_{T^\alpha}\Big),$$

provided that the two dual $\mathcal{A}_2^\alpha$ conditions hold, and the two dual testing conditions for T^α hold, the weak boundedness property for T^α holds for a sufficiently large constant C depending on the goodness parameter $\mathbf{r}$, and provided that the two dual energy conditions $\mathcal{E}_\alpha+\mathcal{E}^*_\alpha<\infty$ hold uniformly over all dyadic grids $\mathcal{D}^n$, and where the goodness parameters $\mathbf{r}$ and ε implicit in the definition of $\mathcal{M}^\ell_{\mathbf{r}\text{-deep}}(K)$ are fixed sufficiently large and small respectively depending on n, α and γ.

(2) *Conversely, suppose $0\le\alpha<n$ and that $\mathbf{T}^\alpha=\{T^\alpha_j\}_{j=1}^J$ is a vector of Calderón–Zygmund operators with standard kernels $\{K^\alpha_j\}_{j=1}^J$. In the range*

$0 \leq \alpha < n/2$, *we assume the following* ellipticity *condition: there is* $c > 0$ *such that for* each *unit vector* $\mathbf{u}$ *there is* j *satisfying*

$$|K_j^\alpha(x, x + t\mathbf{u})| \geq c\, t^{\alpha - n}, \quad t \in \mathbb{R}. \tag{2.10}$$

For the range $n/2 \leq \alpha < n$, *we asume the following* strong ellipticity *condition: for each* $m \in \{1, -1\}^n$, *there is a sequence of coefficients* $\{\lambda_j^m\}_{j=1}^J$ *such that*

$$\Big| \sum_{j=1}^{J} \lambda_j^m\, K_j^\alpha(x, x + t\mathbf{u}) \Big| \geq c\, t^{\alpha - n}, \quad t \in \mathbb{R} \tag{2.11}$$

holds for all *unit vectors* $\mathbf{u}$ *in the* n*-ant*

$$V_m = \{x \in \mathbb{R}^n : m_i\, x_i > 0 \text{ for } 1 \leq i \leq n\}, \quad m \in \{1, -1\}^n.$$

Furthermore, assume that each operator T_j^α *is bounded from* $L^2(\sigma)$ *to* $L^2(\omega)$,

$$\big\| (T_j^\alpha)_\sigma f \big\|_{L^2(\omega)} \leq \mathfrak{N}_{T_j^\alpha} \|f\|_{L^2(\sigma)}.$$

Then the fractional $\mathcal{A}_2^\alpha$ *condition holds, and moreover,*

$$\sqrt{\mathcal{A}_2^\alpha + \mathcal{A}_2^{\alpha,*}} \leq C\, \mathfrak{N}_{\mathbf{T}^\alpha}.$$

Problem 2.7. Given any strongly elliptic vector $\mathbf{T}^\alpha$ of classical α-fractional Calderón–Zygmund operators, it is an open question whether or not the energy conditions are necessary for boundedness of $\mathbf{T}^\alpha$. See [19] for a failure of *energy reversal* in higher dimensions −such an energy reversal was used in dimension $n = 1$ to prove the necessity of the energy condition for the Hilbert transform.

Remark 2.8. The boundedness of an individual operator T^α cannot in general imply the finiteness of either A_2^α or $\mathcal{E}_\alpha$. For a trivial example, if σ and ω are supported on the x-axis in the plane, then the second Riesz transform R_2 is the zero operator from $L^2(\sigma)$ to $L^2(\omega)$, simply because the kernel $K_2(x, y)$ of R_2 satisfies $K_2((x_1, 0), (y_1, 0)) = \frac{0-0}{|x_1 - y_1|^{3-\alpha}} = 0$.

Remark 2.9. In [8], M. Lacey and B. Wick use the NTV technique of surgery to show that the weak boundedness property for the Riesz transform vector $\mathbf{R}^{\alpha,n}$ is implied by the $\mathcal{A}_2^\alpha$ and cube testing conditions, and this has the consequence of eliminating the weak boundedness property as a condition. Their proof of this implication extends to the more general operators T^α considered here, and so the weak boundedness property can be dropped from the statement of Theorem 2.6.

3. Proof of Theorem 2.6

We now give the proof of Theorem 2.6 in the following 8 sections. Using the good random grids of Nazarov, Treil and Volberg, a standard argument of NTV, see e.g. [23], reduces the two weight inequality (1.1) for T^α to proving boundedness of a bilinear form $\mathcal{T}^\alpha(f, g)$ with uniform constants over dyadic grids, and where the Haar supports of the functions f and g are contained in good cubes, whose children are all good as well, with goodness parameters $\mathbf{r} < \infty$ and $\varepsilon > 0$ chosen sufficiently large and small respectively. Here the Haar support of f is $\operatorname{Haarsupp} \widehat{f} = \{I \in \mathcal{D} : \triangle_I^\sigma f \neq 0\}$, and similarly for g.

In fact we can assume even more, namely that the Haar supports of f and g are contained in the collection of $\boldsymbol{\tau}$-*good* cubes

$$\text{(3.1)} \quad \mathcal{D}^{\boldsymbol{\tau}}_{(\mathbf{r},\varepsilon)\text{-good}} \equiv \{K \in \mathcal{D} : \mathfrak{C}_K \subset \mathcal{D}_{(\mathbf{r},\varepsilon)\text{-good}} \text{ and } \pi^\ell_{\mathcal{D}} K \text{ are in } \mathcal{D}_{(\mathbf{r},\varepsilon)\text{-good}} \text{ for all } 0 \leq \ell \leq \boldsymbol{\tau}\},$$

that are $(\mathbf{r},\varepsilon)$-good, whose children are also $(\mathbf{r},\varepsilon)$-good, and whose ℓ-parents up to level $\boldsymbol{\tau}$ are also $(\mathbf{r},\varepsilon)$-good. Here $\boldsymbol{\tau} > \mathbf{r}$ is a parameter to be fixed in Definition 8.6 below. We may assume this restriction on the Haar supports of f and g by choosing $(\mathbf{r},\varepsilon)$ appropriately and using the following lemma.

Lemma 3.1. *Given* $\mathbf{s} \geq 1$, $\mathbf{t} \geq 2$ *and* $0 < \varepsilon < 1$, *we have*

$$\mathcal{D}^{\mathbf{s}}_{(\mathbf{s}+\mathbf{t},\varepsilon)\text{-good}} \subset \mathcal{D}_{(\mathbf{t},\delta)\text{-good}}\,,$$

provided

$$\mathbf{s}\,\varepsilon < \mathbf{t}\,(1-\varepsilon) - 2 \quad \text{and} \quad \delta = \varepsilon + \frac{\mathbf{s}\,\varepsilon + 1}{\mathbf{t}}\,.$$

Proof. Fix goodness parameters $\mathbf{r} = \mathbf{s}+\mathbf{t}$ and ε, and suppose that $\mathbf{s} < \mathbf{r}(1-\varepsilon) - 2$. Choose a good cube I and a supercube K with $|I|^{1/n} \leq 2^{-\mathbf{r}}|K|^{1/n}$. Set $J \equiv \pi^{\mathbf{s}} I$. Then we have

$$J = \pi^s I \subset K \quad \text{and} \quad |J|^{1/n} \leq 2^{-\mathbf{t}}\,|K|^{1/n}.$$

Because I is good we have

$$\operatorname{dist}(I, K^c) \geq \frac{1}{2}\,|I|^{\varepsilon/n}\,|K|^{(1-\varepsilon)/n},$$

and hence also

$$\begin{aligned}
\operatorname{dist}(J, K^c) &\geq \operatorname{dist}(I, K^c) - |J|^{1/n} \geq \frac{1}{2}\,|I|^{\varepsilon/n}\,|K|^{(1-\varepsilon)/n} - 2^{\mathbf{s}}\,|I|^{1/n} \\
&= \frac{1}{2}\,|I|^{\varepsilon/n}\,|K|^{(1-\varepsilon)/n}\left\{1 - 2^{1+\mathbf{s}}\left(\frac{|I|^{1/n}}{|K|^{1/n}}\right)^{1-\varepsilon}\right\} \geq \frac{1}{4}\,|I|^{\varepsilon/n}\,|K|^{(1-\varepsilon)/n},
\end{aligned}$$

which follows from $|I|^{1/n} \leq 2^{-\mathbf{r}}|K|^{1/n}$ provided we take $2^{1+\mathbf{s}}\,2^{-\mathbf{r}(1-\varepsilon)} \leq 1/2$, i.e.,

$$\mathbf{s} < \mathbf{r}(1-\varepsilon) - 2.$$

Finally we choose $\delta > \varepsilon$ so that

$$\frac{1}{4}\,|I|^{\varepsilon/n}\,|K|^{(1-\varepsilon)/n} = 2^{-\mathbf{s}\varepsilon-2}\,|J|^{\varepsilon/n}\,|K|^{(1-\varepsilon)/n} \geq \frac{1}{2}\,|J|^{\delta/n}\,|K|^{(1-\delta)/n}$$

when $|J|^{1/n} \leq 2^{-\mathbf{t}}\,|K|^{1/n}$, which follows if we choose δ to satisfy

$$\begin{aligned}
&2^{-\mathbf{s}\varepsilon-2}\,(2^{-\mathbf{t}}\,|K|^{1/n})^{\varepsilon}|K|^{(1-\varepsilon)/n} = \frac{1}{2}\,(2^{-\mathbf{t}}\,|K|^{1/n})^{\delta}\,|K|^{(1-\delta)/n}; \\
&2^{-\mathbf{s}\varepsilon-2} = \frac{1}{2}\,(2^{-\mathbf{t}})^{\delta-\varepsilon}; \quad -\mathbf{s}\varepsilon - 1 = -\mathbf{t}(\delta-\varepsilon); \quad \delta = \varepsilon + \frac{\mathbf{s}\varepsilon+1}{\mathbf{t}}.
\end{aligned}$$

□

For convenience in notation we will sometimes suppress the dependence on α in our nonlinear forms, but will retain it in the operators, Poisson integrals and constants. More precisely, let $\mathcal{D}^\sigma = \mathcal{D}^\omega$ be an $(\mathbf{r},\varepsilon)$-good grid on $\mathbb{R}^n$, and let $\{h_I^{\sigma,a}\}_{I\in\mathcal{D}^\sigma,\, a\in\Gamma_n}$ and $\{h_J^{\omega,b}\}_{J\in\mathcal{D}^\omega,\, b\in\Gamma_n}$ be corresponding Haar bases as described below, so that

$$f = \sum_{I\in\mathcal{D}^\sigma} \triangle_I^\sigma f = \sum_{I\in\mathcal{D}^\sigma,\, a\in\Gamma_n} \left\langle f, h_I^{\sigma,a}\right\rangle h_I^{\sigma,a} = \sum_{I\in\mathcal{D}^\sigma,\, a\in\Gamma_n} \widehat{f}(I;a)\, h_I^{\sigma,a},$$
$$g = \sum_{J\in\mathcal{D}^\omega} \triangle_J^\omega g = \sum_{J\in\mathcal{D}^\omega,\, b\in\Gamma_n} \left\langle g, h_J^{\omega,b}\right\rangle h_J^{\omega,b} = \sum_{J\in\mathcal{D}^\omega,\, b\in\Gamma_n} \widehat{g}(J;b)\, h_J^{\omega,b},$$

where the appropriate measure is understood in the notation $\widehat{f}(I;a)$ and $\widehat{g}(J;b)$, and where these Haar coefficients $\widehat{f}(I;a)$ and $\widehat{g}(J;b)$ vanish if the cubes I and J are not good. Inequality (2.9) is equivalent to boundedness of the bilinear form

$$\mathcal{T}^\alpha(f,g) \equiv \langle T_\sigma^\alpha(f), g\rangle_\omega = \sum_{I\in\mathcal{D}^\sigma \text{ and } J\in\mathcal{D}^\omega} \langle T_\sigma^\alpha(\triangle_I^\sigma f), \triangle_J^\omega g\rangle_\omega$$

on $L^2(\sigma)\times L^2(\omega)$, i.e.,

$$|\mathcal{T}^\alpha(f,g)| \leq \mathfrak{N}_{T_\sigma^\alpha} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}.$$

We may assume the two grids $\mathcal{D}^\sigma$ and $\mathcal{D}^\omega$ are equal here, and this we will do throughout the paper, although we sometimes continue to use the measure as a superscript on $\mathcal{D}$ for clarity of exposition. Roughly speaking, we analyze the form $\mathcal{T}^\alpha(f,g)$ by splitting it in a nonlinear way into three main pieces, following in part the approach in [5] and [6]. The first piece consists of cubes I and J that are either disjoint or of comparable side length, and this piece is handled using the section on preliminaries of NTV type. The second piece consists of cubes I and J that overlap, but are 'far apart' in a nonlinear way, and this piece is handled using the sections on the intertwining proposition and the control of the functional energy condition by the energy condition. Finally, the remaining local piece where the overlapping cubes are 'close' is handled by generalizing methods of NTV as in [4], and then splitting the stopping form into two sublinear stopping forms, one of which is handled using techniques of [3], and the other using the stopping time and recursion of M. Lacey [1]. See the schematic diagram in Subsection 8.4 below.

4. Necessity of the $\mathcal{A}_2^\alpha$ conditions

Here we prove in particular the necessity of the fractional $\mathcal{A}_2^\alpha$ condition when $0\leq\alpha<n$, for the α-fractional Riesz vector transform $\mathbf{R}^\alpha$ defined by

$$\mathbf{R}^\alpha(f\sigma)(x) = \int_{\mathbb{R}^n} K_j^\alpha(x,y)\, f(y)\, d\sigma(y), \quad K_j^\alpha(x,y) = \frac{x^j - y^j}{|x-y|^{n+1-\alpha}},$$

whose kernel $K_j^\alpha(x,y)$ satisfies (2.1) for $0\leq\alpha<n$. Parts of the following argument are taken from unpublished material obtained in joint work with M. Lacey. Note

also that the necessity of the classical A_2^α condition, for many singular integral operators, including among others the vector Riesz transforms, the Cauchy transform and the Beurling transform was obtained previously by Liaw and Treil [9].

Lemma 4.1. *Suppose $0 \leq \alpha < n$. Let T^α be any collection of operators with α-standard fractional kernel satisfying the ellipticity condition (2.10), and in the case $n/2 \leq \alpha < n$, we also assume the more restrictive condition (2.11). Then for $0 \leq \alpha < n$ we have*

$$\sqrt{\mathcal{A}_2^\alpha} \lesssim \mathfrak{N}_\alpha(T^\alpha).$$

Remark 4.2. Cancellation properties of T^α play no role in the proof below. Indeed the proof shows that $\mathcal{A}_2^\alpha$ is dominated by the best constant C in the restricted inequality

$$\|\chi_E T^\alpha(f\sigma)\|_{L^{2,\infty}(\omega)} \leq C\,\|f\|_{L^2(\sigma)}, \quad E = \mathbb{R}^n \setminus \operatorname{supp} f.$$

Proof. First we give the proof for the case when T^α is the α-fractional Riesz transform $\mathbf{R}^\alpha$, whose kernel is $\mathbf{K}^\alpha(x,y) = \frac{x-y}{|x-y|^{n+1-\alpha}}$. Define the 2^n generalized n-ants $\mathcal{Q}_m$ for $m \in \{-1,1\}^n$, and their translates $\mathcal{Q}_m(w)$ for $w \in \mathbb{R}^n$ by

$$\mathcal{Q}_m = \{(x_1,\ldots,x_n) : m_k x_k > 0\},$$
$$\mathcal{Q}_m(w) = \{z : z - w \in \mathcal{Q}_m\}, \quad w \in \mathbb{R}^n.$$

Fix $m \in \{-1,1\}^n$ and a cube I. For $a \in \mathbb{R}^n$ and $r > 0$ let

$$s_I(x) = \frac{\ell(I)}{\ell(I) + |x - \zeta_I|}, \quad \text{and} \quad f_{a,r}(y) = \mathbf{1}_{\mathcal{Q}_{-m}(a)\cap B(0,r)}(y)\, s_I(y)^{n-\alpha},$$

where ζ_I is the center of the cube I. Now

$$\ell(I)\,|x-y| \leq \ell(I)\,|x-\zeta_I| + \ell(I)\,|\zeta_I - y| \leq [\,\ell(I) + |x-\zeta_I|\,]\,[\,\ell(I) + |\zeta_I - y|\,]$$

implies

$$\frac{1}{|x-y|} \geq \frac{1}{\ell(I)}\, s_I(x)\, s_I(y), \quad x, y \in \mathbb{R}^n.$$

Now the key observation is that with $L\zeta \equiv m \cdot \zeta$, we have

$$L(x-y) = m \cdot (x-y) \geq |x-y|, \quad x \in \mathcal{Q}_m(y),$$

which yields

$$\text{(4.1)} \qquad L(\mathbf{K}^\alpha(x,y)) = \frac{L(x-y)}{|x-y|^{n+1-\alpha}} \geq \frac{1}{|x-y|^{n-\alpha}} \geq \ell(I)^{\alpha-n}\, s_I(x)^{n-\alpha}\, s_I(y)^{n-\alpha},$$

provided $x \in \mathcal{Q}_{+,+}(y)$.

Now we note that $x \in \mathcal{Q}_m(y)$ when $x \in \mathcal{Q}_m(a)$ and $y \in \mathcal{Q}_{-m}(a)$ to obtain that for $x \in \mathcal{Q}_m(a)$,

$$\begin{aligned} L(T^\alpha(f_{a,r}\sigma)(x)) &= \int_{\mathcal{Q}_{-m}(a)\cap B(0,r)} \frac{L(x-y)}{|x-y|^{n+1-\alpha}} s_I(y)\, d\sigma(y) \\ &\geq \ell(I)^{\alpha-n} s_I(x)^{n-\alpha} \int_{\mathcal{Q}_{-m}(a)\cap B(0,r)} s_I(y)^{2n-2\alpha}\, d\sigma(y). \end{aligned}$$

Applying $|L\zeta| \leq \sqrt{n}\,|\zeta|$ and our assumed two weight inequality for the fractional Riesz transform, we see that for $r > 0$ large,

$$\begin{aligned} \ell(I)^{2\alpha-2n} \int_{\mathcal{Q}_m(a)} s_I(x)^{2n-2\alpha} \Big(\int_{\mathcal{Q}_{-m}(a)\cap B(0,r)} s_I(y)^{2n-2\alpha} d\sigma(y) \Big)^2 d\omega(x) \\ \leq \|LT(\sigma f_{a,r})\|^2_{L^2(\omega)} \lesssim \mathfrak{N}_\alpha(\mathbf{R}^\alpha)^2 \|f_{a,r}\|^2_{L^2(\sigma)} \\ = \mathfrak{N}_\alpha(\mathbf{R}^\alpha)^2 \int_{\mathcal{Q}_{-m}(a)\cap B(0,r)} s_I(y)^{2n-2\alpha} d\sigma(y). \end{aligned}$$

Rearranging the last inequality, we obtain

$$\ell(I)^{2\alpha-2n} \int_{\mathcal{Q}_m(a)} s_I(x)^{2n-2\alpha}\, d\omega(x) \int_{\mathcal{Q}_{-m}(a)\cap B(0,r)} s_I(y)^{2n-2\alpha}\, d\sigma(y) \lesssim \mathfrak{N}_\alpha(\mathbf{R}^\alpha)^2,$$

and upon letting $r \to \infty$,

$$\int_{\mathcal{Q}_m(a)} \frac{\ell(I)^{2-\alpha}}{(\ell(I)+|x-\zeta_I|)^{4-2\alpha}}\, d\omega(x) \int_{\mathcal{Q}_{-m}(a)} \frac{\ell(I)^{2-\alpha}}{(\ell(I)+|y-\zeta_I|)^{4-2\alpha}}\, d\sigma(y) \lesssim \mathfrak{N}_\alpha(\mathbf{R}^\alpha)^2.$$

Note that the ranges of integration above are pairs of opposing n-ants.

Fix a cube Q, which without loss of generality can be taken to be centered at the origin, $\zeta_Q = 0$. Then choose $a = (2\ell(Q), 2\ell(Q))$ and $I = Q$ so that we have

$$\begin{aligned} &\Big(\int_{\mathcal{Q}_m(a)} \frac{\ell(Q)^{n-\alpha}}{(\ell(Q)+|x|)^{2n-2\alpha}}\, d\omega(x) \Big) \Big(\ell(Q)^{\alpha-n} \int_Q d\sigma \Big) \\ &\qquad \leq C_\alpha \int_{\mathcal{Q}_m(a)} \frac{\ell(Q)^{n-\alpha}}{(\ell(Q)+|x|)^{2n-2\alpha}}\, d\omega(x) \int_{\mathcal{Q}_{-m}(a)} \frac{\ell(Q)^{n-\alpha}}{(\ell(Q)+|y|)^{2n-2\alpha}}\, d\sigma(y) \\ &\qquad \lesssim \mathfrak{N}_\alpha(\mathbf{R}^\alpha)^2. \end{aligned}$$

Now fix $m = (1, 1, \dots, 1)$ and note that there is a fixed N (independent of $\ell(Q)$) and a fixed collection of rotations $\{\rho_k\}_{k=1}^N$, such that the rotates $\rho_k \mathcal{Q}_m(a)$, $1 \leq k \leq N$, of the n-ant $\mathcal{Q}_m(a)$ cover the complement of the ball $B(0, 4\sqrt{n}\,\ell(Q))$:

$$B(0, 4\sqrt{n}\,\ell(Q))^c \subset \bigcup_{k=1}^N \rho_k\, \mathcal{Q}_m(a).$$

Then we obtain, upon applying the same argument to these rotated pairs of n-ants,

$$(4.2)\quad \Big(\int_{B(0,4\sqrt{n}\,\ell(Q))^c} \frac{\ell(Q)^{n-\alpha}}{(\ell(Q)+|x|)^{2n-2\alpha}}\,d\omega(x)\Big)\Big(\ell(Q)^{\alpha-n}\int_Q d\sigma\Big) \lesssim \mathfrak{N}_\alpha(\mathbf{R}^\alpha)^2.$$

Now we assume for the moment the tailless A_2^α condition

$$\ell(Q')^{2(\alpha-n)}\Big(\int_{Q'} d\omega\Big)\Big(\int_{Q'} d\sigma\Big) \leq A_2^\alpha.$$

If we use this with $Q' = 4\sqrt{n}Q$, together with (4.2), we obtain

$$\Big(\int \frac{\ell(Q)^{n-\alpha}}{(\ell(Q)+|x|)^{2n-2\alpha}}\,d\omega(x)\Big)^{1/2}\Big(\ell(Q)^{\alpha-n}\int_Q d\sigma\Big)^{1/2} \lesssim \mathfrak{N}_\alpha(\mathbf{R}^\alpha)$$

or

$$\ell(Q)^\alpha\Big(\frac{1}{|Q|}\int \frac{1}{\big(1+\frac{|x-\zeta_Q|}{\ell(Q)}\big)^{2n-2\alpha}} d\omega(x)\Big)^{1/2}\Big(\frac{1}{|Q|}\int_Q d\sigma\Big)^{1/2} \lesssim \mathfrak{N}_\alpha(\mathbf{R}^\alpha).$$

Clearly we can reverse the roles of the measures ω and σ and obtain

$$\sqrt{\mathcal{A}_2^\alpha} \lesssim \mathfrak{N}_\alpha(\mathbf{R}^\alpha) + \sqrt{A_2^\alpha}$$

for the kernels $\mathbf{K}^\alpha$, $0 \leq \alpha < n$.

More generally, to obtain the case when T^α is elliptic and the tailless A_2^α condition holds, we note that the key estimate (4.1) above extends to the kernel $\sum_{j=1}^J \lambda_j^m K_j^\alpha$ of $\sum_{j=1}^J \lambda_j^m T_j^\alpha$ in (2.11) if the n-ants above are replaced by thin cones of sufficiently small aperture, and there is in addition sufficient separation between opposing cones, which in turn may require a larger constant than $4\sqrt{n}$ in the choice of Q' above.

Finally, we turn to showing that the tailless A_2^α condition is implied by the norm inequality, i.e.,

$$\sqrt{A_2^\alpha} \equiv \sup_{Q'} \ell(Q')^\alpha\Big(\frac{1}{|Q'|}\int_{Q'} d\omega\Big)^{1/2}\Big(\frac{1}{|Q'|}\int_{Q'} d\sigma\Big)^{1/2} \lesssim \mathfrak{N}_\alpha(\mathbf{R}^\alpha);$$

$$\text{i.e.,}\quad \Big(\int_{Q'} d\omega\Big)\Big(\int_{Q'} d\sigma\Big) \lesssim \mathfrak{N}_\gamma(\mathbf{R}^\gamma)^2\,|Q'|^{2-2\alpha/n}.$$

In the range $0 \leq \alpha < n/2$ where we only assume (2.10), we invoke the corresponding argument in [2]. Indeed, with notation as in that proof, and suppressing some of the initial work there, then $\mathcal{A}_2(\omega,\sigma;\mathsf{Q}) = |\mathsf{Q}|_{\omega\times\sigma}$ where $\omega\times\sigma$ denotes product measure on $\mathbb{R}^n\times\mathbb{R}^n$, and we have

$$\mathcal{A}_2(\omega,\sigma;\mathsf{Q}_0) = \sum_\zeta \mathcal{A}_2(\omega,\sigma;\mathsf{Q}_\zeta) + \sum_\beta \mathcal{A}_2(\omega,\sigma;\mathsf{P}_\beta).$$

Now we have

$$\sum_\zeta \mathcal{A}_2(\omega,\sigma;\mathsf{Q}_\zeta) = \sum_\zeta |\mathsf{Q}_\zeta|_{\omega\times\sigma} \leq \sum_\zeta \mathfrak{N}_\alpha(\mathbf{R}^\alpha)^2\,|\mathsf{Q}_\zeta|^{1-\alpha/n},$$

and

$$\begin{aligned}\sum_{\zeta}|\mathsf{Q}_\zeta|^{1-\alpha/n} &= \sum_{k\in\mathbb{Z}\,:\,2^k\le\ell(Q_0)}\ \sum_{\zeta\,:\,\ell(Q_\zeta)=2^k}(2^{2nk})^{1-\alpha/n}\\ &\approx \sum_{k\in\mathbb{Z}\,:\,2^k\le\ell(Q_0)}\left(\frac{2^k}{\ell(Q_0)}\right)^{-n}(2^{2nk})^{1-\alpha/n}\quad\text{(Whitney)}\\ &= \ell(Q_0)^n\sum_{k\in\mathbb{Z}\,:\,2^k\le\ell(Q_0)}2^{nk(-1+2-2\alpha/n)}\\ &\le C_\alpha\,\ell(Q_0)^n\,\ell(Q_0)^{n(1-2\alpha/n)} = C_\alpha|Q_0\times Q_0|^{2-2\alpha/n} = C_\alpha\,|\mathsf{Q}_0|^{1-\alpha/n},\end{aligned}$$

provided $0\le\alpha<n/2$. Since ω and σ have no point masses in common, it is not hard to show, using that the side length of $\mathsf{P}_\beta = P_\beta\times P'_\beta$ is 2^{-N} and $\operatorname{dist}(\mathsf{P}_\beta,\mathcal{D})\le C2^{-N}$, that we have the following limit:

$$\sum_\beta \mathcal{A}_2(\omega,\sigma;\mathsf{P}_\beta)\to 0\quad\text{as } N\to\infty.$$

Indeed, if σ has no point masses at all, then

$$\begin{aligned}\sum_\beta \mathcal{A}_2(\omega,\sigma;\mathsf{P}_\beta) = \sum_\beta |P_\beta|_\omega\,|P'_\beta|_\sigma &\le \Big(\sum_\beta|P_\beta|_\omega\Big)\sup_\beta|P'_\beta|_\sigma\\ &\le C\,|Q_0|_\omega\,\sup_\beta|P'_\beta|_\sigma\to 0\quad\text{as } N\to\infty,\end{aligned}$$

while if σ contains a point mass $c\delta_x$, then

$$\begin{aligned}\sum_{\beta:\,x\in P'_\beta}\mathcal{A}_2(\omega,\sigma;\mathsf{P}_\beta) &\le \Big(\sum_{\beta:\,x\in P'_\beta}|P_\beta|_\omega\Big)\sup_{\beta:\,x\in P'_\beta}|P'_\beta|_\sigma\\ &\le C\Big(\sum_{\beta:\,x\in P'_\beta}|P_\beta|_\omega\Big)\to 0\quad\text{as } N\to\infty,\end{aligned}$$

since ω has no point mass at x. This continues to hold if σ contains finitely many point masses disjoint from those of ω, and a limiting argument finally applies. This completes the proof that $\sqrt{A_2^\alpha}\lesssim\mathfrak{N}_\alpha(\mathbf{R}^\alpha)$ for the range $0\le\alpha<n/2$.

Now we turn to proving $\sqrt{A_2^\alpha}\lesssim\mathfrak{N}_\alpha(\mathbf{R}^\alpha)$ for the range $n/2\le\alpha<n$, where we assume the stronger ellipticity condition (2.11). So fix a cube $Q=\prod_{i=1}^n Q_i$, where $Q_i=[a_i,b_i]$. Choose $\theta_1\in[a_1,b_1]$ so that both

$$\Big|[a_1,\theta_1]\times\prod_{i=2}^n Q_i\Big|_\omega\quad\text{and}\quad\Big|[\theta_1,b_1]\times\prod_{i=2}^n Q_i\Big|_\omega\ge\frac12\,|Q|_\omega.$$

Now denote the two intervals $[a_1,\theta_1]$ and $[\theta_1,b_1]$ by $[a_1^*,b_1^*]$ and $[a_1^{**},b_1^{**}]$ where the order is chosen so that

$$\Big|[a_1^*,b_1^*]\times\prod_{i=2}^n Q_i\Big|_\sigma\le\Big|[a_1^{**},b_1^{**}]\times\prod_{i=2}^n Q_i\Big|_\sigma.$$

Then we have both

$$\Big|[a_1^*,b_1^*]\times\prod_{i=2}^n Q_i\Big|_\omega \ge \frac{1}{2}|Q|_\omega \quad \text{and} \quad \Big|[a_1^{**},b_1^{**}]\times\prod_{i=2}^n Q_i\Big|_\sigma \ge \frac{1}{2}\,|Q|_\sigma\,.$$

Now choose $\theta_2\in[a_2,b_2]$ so that both

$$\Big|[a_1^*,b_1^*]\times[a_2,\theta_2]\times\prod_{i=3}^n Q_i\Big|_\omega \quad \text{and} \quad \Big|[a_1^*,b_1^*]\times[\theta_2,b_2]\times\prod_{i=3}^n Q_i\Big|_\omega \ge \frac{1}{4}\,|Q|_\omega,$$

and denote the two intervals $[a_2,\theta_2]$ and $[\theta_2,b_2]$ by $[a_2^*,b_2^*]$ and $[a_2^{**},b_2^{**}]$ where the order is chosen so that

$$\Big|[a_1^{**},b_1^{**}]\times[a_2^*,b_2^*]\times\prod_{i=2}^n Q_i\Big|_\sigma \le \Big|[a_1^{**},b_1^{**}]\times[a_2^{**},b_2^{**}]\times\prod_{i=2}^n Q_i\Big|_\sigma.$$

Then we have both

$$\Big|[a_1^*,b_1^*]\times[a_2^*,b_2^*]\times\prod_{i=3}^n Q_i\Big|_\omega \ge \frac{1}{4}\,|Q|_\omega\,,$$
$$\Big|[a_1^{**},b_1^{**}]\times[a_2^{**},b_2^{**}]\times\prod_{i=3}^n Q_i\Big|_\sigma \ge \frac{1}{4}\,|Q|_\sigma\,.$$

Then we choose $\theta_3\in[a_3,b_3]$ so that both

$$\Big|[a_1^*,b_1^*]\times[a_2^*,b_2^*]\times[a_3,\theta_3]\times\prod_{i=4}^n Q_i\Big|_\omega \ge \frac{1}{8}\,|Q|_\omega,$$
$$\Big|[a_1^*,b_1^*]\times[a_2^*,b_2^*]\times[\theta_3,b_3]\times\prod_{i=4}^n Q_i\Big|_\omega \ge \frac{1}{8}\,|Q|_\omega,$$

and continuing in this way we end up with two rectangles,

$$G\equiv[a_1^*,b_1^*]\times[a_2^*,b_2^*]\times\cdots\times[a_n^*,b_n^*],$$
$$H\equiv[a_1^{**},b_1^{**}]\times[a_2^{**},b_2^{**}]\times\cdots\times[a_n^{**},b_n^{**}],$$

that satisfy

$$|G|_\omega=\big|[a_1^*,b_1^*]\times[a_2^*,b_2^*]\times\cdots\times[a_n^*,b_n^*]\big|_\omega \ge \frac{1}{2^n}\,|Q|_\omega,$$
$$|H|_\sigma=\big|[a_1^{**},b_1^{**}]\times[a_2^{**},b_2^{**}]\times\cdots\times[a_n^{**},b_n^{**}]\big|_\sigma \ge \frac{1}{2^n}\,|Q|_\sigma.$$

However, the rectangles G and H lie in opposing n-ants at the vertex $\theta=(\theta_1,\theta_2,\ldots,\theta_n)$, and so we can apply (2.11) to obtain that for $x\in G$,

$$\Big|\sum_{j=1}^J \lambda_j^m\,T_j^\alpha(\mathbf{1}_H\sigma)(x)\Big| = \Big|\int_H\sum_{j=1}^J\lambda_j^m\,K_j^\alpha(x,y)\,d\sigma(y)\Big|$$
$$\gtrsim \int_H |x-y|^{\alpha-n}\,d\sigma(y)\gtrsim |Q|^{\alpha/n-1}|\,H|_\sigma.$$

Then from the norm inequality we get

$$|G|_\omega (|Q|^{\alpha/n-1} |H|_\sigma)^2 \lesssim \int_G \Big| \sum_{j=1}^J \lambda_j^m T_j^\alpha (\mathbf{1}_H \sigma) \Big|^2 d\omega$$
$$\lesssim \mathfrak{N}^2_{\sum_{j=1}^J \lambda_j^m T_j^\alpha} \int \mathbf{1}_H^2 \, d\sigma = \mathfrak{N}^2_{\sum_{j=1}^J \lambda_j^m T_j^\alpha} |H|_\sigma ,$$

from which we deduce that

$$|Q|^{2(\alpha/n-1)} |Q|_\omega |Q|_\sigma \lesssim 2^{2n} |Q|^{2(\alpha/n-1)} |G|_\omega |H|_\sigma \lesssim 2^{2n} \mathfrak{N}^2_{\sum_{j=1}^J \lambda_j^m T_j^\alpha},$$

and hence

$$A_2^\alpha \lesssim 2^{2n} \mathfrak{N}^2_{\sum_{j=1}^J \lambda_j^m T_j^\alpha} .$$

This completes the proof of Lemma 4.1. □

5. A weighted Haar basis

We will use a construction of the Haar basis in $\mathbb{R}^n$ that is adapted to a measure μ (cf. [12]). Given a dyadic cube $Q \in \mathcal{D}$ let $\triangle_Q^\mu$ denote orthogonal projection onto the finite dimensional subspace $L_Q^2(\mu)$ of $L^2(\mu)$ that consists of linear combinations of the indicators of the children $\mathfrak{C}(Q)$ of Q that have μ-mean zero over Q:

$$L_Q^2(\mu) \equiv \Big\{ f = \sum_{Q' \in \mathfrak{C}(Q)} a_{Q'} \mathbf{1}_{Q'} \ : \ a_{Q'} \in \mathbb{R}, \int_Q f \, d\mu = 0 \Big\}.$$

Then we have the important telescoping property for dyadic cubes $Q_1 \subset Q_2$:
(5.1)

$$\mathbf{1}_{Q_0}(x) \Big(\sum_{Q \in [Q_1, Q_2]} \triangle_Q^\mu f(x) \Big) = \mathbf{1}_{Q_0}(x) \big(\mathbb{E}_{Q_0}^\mu f - \mathbb{E}_{Q_2}^\mu f \big), \quad Q_0 \in \mathfrak{C}(Q_1), \ f \in L^2(\mu).$$

We will at times find it convenient to use a fixed orthonormal basis $\{h_Q^{\mu,a}\}_{a \in \Gamma_n}$ of $L_Q^2(\mu)$ where $\Gamma_n \equiv \{0,1\}^n \setminus \{\mathbf{1}\}$ is a convenient index set with $\mathbf{1} = (1,1,\ldots,1)$. Then $\{h_Q^{\mu,a}\}_{a \in \Gamma_n \text{ and } Q \in \mathcal{D}}$ is an orthonormal basis for $L^2(\mu)$, with the understanding that we add the constant function $\mathbf{1}$ if μ is a finite measure. In particular we have

$$\|f\|_{L^2(\mu)}^2 = \sum_Q \big\| \triangle_Q^\mu f \big\|_{L^2(\mu)}^2 = \sum_Q \sum_{a \in \Gamma_n} \big| \widehat{f}(Q) \big|^2,$$

where

$$|\widehat{f}(Q)|^2 \equiv \sum_{a \in \Gamma_n} \big| \langle f, h_Q^{\mu,a} \rangle_\mu \big|^2,$$

and the measure is suppressed in the notation. We also record the following useful estimate. If I' is any of the 2^n $\mathcal{D}$-children of I, and $a \in \Gamma_n$, then

$$|\mathbb{E}_{I'}^\mu h_I^{\mu,a}| \le \sqrt{\mathbb{E}_{I'}^\mu (h_I^{\mu,a})^2} \le \frac{1}{\sqrt{|I'|_\mu}}. \tag{5.2}$$

6. Monotonicity lemma and energy lemma

The monotonicity lemma below will be used to prove the energy lemma, which is then used in several places in the proof of Theorem 2.6. The formulation of the monotonicity lemma with $m = 2$ is due to M. Lacey and B. Wick [8], and corrects that used in previous versions of this paper.

6.1. The monotonicity lemma

For $0 \le \alpha < n$ and $m \in \mathbb{R}_+$, we recall the m-weighted fractional Poisson integral

$$\mathrm{P}_m^\alpha(J,\mu) \equiv \int_{\mathbb{R}^n} \frac{|J|^{m/n}}{(|J|^{1/n} + |y - c_J|)^{n+m-\alpha}}\, d\mu(y),$$

where $\mathrm{P}_1^\alpha(J,\mu) = \mathrm{P}^\alpha(J,\mu)$ is the standard Poisson integral.

Lemma 6.1 (Monotonicity). *Suppose that I, J and J^* are cubes in $\mathbb{R}^n$ such that $J \subset J^* \subset 2J^* \subset I$, and that μ is a signed measure on $\mathbb{R}^n$ supported outside I. Finally suppose that T^α is a standard fractional singular integral on $\mathbb{R}^n$ as defined in Definition 2.1 with $0 < \alpha < n$. Then we have the estimate*

$$\| \triangle_J^\omega T^\alpha \mu\|_{L^2(\omega)} \lesssim \Phi^\alpha(J,|\mu|), \tag{6.1}$$

where for a positive measure ν,

$$\Phi^\alpha(J,\nu)^2 \equiv \left(\frac{\mathrm{P}^\alpha(J,\nu)}{|J|^{1/n}}\right)^2 \| \triangle_J^\omega \mathbf{x}\|_{L^2(\omega)}^2 + \left(\frac{\mathrm{P}_{1+\delta}^\alpha(J,\nu)}{|J|^{1/n}}\right)^2 \|\mathbf{x} - \mathbf{m}_J\|_{L^2(\mathbf{1}_J\omega)}^2 ,$$
$$\mathbf{m}_J \equiv \mathbb{E}_J^\omega \mathbf{x} = \frac{1}{|J|_\omega}\int_J \mathbf{x}\, d\omega .$$

Proof. The general case follows easily from the case $J^* = J$, so we assume this restriction.

Let $\{h_J^{\omega,a}\}_{a\in\Gamma}$ be an orthonormal basis of $L_J^2(\mu)$ as in the previous section. Now we use the smoothness estimate (2.1), together with Taylor's formula and the vanishing mean of the Haar functions $h_J^{\omega,a}$ and $\mathbf{m}_J \equiv \frac{1}{|J|_\mu}\int_J \mathbf{x}\, d\mu(x) \in J$, to obtain

$$\begin{aligned}
|\langle T^\alpha \mu, h_J^{\omega,a}\rangle_\omega| &= \left| \int \left\{ \int K^\alpha(x,y) h_J^{\omega,a}(x) d\omega(x) \right\} d\mu(y)\right| = \left| \int \langle K_y^\alpha, h_J^{\omega,a}\rangle_\omega\, d\mu(y)\right| \\
&= \left| \int \langle K_y^\alpha(x) - K_y^\alpha(\mathbf{m}_J), h_J^{\omega,a}\rangle_\omega\, d\mu(y)\right| \\
&\le \left| \left\langle \left[\int \nabla K_y^\alpha(\mathbf{m}_J) d\mu(y)\right] (\mathbf{x} - \mathbf{m}_J), h_J^{\omega,a}\right\rangle_\omega \right| \\
&\quad + \left\langle \left[\int \sup_{\theta_J \in J} |\nabla K_y^\alpha(\theta_J) - \nabla K_y^\alpha(\mathbf{m}_J)|\, d\mu(y)\right] |x - \mathbf{m}_J|, |h_J^{\omega,a}| \right\rangle_\omega \\
&\lesssim C_{CZ} \frac{\mathrm{P}^\alpha(J,|\mu|)}{|J|^{1/n}} \| \triangle_J^\omega \mathbf{x}\|_{L^2(\omega)} + C_{CZ} \frac{\mathrm{P}_{1+\delta}^\alpha(J,|\mu|)}{|J|^{1/n}} \|\mathbf{x} - \mathbf{m}_J\|_{L^2(\mathbf{1}_J\omega)}.
\end{aligned}$$

□

6.2. The energy lemma

Suppose now we are given a subset $\mathcal{H}$ of the dyadic grid $\mathcal{D}^\omega$. Let $\mathsf{P}^\omega_{\mathcal{H}} = \sum_{J\in\mathcal{H}} \triangle^\omega_J$ be the ω-Haar projection onto $\mathcal{H}$. For μ, ω positive locally finite Borel measures on $\mathbb{R}^n$, and $\mathcal{H}$ a subset of the dyadic grid $\mathcal{D}^\omega$, we define

$$\mathcal{H}^* \equiv \bigcup_{J\in\mathcal{H}} \{J' \in \mathcal{D}^\omega : J' \subset J\}.$$

Lemma 6.2 (Energy lemma). *Let J be a cube in $\mathcal{D}^\omega$. Let Ψ_J be an $L^2(\omega)$ function supported in J and with ω-integral zero, and denote its Haar support by $\mathcal{H} = \operatorname{supp}\widehat{\Psi_J}$. Let ν be a positive measure supported in $\mathbb{R}^n \setminus \gamma J$ with $\gamma \geq 2$, and for each $J' \in \mathcal{H}$, let $d\nu_{J'} = \varphi_{J'}\, d\nu$ with $|\varphi_{J'}| \leq 1$. Let T^α be a standard α-fractional Calderón–Zygmund operator with $0 \leq \alpha < n$. Then with $\delta' = \delta/2$ we have*

$$\begin{aligned}\Big|\sum_{J'\in\mathcal{H}} \langle T^\alpha(\nu_{J'}), \triangle^\omega_{J'}\Psi_J\rangle_\omega\Big| &\lesssim \|\Psi_J\|_{L^2(\omega)} \left(\frac{\mathrm{P}^\alpha(J,\nu)}{|J|^{1/n}}\right) \|\mathsf{P}^\omega_{\mathcal{H}}\mathbf{x}\|_{L^2(\omega)} \\ &\quad + \|\Psi_J\|_{L^2(\omega)} \frac{1}{\gamma^{\delta'}} \left(\frac{\mathrm{P}^\alpha_{1+\delta'}(J,\nu)}{|J|^{1/n}}\right) \|\mathsf{P}^\omega_{\mathcal{H}^*}\mathbf{x}\|_{L^2(\omega)} \\ &\lesssim \|\Psi_J\|_{L^2(\omega)} \left(\frac{\mathrm{P}^\alpha(J,\nu)}{|J|^{1/n}}\right) \|\mathsf{P}^\omega_{\mathcal{H}^*}\mathbf{x}\|_{L^2(\omega)},\end{aligned}$$

and in particular the 'pivotal' bound

$$|\langle T^\alpha(\nu), \Psi_J\rangle_\omega| \leq C\,\|\Psi_J\|_{L^2(\omega)}\, \mathrm{P}^\alpha(J,|\nu|)\sqrt{|J|_\omega}\,.$$

Remark 6.3. The first term on the right side of the energy inequality above is the 'big' Poisson integral P^α times the 'small' energy term $\|\mathsf{P}^\omega_{\mathcal{H}}\mathbf{x}\|^2_{L^2(\omega)}$ that is additive in $\mathcal{H}$, while the second term on the right is the 'small' Poisson integral $\mathrm{P}^\alpha_{1+\delta'}$ times the 'big' energy term $\|\mathsf{P}^\omega_{\mathcal{H}^*}\mathbf{x}\|_{L^2(\omega)}$ that is no longer additive in $\mathcal{H}$. The first term presents no problems in subsequent analysis due solely to the additivity of the 'small' energy term. It is the second term that must be handled by special methods. For example, in the intertwining proposition below, the interaction of the singular integral occurs with a pair of cubes $J \subset I$ at *highly separated* levels, where the goodness of J can exploit the decay δ' in the kernel of the 'small' Poisson integral $\mathrm{P}^\alpha_{1+\delta'}$ relative to the 'big' Poisson integral P^α, and results in a bound directly by the energy condition. On the other hand, in the local recursion of M. Lacey at the end of the paper, the separation of levels in the pairs $J \subset I$ can be as *little* as a fixed parameter $\boldsymbol{\rho}$, and here we must first separate the stopping form into two sublinear forms that involve the two estimates respectively. The form corresponding to the smaller Poisson integral $\mathrm{P}^\alpha_{1+\delta'}$ is again handled using goodness and the decay δ' in the kernel, while the form corresponding to the larger Poisson integral P^α requires the full force of the stopping time and recursion argument of M. Lacey.

Proof. Using the monotonicity Lemma 6.1, followed by $|\nu_{J'}| \leq \nu$ and the Poisson equivalence

$$\frac{\mathrm{P}^\alpha_m(J',\nu)}{|J'|^{m/n}} \approx \frac{\mathrm{P}^\alpha_m(J,\nu)}{|J|^{m/n}}, \quad J' \subset J \subset 2J, \quad \operatorname{supp}\nu \cap 2J = \emptyset, \tag{6.2}$$

we have

$$\begin{aligned}
&\Big|\sum_{J'\in\mathcal{H}}\langle T^\alpha(\nu_{J'}),\triangle^\omega_{J'}\Psi_J\rangle_\omega\Big| = \Big|\sum_{J'\in\mathcal{H}}\langle\triangle^\omega_{J'}T^\alpha(\nu_{J'}),\triangle^\omega_{J'}\Psi_J\rangle_\omega\Big|\\
&\lesssim \Big|\sum_{J'\in\mathcal{H}}\Phi^\alpha(J',|\nu_{J'}|)\|\triangle^\omega_{J'}\Psi_J\|_{L^2(\omega)}\Big|\\
&\lesssim \Big(\sum_{J'\in\mathcal{H}}\Big(\frac{\mathrm{P}^\alpha(J',\nu)}{|J'|^{1/n}}\Big)^2\|\triangle^\omega_{J'}\mathbf{x}\|^2_{L^2(\omega)}\Big)^{1/2}\Big(\sum_{J'\in\mathcal{H}}\|\triangle^\omega_{J'}\Psi_J\|^2_{L^2(\omega)}\Big)^{1/2}\\
&\quad+\Big(\sum_{J'\in\mathcal{H}}\Big(\frac{\mathrm{P}^\alpha_{1+\delta}(J',\nu)}{|J'|^{1/n}}\Big)^2\sum_{J''\subset J'}\|\triangle^\omega_{J''}\mathbf{x}\|^2_{L^2(\omega)}\Big)^{1/2}\Big(\sum_{J'\in\mathcal{H}}\|\triangle^\omega_{J'}\Psi_J\|^2_{L^2(\omega)}\Big)^{1/2}\\
&\lesssim \Big(\frac{\mathrm{P}^\alpha(J,\nu)}{|J|^{1/n}}\Big)\|\mathsf{P}^\omega_{\mathcal{H}}\mathbf{x}\|_{L^2(\omega)}\|\Psi_J\|_{L^2(\omega)}+\frac{1}{\gamma^{\delta'}}\Big(\frac{\mathrm{P}^\alpha_{1+\delta'}(J,\nu)}{|J|^{1/n}}\Big)\|\mathsf{P}^\omega_{\mathcal{H}^*}\mathbf{x}\|_{L^2(\omega)}\|\Psi_J\|_{L^2(\omega)}\,.
\end{aligned}$$

The last inequality follows from

$$\begin{aligned}
&\sum_{J'\in\mathcal{H}}\Big(\frac{\mathrm{P}^\alpha_{1+\delta}(J',\nu)}{|J'|^{1/n}}\Big)^2\sum_{J''\subset J'}\|\triangle^\omega_{J''}\mathbf{x}\|^2_{L^2(\omega)}\\
&\qquad=\sum_{J''\subset J}\Big\{\sum_{J':\,J''\subset J'\subset J}\Big(\frac{\mathrm{P}^\alpha_{1+\delta}(J',\nu)}{|J'|^{1/n}}\Big)^2\Big\}\|\triangle^\omega_{J''}\mathbf{x}\|^2_{L^2(\omega)}\\
&\qquad\lesssim\frac{1}{\gamma^{2\delta'}}\sum_{J''\in\mathcal{H}^*}\Big(\frac{\mathrm{P}^\alpha_{1+\delta'}(J'',\nu)}{|J''|^{1/n}}\Big)^2\|\triangle^\omega_{J''}\mathbf{x}\|^2_{L^2(\omega)}\,,
\end{aligned}$$

which in turn follows from (recalling $\delta=2\delta'$)

$$\begin{aligned}
&\sum_{J':\,J''\subset J'\subset J}\Big(\frac{\mathrm{P}^\alpha_{1+\delta}(J',\nu)}{|J'|^{1/n}}\Big)^2\\
&\quad=\sum_{J':\,J''\subset J'\subset J}|J'|^{2\delta/n}\Big(\int_{\mathbb{R}^n\setminus\gamma J}\frac{1}{(|J'|^{1/n}+|y-c_{J'}|)^{n+1+\delta-\alpha}}\,d\nu(y)\Big)^2\\
&\quad\lesssim\sum_{J':\,J''\subset J'\subset J}\frac{1}{\gamma^{2\delta'}}\frac{|J'|^{2\delta/n}}{|J|^{2\delta/n}}\Big(\int_{\mathbb{R}^n\setminus\gamma J}\frac{|J|^{\delta'/n}}{\big(|J|^{1/n}+|y-c_J|\big)^{n+1+\delta'-\alpha}}\,d\nu(y)\Big)^2\\
&\quad=\frac{1}{\gamma^{2\delta'}}\Big(\sum_{J':\;J''\subset J'\subset J}\frac{|J'|^{2\delta/n}}{|J|^{2\delta/n}}\Big)\Big(\frac{\mathrm{P}^\alpha_{1+\delta'}(J,\nu)}{|J|^{1/n}}\Big)^2\\
&\quad\lesssim\frac{1}{\gamma^{2\delta'}}\Big(\frac{\mathrm{P}^\alpha_{1+\delta'}(J,\nu)}{|J|^{1/n}}\Big)^2.
\end{aligned}$$

Finally we have the 'pivotal' bound from (6.2) and

$$\sum_{J''\subset J}\|\triangle^\omega_{J''}\mathbf{x}\|^2_{L^2(\omega)}=\|\mathbf{x}-\mathbf{m}_J\|^2_{L^2(\mathbf{1}_J\omega)}\leq|J|^{2/n}\,|J|_\omega\,. \qquad\square$$

7. Preliminaries of NTV type

An important reduction of our theorem is delivered by the following two lemmas, that in the case of one dimension are due to Nazarov, Treil and Volberg (see [13] and [23]). The proofs given there do not extend in standard ways to higher dimensions, and we use the weak boundedness property to handle the case of touching cubes, and an application of Schur's lemma to handle the case of separated cubes. The first lemma below is Lemmas 8.1 and 8.7 in [8] but with the larger constant $\mathcal{A}_2^\alpha$ there in place of A_2^α.

Lemma 7.1. *Suppose T^α is a standard fractional singular integral with $0 \leq \alpha < n$, and that all of the cubes $I \in \mathcal{D}^\sigma$, $J \in \mathcal{D}^\omega$ below are good with goodness parameters ε and $\mathbf{r}$. Fix a positive integer $\boldsymbol{\rho} > \mathbf{r}$. For $f \in L^2(\sigma)$ and $g \in L^2(\omega)$ we have*

$$\sum_{\substack{(I,J)\in\mathcal{D}^\sigma\times\mathcal{D}^\omega \\ 2^{-\boldsymbol{\rho}}|I|^{1/n}\leq|J|^{1/n}\leq 2^{\boldsymbol{\rho}}|I|^{1/n}}} |\langle T_\sigma^\alpha(\triangle_I^\sigma f), \triangle_J^\omega g\rangle_\omega|$$

$$\lesssim \left(\mathfrak{T}_\alpha + \mathfrak{T}_\alpha^* + \mathcal{WBP}_{T^\alpha} + \sqrt{A_2^\alpha}\,\right)\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)} \tag{7.1}$$

and

$$\sum_{\substack{(I,J)\in\mathcal{D}^\sigma\times\mathcal{D}^\omega \\ I\cap J=\emptyset \text{ and } \frac{|J|^{1/n}}{|I|^{1/n}}\notin[2^{-\boldsymbol{\rho}},2^{\boldsymbol{\rho}}]}} |\langle T_\sigma^\alpha(\triangle_I^\sigma f), \triangle_J^\omega g\rangle_\omega| \lesssim \sqrt{A_2^\alpha}\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}. \tag{7.2}$$

Lemma 7.2. *Suppose T^α is a standard fractional singular integral with $0 \leq \alpha < n$, that all of the cubes $I \in \mathcal{D}^\sigma$, $J \in \mathcal{D}^\omega$ below are good, that $\boldsymbol{\rho} > \mathbf{r}$, that $f \in L^2(\sigma)$ and $g \in L^2(\omega)$, that $\mathcal{F} \subset \mathcal{D}^\sigma$ and $\mathcal{G} \subset \mathcal{D}^\omega$ are σ-Carleson and ω-Carleson collections respectively, i.e.,*

$$\sum_{F'\in\mathcal{F}:\,F'\subset F} |F'|_\sigma \lesssim |F|_\sigma, \quad F \in \mathcal{F}, \quad \text{and} \quad \sum_{G'\in\mathcal{G}:\,G'\subset G} |G'|_\omega \lesssim |G|_\omega, \quad G \in \mathcal{G},$$

that there are numerical sequences $\{\alpha_{\mathcal{F}}(F)\}_{F\in\mathcal{F}}$ and $\{\beta_{\mathcal{G}}(G)\}_{G\in\mathcal{G}}$ such that

$$\sum_{F\in\mathcal{F}} \alpha_{\mathcal{F}}(F)^2\,|F|_\sigma \leq \|f\|_{L^2(\sigma)}^2 \quad \text{and} \quad \sum_{G\in\mathcal{G}} \beta_{\mathcal{G}}(G)^2\,|G|_\sigma \leq \|g\|_{L^2(\sigma)}^2\,, \tag{7.3}$$

and finally that for each pair of cubes $(I,J) \in \mathcal{D}^\sigma\times\mathcal{D}^\omega$, there are bounded functions $\beta_{I,J}$ and $\gamma_{I,J}$ supported in $I\setminus 2J$ and $J\setminus 2I$ respectively, satisfying

$$\|\beta_{I,J}\|_\infty, \|\gamma_{I,J}\|_\infty \leq 1.$$

Then

$$\sum_{\substack{(F,J)\in\mathcal{F}\times\mathcal{D}^\omega \\ F\cap J=\emptyset \text{ and } |J|^{1/n}\leq 2^{-\rho}|F|^{1/n}}} \left|\left\langle T_\sigma^\alpha\left(\beta_{F,J}\mathbf{1}_F\alpha_{\mathcal{F}}(F)\right),\triangle_J^\omega g\right\rangle_\omega\right| + \sum_{\substack{(I,G)\in\mathcal{D}^\sigma\times\mathcal{G} \\ I\cap G=\emptyset \text{ and } |I|^{1/n}\leq 2^{-\rho}|G|^{1/n}}} \left|\left\langle T_\sigma^\alpha\left(\triangle_I^\sigma f\right),\gamma_{I,G}\mathbf{1}_G\beta_{\mathcal{G}}(G)\right\rangle_\omega\right| \lesssim \sqrt{A_2^\alpha}\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}. \tag{7.4}$$

Remark 7.3. If $\mathcal{F}$ and $\mathcal{G}$ are σ-Carleson and ω-Carleson collections respectively, and if $\alpha_{\mathcal{F}}(F)=\mathbb{E}_F^\sigma|f|$ and $\beta_{\mathcal{G}}(G)=\mathbb{E}_G^\omega|g|$, then the quasiorthogonality condition (7.3) holds, and this special case of Lemma 7.2 serves as a basic example.

Remark 7.4. Lemmas 7.1 and 7.2 differ mainly in that an orthogonal collection of Haar projections is replaced by a quasiorthogonal collection of indicators $\{\mathbf{1}_F\alpha_{\mathcal{F}}(F)\}_{F\in\mathcal{F}}$. More precisely, the main difference between (7.2) and (7.4) is that a Haar projection $\triangle_I^\sigma f$ or $\triangle_J^\omega g$ has been replaced with a constant multiple of an indicator $\mathbf{1}_F\alpha_{\mathcal{F}}(F)$ or $\mathbf{1}_G\beta_{\mathcal{G}}(G)$, and in addition, a bounded function is permitted to multiply the indicator of the cube having larger sidelength.

Proof. Note that in (7.1) we have used the parameter $\boldsymbol{\rho}$ in the exponent rather than $\mathbf{r}$, and this is possible because the arguments we use here only require that there are finitely many levels of scale separating I and J. To handle this term we first decompose it into

$$\left\{\sum_{\substack{(I,J)\in\mathcal{D}^\sigma\times\mathcal{D}^\omega:\, J\subset 3I \\ 2^{-\rho}|I|^{1/n}\leq|J|^{1/n}\leq 2^\rho|I|^{1/n}}} + \sum_{\substack{(I,J)\in\mathcal{D}^\sigma\times\mathcal{D}^\omega:\, I\subset 3J \\ 2^{-\rho}|I|^{1/n}\leq|J|^{1/n}\leq 2^\rho|I|^{1/n}}} + \sum_{\substack{(I,J)\in\mathcal{D}^\sigma\times\mathcal{D}^\omega \\ 2^{-\rho}|I|^{1/n}\leq|J|^{1/n}\leq 2^\rho|I|^{1/n} \\ J\not\subset 3I \text{ and } I\not\subset 3J}}\right\} \left|\left\langle T_\sigma^\alpha\left(\triangle_I^\sigma f\right),\triangle_J^\omega g\right\rangle_\omega\right| \equiv A_1+A_2+A_3.$$

The proof of the bound for term A_3 is similar to that of the bound for the left side of (7.2), and so we will defer the bound for A_3 until after (7.2) has been proved.

We now consider term A_1 as term A_2 is symmetric. To handle this term we will write the Haar functions h_I^σ and h_J^ω as linear combinations of the indicators of the children of their supporting cubes, denoted I_θ and $J_{\theta'}$ respectively. Then we use the testing condition on I_θ and $J_{\theta'}$ when they *overlap*, i.e. their interiors intersect; we use the weak boundedness property on I_θ and $J_{\theta'}$ when they *touch*, i.e., their interiors are disjoint but their closures intersect (even in just a point); and finally we use the A_2^α condition when I_θ and $J_{\theta'}$ are *separated*, i.e., their closures are disjoint. We will suppose initially that the side length of J is at most the side length I, i.e., $|J|^{1/n}\leq|I|^{1/n}$, the proof for $J=\pi I$ being similar but for one point mentioned below. So suppose that I_θ is a child of I and that $J_{\theta'}$ is a child of J. If $J_{\theta'} \subset I_\theta$ we have from (5.2) that

$$\begin{aligned}
|\langle T_\sigma^\alpha(\mathbf{1}_{I_\theta} \triangle_I^\sigma f), \mathbf{1}_{J_{\theta'}} \triangle_J^\omega g\rangle_\omega| &\lesssim \sup_{a,a'\in\Gamma_n} \frac{|\langle f, h_I^{\sigma,a}\rangle_\sigma|}{\sqrt{|I_\theta|_\sigma}} |\langle T_\sigma^\alpha(\mathbf{1}_{I_\theta}), \mathbf{1}_{J_{\theta'}}\rangle_\omega| \frac{|\langle g, h_J^{\omega,a'}\rangle_\omega|}{\sqrt{|J_{\theta'}|_\omega}} \\
&\lesssim \sup_{a,a'\in\Gamma_n} \frac{|\langle f, h_I^{\sigma,a}\rangle_\sigma|}{\sqrt{|I_\theta|_\sigma}} \left(\int_{J_{\theta'}} |T_\sigma^\alpha(\mathbf{1}_{I_\theta})|^2 d\omega\right)^{1/2} |\langle g, h_J^{\omega,a'}\rangle_\omega| \\
&\lesssim \sup_{a,a'\in\Gamma_n} \frac{|\langle f, h_I^{\sigma,a}\rangle_\sigma|}{\sqrt{|I_\theta|_\sigma}} \mathfrak{T}_{T_\alpha} |I_\theta|_\sigma^{1/2} |\langle g, h_J^{\omega,a'}\rangle_\omega| \\
&\lesssim \sup_{a,a'\in\Gamma_n} \mathfrak{T}_{T_\alpha} |\langle f, h_I^{\sigma,a}\rangle_\sigma| \, |\langle g, h_J^{\omega,a'}\rangle_\omega| .
\end{aligned}$$

The point referred to above is that when $J = \pi I$ we write $\langle T_\sigma^\alpha(\mathbf{1}_{I_\theta}), \mathbf{1}_{J_{\theta'}}\rangle_\omega = \langle \mathbf{1}_{I_\theta}, T_\omega^{\alpha,*}(\mathbf{1}_{J_{\theta'}})\rangle_\sigma$ and get the dual testing constant $\mathfrak{T}_{T_\alpha}^*$. If $J_{\theta'}$ and I_θ touch, then $|J_{\theta'}|^{1/n} \le |I_\theta|^{1/n}$ and we have $J_{\theta'} \subset 3I_\theta \setminus I_\theta$, and so

$$\begin{aligned}
|\langle T_\sigma^\alpha(\mathbf{1}_{I_\theta} \triangle_I^\sigma f), \mathbf{1}_{J_{\theta'}} \triangle_J^\omega g\rangle_\omega| &\lesssim \sup_{a,a'\in\Gamma_n} \frac{|\langle f, h_I^{\sigma,a}\rangle_\sigma|}{\sqrt{|I_\theta|_\sigma}} |\langle T_\sigma^\alpha(\mathbf{1}_{I_\theta}), \mathbf{1}_{J_{\theta'}}\rangle_\omega| \frac{|\langle g, h_J^{\omega,a'}\rangle_\omega|}{\sqrt{|J_{\theta'}|_\omega}} \\
&\lesssim \sup_{a,a'\in\Gamma_n} \frac{|\langle f, h_I^{\sigma,a}\rangle_\sigma|}{\sqrt{|I_\theta|_\sigma}} \mathcal{WBP}_{T^\alpha} \sqrt{|I_\theta|_\sigma |J_{\theta'}|_\omega} \frac{|\langle g, h_J^{\omega,a'}\rangle_\omega|}{\sqrt{|J_{\theta'}|_\omega}} \\
&= \sup_{a,a'\in\Gamma_n} \mathcal{WBP}_{T^\alpha} |\langle f, h_I^{\sigma,a}\rangle_\sigma| \, |\langle g, h_J^{\omega,a'}\rangle_\omega| .
\end{aligned}$$

Finally, if $J_{\theta'}$ and I_θ are separated, and if K is the smallest (not necessarily dyadic) cube containing both $J_{\theta'}$ and I_θ, then $\operatorname{dist}(I_\theta, J_{\theta'}) \approx |K|^{1/n}$ and we have

$$\begin{aligned}
|\langle T_\sigma^\alpha(\mathbf{1}_{I_\theta} \triangle_I^\sigma f), \mathbf{1}_{J_{\theta'}} \triangle_J^\omega g\rangle_\omega| &\lesssim \sup_{a,a'\in\Gamma_n} \frac{|\langle f, h_I^{\sigma,a}\rangle_\sigma|}{\sqrt{|I_\theta|_\sigma}} |\langle T_\sigma^\alpha(\mathbf{1}_{I_\theta}), \mathbf{1}_{J_{\theta'}}\rangle_\omega| \frac{|\langle g, h_J^{\omega,a'}\rangle_\omega|}{\sqrt{|J_{\theta'}|_\omega}} \\
&\lesssim \sup_{a,a'\in\Gamma_n} \frac{|\langle f, h_I^{\sigma,a}\rangle_\sigma|}{\sqrt{|I_\theta|_\sigma}} \frac{1}{\operatorname{dist}(I_\theta, J_{\theta'})^{n-\alpha}} |I_\theta|_\sigma \, |J_{\theta'}|_\omega \frac{|\langle g, h_J^{\omega,a'}\rangle_\omega|}{\sqrt{|J_{\theta'}|_\omega}} \\
&= \sup_{a,a'\in\Gamma_n} \frac{\sqrt{|I_\theta|_\sigma |J_{\theta'}|_\omega}}{\operatorname{dist}(I_\theta, J_{\theta'})^{n-\alpha}} |\langle f, h_I^{\sigma,a}\rangle_\sigma| \, |\langle g, h_J^{\omega,a'}\rangle_\omega| \\
&\lesssim \sup_{a,a'\in\Gamma_n} \frac{\sqrt{|K|_\sigma |K|_\omega}}{|K|^{\frac{1}{n}(n-\alpha)}} |\langle f, h_I^{\sigma,a}\rangle_\sigma| \, |\langle g, h_J^{\omega,a'}\rangle_\omega| \\
&\lesssim \sqrt{A_2^\alpha} \sup_{a,a'\in\Gamma_n} |\langle f, h_I^{\sigma,a}\rangle_\sigma| \, |\langle g, h_J^{\omega,a'}\rangle_\omega| .
\end{aligned}$$

Now we sum over all the children of J and I satisfying $2^{-\rho}|I|^{1/n} \le |J|^{1/n} \le 2^{\rho}|I|^{1/n}$ for which $J \subset 3I$ to obtain that

$$A_1 \lesssim \left(\mathfrak{T}_{T_\alpha} + \mathfrak{T}_{T_\alpha}^* + \mathcal{WBP}_{T^\alpha} + \sqrt{A_2^\alpha}\right) \sup_{a,a'\in\Gamma_n} \sum_{\substack{(I,J)\in\mathcal{D}^\sigma\times\mathcal{D}^\omega:\, J\subset 3I \\ 2^{-\rho}|I|^{1/n} \le |J|^{1/n} \le 2^{\rho}|I|^{1/n}}} |\langle f, h_I^{\sigma,a}\rangle_\sigma| |\langle g, h_J^{\omega,a'}\rangle_\omega| .$$

Now Cauchy–Schwarz gives the estimate

$$\sum_{a,a'\in\Gamma_n}\sum_{\substack{(I,J)\in\mathcal{D}^\sigma\times\mathcal{D}^\omega:\, J\subset 3I\\ 2^{-\rho}|I|^{1/n}\le|J|^{1/n}\le 2^{\rho}|I|^{1/n}}}|\langle f,h_I^\sigma\rangle_\sigma||\langle g,h_I^\omega\rangle_\omega|$$

$$\le\sup_{a,a'\in\Gamma_n}\Bigg(\sum_{\substack{(I,J)\in\mathcal{D}^\sigma\times\mathcal{D}^\omega:\, J\subset 3I\\ 2^{-\rho}|I|^{1/n}\le|J|^{1/n}\le 2^{\rho}|I|^{1/n}}}|\langle f,h_I^\sigma\rangle_\sigma|^2\Bigg)^{1/2}\Bigg(\sum_{\substack{(I,J)\in\mathcal{D}^\sigma\times\mathcal{D}^\omega:\, J\subset 3I\\ 2^{-\rho}|I|^{1/n}\le|J|^{1/n}\le 2^{\rho}|I|^{1/n}}}|\langle g,h_J^\omega\rangle_\omega|^2\Bigg)^{1/2}$$

$$\lesssim\|f\|_{L^2(\sigma)}\,\|g\|_{L^2(\omega)}\,,$$

This completes our proof of (7.1) save for the deferral of term A_3, which we bound below.

Now we turn to the sum of separated cubes in (7.2) and (7.4). In each of these inequalities we have either orthogonality or quasiorthogonality, due either to the presence of a Haar projection such as $\triangle_I^\sigma f$, or the presence of an appropriate Carleson indicator such as $\beta_{F,J}\mathbf{1}_F\alpha_{\mathcal{F}}(F)$. We will prove below the estimate for the separated sum corresponding to (7.2). The corresponding estimates for (7.4) are handled in a similar way, the only difference being that the quasiorthogonality of Carleson indicators such as $\beta_{F,J}\mathbf{1}_F\alpha_{\mathcal{F}}(F)$ is used in place of the orthogonality of Haar functions such as $\triangle_I^\sigma f$. The bounded functions $\beta_{F,J}$ are replaced with constants after an application of the energy lemma, and then the arguments proceed as below.

We split the pairs $(I,J)\in\mathcal{D}^\sigma\times\mathcal{D}^\omega$ occurring in (7.2) into two groups, those with side length of J smaller than side length of I, and those with side length of I smaller than side length of J, treating only the former case, the latter being symmetric. Thus we prove the following bound:

$$\mathcal{A}(f,g)\equiv\sum_{\substack{(I,J)\in\mathcal{D}^\sigma\times\mathcal{D}^\omega\\ I\cap J=\emptyset\text{ and }|J|^{1/n}\le 2^{-\rho}|I|^{1/n}}}|\langle T_\sigma^\alpha(\triangle_I^\sigma f),\triangle_J^\omega g\rangle_\omega|\lesssim\sqrt{A_2^\alpha}\,\|f\|_{L^2(\sigma)}\,\|g\|_{L^2(\omega)}\,.$$

We apply the 'pivotal' bound from the energy Lemma 6.2 to estimate the inner product $\langle T_\sigma^\alpha(\triangle_I^\sigma f),\triangle_J^\omega g\rangle_\omega$ and obtain,

$$|\langle T_\sigma^\alpha(\triangle_I^\sigma f),\triangle_J^\omega g\rangle_\omega|\lesssim\|\triangle_J^\omega g\|_{L^2(\omega)}\,\mathrm{P}^\alpha(J,|\triangle_I^\sigma f|\sigma)\sqrt{|J|_\omega}\,,$$

Denote by dist the ℓ^∞ distance in $\mathbb{R}^n$: $\mathrm{dist}(x,y)=\max_{1\le j\le n}|x_j-y_j|$. We now estimate separately the long-range and mid-range cases where $\mathrm{dist}(J,I)\ge|I|^{1/n}$ holds or not, and we decompose A accordingly:

$$\mathcal{A}(f,g)\equiv\mathcal{A}^{\mathrm{long}}(f,g)+\mathcal{A}^{\mathrm{mid}}(f,g).$$

The long-range case. We begin with the case where $\mathrm{dist}(J,I)$ is at least $|I|^{1/n}$, i.e., $J\cap 3I=\emptyset$. Since J and I are separated by at least $\max\{|J|^{1/n},|I|^{1/n}\}$, we have the inequality

$$\mathrm{P}^\alpha(J,|\triangle_I^\sigma f|\sigma)\approx\int_I\frac{|J|^{1/n}}{|y-c_J|^{n+1-\alpha}}|\triangle_I^\sigma f(y)|\,d\sigma(y)\le\|\triangle_I^\sigma f\|_{L^2(\sigma)}\frac{|J|^{1/n}\sqrt{|I|_\sigma}}{\mathrm{dist}(I,J)^{n+1-\alpha}},$$

since

$$\int_I |\triangle_I^\sigma f(y)|\, d\sigma(y) \leq \| \triangle_I^\sigma f\|_{L^2(\sigma)} \sqrt{|I|_\sigma}.$$

Thus with $A(f,g) = \mathcal{A}^{\mathrm{long}}(f,g)$ we have

$$\begin{aligned} A(f,g) &\leq \sum_{I\in\mathcal{D}} \sum_{J\,:\,|J|^{1/n}\leq|I|^{1/n}:\,\mathrm{dist}(I,J)\geq|I|^{1/n}} \| \triangle_I^\sigma f\|_{L^2(\sigma)} \| \triangle_J^\omega g\|_{L^2(\omega)} \\ &\qquad \times \frac{|J|^{1/n}}{\mathrm{dist}(I,J)^{n+1-\alpha}} \sqrt{|I|_\sigma}\sqrt{|J|_\omega} \\ &\equiv \sum_{(I,J)\in\mathcal{P}} \| \triangle_I^\sigma f\|_{L^2(\sigma)} \| \triangle_J^\omega g\|_{L^2(\omega)} A(I,J); \end{aligned}$$

$$\text{with} \quad A(I,J) \equiv \frac{|J|^{1/n}}{\mathrm{dist}(I,J)^{n+1-\alpha}} \sqrt{|I|_\sigma}\sqrt{|J|_\omega};$$

$$\text{and} \qquad \mathcal{P} \equiv \left\{ (I,J) \in \mathcal{D}\times\mathcal{D} : |J|^{1/n} \leq |I|^{1/n} \text{ and } \mathrm{dist}(I,J) \geq |I|^{1/n} \right\}.$$

Now let $\mathcal{D}_N \equiv \{K \in \mathcal{D} : |K|^{1/n} = 2^N\}$ for each $N \in \mathbb{Z}$. For $N \in \mathbb{Z}$ and $s \in \mathbb{Z}_+$, we further decompose $A(f,g)$ by pigeonholing the side lengths of I and J by 2^N and 2^{N-s} respectively:

$$\begin{aligned} A(f,g) &= \sum_{s=0}^{\infty} \sum_{N\in\mathbb{Z}} A_N^s(f,g); \\ A_N^s(f,g) &\equiv \sum_{(I,J)\in\mathcal{P}_N^s} \| \triangle_I^\sigma f\|_{L^2(\sigma)} \| \triangle_J^\omega g\|_{L^2(\omega)} A(I,J), \\ \text{where } \mathcal{P}_N^s &\equiv \left\{ (I,J) \in \mathcal{D}_N \times \mathcal{D}_{N-s} : \mathrm{dist}(I,J) \geq |I|^{1/n} \right\}. \end{aligned}$$

Now $A_N^s(f,g) = A_N^s(\mathsf{P}_N^\sigma f, \mathsf{P}_{N-s}^\omega g)$ where $\mathsf{P}_M^\mu = \sum_{K\in\mathcal{D}_M} \triangle_K^\mu$ denotes Haar projection onto $\mathrm{Span}\{h_K^{\mu,a}\}_{K\in\mathcal{D}_M, a\in\Gamma_n}$, and so by orthogonality of the projections $\{\mathsf{P}_M^\mu\}_{M\in\mathbb{Z}}$ we have

$$\begin{aligned} \Big| \sum_{N\in\mathbb{Z}} A_N^s(f,g) \Big| &= \sum_{N\in\mathbb{Z}} |A_N^s(\mathsf{P}_N^\sigma f, \mathsf{P}_{N-s}^\omega g)| \leq \sum_{N\in\mathbb{Z}} \|A_N^s\| \|\mathsf{P}_N^\sigma f\|_{L^2(\sigma)} \|\mathsf{P}_{N-s}^\omega g\|_{L^2(\omega)} \\ &\leq \Big\{ \sup_{N\in\mathbb{Z}} \|A_N^s\| \Big\} \Big(\sum_{N\in\mathbb{Z}} \|\mathsf{P}_N^\sigma f\|_{L^2(\sigma)}^2 \Big)^{1/2} \Big(\sum_{N\in\mathbb{Z}} \|\mathsf{P}_{N-s}^\omega g\|_{L^2(\omega)}^2 \Big)^{1/2} \\ &\leq \Big\{ \sup_{N\in\mathbb{Z}} \|A_N^s\| \Big\} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}. \end{aligned}$$

Thus it suffices to show an estimate uniform in N with geometric decay in s, and we will show

$$|A_N^s(f,g)| \leq C\, 2^{-s} \sqrt{\mathcal{A}_2^\alpha}\, \|f\|_{L^2(\sigma)}\, \|g\|_{L^2(\omega)}, \quad \text{for } s \geq 0 \text{ and } N \in \mathbb{Z}. \tag{7.5}$$

We now pigeonhole the distance between I and J:

$$A_N^s(f,g) = \sum_{\ell=0}^{\infty} A_{N,\ell}^s(f,g);$$
$$A_{N,\ell}^s(f,g) \equiv \sum_{(I,J)\in\mathcal{P}_{N,\ell}^s} \| \triangle_I^\sigma f\|_{L^2(\sigma)} \| \triangle_J^\omega g\|_{L^2(\omega)} A(I,J),$$
$$\text{where } \mathcal{P}_{N,\ell}^s \equiv \left\{(I,J) \in \mathcal{D}_N \times \mathcal{D}_{N-s} : \operatorname{dist}(I,J) \approx 2^{N+\ell}\right\}.$$

If we define $\mathcal{H}(A_{N,\ell}^s)$ to be the bilinear form on $\ell^2\times\ell^2$ with matrix $[A(I,J)]_{(I,J)\in\mathcal{P}_{N,\ell}^s}$, then it remains to show that the norm $\|\mathcal{H}(A_{N,\ell}^s)\|_{\ell^2\to\ell^2}$ of $\mathcal{H}(A_{N,\ell}^s)$ on the sequence space ℓ^2 is bounded by $C2^{-s-\ell}\sqrt{A_2^\alpha}$. In turn, this is equivalent to showing that the norm $\|\mathcal{H}(B_{N,\ell}^s)\|_{\ell^2\to\ell^2}$ of the bilinear form $\mathcal{H}(B_{N,\ell}^s) \equiv \mathcal{H}(A_{N,\ell}^s)^{\mathrm{tr}}\mathcal{H}(A_{N,\ell}^s)$ on the sequence space ℓ^2 is bounded by $C^2 2^{-2s-2\ell}A_2^\alpha$. Here $\mathcal{H}(B_{N,\ell}^s)$ is the quadratic form with matrix kernel $[B_{N,\ell}^s(J,J')]_{J,J'\in\mathcal{D}_{N-s}}$ having entries:

$$B_{N,\ell}^s(J,J') \equiv \sum_{I\in\mathcal{D}_N:\operatorname{dist}(I,J)\approx\operatorname{dist}(I,J')\approx 2^{N+\ell}} A(I,J)A(I,J'), \quad \text{for } J,J'\in\mathcal{D}_{N-s}.$$

We are reduced to showing

$$\|\mathcal{H}(B_{N,\ell}^s)\|_{\ell^2\to\ell^2} \le C\,2^{-2s-2\ell}A_2^\alpha \quad \text{for } s\ge 0,\ \ell\ge 0 \text{ and } N\in\mathbb{Z}.$$

For this we begin by computing $B_{N,\ell}^s(J,J')$:

$$\begin{aligned} B_{N,\ell}^s(J,J') &= \sum_{\substack{I\in\mathcal{D}_N\\ \operatorname{dist}(I,J)\approx\operatorname{dist}(I,J')\approx 2^{N+\ell}}} \frac{|J|^{1/n}}{\operatorname{dist}(I,J)^{n+1-\alpha}}\sqrt{|I|_\sigma}\sqrt{|J|_\omega} \\ &\qquad\qquad \times \frac{|J'|^{1/n}}{\operatorname{dist}(I,J')^{n+1-\alpha}}\sqrt{|I|_\sigma}\sqrt{|J'|_\omega} \\ &= \Bigg\{ \sum_{\substack{I\in\mathcal{D}_N\\ \operatorname{dist}(I,J)\approx\operatorname{dist}(I,J')\approx 2^{N+\ell}}} |I|_\sigma \frac{1}{\operatorname{dist}(I,J)^{n+1-\alpha}\operatorname{dist}(I,J')^{n+1-\alpha}} \Bigg\} \\ &\qquad\qquad \times |J|^{1/n}\,|J'|^{1/n}\sqrt{|J|_\omega}\sqrt{|J'|_\omega}. \end{aligned}$$

Now we show that

$$\|B_{N,\ell}^s\|_{\ell^2\to\ell^2} \lesssim 2^{-2s-2\ell}A_2^\alpha, \tag{7.6}$$

by applying the proof of Schur's lemma. Fix $\ell\ge 0$ and $s\ge 0$. Choose the Schur function $\beta(K) = 1/\sqrt{|K|_\omega}$. Fix $J\in\mathcal{D}_{N-s}$. We have

$$\begin{aligned} \sum_{J'\in\mathcal{D}_{N-s}} \frac{\beta(J)}{\beta(J')} B_{N,\ell}^s(J,J') &\lesssim \sum_{\substack{J'\in\mathcal{D}_{N-s}\\ \operatorname{dist}(J,J')\le 2^{N+\ell+2}}} \Bigg\{ \sum_{\substack{I\in\mathcal{D}_N\\ \operatorname{dist}(I,J)\approx 2^{N+\ell}}} |I|_\sigma \Bigg\} \frac{2^{2(N-s)}}{2^{2(\ell+N)(n+1-\alpha)}}\,|J'|_\omega \\ &\lesssim 2^{-2s-2\ell}\frac{|2^{10+\ell+s}J|_\sigma}{2^{(\ell+N)(n-\alpha)}}\frac{|2^{12+\ell+s}J|_\omega}{2^{(\ell+N)(n-\alpha)}} \lesssim 2^{-2s-2\ell}A_2^\alpha, \end{aligned}$$

since $I \in \mathcal{D}_N$ and $\operatorname{dist}(I,J) \approx 2^{N+\ell}$ imply that $I \subset 2^{10+\ell+s}J$ which has side length comparable to $2^{(\ell+N)}$, and similarly $J' \subset 2^{12+\ell+s}J$. Thus we can now apply Schur's argument with $\sum_J (a_J)^2 = \sum_{J'} (b_{J'})^2 = 1$ to obtain

$$\begin{aligned}\sum_{J,J'\in\mathcal{D}_{N-s}} a_J b_{J'} B^s_{N,\ell}(J,J') &= \sum_{J,J'\in\mathcal{D}_{N-s}} a_J \beta(J) b_{J'} \beta(J') \frac{B^s_{N,\ell}(J,J')}{\beta(J)\beta(J')} \\ &\le \sum_J (a_J\beta(J))^2 \sum_{J'} \frac{B^s_{N,\ell}(J,J')}{\beta(J)\beta(J')} + \sum_{J'} (b_{J'}\beta(J'))^2 \frac{B^s_{N,\ell}(J,J')}{\beta(J)\beta(J')} \\ &= \sum_J (a_J)^2 \Big\{ \sum_{J'} \frac{\beta(J)}{\beta(J')} B^s_{N,\ell}(J,J') \Big\} + \sum_{J'} (b_{J'})^2 \Big\{ \sum_J \frac{\beta(J')}{\beta(J)} B^s_{N,\ell}(J,J') \Big\} \\ &\lesssim 2^{-2s-2\ell} A_2^\alpha \Big(\sum_J (a_J)^2 + \sum_{J'} (b_{J'})^2 \Big) = 2^{1-2s-2\ell} A_2^\alpha \,.\end{aligned}$$

This completes the proof of (7.6). We can now sum in ℓ to get (7.5) and we are done. This completes our proof of the long-range estimate

$$\mathcal{A}^{\text{long}}(f,g) \lesssim \sqrt{A_2^\alpha}\, \|f\|_{L^2(\sigma)}\, \|g\|_{L^2(\omega)} \,.$$

At this point we pause to complete the proof of (7.1). Indeed, the deferred term A_3 can be handled using the above argument since $3J \cap I = \emptyset = J \cap 3I$ implies that we can use the energy Lemma 6.2 as we did above.

The mid range case. Let

$$\mathcal{P} \equiv \big\{ (I,J) \in \mathcal{D}\times\mathcal{D} : J \text{ is good},\ |J|^{1/n} \le 2^{-\boldsymbol{\rho}} |I|^{1/n},\ J \subset 3I \setminus I \big\} \,.$$

For $(I,J) \in \mathcal{P}$, the 'pivotal' estimate from the energy Lemma 6.2 gives

$$|\langle T^\alpha_\sigma(\triangle^\sigma_I f), \triangle^\omega_J g\rangle_\omega| \lesssim \| \triangle^\omega_J g\|_{L^2(\omega)}\, \mathrm{P}^\alpha(J, |\triangle^\sigma_I f|\sigma) \sqrt{|J|_\omega} \,.$$

Now we pigeonhole the lengths of I and J and the distance between them by defining

$$\begin{aligned}\mathcal{P}^s_{N,d} \equiv \big\{ (I,J) \in \mathcal{D}\times\mathcal{D} : J \text{ is good},\ |I|^{1/n} = 2^N,\ |J|^{1/n} = 2^{N-s}, \\ J \subset 3I\setminus I,\ 2^{d-1} \le \operatorname{dist}(I,J) \le 2^d \big\} \,.\end{aligned}$$

Note that the closest a good cube J can come to I is determined by the goodness inequality, which gives this bound for $2^d \ge \operatorname{dist}(I,J)$:

$$2^d \ge \frac{1}{2} |I|^{\frac{1-\varepsilon}{n}} |J|^{\varepsilon/n} = \frac{1}{2} 2^{N(1-\varepsilon)} 2^{(N-s)\varepsilon} = \frac{1}{2} 2^{N-\varepsilon s}; \text{ which implies } N-\varepsilon s-1 \le d \le N,$$

where the last inequality holds because we are in the case of the mid-range term. Thus we have

$$\begin{aligned}\sum_{(I,J)\in\mathcal{P}} |\langle T^\alpha_\sigma(\triangle^\sigma_I f), \triangle^\omega_J g\rangle_\omega| &\lesssim \sum_{(I,J)\in\mathcal{P}} \| \triangle^\omega_J g\|_{L^2(\omega)}\, \mathrm{P}^\alpha(J, |\triangle^\sigma_I f|\sigma) \sqrt{|J|_\omega} \\ &= \sum_{s=\boldsymbol{\rho}}^\infty \sum_{N\in\mathbb{Z}} \sum_{d=N-\varepsilon s-1}^{N} \sum_{(I,J)\in\mathcal{P}^s_{N,d}} \| \triangle^\omega_J g\|_{L^2(\omega)}\, \mathrm{P}^\alpha(J, |\triangle^\sigma_I f|\sigma) \sqrt{|J|_\omega} \,.\end{aligned}$$

Now we use

$$\begin{aligned}\mathrm{P}^{\alpha}(J,|\triangle_I^{\sigma} f|\sigma) &= \int_I \frac{|J|^{1/n}}{\left(|J|^{1/n}+|y-c_J|\right)^{n+1-\alpha}}\,|\triangle_I^{\sigma} f(y)|\,d\sigma(y)\\ &\lesssim \frac{2^{N-s}}{2^{d(n+1-\alpha)}}\|\triangle_I^{\sigma} f\|_{L^2(\sigma)}\sqrt{|I|_\sigma}\end{aligned}$$

and apply Cauchy–Schwarz in J and use $J\subset 3I$ to get

$$\begin{aligned}&\sum_{(I,J)\in\mathcal{P}} |\langle T_\sigma^\alpha(\triangle_I^\sigma f),\triangle_J^\omega g\rangle_\omega|\\ &\lesssim \sum_{s=\boldsymbol{\rho}}^{\infty}\sum_{N\in\mathbb{Z}}\sum_{d=N-\varepsilon s-1}^{N}\sum_{I\in\mathcal{D}_N}\frac{2^{N-s}2^{N(n-\alpha)}}{2^{d(n+1-\alpha)}}\|\triangle_I^{\sigma} f\|_{L^2(\sigma)}\frac{\sqrt{|I|_\sigma}\sqrt{|3I|_\omega}}{2^{N(n-\alpha)}}\\ &\qquad\times\Bigg(\sum_{\substack{J\in\mathcal{D}_{N-s}\\ J\subset 3I\setminus I \text{ and } \operatorname{dist}(I,J)\approx 2^d}}\|\triangle_J^{\omega} g\|_{L^2(\omega)}\Bigg)^{1/2}\\ &\lesssim \sum_{s=\boldsymbol{\rho}}^{\infty}\sum_{N\in\mathbb{Z}}\frac{2^{N-s}\,2^{N(n-\alpha)}}{2^{(N-\varepsilon s)(n+1-\alpha)}}\sqrt{A_2^\alpha}\sum_{I\in\mathcal{D}_N}\|\triangle_I^{\sigma} f\|_{L^2(\sigma)}\Bigg(\sum_{\substack{J\in\mathcal{D}_{N-s}\\ J\subset 3I\setminus I}}\|\triangle_J^{\omega} g\|_{L^2(\omega)}^2\Bigg)^{1/2}\\ &\lesssim \sum_{s=\boldsymbol{\rho}}^{\infty}2^{-s[1-\varepsilon(n+1-\alpha)]}\sqrt{A_2^\alpha}\,\|f\|_{L^2(\sigma)}\,\|g\|_{L^2(\omega)}\lesssim\sqrt{A_2^\alpha}\,\|f\|_{L^2(\sigma)}\,\|g\|_{L^2(\omega)}\,,\end{aligned}$$

where in the third line above we have used $\sum_{d=N-\varepsilon s-1}^{N}\frac{1}{2^{d(n+1-\alpha)}}\approx\frac{1}{2^{(N-\varepsilon s)(n+1-\alpha)}}$, and in the last line $\frac{2^{N-s}2^{N(n-\alpha)}}{2^{(N-\varepsilon s)(n+1-\alpha)}}=2^{-s[1-\varepsilon(n+1-\alpha)]}$ followed by Cauchy–Schwarz in I and N, using that we have bounded overlap in the triples of I for $I\in\mathcal{D}_N$. More precisely, if we define $f_k\equiv\sum_{I\in\mathcal{D}_k}\triangle_I^\sigma f h_I^\sigma$ and $g_k\equiv\sum_{I\in\mathcal{D}_k}\triangle_J^\omega g h_J^\omega$, then we have the orthogonality inequality

$$\begin{aligned}\sum_{N\in\mathbb{Z}}\|f_N\|_{L^2(\sigma)}\,\|g_{N-s}\|_{L^2(\omega)} &\le \Big(\sum_{N\in\mathbb{Z}}\|f_N\|_{L^2(\sigma)}^2\Big)^{1/2}\Big(\sum_{N\in\mathbb{Z}}\|g_{N-s}\|_{L^2(\omega)}^2\Big)^{1/2}\\ &= \|f\|_{L^2(\sigma)}\,\|g\|_{L^2(\omega)}\,.\end{aligned}$$

We have assumed that $0<\varepsilon<1/(n+1-\alpha)$ in the calculations above, and this completes the proof of Lemma 7.1. □

8. Corona decompositions and splittings

We will use two different corona constructions to reduce matters to the stopping form, the main part of which is handled by Lacey's recursion argument, namely a Calderón–Zygmund decomposition and an energy decomposition of NTV type. We will then iterate these coronas into a double corona. We first recall our basic setup. For convenience in notation we will sometimes suppress the dependence on α in our nonlinear forms, but will retain it in the operators, Poisson integrals and constants. We will assume that the good/bad cube machinery of Nazarov, Treil and Volberg [23] is in force here. Let $\mathcal{D}^\sigma = \mathcal{D}^\omega$ be an $(\mathbf{r},\varepsilon)$-good grid on $\mathbb{R}^n$, and let $\{h_I^{\sigma,a}\}_{I\in\mathcal{D}^\sigma,\ a\in\Gamma_n}$ and $\{h_J^{\omega,b}\}_{J\in\mathcal{D}^\omega,\ b\in\Gamma_n}$ be corresponding Haar bases as described above, so that

$$f=\sum_{I\in\mathcal{D}^\sigma}\triangle_I^\sigma f \quad \text{and} \quad g=\sum_{J\in\mathcal{D}^\omega}\triangle_J^\omega g\,,$$

where the Haar projections $\triangle_I^\sigma f$ and $\triangle_J^\omega g$ vanish if the cubes I and J are not good. Inequality (2.9) is equivalent to boundedness of the bilinear form

$$\mathcal{T}^\alpha(f,g)\equiv\langle T_\sigma^\alpha(f),g\rangle_\omega=\sum_{I\in\mathcal{D}^\sigma\text{ and }J\in\mathcal{D}^\omega}\langle T_\sigma^\alpha(\triangle_I^\sigma f),\triangle_J^\omega g\rangle_\omega$$

on $L^2(\sigma)\times L^2(\omega)$, i.e.,

$$|\mathcal{T}^\alpha(f,g)|\leq\mathfrak{N}_{T^\alpha}\|f\|_{L^2(\sigma)}\,\|g\|_{L^2(\omega)}.$$

8.1. The Calderón–Zygmund corona

We now introduce a stopping tree $\mathcal{F}$ for the function $f\in L^2(\sigma)$. Let $\mathcal{F}$ be a collection of Calderón–Zygmund stopping cubes for f, and let $\mathcal{D}^\sigma=\bigcup_{F\in\mathcal{F}}\mathcal{C}_F$ be the associated corona decomposition of the dyadic grid $\mathcal{D}^\sigma$.

For a cube $I\in\mathcal{D}^\sigma$ let $\pi_{\mathcal{D}^\sigma}I$ be the $\mathcal{D}^\sigma$-parent of I in the grid $\mathcal{D}^\sigma$, and let $\pi_{\mathcal{F}}I$ be the smallest member of $\mathcal{F}$ that contains I. For $F,F'\in\mathcal{F}$, we say that F' is an $\mathcal{F}$-child of F if $\pi_{\mathcal{F}}(\pi_{\mathcal{D}^\sigma}F')=F$ (it could be that $F=\pi_{\mathcal{D}^\sigma}F'$), and we denote by $\mathfrak{C}_{\mathcal{F}}(F)$ the set of $\mathcal{F}$-children of F. For $F\in\mathcal{F}$, define the projection $\mathsf{P}_{\mathcal{C}_F}^\sigma$ onto the linear span of the Haar functions $\{h_I^{\sigma,a}\}_{I\in\mathcal{C}_F,\ a\in\Gamma_n}$ by

$$\mathsf{P}_{\mathcal{C}_F}^\sigma f=\sum_{I\in\mathcal{C}_F}\triangle_I^\sigma f=\sum_{I\in\mathcal{C}_F,\ a\in\Gamma_n}\langle f,h_I^{\sigma,a}\rangle_\sigma\,h_I^{\sigma,a}.$$

The standard properties of these projections are

$$f=\sum_{F\in\mathcal{F}}\mathsf{P}_{\mathcal{C}_F}^\sigma f\,,\qquad \int(\mathsf{P}_{\mathcal{C}_F}^\sigma f)\sigma=0\,,\qquad \|f\|_{L^2(\sigma)}^2=\sum_{F\in\mathcal{F}}\|\mathsf{P}_{\mathcal{C}_F}^\sigma f\|_{L^2(\sigma)}^2\,.$$

8.2. The energy corona

We must also impose an energy corona decomposition as in [13] and [3].

Definition 8.1. Given a cube S_0, define $\mathcal{S}(S_0)$ to be the maximal subcubes $I\subset S_0$ such that
(8.1)

$$\sum_{J\in\mathcal{M}_{\boldsymbol{\tau}\text{-deep}}(I)}\left(\frac{\mathrm{P}^\alpha(J,\mathbf{1}_{S_0\setminus\gamma J}\sigma)}{|J|^{1/n}}\right)^2\left\|\mathsf{P}_J^{\mathrm{subgood},\omega}\mathbf{x}\right\|_{L^2(\omega)}^2\geq C_{\mathrm{energy}}\left[(\mathcal{E}_\alpha^{\mathrm{deep}})^2+A_2^\alpha\right]|I|_\sigma,$$

where $\mathcal{E}_\alpha^{\text{deep}}$ is the constant in the deep energy condition defined in Definition 2.4, and C_{energy} is a sufficiently large positive constant depending only on $\boldsymbol{\tau}, \mathbf{r}, n$ and α. Then define the σ-energy stopping cubes of S_0 to be the collection $\mathcal{S} = \{S_0\} \cup \bigcup_{n=0}^{\infty} \mathcal{S}_n$, where $\mathcal{S}_0 = \mathcal{S}(S_0)$ and $\mathcal{S}_{n+1} = \bigcup_{S \in \mathcal{S}_n} \mathcal{S}(S)$ for $n \geq 0$.

From the energy condition in Definition 2.4 we obtain the σ-Carleson estimate

$$\sum_{S \in \mathcal{S}:\, S \subset I} |S|_\sigma \leq 2\,|I|_\sigma, \quad I \in \mathcal{D}^\sigma. \tag{8.2}$$

Indeed, using the deep energy condition, the first generation satisfies

$$\begin{aligned}
(8.3)\quad & \sum_{S \in \mathcal{S}_1} |S|_\sigma \\
&\leq \frac{1}{C_{\text{energy}}[(\mathcal{E}_\alpha^{\text{deep}})^2 + A_2^\alpha]} \sum_{S \in \mathcal{S}_1} \sum_{J \in \mathcal{M}_{\boldsymbol{\tau}\text{-deep}}(S)} \left(\frac{\mathrm{P}^\alpha(J, \mathbf{1}_{S_0 \setminus \gamma J}\sigma)}{|J|^{1/n}} \right)^2 \|\mathsf{P}_J^{\text{subgood},\omega} \mathbf{x}\|_{L^2(\omega)}^2 \\
&\leq \frac{1}{C_{\text{energy}}[(\mathcal{E}_\alpha^{\text{deep}})^2 + A_2^\alpha]} \sum_{S \in \mathcal{S}_1} \sum_{J \in \mathcal{M}_{\boldsymbol{\tau}\text{-deep}}(S)} \left(\frac{\mathrm{P}^\alpha(J, \mathbf{1}_{S_0}\sigma)}{|J|^{1/n}} \right)^2 \|\mathsf{P}_J^{\text{subgood},\omega} \mathbf{x}\|_{L^2(\omega)}^2 \\
&\leq \frac{C_{\boldsymbol{\tau},\mathbf{r},n,\alpha}}{C_{\text{energy}}[(\mathcal{E}_\alpha^{\text{deep}})^2 + A_2^\alpha]} \sum_{S \in \mathcal{S}_1} \sum_{J \in \mathcal{M}_{\mathbf{r}\text{-deep}}(S)} \left(\frac{\mathrm{P}^\alpha(J, \mathbf{1}_{S_0}\sigma)}{|J|^{1/n}} \right)^2 \|\mathsf{P}_J^{\text{subgood},\omega} \mathbf{x}\|_{L^2(\omega)}^2 \\
&\leq \frac{C_{\boldsymbol{\tau},\mathbf{r},n,\alpha}}{C_{\text{energy}}[(\mathcal{E}_\alpha^{\text{deep}})^2 + A_2^\alpha]} (\mathcal{E}_\alpha^{\text{deepplug}})^2\ |S_0|_\sigma = \frac{1}{2}|S_0|_\sigma\,,
\end{aligned}$$

provided we take $C_{\text{energy}} = 2C_{\boldsymbol{\tau},\mathbf{r},n,\alpha} \frac{(\mathcal{E}_\alpha^{\text{deepplug}})^2}{(\mathcal{E}_\alpha^{\text{deep}})^2 + A_2^\alpha}$. The third inequality above, in which $\boldsymbol{\tau}$ is replaced by $\mathbf{r}$ (but the goodness parameter $\varepsilon > 0$ is unchanged), follows because if $J_1 \in \mathcal{M}_{\boldsymbol{\tau}\text{-deep}}(S)$, then $J_1 \subset J_2$ for a unique $J_2 \in \mathcal{M}_{\mathbf{r}\text{-deep}}(S)$ and we have $|J_2|^{1/n} \leq 2^{\boldsymbol{\tau}-\mathbf{r}} |J_1|^{1/n}$, hence $\frac{\mathrm{P}^\alpha(J_1, \mathbf{1}_{S_0}\sigma)}{|J_1|^{1/n}} \leq C_{\boldsymbol{\tau},\mathbf{r},n,\alpha} \frac{\mathrm{P}^\alpha(J_2, \mathbf{1}_{S_0}\sigma)}{|J_2|^{1/n}}$. Subsequent generations satisfy a similar estimate, which then easily gives (8.2). We emphasize that this collection of stopping times depends only on S_0 and the weight pair (σ, ω), and not on any functions at hand.

Finally, we record the reason for introducing energy stopping times. If

$$X_\alpha(\mathcal{C}_S)^2 \equiv \sup_{I \in \mathcal{C}_S} \frac{1}{|I|_\sigma} \sum_{J \in \mathcal{M}_{\boldsymbol{\tau}\text{-deep}}(I)} \left(\frac{\mathrm{P}^\alpha(J, \mathbf{1}_{S \setminus \gamma J}\sigma)}{|J|^{1/n}} \right)^2 \|\mathsf{P}_J^{\text{subgood},\omega} \mathbf{x}\|_{L^2(\omega)}^2 \tag{8.4}$$

is (the square of) the *α-stopping energy* of the weight pair (σ, ω) with respect to the corona $\mathcal{C}_S$, then we have the *stopping energy bounds*

$$X_\alpha(\mathcal{C}_S) \leq \sqrt{C_{\text{energy}}}\ \sqrt{(\mathcal{E}_\alpha^{\text{deep}})^2 + A_2^\alpha}, \quad S \in \mathcal{S}, \tag{8.5}$$

where the deep energy constant $\mathcal{E}_\alpha^{\text{deep}}$ is controlled by assumption.

8.3. General stopping data

It is useful to extend our notion of corona decomposition to more general stopping data. Our general definition of stopping data will use a positive constant $C_0 \geq 4$.

Definition 8.2. Suppose we are given a positive constant $C_0 \geq 4$, a subset $\mathcal{F}$ of the dyadic grid $\mathcal{D}^\sigma$ (called the stopping times), and a corresponding sequence $\alpha_{\mathcal{F}} \equiv \{\alpha_{\mathcal{F}}(F)\}_{F\in\mathcal{F}}$ of nonnegative numbers $\alpha_{\mathcal{F}}(F) \geq 0$ (called the stopping data). Let $(\mathcal{F}, \prec, \pi_{\mathcal{F}})$ be the tree structure on $\mathcal{F}$ inherited from $\mathcal{D}^\sigma$, and for each $F \in \mathcal{F}$ denote by $\mathcal{C}_F = \{I \in \mathcal{D}^\sigma : \pi_{\mathcal{F}} I = F\}$ the corona associated with F:

$$\mathcal{C}_F = \{I \in \mathcal{D}^\sigma : I \subset F \text{ and } I \not\subset F' \text{ for any } F' \prec F\}.$$

We say the triple $(C_0, \mathcal{F}, \alpha_{\mathcal{F}})$ constitutes *stopping data* for a function $f \in L^1_{\mathrm{loc}}(\sigma)$ if

(1) $\mathbb{E}^\sigma_I |f| \leq \alpha_{\mathcal{F}}(F)$ for all $I \in \mathcal{C}_F$ and $F \in \mathcal{F}$,

(2) $\sum_{F' \preceq F} |F'|_\sigma \leq C_0 \, |F|_\sigma$ for all $F \in \mathcal{F}$,

(3) $\sum_{F\in\mathcal{F}} \alpha_{\mathcal{F}}(F)^2 |F|_\sigma \leq C_0^2 \, \|f\|^2_{L^2(\sigma)}$,

(4) $\alpha_{\mathcal{F}}(F) \leq \alpha_{\mathcal{F}}(F')$ whenever $F', F \in \mathcal{F}$ with $F' \subset F$.

Definition 8.3. If $(C_0, \mathcal{F}, \alpha_{\mathcal{F}})$ constitutes (general) *stopping data* for a function $f \in L^1_{\mathrm{loc}}(\sigma)$, we refer to the orthogonal decomposition

$$f = \sum_{F\in\mathcal{F}} \mathsf{P}^\sigma_{\mathcal{C}_F} f; \quad \mathsf{P}^\sigma_{\mathcal{C}_F} f \equiv \sum_{I\in\mathcal{C}_F} \triangle^\sigma_I f,$$

as the (general) *corona decomposition* of f associated with the stopping times $\mathcal{F}$.

Property (1) says that $\alpha_{\mathcal{F}}(F)$ bounds the averages of f in the corona $\mathcal{C}_F$, and property (2) says that the cubes at the tops of the coronas satisfy a Carleson condition relative to the weight σ. Note that a standard 'maximal cube' argument extends the Carleson condition in property (2) to the inequality

$$\sum_{F'\in\mathcal{F}:\, F'\subset A} |F'|_\sigma \leq C_0 \, |A|_\sigma \qquad \text{for all open sets } A \subset \mathbb{R}\,.$$

Property (3) is the quasiorthogonality condition that says the sequence of functions $\{\alpha_{\mathcal{F}}(F)\mathbf{1}_F\}_{F\in\mathcal{F}}$ is in the vector-valued space $L^2(\ell^2;\sigma)$, and property (4) says that the control on averages is nondecreasing on the stopping tree $\mathcal{F}$. We emphasize that we are *not* assuming in this definition the stronger property that there is $C > 1$ such that $\alpha_{\mathcal{F}}(F') > C\alpha_{\mathcal{F}}(F)$ whenever $F', F \in \mathcal{F}$ with $F' \subsetneqq F$. Instead, the properties (2) and (3) substitute for this lack. Of course the stronger property *does* hold for the familiar *Calderón–Zygmund* stopping data determined by the following requirements for $C > 1$,

$$\begin{aligned} \mathbb{E}^\sigma_{F'}|f| &> C\,\mathbb{E}^\sigma_F|f| \quad \text{whenever } F', F \in \mathcal{F} \text{ with } F' \subsetneqq F\,, \\ \mathbb{E}^\sigma_I|f| &\leq C\,\mathbb{E}^\sigma_F|f| \quad \text{for } I \in \mathcal{C}_F\,, \end{aligned}$$

which are themselves sufficiently strong to automatically force properties (2) and (3) with $\alpha_{\mathcal{F}}(F) = \mathbb{E}^\sigma_F|f|$.

We have the following useful consequence of (2) and (3) that says the sequence $\{\alpha_{\mathcal{F}}(F)\mathbf{1}_F\}_{F\in\mathcal{F}}$ has a *quasiorthogonal* property relative to f with a constant C_0' depending only on C_0:

$$\left\| \sum_{F\in\mathcal{F}} \alpha_{\mathcal{F}}(F)\mathbf{1}_F \right\|_{L^2(\sigma)}^2 \leq C_0' \left\|f\right\|_{L^2(\sigma)}^2 . \tag{8.6}$$

Indeed, the Carleson condition (2) implies a geometric decay in levels of the tree $\mathcal{F}$, namely that there are positive constants C_1 and ε, depending on C_0, such that if $\mathfrak{C}_{\mathcal{F}}^{(n)}(F)$ denotes the set of n^{th} generation children of F in $\mathcal{F}$,

$$\sum_{F'\in\mathfrak{C}_{\mathcal{F}}^{(n)}(F):} |F'|_\sigma \leq \left(C_1 2^{-\varepsilon n}\right)^2 |F|_\sigma, \quad \text{for all } n\geq 0 \text{ and } F\in\mathcal{F}.$$

From this we obtain that

$$\begin{aligned}
\sum_{n=0}^{\infty} \sum_{F'\in\mathfrak{C}_{\mathcal{F}}^{(n)}(F):} \alpha_{\mathcal{F}}(F')|F'|_\sigma &\leq \sum_{n=0}^{\infty}\Big(\sum_{F'\in\mathfrak{C}_{\mathcal{F}}^{(n)}(F)} \alpha_{\mathcal{F}}(F')^2|F'|_\sigma\Big)^{1/2} C_1 2^{-\varepsilon n}\sqrt{|F|_\sigma} \\
&\leq C_1\sqrt{|F|_\sigma}\, C_\varepsilon \Big(\sum_{n=0}^{\infty} 2^{-\varepsilon n} \sum_{F'\in\mathfrak{C}_{\mathcal{F}}^{(n)}(F)} \alpha_{\mathcal{F}}(F')^2|F'|_\sigma\Big)^{1/2},
\end{aligned}$$

and hence that

$$\begin{aligned}
&\sum_{F\in\mathcal{F}} \alpha_{\mathcal{F}}(F)\Big\{\sum_{n=0}^{\infty} \sum_{F'\in\mathfrak{C}_{\mathcal{F}}^{(n)}(F)} \alpha_{\mathcal{F}}(F')|F'|_\sigma\Big\} \\
&\qquad\lesssim \sum_{F\in\mathcal{F}} \alpha_{\mathcal{F}}(F)\sqrt{|F|_\sigma}\Big(\sum_{n=0}^{\infty} 2^{-\varepsilon n}\sum_{F'\in\mathfrak{C}_{\mathcal{F}}^{(n)}(F)} \alpha_{\mathcal{F}}(F')^2|F'|_\sigma\Big)^{1/2} \\
&\qquad\lesssim \Big(\sum_{F\in\mathcal{F}} \alpha_{\mathcal{F}}(F)^2|F|_\sigma\Big)^{1/2}\Big(\sum_{n=0}^{\infty} 2^{-\varepsilon n}\sum_{F\in\mathcal{F}}\sum_{F'\in\mathfrak{C}_{\mathcal{F}}^{(n)}(F)} \alpha_{\mathcal{F}}(F')^2|F'|_\sigma\Big)^{1/2} \\
&\qquad\lesssim \|f\|_{L^2(\sigma)}\Big(\sum_{F'\in\mathcal{F}} \alpha_{\mathcal{F}}(F')^2|F'|_\sigma\Big)^{1/2} \lesssim \|f\|_{L^2(\sigma)}^2 .
\end{aligned}$$

This proves (8.6) since $\|\sum_{F\in\mathcal{F}} \alpha_{\mathcal{F}}(F)\mathbf{1}_F\|_{L^2(\sigma)}^2$ is dominated by twice the left hand side above.

We will use a construction that permits *iteration* of general corona decompositions.

Lemma 8.4. *Suppose that* $(C_0,\mathcal{F},\alpha_{\mathcal{F}})$ *constitutes* stopping data *for a function* $f\in L^1_{\text{loc}}(\sigma)$*, and that for each* $F\in\mathcal{F}$*,* $(C_0,\mathcal{K}(F),\alpha_{\mathcal{K}(F)})$ *constitutes* stopping data *for the corona projection* $\mathsf{P}^\sigma_{\mathcal{C}_F} f$*, where, in addition,* $F\in\mathcal{K}(F)$*. There is a positive* *constant* C_1*, depending only on* C_0*, such that if*

$$\mathcal{K}^*(F) \equiv \left\{K \in \mathcal{K}(F) \cap \mathcal{C}_F : \alpha_{\mathcal{K}(F)}(K) \geq \alpha_{\mathcal{F}}(F)\right\}$$

$$\mathcal{K} \equiv \bigcup_{F \in \mathcal{F}} \mathcal{K}^*(F) \cup \{F\},$$

$$\alpha_{\mathcal{K}}(K) \equiv \begin{cases} \alpha_{\mathcal{K}(F)}(K) & \text{for} \quad K \in \mathcal{K}^*(F) \setminus \{F\} \\ \max\{\alpha_{\mathcal{F}}(F), \alpha_{\mathcal{K}(F)}(F)\} & \text{for} \quad K = F \end{cases}, \quad \text{for } F \in \mathcal{F},$$

the triple $(C_1, \mathcal{K}, \alpha_{\mathcal{K}})$ *constitutes* stopping data *for* f*. We refer to the collection of cubes* $\mathcal{K}$ *as the* iterated *stopping times, and to the orthogonal decomposition* $f = \sum_{K \in \mathcal{K}} P_{\mathcal{C}_K^{\mathcal{K}}} f$ *as the* iterated *corona decomposition of* f*, where*

$$\mathcal{C}_K^{\mathcal{K}} \equiv \{I \in \mathcal{D} : I \subset K \text{ and } I \not\subset K' \text{ for } K' \prec_{\mathcal{K}} K\}.$$

Note that in our definition of $(C_1, \mathcal{K}, \alpha_{\mathcal{K}})$ we have 'discarded' from $\mathcal{K}(F)$ all of those $K \in \mathcal{K}(F)$ that are not in the corona $\mathcal{C}_F$, and also all of those $K \in \mathcal{K}(F)$ for which $\alpha_{\mathcal{K}(F)}(K)$ is strictly less than $\alpha_{\mathcal{F}}(F)$. Then the union of over F of what remains is our new collection of stopping times. We then define stopping data $\alpha_{\mathcal{K}}(K)$ according to whether or not $K \in \mathcal{F}$: if $K \notin \mathcal{F}$ but $K \in \mathcal{C}_F$ then $\alpha_{\mathcal{K}}(K)$ equals $\alpha_{\mathcal{K}(F)}(K)$, while if $K \in \mathcal{F}$, then $\alpha_{\mathcal{K}}(K)$ is the larger of $\alpha_{\mathcal{K}(F)}(F)$ and $\alpha_{\mathcal{F}}(K)$.

Proof. The monotonicity property (4) for the triple $(C_1, \mathcal{K}, \alpha_{\mathcal{K}})$ is obvious from the construction of $\mathcal{K}$ and $\alpha_{\mathcal{K}}(K)$. To establish property (1), we must distinguish between the various coronas $\mathcal{C}_K^{\mathcal{K}}$, $\mathcal{C}_K^{\mathcal{K}(F)}$ and $\mathcal{C}_K^{\mathcal{F}}$ that could be associated with $K \in \mathcal{K}$, when K belongs to any of the stopping trees $\mathcal{K}$, $\mathcal{K}(F)$ or $\mathcal{F}$. Suppose now that $I \in \mathcal{C}_K^{\mathcal{K}}$ for some $K \in \mathcal{K}$. Then there is a unique $F \in \mathcal{F}$ such that $\mathcal{C}_K^{\mathcal{K}} \subset \mathcal{C}_K^{\mathcal{K}(F)} \subset C_F^{\mathcal{F}}$, and so $\mathbb{E}_I^\sigma |f| \leq \alpha_{\mathcal{F}}(F)$ by property (1) for the triple $(C_0, \mathcal{F}, \alpha_{\mathcal{F}})$. Then $\alpha_{\mathcal{F}}(F) \leq \alpha_{\mathcal{K}}(K)$ follows from the definition of $\alpha_{\mathcal{K}}(K)$, and we have property (1) for the triple $(C_1, \mathcal{K}, \alpha_{\mathcal{K}})$. Property (2) holds for the triple $(C_1, \mathcal{K}, \alpha_{\mathcal{K}})$ since if $K \in \mathcal{C}_F^{\mathcal{F}}$, then

$$\begin{aligned}\sum_{K' \preceq_{\mathcal{K}} K} |K'|_\sigma &= \sum_{K' \in \mathcal{K}(F) \colon K' \subset K} |K'|_\sigma + \sum_{F' \prec_{\mathcal{F}} F \colon F' \subset K} \sum_{K' \in \mathcal{K}(F')} |K'|_\sigma \\ &\leq C_0 \, |K|_\sigma + \sum_{F' \prec_{\mathcal{F}} F \colon F' \subset K} C_0 \, |F'|_\sigma \leq 2 \, C_0^2 \, |K|_\sigma .\end{aligned}$$

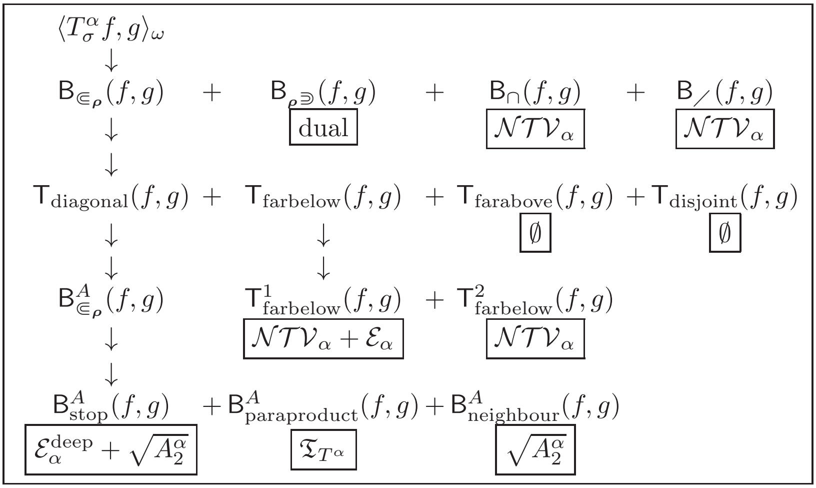
Finally, property (3) holds for the triple $(C_1, \mathcal{K}, \alpha_{\mathcal{K}})$ since

$$\begin{aligned}\sum_{K \in \mathcal{K}} \alpha_{\mathcal{K}}(K)^2 |K|_\sigma &= \sum_{F \in \mathcal{F}} \sum_{K \in \mathcal{K}(F)} \alpha_{\mathcal{K}(F)}(K)^2 \, |K|_\sigma + \sum_{F \in \mathcal{F}} \alpha_{\mathcal{F}}(F)^2 \, |F|_\sigma \\ &\leq \sum_{F \in \mathcal{F}} C_0^2 \, \|\mathsf{P}_{\mathcal{C}_F}^\sigma f\|_{L^2(\sigma)}^2 + C_0^2 \, \|f\|_{L^2(\sigma)}^2 \leq 2 \, C_0^2 \, \|f\|_{L^2(\sigma)}^2.\end{aligned}$$

□

8.4. Doubly iterated coronas and the NTV cube size splitting

Here is a brief schematic diagram of the decompositions, with bounds in $\Box$, used in this subsection:



We begin with the NTV *cube size splitting* of the inner product $\langle T^\alpha_\sigma f, g\rangle_\omega$ −and later apply the iterated corona construction− that splits the pairs of cubes (I,J) in a simultaneous Haar decomposition of f and g into four groups, namely those pairs that:

1. are below the size diagonal and $\boldsymbol{\rho}$-deeply embedded,
2. are above the size diagonal and $\boldsymbol{\rho}$-deeply embedded,
3. are disjoint, and
4. are of $\boldsymbol{\rho}$-comparable size.

More precisely we have

$$\begin{aligned}
\langle T^\alpha_\sigma f, g\rangle_\omega &= \sum_{I\in\mathcal{D}^\sigma,\ J\in\mathcal{D}^\omega} \langle T^\alpha_\sigma(\triangle^\sigma_I f),(\triangle^\omega_I g)\rangle_\omega \\
&= \sum_{\substack{I\in\mathcal{D}^\sigma,\ J\in\mathcal{D}^\omega\\ J\Subset_{\boldsymbol{\rho}} I}} \langle T^\alpha_\sigma(\triangle^\sigma_I f),(\triangle^\omega_J g)\rangle_\omega + \sum_{\substack{I\in\mathcal{D}^\sigma,\ J\in\mathcal{D}^\omega\\ J_{\boldsymbol{\rho}}\Supset I}} \langle T^\alpha_\sigma(\triangle^\sigma_I f),(\triangle^\omega_J g)\rangle_\omega \\
&\quad + \sum_{\substack{I\in\mathcal{D}^\sigma,\ J\in\mathcal{D}^\omega\\ J\cap I=\emptyset}} \langle T^\alpha_\sigma(\triangle^\sigma_I f),(\triangle^\omega_J g)\rangle_\omega + \sum_{\substack{I\in\mathcal{D}^\sigma,\ J\in\mathcal{D}^\omega\\ 2^{-n\boldsymbol{\rho}}\le |J|\diagup|I|\le 2^{n\,\boldsymbol{\rho}}}} \langle T^\alpha_\sigma(\triangle^\sigma_I f),(\triangle^\omega_J g)\rangle_\omega \\
&= \mathsf{B}_{\Subset_{\boldsymbol{\rho}}}(f,g) + \mathsf{B}_{\boldsymbol{\rho}\Supset}(f,g) + \mathsf{B}_{\cap}(f,g) + \mathsf{B}_{\diagup}(f,g)\,.
\end{aligned}$$

Lemma 7.1 in the section on NTV preliminaries show that the *disjoint* and *comparable* forms $\mathsf{B}_{\cap}(f,g)$ and $\mathsf{B}_{\diagup}(f,g)$ are both bounded by the $\mathcal{A}^\alpha_2$, testing and weak boundedness property constants. The *below* and *above* forms are clearly symmetric, so we need only consider the form $\mathsf{B}_{\Subset_{\boldsymbol{\rho}}}(f,g)$, to which we turn for the remainder of the proof.

In order to bound the below form $\mathsf{B}_{\Subset_{\boldsymbol{\rho}}}(f,g)$, we will apply two different corona decompositions in succession to the function $f \in L^2(\sigma)$, gaining structure with each application; first to a boundedness property for f, and then to a regularizing property of the weight σ. We first apply the Calderón–Zygmund corona decomposition to the function $f \in L^2(\sigma)$ obtain

$$f = \sum_{F \in \mathcal{F}} \mathsf{P}^{\sigma}_{\mathcal{C}^{\sigma}_F} f \,.$$

Then for each fixed $F \in \mathcal{F}$, construct the *energy* corona decomposition $\{\mathcal{C}^{\sigma}_S\}_{S \in \mathcal{S}(F)}$ corresponding to the weight pair (σ,ω) with top cube $S_0 = F$, as given in Definition 8.1. At this point we apply Lemma 8.4 to obtain iterated stopping times $\mathcal{S}$ and iterated stopping data $\{\alpha_{\mathcal{S}(\mathcal{F})}(S)\}_{S \in \mathcal{S}(\mathcal{F})}$. This gives us the following *double corona decomposition* of f:

$$f = \sum_{F \in \mathcal{F}} \mathsf{P}^{\sigma}_{\mathcal{C}^{\sigma}_F} f = \sum_{F \in \mathcal{F}} \sum_{S \in \mathcal{S}^*(F) \cup \{F\}} \mathsf{P}^{\sigma}_{\mathcal{C}^{\sigma}_S} \mathsf{P}^{\sigma}_{\mathcal{C}^{\sigma}_F} f = \sum_{S \in \mathcal{S}} \mathsf{P}^{\sigma}_{\mathcal{C}^{\sigma}_S} f \equiv \sum_{A \in \mathcal{A}} \mathsf{P}^{\sigma}_{\mathcal{C}_A} f \,, \tag{8.7}$$

where $\mathcal{A} \equiv \mathcal{S}$ is the double stopping collection for f. We are relabeling the double corona as $\mathcal{A}$ here so as to minimize confusion. We now record the main facts proved above for the double corona.

Lemma 8.5. *The data $\mathcal{A}$ and $\{\alpha_{\mathcal{A}}(A)\}_{A \in \mathcal{A}}$ satisfy properties* (1), (2), (3) *and* (4) *in Definition 8.2.*

To bound $\mathsf{B}_{\Subset_{\boldsymbol{\rho}}}(f,g)$ we fix the stopping data $\mathcal{A}$ and $\{\alpha_{\mathcal{A}}(A)\}_{A \in \mathcal{A}}$ constructed above with the double iterated corona. We now consider the following *canonical splitting* of the form $\mathsf{B}_{\Subset_{\boldsymbol{\rho}}}(f,g)$ that involves the Haar corona projections $\mathsf{P}^{\sigma}_{\mathcal{C}_A}$ acting on f and the $\boldsymbol{\tau}$-*shifted* Haar corona projections $\mathsf{P}^{\omega}_{\mathcal{C}^{\boldsymbol{\tau}\text{-shift}}_B}$ acting on g. Here the $\boldsymbol{\tau}$-shifted corona $\mathcal{C}^{\boldsymbol{\tau}\text{-shift}}_B$ is defined to include only those cubes $J \in \mathcal{C}_B$ that are *not* $\boldsymbol{\tau}$-nearby B, and to include also such cubes J which in addition *are* $\boldsymbol{\tau}$-nearby in the children B' of B.

Definition 8.6. The parameters $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ are now fixed to satisfy

$$\boldsymbol{\tau} > \mathbf{r} \quad \text{and} \quad \boldsymbol{\rho} > \mathbf{r} + \boldsymbol{\tau} \,,$$

where $\mathbf{r}$ is the goodness parameter already fixed.

Definition 8.7. For $B \in \mathcal{A}$ we define

$$\mathcal{C}^{\boldsymbol{\tau}\text{-shift}}_B = \{J \in \mathcal{C}_B : J \Subset_{\boldsymbol{\tau}} B\} \bigcup \bigcup_{B' \in \mathfrak{C}_{\mathcal{A}}(B)} \{J \in \mathcal{D} : J \Subset_{\boldsymbol{\tau}} B \text{ and } J \textit{ is } \boldsymbol{\tau}\text{-nearby in } B'\} \,.$$

We will use repeatedly the fact that the $\boldsymbol{\tau}$-shifted coronas $\mathcal{C}^{\boldsymbol{\tau}\text{-shift}}_B$ have overlap bounded by $\boldsymbol{\tau}$:

$$\sum_{B \in \mathcal{A}} \mathbf{1}_{\mathcal{C}^{\boldsymbol{\tau}\text{-shift}}_B}(J) \leq \boldsymbol{\tau}, \quad J \in \mathcal{D}. \tag{8.8}$$

The forms $\mathsf{B}_{\Subset_{\boldsymbol{\rho}}}(f,g)$ are no longer linear in f and g as the 'cut' is determined by the coronas $\mathcal{C}_F$ and $\mathcal{C}_G^{\boldsymbol{\tau}\text{-shift}}$, which depend on f as well as the measures σ and ω. However, if the coronas are held fixed, then the forms can be considered bilinear in f and g. It is convenient at this point to introduce the following shorthand notation:

$$\left\langle T_\sigma^\alpha(\mathsf{P}_{\mathcal{C}_A}^\sigma f), \mathsf{P}_{\mathcal{C}_B^{\boldsymbol{\tau}\text{-shift}}}^\omega g \right\rangle_\omega^{\Subset_{\boldsymbol{\rho}}} \equiv \sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_B^{\boldsymbol{\tau}\text{-shift}} \\ J\Subset_{\boldsymbol{\rho}} I}} \langle T_\sigma^\alpha(\triangle_I^\sigma f), (\triangle_J^\omega g)\rangle_\omega \,.$$

We then have the canonical splitting

$$\begin{aligned}
\mathsf{B}_{\Subset_{\boldsymbol{\rho}}}(f,g) &= \sum_{A,B\in\mathcal{A}} \left\langle T_\sigma^\alpha(\mathsf{P}_{\mathcal{C}_A}^\sigma f), \mathsf{P}_{\mathcal{C}_B^{\boldsymbol{\tau}\text{-shift}}}^\omega g \right\rangle_\omega^{\Subset_{\boldsymbol{\rho}}} \\
&= \sum_{A\in\mathcal{A}} \left\langle T_\sigma^\alpha(\mathsf{P}_{\mathcal{C}_A}^\sigma f), \mathsf{P}_{\mathcal{C}_A^{\boldsymbol{\tau}\text{-shift}}}^\omega g \right\rangle_\omega^{\Subset_{\boldsymbol{\rho}}} + \sum_{\substack{A,B\in\mathcal{A} \\ B\subsetneqq A}} \left\langle T_\sigma^\alpha(\mathsf{P}_{\mathcal{C}_A}^\sigma f), \mathsf{P}_{\mathcal{C}_B^{\boldsymbol{\tau}\text{-shift}}}^\omega g \right\rangle_\omega^{\Subset_{\boldsymbol{\rho}}} \\
&\quad + \sum_{\substack{A,B\in\mathcal{A} \\ B\supsetneqq A}} \left\langle T_\sigma^\alpha(\mathsf{P}_{\mathcal{C}_A}^\sigma f), \mathsf{P}_{\mathcal{C}_B^{\boldsymbol{\tau}\text{-shift}}}^\omega g \right\rangle_\omega^{\Subset_{\boldsymbol{\rho}}} + \sum_{\substack{A,B\in\mathcal{A} \\ A\cap B=\emptyset}} \left\langle T_\sigma^\alpha(\mathsf{P}_{\mathcal{C}_A}^\sigma f), \mathsf{P}_{\mathcal{C}_B^{\boldsymbol{\tau}\text{-shift}}}^\omega g \right\rangle_\omega^{\Subset_{\boldsymbol{\rho}}} \\
&\equiv \mathsf{T}_{\text{diagonal}}(f,g) + \mathsf{T}_{\text{farbelow}}(f,g) + \mathsf{T}_{\text{farabove}}(f,g) + \mathsf{T}_{\text{disjoint}}(f,g) \,.
\end{aligned} \tag{8.9}$$

Now the final two terms $\mathsf{T}_{\text{farabove}}(f,g)$ and $\mathsf{T}_{\text{disjoint}}(f,g)$ each vanish since there are no pairs $(I,J)\in\mathcal{C}_A\times\mathcal{C}_B^{\boldsymbol{\tau}\text{-shift}}$ with both (i) $J\Subset_{\boldsymbol{\rho}} I$ and (ii) either $B\subsetneqq A$ or $B\cap A=\emptyset$.

The *far below* term $\mathsf{T}_{\text{farbelow}}(f,g)$ is bounded using the intertwining proposition and the control of functional energy condition by the energy condition given in the next two sections. Indeed, assuming these two results, we have from $\boldsymbol{\tau}<\boldsymbol{\rho}$ that

$$\begin{aligned}
\mathsf{T}_{\text{farbelow}}(f,g) &= \sum_{\substack{A,B\in\mathcal{A} \\ B\subsetneqq A}} \sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_B^{\boldsymbol{\tau}-\text{shift}} \\ J\Subset_{\boldsymbol{\rho}} I}} \langle T_\sigma^\alpha(\triangle_I^\sigma f), (\triangle_J^\omega g)\rangle_\omega \\
&= \sum_{B\in\mathcal{A}} \sum_{A\in\mathcal{A}:\, B\subsetneqq A} \sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_B^{\boldsymbol{\tau}\text{-shift}} \\ J\Subset_{\boldsymbol{\rho}} I}} \langle T_\sigma^\alpha(\triangle_I^\sigma f), (\triangle_J^\omega g)\rangle_\omega \\
&= \sum_{B\in\mathcal{A}} \sum_{A\in\mathcal{A}:\, B\subsetneqq A} \sum_{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_B^{\boldsymbol{\tau}-\text{shift}}} \langle T_\sigma^\alpha(\triangle_I^\sigma f), (\triangle_J^\omega g)\rangle_\omega \\
&\quad - \sum_{B\in\mathcal{A}} \sum_{A\in\mathcal{A}:\, B\subsetneqq A} \sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_B^{\boldsymbol{\tau}\text{-shift}} \\ J\not\Subset_{\boldsymbol{\rho}} I}} \langle T_\sigma^\alpha(\triangle_I^\sigma f), (\triangle_J^\omega g)\rangle_\omega \\
&= \mathsf{T}_{\text{farbelow}}^1(f,g) - \mathsf{T}_{\text{farbelow}}^2(f,g) \,.
\end{aligned}$$

Now $\mathsf{T}_{\text{farbelow}}^2(f,g)$ is bounded by $\mathcal{NTV}_\alpha$ by Lemma 7.1, since J is good if $\triangle_J^\omega g\neq 0$.

The form $\mathsf{T}_{\text{farbelow}}^1(f,g)$ can be written as

$$\mathsf{T}_{\text{farbelow}}^1(f,g) = \sum_{B\in\mathcal{A}} \sum_{I\in\mathcal{D}:\, B\subsetneqq I} \langle T_\sigma^\alpha(\triangle_I^\sigma f), g_B\rangle_\omega \,; \quad \text{where } g_B \equiv \sum_{J\in\mathcal{C}_B^{\boldsymbol{\tau}\text{-shift}}} \triangle_J^\omega g \,.$$

The intertwining Proposition 9.4 applies to this latter form and shows that it is bounded by $\mathcal{NTV}_\alpha + \mathfrak{F}_\alpha$. Then Proposition 10.1 shows that $\mathfrak{F}_\alpha \lesssim \mathcal{A}_2^\alpha + \mathcal{E}_\alpha$, which completes the proof that

$$|\mathsf{T}_{\mathrm{farbelow}}(f,g)| \lesssim (\mathcal{NTV}_\alpha + \mathcal{E}_\alpha) \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)} . \tag{8.10}$$

The boundedness of the diagonal term $\mathsf{T}_{\mathrm{diagonal}}(f,g)$ will then be reduced to the forms in the paraproduct/neighbour/stopping form decomposition of NTV. The stopping form is then further split into two sublinear forms in (11.6) below, where the boundedness of the more difficult of the two is treated by adapting the stopping time and recuresion of M. Lacey [1]. More precisely, to handle the diagonal term $\mathsf{T}_{\mathrm{diagonal}}(f,g)$, it is enough to consider the individual corona pieces

$$\mathsf{B}^A_{\Subset_\rho}(f,g) \equiv \left\langle T^\alpha_\sigma(\mathsf{P}^\sigma_{\mathcal{C}_A} f), \mathsf{P}^\omega_{\mathcal{C}_A^{\tau\text{-shift}}} g \right\rangle^{\Subset}_\omega ,$$

and to prove the following estimate:

$$|\mathsf{B}^A_{\Subset_\rho}(f,g)| \lesssim (\mathcal{NTV}_\alpha + \mathcal{E}_\alpha) \left(\alpha_{\mathcal{A}}(A)\sqrt{|A|_\sigma} + \|\mathsf{P}^\sigma_{\mathcal{C}_A} f\|_{L^2(\sigma)} \right) \left\| \mathsf{P}^\omega_{\mathcal{C}_A^{\tau\text{-shift}}} g \right\|_{L^2(\omega)} .$$

Indeed, we then have from Cauchy–Schwarz that

$$\begin{aligned} \sum_{A\in\mathcal{A}} |\mathsf{B}^A_{\Subset_\rho}(f,g)| &= \sum_{A\in\mathcal{A}} \left|\mathsf{B}^A_{\Subset_\rho}(\mathsf{P}^\sigma_{\mathcal{C}_A} f, \mathsf{P}^\omega_{\mathcal{C}_A^{\tau\text{-shift}}} g)\right| \\ &\lesssim (\mathcal{NTV}_\alpha + \mathcal{E}_\alpha) \left(\sum_{A\in\mathcal{A}} \alpha_{\mathcal{A}}(A)^2 |A|_\sigma + \|\mathsf{P}^\sigma_{\mathcal{C}_A} f\|^2_{L^2(\sigma)} \right)^{1/2} \left(\sum_{A\in\mathcal{A}} \|\mathsf{P}^\omega_{\mathcal{C}_A^{\tau\text{-shift}}} g\|^2_{L^2(\omega)} \right)^{1/2} \\ &\lesssim (\mathcal{NTV}_\alpha + \mathcal{E}_\alpha) \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)} , \end{aligned}$$

where the last line uses quasiorthogonality in f and orthogonality in both f and g.

Following arguments in [13], [23] and [4], we now use the paraproduct/neighbour/stopping splitting of NTV to reduce boundedness of $\mathsf{B}^A_{\Subset_\rho}(f,g)$ to boundedness of the associated stopping form

$$\mathsf{B}^A_{\mathrm{stop}}(f,g) \equiv \sum_{I\in\mathrm{supp}\widehat{f}} \ \sum_{J:\, J\Subset_\rho I \text{ and } I_J\notin\mathcal{A}} \left(\mathbb{E}^\sigma_{I_J} \triangle^\sigma_I f\right) \left\langle T^\alpha_\sigma \mathbf{1}_{A\setminus I_J}, \triangle^\omega_J g\right\rangle_\omega , \tag{8.11}$$

where f is supported in the cube A and its expectations $\mathbb{E}^\sigma_I |f|$ are bounded by $\alpha_{\mathcal{A}}(A)$ for $I \in \mathcal{C}^\sigma_A$, the Haar support of f is contained in the corona $\mathcal{C}^\sigma_A$, and the Haar support of g is contained in $\mathcal{C}_A^{\tau\text{-shift}}$, and where I_J is the $\mathcal{D}$-child of I that contains J. Indeed, to see this, we note that $\triangle^\sigma_I f = \mathbf{1}_I \triangle^\sigma_I f$ and write both

$$\mathbf{1}_I = \mathbf{1}_{I_J} + \sum_{\theta(I_J)\in\mathfrak{C}_{\mathcal{D}}(I)\setminus\{I_J\}} \mathbf{1}_{\theta(I_J)} , \qquad \mathbf{1}_{I_J} = \mathbf{1}_A - \mathbf{1}_{A\setminus I_J} ,$$

where $\theta(I_J) \in \mathfrak{C}_{\mathcal{D}}(I) \setminus \{I_J\}$ ranges over the $2^n - 1$ $\mathcal{D}$-children of I other than the child I_J that contains J. Then we obtain

$$\begin{aligned}\langle T_\sigma^\alpha \triangle_I^\sigma f, \triangle_J^\omega g\rangle_\omega\, \langle T_\sigma^\alpha(\mathbf{1}_{I_J} \triangle_I^\sigma f), \triangle_J^\omega g\rangle_\omega + \sum_{\theta(I_J)\in\mathfrak{C}_{\mathcal{D}}(I)\setminus\{I_J\}} \langle T_\sigma^\alpha(\mathbf{1}_{\theta(I_J)} \triangle_I^\sigma f), \triangle_J^\omega g\rangle_\omega \\
&= (\mathbb{E}_{I_J}^\sigma \triangle_I^\sigma f)\langle T_\sigma^\alpha(\mathbf{1}_{I_J}), \triangle_J^\omega g\rangle_\omega + \sum_{\theta(I_J)\in\mathfrak{C}_{\mathcal{D}}(I)\setminus\{I_J\}} \langle T_\sigma^\alpha(\mathbf{1}_{\theta(I_J)} \triangle_I^\sigma f), \triangle_J^\omega g\rangle_\omega \\
&= (\mathbb{E}_{I_J}^\sigma \triangle_I^\sigma f)\langle T_\sigma^\alpha \mathbf{1}_A, \triangle_J^\omega g\rangle_\omega - (\mathbb{E}_{I_J}^\sigma \triangle_I^\sigma f)\langle T_\sigma^\alpha \mathbf{1}_{A\setminus I_J}, \triangle_J^\omega g\rangle_\omega \\
&\quad + \sum_{\theta(I_J)\in\mathfrak{C}_{\mathcal{D}}(I)\setminus\{I_J\}} \langle T_\sigma^\alpha(\mathbf{1}_{\theta(I_J)} \triangle_I^\sigma f), \triangle_J^\omega g\rangle_\omega ,\end{aligned}$$

and the corresponding NTV splitting of $\mathsf{B}_{\Subset_{\boldsymbol{\rho}}}^A(f,g)$:

$$\begin{aligned}\mathsf{B}_{\Subset_{\boldsymbol{\rho}}}^A(f,g) &= \left\langle T_\sigma^\alpha(\mathsf{P}_{\mathcal{C}_A}^\sigma f), \mathsf{P}_{\mathcal{C}_A^{\boldsymbol{\tau}\text{-shift}}}^\omega g\right\rangle_\omega^{\Subset_{\boldsymbol{\rho}}} = \sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_A^{\boldsymbol{\tau}\text{-shift}} \\ J\Subset_{\boldsymbol{\rho}} I}} \langle T_\sigma^\alpha(\triangle_I^\sigma f), \triangle_J^\omega g\rangle_\omega \\
&= \sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_A^{\boldsymbol{\tau}\text{-shift}} \\ J\Subset_{\boldsymbol{\rho}} I}} (\mathbb{E}_{I_J}^\sigma \triangle_I^\sigma f)\langle T_\sigma^\alpha \mathbf{1}_A, \triangle_J^\omega g\rangle_\omega \\
&\quad - \sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_A^{\boldsymbol{\tau}\text{-shift}} \\ J\Subset_{\boldsymbol{\rho}} I}} (\mathbb{E}_{I_J}^\sigma \triangle_I^\sigma f)\langle T_\sigma^\alpha \mathbf{1}_{A\setminus I_J}, \triangle_J^\omega g\rangle_\omega \\
&\quad + \sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_A^{\boldsymbol{\tau}\text{-shift}} \\ J\Subset_{\boldsymbol{\rho}} I}} \sum_{\theta(I_J)\in\mathfrak{C}_{\mathcal{D}}(I)\setminus\{I_J\}} \langle T_\sigma^\alpha(\mathbf{1}_{\theta(I_J)} \triangle_I^\sigma f), \triangle_J^\omega g\rangle_\omega \\
&\equiv \mathsf{B}_{\text{paraproduct}}^A(f,g) - \mathsf{B}_{\text{stop}}^A(f,g) + \mathsf{B}_{\text{neighbour}}^A(f,g) .\end{aligned}$$

The paraproduct form $\mathsf{B}_{\text{paraproduct}}^A(f,g)$ is easily controlled by the testing condition for T^α. Indeed, we have

$$\begin{aligned}\mathsf{B}_{\text{paraproduct}}^A(f,g) &= \sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_A^{\boldsymbol{\tau}\text{-shift}} \\ J\Subset I}} (\mathbb{E}_{I_J}^\sigma \triangle_I^\sigma f)\langle T_\sigma^\alpha \mathbf{1}_A, \triangle_J^\omega g\rangle_\omega \\
&= \sum_{J\in\mathcal{C}_A^{\boldsymbol{\tau}\text{-shift}}} \langle T_\sigma^\alpha \mathbf{1}_A, \triangle_J^\omega g\rangle_\omega \left\{ \sum_{I\in\mathcal{C}_A:\, J\Subset I} (\mathbb{E}_{I_J}^\sigma \triangle_I^\sigma f)\right\} \\
&= \sum_{J\in\mathcal{C}_A^{\boldsymbol{\tau}\text{-shift}}} \langle T_\sigma^\alpha \mathbf{1}_A, \triangle_J^\omega g\rangle_\omega \{\mathbb{E}_{I^\natural(J)_J}^\sigma f - \mathbb{E}_A^\sigma f\} \\
&= \left\langle T_\sigma^\alpha \mathbf{1}_A, \sum_{J\in\mathcal{C}_A^{\boldsymbol{\tau}\text{-shift}}} \{\mathbb{E}_{I^\natural(J)_J}^\sigma f - \mathbb{E}_A^\sigma f\} \triangle_J^\omega g\right\rangle_\omega ,\end{aligned}$$

where $I^\natural(J)$ denotes the smallest cube $I\in\mathcal{C}_A$ such that $J\Subset_{\boldsymbol{\rho}} I$, and of course $I^\natural(J)_J$ denotes its child containing J. We claim that, by construction of the corona, we have $I^\natural(J)_J\notin\mathcal{A}$, and so $\left|\mathbb{E}_{I^\natural(J)_J}^\sigma f\right| \lesssim \mathbb{E}_A^\sigma|f| \leq \alpha_{\mathcal{A}}(A)$. Indeed, in our application of the stopping form we have $f = \mathsf{P}_{\mathcal{C}_A}^\sigma f$ and $g = \mathsf{P}_{\mathcal{C}_A^{\boldsymbol{\tau}\text{-shift}}}^\omega g$, and the definitions of the coronas $\mathcal{C}_A$ and $\mathcal{C}_A^{\boldsymbol{\tau}\text{-shift}}$ together with $\mathbf{r} < \boldsymbol{\tau} < \boldsymbol{\rho}$ imply that $I^\natural(J)_J \notin \mathcal{A}$ for $J \in \mathcal{C}_A^{\boldsymbol{\tau}\text{-shift}}$.

Thus from the orthogonality of the Haar projections $\triangle_J^\omega g$ and the bound on the coefficients $\left|\mathbb{E}_{I^\natural(J)_J}^\sigma f - \mathbb{E}_A^\sigma f\right| \lesssim \alpha_{\mathcal{A}}(A)$ we have

$$\begin{aligned}|\mathsf{B}_{\text{paraproduct}}^A(f,g)| &= \left|\left\langle T_\sigma^\alpha \mathbf{1}_A, \sum_{J\in\mathcal{C}_A^{\boldsymbol{\tau}\text{-shift}}} \{\mathbb{E}_{I^\natural(J)_J}^\sigma f - \mathbb{E}_A^\sigma f\} \triangle_J^\omega g\right\rangle_\omega\right| \\ &\lesssim \alpha_{\mathcal{A}}(A)\ \|\mathbf{1}_A T_\sigma^\alpha \mathbf{1}_A\|_{L^2(\omega)}\ \left\|\mathsf{P}_{\mathcal{C}_A^{\boldsymbol{\tau}\text{-shift}}}^\omega g\right\|_{L^2(\omega)} \\ &\le \mathfrak{T}_{T^\alpha}\ \alpha_{\mathcal{A}}(A)\ \sqrt{|A|_\sigma}\ \left\|\mathsf{P}_{\mathcal{C}_A^{\boldsymbol{\tau}\text{-shift}}}^\omega g\right\|_{L^2(\omega)},\end{aligned}$$

because

$$\left\|\sum_{J\in\mathcal{C}_A^{\boldsymbol{\tau}\text{-shift}}} \lambda_J \triangle_J^\omega g\right\|_{L^2(\omega)} \le \left(\sup_J |\lambda_J|\right) \left\|\sum_{J\in\mathcal{C}_A^{\boldsymbol{\tau}\text{-shift}}} \triangle_J^\omega g\right\|_{L^2(\omega)}.$$

Next, the neighbour form $\mathsf{B}_{\text{neighbour}}^A(f,g)$ is easily controlled by the A_2^α condition using the Energy Lemma 6.2 and the fact that the cubes J are good. In particular, the information encoded in the stopping tree $\mathcal{A}$ plays no role here. We have

$$\mathsf{B}_{\text{neighbour}}^A(f,g) = \sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_A^{\boldsymbol{\tau}\text{-shift}} \\ J\Subset_{\boldsymbol{\rho}} I}} \sum_{\theta(I_J)\in\mathfrak{C}_{\mathcal{D}}(I)\setminus\{I_J\}} \langle T_\sigma^\alpha(\mathbf{1}_{\theta(I_J)} \triangle_I^\sigma f), \triangle_J^\omega g\rangle_\omega .$$

Recall that I_J is the child of I that contains J. Fix $\theta(I_J) \in \mathfrak{C}_{\mathcal{D}}(I) \setminus \{I_J\}$ momentarily, and an integer $s \ge \mathbf{r}$. The inner product to be estimated is

$$\langle T_\sigma^\alpha(\mathbf{1}_{\theta(I_J)}\sigma\Delta_I^\sigma f), \Delta_J^\omega g\rangle_\omega ,$$

i.e.,

$$\langle T_\sigma^\alpha(\mathbf{1}_{\theta(I_J)} \triangle_I^\sigma f), \triangle_J^\omega g\rangle_\omega = \mathbb{E}_{\theta(I_J)}^\sigma \Delta_I^\sigma f \cdot \langle T_\sigma^\alpha(\mathbf{1}_{\theta(I_J)}), \triangle_J^\omega g\rangle_\omega .$$

Thus we can write

$$\begin{aligned}&\mathsf{B}_{\text{neighbour}}^A(f,g) \\ &= \sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_A^{\boldsymbol{\tau}\text{-shift}} \\ J\Subset_{\boldsymbol{\rho}} I}} \sum_{\theta(I_J)\in\mathfrak{C}_{\mathcal{D}}(I)\setminus\{I_J\}} \left(\mathbb{E}_{\theta(I_J)}^\sigma \Delta_I^\sigma f\right) \langle T_\sigma^\alpha(\mathbf{1}_{\theta(I_J)}\sigma), \Delta_J^\omega g\rangle_\omega .\end{aligned} \tag{8.12}$$

Now we will use the following fractional analogue of the Poisson inequality in [23].

Lemma 8.8. *Suppose that $J \subset I \subset K$ and that* $\operatorname{dist}(J,\partial I) > \frac{1}{2}|J|^{\varepsilon/n}|I|^{(1-\varepsilon)/n}$. *Then*

$$\mathrm{P}^\alpha(J,\sigma\mathbf{1}_{K\setminus I}) \lesssim \left(\frac{|J|^{1/n}}{|I|^{1/n}}\right)^{1-\varepsilon(n+1-\alpha)} \mathrm{P}^\alpha(I,\sigma\mathbf{1}_{K\setminus I}). \tag{8.13}$$

Proof. We have

$$\mathrm{P}^{\alpha}(J,\sigma\chi_{K\setminus I})\approx\sum_{k=0}^{\infty}2^{-k}\frac{1}{|2^kJ|^{1-\alpha/n}}\int_{(2^kJ)\cap(K\setminus I)}d\sigma,$$

and $(2^kJ)\cap(K\setminus I)\neq\emptyset$ requires $\operatorname{dist}(J,K\setminus I)\leq c\,2^k\,\ell(J)$ for some dimensional constant $c>0$.

Let k_0 be the smallest such k. By our distance assumption we must then have

$$\frac{1}{2}\,|J|^{\varepsilon/n}\,|I|^{(1-\varepsilon)/n}\leq\operatorname{dist}(J,\partial I)\leq c\,2^{k_0}\,|J|^{1/n},$$

or

$$2^{-k_0-1}\leq c\left(\frac{|J|^{1/n}}{|I|^{1/n}}\right)^{1-\varepsilon}.$$

Now let k_1 be defined by $2^{k_1}\equiv|I|^{1/n}/|J|^{1/n}$. Then assuming $k_1>k_0$ (the case $k_1\leq k_0$ is similar) we have

$$\begin{aligned}\mathrm{P}^{\alpha}(J,\sigma\chi_{K\setminus I})&\approx\left\{\sum_{k=k_0}^{k_1}+\sum_{k=k_1}^{\infty}\right\}2^{-k}\frac{1}{|2^kJ|^{1-\alpha/n}}\int_{(2^kJ)\cap(K\setminus I)}d\sigma\\&\lesssim 2^{-k_0}\frac{|I|^{1-\alpha/n}}{|2^{k_0}J|^{1-\alpha/n}}\left(\frac{1}{|I|^{1-\alpha/n}}\int_{(2^{k_1}J)\cap(K\setminus I)}d\sigma\right)+2^{-k_1}\mathrm{P}^{\alpha}(I,\sigma\chi_{K\setminus I})\\&\lesssim\left(\frac{|J|^{1/n}}{|I|^{1/n}}\right)^{(1-\varepsilon)(n+1-\alpha)}\left(\frac{|I|^{1/n}}{|J|^{1/n}}\right)^{n-\alpha}\mathrm{P}^{\alpha}(I,\sigma\chi_{K\setminus I})+\frac{|J|^{1/n}}{|I|^{1/n}}\mathrm{P}^{\alpha}(I,\sigma\chi_{K\setminus I}),\end{aligned}$$

which is the inequality (8.13). □

Now fix $I_0,I_\theta\in\mathfrak{C}_{\mathcal{D}}(I)$ with $I_0\neq I_\theta$ and assume that $J\Subset_{\mathbf{r}}I_0$. Let $|J|^{1/n}/|I_0|^{1/n}=2^{-s}$ in the pivotal estimate in the energy Lemma 6.2 with $J\subset I_0\subset I$ to obtain

$$\begin{aligned}|\langle T_\sigma^\alpha(\mathbf{1}_{I_\theta}\sigma),\Delta_J^\omega g\rangle_\omega|&\lesssim\|\Delta_J^\omega g\|_{L^2(\omega)}\sqrt{|J|_\omega}\,\mathrm{P}^{\alpha}(J,\mathbf{1}_{I_\theta}\sigma)\\&\lesssim\|\Delta_J^\omega g\|_{L^2(\omega)}\sqrt{|J|_\omega}\cdot 2^{-(1-\varepsilon(n+1-\alpha))s}\,\mathrm{P}^{\alpha}(I_0,\mathbf{1}_{I_\theta}\sigma).\end{aligned}$$

Here we are using (8.13), which applies since $J\subset I_0$.

In the sum below, we keep the side length of the cubes J fixed, and of course $J\subset I_0$. We estimate

$$\begin{aligned}A(I,I_0,I_\theta,s)&\equiv\sum_{J\,:\,2^s\,\ell(J)=\ell(I):J\subset I_0}|\langle T_\sigma^\alpha(\mathbf{1}_{I_\theta}\sigma\Delta_I^\sigma f),\Delta_J^\omega g\rangle_\omega|\\&\leq 2^{-(1-\varepsilon(n+1-\alpha))s}\,|\mathbb{E}_{I_\theta}^\sigma\Delta_I^\sigma f|\,\mathrm{P}^{\alpha}(I_0,\mathbf{1}_{I_\theta}\sigma)\sum_{J\,:\,2^s\,\ell(J)=\ell(I):\,J\subset I_0}\|\Delta_J^\omega g\|_{L^2(\omega)}\sqrt{|J|_\omega}\\&\leq 2^{-(1-\varepsilon(n+1-\alpha))s}\,|\mathbb{E}_{I_\theta}^\sigma\Delta_I^\sigma f|\,\mathrm{P}^{\alpha}(I_0,\mathbf{1}_{I_\theta}\sigma)\sqrt{|I_0|_\omega}\,\Lambda(I,I_0,I_\theta,s),\end{aligned}$$

$$\Lambda(I,I_0,I_\theta,s)^2\equiv\sum_{J\in\mathcal{C}_A^{\tau\text{-shift}}:\,2^s\,\ell(J)=\ell(I):\,J\subset I_0}\|\Delta_J^\omega g\|_{L^2(\omega)}^2.$$

The last line follows upon using the Cauchy–Schwarz inequality and the fact that $\Delta_J^\omega g = 0$ if $J \notin \mathcal{C}_A^{\tau\text{-shift}}$. We also note that since $2^{s+1}\ell(J) = \ell(I)$,

$$\sum_{I_0 \in \mathfrak{C}_{\mathcal{D}}(I)} \Lambda(I, I_0, I_\theta, s)^2 \equiv \sum_{J \in \mathcal{C}_A^{\tau\text{-shift}}:\ 2^{s+1}\ \ell(J)=\ell(I):\ J\subset I} \|\Delta_J^\omega g\|_{L^2(\omega)}^2 \ ; \tag{8.14}$$

$$\sum_{I \in \mathcal{C}_A} \sum_{I_0 \in \mathfrak{C}_{\mathcal{D}}(I)} \Lambda(I, I_0, I_\theta, s)^2 \leq \left\|\mathsf{P}_{\mathcal{C}_A^{\tau\text{-shift}}}^\omega g\right\|_{L^2(\omega)}^2 .$$

Using

$$|\mathbb{E}_{I_\theta}^\sigma \Delta_I^\sigma f| \leq \sqrt{\mathbb{E}_{I_\theta}^\sigma |\Delta_I^\sigma f|^2} \leq \|\Delta_I^\sigma f\|_{L^2(\sigma)}\ |I_\theta|_\sigma^{-1/2}, \tag{8.15}$$

we can thus estimate $A(I, I_0, I_\theta, s)$ as follows, in which we use the A_2^α hypothesis $\sup_I \frac{|I|_\sigma |I|_\omega}{|I|^{2(1-\alpha/n)}} = A_2^\alpha < \infty$:

$$\begin{aligned}
&A(I, I_0, I_\theta, s)\\
&\qquad \lesssim 2^{-(1-\varepsilon(n+1-\alpha))s}\ \|\Delta_I^\sigma f\|_{L^2(\sigma)}\ \Lambda(I, I_0, I_\theta, s) \cdot |I_\theta|_\sigma^{-1/2}\, \mathrm{P}^\alpha(I_0, \mathbf{1}_{I_\theta}\sigma) \sqrt{|I_0|_\omega}\\
&\qquad \lesssim \sqrt{A_2^\alpha}\, 2^{-(1-\varepsilon(n+1-\alpha))s}\ \|\Delta_I^\sigma f\|_{L^2(\sigma)}\ \Lambda(I, I_0, I_\theta, s)\,,
\end{aligned}$$

since $\mathrm{P}^\alpha(I_0, \mathbf{1}_{I_\theta}\sigma) \lesssim |I_\theta|_\sigma / |I_\theta|^{1-\alpha/n}$ shows that

$$|I_\theta|_\sigma^{-1/2}\, \mathrm{P}^\alpha(I_0, \mathbf{1}_{I_\theta}\sigma)\ \sqrt{|I_0|_\omega} \lesssim \frac{\sqrt{|I_\theta|_\sigma}\sqrt{|I_0|_\omega}}{|I|^{1-\alpha/n}} \lesssim \sqrt{A_2^\alpha}.$$

An application of Cauchy–Schwarz in the sum over I using (8.14) then shows that

$$\begin{aligned}
&\sum_{I \in \mathcal{C}_A} \sum_{\substack{I_0, I_\theta \in \mathfrak{C}_{\mathcal{D}}(I)\\ I_0 \neq I_\theta}} A(I, I_0, I_\theta, s)\\
&\lesssim \sqrt{A_2^\alpha}\ 2^{-(1-\varepsilon(n+1-\alpha))s} \left(\sum_{I \in \mathcal{C}_A} \|\Delta_I^\sigma f\|_{L^2(\sigma)}^2\right)^{1/2}\\
&\qquad\qquad \times \left(\sum_{I \in \mathcal{C}_A} \left(\sum_{\substack{I_0, I_\theta \in \mathfrak{C}_{\mathcal{D}}(I)\\ I_0 \neq I_\theta}} \Lambda(I, I_0, I_\theta, s)\right)^2\right)^{1/2}\\
&\lesssim \sqrt{A_2^\alpha}\ 2^{-(1-\varepsilon(n+1-\alpha))s} \|\mathsf{P}_{\mathcal{C}_A}^\sigma f\|_{L^2(\sigma)} \sqrt{2^n} \left(\sum_{I \in \mathcal{C}_A} \left(\sum_{\substack{I_0 \in \mathfrak{C}_{\mathcal{D}}(I)\\ I_0 \neq I_\theta}} \Lambda(I, I_0, I_\theta, s)\right)^2\right)^{1/2}\\
&\lesssim \sqrt{A_2^\alpha}\ 2^{-(1-\varepsilon(n+1-\alpha))s} \|\mathsf{P}_{\mathcal{C}_A}^\sigma\ f\|_{L^2(\sigma)} \left\|\mathsf{P}_{\mathcal{C}_A^{\tau\text{-shift}}}^\omega g\right\|_{L^2(\omega)} .
\end{aligned}$$

This estimate is summable in $s \geq \mathbf{r}$, and so the proof of

$$\left|\mathsf{B}^{A}_{\mathrm{neighbour}}(f,g)\right| = \left|\sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_A^{\tau\text{-shift}} \\ J\Subset_\rho I}} \sum_{\theta(I_J)\in\mathfrak{C}_{\mathcal{D}}(I)\setminus\{I_J\}} \left\langle T^\alpha_\sigma\left(\mathbf{1}_{\theta(I_J)}\bigtriangleup^\sigma_I f\right), \bigtriangleup^\omega_J g\right\rangle_\omega\right|$$

$$= \left|\sum_{I\in\mathcal{C}_A}\sum_{\substack{I_0,I_\theta\in\mathfrak{C}_{\mathcal{D}}(I)\\ I_0\neq I_\theta}}\sum_{s=\mathbf{r}}^{\infty} A(I,I_0,I_\theta,s)\right| \lesssim \sqrt{A^\alpha_2}\,\left\|\mathsf{P}^\sigma_{\mathcal{C}_A}f\right\|_{L^2(\sigma)}\left\|\mathsf{P}^\omega_{\mathcal{C}_A^{\tau\text{-shift}}}g\right\|_{L^2(\omega)}$$

is complete.

It is to the sublinear form on the left side of (11.7) below, derived from the stopping form $\mathsf{B}^{A}_{\mathrm{stop}}(f,g)$, that the argument of M. Lacey in [1] will be adapted. This will result in the inequality

$$(8.16)\quad \left|\mathsf{B}^{A}_{\mathrm{stop}}(f,g)\right| \lesssim \left(\mathcal{E}^{\mathrm{deep}}_\alpha + \sqrt{A^\alpha_2}\right)\left(\alpha_{\mathcal{A}}(A)\sqrt{|A|_\sigma} + \|f\|_{L^2(\sigma)}\right)\|g\|_{L^2(\omega)}\,,\ A\in\mathcal{A},$$

where the bounded averages of f in $\mathsf{B}^{A}_{\mathrm{stop}}(f,g)$ will prove crucial. But first we turn to completing the proof of the bound (8.10) for the far below form $\mathsf{T}_{\mathrm{farbelow}}(f,g)$ using the intertwining proposition and the control of functional energy by the $\mathcal{A}^\alpha_2$ condition and the energy condition $\mathcal{E}_\alpha$.

9. Intertwining proposition

Here we generalize the intertwining proposition (see e.g. [15]) to higher dimensions. The main principle here says that, modulo terms that are controlled by the functional energy constant $\mathfrak{F}_\alpha$ and the NTV constant $\mathcal{NTV}_\alpha$ (see below), we can pass the shifted ω-corona projection $\mathsf{P}^\omega_{\mathcal{C}_B^{\tau\text{-shift}}}$ through the operator T^α to become the shifted corona projection σ-corona projection $\mathsf{P}^\sigma_{\mathcal{C}_B^{\tau\text{-shift}}}$.

More precisely, the idea is that with $T^\alpha_\sigma f \equiv T^\alpha(f\sigma)$, the intertwining operator

$$\mathsf{P}^\omega_{\mathcal{C}_B^{\tau\text{-shift}}}\left[\mathsf{P}^\omega_{\mathcal{C}_B^{\tau\text{-shift}}}T^\alpha_\sigma - T^\alpha_\sigma\mathsf{P}^\sigma_{\mathcal{C}_B^{\tau\text{-shift}}}\right]\mathsf{P}^\sigma_{\mathcal{C}_A}$$

is bounded with constant $\mathfrak{F}_\alpha + \mathcal{NTV}_\alpha$. In those cases where the coronas $\mathcal{C}_B^{\tau\text{-shift}}$ and $\mathcal{C}_A$ are (almost) disjoint, the intertwining operator reduces (essentially) to $\mathsf{P}^\omega_{\mathcal{C}_B^{\tau\text{-shift}}}T^\alpha_\sigma\mathsf{P}^\sigma_{\mathcal{C}_A}$, and then combined with the control of the functional energy constant $\mathfrak{F}_\alpha$ by the energy condition constant $\mathcal{E}_\alpha$ and $\mathcal{A}^\alpha_2 + \mathcal{A}^{\alpha,*}_2$, we obtain the required bound (8.10) for $\mathsf{T}_{\mathrm{farbelow}}(f,g)$ above.

To describe the quantities we use to bound these forms, we need to adapt to higher dimensions three definitions used for the Hilbert transform that are relevant to functional energy.

Definition 9.1. A collection $\mathcal{F}$ of dyadic cubes is σ-*Carleson* if

$$\sum_{F\in\mathcal{F}:\,F\subset S} |F|_\sigma \leq C_{\mathcal{F}}\,|S|_\sigma, \quad S\in\mathcal{F}.$$

The constant $C_{\mathcal{F}}$ is referred to as the Carleson norm of $\mathcal{F}$.

Definition 9.2. Let $\mathcal{F}$ be a collection of dyadic cubes. The good $\boldsymbol{\tau}$-shifted corona corresponding to F is defined by

$$\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}\text{-shift}} \equiv \left\{ J \in \mathcal{D}_{\mathrm{good}}^{\omega} : J \Subset_{\boldsymbol{\tau}} F \text{ and } J \not\Subset_{\boldsymbol{\tau}} F' \text{ for any } F' \in \mathfrak{C}_{\mathcal{F}}(F) \right\}.$$

Note that $\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}\text{-shift}} = \mathcal{C}_F^{\boldsymbol{\tau}\text{-shift}} \cap \mathcal{D}_{\mathrm{good}}^{\omega}$ and the collections $\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}\text{-shift}}$ have bounded overlap $\boldsymbol{\tau}$ since, for fixed J, there are at most $\boldsymbol{\tau}$ cubes $F \in \mathcal{F}$ with the property that J is good, $J \Subset_{\boldsymbol{\tau}} F$ and $J \not\Subset_{\boldsymbol{\tau}} F'$ for any $F' \in \mathfrak{C}_{\mathcal{F}}(F)$. Here $\mathfrak{C}_{\mathcal{F}}(F)$ denotes the set of $\mathcal{F}$-children of F. Given any collection $\mathcal{H} \subset \mathcal{D}$ of cubes, and a dyadic cube J, we define the corresponding Haar projection $\mathsf{P}_{\mathcal{H}}^{\omega}$ and its localization $\mathsf{P}_{\mathcal{H};J}^{\omega}$ to J by

$$\mathsf{P}_{\mathcal{H}}^{\omega} = \sum_{H \in \mathcal{H}} \triangle_H^{\omega} \quad \text{and} \quad \mathsf{P}_{\mathcal{H};J}^{\omega} = \sum_{H \in \mathcal{H}:\, H \subset J} \triangle_H^{\omega} . \tag{9.1}$$

Definition 9.3. Let $\mathfrak{F}_\alpha$ be the smallest constant in the 'functional energy' inequality below, holding for all $h \in L^2(\sigma)$, $g \in L^2(\omega)$ and all σ-Carleson collections $\mathcal{F}$ with Carleson norm $C_{\mathcal{F}}$ bounded by a fixed constant C:

$$\sum_{F \in \mathcal{F}} \sum_{J \in \mathcal{M}_{\mathbf{r}\text{-deep}}(F)} \left(\frac{\mathrm{P}^{\alpha}(J, h\sigma)}{|J|^{1/n}} \right)^2 \left\| \mathsf{P}^{\omega}_{\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}\text{-shift}};J} \mathbf{x} \right\|^2_{L^2(\omega)} \leq \mathfrak{F}_\alpha \|h\|_{L^2(\sigma)} . \tag{9.2}$$

This definition of $\mathfrak{F}_\alpha$ depends on the choice of the fixed constant C, but it will be clear from the arguments below that C may be taken to depend only on n and α, and we do not compute its value here.

There is a similar definition of the dual constant $\mathfrak{F}_\alpha^*$.

We now show that the functional energy inequality (9.2) suffices to prove an α-fractional n-dimensional analogue of the intertwining proposition (see e.g. [15]). Let $\mathcal{F}$ be any subset of $\mathcal{D}$. For any $J \in \mathcal{D}$, we define $\pi_{\mathcal{F}}^0 J$ to be the smallest $F \in \mathcal{F}$ that contains J. Then for $s \geq 1$, we recursively define $\pi_{\mathcal{F}}^s J$ to be the smallest $F \in \mathcal{F}$ that *strictly* contains $\pi_{\mathcal{F}}^{s-1} J$. This definition satisfies $\pi_{\mathcal{F}}^{s+t} J = \pi_{\mathcal{F}}^s \pi_{\mathcal{F}}^t J$ for all $s, t \geq 0$ and $J \in \mathcal{D}$. In particular $\pi_{\mathcal{F}}^s J = \pi_{\mathcal{F}}^s F$ where $F = \pi_{\mathcal{F}}^0 J$. In the special case $\mathcal{F} = \mathcal{D}$ we often suppress the subscript $\mathcal{F}$ and simply write π^s for $\pi_{\mathcal{D}}^s$. Finally, for $F \in \mathcal{F}$, we write $\mathfrak{C}_{\mathcal{F}}(F) \equiv \{F' \in \mathcal{F} : \pi_{\mathcal{F}}^1 F' = F\}$ for the collection of $\mathcal{F}$-children of F. Let

$$\mathcal{NTV}_\alpha \equiv \sqrt{\mathcal{A}_2^\alpha} + \mathfrak{T}_\alpha + \mathcal{WBP}_\alpha .$$

Proposition 9.4 (The intertwining proposition). *Suppose that $\mathcal{F}$ is σ-Carleson. Then*

$$\left| \sum_{F \in \mathcal{F}} \sum_{I:\, I \supsetneqq F} \left\langle T_\sigma^\alpha \triangle_I^\sigma f, \mathsf{P}^{\omega}_{\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}\text{-shift}}} g \right\rangle_\omega \right| \lesssim \left(\mathfrak{F}_\alpha + \mathcal{E}_\alpha + \mathcal{NTV}_\alpha \right) \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}.$$

Proof. We let $g_F = \mathsf{P}^{\omega}_{\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}\text{-shift}}} g$, which is supported in F, and write the left-hand side of the display above as

$$\sum_{F \in \mathcal{F}} \sum_{I:\, I \supsetneqq F} \langle T_\sigma^\alpha \triangle_I^\sigma f, g_F \rangle_\omega = \sum_{F \in \mathcal{F}} \left\langle T_\sigma^\alpha \Big(\sum_{I:\, I \supsetneqq F} \triangle_I^\sigma f \Big), g_F \right\rangle_\omega \equiv \sum_{F \in \mathcal{F}} \langle T_\sigma^\alpha f_F, g_F \rangle_\omega ,$$

where

$$f_F \equiv \sum_{I:\, I \supsetneqq F} \triangle_I^\sigma f \,.$$

Here, f_F is constant on F. We note that the cubes I occurring in this sum are linearly and consecutively ordered by inclusion, along with the cubes $F' \in \mathcal{F}$ that contain F. More precisely, we can write

$$F \equiv F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_n \subsetneq F_{n+1} \subsetneq \cdots \subsetneq F_N,$$

where $F_m = \pi_{\mathcal{F}}^m F$ for all $m \geq 1$. We can also write

$$F = F_0 \subsetneqq I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_k \subsetneq I_{k+1} \subsetneq \cdots \subsetneq I_K = F_N,$$

where $I_k = \pi_{\mathcal{D}}^k F$ for all $k \geq 1$. There is a (unique) subsequence $\{k_m\}_{m=1}^N$ such that

$$F_m = I_{k_m}, \quad 1 \leq m \leq N \,.$$

Define

$$f_F(x) = \sum_{\ell=1}^{\infty} \triangle_{I_\ell}^\sigma f(x) \,.$$

Assume now that $k_m \leq k < k_{m+1}$. We denote the $2^n - 1$ siblings of I by $\theta(I)$, $\theta \in \Theta$, i.e., $\{\theta(I)\}_{\theta\in\Theta} = \mathfrak{C}_{\mathcal{D}}(\pi_{\mathcal{D}} I) \setminus \{I\}$. There are two cases to consider here:

$$\theta(I_k) \notin \mathcal{F} \quad \text{and} \quad \theta(I_k) \in \mathcal{F}.$$

Suppose first that $\theta(I_k) \notin \mathcal{F}$. Then $\theta(I_k) \in \mathcal{C}_{F_{m+1}}^\sigma$ and using a telescoping sum, we compute that for

$$x \in \theta(I_k) \subset I_{k+1} \setminus I_k \subset F_{m+1} \setminus F_m \,,$$

we have

$$|f_F(x)| = \Big| \sum_{\ell=k}^{\infty} \triangle_{I_\ell}^\sigma f(x) \Big| = |\mathbb{E}_{\theta(I_k)}^\sigma f - \mathbb{E}_{I_K}^\sigma f| \lesssim \mathbb{E}_{F_{m+1}}^\sigma |f| \,.$$

On the other hand, if $\theta(I_k) \in \mathcal{F}$, then $I_{k+1} \in \mathcal{C}_{F_{m+1}}^\sigma$ and we have

$$|f_F(x) - \triangle_{\theta(I_k)}^\sigma f(x)| = \Big| \sum_{\ell=k+1}^{\infty} \triangle_{I_\ell}^\sigma f(x) \Big| = |\mathbb{E}_{I_{k+1}}^\sigma f - \mathbb{E}_{I_K}^\sigma f| \lesssim \mathbb{E}_{F_{m+1}}^\sigma |f| \,.$$

Now we write

$$\begin{aligned}
f_F &= \varphi_F + \psi_F, \\
\varphi_F &\equiv \sum_{k,\theta:\, \theta(I_k)\in\mathcal{F}}^{\infty} \mathbf{1}_{\theta(I_k)} \triangle_{I_k}^\sigma f \quad \text{and} \quad \psi_F = f_F - \varphi_F \;; \\
\sum_{F\in\mathcal{F}} \langle T_\sigma^\alpha f_F, g_F \rangle_\omega &= \sum_{F\in\mathcal{F}} \langle T_\sigma^\alpha \varphi_F, g_F \rangle_\omega + \sum_{F\in\mathcal{F}} \langle T_\sigma^\alpha \psi_F, g_F \rangle_\omega \,,
\end{aligned}$$

and note that both φ_F and ψ_F are constant on F. We can apply (7.4), using $\theta(I_k) \in \mathcal{F}$, to the first sum here to obtain

$$\Big| \sum_{F\in\mathcal{F}} \langle T^{\alpha}_{\sigma}\varphi_F, g_F\rangle_{\omega} \Big| \lesssim \mathcal{NTV}_{\alpha} \ \Big\| \sum_{F\in\mathcal{F}} \varphi_F \Big\|_{L^2(\sigma)} \Big\| \sum_{F\in\mathcal{F}} g_F \Big\|^2_{L^2(\omega)}$$
$$\lesssim \mathcal{NTV}_{\alpha} \ \|f\|_{L^2(\sigma)} \Big[\sum_{F\in\mathcal{F}} \|g_F\|^2_{L^2(\omega)} \Big]^{1/2}.$$

Turning to the second sum we note that

$$\begin{aligned}
|\psi_F| &\leq \sum_{m=0}^{N} \left(\mathbb{E}^{\sigma}_{F_{m+1}}|f|\right) \mathbf{1}_{F_{m+1}\setminus F_m} = \left(\mathbb{E}^{\sigma}_{F}|f|\right) \mathbf{1}_F + \sum_{m=0}^{N} \left(\mathbb{E}^{\sigma}_{\pi^{m+1}_{\mathcal{F}}F}|f|\right) \mathbf{1}_{\pi^{m+1}_{\mathcal{F}}F\setminus \pi^{m}_{\mathcal{F}}F} \\
&= \left(\mathbb{E}^{\sigma}_{F}|f|\right) \mathbf{1}_F + \sum_{F'\in\mathcal{F}\colon F\subset F'} \left(\mathbb{E}^{\sigma}_{\pi_{\mathcal{F}}F'}|f|\right) \mathbf{1}_{\pi_{\mathcal{F}}F'\setminus F'} \\
&\leq \alpha_{\mathcal{F}}(F) \ \mathbf{1}_F + \sum_{F'\in\mathcal{F}\colon F\subset F'} \alpha_{\mathcal{F}}(\pi_{\mathcal{F}}F') \ \mathbf{1}_{\pi_{\mathcal{F}}F'\setminus F'} \\
&\leq \alpha_{\mathcal{F}}(F) \ \mathbf{1}_F + \sum_{F'\in\mathcal{F}\colon F\subset F'} \alpha_{\mathcal{F}}(\pi_{\mathcal{F}}F') \ \mathbf{1}_{\pi_{\mathcal{F}}F'} \ \mathbf{1}_{F^c} \\
&= \alpha_{\mathcal{F}}(F) \ \mathbf{1}_F + \Phi \ \mathbf{1}_{F^c}\,, \qquad \text{for all } F\in\mathcal{F},
\end{aligned}$$

where

$$\Phi \equiv \sum_{F''\in\mathcal{F}} \alpha_{\mathcal{F}}(F'') \ \mathbf{1}_{F''}\,.$$

Now we write

$$\sum_{F\in\mathcal{F}} \langle T^{\alpha}_{\sigma}\psi_F, g_F\rangle_{\omega} = \sum_{F\in\mathcal{F}} \langle T^{\alpha}_{\sigma}(\mathbf{1}_F\psi_F), g_F\rangle_{\omega} + \sum_{F\in\mathcal{F}} \langle T^{\alpha}_{\sigma}(\mathbf{1}_{F^c}\psi_F), g_F\rangle_{\omega} \equiv \mathrm{I} + \mathrm{II}.$$

Then cube testing $|\langle T^{\alpha}_{\sigma}\mathbf{1}_F, g_F\rangle_{\omega}| = |\langle \mathbf{1}_F T^{\alpha}_{\sigma}\mathbf{1}_F, g_F\rangle_{\omega}| \leq \mathfrak{T}_{T^{\alpha}} \sqrt{|F|_{\sigma}} \, \|g_F\|_{L^2(\omega)}$ and 'quasi' orthogonality, together with the fact that ψ_F is a constant on F bounded by $\alpha_{\mathcal{F}}(F)$, give

$$\begin{aligned}
|\,\mathrm{I}\,| &\leq \sum_{F\in\mathcal{F}} |\langle T^{\alpha}_{\sigma}\mathbf{1}_F \,\psi_F, g_F\rangle_{\omega}| \lesssim \sum_{F\in\mathcal{F}} \alpha_{\mathcal{F}}(F) \ |\langle T^{\alpha}_{\sigma}\mathbf{1}_F, g_F\rangle_{\omega}| \\
&\lesssim \sum_{F\in\mathcal{F}} \alpha_{\mathcal{F}}(F) \mathcal{NTV}_{\alpha} \sqrt{|F|_{\sigma}} \|g_F\|_{L^2(\omega)} \lesssim \mathcal{NTV}_{\alpha} \|f\|_{L^2(\sigma)} \Big[\sum_{F\in\mathcal{F}} \|g_F\|^2_{L^2(\omega)} \Big]^{1/2}.
\end{aligned}$$

Now $\mathbf{1}_{F^c}\psi_F$ is supported outside F, and each J in the Haar support of g_F is $\mathbf{r}$-deeply embedded in F, i.e., $J \Subset_{\mathbf{r}} F$. Thus we can apply the energy Lemma 6.2

to obtain

$$\begin{aligned}
|\,\mathrm{II}\,| &= \Big|\sum_{F\in\mathcal{F}}\langle T_\sigma^\alpha(\mathbf{1}_{F^c}\psi_F),g_F\rangle_\omega\Big| \\
&\lesssim \sum_{F\in\mathcal{F}}\sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)}\frac{\mathrm{P}^\alpha(J,\mathbf{1}_{F^c}\Phi\sigma)}{|J|^{1/n}}\left\|\mathsf{P}^\omega_{\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}\text{-shift}};J}\mathbf{x}\right\|_{L^2(\omega)}\|\mathsf{P}^\omega_J g_F\|_{L^2(\omega)} \\
&\quad+\sum_{F\in\mathcal{F}}\sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)}\frac{\mathrm{P}^\alpha_{1+\delta'}(J,\mathbf{1}_{F^c}\Phi\sigma)}{|J|^{1/n}}\left\|\mathsf{P}^\omega_{(\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}\text{-shift}})^*;J}\mathbf{x}\right\|_{L^2(\omega)}\|\mathsf{P}^\omega_J g_F\|_{L^2(\omega)} \\
&\equiv \mathrm{II}_G+\mathrm{II}_B\,.
\end{aligned}$$

Then from Cauchy–Schwarz, the functional energy condition, and $\|\Phi\|_{L^2(\sigma)}\lesssim\|f\|_{L^2(\sigma)}$ we obtain

$$\begin{aligned}
|\,\mathrm{II}_G| &\le \Big(\sum_{F\in\mathcal{F}}\sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)}\Big(\frac{\mathrm{P}^\alpha(J,\mathbf{1}_{F^c}\Phi\sigma)}{|J|^{1/n}}\Big)^2\|\mathsf{P}^\omega_{\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}\text{-shift}};J}\,\mathbf{x}\|^2_{L^2(\omega)}\Big)^{1/2} \\
&\qquad\times\Big(\sum_{F\in\mathcal{F}}\sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)}\|\mathsf{P}^\omega_J g_F\|^2_{L^2(\omega)}\Big)^{1/2} \\
&\qquad\lesssim \mathfrak{F}_\alpha\|\Phi\|_{L^2(\sigma)}\Big[\sum_{F\in\mathcal{F}}\|g_F\|^2_{L^2(\omega)}\Big]^{1/2}\lesssim\mathfrak{F}_\alpha\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)},
\end{aligned}$$

by the bounded overlap by $\boldsymbol{\tau}$ of the shifted coronas $\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}\text{-shift}}$.

In term II_B the projections $\mathsf{P}^\omega_{(\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}\text{-shift}})^*;J}$ are no longer almost orthogonal, and we must instead exploit the decay in the Poisson integral $\mathrm{P}^\alpha_{1+\delta'}$ along with goodness of the cubes J. This idea was already used by M. Lacey and B. Wick in [8] in a similar situation. As a consequence of this decay we will be able to bound II_B *directly* by the energy condition, without having to invoke the more difficult functional energy condition. For the decay we compute

$$\begin{aligned}
\frac{\mathrm{P}^\alpha_{1+\delta'}(J,\Phi\sigma)}{|J|^{1/n}} &= \int_{F^c}\frac{|J|^{\delta'/n}}{|y-c_J|^{n+1+\delta-\alpha}}\,\Phi(y)\,d\sigma(y) \\
&\le \sum_{t=0}^\infty\int_{\pi_{\mathcal{F}}^{t+1}F\setminus\pi_{\mathcal{F}}^tF}\Big(\frac{|J|^{1/n}}{\mathrm{dist}(c_J,(\pi_{\mathcal{F}}^tF)^c)}\Big)^{\delta'}\frac{1}{|y-c_J|^{n+1-\alpha}}\,\Phi(y)\,d\sigma(y) \\
&\le \sum_{t=0}^\infty\Big(\frac{|J|^{1/n}}{\mathrm{dist}(c_J,(\pi_{\mathcal{F}}^tF)^c)}\Big)^{\delta'}\frac{\mathrm{P}^\alpha(J,\mathbf{1}_{\pi_{\mathcal{F}}^{t+1}F\setminus\pi_{\mathcal{F}}^tF}\Phi\sigma)}{|J|^{1/n}},
\end{aligned}$$

and then use the goodness inequality

$$\mathrm{dist}(c_J,(\pi_{\mathcal{F}}^tF)^c)\ge\frac12\,|\pi_{\mathcal{F}}^tF|^{(1-\varepsilon)/n}\,|J|^{\varepsilon/n}\ge\frac12 2^{t(1-\varepsilon)}|F|^{\frac{1-\varepsilon}{n}}|J|^{\varepsilon/n}\ge 2^{t(1-\varepsilon)-1}|J|^{1/n},$$

to conclude that

$$\left(\frac{\mathrm{P}^{\alpha}_{1+\delta'}(J,\mathbf{1}_{F^c}\Phi\sigma)}{|J|^{1/n}}\right)^2 \lesssim \left(\sum_{t=0}^{\infty} 2^{-t\delta'(1-\varepsilon)} \frac{\mathrm{P}^{\alpha}(J,\mathbf{1}_{\pi_{\mathcal{F}}^{t+1}F\setminus\pi_{\mathcal{F}}^{t}F}\Phi\sigma)}{|J|^{1/n}}\right)^2 \lesssim \sum_{t=0}^{\infty} 2^{-t\delta'(1-\varepsilon)} \left(\frac{\mathrm{P}^{\alpha}(J,\mathbf{1}_{\pi_{\mathcal{F}}^{t+1}F\setminus\pi_{\mathcal{F}}^{t}F}\Phi\sigma)}{|J|^{1/n}}\right)^2 . \tag{9.3}$$

Now we apply Cauchy–Schwarz to obtain

$$\begin{aligned}
\mathrm{II}_B &= \sum_{F\in\mathcal{F}} \sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)} \frac{\mathrm{P}^{\alpha}_{1+\delta'}(J,\mathbf{1}_{F^c}\Phi\sigma)}{|J|^{1/n}} \left\| \mathsf{P}^{\omega}_{(\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}\text{-shift}})^*;J} \mathbf{x} \right\|_{L^2(\omega)} \|\mathsf{P}^{\omega}_J g_F\|_{L^2(\omega)} \\
&\le \left(\sum_{F\in\mathcal{F}} \sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)} \left(\frac{\mathrm{P}^{\alpha}_{1+\delta'}(J,\mathbf{1}_{F^c}\Phi\sigma)}{|J|^{1/n}}\right)^2 \left\| \mathsf{P}^{\omega}_{(\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}\text{-shift}})^*;J} \mathbf{x} \right\|^2_{L^2(\omega)} \right)^{1/2} \\
&\qquad \times \left[\sum_F \|g_F\|^2_{L^2(\omega)} \right]^{1/2} \\
&\equiv \sqrt{\mathrm{II}_{\mathrm{energy}}} \left[\sum_F \|g_F\|^2_{L^2(\omega)} \right]^{1/2},
\end{aligned}$$

and it remains to estimate $\mathrm{II}_{\mathrm{energy}}$. From (9.3) and the deep plugged energy condition we have

$$\begin{aligned}
&\mathrm{II}_{\mathrm{energy}} \\
&\le \sum_{F\in\mathcal{F}} \sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)} \sum_{t=0}^{\infty} 2^{-t\delta'(1-\varepsilon)} \left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{\pi_{\mathcal{F}}^{t+1}F\setminus\pi_{\mathcal{F}}^{t}F}\Phi\sigma\right)}{|J|^{1/n}}\right)^2 \\
&\qquad\qquad \times \left\| \mathsf{P}^{\omega}_{(\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}\text{-shift}})^*;J} \mathbf{x} \right\|^2_{L^2(\omega)} \\
&= \sum_{t=0}^{\infty} 2^{-t\delta'(1-\varepsilon)} \sum_{G\in\mathcal{F}} \sum_{F\in\mathfrak{C}_{\mathcal{F}}^{(t+1)}(G)} \sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)} \left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{G\setminus\pi_{\mathcal{F}}^{t}F}\Phi\sigma\right)}{|J|^{1/n}}\right)^2 \\
&\qquad\qquad \times \left\| \mathsf{P}^{\omega}_{(\mathcal{C}_F^{\mathrm{good},\boldsymbol{\tau}\text{-shift}})^*;J} \mathbf{x} \right\|^2_{L^2(\omega)} \\
&\lesssim \sum_{t=0}^{\infty} 2^{-t\delta'(1-\varepsilon)} \sum_{G\in\mathcal{F}} \alpha_{\mathcal{F}}(G)^2 \sum_{F\in\mathfrak{C}_{\mathcal{F}}^{(t+1)}(G)} \sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)} \left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{G\setminus\pi_{\mathcal{F}}^{t}F}\sigma\right)}{|J|^{1/n}}\right)^2 \\
&\qquad\qquad \times \|\mathsf{P}^{\mathrm{subgood},\omega}_J \mathbf{x}\|^2_{L^2(\omega)} \\
&\lesssim \sum_{t=0}^{\infty} 2^{-t\delta'(1-\varepsilon)} \sum_{G\in\mathcal{F}} \alpha_{\mathcal{F}}(G)^2 \left(\mathcal{E}_\alpha^2 + A_2^\alpha\right) |G|_\sigma \lesssim \left(\mathcal{E}_\alpha^2 + A_2^\alpha\right) \|f\|^2_{L^2(\sigma)} .
\end{aligned}$$

This completes the proof of the intertwining Proposition 9.4. □

10. Control of functional energy by energy modulo $\mathcal{A}_2^\alpha$

Now we show that the functional energy constants $\mathfrak{F}_\alpha$ are controlled by $\mathcal{A}_2^\alpha$ and both the *deep* and *refined* energy constants $\mathcal{E}_\alpha^{\text{deep}}$ and $\mathcal{E}_\alpha^{\text{refined}}$ defined in Definition 2.4. Recall $(\mathcal{E}_\alpha)^2 = (\mathcal{E}_\alpha^{\text{deep}})^2 + (\mathcal{E}_\alpha^{\text{refined}})^2$.

Proposition 10.1.

$$\mathfrak{F}_\alpha \lesssim \mathcal{E}_\alpha + \sqrt{\mathcal{A}_2^\alpha} + \sqrt{\mathcal{A}_2^{\alpha,*}} \quad \text{and} \quad \mathfrak{F}_\alpha^* \lesssim \mathcal{E}_\alpha^* + \sqrt{\mathcal{A}_2^\alpha} + \sqrt{\mathcal{A}_2^{\alpha,*}}\,.$$

To prove this proposition, we fix $\mathcal{F}$ as in (9.2) and set

$$\mu \equiv \sum_{F\in\mathcal{F}} \sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)} \left\| \mathsf{P}_{F,J}^\omega \frac{x}{|J|^{1/n}} \right\|_{L^2(\omega)}^2 \cdot \delta_{(c(J),|J|^{1/n})}\,, \tag{10.1}$$

where $\mathcal{M}_{\mathbf{r}\text{-deep}}(F)$ consists of the maximal $\mathbf{r}$-deeply embedded subcubes of F. For convenience in notation, we denote for any dyadic cube J the localized projection $\mathsf{P}^\omega_{\mathcal{C}_F^{\text{good},\boldsymbol{\tau}\text{-shift}};J}$ given in (9.1) by

$$\mathsf{P}_{F,J}^\omega \equiv \mathsf{P}^\omega_{\mathcal{C}_F^{\text{good},\boldsymbol{\tau}\text{-shift}};J} = \sum_{J'\subset J:\ J'\in\mathcal{C}_F^{\text{good},\boldsymbol{\tau}\text{-shift}}} \triangle_{J'}^\omega\,.$$

We emphasize that the cubes $J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)$ are not necessarily good, but that the subcubes $J'\subset J$ arising in the projection $\mathsf{P}_{F,J}^\omega$ are good. We can replace x by $x-c$ inside the projection for any choice of c we wish; the projection is unchanged. Here δ_q denotes a Dirac unit mass at a point q in the upper half plane $\mathbb{R}_+^2$.

We prove the two-weight inequality

$$\|\mathbb{P}^\alpha(f\sigma)\|_{L^2(\mathbb{R}_+^{n+1},\,\mu)} \lesssim \left(\mathcal{E}_\alpha + \sqrt{\mathcal{A}_2^\alpha} + \sqrt{\mathcal{A}_2^{\alpha,*}}\right) \|f\|_{L^2(\sigma)}\,, \tag{10.2}$$

for all nonnegative f in $L^2(\sigma)$, noting that $\mathcal{F}$ and f are *not* related here. Above, $\mathbb{P}^\alpha(\cdot)$ denotes the α-fractional Poisson extension to the upper half-space $\mathbb{R}_+^{n+1}$,

$$\mathbb{P}^\alpha\nu(x,t) \equiv \int_{\mathbb{R}^n} \frac{t}{(t^2+|x-y|^2)^{\frac{n+1-\alpha}{2}}}\, d\nu(y)\,,$$

so that in particular

$$\|\mathbb{P}^\alpha(f\sigma)\|^2_{L^2(\mathbb{R}_+^{n+1},\,\mu)} = \sum_{F\in\mathcal{F}} \sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)} \mathbb{P}^\alpha(f\sigma)(c(J),|J|^{1/n})^2 \left\| \mathsf{P}_{F,J}^\omega \frac{x}{|J|^{1/n}} \right\|_{L^2(\omega)}^2,$$

and so (10.2) proves the first line in Proposition 10.1 upon inspecting (9.2).

By the two-weight inequality for the Poisson operator in [14], inequality (10.2) requires checking these two inequalities

$$\int_{\mathbb{R}_+^{n+1}} \mathbb{P}^\alpha(\mathbf{1}_I\sigma)(x,t)^2\, d\mu(x,t) \equiv \|\mathbb{P}^\alpha(\mathbf{1}_I\sigma)\|^2_{L^2(\widehat{I},\mu)} \lesssim (\mathcal{A}_2^{\alpha,*} + \mathcal{E}_\alpha^2)\,\sigma(I)\,, \tag{10.3}$$

$$\int_{\mathbb{R}} [\mathbb{P}^{\alpha*}(t\,\mathbf{1}_{\widehat{I}}\mu)]^2\, d\sigma(x) \lesssim (\mathcal{A}_2^\alpha + \mathcal{E}_\alpha\sqrt{\mathcal{A}_2^\alpha}) \int_{\widehat{I}} t^2\, d\mu(x,t)\,, \tag{10.4}$$

for all *dyadic* cubes $I \in \mathcal{D}$, where $\widehat{I} = I \times [0, |I|]$ is the box over I in the upper half-space, and

$$\mathbb{P}^{\alpha *}(t\,\mathbf{1}_{\widehat{I}}\,\mu)(x) = \int_{\widehat{I}} \frac{t^2}{(t^2 + |x-y|^2)^{\frac{n+1-\alpha}{2}}}\, d\mu(y,t)\,.$$

It is important to note that we can choose for $\mathcal{D}$ any fixed dyadic grid, the compensating point being that the integrations on the left sides of (10.3) and (10.4) are taken over the entire spaces $\mathbb{R}^n_+$ and $\mathbb{R}^n$ respectively.

Remark 10.2. There is a gap in the proof of the Poisson inequality at the top of page 542 in [14]. However, this gap can be fixed as in [22] or [2].

The following elementary Poisson inequalities will be used extensively.

Lemma 10.3. *Suppose that J, K, I are cubes satisfying $J \subset K \subset 2K \subset I$, and that$\mu$ is a positive measure supported in $\mathbb{R}^n \setminus I$. Then*

$$\frac{\mathrm{P}^\alpha(J,\mu)}{|J|^{1\ /n}} \lesssim \frac{\mathrm{P}^\alpha(K,\mu)}{|K|^{1/n}} \lesssim \frac{\mathrm{P}^\alpha(J,\mu)}{|J|^{1/n}}.$$

Proof. We have

$$\frac{\mathrm{P}^\alpha(J,\mu)}{|J|^{1/n}} = \frac{1}{|J|^{1/n}} \int \frac{|J|^{1/n}}{(|J|^{1/n} + |x - c_J|)^{n+1-\alpha}}\, d\mu(x)\,,$$

where $J \subset K \subset 2K \subset I$ implies that

$$|J|^{1/n} + |x - c_J| \approx |K|^{1/n} + |x - c_K|, \qquad x \in \mathbb{R}^n \setminus I. \qquad \square$$

Now we record the bounded overlap of the projections $\mathsf{P}^\omega_{F,J}$.

Lemma 10.4. *Suppose $\mathsf{P}^\omega_{F,J}$ is as above and fix any $I_0 \in \mathcal{D}$. If $J \in \mathcal{M}_{\mathbf{r}\text{-deep}}(F)$ for some $F \in \mathcal{F}$ with $F \supsetneqq I_0$ and $\mathsf{P}^\omega_{F,J} \neq 0$, then*

$$F = \pi^{(\ell)}_{\mathcal{F}} I_0 \quad \text{for some } 0 \leq \ell \leq \boldsymbol{\tau}\,.$$

As a consequence we have the bounded overlap,

$$\#\{F \in \mathcal{F} : J \subset I_0 \subsetneqq F \text{ for some } J \in \mathcal{M}_{\mathbf{r}\text{-deep}}(F) \text{ with } \mathsf{P}^\omega_{F,J} \neq 0\} \leq \boldsymbol{\tau}.$$

Proof. Indeed, if $J' \in \mathcal{C}^{\mathrm{good},\boldsymbol{\tau}\text{-shift}}_{\pi^{(\ell)}_{\mathcal{F}} I_0}$ for some $\ell > \boldsymbol{\tau}$, then either $J' \cap \pi^{(0)}_{\mathcal{F}} I_0 = \emptyset$ or $J' \supset \pi^{(0)}_{\mathcal{F}} I_0$. Since $J \subset I_0 \subset \pi^{(0)}_{\mathcal{F}} I_0$, we cannot have J' contained in J, and this shows that $\mathsf{P}^\omega_{\pi^{(\ell)}_{\mathcal{F}} I_0, J} = 0$. $\square$

Finally we record the only place in the proof where the refined energy condition is used. This lemma will be used in bounding both of the Poisson testing conditions.

Lemma 10.5. *Let $\mathcal{F}$ and $\{\mathsf{P}^{\omega}_{F,J}\}_{F\in\mathcal{F},\,J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)}$ be as above. For any shifted $\mathcal{D}$-dyadic cube I_0, define*

$$B(I_0)\equiv\sum_{F\in\mathcal{F}:\,F\supsetneqq I_0}\ \sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F):\ J\subset I_0}\left(\frac{\mathrm{P}^{\alpha}(J,\mathbf{1}_{I_0}\sigma)}{|J|^{1/n}}\right)^2\|\mathsf{P}^{\omega}_{F,J}\mathbf{x}\|^2_{L^2(\omega)}\,. \tag{10.5}$$

Then

$$B(I_0)\lesssim\boldsymbol{\tau}\left((\mathcal{E}^{\mathrm{refinedplug}}_{\alpha})^2+(\mathcal{E}^{\mathrm{deepplug}}_{\alpha})^2\right)|I_0|_{\sigma}\lesssim\boldsymbol{\tau}\left((\mathcal{E}_{\alpha})^2+\beta A^{\alpha}_2\right)|I_0|_{\sigma}\,. \tag{10.6}$$

Proof. Define, for I_0 a dyadic cube,

$$\Lambda(I_0)\equiv\{J\subset I_0: J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)\text{ for some }F\supsetneqq I_0\text{ with }\mathsf{P}^{\omega}_{F,J}\neq 0\}.$$

By Lemma 10.4 we may pigeonhole the cubes J in $\Lambda(I_0)$ as follows:

$$\Lambda(I_0)=\bigcup_{\ell=0}^{\boldsymbol{\tau}}\Lambda_{\ell}(I_0);\quad \Lambda_{\ell}(I_0)\equiv\left\{J\subset I_0: J\in\mathcal{M}_{\mathbf{r}\text{-deep}}\left(\pi^{(\ell)}_{\mathcal{F}}I_0\right)\right\}.$$

Now fix ℓ, and for each J in the pairwise disjoint decomposition $\Lambda_{\ell}(I_0)$ of I_0, note that *either* J must contain some $K\in\mathcal{M}_{\mathbf{r}\text{-deep}}(I_0)$ or $J\subset K$ for some $K\in\mathcal{M}_{\mathbf{r}\text{-deep}}(I_0)$;

$$\begin{aligned}\Lambda_{\ell}(I_0)&=\Lambda^{\mathrm{big}}_{\ell}(I_0)\cup\Lambda^{\mathrm{small}}_{\ell}(I_0)\,;\\ \Lambda^{\mathrm{small}}_{\ell}(I_0)&\equiv\{J\in\Lambda_{\ell}(I_0): J\subset K\text{ for some }K\in\mathcal{M}_{\mathbf{r}\text{-deep}}(I_0)\}\,,\end{aligned}$$

and we make the corresponding decomposition $B(I_0)=B^{\mathrm{big}}(I_0)+B^{\mathrm{small}}(I_0)$, where

$$\begin{aligned}&B^{\mathrm{big/small}}(I_0)\\ &\equiv\sum_{\ell=0}^{\boldsymbol{\tau}}\ \sum_{J\in\Lambda^{\mathrm{big/small}}_{\ell}(I_0)}\left(\frac{\mathrm{P}^{\alpha}(J,\mathbf{1}_{I_0}\sigma)}{|J|^{1/n}}\right)^2\sum_{F\in\mathcal{F}:\,F\supsetneqq I_0\text{ and }J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)}\left\|\mathsf{P}^{\omega}_{F,J}\mathbf{x}\right\|^2_{L^2(\omega)}\,.\end{aligned}$$

Turning first to $B^{\mathrm{small}}(I_0)$, we use the $\boldsymbol{\tau}$-overlap of the projections $\mathsf{P}^{\omega}_{F,J}$, together with Lemma 10.3, to obtain

$$\begin{aligned}B^{\mathrm{small}}(I_0)&\leq\boldsymbol{\tau}\sum_{\ell=0}^{\boldsymbol{\tau}}\ \sum_{J\in\Lambda^{\mathrm{small}}_{\ell}(I_0)}\left(\frac{\mathrm{P}^{\alpha}(J,\mathbf{1}_{I_0}\sigma)}{|J|^{1/n}}\right)^2\|\mathsf{P}^{\omega}_{J}\mathbf{x}\|^2_{L^2(\omega)}\\ &\lesssim\boldsymbol{\tau}^2\,(\mathcal{E}^{\mathrm{refinedplug}}_{\alpha})^2\,|I_0|_{\sigma}\lesssim\boldsymbol{\tau}^2\,[(\mathcal{E}_{\alpha})^2+\beta A^{\alpha}_2]\,|I_0|_{\sigma}\,,\end{aligned} \tag{10.7}$$

where the final estimate follows from (2.8), and this, for both I_0 $\mathcal{D}$-dyadic and I_0 shifted $\mathcal{D}$-dyadic, is the only point in the proof of Theorem 2.6 that the refined energy condition is used. Indeed, each cube $\pi^{(\ell)}_{\mathcal{F}}I_0$ equals $\pi^{(\ell')}_{\mathcal{D}}I_0$ for some ℓ', and it is with this ℓ' that we apply the plugged refined energy condition.

Turning now to the more delicate term $B^{\mathrm{big}}(I_0)$, we write for $J \in \Lambda_\ell^{\mathrm{big}}(I_0)$,

$$\begin{aligned}\left\| \mathsf{P}_J^{\mathrm{good},\omega} \mathbf{x} \right\|_{L^2(\omega)}^2 &= \sum_{J' \subset J:\ J' \text{ good}} \|\triangle_{J'}^\omega \mathbf{x}\|_{L^2(\omega)}^2 \\ &= \sum_{J' \in \mathcal{N}_{\mathbf{r}}(I):\ J' \subset J} \|\triangle_{J'}^\omega \mathbf{x}\|_{L^2(\omega)}^2 + \sum_{K \in \mathcal{M}_{\mathbf{r}\text{-deep}}(I_0):\ K \subset J} \left\| \mathsf{P}_K^{\mathrm{good},\omega} \mathbf{x} \right\|_{L^2(\omega)}^2 ,\end{aligned}$$

where $\mathcal{N}_{\mathbf{r}}(I) \equiv \{J' \subset I : \ell(J') \geq 2^{-\mathbf{r}} \ell(I)\}$ denotes the set of $\mathbf{r}$-near cubes in I, and then using the $\boldsymbol{\tau}$-overlap of the projections $\mathsf{P}_{F,J}^\omega$, we estimate

$$\begin{aligned}B^{\mathrm{big}}(I_0) &= \sum_{\ell=0}^{\boldsymbol{\tau}} \sum_{J \in \Lambda_\ell^{\mathrm{big}}(I_0)} \left(\frac{\mathrm{P}^\alpha(J, \mathbf{1}_{I_0}\sigma)}{|J|^{1/n}} \right)^2 \sum_{F \in \mathcal{F}:\ F \supsetneqq I_0 \text{ and } J \in \mathcal{M}_{\mathbf{r}\text{-deep}}(F)} \left\| \mathsf{P}_{F,J}^\omega \mathbf{x} \right\|_{L^2(\omega)}^2 \\ &\leq \boldsymbol{\tau} \sum_{\ell=0}^{\boldsymbol{\tau}} \sum_{J \in \Lambda_\ell^{\mathrm{big}}(I_0)} \left(\frac{\mathrm{P}^\alpha(J, \mathbf{1}_{I_0}\sigma)}{|J|^{1/n}} \right)^2 \|\mathsf{P}_J^{\mathrm{good},\omega} \mathbf{x}\|_{L^2(\omega)}^2 \\ &= \boldsymbol{\tau} \sum_{\ell=0}^{\boldsymbol{\tau}} \sum_{J \in \Lambda_\ell^{\mathrm{big}}(I_0)} \left(\frac{\mathrm{P}^\alpha(J, \mathbf{1}_{I_0}\sigma)}{|J|^{1/n}} \right)^2 \sum_{J' \in \mathcal{N}_{\mathbf{r}}(I_0):\ J' \subset J} \| \triangle_{J'}^\omega \mathbf{x}\|_{L^2(\omega)}^2 \\ &\quad + \boldsymbol{\tau} \sum_{\ell=0}^{\boldsymbol{\tau}} \sum_{J \in \Lambda_\ell^{\mathrm{big}}(I_0)} \left(\frac{\mathrm{P}^\alpha(J, \mathbf{1}_{I_0}\sigma)}{|J|^{1/n}} \right)^2 \sum_{K \in \mathcal{M}_{\mathbf{r}\text{-deep}}(I_0):\ K \subset J} \|\mathsf{P}_K^{\mathrm{subgood},\omega} \mathbf{x}\|_{L^2(\omega)L^2(\omega)}^2 \\ &\equiv \boldsymbol{\tau} \left(B_1^{\mathrm{big}}(I_0) + B_2^{\mathrm{big}}(I_0) \right).\end{aligned}$$

Now we have, using that the $J \in \Lambda_l^{\mathrm{big}}(I_0)$ are pairwise disjoint,

$$\begin{aligned}B_1^{\mathrm{big}}(I_0) &\approx \sum_{\ell=0}^{\boldsymbol{\tau}} \left(\frac{\mathrm{P}^\alpha(I_0, \mathbf{1}_{I_0}\sigma)}{|I_0|^{1/n}} \right)^2 \sum_{J' \in \mathcal{N}_{\mathbf{r}}(I_0)} \| \triangle_J^\omega \mathbf{x}\|_{L^2(\omega)}^2 \\ &\lesssim \sum_{\ell=0}^{\boldsymbol{\tau}} (\#\mathcal{N}(I_0)) \left(\frac{\mathrm{P}^\alpha(I_0, \mathbf{1}_{I_0}\sigma)}{|I_0|^{1/n}} \right)^2 |I_0|^{2/n} \, |I_0|_\omega \lesssim \boldsymbol{\tau}\, 2^{n\mathbf{r}} \, A_2^\alpha \, |I_0|_\sigma .\end{aligned}$$

Using $\mathrm{P}^\alpha(J, \mathbf{1}_{I_0}\sigma) = \mathrm{P}^\alpha(J, \mathbf{1}_J\sigma) + \mathrm{P}^\alpha(J, \mathbf{1}_{I_0 \setminus J}\sigma)$, we have

$$\begin{aligned}B_2^{\mathrm{big}}(I_0) &\approx \sum_{\ell=0}^{\boldsymbol{\tau}} \sum_{J \in \Lambda_\ell^{\mathrm{big}}(I_0)} \left(\frac{\mathrm{P}^\alpha(J, \mathbf{1}_J\sigma)}{|J|^{1/n}} \right)^2 \sum_{K \in \mathcal{M}_{\mathbf{r}\text{-deep}}(I_0):\ K \subset J} \|\mathsf{P}_K^{\mathrm{subgood},\omega} \mathbf{x}\|_{L^2(\omega)}^2 \\ &\quad + \sum_{\ell=0}^{\boldsymbol{\tau}} \sum_{J \in \Lambda_\ell^{\mathrm{big}}(I_0)} \left(\frac{\mathrm{P}^\alpha(J, \mathbf{1}_{I_0 \setminus J}\sigma)}{|J|^{1/n}} \right)^2 \sum_{K \in \mathcal{M}_{\mathbf{r}\text{-deep}}(I_0):\ K \subset J} \|\mathsf{P}_K^{\mathrm{subgood},\omega} \mathbf{x}\|_{L^2(\omega)}^2 \\ &\equiv B_3^{\mathrm{big}}(I_0) + B_4^{\mathrm{big}}(I_0).\end{aligned}$$

Now, since the $J \in \Lambda_l^{\mathrm{big}}(I_0)$ are pairwise disjoint,

$$B_3^{\mathrm{big}}(I_0) \lesssim \sum_{\ell=0}^{\boldsymbol{\tau}} \sum_{J \in \Lambda_\ell^{\mathrm{big}}(I_0)} \left(\frac{|J|_\sigma}{|J|^{1/n}} \right)^2 |J|^{2/n} \, |J|_\omega \lesssim \boldsymbol{\tau} \, A_2^\alpha \, |I_0|_\sigma ,$$

and since, for $K \subset J$,

$$\frac{\mathrm{P}^{\alpha}(J, \mathbf{1}_{I_0 \setminus J}\sigma)}{|J|^{1/n}} \lesssim \frac{\mathrm{P}^{\alpha}(K, \mathbf{1}_{I_0 \setminus J}\sigma)}{|K|^{1/n}},$$

we have

$$\begin{aligned}
B_4^{\mathrm{big}}(I_0) &= \sum_{\ell=0}^{\boldsymbol{\tau}} \sum_{J \in \Lambda_\ell^{\mathrm{big}}(I_0)} \sum_{K \in \mathcal{M}_{\mathbf{r}\text{-deep}}(I_0):\, K \subset J} \left(\frac{\mathrm{P}^{\alpha}(J, \mathbf{1}_{I_0 \setminus J}\sigma)}{|J|^{1/n}} \right)^2 \|\mathsf{P}_K^{\mathrm{subgood},\omega} \mathbf{x}\|_{L^2(\omega)}^2 \\
&\lesssim \sum_{\ell=0}^{\boldsymbol{\tau}} \sum_{J \in \Lambda_\ell^{\mathrm{big}}(I_0)} \sum_{K \in \mathcal{M}_{\mathbf{r}\text{-deep}}(I_0):\, K \subset J} \left(\frac{\mathrm{P}^{\alpha}(K, \mathbf{1}_{I_0 \setminus J}\sigma)}{|K|^{1/n}} \right)^2 \|\mathsf{P}_K^{\mathrm{subgood},\omega} \mathbf{x}\|_{L^2(\omega)}^2 \\
&\leq \sum_{\ell=0}^{\boldsymbol{\tau}} \sum_{J \in \Lambda_\ell^{\mathrm{big}}(I_0)} \sum_{K \in \mathcal{M}_{\mathbf{r}\text{-deep}}(I_0):\, K \subset J} \left(\frac{\mathrm{P}^{\alpha}(K, \mathbf{1}_{I_0 \setminus K}\sigma)}{|K|^{1/n}} \right)^2 \|\mathsf{P}_K^{\mathrm{subgood},\omega} \mathbf{x}\|_{L^2(\omega)}^2 \\
&\leq \sum_{\ell=0}^{\boldsymbol{\tau}} \sum_{K \in \mathcal{M}_{\mathbf{r}\text{-deep}}(I_0)} \left(\frac{\mathrm{P}^{\alpha}(K, \mathbf{1}_{I_0 \setminus K}\sigma)}{|K|^{1/n}} \right)^2 \|\mathsf{P}_K^{\mathrm{subgood},\omega} \mathbf{x}\|_{L^2(\omega)}^2 \\
&\lesssim \boldsymbol{\tau} (\mathcal{E}_\alpha^{\mathrm{deepplug}})^2 \, |I_0|_\sigma \lesssim \boldsymbol{\tau} ((\mathcal{E}_\alpha^{\mathrm{deep}})^2 + \beta A_2^\alpha) \, |I_0|_\sigma \,,
\end{aligned}$$

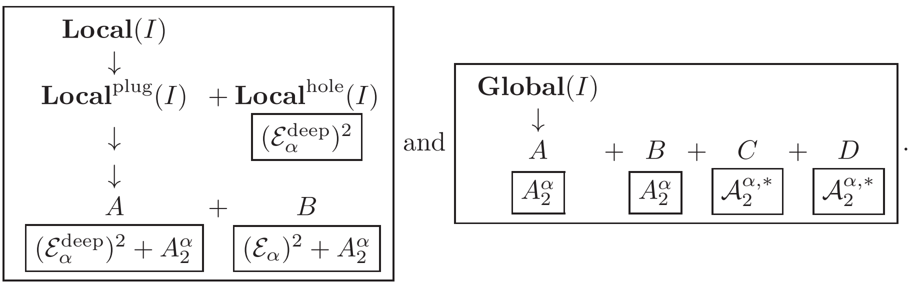
where the final line follows from (2.7). Finally, the case when I_0 is a shifted $\mathcal{D}$-dyadic cube is easy and left for the reader. □

10.1. The Poisson testing inequality

Fix $I \in \mathcal{D}$. We split the integration on the left side of (10.3) into a local and global piece:

$$\begin{aligned}
\int_{\mathbb{R}_+^{n+1}} \mathbb{P}^\alpha (\mathbf{1}_I \sigma)^2 \, d\mu &= \int_{\widehat{I}} \mathbb{P}^\alpha (\mathbf{1}_I \sigma)^2 \, d\mu + \int_{\mathbb{R}_+^{n+1} \setminus \widehat{I}} \mathbb{P}^\alpha (\mathbf{1}_I \sigma)^2 \, d\mu \\
&\equiv \mathbf{Local}(I) + \mathbf{Global}(I).
\end{aligned}$$

Here is a brief schematic diagram of the decompositions, with bounds in □, used in this subsection:



We turn first to estimating the local term $\mathbf{Local}(I)$.

An important consequence of the fact that I and J lie in the same grid $\mathcal{D} = \mathcal{D}^\omega$, is that $(c(J), |J|) \in \widehat{I}$ if and only if $J \subset I$.

Thus we have

$$\begin{aligned}\int_{\widehat{I}} \mathbb{P}^{\alpha}(\mathbf{1}_I\sigma)(x,t)^2\,d\mu(x,t) &= \sum_{F\in\mathcal{F}}\ \sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F):\,J\subset I} \mathbb{P}^{\alpha}(\mathbf{1}_I\sigma)(c_J,|J|^{1/n})^2 \left\|\mathsf{P}^{\omega}_{F,J}\,\frac{\mathbf{x}}{|J|^{1/n}}\right\|^2_{L^2(\omega)}\\ &= \sum_{F\in\mathcal{F}}\ \sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F):\,J\subset I} \mathrm{P}^{\alpha}(J,\mathbf{1}_I\sigma)^2 \left\|\mathsf{P}^{\omega}_{F,J}\,\frac{\mathbf{x}}{|J|^{1/n}}\right\|^2_{L^2(\omega)}.\end{aligned}$$

In the first stage of the proof, we 'create some holes' by restricting the support of σ to the cube F in the 'plugged' local sum below:

$$\begin{aligned}\mathbf{Local}^{\mathrm{plug}}(I) &\equiv \sum_{F\in\mathcal{F}}\ \sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F):\,J\subset I}\left(\frac{\mathrm{P}^{\alpha}(J,\mathbf{1}_{F\cap I}\sigma)}{|J|^{1/n}}\right)^2\|\mathsf{P}^{\omega}_{F,J}\mathbf{x}\|^2_{L^2(\omega)}\\ &= \left\{\sum_{F\in\mathcal{F}:\,F\subset I}+\sum_{F\in\mathcal{F}:\,F\supsetneqq I}\right\}\sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F):\,J\subset I}\left(\frac{\mathrm{P}^{\alpha}(J,\mathbf{1}_{F\cap I}\sigma)}{|J|^{1/n}}\right)^2\|\mathsf{P}^{\omega}_{F,J}\mathbf{x}\|^2_{L^2(\omega)}\\ &= A+B.\end{aligned}$$

Then a *trivial* application of the deep energy condition (where 'trivial' means that the outer decomposition is just a single cube) gives

$$\begin{aligned}A &\le \sum_{F\in\mathcal{F}:\,F\subset I}\ \sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)}\left(\frac{\mathrm{P}^{\alpha}(J,\mathbf{1}_{F}\sigma)}{|J|^{1/n}}\right)^2\|\mathsf{P}^{\omega}_{F,J}\mathbf{x}\|^2_{L^2(\omega)}\\ &\le \sum_{F\in\mathcal{F}:\,F\subset I}(\mathcal{E}^{\mathrm{deepplug}}_{\alpha})^2\,|F|_\sigma \lesssim (\mathcal{E}^2_\alpha+A^\alpha_2)\,|I|_\sigma\,,\end{aligned}$$

since $\|\mathsf{P}^{\omega}_{F,J}x\|^2_{L^2(\omega)}\le\|\mathsf{P}^{\omega}_{J}\mathbf{x}\|^2_{L^2(\omega)}$, where we recall that the energy constant $\mathcal{E}^{\mathrm{deepplug}}_{\alpha}$ is defined in (2.6). We also used that the stopping cubes $\mathcal{F}$ satisfy a σ-Carleson measure estimate,

$$\sum_{F\in\mathcal{F}:\,F\subset F_0}|F|_\sigma\lesssim|F_0|_\sigma.$$

Lemma 10.5 applies with $I_0=I$ to the remaining term B to obtain the bound

$$B\le\boldsymbol{\tau}((\mathcal{E}_\alpha)^2+\beta A^\alpha_2)\,|I|_\sigma\,.$$

It remains then to show the inequality with 'holes', where the support of σ is restricted to the complement of the cube F. For $I\in\mathcal{D}$ we define

$$\mathcal{F}_I\equiv\{F\in\mathcal{F}:F\subsetneqq I\}\cup\{I\}\,,$$

so that the term $\mathbf{Local}^{\mathrm{hole}}(I)$ is the left-hand side of (10.8) below.

Lemma 10.6. *We have*

$$\begin{aligned}&\mathbf{Local}^{\mathrm{hole}}(I)\\ &= \sum_{F\in\mathcal{F}_I}\ \sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)}\left(\frac{\mathrm{P}^{\alpha}(J,\mathbf{1}_{I\setminus F}\sigma)}{|J|^{1/n}}\right)^2\left\|\mathsf{P}^{\omega}_{F,J}x\right\|^2_{L^2(\omega)}\lesssim(\mathcal{E}^{\mathrm{deep}}_{\alpha})^2\,|I|_\sigma\,.\end{aligned}\tag{10.8}$$

Proof. We estimate

$$S \equiv \sum_{F\in\mathcal{F}_I} \sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)} \left(\frac{\mathrm{P}^\alpha(J,\mathbf{1}_{I\setminus F}\sigma)}{|J|^{1/n}}\right)^2 \left\|\mathsf{P}^\omega_{F,J}\mathbf{x}\right\|^2_{L^2(\omega)}$$

by

$$\begin{aligned}
S &= \sum_{F\in\mathcal{F}_I} \sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)} \sum_{F'\in\mathcal{F}:\ F\subset F'\subsetneqq I} \left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{\pi_{\mathcal{F}_I}F'\setminus F'}\sigma\right)}{|J|^{1/n}}\right)^2 \left\|\mathsf{P}^\omega_{F,J}\mathbf{x}\right\|^2_{L^2(\omega)}\\
&= \sum_{F'\in\mathcal{F}_I} \sum_{F\in\mathcal{F}:\ F\subset F'} \sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)} \left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{\pi_{\mathcal{F}_I}F'\setminus F'}\sigma\right)}{|J|^{1/n}}\right)^2 \left\|\mathsf{P}^\omega_{F,J}\mathbf{x}\right\|^2_{L^2(\omega)}\\
&= \sum_{F'\in\mathcal{F}_I} \sum_{K\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F')} \sum_{F\in\mathcal{F}:\ F\subset F'} \sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F):\ J\subset K} \left(\frac{\mathrm{P}^\alpha\left(J,\mathbf{1}_{\pi_{\mathcal{F}_I}F'\setminus F'}\sigma\right)}{|J|^{1/n}}\right)^2\\
&\qquad\qquad \times \left\|\mathsf{P}^\omega_{F,J}\mathbf{x}\right\|^2_{L^2(\omega)}\\
&\lesssim \sum_{F'\in\mathcal{F}_I} \sum_{K\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F')} \left(\frac{\mathrm{P}^\alpha\left(K,\mathbf{1}_{\pi_{\mathcal{F}_I}F'\setminus F'}\sigma\right)}{|K|^{1/n}}\right)^2\\
&\qquad\qquad \times \sum_{F\in\mathcal{F}:\ F\subset F'} \sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F):\ J\subset K} \left\|\mathsf{P}^\omega_{F,J}\mathbf{x}\right\|^2_{L^2(\omega)},
\end{aligned}$$

by the Poisson inequalities in Lemma 10.3. We now invoke

$$\sum_{F\in\mathcal{F}:\ F\subset F'} \sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F):\ J\subset K} \left\|\mathsf{P}^\omega_{F,J}\mathbf{x}\right\|^2_{L^2(\omega)} \lesssim \left\|\widehat{\mathsf{P}}^\omega_{F',K}\mathbf{x}\right\|^2_{L^2(\omega)},$$

where the implied constant depends on $\boldsymbol{\tau}$ and for $K\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F')$,

$$\widehat{\mathsf{P}}^\omega_{F',K} \equiv \sum_{F\in\mathcal{F}:\ F\subset F'} \sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F):\ J\subset K} \mathsf{P}^\omega_{F,J}.$$

Now denote by $d(F) \equiv d_{\mathcal{F}_I}(F,I)$ the distance from F to I in the tree $\mathcal{F}_I$. Since the collection $\mathcal{F}$ satisfies a Carleson condition, we have geometric decay in generations:

$$\sum_{F\in\mathcal{F}_I:\ d(F)=k} |F|_\sigma \lesssim 2^{-\delta k}|I|_\sigma\ , \quad k\geq 0.$$

Thus we can write

$$\begin{aligned}
|S| &\lesssim \sum_{F'\in\mathcal{F}_I} \sum_{K\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F')} \left(\frac{\mathrm{P}^\alpha\left(K,\mathbf{1}_{\pi_{\mathcal{F}_I}F'\setminus F'}\sigma\right)}{|K|^{1/n}}\right)^2 \left\|\widehat{\mathsf{P}}^\omega_{F',K}\mathbf{x}\right\|^2_{L^2(\omega)}\\
&= \sum_{k=0}^{\infty} \sum_{F'\in\mathcal{F}_I:\ d(F')=k} \sum_{K\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F')} \left(\frac{\mathrm{P}^\alpha\left(K,\mathbf{1}_{\pi_{\mathcal{F}_I}F'\setminus F'}\sigma\right)}{|K|^{1/n}}\right)^2 \left\|\widehat{\mathsf{P}}^\omega_{F',K}\mathbf{x}\right\|^2_{L^2(\omega)}\\
&\equiv \sum_{k=0}^{\infty} A_k,
\end{aligned}$$

where by the deep energy condition,

$$A_k = \sum_{F' \in \mathcal{F}_I:\ d(F')=k} \sum_{K \in \mathcal{M}_{\mathbf{r}\text{-deep}}(F')} \Big(\frac{\mathrm{P}^{\alpha}\big(K, \mathbf{1}_{\pi_{\mathcal{F}_I} F' \setminus F'}\sigma\big)}{|K|^{1/n}}\Big)^2 \big\|\widehat{\mathsf{P}}^{\omega}_{F',K}\mathbf{x}\big\|^2_{L^2(\omega)}$$
$$\lesssim \big(\mathcal{E}^{\mathrm{deep}}_{\alpha}\big)^2 \sum_{F'' \in \mathcal{F}_I:\ d(F'')=k-1} |F''|_{\sigma} \lesssim \big(\mathcal{E}^{\mathrm{deep}}_{\alpha}\big)^2\, 2^{-\delta k}\, |I|_{\sigma}\ ,$$

and we finally obtain

$$|S| \lesssim \sum_{k=0}^{\infty} \big(\mathcal{E}^{\mathrm{deep}}_{\alpha}\big)^2\, 2^{-\delta k}\, |I|_{\sigma} \lesssim \big(\mathcal{E}^{\mathrm{deep}}_{\alpha}\big)^2\, |I|_{\sigma}\,,$$

which is (10.8). □

Altogether, then, we have proved that $\mathbf{Local}(I_0) \lesssim ((\mathcal{E}_{\alpha})^2 + A_2^{\alpha})\, |I_0|_{\sigma}$ when I_0 is a $\mathcal{D}$-dyadic cube. We leave the straightforward extension of this inequality to shifted $\mathcal{D}$-dyadic cubes I_0 for the reader.

Now we turn to proving the following estimate for the global part of the first testing condition (10.3):

$$\int_{\mathbb{R}^{n+1}_+ \setminus \widehat{I}} \mathbb{P}^{\alpha}(\mathbf{1}_I\sigma)^2\, d\mu \lesssim \mathcal{A}_2^{\alpha,*}\, |I|_{\sigma}\,.$$

We begin by decomposing the integral on the left into four pieces where we use $F \sim J$ to denote the sum over those $F \in \mathcal{F}$ such that $J \in \mathcal{M}_{\mathbf{r}\text{-deep}}(F)$. Note that given J, there are at most a fixed number C of $F \in \mathcal{F}$ such that $F \sim J$. We have:

$$\int_{\mathbb{R}^{n+1}_+ \setminus \widehat{I}} \mathbb{P}^{\alpha}(\mathbf{1}_I\sigma)^2 d\mu$$
$$= \sum_{J:\, (c_J, |J|^{1/n}) \in \mathbb{R}^{n+1}_+ \setminus \widehat{I}} \mathbb{P}^{\alpha}(\mathbf{1}_I\sigma)(c_J, |J|^{1/n})^2 \sum_{\substack{F \in \mathcal{F}:\\ J \in \mathcal{M}_{\mathbf{r}\text{-deep}}(F)}} \left\|\mathsf{P}^{\omega}_{F,J}\frac{\mathbf{x}}{|J|^{1/n}}\right\|^2_{L^2(\omega)}$$
$$= \Bigg\{ \sum_{\substack{J \cap 3I = \emptyset\\ |J|^{1/n} \le |I|^{1/n}}} + \sum_{J \subset 3I \setminus I} + \sum_{\substack{J \cap I = \emptyset\\ |J|^{1/n} > |I|^{1/n}}} + \sum_{J \supsetneqq I} \Bigg\} \mathbb{P}^{\alpha}(\mathbf{1}_I\sigma)(c_J, |J|^{1/n})^2$$
$$\times \sum_{\substack{F \in \mathcal{F}:\\ J \in \mathcal{M}_{\mathbf{r}\text{-deep}}(F)}} \left\|\mathsf{P}^{\omega}_{F,J}\frac{\mathbf{x}}{|J|^{1/n}}\right\|^2_{L^2(\omega)}$$
$$= A + B + C + D\,.$$

We further decompose term A according to the length of J and its distance from I, and then use Lemma 10.4, with $I_0 = J$, to obtain:

$$A \lesssim \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \sum_{\substack{J \subset 3^{k+1}I \setminus 3^k I \\ |J|^{1/n} = 2^{-m}|I|^{1/n}}} \left(\frac{2^{-m}|I|^{1/n}}{\operatorname{dist}(J,I)^{n+1-\alpha}} \, |I|_\sigma \right)^2 \boldsymbol{\tau} \, |J|_\omega$$
$$\lesssim \sum_{m=0}^{\infty} 2^{-2m} \sum_{k=1}^{\infty} \frac{|I|^{2/n} \, |I|_\sigma \, |3^{k+1}I \setminus 3^k I|_\omega}{|3^k I|^{2(1+1/n-\alpha/n)}} \, |I|_\sigma$$
$$\lesssim \sum_{m=0}^{\infty} 2^{-2m} \sum_{k=1}^{\infty} 3^{-2k} \left\{ \frac{|3^{k+1}I|_\sigma \, |3^{k+1}I|_\omega}{|3^k I|^{2(1-\alpha/n)}} \right\} |I|_\sigma \lesssim A_2^\alpha \, |I|_\sigma \, .$$

For term B we let

$$\mathcal{J}^* \equiv \bigcup_{F \in \mathcal{F}} \bigcup_{J \in \mathcal{M}_{r\text{-deep}}(F)} \{K \in \mathcal{C}_F^{\text{good},\boldsymbol{\tau}\text{-shift}} : K \subset J\},$$

which is the union of all K occurring in the projections $\mathsf{P}_{F,J}^\omega$. We further decompose term B according to the length of J and use the fractional version of the Poisson inequality (8.13) in Lemma 8.8 on the neighbour I' of I containing K (essentially in [23]),

$$\mathrm{P}^\alpha(K, \mathbf{1}_I \sigma)^2 \lesssim \left(\frac{|K|^{1/n}}{|I|^{1/n}} \right)^{2-2(n+1-\alpha)\varepsilon} \mathrm{P}^\alpha(I, \mathbf{1}_I \sigma)^2, \quad K \in \mathcal{J}^*, K \subset 3I \setminus I,$$

where we have used that $\mathrm{P}^\alpha(I', \mathbf{1}_I \sigma) \approx \mathrm{P}^\alpha(I, \mathbf{1}_I \sigma)$ and that the cubes $K \in \mathcal{J}^*$ are good.

We then obtain from Lemma 10.4, with $I_0 = J$,

$$B = \sum_{J \subset 3I \setminus I} \left(\frac{\mathrm{P}^\alpha(J, \mathbf{1}_I \sigma)}{|J|^{1/n}} \right)^2 \sum_{\substack{F \in \mathcal{F}: \\ J \in \mathcal{M}_{\mathbf{r}\text{-deep}}(F)}} \|\mathsf{P}_{F,J}^\omega x\|_{L^2(\omega)}^2$$
$$\lesssim \sum_{m=0}^{\infty} \sum_{\substack{K \subset 3I \setminus I \\ |K|^{1/n} = 2^{-m}|I|^{1/n}}} (2^{-m})^{2-2(n+1-\alpha)\varepsilon} \left(\frac{|I|_\sigma}{|I|^{1-\alpha/n}} \right)^2 \boldsymbol{\tau} \, |K|_\omega$$
$$\lesssim \boldsymbol{\tau} \sum_{m=0}^{\infty} (2^{-m})^{2-2(n+1-\alpha)\varepsilon} \, \frac{|3I|_\sigma \, |3I|_\omega}{|3I|^{2(1-\alpha/n)}} \, |I|_\sigma \lesssim \boldsymbol{\tau} \, A_2^\alpha \, |I|_\sigma \, .$$

For term C we will have to group the cubes J into blocks B_i, and then exploit Lemma 10.4. We first split the sum according to whether or not I intersects the triple of J:

$$C \approx \Big\{ \sum_{\substack{J:\, I\cap 3J=\emptyset \\ |J|^{1/n}>|I|^{1/n}}} + \sum_{\substack{J:\, I\subset 3J\setminus J \\ |J|^{1/n}>|I|^{1/n}}} \Big\} \Big(\frac{|J|^{1/n}}{(|J|^{1/n}+\operatorname{dist}(J,I))^{n+1-\alpha}} |I|_\sigma \Big)^2 \times \sum_{\substack{F\in\mathcal{F}: \\ J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)}} \Big\| \mathsf{P}^\omega_{F,J} \frac{\mathbf{x}}{|J|^{1/n}} \Big\|^2_{L^2(\omega)}$$

$$= C_1 + C_2 .$$

We first consider C_1. Let $\mathcal{M}$ be the maximal dyadic cubes in $\{Q : 3Q \cap I = \emptyset\}$, and then let $\{B_i\}_{i=1}^\infty$ be an enumeration of those $Q \in \mathcal{M}$ whose side length is at least $|I|^{1/n}$. Now we further decompose the sum in C_1 by grouping the cubes J into the Whitney cubes B_i, and then using Lemma 10.4, with $I_0 = J$,

$$\begin{aligned}
C_1 &\le \sum_{i=1}^\infty \sum_{J:\, J\subset B_i} \Big(\frac{1}{(|J|^{1/n}+\operatorname{dist}(J,I))^{n+1-\alpha}} |I|_\sigma \Big)^2 \sum_{\substack{F\in\mathcal{F}: \\ J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)}} \|\mathsf{P}^\omega_{F,J}\mathbf{x}\|^2_{L^2(\omega)} \\
&\lesssim \sum_{i=1}^\infty \Big(\frac{1}{(|B_i|^{1/n}+\operatorname{dist}(B_i,I))^{n+1-\alpha}} |I|_\sigma \Big)^2 \sum_{J:\, J\subset B_i} \sum_{\substack{F\in\mathcal{F}: \\ J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)}} \|\mathsf{P}^\omega_{F,J}\mathbf{x}\|^2_{L^2(\omega)} \\
&\lesssim \sum_{i=1}^\infty \Big(\frac{1}{(|B_i|^{1/n}+\operatorname{dist}(B_i,I))^{n+1-\alpha}} |I|_\sigma \Big)^2 \sum_{J:\, J\subset B_i} \boldsymbol{\tau}\, |J|^{2/n} |J|_\omega \\
&\lesssim \sum_{i=1}^\infty \Big(\frac{1}{(|B_i|^{1/n}+\operatorname{dist}(B_i,I))^{n+1-\alpha}} |I|_\sigma \Big)^2 \boldsymbol{\tau}\, |B_i|^{2/n} |B_i|_\omega \\
&\lesssim \boldsymbol{\tau} \Big\{ \sum_{i=1}^\infty \frac{|B_i|_\omega\, |I|_\sigma}{|B_i|^{2(1-\alpha/n)}} \Big\} |I|_\sigma ,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=1}^\infty \frac{|B_i|_\omega\, |I|_\sigma}{|B_i|^{2(1-\alpha/n)}} &= \frac{|I|_\sigma}{|I|^{1-\alpha/n}} \sum_{i=1}^\infty \frac{|I|^{1-\alpha/n}}{|B_i|^{2(1-\alpha/n)}} |B_i|_\omega \\
&\approx \frac{|I|_\sigma}{|I|^{1-\alpha/n}} \sum_{i=1}^\infty \int_{B_i} \frac{|I|^{1-\alpha/n}}{\operatorname{dist}(x,I)^{2(n-\alpha)}} \, d\omega(x) \\
&\approx \frac{|I|_\sigma}{|I|^{1-\alpha/n}} \sum_{i=1}^\infty \int_{B_i} \Big(\frac{|I|^{1/n}}{\big[|I|^{1/n}+\operatorname{dist}(x,I)\big]^2} \Big)^{n-\alpha} d\omega(x) \\
&\le \frac{|I|_\sigma}{|I|^{1-\alpha/n}} \mathcal{P}^\alpha(I,\omega) \le \mathcal{A}_2^{\alpha,*}.
\end{aligned}$$

We obtain $C_1 \lesssim \boldsymbol{\tau}\, \mathcal{A}_2^{\alpha,*} |I|_\sigma$.

Next we turn to estimating term C_2 where the triple of J contains I but J itself does not. Note that there are at most 2^n such cubes J of a given side length, one in each 'generalized octant' relative to I. So with this in mind we sum over the cubes J according to their lengths to obtain

$$\begin{aligned} C_2 &= \sum_{m=0}^{\infty} \sum_{\substack{J:\, I\subset 3J\setminus J \\ |J|^{1/n}=2^m|I|^{1/n}}} \left(\frac{|J|^{1/n}}{(|J|^{1/n}+\mathrm{dist}(J,I))^{n+1-\alpha}}|I|_\sigma\right)^2 \sum_{\substack{F\in\mathcal{F}: \\ J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)}} \left\|\mathsf{P}^\omega_{F,J}\frac{\mathbf{x}}{|J|^{1/n}}\right\|^2_{L^2(\omega)} \\ &\lesssim \sum_{m=0}^{\infty} \left(\frac{|I|_\sigma}{|2^m I|^{1-\alpha/n}}\right)^2 \boldsymbol{\tau}\, |3\cdot 2^m I|_\omega = \boldsymbol{\tau}\left\{\frac{|I|_\sigma}{|I|^{1-\alpha/n}} \sum_{m=0}^{\infty} \frac{|I|^{1-\alpha/n}\,|3\cdot 2^m I|_\omega}{|2^m I|^{2(1-\alpha/n)}}\right\}|I|_\sigma \\ &\lesssim \boldsymbol{\tau}\left\{\frac{|I|_\sigma}{|I|^{1-\alpha/n}}\,\mathcal{P}^\alpha(I,\omega)\right\}|I|_\sigma \le \boldsymbol{\tau}\,\mathcal{A}_2^{\alpha,*}\,|I|_\sigma, \end{aligned}$$

since, in analogy with the corresponding estimate above,

$$\sum_{m=0}^{\infty} \frac{|I|^{1-\alpha/n}\,|3\cdot 2^n I|_\omega}{|2^m I|^{2(1-\alpha/n)}} = \int \sum_{m=0}^{\infty} \frac{|I|^{1-\alpha/n}}{|2^m I|^{2(1-\alpha/n)}}\ \mathbf{1}_{3\cdot 2^m I}(x)\ d\omega(x) \lesssim \mathcal{P}^\alpha(I,\omega)\,.$$

Finally, we turn to term D, which is handled in the same way as term C_2. The cubes J occurring here are included in the set of ancestors $A_k \equiv \pi^{(k)}_{\mathcal{D}} I$ of I, $1\le k<\infty$. We thus have from Lemma 10.4 again,

$$\begin{aligned} D &= \sum_{k=1}^{\infty} \mathbb{P}^\alpha(\mathbf{1}_I\sigma)(c(A_k),|A_k|^{1/n})^2 \sum_{\substack{F\in\mathcal{F}: \\ A_k\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)}} \left\|\mathsf{P}^\omega_{F,A_k}\frac{\mathbf{x}}{|A_k|^{1/n}}\right\|^2_{L^2(\omega)} \\ &\lesssim \sum_{k=1}^{\infty} \left(\frac{|I|_\sigma\,|A_k|^{1/n}}{|A_k|^{1+(1-\alpha)/n}}\right)^2 \boldsymbol{\tau}\,|A_k|_\omega = \boldsymbol{\tau}\left\{\frac{|I|_\sigma}{|I|^{1-\alpha/n}} \sum_{k=1}^{\infty} \frac{|I|^{1-\alpha/n}}{|A_k|^{2(1-\alpha/n)}}\,|A_k|_\omega\right\}|I|_\sigma \\ &\lesssim \left\{\frac{|I|_\sigma}{|I|^{1-\alpha/n}}\,\mathcal{P}^\alpha(I,\omega)\right\}|I|_\sigma \lesssim \mathcal{A}_2^{\alpha,*}\,|I|_\sigma, \end{aligned}$$

since

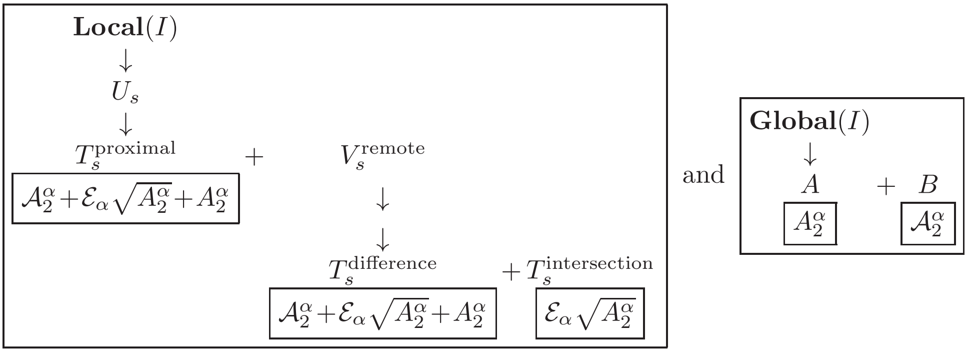
$$\begin{aligned} \sum_{k=1}^{\infty} \frac{|I|^{1-\alpha/n}}{|A_k|^{2(1-\alpha/n)}}\,|A_k|_\omega &= \int \sum_{k=1}^{\infty} \frac{|I|^{1-\alpha/n}}{|A_k|^{2(1-\alpha/n)}}\ \mathbf{1}_{A_k(x)}\,d\omega(x) \\ &= \int \sum_{k=1}^{\infty} \frac{1}{2^{2(1-\alpha/n)k}}\ \frac{|I|^{1-\alpha/n}}{|I|^{2(1-\alpha/n)}}\ \mathbf{1}_{A_k(x)}\,d\omega(x) \\ &\lesssim \int \left(\frac{|I|^{1/n}}{(|I|^{1/n}+\mathrm{dist}(x,I))^2}\right)^{n-\alpha} d\omega(x) = \mathcal{P}^\alpha(I,\omega)\,. \end{aligned}$$

10.2. The dual Poisson testing inequality

Again we split the integration on the left side of (10.4) into local and global parts:

$$\begin{aligned} \int_{\mathbb{R}} [\,\mathbb{P}^{\alpha*}(t\,\mathbf{1}_{\widehat{I}}\,\mu)\,]^2\,\sigma &= \int_I [\,\mathbb{P}^{\alpha*}(t\,\mathbf{1}_{\widehat{I}}\,\mu)\,]^2\,\sigma + \int_{\mathbb{R}\setminus I} [\,\mathbb{P}^{\alpha*}(t\,\mathbf{1}_{\widehat{I}}\,\mu)\,]^2\,\sigma \\ &\equiv \mathbf{Local}(I) + \mathbf{Global}(I). \end{aligned} \tag{10.9}$$

Here is a brief schematic diagram of the decompositions, with bounds in $\square$, used in this subsection:



We begin with the local part **Local**(I). Note that the right hand side of (10.4) is

$$\int_{\widehat{I}} t^2 \, d\mu = \sum_{F \in \mathcal{F}} \sum_{\substack{J \in \mathcal{M}_{\mathbf{r}\text{-deep}}(F) \\ J \subset I}} \left\| \mathsf{P}_{F,J}^{\omega} \mathbf{x} \right\|_{L^2(\omega)}^2 . \tag{10.10}$$

We now compute

$$\mathbb{P}^{\alpha *}(t \, \mathbf{1}_{\widehat{I}} \, \mu)(y) = \sum_{F \in \mathcal{F}} \sum_{\substack{J \in \mathcal{M}_{\mathbf{r}\text{-deep}}(F) \\ J \subset I}} \frac{\left\| \mathsf{P}_{F,J}^{\omega} \mathbf{x} \right\|_{L^2(\omega)}^2}{\left(|J|^{1/n} + |y - c_J| \right)^{n+1-\alpha}} , \tag{10.11}$$

and then expand the square and integrate to obtain that the local term **Local** is

$$\sum_{\substack{F \in \mathcal{F} \\ J \in \mathcal{M}_{\mathbf{r}-\text{deep}}(F) \\ J \subset I}} \sum_{\substack{F' \in \mathcal{F} \\ J' \in \mathcal{M}_{\mathbf{r}\text{-deep}}(F') \\ J' \subset I}} \int_I \frac{\left\| \mathsf{P}_{F,J}^{\omega} \mathbf{x} \right\|_{L^2(\omega)}^2}{\left(|J|^{1/n} + |y - c_J| \right)^{n+1-\alpha}} \frac{\left\| \mathsf{P}_{F',J'}^{\omega} \mathbf{x} \right\|_{L^2(\omega)}^2}{\left(|J'|^{1/n} + |y - c_{J'}| \right)^{n+1-\alpha}} \, d\sigma(y) .$$

By symmetry we may assume that $|J'|^{1/n} \leq |J|^{1/n}$. We fix an integer s, and consider those cubes J and J' with $|J'|^{1/n} = 2^{-s} |J|^{1/n}$. For fixed s we will control the expression

$$\begin{aligned} U_s \equiv & \sum_{F, F' \in \mathcal{F}} \sum_{\substack{J \in \mathcal{M}_{\mathbf{r}\text{-deep}}(F), \ J' \in \mathcal{M}_{\mathbf{r}\text{-deep}}(F') \\ J, J' \subset I, \ |J'|^{1/n} = 2^{-s} |J|^{1/n}}} \\ & \times \int_I \frac{\left\| \mathsf{P}_{F,J}^{\omega} \mathbf{x} \right\|_{L^2(\omega)}^2}{\left(|J|^{1/n} + |y - c_J| \right)^{n+1-\alpha}} \frac{\left\| \mathsf{P}_{F',J'}^{\omega} \mathbf{x} \right\|_{L^2(\omega)}^2}{\left(|J'|^{1/n} + |y - c_{J'}| \right)^{n+1-\alpha}} \, d\sigma(y) , \end{aligned}$$

by proving that

$$U_s \lesssim 2^{-\varepsilon s} \left(\mathcal{A}_2^{\alpha} + \mathcal{E}_{\alpha} \sqrt{A_2^{\alpha}} \right) . \tag{10.12}$$

With this accomplished, we can sum in $s \geq 0$ to control the local term **Local**.

Our first decomposition is to write

$$U_s = T_s^{\text{proximal}} + V_s^{\text{remote}}, \tag{10.13}$$

where we fix $\varepsilon > 0$ to be chosen later ($\varepsilon = \frac{1}{2n}$ works), and in the 'proximal' term T_s^{proximal} we restrict the summation over pairs of cubes J, J' to those satisfying $|c(J) - c(J')| < 2^{s\varepsilon}|J|^{1/n}$; while in the 'remote' term V_s^{remote} we restrict the summation over pairs of cubes J, J' to those satisfying the opposite inequality $|c(J) - c(J')| \geq 2^{s\varepsilon}|J|^{1/n}$. Then we further decompose

$$V_s^{\text{remote}} = T_s^{\text{difference}} + T_s^{\text{intersection}},$$

where in the 'difference' term $T_s^{\text{difference}}$ we restrict integration in y to the difference $I \setminus B(J, J')$ of I and

$$B(J, J') \equiv B\left(c_J, \tfrac{1}{2}|c_J - c_{J'}|\right),$$

the ball centered at c_J with radius $\frac{1}{2}|c_J - c_{J'}|$; while in the 'intersection' term $T_s^{\text{intersection}}$ we restrict integration in y to the intersection $I \cap B(J, J')$ of I with the ball $B(J, J')$; i.e.,

$$T_s^{\text{intersection}} \equiv \sum_{F,F'\in\mathcal{F}} \sum_{\substack{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F),\ J'\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F') \\ J,J'\subset I,\ |J'|^{1/n}=2^{-s}|J|^{1/n} \\ |c(J)-c(J')|\geq 2^{s(1+\varepsilon)}|J'|^{1/n}}} \times \int_{I\cap B(J,J')} \frac{\|\mathsf{P}^\omega_{F,J}\mathbf{x}\|^2_{L^2(\omega)}}{(|J|^{1/n} + |y - c_J|)^{n+1-\alpha}} \frac{\|\mathsf{P}^\omega_{F',J'}\mathbf{x}\|^2_{L^2(\omega)}}{(|J'|^{1/n} + |y - c_{J'}|)^{n+1-\alpha}}\, d\sigma(y),$$

We will exploit the restriction of integration to $I \cap B(J, J')$, together with the condition

$$|c_J - c_{J'}| \geq 2^{s(1+\varepsilon)}|J'|^{1/n} = 2^{s\varepsilon}|J|^{1/n},$$

in establishing (10.17) below, which will then give an estimate for the term $T_s^{\text{intersection}}$ using an argument dual to that used for the other terms T_s^{proximal} and $T_s^{\text{difference}}$. We now turn to estimating the proximal and difference terms.

10.2.1. The proximal and difference terms. We have, using (10.10), that

$$\begin{aligned} T_s^{\text{proximal}} &\equiv \sum_{F,F'\in\mathcal{F}} \sum_{\substack{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F),\ J'\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F') \\ J,J'\subset I,\ |J'|^{1/n}=2^{-s}|J|^{1/n} \text{ and } |c_J-c_{J'}|<2^{s\varepsilon}|J|^{1/n}}} \\ &\quad \times \int_I \frac{\|\mathsf{P}^\omega_{F,J}\mathbf{x}\|^2_{L^2(\omega)}}{(|J|^{1/n} + |y - c_J|)^{n+1-\alpha}} \frac{\|\mathsf{P}^\omega_{F',J'}\mathbf{x}\|^2_{L^2(\omega)}}{(|J'|^{1/n} + |y - c_{J'}|)^{n+1-\alpha}}\, d\sigma(y) \\ &\leq M_s^{\text{proximal}} \sum_{F\in\mathcal{F}} \sum_{\substack{\mathcal{M}_{\mathbf{r}\text{-deep}}(F) \\ J\subset I}} \|\mathsf{P}^\omega_{F,J} z\|^2_\omega = M_s^{\text{proximal}} \int_{\hat{I}} t^2\, d\mu, \end{aligned}$$

where

$$M_s^{\text{proximal}} \equiv \sup_{F\in\mathcal{F}} \sup_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)} A_s^{\text{proximal}}(J);$$

$$A_s^{\text{proximal}}(J) \equiv \sum_{F'\in\mathcal{F}} \sum_{\substack{J'\in\mathcal{M}_{\mathbf{r}-\text{deep}}(F') \\ J'\subset I,\ |J'|^{1/n}=2^{-s}|J|^{1/n} \text{ and } |c_J-c_{J'}|<2^{s\varepsilon}|J|^{1/n}}} \int_I S_{(J',J)}^{F'}(y)\,d\sigma(y);$$

$$S_{(J',J)}^{F'}(x) \equiv \frac{1}{(|J|^{1/n}+|y-c_J|)^{n+1-\alpha}} \frac{\|\mathsf{P}_{F',J'}^{\omega}\mathbf{x}\|_{L^2(\omega)}^2}{(|J'|^{1/n}+|y-c_{J'}|)^{n+1-\alpha}},$$

and similarly

$$\begin{aligned} T_s^{\text{difference}} &\equiv \sum_{F,F'\in\mathcal{F}} \sum_{\substack{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F),\ J'\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F') \\ J,J'\subset I,\ |J'|^{1/n}=2^{-s}|J|^{1/n} \text{ and } |c_J-c_{J'}|\geq 2^{s\varepsilon}|J|^{1/n}}} \\ &\quad\times \int_{I\setminus B(J,J')} \frac{\|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\|_{L^2(\omega)}^2}{(|J|^{1/n}+|y-c_J|)^{n+1-\alpha}} \frac{\|\mathsf{P}_{F',J'}^{\omega}\mathbf{x}\|_{L^2(\omega)}^2}{(|J'|^{1/n}+|y-c_{J'}|)^{n+1-\alpha}}\,d\sigma(y) \\ &\leq M_s^{\text{difference}} \sum_{F\in\mathcal{F}} \sum_{\substack{\mathcal{M}_{\mathbf{r}\text{-deep}}(F) \\ J\subset I}} \|\mathsf{P}_{F,J}^{\omega}z\|_{\omega}^2 = M_s^{\text{difference}} \int_{\hat{I}} t^2\,d\mu; \end{aligned}$$

where

$$M_s^{\text{difference}} \equiv \sup_{F\in\mathcal{F}} \sup_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)} A_s^{\text{remote}}(J)$$

and

$$A_s^{\text{difference}}(J) \equiv \sum_{F'\in\mathcal{F}} \sum_{\substack{J'\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F') \\ J'\subset I,\ |J'|^{1/n}=2^{-s}|J|^{1/n} \text{ and } |c_J-c_{J'}|\geq 2^{s\varepsilon}|J|^{1/n}}} \int_{I\setminus B(J,J')} S_{(J',J)}^{F'}(y)\,d\sigma(y).$$

The restriction of integration in $A_s^{\text{difference}}$ to $I\setminus B(J,J')$ will be used to establish (10.15) below.

Notation. Since the cubes F, J, F' and J' that arise in all of the sums here satisfy

$$J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F),\, J'\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F') \quad \text{and} \quad \ell(J')=2^{-s}\ell(J),$$

we will often employ the notation $\sum^{*}$ to remind the reader that, as applicable, these three conditions are in force even when they are not explicitly mentioned.

Now fix J as in M_s^{proximal} respectively $M_s^{\text{difference}}$, and decompose the sum over J' in $A_s^{\text{proximal}}(J)$ respectively $A_s^{\text{difference}}(J)$ by

$$\begin{aligned}
A_s^{\text{proximal}}(J) &= \sum_{F'\in\mathcal{F}} \sum_{\substack{J'\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F') \\ J'\subset I,\ |J'|^{1/n}=2^{-s}|J|^{1/n} \text{ and } |c_J-c_{J'}|<2^{s\varepsilon}|J|^{1/n}}} \int_I S_{(J',J)}^{F'}(y)\,d\sigma(y) \\
&= \sum_{F'\in\mathcal{F}} \sum^{*}_{\substack{c_{J'}\in 2J \\ |c_J-c_{J'}|<2^{s\varepsilon}|J|^{1/n}}} \int_I S_{(J',J)}^{F'}(y)\,d\sigma(y) \\
&\quad + \sum_{F'\in\mathcal{F}} \sum_{\ell=1}^{\infty} \sum^{*}_{\substack{c_{J'}\in 2^{\ell+1}J\setminus 2^{\ell}J \\ |c_J-c_{J'}|<2^{s\varepsilon}|J|^{1/n}}} \int_I S_{(J',J)}^{F'}(y)\,d\sigma(y) \\
&\equiv \sum_{\ell=0}^{\infty} A_s^{\text{proximal},\ell}(J)\,,
\end{aligned}$$

respectively,

$$\begin{aligned}
A_s^{\text{difference}}(J) &= \sum_{F'\in\mathcal{F}} \sum_{\substack{J'\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F') \\ J'\subset I,\ |J'|^{1/n}=2^{-s}|J|^{1/n} \\ \text{and } |c_J-c_{J'}|\geq 2^{s\varepsilon}|J|^{1/n}}} \int_{I\setminus B(J,J')} S_{(J',J)}^{F'}(y)\,d\sigma(y) \\
&= \sum_{F'\in\mathcal{F}} \sum^{*}_{\substack{c_{J'}\in 2J \\ |c_J-c_{J'}|\geq 2^{s\varepsilon}|J|^{1/n}}} \int_{I\setminus B(J,J')} S_{(J',J)}^{F'}(y)\,d\sigma(y) \\
&\quad + \sum_{\ell=1}^{\infty} \sum_{F'\in\mathcal{F}} \sum^{*}_{\substack{c_{J'}\in 2^{\ell+1}J\setminus 2^{\ell}J \\ |c_J-c_{J'}|\geq 2^{s\varepsilon}|J|^{1/n}}} \int_{I\setminus B(J,J')} S_{(J',J)}^{F'}(y)\,d\sigma(y) \\
&\equiv \sum_{\ell=0}^{\infty} A_s^{\text{difference},\ell}(J)\,.
\end{aligned}$$

Let m be the smallest integer for which

$$2^{-m}\sqrt{n} \leq \frac{1}{3}. \tag{10.14}$$

Now decompose the integrals over I in $A_s^{\text{proximal},\ell}(J)$ by

$$\begin{aligned}
A_s^{\text{proximal},0}(J) &= \sum_{F'\in\mathcal{F}} \sum^{*}_{\substack{c_{J'}\in 2J \\ |c_J-c_{J'}|<2^{s\varepsilon}|J|^{1/n}}} \int_{I\setminus 4J} S_{(J',J)}^{F'}(y)\,d\sigma(y) \\
&\quad + \sum_{F'\in\mathcal{F}} \sum^{*}_{\substack{c_{J'}\in 2J \\ |c_J-c_{J'}|<2^{s\varepsilon}|J|^{1/n}}} \int_{I\cap 4J} S_{(J',J)}^{F'}(y)\,d\sigma(y) \\
&\equiv A_{s,\text{far}}^{\text{proximal},0}(J) + A_{s,\text{near}}^{\text{proximal},0}(J),
\end{aligned}$$

and

$$A_s^{\text{proximal},\ell}(J) = \sum_{F'\in\mathcal{F}} \sideset{}{^*}\sum_{\substack{c_{J'}\in 2^{\ell+1}J\setminus 2^\ell J \\ |c_J-c_{J'}|<2^{s\varepsilon}|J|^{1/n}}} \int_{I\setminus 2^{\ell+2}J} S_{(J',J)}^{F'}(y)\,d\sigma(y)$$
$$+ \sum_{F'\in\mathcal{F}} \sideset{}{^*}\sum_{\substack{c_{J'}\in 2^{\ell+1}J\setminus 2^\ell J \\ |c_J-c_{J'}|<2^{s\varepsilon}|J|^{1/n}}} \int_{I\cap(2^{\ell+2}J\setminus 2^{\ell-m}J)} S_{(J',J)}^{F'}(y)\,d\sigma(y)$$
$$+ \sum_{F'\in\mathcal{F}} \sideset{}{^*}\sum_{\substack{c_{J'}\in 2^{\ell+1}J\setminus 2^\ell J \\ |c_J-c_{J'}|<2^{s\varepsilon}|J|^{1/n}}} \int_{I\cap 2^{\ell-m}J} S_{(J',J)}^{F'}(y)\,d\sigma(y)$$
$$\equiv A_{s,\text{far}}^{\text{proximal},\ell}(J) + A_{s,\text{near}}^{\text{proximal},\ell}(J) + A_{s,\text{close}}^{\text{proximal},\ell}(J), \quad \ell\geq 1.$$

Similarly we decompose the integrals over $I^*\equiv I\setminus B(J,J')$ in $A_s^{\text{difference},\ell}(J)$ by

$$A_s^{\text{difference},0}(J) = \sum_{F'\in\mathcal{F}} \sideset{}{^*}\sum_{\substack{c_{J'}\in 2J \\ |c_J-c_{J'}|\geq 2^{s\varepsilon}|J|^{1/n}}} \int_{I^*\setminus 4J} S_{(J',J)}^{F'}(y)\,d\sigma(y)$$
$$+ \sum_{F'\in\mathcal{F}} \sideset{}{^*}\sum_{\substack{c_{J'}\in 2J \\ |c_J-c_{J'}|\geq 2^{s\varepsilon}|J|^{1/n}}} \int_{I^*\cap 4J} S_{(J',J)}^{F'}(y)\,d\sigma(y)$$
$$\equiv A_{s,\text{far}}^{\text{difference},0}(J) + A_{s,\text{near}}^{\text{difference},0}(J),$$

and

$$A_s^{\text{difference},\ell}(J) = \sum_{F'\in\mathcal{F}} \sideset{}{^*}\sum_{\substack{c_{J'}\in 2^{\ell+1}J\setminus 2^\ell J \\ |c_J-c_{J'}|\geq 2^{s\varepsilon}|J|^{1/n}}} \int_{I^*\setminus 2^{\ell+2}J} S_{(J',J)}^{F'}(y)\,d\sigma(y)$$
$$+ \sum_{F'\in\mathcal{F}} \sideset{}{^*}\sum_{\substack{c_{J'}\in 2^{\ell+1}J\setminus 2^\ell J \\ |c_J-c_{J'}|\geq 2^{s\varepsilon}|J|^{1/n}}} \int_{I^*\cap(2^{\ell+2}J\setminus 2^{\ell-m}J)} S_{(J',J)}^{F'}(y)\,d\sigma(y)$$
$$+ \sum_{F'\in\mathcal{F}} \sideset{}{^*}\sum_{\substack{c_{J'}\in 2^{\ell+1}J\setminus 2^\ell J \\ |c_J-c_{J'}|\geq 2^{s\varepsilon}|J|^{1/n}}} \int_{I^*\cap 2^{\ell-m}J} S_{(J',J)}^{F'}(y)\,d\sigma(y)$$
$$\equiv A_{s,\text{far}}^{\text{difference},\ell}(J) + A_{s,\text{near}}^{\text{difference},\ell}(J) + A_{s,\text{close}}^{\text{difference},\ell}(J), \quad \ell\geq 1.$$

We now note the important point that the close terms $A_{s,\text{close}}^{\text{proximal},\ell}(J)$ and $A_{s,\text{close}}^{\text{difference},\ell}(J)$ both *vanish* for $\ell>\varepsilon s$ because of the decomposition (10.13):

$$A_{s,\text{close}}^{\text{proximal},\ell}(J) = A_{s,\text{close}}^{\text{difference},\ell}(J) = 0, \quad \ell > 1+\varepsilon s\,. \tag{10.15}$$

Indeed, if $c_{J'} \in 2^{\ell+1}J \setminus 2^{\ell}J$, then we have

$$\frac{1}{2}\,2^{\ell}\,|J|^{1/n} \leq |c_J - c_{J'}|\,, \tag{10.16}$$

and if $\ell > 1+\varepsilon s$, then

$$|c_J - c_{J'}| \geq 2^{\varepsilon s}|J|^{1/n} = 2^{(1+\varepsilon)s}|J'|^{1/n}.$$

It now follows from the definition of V_s and T_s in (10.13), that $A^{\text{proximal},\ell}_{s,\text{close}}(J) = 0$, and so we are left to consider the term $A^{\text{difference},\ell}_{s,\text{close}}(J)$, where the integration is taken over the set $I \setminus B(J,J')$. But we are also restricted in $A^{\text{difference},\ell}_{s,\text{close}}(J)$ to integrating over the cube $2^{\ell-m}J$, which is contained in $B(J,J')$ by (10.16). Indeed, the smallest ball centered at $c(J)$ that contains $2^{\ell-m}J$ has radius $\sqrt{n}\frac{1}{2}2^{\ell-m}|J|^{1/n}$, which by (10.14) and (10.16) is at most

$$\frac{1}{4}\,2^{\ell}\,|J|^{1/n} \leq \frac{1}{2}\,|c_J - c_{J'}|,$$

the radius of $B(J,J')$. Thus the range of integration in the term $A^{\text{difference},\ell}_{s,\text{close}}(J)$ is the empty set, and so $A^{\text{difference},\ell}_{s,\text{close}}(J) = 0$ as well as $A^{\text{proximal},\ell}_{s,\text{close}}(J) = 0$. This proves (10.15).

Thus from now on in this subsection we may replace $I \setminus B(J,J')$ by I since all the terms are positive, and we treat T^{proximal}_s and $T^{\text{difference}}_s$ in the same way now that the terms $A^{\text{proximal},\ell}_{s,\text{close}}(J)$ and $A^{\text{difference},\ell}_{s,\text{close}}(J)$ both vanish for $\ell > 1+\varepsilon s$. Thus we will suppress the superscripts proximal and difference in the far, near and close decomposition of $A^{\text{proximal},\ell}_{s,\text{close}}(J)$ and $A^{\text{difference},\ell}_{s,\text{close}}(J)$, and we will also suppress the conditions $|c_J - c_{J'}| < 2^{\varepsilon s}|J|^{1/n}$ and $|c_J - c_{J'}| \geq 2^{\varepsilon s}|J|^{1/n}$ in the proximal and difference terms since they no longer play a role. Using the bounded overlap of the shifted coronas $\mathcal{C}^{\text{good},\boldsymbol{\tau}\text{-shift}}_F$, we have

$$\sum_{F' \in \mathcal{F}} \|\mathsf{P}^{\omega}_{F',J'}\mathbf{x}\|^2_{L^2(\omega)} \lesssim \boldsymbol{\tau}|J'|^{2/n}|J'|_{\omega},$$

and so

$$\begin{aligned}
A^{0}_{s,\text{far}}(J) &= \sum_{F' \in \mathcal{F}} \sideset{}{^*}\sum_{c_{J'} \in 2J} \int_{I \setminus (3J)} S^{F'}_{(J',J)}(y)\,d\sigma(y) \\
&\lesssim \boldsymbol{\tau} \sideset{}{^*}\sum_{c_{J'} \in 2J} \int_{I \setminus (3J)} \frac{|J'|^{2/n}\,|J'|_{\omega}}{(|J|^{1/n} + |y-c_J|)^{2(n+1-\alpha)}}\,d\sigma(y) \\
&= \boldsymbol{\tau}\,2^{-2s}\Big(\sideset{}{^*}\sum_{c_{J'} \in 2J} |J'|_{\omega}\Big) \int_{I \setminus (3J)} \frac{|J|^{2/n}}{(|J|^{1/n} + |y-c_J|)^{2(n+1-\alpha)}}\,d\sigma(y),
\end{aligned}$$

which is dominated by

$$\begin{aligned}\boldsymbol{\tau}\, 2^{-2s}\, |3J|_\omega \int_{I\setminus(3J)} \frac{1}{(|J|^{1/n}+|y-c_J|)^{2(n-\alpha)}}\, d\sigma(y) \\ \approx \boldsymbol{\tau}\, 2^{-2s} \frac{|3J|_\omega}{|4J|^{1-\alpha/n}} \int_{I\setminus(3J)} \left(\frac{|J|^{1/n}}{(|J|^{1/n}+|y-c_J|)^2}\right)^{n-\alpha} d\sigma(y) \\ \lesssim \boldsymbol{\tau}\, 2^{-2s} \frac{|3J|_\omega}{|3J|^{1-\alpha/n}}\, \mathcal{P}^\alpha(3J,\sigma) \lesssim \boldsymbol{\tau}\, 2^{-2s}\, \mathcal{A}_2^\alpha .\end{aligned}$$

To estimate the near term $A^0_{s,\mathrm{near}}(J)$, we initially keep the energy $\|\mathsf{P}^\omega_{F',J'} z\|^2_{L^2(\omega)}$ and write

$$\begin{aligned} A^0_{s,\mathrm{near}}(J) &= \sum_{F'\in\mathcal{F}} \sum^{*}_{c_{J'}\in 2J} \int_{I\cap(3J)} S^{F'}_{(J',J)}(y)\, d\sigma(y) \\ &\approx \sum_{F'\in\mathcal{F}} \sum^{*}_{c_{J'}\in 2J} \int_{I\cap(3J)} \frac{1}{|J|^{\frac{1}{n}(n+1-\alpha)}} \frac{\|\mathsf{P}^\omega_{F',J'}\mathbf{x}\|^2_{L^2(\omega)}}{(|J'|^{1/n}+|y-c_{J'}|)^{n+1-\alpha}}\, d\sigma(y) \\ &= \sum_{F'\in\mathcal{F}} \frac{1}{|J|^{\frac{1}{n}(n+1-\alpha)}} \sum^{*}_{c_{J'}\in 2J} \|\mathsf{P}^\omega_{F',J'}\mathbf{x}\|^2_{L^2(\omega)} \int_{I\cap(3J)} \frac{1}{(|J'|^{1/n}+|y-c_{J'}|)^{n+1-\alpha}}\, d\sigma(y) \\ &= \sum_{F'\in\mathcal{F}} \frac{1}{|J|^{\frac{1}{n}(n+1-\alpha)}} \sum^{*}_{c_{J'}\in 2J} \|\mathsf{P}^\omega_{F',J'}\mathbf{x}\|^2_{L^2(\omega)} \frac{\mathrm{P}^\alpha(J',\mathbf{1}_{I\cap(3J)}\sigma)}{|J'|^{1/n}} .\end{aligned}$$

Now by Cauchy–Schwarz and Lemma 10.5, this is dominated by

$$\begin{aligned} &\frac{1}{|J|^{\frac{1}{n}(n+1-\alpha)}} \Big(\sum_{F'\in\mathcal{F}} \sum^{*}_{c(J')\in 2J \text{ and } J'\subset I} \|\mathsf{P}^\omega_{F',J'}\mathbf{x}\|^2_{L^2(\omega)} \Big)^{1/2} \\ &\qquad \times \Big(\sum_{F'\in\mathcal{F}} \sum^{*}_{c_{J'}\in 2J \text{ and } J'\subset I} \|\mathsf{P}^\omega_{F',J'}\mathbf{x}\|^2_{L^2(\omega)} \Big(\frac{\mathrm{P}^\alpha(J',\mathbf{1}_{I\cap(4J)}\sigma)}{|J'|^{1/n}} \Big)^2 \Big)^{1/2} \\ &\lesssim \frac{1}{|J|^{\frac{1}{n}(n+1-\alpha)}} \big(\boldsymbol{\tau} \sum^{*}_{c_{J'}\in 2J} |J'|^{2/n} |J'|_\omega\big)^{1/2}\, \mathcal{E}_\alpha \sqrt{\boldsymbol{\tau}|4J|_\sigma} \\ &\lesssim \boldsymbol{\tau} \frac{2^{-s}\, |J|^{1/n}}{|J|^{\frac{1}{n}(n+1-\alpha)}} \sqrt{|3J|_\omega}\, \mathcal{E}_\alpha \sqrt{|4J|_\sigma} \\ &\lesssim \boldsymbol{\tau}\, 2^{-s}\, \mathcal{E}_\alpha \sqrt{\frac{|4J|_\omega}{|J|^{\frac{1}{n}(n-\alpha)}} \frac{|4J|_\sigma}{|J|^{\frac{1}{n}(n-\alpha)}}} \\ &\lesssim \boldsymbol{\tau}\, 2^{-s}\, \mathcal{E}_\alpha \sqrt{A_2^\alpha} .\end{aligned}$$

Here the estimate for $\mathbf{Local}(I_0)$ applies to the expression

$$\sum_{F'\in\mathcal{F}} \sum^{*}_{c_{J'}\in 2J \text{ and } J'\subset I} \|\mathsf{P}^\omega_{F',J'}\mathbf{x}\|^2_{L^2(\omega)} \Big(\frac{\mathrm{P}^\alpha(J',\mathbf{1}_{I\cap(4J)}\sigma)}{|J'|^{1/n}} \Big)^2 ,$$

with $I_0 = \widehat{J}$, and where $\widehat{J}$ is a shifted $\mathcal{D}$-dyadic cube satisfying $\bigcup_{c_{J'}\in 2J} J' \subset \widehat{J}$ and $|\widehat{J}|^{1/n} \leq C\,|J|^{1/n}$.

Similarly, for $\ell \geq 1$, we can estimate the far term

$$\begin{aligned}
A^{\ell}_{s,\mathrm{far}}(J) &= \sum_{F'\in\mathcal{F}} \sideset{}{^*}\sum_{c_{J'}\in(2^{\ell+1}J)\setminus(2^{\ell}J)} \int_{I\setminus(2^{\ell+2}J)} S^{F'}_{(J',J)}(y)\,d\sigma(y) \\
&\lesssim \boldsymbol{\tau} \sideset{}{^*}\sum_{c_{J'}\in(2^{\ell+1}J)\setminus(2^{\ell}J)} \int_{I\setminus(2^{\ell+2}J)} \frac{|J'|^{2/n}|J'|_{\omega}}{(|J|^{1/n}+|y-c_J|)^{2(n+1-\alpha)}}\,d\sigma(y) \\
&= \boldsymbol{\tau}\, 2^{-2s}\Big(\sideset{}{^*}\sum_{c_{J'}\in(2^{\ell+1}J)} |J'|_{\omega}\Big) \int_{I\setminus(2^{\ell+2}J)} \frac{|J|^{2/n}}{(|J|^{1/n}+|y-c_J|)^{2(n+1-\alpha)}}\,d\sigma(y) \\
&\approx \boldsymbol{\tau}\, 2^{-2s}\, 2^{-2\ell/n}\Big(\sideset{}{^*}\sum_{c_{J'}\in(2^{\ell+1}J)} |J'|_{\omega}\Big) \int_{I\setminus(2^{\ell+2}J)} \frac{|2^{\ell}J|^{2/n}}{(|2^{\ell}J|^{1/n}+|y-c_{2^{\ell}J}|)^{2(n+1-\alpha)}}\,d\sigma(y),
\end{aligned}$$

which is at most

$$\begin{aligned}
&\boldsymbol{\tau}\, 2^{-2s}\, 2^{-2\ell/n}\,|2^{\ell+2}J|_{\omega} \int_{I\setminus(2^{\ell+2}J)} \frac{1}{(|2^{\ell}J|^{1/n}+|y-c_{2^{\ell}J}|)^{2(n-\alpha)}}\,d\sigma(y) \\
&\quad\approx \boldsymbol{\tau}\, 2^{-2s}\, 2^{-2\ell/n}\,\frac{|3^{\ell+2}J|_{\omega}}{|3^{\ell}J|^{1-\alpha/n}} \int_{I\setminus(3^{\ell+2}J)} \Big(\frac{|2^{\ell}J|^{1/n}}{(|2^{\ell}J|^{1/n}+|y-c_{2^{\ell}J}|)^{2}}\Big)^{n-\alpha} d\sigma(y) \\
&\quad\lesssim \boldsymbol{\tau}\, 2^{-2s}\, 2^{-2\ell/n}\Big\{\frac{|2^{\ell+2}J|_{\omega}}{|2^{\ell}J|^{1-\alpha/n}}\,\mathcal{P}^{\alpha}(2^{\ell+2}J,\sigma)\Big\} \lesssim \boldsymbol{\tau}\, 2^{-2s}\, 2^{-2\ell/n}\,\mathcal{A}^{\alpha}_{2}\,.
\end{aligned}$$

The near term $A^{\ell}_{s,\mathrm{near}}(J)$ is

$$\begin{aligned}
&\sum_{F'\in\mathcal{F}} \sideset{}{^*}\sum_{c_{J'}\in 2^{\ell+1}J\setminus 2^{\ell}J} \int_{I\cap(2^{\ell+2}J\setminus 2^{\ell-m}J)} S^{F'}_{(J',J)}(y)\,d\sigma(y) \\
&\approx \sum_{F'\in\mathcal{F}} \sideset{}{^*}\sum_{c_{J'}\in 2^{\ell+1}J\setminus 2^{\ell}J} \int_{I\cap(2^{\ell+2}J\setminus 2^{\ell-m}J)} \frac{1}{|2^{\ell(1-\varepsilon)}J|^{\frac{1}{n}(n+1-\alpha)}} \\
&\qquad\qquad\qquad \times \frac{\|\mathsf{P}^{\omega}_{F',J'}\mathbf{x}\|^2_{L^2(\omega)}}{(|J'|^{1/n}+|y-c_{J'}|)^{n+1-\alpha}}\,d\sigma(y) \\
&= \frac{1}{|2^{\ell-1}J|^{\frac{1}{n}(n+1-\alpha)}} \sum_{F'\in\mathcal{F}} \sideset{}{^*}\sum_{c_{J'}\in 2^{\ell+1}J\setminus 2^{\ell}J} \|\mathsf{P}^{\omega}_{F',J'}\mathbf{x}\|^2_{L^2(\omega)} \\
&\qquad\qquad\qquad \times \int_{I\cap(2^{\ell+2}J\setminus 2^{\ell-m}J)} \frac{1}{(|J'|^{1/n}+|y-c_{J'}|)^{n+1-\alpha}}\,d\sigma(y),
\end{aligned}$$

and is dominated by

$$\begin{aligned}
&\frac{1}{|2^{\ell-m}J|^{\frac{1}{n}(n+1-\alpha)}}\sum_{F'\in\mathcal{F}}\sum_{c_{J'}\in 2^{\ell+1}J\setminus 2^{\ell}J}^{*}\|\mathsf{P}^{\omega}_{F',J'}\mathbf{x}\|^2_{L^2(\omega)}\frac{\mathrm{P}^{\alpha}(J',\mathbf{1}_{I\cap(2^{\ell+2}J)}\sigma)}{|J'|^{1/n}}\\
&\quad\le\frac{1}{|2^{\ell-m}J|^{\frac{1}{n}(n+1-\alpha)}}\Big(\sum_{F'\in\mathcal{F}}\sum_{c_{J'}\in 2^{\ell+1}J\setminus 2^{\ell}J}^{*}\|\mathsf{P}^{\omega}_{F',J'}\mathbf{x}\|^2_{L^2(\omega)}\Big)^{1/2}\\
&\qquad\times\Big(\sum_{F'\in\mathcal{F}}\sum_{c_{J'}\in 2^{\ell+1}J\setminus 2^{\ell}J}^{*}\|\mathsf{P}^{\omega}_{F',J'}\mathbf{x}\|^2_{L^2(\omega)}\Big(\frac{\mathrm{P}^{\alpha}(J',\mathbf{1}_{I\cap(2^{\ell+2}J)}\sigma)}{|J'|^{1/n}}\Big)^2\Big)^{1/2}.
\end{aligned}$$

This can now be estimated by $\mathcal{E}_\alpha$ using $\sum_{F'\in\mathcal{F}}\|\mathsf{P}^{\omega}_{F',J'}z\|^2_{L^2(\omega)}\le\boldsymbol{\tau}\,|J'|^{2/n}|J'|_\omega$ and the estimate for $\mathbf{Local}(I_0)$ to get

$$\begin{aligned}
A^{\ell}_{s,\mathrm{near}}(J)&\lesssim 2^{-s}\,2^{-\ell/n}\,\frac{|2^{\ell}J|^{1/n}}{|2^{\ell-m}J|^{\frac{1}{n}(n+1-\alpha)}}\sqrt{|2^{\ell+3}J|_\omega}\,\mathcal{E}_\alpha\sqrt{|2^{\ell+2}J|_\sigma}\\
&\lesssim 2^{-s}\,2^{-\ell/n}\,\mathcal{E}_\alpha\sqrt{\frac{|2^{\ell+3}J|_\omega}{|2^{\ell+3}J|^{1-\alpha/n}}\frac{|2^{\ell+3}J|_\sigma}{|2^{\ell+3}J|^{1-\alpha/n}}}\lesssim 2^{-s}\,2^{-\ell/n}\,\mathcal{E}_\alpha\sqrt{A^{\alpha}_2}\,,
\end{aligned}$$

where $I_0=\widehat{J}$ is a shifted $\mathcal{D}$-dyadic cube satisfying $\bigcup_{c_{J'}\in 2^{\ell+1}J\setminus 2^{\ell}J}J'\subset\widehat{J}$ and $|\widehat{J}|^{1/n}\le 3\cdot 2^{\ell+1}|J|^{1/n}$. We are also using here that $m\approx 1+\frac{1}{2}\log_2 n$ is harmless. These estimates are summable in both s and ℓ.

Now we turn to the terms $A^{\ell}_{s,\mathrm{close}}(J)$, and recall from (10.15) that $A^{\ell}_{s,\mathrm{close}}(J)=0$ if $\ell>1+\varepsilon s$. So we now suppose that $\ell\le 1+\varepsilon s$. We have, with m as in (10.14),

$$\begin{aligned}
A^{\ell}_{s,\mathrm{close}}(J)&=\sum_{F'\in\mathcal{F}}\sum_{c_{J'}\in 2^{\ell+1}J\setminus 2^{\ell}J}^{*}\int_{I\cap(2^{\ell-m}J)}S_{(J',J)}(y)\,d\sigma(y)\\
&\approx\sum_{F'\in\mathcal{F}}\sum_{c_{J'}\in 2^{\ell+1}J\setminus 2^{\ell}J}^{*}\int_{I\cap(2^{\ell-m}J)}\frac{1}{(|J|^{1/n}+|y-c_J|)^{n+1-\alpha}}\frac{\|\mathsf{P}^{\omega}_{F',J'}\mathbf{x}\|^2_{L^2(\omega)}}{|2^{\ell}J|^{\frac{1}{n}(n+1-\alpha)}}\,d\sigma(y)\\
&\approx\Big(\sum_{F'\in\mathcal{F}}\sum_{c_{J'}\in 2^{\ell+1}J\setminus 2^{\ell}J}^{*}\|\mathsf{P}^{\omega}_{F',J'}\mathbf{x}\|^2_{L^2(\omega)}\Big)\frac{1}{|2^{\ell}J|^{\frac{1}{n}(n+1-\alpha)}}\\
&\quad\times\int_{I\cap(2^{\ell-m}J)}\frac{1}{(|J|^{1/n}+|y-c_J|)^{n+1-\alpha}}\,d\sigma(y).
\end{aligned}$$

Now we use the inequality $\sum_{F'\in\mathcal{F}}\|\mathsf{P}^{\omega}_{F',J'}z\|^2_{L^2(\omega)}\le\boldsymbol{\tau}\,|J'|^{2/n}\,|J'|_\omega$ to get the relatively crude estimate

$$\begin{aligned}
&A^{\ell}_{s,\mathrm{close}}(J)\\
&\lesssim \boldsymbol{\tau}\, 2^{-2s}\, |J|^{2/n}\, |2^{\ell+1}J|_{\omega}\, \frac{1}{|2^{\ell}J|^{\frac{1}{n}(n+1-\alpha)}} \int_{I\cap(2^{\ell-m}J)} \frac{1}{(|J|^{1/n}+|y-c_J|)^{n+1-\alpha}}\, d\sigma(y)\\
&\lesssim \boldsymbol{\tau}\, 2^{-2s}\, |J|^{2/n}\, \frac{|2^{\ell+1}J|_{\omega}}{|2^{\ell}J|^{\frac{1}{n}(n+1-\alpha)}}\, \frac{|2^{\ell-m}J|_{\sigma}}{|J|^{\frac{1}{n}(n+1-\alpha)}}\\
&\lesssim \boldsymbol{\tau}\, 2^{-2s} \frac{|2^{\ell+1}J|_{\omega}}{|2^{\ell+1}J|^{1-\alpha/n}}\, \frac{|2^{\ell+1}J|_{\sigma}}{|2^{\ell+1}J|^{1-\alpha/n}}\, 2^{\ell(n-1-\alpha)} \lesssim \boldsymbol{\tau}\, 2^{-2s}\, 2^{\ell(n-1-\alpha)}\, A^{\alpha}_2 \lesssim \boldsymbol{\tau}\, 2^{-s}\, A^{\alpha}_2
\end{aligned}$$

provided that $\ell \le s/n$. But we are assuming $\ell \le 1+\varepsilon s$ here and so we obtain a suitable estimate for $A^{\ell}_{s,\mathrm{close}}(J)$ provided we choose $0<\varepsilon<1/n$.

Remark 10.7. We cannot simply sum the estimate

$$A^{\ell}_{s,\mathrm{close}}(J) \lesssim 2^{-2s}\, |J|^{2/n}\, |2^{\ell+1}J|_{\omega}\, \frac{1}{|2^{\ell}J|^{\frac{1}{n}(n+1-\alpha)}}\, \frac{\mathrm{P}^{\alpha}(J, \mathbf{1}_{2^{\ell-1}J}\sigma)}{|J|^{1/n}}$$

over all $\ell \ge 1$ to get

$$\sum_{\ell} A^{\ell}_{s,\mathrm{close}}(J) \lesssim 2^{-2s}\, \mathrm{P}^{\alpha}(J,\sigma) \sum_{\ell} \frac{|J|^{1/n}\, |2^{\ell+1}J|_{\omega}}{|2^{\ell}J|^{\frac{1}{n}(n+1-\alpha)}} \lesssim 2^{-2s}\, \mathrm{P}^{\alpha}(J,\sigma)\, \mathrm{P}^{\alpha}(J,\omega),$$

since we only have control of the product $\mathrm{P}(J,\sigma)\mathrm{P}(J,\omega)$ in dimension $n=1$, where the two Poisson kernels P and $\mathcal{P}$ coincide, and the two-tailed $\mathcal{A}_2$ condition is known to hold.

The above estimates prove

$$T^{\mathrm{proximal}}_s + T^{\mathrm{difference}}_s \lesssim 2^{-s}(\mathcal{A}^{\alpha}_2 + \mathcal{E}_{\alpha}\sqrt{A^{\alpha}_2} + A^{\alpha}_2) \lesssim 2^{-s}(\mathcal{A}^{\alpha}_2 + \mathcal{E}_{\alpha}\sqrt{A^{\alpha}_2}).$$

10.2.2. The intersection term. Now we return to the term

$$\begin{aligned}
T^{\mathrm{intersection}}_s \equiv \sum_{F,F'\in\mathcal{F}}\ & \sum_{\substack{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F),\ J'\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F')\\ J,J'\subset I,\ |J'|^{1/n}=2^{-s}|J|^{1/n}\\ |c(J)-c(J')|\ge 2^{s(1+\varepsilon)}|J'|^{1/n}}}\\
&\times \int_{I\cap B(J,J')} \frac{\|\mathsf{P}^{\omega}_{F,J}\mathbf{x}\|^2_{L^2(\omega)}}{(|J|^{1/n}+|y-c_J|)^{n+1-\alpha}}\, \frac{\|\mathsf{P}^{\omega}_{F',J'}\mathbf{x}\|^2_{L^2(\omega)}}{(|J'|^{1/n}+|y-c_{J'}|)^{n+1-\alpha}}\, d\sigma(y)\,.
\end{aligned}$$

It will suffice to show that $T^{\mathrm{intersection}}_s$ satisfies the estimate

$$T^{\mathrm{intersection}}_s \lesssim 2^{-s\varepsilon}\, \mathcal{E}_{\alpha}\sqrt{A^{\alpha}_2} \sum_{F\in\mathcal{F}} \sum_{\substack{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)\\ J\subset I}} \|\mathsf{P}^{\omega}_{F,J}\mathbf{x}\|^2_{L^2(\omega)} = 2^{-s\varepsilon}\, \mathcal{E}_{\alpha}\sqrt{A^{\alpha}_2} \int_{\widehat{I}} t^2\, d\mu\,.$$

Using $B(J,J') = B(c_J, \frac{1}{2}|c_J - c_{J'}|)$, we can write (suppressing some notation for clarity),

$$\begin{aligned}
&T_s^{\text{intersection}} \\
&= \sum_{F,F'} \sum_{J,J'} \int_{I\cap B(J,J')} \frac{\|\mathsf{P}^{\omega}_{F,J}\mathbf{x}\|^2_{L^2(\omega)}}{(|J|^{1/n}+|y-c_J|)^{n+1-\alpha}} \frac{\|\mathsf{P}^{\omega}_{F',J'}\mathbf{x}\|^2_{L^2(\omega)}}{(|J'|^{1/n}+|y-c_{J'}|)^{n+1-\alpha}}\, d\sigma(y) \\
&\approx \sum_{F,F'} \sum_{J,J'} \|\mathsf{P}^{\omega}_{F,J}\mathbf{x}\|^2_{L^2(\omega)} \|\mathsf{P}^{\omega}_{F',J'}\mathbf{x}\|^2_{L^2(\omega)} \frac{1}{|c_J - c_{J'}|^{n+1-\alpha}} \\
&\qquad\qquad\qquad \times \int_{I\cap B(J,J')} \frac{1}{(|J|^{1/n}+|y-c_J|)^{n+1-\alpha}}\, d\sigma(y) \\
&\approx \sum_{F,F'} \sum_{J,J'} \|\mathsf{P}^{\omega}_{F,J}\mathbf{x}\|^2_{L^2(\omega)} \|\mathsf{P}^{\omega}_{F',J'}\mathbf{x}\|^2_{L^2(\omega)} \frac{1}{|c_J - c_{J'}|^{n+1-\alpha}} \frac{\mathrm{P}^{\alpha}(J, \mathbf{1}_{I\cap B(J,J')}\sigma)}{|J|^{1/n}} \\
&\leq \sum_{F'} \sum_{J'} \|\mathsf{P}^{\omega}_{F',J'}\mathbf{x}\|^2_{L^2(\omega)} \sum_{F} \sum_{J} \frac{1}{|c_J - c_{J'}|^{n+1-\alpha}} \|\mathsf{P}^{\omega}_{F,J}\mathbf{x}\|^2_{L^2(\omega)} \frac{\mathrm{P}^{\alpha}(J, \mathbf{1}_{I\cap B(J,J')}\sigma)}{|J|^{1/n}},
\end{aligned}$$

and it remains to show that for each J',

$$\begin{aligned}
S_s(J') &\equiv \sum_{F} \sideset{}{^*}\sum_{J:\,|c(J)-c(J')|\geq 2^{s(1+\varepsilon)}|J'|^{1/n}} \frac{\|\mathsf{P}^{\omega}_{F,J}\mathbf{x}\|^2_{L^2(\omega)}}{|c_J - c_{J'}|^{n+1-\alpha}} \frac{\mathrm{P}^{\alpha}(J, \mathbf{1}_{I\cap B(J,J')}\sigma)}{|J|^{1/n}} \\
&\lesssim 2^{-\varepsilon s}\, \mathcal{E}_\alpha \sqrt{A_2^{\alpha}}\,.
\end{aligned}$$

We write

$$\begin{aligned}
S_s(J') &\approx \sum_{k\geq s(1+\varepsilon)-m} \frac{1}{(2^k|J'|^{1/n})^{n+1-\alpha}} \\
&\qquad \times \sum_{F} \sideset{}{^*}\sum_{J:\,|c_J-c_{J'}|\approx 2^k|J'|^{1/n}} \|\mathsf{P}^{\omega}_{F,J}\mathbf{x}\|^2_{L^2(\omega)} \frac{\mathrm{P}^{\alpha}(J, \mathbf{1}_{I\cap B(J,J')}\sigma)}{|J|^{1/n}} \\
&\equiv \sum_{k\geq s(1+\varepsilon)-m} \frac{1}{(2^k|J'|^{1/n})^{n+1-\alpha}} S_s^k(J')\,,
\end{aligned}$$

where by $|c_J - c_{J'}| \approx 2^k|J'|^{1/n}$ we mean $2^k|J'|^{1/n} \leq |c_J - c_{J'}| \leq 2^{k+1}|J'|^{1/n}$. Here m is as in (10.14), and we are using the inequality,

$$k + m \geq (1+\varepsilon)s. \tag{10.17}$$

Indeed, in the term V_s we have $|c_J - c_{J'}| \geq 2^{(1+\varepsilon)s}|J'|^{1/n}$, and combined with $|c_J - c_{J'}| \leq \sqrt{n}\, 2^k|J'|^{1/n}$, we obtain (10.17).

Now we apply Cauchy–Schwarz and Lemma 10.5 to get

$$\begin{aligned} S_s^k(J') &\leq \Big(\sum_F \sum_{J:\ |c_J - c_{J'}| \approx 2^k |J'|^{1/n}}^{*} \|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\|_{L^2(\omega)}^2\Big)^{1/2} \\ &\qquad \times \Big(\sum_F \sum_{J:\,|c_J - c_{J'}| \approx 2^k |J'|^{1/n}}^{*} \|\mathsf{P}_{F,J}^{\omega}\mathbf{x}\|_{L^2(\omega)}^2 \Big(\frac{\mathrm{P}^{\alpha}(J, \mathbf{1}_{I\cap B(J,J')}\sigma)}{|J|^{1/n}}\Big)^2\Big)^{1/2} \\ &\lesssim \Big(\boldsymbol{\tau} \sum_{J:\,|c_J - c_{J'}| \approx 2^k |J'|^{1/n}}^{*} |J|^{2/n}\,|J|_{\omega}\Big)^{1/2} (\boldsymbol{\tau}\,\mathcal{E}_{\alpha}^2\,|2^k J'|_{\sigma})^{1/2} \\ &\lesssim \boldsymbol{\tau}\,\mathcal{E}_{\alpha}\,2^s\,|J'|^{1/n} \sqrt{|C2^k J'|_{\omega}}\sqrt{|2^k J'|_{\sigma}} \lesssim \boldsymbol{\tau}\,\mathcal{E}_{\alpha}\sqrt{A_2^{\alpha}}\,2^s\,|J'|^{1/n}\,|2^k J'|^{1-\alpha/n} \\ &= \boldsymbol{\tau}\,\mathcal{E}_{\alpha}\sqrt{A_2^{\alpha}}\,2^s\,2^{k(n-\alpha)}\,|J'|^{\frac{1}{n}(n+1-\alpha)}, \end{aligned}$$

provided

$$B(J,J') \subset C\,2^k\,J'.$$

But this follows from $|c_J - c_{J'}| \approx 2^k |J'|^{1/n}$ and (10.17), which shows in particular that $k \geq s + c$.

Then we have

$$\begin{aligned} S_s(J') &= \sum_{k\geq(1+\varepsilon)s-m} \frac{1}{(2^k|J'|^{1/n})^{n+1-\alpha}}\,S_s^k(J') \\ &\lesssim \boldsymbol{\tau}\,\mathcal{E}_{\alpha}\sqrt{A_2^{\alpha}} \sum_{k\geq(1+\varepsilon)s-m} \frac{1}{(2^k|J'|^{1/n})^{n+1-\alpha}}\,2^s\,2^{k(n-\alpha)}\,|J'|^{\frac{1}{n}(n+1-\alpha)} \\ &\lesssim \boldsymbol{\tau}\,\mathcal{E}_{\alpha}\sqrt{A_2^{\alpha}} \sum_{k\geq(1+\varepsilon)s-m} 2^{s-k} \lesssim \boldsymbol{\tau}\,2^{-\varepsilon s}\,\mathcal{E}_{\alpha}\sqrt{A_2^{\alpha}}, \end{aligned}$$

which is summable in s. This completes the proof of (10.12), and hence of the estimate for the local part $\mathbf{Local}(I)$ in (10.9) of the second testing condition (10.4).

10.2.3. The global estimate. It remains to prove the following estimate for the global part $\mathbf{Global}(I)$ in (10.9) of the second testing condition (10.4):

$$\int_{\mathbb{R}\setminus I} [\,\mathbb{P}^{\alpha *}(t\,\mathbf{1}_{\widehat{I}}\,\mu)\,]^2\,\sigma \lesssim \mathcal{A}_2^{\alpha}\,|I|_{\sigma}.$$

We decompose the integral on the left into two pieces:

$$\int_{\mathbb{R}\setminus I} [\,\mathbb{P}^{\alpha *}(t\,\mathbf{1}_{\widehat{I}}\,\mu)\,]^2\,\sigma = \int_{\mathbb{R}\setminus 3I} [\,\mathbb{P}^{\alpha *}(t\,\mathbf{1}_{\widehat{I}}\,\mu)\,]^2\,\sigma + \int_{3I\setminus I} [\,\mathbb{P}^{\alpha *}(t\,\mathbf{1}_{\widehat{I}}\,\mu)]^2\,\sigma = A + B.$$

We further decompose term A in annuli and use (10.11) to obtain

$$
\begin{aligned}
A &= \sum_{m=1}^{\infty} \int_{3^{m+1}I\setminus 3^m I} \left[\mathbb{P}^{\alpha *}(t\,\mathbf{1}_{\widehat{I}}\,\mu)\right]^2 \sigma \\
&= \sum_{m=1}^{\infty} \int_{3^{m+1}I\setminus 3^m I} \left[\sum_{F\in\mathcal{F}} \sum_{\substack{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)\\ J\subset I}} \frac{\|\mathsf{P}^{\omega}_{F,J}\mathbf{x}\|^2_{L^2(\omega)}}{(|J|+|y-c_J|)^{n+1-\alpha}}\right]^2 d\sigma(y) \\
&\lesssim \sum_{m=1}^{\infty} \int_{3^{m+1}I\setminus 3^m I} \left[\sum_{F\in\mathcal{F}} \sum_{\substack{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)\\ J\subset I}} \|\mathsf{P}^{\omega}_{F,J}\mathbf{x}\|^2_{L^2(\omega)}\right]^2 \frac{1}{(3^m|I|^{1/n})^{2(n+1-\alpha)}}\, d\sigma(y)\,.
\end{aligned}
$$

Now use (10.10) to get

$$
\int_{\widehat{I}} t^2\, d\mu = \sum_{F\in\mathcal{F}} \sum_{\substack{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)\\ J\subset I}} \|\mathsf{P}^{\omega}_{F,J}\mathbf{x}\|^2_{L^2(\omega)} \lesssim \boldsymbol{\tau}\, \|\mathbf{1}_I(x-c_I)\|^2_{L^2(\omega)} \lesssim |I|^{2/n}\,|I|_{\omega},
$$

and to obtain that

$$
\begin{aligned}
A &\lesssim \sum_{m=1}^{\infty} \int_{3^{m+1}I\setminus 3^m I} \left[\int_{\widehat{I}} t^2\, d\mu\right] \left[\,|I|^{2/n}\,|I|_{\omega}\right] \frac{1}{(3^m|I|^{1/n})^{2(n+1-\alpha)}}\, d\sigma(y) \\
&\lesssim \left\{\sum_{m=1}^{\infty} 3^{-2m}\, \frac{|3^{m+1}I|_{\omega}\,|3^{m+1}I|_{\sigma}}{|3^{m+1}I|^{2(1-\alpha/n)}}\right\} \left[\int_{\widehat{I}} t^2\, d\mu\right] \lesssim A_2^{\alpha} \int_{\widehat{I}} t^2\, d\mu\,.
\end{aligned}
$$

Finally, we estimate term B by using (10.11) to write

$$
B = \int_{3I\setminus I} \left[\sum_{F\in\mathcal{F}} \sum_{\substack{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(F)\\ J\subset I}} \frac{\|\mathsf{P}^{\omega}_{F,J}\mathbf{x}\|^2_{L^2(\omega)}}{(|J|^{1/n}+|y-c_J|)^{n+1-\alpha}}\right]^2 d\sigma(y)\,,
$$

and then expanding the square and calculating as in the proof of the local part given earlier to obtain the bound $\mathcal{A}_2^{\alpha}$. The details are similar, but easier in that the energy condition is not needed, and they are left to the reader.

11. The stopping form

In this section we adapt the argument of M. Lacey in [1] to apply in the setting of a general α-fractional Calderón–Zygmund operator T^{α} in $\mathbb{R}^n$ using the monotonicity Lemma 6.1 and our energy condition in Definition 2.4. We will prove the bound (8.16) for the stopping form

$$
\begin{aligned}
\mathsf{B}^{A}_{\text{stop}}(f,g) &\equiv \sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}^{\boldsymbol{\tau}\text{-shift}}_A\\ J\Subset_{\boldsymbol{\rho}} I_J}} (\mathbb{E}^{\sigma}_{I_J}\,\triangle^{\sigma}_I f)\langle T^{\alpha}_{\sigma}\mathbf{1}_{A\setminus I_J}, \triangle^{\omega}_J g\rangle_{\omega} \qquad (11.1)\\
&= \sum_{\substack{I:\,\pi I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}^{\boldsymbol{\tau}\text{-shift}}_A\\ J\Subset_{\boldsymbol{\rho}} I}} (\mathbb{E}^{\sigma}_{I}\,\triangle^{\sigma}_{\pi I} f)\langle T^{\alpha}_{\sigma}\mathbf{1}_{A\setminus I}, \triangle^{\omega}_J g\rangle_{\omega}\,,
\end{aligned}
$$

where we have made the 'change of dummy variable' $I_J \to I$ for convenience in notation (recall that the child of I that contains J is denoted I_J).

However, the monotonicity lemma of Lacey and Wick has an additional term on the right hand side, and our energy condition is not a direct generalization of the one-dimensional energy condition. These differences in higher dimension result in changes and complications that must be tracked throughout the argument. In particular, we find it necessary to separate the interaction of the two terms on the right side of the monotonicity lemma by splitting the stopping form into the two corresponding sublinear forms in (11.6) below. Recall that for $A \in \mathcal{A}$ the *shifted* corona is given in Definition 8.7 by

$$\mathcal{C}_A^{\boldsymbol{\tau}\text{-shift}} = \{J \in \mathcal{C}_A : J \Subset_{\boldsymbol{\tau}} A\} \bigcup \bigcup_{A' \in \mathfrak{C}_{\mathcal{A}}(A)} \{J \in \mathcal{D} : J \Subset_{\boldsymbol{\tau}} A \text{ and } J \text{ is } \boldsymbol{\tau}\text{-nearby in } A'\},$$

and in particular the $\mathbf{1}$-shifted corona is given by $\mathcal{C}_A^{\mathbf{1}\text{-shift}} = (\mathcal{C}_A \setminus \{A\}) \cup \mathfrak{C}_{\mathcal{A}}(A)$.

Definition 11.1. Suppose that $A \in \mathcal{A}$ and that $\mathcal{P} \subset \mathcal{C}_A^{\mathbf{1}\text{-shift}} \times \mathcal{C}_A^{\boldsymbol{\tau}\text{-shift}}$. We say that the collection of pairs $\mathcal{P}$ is A-*admissible* if

- (good and $(\boldsymbol{\rho}-1)$-deeply embedded) J is good and $J \Subset_{\boldsymbol{\rho}-1} I \subsetneqq A$ for every $(I,J) \in \mathcal{P}$,
- (tree-connected in the first component) if $I_1 \subset I_2$ and both $(I_1,J) \in \mathcal{P}$ and $(I_2,J) \in \mathcal{P}$, then $(I,J) \in \mathcal{P}$ for every I in the geodesic $[I_1,I_2] = \{I \in \mathcal{D} : I_1 \subset I \subset I_2\}$.

However, since $(I,J) \in \mathcal{P}$ implies both $J \in \mathcal{C}_A^{\boldsymbol{\tau}\text{-shift}}$ and $J \Subset_{\boldsymbol{\rho}-1} I$, the assumption $\boldsymbol{\rho} > \boldsymbol{\tau}$ in Definition 8.6 shows that I is in the corona $\mathcal{C}_A$, and hence we may replace $\mathcal{C}_A^{\mathbf{1}\text{-shift}}$ with the restricted corona $\mathcal{C}'_A \equiv \mathcal{C}_A \setminus \{A\}$ in the above definition of A-*admissible*. The basic example of an admissible collection of pairs is obtained from the pairs of cubes summed in the stopping form $\mathsf{B}_{\text{stop}}^A(f,g)$ in (11.1), which occurs in (8.16) above:

$$\mathcal{P}^A \equiv \{(I,J) : I \in \mathcal{C}'_A \text{ and } J \in \mathcal{C}_A^{\boldsymbol{\tau}\text{-shift}} \text{ where } J \text{ is } \boldsymbol{\tau}\text{-good}, J \Subset_{\boldsymbol{\rho}-1} I \text{ and } I \notin \mathcal{A}\}. \tag{11.2}$$

Recall that J is $\boldsymbol{\tau}$-good if $J \in \mathcal{D}^{\boldsymbol{\tau}}_{(\mathbf{r},\varepsilon)\text{-good}}$ as in (3.1), i.e., if J and its ℓ-parents up to level $\boldsymbol{\tau}$ are all good. Recall also that the Haar support of g is contained in the collection of $\boldsymbol{\tau}$-good cubes.

Definition 11.2. Suppose that $A \in \mathcal{A}$ and that $\mathcal{P}$ is an A-*admissible* collection of pairs. Define the associated *stopping* form $\mathsf{B}_{\text{stop}}^{A,\mathcal{P}}$ by

$$\mathsf{B}_{\text{stop}}^{A,\mathcal{P}}(f,g) \equiv \sum_{(I,J)\in\mathcal{P}} \left(\mathbb{E}_I^\sigma \triangle_{\pi I}^\sigma f\right) \langle T_\sigma^\alpha \mathbf{1}_{A\setminus I}, \triangle_J^\omega g\rangle_\omega ,$$

where we may of course further restrict I to $\pi I \in \operatorname{supp} \hat{f}$ if we wish.

Given an A-admissible collection $\mathcal{P}$ of pairs define the reduced collection $\mathcal{P}^{\text{red}}$ as follows. For each fixed J let I_J^{red} be the largest good cube I such that $(I,J) \in \mathcal{P}$. Then set

$$\mathcal{P}^{\text{red}} \equiv \left\{ (I,J) \in \mathcal{P} : I \subset I_J^{\text{red}} \right\}.$$

Clearly $\mathcal{P}^{\text{red}}$ is A-admissible. Now recall our assumption that the Haar support of f is contained in the set of $\boldsymbol{\tau}$-good cubes, which in particular requires that their children are all good as well. This assumption has the important implication that

$$\mathsf{B}_{\text{stop}}^{A,\mathcal{P}}(f,g) = \mathsf{B}_{\text{stop}}^{A,\mathcal{P}^{\text{red}}}(f,g).$$

Indeed, if $(I,J) \in \mathcal{P} \setminus \mathcal{P}^{\text{red}}$ then $\pi I \notin \text{Haarsupp} f$ and so $\mathbb{E}_I^{\sigma} \triangle_{\pi I}^{\sigma} f = 0$. Thus for the purpose of bounding the stopping form, we may assume that the following additional property holds for any A-admissible collection of pairs $\mathcal{P}$:

- if $(I,J) \in \mathcal{P}$ is maximal in the sense that $I \supset I'$ for all I' satisfying $(I',J) \in \mathcal{P}$, then I is good.

Note that there is an asymmetry in our definition of $\mathcal{P}^{\text{red}}$ here, namely that the second components J are required to be $\boldsymbol{\tau}$-good, while the maximal first components I are required to be good. Of course the treatment of the dual stopping forms will use the reversed requirements, and this accounts for our symmetric restrictions imposed on the Haar supports of f and g at the outset of the proof.

Definition 11.3. We say that an admissible collection $\mathcal{P}$ is *reduced* if $\mathcal{P} = \mathcal{P}^{\text{red}}$, so that the additional property above holds.

Note that

$$\mathsf{B}_{\text{stop}}^{A,\mathcal{P}}(f,g) = \mathsf{B}_{\text{stop}}^{A,\mathcal{P}}(\mathsf{P}_{\mathcal{C}_A}^{\sigma} f, \mathsf{P}_{\mathcal{C}_A^{\tau\text{-shift}}}^{\omega} g).$$

Recall that the deep energy condition constant $\mathcal{E}_{\alpha}^{\text{deep}}$ is given by

$$(\mathcal{E}_{\alpha}^{\text{deep}})^2 \equiv \sup_{I=\dot{\cup} I_r} \frac{1}{|I|_{\sigma}} \sum_{r=1}^{\infty} \sum_{J \in \mathcal{M}_{\mathbf{r}\text{-deep}}(I_r)} \left(\frac{\mathrm{P}^{\alpha}(J, \mathbf{1}_{I \setminus \gamma J}\sigma)}{|J|^{1/n}} \right)^2 \|\mathsf{P}_J^{\text{subgood},\omega} \mathbf{x}\|_{L^2(\omega)}^2 .$$

Proposition 11.4. *Suppose that $A \in \mathcal{A}$ and that $\mathcal{P}$ is an A-*admissible *collection of pairs. Then the stopping form $\mathsf{B}_{\text{stop}}^{A,\mathcal{P}}$ satisfies the bound*

$$|\mathsf{B}_{\text{stop}}^{A,\mathcal{P}}(f,g)| \lesssim (\mathcal{E}_{\alpha}^{\text{deep}} + \sqrt{A_2^{\alpha}}) \left(\|f\|_{L^2(\sigma)} + \alpha_{\mathcal{A}}(f)\sqrt{|A|_{\sigma}} \right) \|g\|_{L^2(\omega)} . \tag{11.3}$$

With this proposition in hand, we can complete the proof of (8.16), and hence of Theorem 2.6, by summing over the stopping cubes $A \in \mathcal{A}$ with the choice $\mathcal{P}^A$

of A-admissible pairs for each A:

$$\begin{aligned}
&\sum_{A\in\mathcal{A}} \left|\mathsf{B}_{\mathrm{stop}}^{A,\mathcal{P}^A}(f,g)\right| \\
&\qquad\lesssim \sum_{A\in\mathcal{A}} (\mathcal{E}_\alpha^{\mathrm{deep}}+\sqrt{A_2^\alpha})\left(\|\mathsf{P}_{\mathcal{C}_A} f\|_{L^2(\sigma)}+\alpha_{\mathcal{A}}(f)\sqrt{|A|_\sigma}\right)\left\|\mathsf{P}_{\mathcal{C}_A^{\tau-\mathrm{shift}}} g\right\|_{L^2(\omega)} \\
&\qquad\lesssim (\mathcal{E}_\alpha^{\mathrm{deep}}+\sqrt{A_2^\alpha})\left(\sum_{A\in\mathcal{A}}\left(\|\mathsf{P}_{\mathcal{C}_A} f\|_{L^2(\sigma)}^2+\alpha_{\mathcal{A}}(f)^2|A|_\sigma\right)\right)^{1/2} \\
&\qquad\qquad\qquad\qquad\times\left(\sum_{A\in\mathcal{A}}\left\|\mathsf{P}_{\mathcal{C}_A^{\tau\text{-shift}}} g\right\|_{L^2(\omega)}^2\right)^{1/2} \\
&\qquad\lesssim (\mathcal{E}_\alpha^{\mathrm{deep}}+\sqrt{A_2^\alpha})\,\|f\|_{L^2(\sigma)}\,\|g\|_{L^2(\omega)}\ ,
\end{aligned}$$

by orthogonality $\sum_{A\in\mathcal{A}}\|\mathsf{P}_{\mathcal{C}_A} f\|_{L^2(\sigma)}^2\le\|f\|_{L^2(\sigma)}^2$ in corona projections $\mathcal{C}_A^\sigma$, 'quasi' orthogonality $\sum_{A\in\mathcal{A}}\alpha_{\mathcal{A}}(f)^2|A|_\sigma\lesssim\|f\|_{L^2(\sigma)}^2$ in the stopping cubes $\mathcal{A}$, and by the bounded overlap of the shifted coronas $\mathcal{C}_A^{\tau\text{-shift}}$:

$$\sum_{A\in\mathcal{A}}\mathbf{1}_{\mathcal{C}_A^{\tau\text{-shift}}}\le\boldsymbol{\tau}\,\mathbf{1}_{\mathcal{D}}\,.$$

To prove Proposition 11.4, we begin by letting $\Pi_2\mathcal{P}$ consist of the second components of the pairs in $\mathcal{P}$ and writing

$$\mathsf{B}_{\mathrm{stop}}^{A,\mathcal{P}}(f,g)=\sum_{J\in\Pi_2\mathcal{P}}\langle T_\sigma^\alpha\varphi_J^{\mathcal{P}},\triangle_J^\omega g\rangle_\omega\,;\quad\text{where }\varphi_J^{\mathcal{P}}\equiv\sum_{I\in\mathcal{C}'_A:\,(I,J)\in\mathcal{P}}\mathbb{E}_I^\sigma(\triangle_{\pi I}^\sigma f)\ \mathbf{1}_{A\setminus I}\,.$$

By the tree-connected property of $\mathcal{P}$, and the telescoping property of martingale differences, together with the bound $\alpha_{\mathcal{A}}(A)$ on the averages of f in the corona $\mathcal{C}_A$, we have

$$|\varphi_J^{\mathcal{P}}|\lesssim\alpha_{\mathcal{A}}(A)\,\mathbf{1}_{A\setminus I_{\mathcal{P}}(J)}\,,\tag{11.4}$$

where $I_{\mathcal{P}}(J)\equiv\cap\{I\colon(I,J)\in\mathcal{P}\}$ is the smallest cube I for which $(I,J)\in\mathcal{P}$. Another important property of these functions is the sublinearity:

$$|\varphi_J^{\mathcal{P}}|\le|\varphi_J^{\mathcal{P}_1}|+|\varphi_J^{\mathcal{P}_2}|,\quad\mathcal{P}=\mathcal{P}_1\,\dot\cup\,\mathcal{P}_2\,.\tag{11.5}$$

Now apply the monotonicity Lemma 6.1 to the inner product $\langle T_\sigma^\alpha\varphi_J,\triangle_J^\omega g\rangle_\omega$ to obtain

$$\begin{aligned}
|\langle T_\sigma^\alpha\varphi_J,\triangle_J^\omega g\rangle_\omega|\lesssim{}&\frac{\mathrm{P}^\alpha(J,|\varphi_J|\mathbf{1}_{A\setminus I_{\mathcal{P}}(J)}\sigma)}{|J|^{1/n}}\,\|\triangle_J^\omega\,\mathbf{x}\|_{L^2(\omega)}\,\|\triangle_J^\omega\,g\|_{L^2(\omega)}\\
&+\frac{\mathrm{P}_{1+\delta}^\alpha(J,|\varphi_J|\mathbf{1}_{A\setminus I_{\mathcal{P}}(J)}\sigma)}{|J|^{1/n}}\,\|\mathsf{P}_J^\omega\mathbf{x}\|_{L^2(\omega)}\,\|\triangle_J^\omega\,g\|_{L^2(\omega)}\,.
\end{aligned}$$

Thus we have

$$(11.6)\quad |\mathsf{B}_{\mathrm{stop}}^{A,\mathcal{P}}(f,g)| \leq \sum_{J\in\Pi_2\mathcal{P}} \frac{\mathrm{P}_1^\alpha(J,|\varphi_J|\mathbf{1}_{A\setminus I_{\mathcal{P}}(J)}\sigma)}{|J|^{1/n}} \| \triangle_J^\omega \mathbf{x}\|_{L^2(\omega)} \| \triangle_J^\omega g\|_{L^2(\omega)}$$
$$+ \sum_{J\in\Pi_2\mathcal{P}} \frac{\mathrm{P}_{1+\delta}^\alpha(J,|\varphi_J|\mathbf{1}_{A\setminus I_{\mathcal{P}}(J)}\sigma)}{|J|^{1/n}} \|\mathsf{P}_J^\omega \mathbf{x}\|_{L^2(\omega)} \| \triangle_J^\omega g\|_{L^2(\omega)}$$
$$\equiv |\mathsf{B}|_{\mathrm{stop},1,\triangle^\omega}^{A,\mathcal{P}}(f,g) + |\mathsf{B}|_{\mathrm{stop},1+\delta,\mathsf{P}^\omega}^{A,\mathcal{P}}(f,g)\,,$$

where we have dominated the stopping form by two sublinear stopping forms that involve the Poisson integrals of order 1 and $1+\delta$ respectively, and where the smaller Poisson integral $\mathrm{P}_{1+\delta}^\alpha$ is multiplied by the larger projection $\|\mathsf{P}_J^\omega \mathbf{x}\|_{L^2(\omega)}$. This splitting turns out to be successful in separating the two energy terms from the right hand side of the energy lemma, because of the two properties (11.4) and (11.5) above. It remains to show the two inequalities:

$$(11.7)\qquad |\mathsf{B}|_{\mathrm{stop},1,\triangle^\omega}^{A,\mathcal{P}}(f,g) \lesssim \left(\mathcal{E}_\alpha^{\mathrm{deep}} + \sqrt{A_2^\alpha}\right) \alpha_{\mathcal{A}}(A) \sqrt{|A|_\sigma}\, \|g\|_{L^2(\omega)}\,,$$

for $f\in L^2(\sigma)$ satisfying where $\mathbb{E}_I^\sigma|f| \leq \alpha_{\mathcal{A}}(A)$ for all $I\in\mathcal{C}_A$; and

$$(11.8)\qquad |\mathsf{B}|_{\mathrm{stop},1+\delta,\mathsf{P}^\omega}^{A,\mathcal{P}}(f,g) \lesssim \left(\mathcal{E}_\alpha^{\mathrm{deep}} + \sqrt{A_2^\alpha}\right) \|f\|_{L^2(\sigma)}\, \|g\|_{L^2(\omega)}\,,$$

where we only need the case $\mathcal{P}=\mathcal{P}^A$ in this latter inequality as there is no recursion involved in treating this second sublinear form. We consider first the easier inequality (11.8) that does not require recursion. In the subsequent subsections we will control the more difficult inequality (11.7) by adapting the stopping time and recursion of M. Lacey to the sublinear form $|\mathsf{B}|_{\mathrm{stop},1,\triangle^\omega}^{A,\mathcal{P}}(f,g)$.

11.1. The second inequality

Now we turn to proving (11.8), i.e.,

$$|\mathsf{B}|_{\mathrm{stop},1+\delta,\mathsf{P}^\omega}^{A,\mathcal{P}}(f,g) \lesssim \left(\mathcal{E}_\alpha^{\mathrm{deep}} + \sqrt{A_2^\alpha}\right) \|f\|_{L^2(\sigma)}\, \|g\|_{L^2(\omega)}\,,$$

where since

$$|\varphi_J| = \Big|\sum_{I\in\mathcal{C}'_A:\ (I,J)\in\mathcal{P}} \mathbb{E}_I^\sigma(\triangle_{\pi I}^\sigma f)\,\mathbf{1}_{A\setminus I}\Big| \leq \sum_{I\in\mathcal{C}'_A:\,(I,J)\in\mathcal{P}} |\mathbb{E}_I^\sigma(\triangle_{\pi I}^\sigma f)\,\mathbf{1}_{A\setminus I}|\,,$$

the sublinear form $|\mathsf{B}|_{\mathrm{stop},1+\delta,\mathsf{P}^\omega}^{A,\mathcal{P}}$ can be dominated and then decomposed by pigeonholing the ratio of side lengths of J and I:

$$|\mathsf{B}|_{\mathrm{stop},1+\delta,\mathsf{P}^\omega}^{A,\mathcal{P}}(f,g) = \sum_{J\in\Pi_2\mathcal{P}} \frac{\mathrm{P}_{1+\delta}^\alpha(J,|\varphi_J|\mathbf{1}_{A\setminus I_{\mathcal{P}}(J)}\sigma)}{|J|^{1/n}} \|\mathsf{P}_J^\omega \mathbf{x}\|_{L^2(\omega)} \| \triangle_J^\omega g\|_{L^2(\omega)}$$
$$\leq \sum_{(I,J)\in\mathcal{P}} \frac{\mathrm{P}_{1+\delta}^\alpha(J,|\mathbb{E}_I^\sigma(\triangle_{\pi I}^\sigma f)|\mathbf{1}_{A\setminus I}\sigma)}{|J|^{1/n}} \|\mathsf{P}_J^\omega \mathbf{x}\|_{L^2(\omega)} \| \triangle_J^\omega g\|_{L^2(\omega)}$$
$$\equiv \sum_{s=0}^\infty |\mathsf{B}|_{\mathrm{stop},1+\delta,\mathsf{P}^\omega}^{A,\mathcal{P};s}(f,g)\,;$$

and

$$|\mathsf{B}|^{A,\mathcal{P};s}_{\mathrm{stop},1+\delta,\mathsf{P}^\omega}(f,g) \equiv \sum_{\substack{(I,J)\in\mathcal{P}\\ |J|^{1/n}=2^{-s}|I|^{1/n}}} \frac{\mathrm{P}^\alpha_{1+\delta}(J,|\mathbb{E}^\sigma_I(\triangle^\sigma_{\pi I}f)|\mathbf{1}_{A\setminus I}\sigma)}{|J|^{1/n}} \|\mathsf{P}^\omega_J\mathbf{x}\|_{L^2(\omega)}\|\triangle^\omega_J g\|_{L^2(\omega)}\,.$$

Here we have the *entire* projection $\mathsf{P}^\omega_J\mathbf{x}$ onto all of the dyadic subintervals of J, but this is offset by the smaller Poisson integral $\mathrm{P}^\alpha_{1+\delta}$. We will now adapt the argument for the stopping term starting on page 42 of [3], where the geometric gain from the assumed energy hypothesis there will be replaced by a geometric gain from the smaller Poisson integral $\mathrm{P}^\alpha_{1+\delta}$ used here.

First, we exploit the additional decay in the Poisson integral $\mathrm{P}^\alpha_{1+\delta}$ as follows. Suppose that $(I,J)\in\mathcal{P}$ with $|J|^{1/n}=2^{-s}|I|^{1/n}$. We then compute

$$\begin{aligned}\frac{\mathrm{P}^\alpha_{1+\delta}(J,\mathbf{1}_{A\setminus I}\sigma)}{|J|^{1/n}} &\approx \int_{A\setminus I}\frac{|J|^{\delta/n}}{|y-c_J|^{n+1+\delta-\alpha}}\,d\sigma(y)\\ &\leq \int_{A\setminus I}\left(\frac{|J|^{1/n}}{\mathrm{dist}(c_J,I^c)}\right)^{\delta}\frac{1}{|y-c_J|^{n+1-\alpha}}\,d\sigma(y) \lesssim \left(\frac{|J|^{1/n}}{\mathrm{dist}(c_J,I^c)}\right)^{\delta}\frac{\mathrm{P}^\alpha(J,\mathbf{1}_{A\setminus I}\sigma)}{|J|^{1/n}}\,,\end{aligned}$$

and use the goodness inequality,

$$\mathrm{dist}(c_J,I^c)\geq \frac{1}{2}\,|I|^{(1-\varepsilon)/n}\,|J|^{\varepsilon/n}\geq \frac{1}{2}\,2^{s(1-\varepsilon)}\,|J|^{1/n},$$

to conclude that

$$\left(\frac{\mathrm{P}^\alpha_{1+\delta}(J,\mathbf{1}_{A\setminus I}\sigma)}{|J|^{1/n}}\right)\lesssim 2^{-s\delta(1-\varepsilon)}\,\frac{\mathrm{P}^\alpha(J,\mathbf{1}_{A\setminus I}\sigma)}{|J|^{1/n}}\,. \tag{11.9}$$

We next claim that for $s\geq 0$ an integer,

$$\begin{aligned}|\mathsf{B}|^{A,\mathcal{P};s}_{\mathrm{stop},1+\delta,\mathsf{P}^\omega}(f,g) &= \sum_{\substack{(I,J)\in\mathcal{P}\\ |J|^{1/n}=2^{-s}|I|^{1/n}}} \frac{\mathrm{P}^\alpha_{1+\delta}(J,|\mathbb{E}^\sigma_I(\triangle^\sigma_{\pi I}f)|\mathbf{1}_{A\setminus I}\sigma)}{|J|^{1/n}} \|\mathsf{P}^\omega_J\mathbf{x}\|_{L^2(\omega)}\|\triangle^\omega_J g\|_{L^2(\omega)}\\ &\lesssim 2^{-s\delta(1-\varepsilon)}\left(\mathcal{E}^{\mathrm{deep}}_\alpha+\sqrt{A^\alpha_2}\right)\|f\|_{L^2(\sigma)}\,\|g\|_{L^2(\omega)}\,,\end{aligned}$$

from which (11.8) follows upon summing in $s\geq 0$. Now using both

$$\begin{aligned}|\mathbb{E}^\sigma_I(\triangle^\sigma_{\pi I}f)| &= \frac{1}{|I|_\sigma}\int_I|\triangle^\sigma_{\pi I}f|\,d\sigma\leq \|\triangle^\sigma_{\pi I}f\|_{L^2(\sigma)}\,\frac{1}{\sqrt{|I|_\sigma}}\,,\\ 2^n\|f\|^2_{L^2(\sigma)} &= \sum_{I\in\mathcal{D}}\|\triangle^\sigma_{\pi I}f\|^2_{L^2(\sigma)}\,,\end{aligned}$$

we apply Cauchy–Schwarz in the I variable above to see that

$$\begin{aligned}&\left[\,|\mathsf{B}|^{A,\mathcal{P};s}_{\mathrm{stop},1+\delta,\mathsf{P}^\omega}(f,g)\,\right]^2\\ &\lesssim \|f\|^2_{L^2(\sigma)}\left[\sum_{I\in\mathcal{C}'_A}\Big(\frac{1}{\sqrt{|I|_\sigma}}\sum_{\substack{J:\,(I,J)\in\mathcal{P}\\ |J|^{1/n}=2^{-s}|I|^{1/n}}}\frac{\mathrm{P}^\alpha_{1+\delta}(J,\mathbf{1}_{A\setminus I}\sigma)}{|J|^{1/n}}\|\mathsf{P}^\omega_J\mathbf{x}\|_{L^2(\omega)}\|\triangle^\omega_J g\|_{L^2(\omega)}\Big)^2\right]^{1/2}.\end{aligned}$$

We can then estimate the sum inside the square brackets by

$$\sum_{I\in\mathcal{C}'_A}\left\{\sum_{\substack{J:\,(I,J)\in\mathcal{P}\\ |J|^{1/n}=2^{-s}|I|^{1/n}}}\|\bigtriangleup_J^{\omega}g\|^2_{L^2(\omega)}\right\}$$
$$\times\sum_{\substack{J:\,(I,J)\in\mathcal{P}\\ |J|^{1/n}=2^{-s}|I|^{1/n}}}\frac{1}{|I|_\sigma}\left(\frac{\mathrm{P}^\alpha_{1+\delta}(J,\mathbf{1}_{A\setminus I}\sigma)}{|J|^{1/n}}\right)^2\|\mathsf{P}^\omega_J\mathbf{x}\|^2_{L^2(\omega)}\lesssim\|g\|^2_{L^2(\omega)}A(s)^2,$$

where

$$A(s)^2\equiv\sup_{I\in\mathcal{C}'_A}\sum_{\substack{J:\ (I,J)\in\mathcal{P}\\ |J|^{1/n}=2^{-s}|I|^{1/n}}}\frac{1}{|I|_\sigma}\left(\frac{\mathrm{P}^\alpha_{1+\delta}(J,\mathbf{1}_{A\setminus I}\sigma)}{|J|^{1/n}}\right)^2\|\mathsf{P}^\omega_J\mathbf{x}\|^2_{L^2(\omega)}\,.$$

Finally then we turn to the analysis of the supremum in last display. From the Poisson decay (11.9) we have

$$\begin{aligned}A(s)^2&\lesssim\sup_{I\in\mathcal{C}'_A}\frac{1}{|I|_\sigma}2^{-2s\delta(1-\varepsilon)}\sum_{\substack{J:\ (I,J)\in\mathcal{P}\\ \ell(J)=2^{-s}\ell(I)}}\left(\frac{\mathrm{P}^\alpha(J,\mathbf{1}_{A\setminus I}\sigma)}{|J|^{1/n}}\right)^2\|\mathsf{P}^\omega_J x\|^2_{L^2(\omega)}\\&\lesssim\sup_{I\in\mathcal{C}'_A}\frac{1}{|I|_\sigma}2^{-2s\delta(1-\varepsilon)}\sum_{K\in\mathcal{M}_{\mathbf{r}\text{-deep}}(I)}\left(\frac{\mathrm{P}^\alpha(K,\mathbf{1}_{A\setminus I}\sigma)}{|K|^{1/n}}\right)^2\sum_{\substack{J\subset K:\ (I,J)\in\mathcal{P}\\ \ell(J)=2^{-s}\ell(I)}}\|\mathsf{P}^\omega_J x\|^2_{L^2(\omega)}\\&\lesssim 2^{-2s\delta(1-\varepsilon)}\left[(\mathcal{E}^{\mathrm{deep}}_\alpha)^2+A_2^\alpha\right],\end{aligned}$$

where the last inequality is the one for which the definition of energy stopping cubes was designed. Indeed, from Definition 8.1, as $(I,J)\in\mathcal{P}$, we have that I is *not* a stopping cube in $\mathcal{A}$, and hence that (8.1) *fails* to hold, delivering the estimate above since $J\Subset_{\boldsymbol{\rho}-1}I$ good must be contained in some $K\in\mathcal{M}_{\mathbf{r}\text{-deep}}(I)$, and since $\mathrm{P}^\alpha(J,\mathbf{1}_{A\setminus I}\sigma)/|J|^{1/n}\approx\mathrm{P}^\alpha(K,\mathbf{1}_{A\setminus I}\sigma)/|K|^{1/n}$. The terms $\|\mathsf{P}^\omega_J x\|^2_{L^2(\omega)}$ are additive since the $J's$ are pigeonholed by $|J|^{1/n}=2^{-s}|I|^{1/n}$.

11.2. The first inequality and the recursion of M. Lacey

Now we turn to proving the more difficult inequality (11.7). Recall that in dimension $n=1$ the energy condition

$$\sum_{n=1}^\infty|J_n|_\omega\,\mathsf{E}(J_n,\omega)^2\,\mathrm{P}(J_n,\mathbf{1}_I\sigma)^2\lesssim(\mathcal{NTV})\ |I|_\sigma,\quad\dot{\bigcup}_{n=1}^\infty J_n\subset I,$$

could not be used in the NTV argument, because the set functional $J\to|J|_\omega\mathsf{E}(J,\omega)^2$ failed to be superadditive. On the other hand, the pivotal condition of NTV,

$$\sum_{n=1}^\infty|J_n|_\omega\,\mathrm{P}(J_n,\mathbf{1}_I\sigma)^2\lesssim|I|_\sigma,\quad\dot{\bigcup}_{n=1}^\infty J_n\subset I,$$

succeeded in the NTV argument because the set functional $J \to |J|_\omega$ is trivially superadditive, indeed additive. The final piece of the argument needed to prove the NTV conjecture was found by M. Lacey in [1], and amounts to first replacing the additivity of the functional $J \to |J|_\omega$ with the additivity of the projection functional $\mathcal{H} \to \|\mathsf{P}^\omega_{\mathcal{H}} x\|^2_{L^2(\omega)}$ defined on subsets $\mathcal{H}$ of the dyadic grid $\mathcal{D}$. Then a stopping time argument relative to this more subtle functional, together with a clever recursion, constitute the main new ingredients in Lacey's argument [1].

To begin the extension to a more general Calderón–Zygmund operator T^α, we also recall the stopping energy generalized to higher dimensions by

$$\mathbf{X}^\alpha(\mathcal{C}_A)^2 \equiv \sup_{I \in \mathcal{C}_A} \frac{1}{|I|_\sigma} \sum_{J \in \mathcal{M}_{\mathbf{r}\text{-deep}}(I)} \left(\frac{\mathrm{P}^\alpha(J, \mathbf{1}_{A \setminus \gamma J}\sigma)}{|J|^{1/n}} \right)^2 \|\mathsf{P}^{\mathrm{subgood},\omega}_J \mathbf{x}\|^2_{L^2(\omega)} ,$$

where $\mathcal{M}_{\mathbf{r}\text{-deep}}(I)$ is the set of maximal $\mathbf{r}$-deeply embedded subcubes of I where $\mathbf{r}$ is the goodness parameter. What now follows is an adaptation to our deep energy condition and the sublinear form $|\mathsf{B}|^{A,\mathcal{P}}_{\mathrm{stop},1,\triangle^\omega}$ of the arguments of M. Lacey in [1]. We have the following Poisson inequality for cubes $B \subset A \subset I$:

$$\frac{\mathrm{P}^\alpha(A, \mathbf{1}_{I \setminus A}\sigma)}{|A|^{1/n}} \approx \int_{I \setminus A} \frac{1}{(|y - c_A|)^{n+1-\alpha}} \, d\sigma(y) \lesssim \int_{I \setminus A} \frac{1}{(|y - c_B|)^{n+1-\alpha}} \, d\sigma(y) \approx \frac{\mathrm{P}^\alpha(B, \mathbf{1}_{I \setminus A}\sigma)}{|B|^{1/n}} . \tag{11.10}$$

11.3. The stopping energy

Fix $A \in \mathcal{A}$. We will use a 'decoupled' modification of the stopping energy $\mathbf{X}(\mathcal{C}_A)$. Suppose that $\mathcal{P}$ is an A-admissible collection of pairs of cubes in the product set $\mathcal{D} \times \mathcal{D}_{\mathrm{good}}$ of pairs of dyadic cubes in $\mathbb{R}^n$ with second component good. For an admissible collection $\mathcal{P}$ let $\Pi_1\mathcal{P}$ and $\Pi_2\mathcal{P}$ be the cubes in the first and second components of the pairs in $\mathcal{P}$ respectively, let $\Pi\mathcal{P} \equiv \Pi_1\mathcal{P} \cup \Pi_2\mathcal{P}$, and for $K \in \Pi\mathcal{P}$ define the $\boldsymbol{\tau}$-deep projection of $\mathcal{P}$ relative to K by

$$\Pi_2^{K,\boldsymbol{\tau}\text{-deep}}\mathcal{P} \equiv \{J \in \Pi_2\mathcal{P} : J \Subset_{\boldsymbol{\tau}} K\} .$$

Now the cubes J in $\Pi_2\mathcal{P}$ are of course *always* good, but this is *not* the case for cubes I in $\Pi_1\mathcal{P}$. Indeed, the collection $\mathcal{P}$ is tree-connected in the first component, and it is clear that there can be many *bad* cubes in a connected geodesic in the tree $\mathcal{D}$. But the Haar support of f is contained in *good* cubes I, and we have also assumed that the children of these cubes I are good. As a consequence we may always assume that our A-admissible collections $\mathcal{P}$ are reduced in the sense of Definition 11.3. Thus we will use as our 'size testing collection' of cubes for $\mathcal{P}$ the collection

$$\Pi^{\mathrm{goodbelow}}\mathcal{P} \equiv \{K' \in \mathcal{D} : K' \text{ is good and } K' \subset K \text{ for some } K \in \Pi\mathcal{P}\} ,$$

which consists of all the good subcubes of any cube in $\Pi\mathcal{P}$. Note that the maximal cubes in $\Pi\mathcal{P} = \Pi\mathcal{P}^{\mathrm{red}}$ are already good themselves, and so we have the important property that

$$(I,J)\in\mathcal{P}=\mathcal{P}^{\mathrm{red}} \text{ implies } I\subset K \text{ for some cube } K\in\Pi^{\mathrm{goodbelow}}\mathcal{P}\,. \tag{11.11}$$

Now define the 'size functional' $\mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{P})$ of $\mathcal{P}$ as follows. Recall that a projection $\mathsf{P}_{\mathcal{H}}^{\omega}$ on $\mathbf{x}$ satisfies

$$\|\mathsf{P}_{\mathcal{H}}^{\omega}\mathbf{x}\|_{L^2(\omega)}^2=\sum_{J\in\mathcal{H}}\|\bigtriangleup_J^{\omega}\mathbf{x}\|_{L^2(\omega)}^2\,.$$

Definition 11.5. If $\mathcal{P}$ is A-admissible, define

$$\mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{P})^2\equiv\sup_{K\in\Pi^{\mathrm{goodbelow}}\mathcal{P}}\frac{1}{|K|_\sigma}\Big(\frac{\mathrm{P}^\alpha(K,\mathbf{1}_{A\setminus K}\sigma)}{|K|^{1/n}}\Big)^2\|\mathsf{P}^{\omega}_{\Pi_2^{K,\boldsymbol{\tau}\text{-deep}}\mathcal{P}}\mathbf{x}\|_{L^2(\omega)}^2\,. \tag{11.12}$$

We should remark that that the cubes K in $\Pi^{\mathrm{goodbelow}}\mathcal{P}$ that fail to contain any $\boldsymbol{\tau}$-parents of cubes from $\Pi_2\mathcal{P}$ will not contribute to the size functional since $\Pi_2^{K,\boldsymbol{\tau}\text{-deep}}\mathcal{P}$ is empty in this case. We note three essential properties of this definition of size functional:

1. **Monotonicity** of size: $\mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{P})\leq\mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{Q})$ if $\mathcal{P}\subset\mathcal{Q}$,
2. **Goodness** of testing cubes: $\Pi^{\mathrm{goodbelow}}\mathcal{P}\subset\mathcal{D}_{\mathrm{good}}$,
3. **Control** by deep energy condition: $\mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{P})\lesssim\mathcal{E}_\alpha^{\mathrm{deep}}+\sqrt{A_2^\alpha}\,$.

The monotonicity property follows from $\Pi^{\mathrm{goodbelow}}\mathcal{P}\subset\Pi^{\mathrm{goodbelow}}\mathcal{Q}$ and $\Pi_2^{K,\boldsymbol{\tau}\text{-deep}}\mathcal{P}\subset\Pi_2^{K,\boldsymbol{\tau}\text{-deep}}\mathcal{Q}$, and the goodness property follows from the definition of $\Pi^{\mathrm{goodbelow}}\mathcal{P}$. The control property is contained in the next lemma, which uses the stopping energy control for the form $\mathsf{B}_{\mathrm{stop}}^A(f,g)$ associated with A.

Lemma 11.6. *If $\mathcal{P}^A$ is as in (11.2) and $\mathcal{P}\subset\mathcal{P}^A$, then*

$$\mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{P})\leq\mathbf{X}_\alpha(\mathcal{C}_A)\lesssim\mathcal{E}_\alpha^{\mathrm{deep}}+\sqrt{A_2^\alpha}\,.$$

Proof. Suppose that $K\in\Pi^{\mathrm{goodbelow}}\mathcal{P}$. To prove the first inequality in the statement we note that

$$\begin{aligned}
&\frac{1}{|K|_\sigma}\Big(\frac{\mathrm{P}^\alpha(K,\mathbf{1}_{A\setminus K}\sigma)}{|K|^{1/n}}\Big)^2\Big\|\mathsf{P}^{\omega}_{(\Pi_2^{K,\boldsymbol{\tau}\text{-deep}}\mathcal{P})^*}\mathbf{x}\Big\|_{L^2(\omega)}^2\\
&\quad\leq\frac{1}{|K|_\sigma}\Big(\frac{\mathrm{P}^\alpha(K,\mathbf{1}_{A\setminus K}\sigma)}{|K|^{1/n}}\Big)^2\sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(K)}\Big\|\mathsf{P}_J^{\mathrm{subgood},\omega}\mathbf{x}\Big\|_{L^2(\omega)}^2\\
&\quad\lesssim\frac{1}{|K|_\sigma}\sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(K)}\Big(\frac{\mathrm{P}^\alpha(J,\mathbf{1}_{A\setminus K}\sigma)}{|J|^{1/n}}\Big)^2\Big\|\mathsf{P}_J^{\mathrm{subgood},\omega}\mathbf{x}\Big\|_{L^2(\omega)}^2\\
&\quad\lesssim\frac{1}{|K|_\sigma}\sum_{J\in\mathcal{M}_{\mathbf{r}\text{-deep}}(K)}\Big(\frac{\mathrm{P}^\alpha(J,\mathbf{1}_{A\setminus\gamma J}\sigma)}{|J|^{1/n}}\Big)^2\Big\|\mathsf{P}_J^{\mathrm{subgood},\omega}\mathbf{x}\Big\|_{L^2(\omega)}^2\leq\mathbf{X}_\alpha(\mathcal{C}_A)\,,
\end{aligned}$$

where the first inequality above follows since every $J' \in \Pi_2^{K,\boldsymbol{\tau}\text{-deep}}\mathcal{P}$ is contained in some $J \in \mathcal{M}_{\mathbf{r}\text{-deep}}(I)$, the second inequality follows from (11.10) with $J \subset K \subset A$, and then the third inequality follows since $J \Subset_{\mathbf{r}} I$ implies $\gamma J \subset I$ by (2.4), and finally since $\Pi_2^{K,\boldsymbol{\tau}\text{-deep}}\mathcal{P} = \emptyset$ if $K \subset A$ and $K \notin \mathcal{C}_A$ by (11.13) below. The second inequality in the statement of the lemma follows from (8.5). □

The following useful fact is needed above and will be used later as well:

$$K \subset A \text{ and } K \notin \mathcal{C}_A \implies \Pi_2^{K,\boldsymbol{\tau}\text{-deep}}\mathcal{P} = \emptyset\,. \tag{11.13}$$

To see this, suppose that $K \in \mathcal{C}_A^{\boldsymbol{\tau}\text{-shift}} \setminus \mathcal{C}_A$. Then $K \subset A'$ for some $A' \in \mathfrak{C}_{\mathcal{A}}(A)$, and so if there is $J \in \Pi_2^{K,\boldsymbol{\tau}\text{-deep}}\mathcal{P}$, then $|J|^{1/n} \leq 2^{-\boldsymbol{\tau}}|K|^{1/n} \leq 2^{-\boldsymbol{\tau}}|A'|^{1/n}$, which implies that $J \notin \mathcal{C}_A^{\boldsymbol{\tau}\text{-shift}}$, which contradicts $\Pi_2^{K,\boldsymbol{\tau}\text{-deep}}\mathcal{P} \subset \mathcal{C}_A^{\boldsymbol{\tau}\text{-shift}}$.

Now define an atomic measure $\omega_{\mathcal{P}}$ in the upper half space $\mathbb{R}_+^{n+1}$ by

$$\omega_{\mathcal{P}} \equiv \sum_{J \in \Pi_2\mathcal{P}} \| \triangle_J^{\omega} \mathbf{x}\|_{L^2(\omega)}^2 \; \delta_{(c_J, |J|^{1/n})}.$$

Define the tent $\mathbf{T}(K)$ over a cube K to be the convex hull of the n-cube $K \times \{0\}$ and the point $(c_K, |K|^{1/n}) \in \mathbb{R}_+^{n+1}$. Define the $\boldsymbol{\tau}$-deep tent $\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(K)$ over a cube K to be the restriction of the tent $\mathbf{T}(K)$ to those points at depth $\boldsymbol{\tau}$ or more below K:

$$\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(K) \equiv \{(y,t) \in \mathbf{T}(K) : t \leq 2^{-\boldsymbol{\tau}}\, |K|^{1/n}\}\,.$$

We can now rewrite the size functional (11.12) of $\mathcal{P}$ as

$$\mathcal{S}_{\text{size}}^{\alpha,A}(\mathcal{P})^2 \equiv \sup_{K \in \Pi^{\text{goodbelow}}\mathcal{P}} \frac{1}{|K|_\sigma} \left(\frac{\mathrm{P}^{\alpha}(K, \mathbf{1}_{A\setminus K}\sigma)}{|K|^{1/n}}\right)^2 \omega_{\mathcal{P}}(\mathbf{T}^{\boldsymbol{\tau}-\text{deep}}(K))\,. \tag{11.14}$$

It will be convenient to write

$$\Psi^{\alpha}(K;\mathcal{P})^2 \equiv \left(\frac{\mathrm{P}^{\alpha}(K, \mathbf{1}_{A\setminus K}\sigma)}{|K|^{1/n}}\right)^2 \omega_{\mathcal{P}}(\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(K))\,,$$

so that we have simply

$$\mathcal{S}_{\text{size}}^{\alpha,A}(\mathcal{P})^2 = \sup_{K \in \Pi^{\text{goodbelow}}\mathcal{P}} \frac{\Psi^{\alpha}(K;\mathcal{P})^2}{|K|_\sigma}\,.$$

Remark 11.7. The functional $\omega_{\mathcal{P}}(\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(K))$ is increasing in K, while the functional $\mathrm{P}^{\alpha}(K, \mathbf{1}_{A\setminus K}\sigma)/|K|^{1/n}$ is 'almost decreasing' in K: if $K_0 \subset K$ then

$$\begin{aligned}
\frac{\mathrm{P}^{\alpha}(K, \mathbf{1}_{A\setminus K}\sigma)}{|K|^{1/n}} &= \int_{A\setminus K} \frac{d\sigma(y)}{(|K|^{1/n} + |y - c_K|)^{n+1-\alpha}} \\
&\lesssim \int_{A\setminus K} \frac{(\sqrt{n})^{n+1-\alpha}\, d\sigma(y)}{(|K_0|^{1/n} + |y - c_{K_0}|)^{n+1-\alpha}} \\
&\leq C_{n,\alpha} \int_{A\setminus K_0} \frac{d\sigma(y)}{(|K_0|^{1/n} + |y - c_{K_0}|)^{n+1-\alpha}} = C_{n,\alpha} \frac{\mathrm{P}^{\alpha}(K_0, \mathbf{1}_{A\setminus K_0}\sigma)}{|K_0|^{1/n}},
\end{aligned}$$

since $|K_0|^{1/n} + |y - c_{K_0}| \leq |K|^{1/n} + |y - c_K| + \frac{1}{2}\text{diam}(K)$ for $y \in A \setminus K$.

11.4. The recursion

Recall that if $\mathcal{P}$ is an admissible collection for a dyadic cube A, the corresponding sublinear form in (11.7) is given in (11.6) by

$$|\mathsf{B}|^{A,\mathcal{P}}_{\operatorname{stop},1,\triangle^\omega}(f,g) \equiv \sum_{J\in\Pi_2\mathcal{P}} \frac{\mathrm{P}^\alpha(J,|\varphi^{\mathcal{P}}_J|\mathbf{1}_{A\setminus I_{\mathcal{P}}(J)}\sigma)}{|J|^{1/n}} \left\| \triangle^\omega_J \mathbf{x}\right\|_{L^2(\omega)} \left\| \triangle^\omega_J g\right\|_{L^2(\omega)} ;$$

$$\text{where } \varphi^{\mathcal{P}}_J \equiv \sum_{I\in\mathcal{C}'_A:\,(I,J)\in\mathcal{P}} \mathbb{E}^\sigma_I(\triangle^\sigma_{\pi I} f)\, \mathbf{1}_{A\setminus I} .$$

In the notation for $|\mathsf{B}|^{A,\mathcal{P}}_{\operatorname{stop},1,\triangle^\omega}$, we are omitting dependence on the parameter α, and to avoid clutter, we will often do so from now on when the dependence on α is inconsequential. Following Lacey [1], we now claim the following proposition, from which we obtain (11.7) as a corollary below. Motivated by the conclusion of Proposition 11.4, we define the *restricted* norm $\mathfrak{N}^{A,\mathcal{P}}_{\operatorname{stop},1,\triangle}$ of the sublinear form $|\mathsf{B}|^{A,\mathcal{P}}_{\operatorname{stop},1,\triangle^\omega}$ to be the best constant $\mathfrak{N}^{A,\mathcal{P}}_{\operatorname{stop},1,\triangle}$ in the inequality

$$|\mathsf{B}|^{A,\mathcal{P}}_{\operatorname{stop},1,\triangle^\omega}(f,g) \leq \mathfrak{N}^{A,\mathcal{P}}_{\operatorname{stop},1,\triangle}\left(\alpha_{\mathcal{A}}(A)\sqrt{|A|_\sigma} + \|f\|_{L^2(\sigma)}\right) \|g\|_{L^2(\omega)} ,$$

where $f\in L^2(\sigma)$ satisfies $\mathbb{E}^\sigma_I|f| \leq \alpha_{\mathcal{A}}(A)$ for all $I\in\mathcal{C}^{\operatorname{good}}_A$.

Proposition 11.8. (*This is a variant for sublinear forms of the size lemma in Lacey* [1]) *Suppose $\varepsilon>0$. Let $\mathcal{P}$ be an* admissible *collection of pairs for a dyadic cube A. Then we can decompose $\mathcal{P}$ into two disjoint collections $\mathcal{P}=\mathcal{P}^{\operatorname{big}}\dot\cup\mathcal{P}^{\operatorname{small}}$, and further decompose $\mathcal{P}^{\operatorname{small}}$ into pairwise disjoint collections $\mathcal{P}^{\operatorname{small}}_1,\mathcal{P}^{\operatorname{small}}_2,\ldots,\mathcal{P}^{\operatorname{small}}_\ell,\ldots$, i.e.,*

$$\mathcal{P}=\mathcal{P}^{\operatorname{big}}\dot\bigcup\left(\dot\bigcup_{\ell=1}^\infty\mathcal{P}^{\operatorname{small}}_\ell\right),$$

such that the collections $\mathcal{P}^{\operatorname{big}}$ and $\mathcal{P}^{\operatorname{small}}_\ell$ are admissible and satisfy

$$\sup_{\ell\geq1}\mathcal{S}^{\alpha,A}_{\operatorname{size}}(\mathcal{P}^{\operatorname{small}}_\ell)^2 \leq \varepsilon\,\mathcal{S}^{\alpha,A}_{\operatorname{size}}(\mathcal{P})^2, \tag{11.15}$$

and

$$\mathfrak{N}^{A,\mathcal{P}}_{\operatorname{stop},1,\triangle} \leq C_\varepsilon\,\mathcal{S}^{\alpha,A}_{\operatorname{size}}(\mathcal{P}) + \sqrt{n\boldsymbol{\tau}}\sup_{\ell\geq1}\mathfrak{N}^{A,\mathcal{P}^{\operatorname{small}}_\ell}_{\operatorname{stop},1,\triangle} . \tag{11.16}$$

Corollary 11.9. *The sublinear stopping form inequality* (11.7) *holds.*

Proof of Corollary 11.9. Set $\mathcal{Q}^0=\mathcal{P}^A$. Apply Proposition 11.8 to obtain a subdecomposition $\{\mathcal{Q}^1_\ell\}_{\ell=1}^\infty$ of $\mathcal{Q}^0$ such that

$$\mathfrak{N}^{A,\mathcal{Q}^0}_{\operatorname{stop},1,\triangle} \leq C_\varepsilon\,\mathcal{S}^{\alpha,A}_{\operatorname{size}}(\mathcal{Q}^0) + \sqrt{n\boldsymbol{\tau}}\sup_{\ell\geq1}\mathfrak{N}^{A,\mathcal{Q}^1_\ell}_{\operatorname{stop},1,\triangle} ,$$

$$\sup_{\ell\geq1}\mathcal{S}^{\alpha,A}_{\operatorname{size}}(\mathcal{Q}^1_\ell) \leq \varepsilon\,\mathcal{S}^{\alpha,A}_{\operatorname{size}}(\mathcal{Q}^0) .$$

Now apply Proposition 11.8 to each $\mathcal{Q}_\ell^1$ to obtain a subdecomposition $\{\mathcal{Q}_{\ell,k}^2\}_{k=1}^\infty$ of $\mathcal{Q}_\ell^1$ such that

$$\mathfrak{N}_{\mathrm{stop},1,\triangle}^{A,\mathcal{Q}_\ell^1} \leq C_\varepsilon\, \mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{Q}_\ell^1) + \sqrt{n\boldsymbol{\tau}} \sup_{k\geq 1} \mathfrak{N}_{\mathrm{stop},1,\triangle}^{A,\mathcal{Q}_{\ell,k}^2},$$

$$\sup_{k\geq 1} \mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{Q}_{\ell,k}^2) \leq \varepsilon\, \mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{Q}_\ell^1).$$

Altogether we have

$$\begin{aligned}\mathfrak{N}_{\mathrm{stop},1,\triangle}^{A,\mathcal{Q}^0} &\leq C_\varepsilon\, \mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{Q}^0) + \sqrt{n\boldsymbol{\tau}} \sup_{\ell\geq 1}\Big\{C_\varepsilon\, \mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{Q}_\ell^1) + \sqrt{n\boldsymbol{\tau}} \sup_{k\geq 1} \mathfrak{N}_{\mathrm{stop},1,\triangle}^{A,\mathcal{Q}_{\ell,k}^2}\Big\}\\ &= C_\varepsilon \left\{\mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{Q}^0) + \sqrt{n\boldsymbol{\tau}}\varepsilon \mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{Q}^0)\right\} + (n\boldsymbol{\tau}) \sup_{\ell,k\geq 1} \mathfrak{N}_{\mathrm{stop},1,\triangle}^{A,\mathcal{Q}_{\ell,k}^2}.\end{aligned}$$

Then with $\zeta \equiv \sqrt{n\boldsymbol{\tau}}$, we obtain by induction for every $N \in \mathbb{N}$,

$$\begin{aligned}\mathfrak{N}_{\mathrm{stop},1,\triangle}^{A,\mathcal{Q}^0} \leq C_\varepsilon &\left\{\mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{Q}^0) + \zeta\,\varepsilon\, \mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{Q}^0) + \cdots + \zeta^N \varepsilon^N \mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{Q}^0)\right\}\\ &+ \zeta^{N+1} \sup_{m\in\mathbb{N}^{N+1}} \mathfrak{N}_{\mathrm{stop},1,\triangle}^{A,\mathcal{Q}_m^{N+1}}.\end{aligned}$$

Now we may assume the collection $\mathcal{Q}^0 = \mathcal{P}^A$ of pairs is finite (simply truncate the corona $\mathcal{C}_A$ and obtain bounds independent of the truncation) and so $\sup_{m\in\mathbb{N}^{N+1}} \mathfrak{N}_{\mathrm{stop},1,\triangle}^{A,\mathcal{Q}_m^{N+1}} = 0$ for N large enough.

Then we obtain (11.7) if we choose $0 < \varepsilon < 1/(1+\zeta)$ and apply Lemma 11.6. □

Proof of Proposition 11.8. We first recall that the 'size testing collection' of cubes $\Pi^{\mathrm{goodbelow}}\mathcal{P}$ is the collection of all *good* subcubes of a cube in $\Pi\mathcal{P}$. We may assume that $\mathcal{P}$ is a finite collection. Begin by defining the collection $\mathcal{L}_0$ to consist of the *minimal* dyadic cubes K in $\Pi^{\mathrm{goodbelow}}\mathcal{P}$ such that

$$\frac{\Psi^\alpha(K;\mathcal{P})^2}{|K|_\sigma} \geq \varepsilon\, \mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{P})^2.$$

where we recall that

$$\Psi^\alpha(K;\mathcal{P})^2 \equiv \left(\frac{\mathrm{P}^\alpha(K,\mathbf{1}_{A\setminus K}\sigma)}{|K|^{1/n}}\right)^2 \omega_{\mathcal{P}}(\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(K)).$$

Note that such minimal cubes exist when $0 < \varepsilon < 1$ because $\mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{P})^2$ is the supremum over $K \in \Pi^{\mathrm{goodbelow}}\mathcal{P}$ of $\Psi^\alpha(K;\mathcal{P})^2/|K|_\sigma$. A key property of the the minimality requirement is that

$$\frac{\Psi^\alpha(K';\mathcal{P})^2}{|K'|_\sigma} < \varepsilon\, \mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{P})^2, \tag{11.17}$$

for all $K' \in \Pi^{\mathrm{goodbelow}}\mathcal{P}$ with $K' \subsetneqq K$ and $K \in \mathcal{L}_0$.

We now perform a stopping time argument 'from the bottom up' with respect to the atomic measure $\omega_{\mathcal{P}}$ in the upper half space. This construction of a stopping time 'from the bottom up' is one of two key innovations in Lacey's argument [1], the other being the recursion described in Proposition 11.8.

We refer to $\mathcal{L}_0$ as the initial or level 0 generation of stopping times. Choose $\rho = 1+\varepsilon$. We then recursively define a sequence of generations $\{\mathcal{L}_m\}_{m=0}^{\infty}$ by letting $\mathcal{L}_m$ consist of the *minimal* dyadic cubes L in $\Pi^{\text{goodbelow}}\mathcal{P}$ that contain a cube from some previous level $\mathcal{L}_\ell$, $\ell < m$, such that

$$\omega_{\mathcal{P}}(\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(L)) \geq \rho\, \omega_{\mathcal{P}}\left(\bigcup_{L' \in \bigcup_{\ell=0}^{m-1} \mathcal{L}_\ell :\, L' \subset L} \mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(L') \right). \tag{11.18}$$

Since $\mathcal{P}$ is finite this recursion stops at some level M. We then let $\mathcal{L}_{M+1}$ consist of all the maximal cubes in $\Pi^{\text{goodbelow}}\mathcal{P}$ that are not already in some $\mathcal{L}_m$. Thus $\mathcal{L}_{M+1}$ will contain either none, some, or all of the maximal cubes in $\Pi^{\text{goodbelow}}\mathcal{P}$. We do not of course have (11.18) for $A' \in \mathcal{L}_{M+1}$ in this case, but we do have that (11.18) fails for subcubes K of $A' \in \mathcal{L}_{M+1}$ that are not contained in any other $L \in \mathcal{L}_m$, and this is sufficient for the arguments below.

We now define the collections $\mathcal{P}^{\text{small}}$ and $\mathcal{P}^{\text{big}}$. The collection $\mathcal{P}^{\text{big}}$ will consist of those pairs $(I,J) \in \mathcal{P}$ for which there is $L \in \bigcup_{m=0}^{M+1} \mathcal{L}_m$, with $J \Subset_{\boldsymbol{\tau}} L \subset I$, and $\mathcal{P}^{\text{small}}$ will consist of the remaining pairs. But a considerable amount of further analysis is required to prove the conclusion of the proposition. First, let $\mathcal{L} \equiv \bigcup_{m=0}^{M+1} \mathcal{L}_m$ be the tree of stopping energy cubes defined above. By our construction above, the maximal elements in $\mathcal{L}$ are the maximal cubes in $\Pi^{\text{goodbelow}}\mathcal{P}$. For $L \in \mathcal{L}$, denote by $\mathcal{C}_L$ the *corona* associated with L in the tree $\mathcal{L}$,

$$\mathcal{C}_L \equiv \{K \in \mathcal{D} : K \subset L \text{ and there is no } L' \in \mathcal{L} \text{ with } K \subset L' \subsetneqq L\},$$

and define the *shifted* corona by

$$\mathcal{C}_L^{\boldsymbol{\tau}\text{-shift}} \equiv \{K \in \mathcal{C}_L : K \Subset_{\boldsymbol{\tau}} L\} \bigcup \bigcup_{L' \in \mathfrak{C}_{\mathcal{L}}(L)} \{K \in \mathcal{D} : K \Subset_{\boldsymbol{\tau}} L \text{ and } K \text{ is } \boldsymbol{\tau}\text{-nearby in } L'\}.$$

Now the parameter m in $\mathcal{L}_m$ refers to the level at which the stopping construction was performed, but for $L \in \mathcal{L}_m$, the corona children L' of L are *not* all necessarily in $\mathcal{L}_{m-1}$, but may be in $\mathcal{L}_{m-t}$ for t large. Thus we need to introduce the notion of geometric depth d in the tree $\mathcal{L}$ by defining

$$\begin{aligned}
\mathcal{G}_0 &\equiv \{L \in \mathcal{L} : L \text{ is maximal}\},\\
\mathcal{G}_1 &\equiv \{L \in \mathcal{L} : L \text{ is maximal with respect to } L \subsetneqq L_0 \text{ for some } L_0 \in \mathcal{G}_0\},\\
&\ \ \vdots\\
\mathcal{G}_{d+1} &\equiv \{L \in \mathcal{L} : L \text{ is maximal with respect to } L \subsetneqq L_d \text{ for some } L_d \in \mathcal{G}_d\},\\
&\ \ \vdots
\end{aligned}$$

We refer to $\mathcal{G}_d$ as the d^{th} generation of cubes in the tree $\mathcal{L}$, and say that the cubes in $\mathcal{G}_d$ are at depth d in the tree $\mathcal{L}$. Thus the cubes in $\mathcal{G}_d$ are the stopping cubes in $\mathcal{L}$ that are d levels in the *geometric* sense below the top level.

Then for $L \in \mathcal{G}_d$ and $t \geq 0$ define

$$\mathcal{P}_{L,t} \equiv \{(I,J) \in \mathcal{P} : I \in \mathcal{C}_L \text{ and } J \in \mathcal{C}_{L'}^{\boldsymbol{\tau}\text{-shift}} \text{ for some } L' \in \mathcal{G}_{d+t} \text{ with } L' \subset L\}.$$

In particular, $(I,J) \in \mathcal{P}_{L,t}$ implies that I is in the corona $\mathcal{C}_L$, and that J is in a shifted corona $\mathcal{C}_{L'}^{\boldsymbol{\tau}\text{-shift}}$ that is t levels of generation *below* $\mathcal{C}_L$. We emphasize the distinction 'generation' as this refers to the depth rather than the level of stopping construction. For $t = 0$ we further decompose $\mathcal{P}_{L,0}$ as

$$\begin{aligned} \mathcal{P}_{L,0} &= \mathcal{P}_{L,0}^{\text{small}} \,\dot{\cup}\, \mathcal{P}_{L,0}^{\text{big}}; \\ \mathcal{P}_{L,0}^{\text{small}} &\equiv \{(I,J) \in \mathcal{P}_{L,0} : I \neq L\}, \\ \mathcal{P}_{L,0}^{\text{big}} &\equiv \{(I,J) \in \mathcal{P}_{L,0} : I = L\}, \end{aligned}$$

with one exeption: if $L \in \mathcal{L}_{M+1}$ we set $\mathcal{P}_{L,0}^{\text{small}} \equiv \mathcal{P}_{L,0}$ since in this case L fails to satisfy (11.18) as pointed out above. Then we set

$$\begin{aligned} \mathcal{P}^{\text{big}} &\equiv \Big\{ \bigcup_{L \in \mathcal{L}} \mathcal{P}_{L,0}^{\text{big}} \Big\} \bigcup \Big\{ \bigcup_{t \geq 1} \bigcup_{L \in \mathcal{L}} \mathcal{P}_{L,t} \Big\}; \\ \{\mathcal{P}_\ell^{\text{small}}\}_{\ell=0}^\infty &\equiv \{\mathcal{P}_{L,0}^{\text{small}}\}_{L \in \mathcal{L}}, \quad \text{after relabelling.} \end{aligned}$$

It is important to note that by (11.11), every pair $(I,J) \in \mathcal{P}$ will be included in either $\mathcal{P}^{\text{small}}$ or $\mathcal{P}^{\text{big}}$. Now we turn to proving the inequalities (11.15) and (11.16).

To prove the inequality (11.15), it suffices with the above relabelling to prove the following claim:

$$\mathcal{S}_{\text{size}}^{\alpha,A}(\mathcal{P}_{L,0}^{\text{small}})^2 \leq (\rho - 1)\, \mathcal{S}_{\text{size}}^{\alpha,A}(\mathcal{P})^2, \quad L \in \mathcal{L}. \tag{11.19}$$

To see (11.19), suppose first that $L \notin \mathcal{L}_{M+1}$. In the case that $L \in \mathcal{L}_0$ is an initial generation cube, then from (11.17) we obtain that

$$\begin{aligned} \mathcal{S}_{\text{size}}^{\alpha,A}(\mathcal{P}_{L,0}^{\text{small}})^2 &\leq \sup_{K' \in \Pi^{\text{goodbelow}}\mathcal{P}:\ K' \subsetneq L} \frac{1}{|K'|_\sigma} \left(\frac{\mathrm{P}^\alpha(K', \mathbf{1}_{A \setminus K'}\sigma)}{|K'|^{1/n}} \right)^2 \omega_{\mathcal{P}}(\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(K')) \\ &\leq \varepsilon\, \mathcal{S}_{\text{size}}^{\alpha,A}(\mathcal{P})^2. \end{aligned}$$

Now suppose that $L \notin \mathcal{L}_0$ and also that $L \notin \mathcal{L}_{M+1}$. Pick a pair $(I,J) \in \mathcal{P}_{L,0}^{\text{small}}$. Then I is in the strict corona $\mathcal{C}_L'$ and J is in the $\boldsymbol{\tau}$*-shifted* corona $\mathcal{C}_L^{\boldsymbol{\tau}\text{-shift}}$. Since $\mathcal{P}_{L,0}^{\text{small}}$ is a finite collection, the definition of $\mathcal{S}_{\text{size}}^{\alpha,A}(\mathcal{P}_{L,0}^{\text{small}})$ shows that there is a cube $K \in \Pi^{\text{goodbelow}}\mathcal{P}_{L,0}^{\text{small}}$ so that

$$\mathcal{S}_{\text{size}}^{\alpha,A}(\mathcal{P}_{L,0}^{\text{small}})^2 = \frac{1}{|K|_\sigma} \left(\frac{\mathrm{P}^\alpha(K, \mathbf{1}_{A \setminus K}\sigma)}{|K|^{1/n}} \right)^2 \omega_{\mathcal{P}}(\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(K)).$$

Now define

$$t' = t'(K) \equiv \max\{s : \text{there is } L' \in \mathcal{L}_s \text{ with } L' \subset K\}.$$

First, suppose that $t'=0$ so that K does not contain any $L'\in\mathcal{L}$. Then it follows from our construction at level $\ell=0$ that

$$\frac{1}{|K|_\sigma}\Big(\frac{\mathrm{P}^\alpha(K,\mathbf{1}_{A\setminus K}\sigma)}{|K|^{1/n}}\Big)^2\omega_{\mathcal{P}}(\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(K))<\varepsilon\,\mathcal{S}_{\text{size}}^{\alpha,A}(\mathcal{P})^2,$$

and hence from $\rho=1+\varepsilon$ we obtain

$$\mathcal{S}_{\text{size}}^{\alpha,A}(\mathcal{P}_{L,0}^{\text{small}})^2<\varepsilon\,\mathcal{S}_{\text{size}}^{\alpha,A}(\mathcal{P})^2=(\rho-1)\,\mathcal{S}_{\text{size}}^{\alpha,A}(\mathcal{P})^2.$$

Now suppose that $t'\geq 1$. Then K fails the stopping condition (11.18) with $m=t'+1$, and so

$$\omega_{\mathcal{P}}(\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(K))<\rho\,\omega_{\mathcal{P}}\Bigg(\bigcup_{L'\in\bigcup_{\ell=0}^{t'}\mathcal{L}_\ell\colon L'\subset K}\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(L')\Bigg).$$

Now we use the crucial fact that $\omega_{\mathcal{P}}$ is *additive* and finite to obtain from this that

$$\begin{aligned}(11.20)\qquad \omega_{\mathcal{P}}\Big(\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(K)\setminus\bigcup_{L'\in\bigcup_{\ell=0}^{t'}\mathcal{L}_\ell\colon L'\subset K}\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(L')\Big)&\\ =\omega_{\mathcal{P}}(\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(K))-\omega_{\mathcal{P}}\Bigg(\bigcup_{L'\in\bigcup_{\ell=0}^{t'}\mathcal{L}_\ell\colon L'\subset K}\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(L')\Bigg)&\\ \leq(\rho-1)\,\omega_{\mathcal{P}}\Bigg(\bigcup_{L'\in\bigcup_{\ell=0}^{t'}\mathcal{L}_\ell\colon L'\subset K}\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(L')\Bigg).&\end{aligned}$$

Thus using

$$\omega_{\mathcal{P}_{L,0}^{\text{small}}}(\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(K))\leq\omega_{\mathcal{P}}\Big(\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(K)\setminus\bigcup_{L'\in\bigcup_{\ell=0}^{t'}\mathcal{L}_\ell\colon L'\subset K}\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(L')\Big),$$

and (11.20) we have

$$\begin{aligned}\mathcal{S}_{\text{size}}^{\alpha,A}(\mathcal{P}_{L,0}^{\text{small}})^2\leq&\sup_{K\in\Pi^{\text{goodbelow}}\mathcal{P}_{L,0}^{\text{small}}}\frac{1}{|K|_\sigma}\Big(\frac{\mathrm{P}^\alpha(K,\mathbf{1}_{A\setminus K}\sigma)}{|K|^{1/n}}\Big)^2\\&\times\omega_{\mathcal{P}}\Big(\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(K)\setminus\bigcup_{L'\in\bigcup_{\ell=0}^{t'}\mathcal{L}_\ell\colon L'\subset K}\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(L')\Big)\\ \leq&(\rho-1)\sup_{K\in\Pi^{\text{goodbelow}}\mathcal{P}_{L,0}^{\text{small}}}\frac{1}{|K|_\sigma}\Big(\frac{\mathrm{P}^\alpha(K,\mathbf{1}_{A\setminus K}\sigma)}{|K|^{1/n}}\Big)^2\\&\times\omega_{\mathcal{P}}\Big(\bigcup_{L'\in\bigcup_{\ell=0}^{t'}\mathcal{L}_\ell\colon L'\subset K}\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(L')\Big).\end{aligned}$$

and we can continue with

$$\begin{aligned}\mathcal{S}_{\text{size}}^{\alpha,A}(\mathcal{P}_{L,0}^{\text{small}})&\leq(\rho-1)\sup_{K\in\Pi^{\text{goodbelow}}\mathcal{P}}\frac{1}{|K|_\sigma}\Big(\frac{\mathrm{P}^\alpha(K,\mathbf{1}_{A\setminus K}\sigma)}{|K|^{1/n}}\Big)^2\omega_{\mathcal{P}}(\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(K))\\&\leq(\rho-1)\,\mathcal{S}_{\text{size}}^{\alpha,A}(\mathcal{P})^2.\end{aligned}$$

In the remaining case where $L \in \mathcal{L}_{M+1}$ we can include L as a testing cube K and the same reasoning applies. This completes the proof of (11.19).

To prove the other inequality (11.16), we need a lemma to bound the norm of certain 'straddled' stopping forms by the size functional $\mathcal{S}_{\text{size}}^{\alpha,A}$, and another lemma to bound sums of 'mutually orthogonal' stopping forms. We interrupt the proof to turn to these matters. □

11.4.1. The straddling lemma. Given an admissible collection of pairs $\mathcal{Q}$ for A, and a subpartition $\mathcal{S} \subset \Pi^{\text{goodbelow}}\mathcal{Q}$ of pairwise disjoint cubes in A, we say that $\mathcal{Q}$ *$\boldsymbol{\tau}$-straddles* $\mathcal{S}$ if for every pair $(I,J) \in \mathcal{Q}$ there is $S \in \mathcal{S} \cap [J,I]$ where $[J,I]$ denotes the geodesic in the dyadic tree $\mathcal{D}$ that connects J to I, and moreover that $J \Subset_{\boldsymbol{\tau}} S$. Denote by $\mathcal{N}_{\boldsymbol{\rho}-1-\boldsymbol{\tau}}^{\text{good}}(S)$ the finite collection of cubes that are both good and $(\boldsymbol{\rho}-1-\boldsymbol{\tau})$-nearby in S. For any good dyadic cube $S \in \mathcal{D}_{\text{good}}$, we will also need the collection $\mathcal{W}^{\text{good}}(S)$ of maximal *good* subcubes I of S whose triples $3I$ are contained in S.

Lemma 11.10. *Let $\mathcal{S}$ be a subpartition of A, and suppose that $\mathcal{Q}$ is an admissible collection of pairs for A such that $\mathcal{S} \subset \Pi^{\text{goodbelow}}\mathcal{Q}$, and such that $\mathcal{Q}$ $\boldsymbol{\tau}$-straddles $\mathcal{S}$. Then we have the sublinear form bound*

$$\mathfrak{N}_{\text{stop},1,\triangle}^{A,\mathcal{Q}} \leq C_{\mathbf{r},\boldsymbol{\tau},\boldsymbol{\rho}} \sup_{S\in\mathcal{S}} \mathcal{S}_{\text{size}}^{\alpha,A;S}(\mathcal{Q}) \leq C_{\mathbf{r},\boldsymbol{\tau},\boldsymbol{\rho}}\, \mathcal{S}_{\text{size}}^{\alpha,A}(\mathcal{Q}),$$

where $\mathcal{S}_{\text{size}}^{\alpha,A;S}$ is an S-localized version of $\mathcal{S}_{\text{size}}^{\alpha,A}$ with an S-hole given by

$$\mathcal{S}_{\text{size}}^{\alpha,A;S}(\mathcal{Q})^2 \equiv \sup_{K\in\mathcal{N}_{\boldsymbol{\rho}-1-\boldsymbol{\tau}}^{\text{good}}(S)\cup\mathcal{W}^{\text{good}}(S)} \frac{1}{|K|_\sigma}\left(\frac{\mathrm{P}^\alpha(K,\mathbf{1}_{A\setminus S}\sigma)}{|K|^{1/n}}\right)^2 \omega_{\mathcal{Q}}(\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(K)). \tag{11.21}$$

Proof. For $S \in S$ let $\mathcal{Q}^S \equiv \{(I,J) \in \mathcal{Q} : J \Subset_{\boldsymbol{\tau}} S \subset I\}$. We begin by using that $\mathcal{Q}$ $\boldsymbol{\tau}$-straddles $\mathcal{S}$, together with the sublinearity property (11.5) of $\varphi_J^{\mathcal{Q}}$, to write

$$\begin{aligned}|\mathsf{B}|_{\text{stop},1,\triangle}^{A,\mathcal{Q}}(f,g) &= \sum_{J\in\Pi_2\mathcal{P}} \frac{\mathrm{P}^\alpha(J,|\varphi_J^{\mathcal{Q}}|\mathbf{1}_{A\setminus I_{\mathcal{Q}}(J)}\sigma)}{|J|^{1/n}} \|\triangle_J^\omega \mathbf{x}\|_{L^2(\omega)} \|\triangle_J^\omega g\|_{L^2(\omega)} \\ &\leq \sum_{S\in\mathcal{S}} \sum_{J\in\Pi_2^{S,\boldsymbol{\tau}\text{-deep}}\mathcal{Q}} \frac{\mathrm{P}^\alpha(J,|\varphi_J^{\mathcal{Q}^S}|\mathbf{1}_{A\setminus I_{\mathcal{Q}}(J)}\sigma)}{|J|^{1/n}} \|\triangle_J^\omega \mathbf{x}\|_{L^2(\omega)} \|\triangle_J^\omega g\|_{L^2(\omega)};\end{aligned}$$

where

$$\varphi_J^{\mathcal{Q}^S} \equiv \sum_{I\in\Pi_1\mathcal{Q}^S:\ (I,J)\in\mathcal{Q}^S} \mathbb{E}_I^\sigma(\triangle_{\pi I}^\sigma f)\,\mathbf{1}_{A\setminus I}.$$

At this point, with S fixed for the moment, we consider separately the finitely many cases $|J|^{1/n} = 2^{-s}|S|^{1/n}$ where $s \geq \boldsymbol{\rho}-1$ and where $\boldsymbol{\tau} \leq s < \boldsymbol{\rho}-1$. More precisely, we pigeonhole the side length of $J \in \Pi_2\mathcal{Q}^S = \Pi_2^{S,\boldsymbol{\tau}\text{-deep}}\mathcal{Q}$ by

$$\begin{aligned}\mathcal{Q}_*^S &\equiv \left\{(I,J) \in \mathcal{Q}^S : J \in \Pi_2\mathcal{Q}^S \text{ and } |J|^{1/n} \leq 2^{-\boldsymbol{\rho}}|S|^{1/n}\right\}, \\ \mathcal{Q}_s^S &\equiv \left\{(I,J) \in \mathcal{Q}^S : J \in \Pi_2\mathcal{Q}^S \text{ and } |J|^{1/n} = 2^{-s}|S|^{1/n}\right\}, \quad \boldsymbol{\tau} \leq s < \boldsymbol{\rho}-1.\end{aligned}$$

Then we have

$$\Pi_2\mathcal{Q}^S_* \equiv \left\{J\in\Pi_2\mathcal{Q}^S : |J|^{1/n}\leq 2^{-\boldsymbol{\rho}}|S|^{1/n}\right\},$$
$$\Pi_2\mathcal{Q}^S_s \equiv \left\{J\in\Pi_2\mathcal{Q}^S : |J|^{1/n}= 2^{-s}|S|^{1/n}\right\},\quad \boldsymbol{\tau}\leq s<\boldsymbol{\rho}-1,$$

and we make the corresponding decomposition for the sublinear form

$$|\mathsf{B}|^{A,\mathcal{Q}}_{\mathrm{stop},1,\triangle}(f,g) = |\mathsf{B}|^{A,\mathcal{Q}_*}_{\mathrm{stop},1,\triangle}(f,g)+\sum_{\boldsymbol{\tau}\leq s<\boldsymbol{\rho}}|\mathsf{B}|^{A,\mathcal{Q}_s}_{\mathrm{stop},1,\triangle}(f,g)$$
$$\equiv \sum_{S\in\mathcal{S}}\sum_{J\in\Pi_2\mathcal{Q}^S_*}\frac{\mathrm{P}^\alpha(J,|\varphi_J^{\mathcal{Q}^S_*}|\mathbf{1}_{A\setminus I_{\mathcal{Q}_*}(J)}\sigma)}{|J|^{1/n}}\|\triangle^\omega_J\mathbf{x}\|_{L^2(\omega)}\|\triangle^\omega_J g\|_{L^2(\omega)}$$
$$+\sum_{\boldsymbol{\tau}\leq s<\boldsymbol{\rho}-1}\sum_{S\in\mathcal{S}}\sum_{J\in\Pi_2\mathcal{Q}^S_s}\frac{\mathrm{P}^\alpha(J,|\varphi_J^{\mathcal{Q}^S_s}|\mathbf{1}_{A\setminus I_{\mathcal{Q}_S}(J)}\sigma)}{|J|^{1/n}}\|\triangle^\omega_J\mathbf{x}\|_{L^2(\omega)}\|\triangle^\omega_J g\|_{L^2(\omega)}.$$

By the tree-connected property of $\mathcal{Q}$, and the telescoping property of martingale differences, together with the bound $\alpha_{\mathcal{A}}(A)$ on the averages of f in the corona $\mathcal{C}_A$, we have

$$|\varphi_J^{\mathcal{Q}^S_*}|,|\varphi_J^{\mathcal{Q}^S_s}|\lesssim\alpha_{\mathcal{A}}(A)\,\mathbf{1}_{A\setminus I_{\mathcal{Q}^S}(J)},\tag{11.22}$$

where $I_{\mathcal{Q}^S}(J)\equiv\bigcap\{I:(I,J)\in\mathcal{Q}^S\}$ is the smallest cube I for which $(I,J)\in\mathcal{Q}^S$.

Case for $|\mathsf{B}|^{A,\mathcal{Q}^S_s}_{\mathrm{stop},1,\triangle}(f,g)$ when $\boldsymbol{\tau}\leq s\leq\boldsymbol{\rho}-1$. Now is a crucial definition that permits us to bound the form by the size functional with a large hole. Let

$$\mathcal{C}^S_s\equiv\pi^{\boldsymbol{\tau}}(\Pi_2\mathcal{Q}^S_s)$$

be the collection of $\boldsymbol{\tau}$-parents of cubes in $\Pi_2\mathcal{Q}^S_s$, and denote by $\mathcal{M}^S_s$ the set of *maximal* cubes in the collection $\mathcal{C}^S_s$. We have that the cubes in $\mathcal{M}^S_s$ are good by our assumption that the Haar support of g is contained in the $\boldsymbol{\tau}$-good grid grid $\mathcal{D}^{\boldsymbol{\tau}}_{(\mathbf{r},\varepsilon)\text{-good}}$, and so $\mathcal{M}^S_s\subset\mathcal{N}_{\boldsymbol{\rho}-\boldsymbol{\tau}}(S)$. Here is the first of two key inclusions:

$$J\Subset_{\boldsymbol{\tau}}K\subset S\quad\text{if } K\in\mathcal{M}^S_s \text{ is the unique cube containing } J.\tag{11.23}$$

Let $I_s\equiv\pi^{\boldsymbol{\rho}-1-s}S$, so that for each J in $\Pi_2\mathcal{Q}^S_s$ we have the second key inclusion:

$$\pi^{\boldsymbol{\rho}-1}J=I_s\subset I_{\mathcal{Q}^S}(J).\tag{11.24}$$

Now each $K\in\mathcal{M}^S_s$ is also $(\boldsymbol{\rho}-1-\boldsymbol{\tau})$-deeply embedded in I_s if $\boldsymbol{\rho}-1\geq\mathbf{r}+\boldsymbol{\tau}$, so that in particular, $3K\subset I_s$. This and (11.24) have the consequence that the following Poisson inequalities hold:

$$\frac{\mathrm{P}^\alpha(J,\mathbf{1}_{A\setminus I_{\mathcal{Q}^S}(J)}\sigma)}{|J|^{m/n}}\lesssim\frac{\mathrm{P}^\alpha(J,\mathbf{1}_{A\setminus I_s}\sigma)}{|J|^{m/n}}\lesssim\frac{\mathrm{P}^\alpha(K,\mathbf{1}_{A\setminus I_s}\sigma)}{|K|^{m/n}}\lesssim\frac{\mathrm{P}^\alpha(K,\mathbf{1}_{A\setminus S}\sigma)}{|K|^{m/n}}.$$

Let

$$\Pi_2\mathcal{Q}^S_s(K)\equiv\{J\in\Pi_2\mathcal{Q}^S_s : J\subset K\},$$

and let

$$\left[\Pi_2 \mathcal{Q}_s^S\right]_\ell \equiv \left\{J \in \Pi_2 \mathcal{Q}_s^S : |J'|^{1/n} = 2^{-\ell} |K|^{1/n}\right\},$$
$$\left[\Pi_2 \mathcal{Q}_s^S\right]_\ell^* \equiv \left\{J' : J' \subset J \in \Pi_2 \mathcal{Q}_s^S : |J'|^{1/n} = 2^{-\ell} |K|^{1/n}\right\}.$$

Now set $\mathcal{Q}_s \equiv \bigcup_{S \in \mathcal{S}} \mathcal{Q}_s^S$. We apply (11.22) and Cauchy–Schwarz in J to bound $|\mathsf{B}|_{\mathrm{stop},1,\triangle}^{A,\mathcal{Q}_s}(f,g)$ by

$$\alpha_{\mathcal{A}}(A) \sum_{S \in \mathcal{S}} \sum_{K \in \mathcal{M}_s^S} \left(\frac{\mathrm{P}^\alpha(K, \mathbf{1}_{A \setminus S}\sigma)}{|K|^{1/n}}\right) \left\|\mathsf{P}^\omega_{\Pi_2^{S,\boldsymbol{\tau}\text{-deep}} \mathcal{Q}_s;K} \mathbf{x}\right\|_{L^2(\omega)} \left\|\mathsf{P}^\omega_{\Pi_2^{S,\boldsymbol{\tau}\text{-deep}} \mathcal{Q}_s;K} g\right\|_{L^2(\omega)},$$

where the localized projections $\mathsf{P}^\omega_{\Pi_2^{S,\boldsymbol{\tau}\text{-deep}} \mathcal{Q}_s;K}$ are defined in (9.1) above.

Thus using Cauchy–Schwarz in K we have that $|\mathsf{B}_{\mathrm{stop},1,\triangle}^{A,\mathcal{Q}_s}(f,g)|$ is bounded by

$$\begin{aligned}
&\alpha_{\mathcal{A}}(A) \sum_{S \in \mathcal{S}} \sum_{K \in \mathcal{M}_s^S} \sqrt{|K|_\sigma} \\
&\qquad\qquad \times \frac{1}{\sqrt{|K|_\sigma}} \left(\frac{\mathrm{P}^\alpha(K, \mathbf{1}_{A \setminus S}\sigma)}{|K|^{1/n}}\right) \left\|\mathsf{P}^\omega_{\Pi_2 \mathcal{Q}_s^S(K)} \mathbf{x}\right\|_{L^2(\omega)} \left\|\mathsf{P}^\omega_{\Pi_2 \mathcal{Q}_s^S(K)} g\right\|_{L^2(\omega)} \\
&\qquad \leq \alpha_{\mathcal{A}}(A) \sup_{S \in \mathcal{S}} \mathcal{S}_{\mathrm{size}}^{\alpha,A;S}(\mathcal{Q}) \left(\sum_{S \in \mathcal{S}} \sum_{K \in \mathcal{N}_{\boldsymbol{\rho}-\boldsymbol{\tau}}(S)} |K|_\sigma\right)^{1/2} \|g\|_{L^2(\omega)} \\
&\qquad \leq \sup_{S \in \mathcal{S}} \mathcal{S}_{\mathrm{size}}^{\alpha,A;S}(\mathcal{Q})\, \alpha_{\mathcal{A}}(A) \sqrt{|A|_\sigma}\, \|g\|_{L^2(\omega)},
\end{aligned}$$

since $J \Subset_{\boldsymbol{\tau}} M \subset K$ by (11.23), since $\mathcal{M}_s^S \subset \mathcal{N}_{\boldsymbol{\rho}-1-\boldsymbol{\tau}}(S)$, and since the collection of cubes $\bigcup_{S \in \mathcal{S}} \mathcal{M}_s^S$ is pairwise disjoint in A.

Case for $|\mathsf{B}|_{\mathrm{stop},1,\triangle}^{A,\mathcal{Q}_*}(f,g)$. This time we let $\mathcal{C}_*^S \equiv \pi^{\boldsymbol{\tau}}(\Pi_2 \mathcal{Q}_*^S)$ and denote by $\mathcal{M}_*^S$ the set of *maximal* cubes in the collection $\mathcal{C}_*^S$. We have the two key inclusions:

$$J \Subset_{\boldsymbol{\tau}} M \Subset_{\boldsymbol{\rho}-1-\boldsymbol{\tau}} S \quad \text{if } M \in \mathcal{M}_*^S \text{ is the unique cube containing } J,$$

and

$$\pi^{\boldsymbol{\rho}} J \subset S \subset I_{\mathcal{Q}}(J).$$

Moreover there is $K \in \mathcal{W}^{\mathrm{good}}(S)$ that contains M. Thus $3K \subset S$ and we have

$$\frac{\mathrm{P}^\alpha(J, \mathbf{1}_{A \setminus S}\sigma)}{|J|^{1/n}} \lesssim \frac{\mathrm{P}^\alpha(K, \mathbf{1}_{A \setminus S}\sigma)}{|K|^{1/n}},$$

and

$$|\varphi_J| \lesssim \alpha_{\mathcal{A}}(A)\, \mathbf{1}_{A \setminus S}.$$

Now set $\mathcal{Q}_* \equiv \bigcup_{S\in\mathcal{S}} \mathcal{Q}_*^S$. Arguing as above, but with $\mathcal{W}^{\mathrm{good}}(S)$ in place of $\mathcal{N}_{\boldsymbol{\rho}-1-\boldsymbol{\tau}}(S)$, and using $J \Subset_{\boldsymbol{\rho}-1} I_{\mathcal{Q}}(J)$, we can bound $|\mathsf{B}|^{A,\mathcal{Q}_*}_{\mathrm{stop},1,\triangle}(f,g)$ by

$$\alpha_{\mathcal{A}}(A) \sum_{S\in\mathcal{S}} \sum_{K\in\mathcal{W}^{\mathrm{good}}(S)} \sqrt{|K|_\sigma}$$
$$\times \frac{1}{\sqrt{|K|_\sigma}} \Big(\frac{\mathrm{P}^\alpha(K, \mathbf{1}_{A\setminus S}\sigma)}{|K|^{1/n}} \Big) \left\| \mathsf{P}^\omega_{\Pi_2 \mathcal{Q}_s^S(K)} \mathbf{x} \right\|_{L^2(\omega)} \| \mathsf{P}^\omega_{\Pi_2 \mathcal{Q}_*^S(K)} g \|_{L^2(\omega)}$$
$$\leq \alpha_{\mathcal{A}}(A) \sup_{S\in\mathcal{S}} \mathcal{S}^{\alpha,A;S}_{\mathrm{size}}(\mathcal{Q}) \Big(\sum_{S\in\mathcal{S}} \sum_{K\in\mathcal{W}^{\mathrm{good}}(S)} |K|_\sigma \Big)^{1/2} \|g\|_{L^2(\omega)}$$
$$\leq \sup_{S\in\mathcal{S}} \mathcal{S}^{\alpha,A;S}_{\mathrm{size}}(\mathcal{Q})\, \alpha_{\mathcal{A}}(A) \sqrt{|A|_\sigma}\, \|g\|_{L^2(\omega)} .$$

We now sum these bounds in s and $*$ and use $\sup_{S\in\mathcal{S}} \mathcal{S}^{\alpha,A;S}_{\mathrm{size}}(\mathcal{Q}) \leq \mathcal{S}^{\alpha,A}_{\mathrm{size}}(\mathcal{Q})$ to complete the proof of Lemma 11.10. □

11.4.2. The orthogonality lemma. Given a set $\{\mathcal{Q}_m\}_{m=0}^\infty$ of admissible collections for A, we say that the collections $\mathcal{Q}_m$ are *mutually orthogonal*, if each collection $\mathcal{Q}_m$ satisfies

$$\mathcal{Q}_m \subset \bigcup_{j=0}^{\infty} \{\mathcal{A}_{m,j} \times \mathcal{B}_{m,j}\} ,$$

where the sets $\{\mathcal{A}_{m,j}\}_{m,j}$ and $\{\mathcal{B}_{m,j}\}_{m,j}$ each have bounded overlap on the dyadic grid $\mathcal{D}$:

$$\sum_{m,j=0}^{\infty} \mathbf{1}_{\mathcal{A}_{m,j}} \leq A\, \mathbf{1}_{\mathcal{D}} \quad \text{and} \quad \sum_{m,j=0}^{\infty} \mathbf{1}_{\mathcal{B}_{m,j}} \leq B\, \mathbf{1}_{\mathcal{D}} .$$

Lemma 11.11. *Suppose that* $\{\mathcal{Q}_m\}_{m=0}^\infty$ *is a set of admissible collections for* A *that are* mutually orthogonal. *Then if* $\mathcal{Q} \equiv \bigcup_{m=0}^\infty \mathcal{Q}_m$, *the sublinear stopping form* $|\mathsf{B}|^{A,\mathcal{Q}}_{\mathrm{stop},1,\triangle}(f,g)$ *has its restricted norm* $\mathfrak{N}^{A,\mathcal{Q}}_{\mathrm{stop},1,\triangle}$ *controlled by the* supremum *of the restricted norms* $\mathfrak{N}^{A,\mathcal{Q}_m}_{\mathrm{stop},1,\triangle}$:

$$\mathfrak{N}^{A,\mathcal{Q}}_{\mathrm{stop},1,\triangle} \leq \sqrt{nAB} \sup_{m\geq 0} \mathfrak{N}^{A,\mathcal{Q}_m}_{\mathrm{stop},1,\triangle} .$$

Proof. If $\mathsf{P}^\sigma_m = \sum_{j\geq 0} \sum_{I\in\mathcal{A}_{m,j}} \triangle^\sigma_{\pi I}$ (note the parent πI in the projection $\triangle^\sigma_{\pi I}$ because of our 'change of dummy variable' in (11.1)) and $\mathsf{P}^\omega_m = \sum_{j\geq 0} \sum_{J\in\mathcal{B}_{m,j}} \triangle^\omega_J$, then we have

$$\mathsf{B}^{A,\mathcal{Q}_m}_{\mathrm{stop}}(f,g) = \mathsf{B}^{A,\mathcal{Q}_m}_{\mathrm{stop}}(\mathsf{P}^\sigma_m f, \mathsf{P}^\omega_m g) ,$$

and

$$\sum_{m\geq 0} \|\mathsf{P}^\sigma_m f\|^2_{L^2(\sigma)} \leq \sum_{m\geq 0} \sum_{j\geq 0} \|\mathsf{P}^\sigma_{\mathcal{A}_{m,j}} f\|^2_{L^2(\sigma)} \leq An\, \|f\|^2_{L^2(\sigma)},$$
$$\sum_{m\geq 0} \|\mathsf{P}^\omega_m g\|^2_{L^2(\sigma)} \leq \sum_{m\geq 0} \sum_{j\geq 0} \|\mathsf{P}^\omega_{\mathcal{B}_{m,j}} g\|^2_{L^2(\omega)} \leq B\, \|g\|^2_{L^2(\omega)} .$$

The sublinear inequality (11.5) and Cauchy–Schwarz now give

$$\begin{aligned}|\mathsf{B}|^{A,\mathcal{Q}}_{\text{stop},1,\triangle}(f,g) &\leq \sum_{m\geq 0} |\mathsf{B}|^{A,\mathcal{Q}_m}_{\text{stop},1,\triangle}(f,g) \leq \sum_{m\geq 0} \mathfrak{N}^{A,\mathcal{Q}_m}_{\text{stop}} \|\mathsf{P}^{\sigma}_m f\|_{L^2(\sigma)} \|\mathsf{P}^{\omega}_m g\|_{L^2(\sigma)} \\ &\leq \Big(\sup_{m\geq 0} \mathfrak{N}^{A,\mathcal{Q}_m}_{\text{stop},1,\triangle}\Big) \Big(\sum_{m\geq 0} \|\mathsf{P}^{\sigma}_m f\|^2_{L^2(\sigma)}\Big)^{1/2} \Big(\sum_{m\geq 0} \|\mathsf{P}^{\omega}_m g\|^2_{L^2(\sigma)}\Big)^{1/2} \\ &\leq \Big(\sup_{m\geq 0} \mathfrak{N}^{A,\mathcal{Q}_m}_{\text{stop},1,\triangle}\Big) \sqrt{nAB}\sqrt{n}\, \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)} \,.\end{aligned}$$

□

11.4.3. Completion of the proof. Now we return to the proof of inequality (11.16) in Proposition 11.8.

Proof of (11.16). Recall that

$$\begin{aligned}\mathcal{P}^{\text{big}} &= \Big\{\bigcup_{L\in\mathcal{L}} \mathcal{P}^{\text{big}}_{L,0}\Big\} \bigcup \Big\{\bigcup_{t\geq 1}\bigcup_{L\in\mathcal{L}} \mathcal{P}_{L,t}\Big\} \equiv \mathcal{Q}^{\text{big}}_0 \bigcup \mathcal{Q}^{big}_1 \,; \\ \mathcal{Q}^{\text{big}}_0 &\equiv \bigcup_{L\in\mathcal{L}} \mathcal{P}^{\text{big}}_{L,0}\,, \quad \mathcal{Q}^{\text{big}}_1 \equiv \bigcup_{t\geq 1} \mathcal{P}^{\text{big}}_t, \quad \mathcal{P}^{\text{big}}_t \equiv \bigcup_{L\in\mathcal{L}} \mathcal{P}_{L,t}\,.\end{aligned}$$

We first consider the collection $\mathcal{Q}^{\text{big}}_0 = \bigcup_{L\in\mathcal{L}} \mathcal{P}^{\text{big}}_{L,0}$, and claim that

$$\mathfrak{N}^{A,\mathcal{P}^{\text{big}}_{L,0}}_{\text{stop},1,\triangle} \leq C\,\mathcal{S}^{\alpha,A}_{\text{size}}(\mathcal{P}^{\text{big}}_{L,0}) \leq C\,\mathcal{S}^{\alpha,A}_{\text{size}}(\mathcal{P}), \quad L\in\mathcal{L}\,. \tag{11.25}$$

To see this we note that $\mathcal{P}^{\text{big}}_{L,0}$ $\boldsymbol{\tau}$-straddles the trivial collection $\{L\}$ consisting of a single cube, since the pairs (I,J) that arise in $\mathcal{P}^{\text{big}}_{L,0}$ have $I=L$ and J in the shifted corona $\mathcal{C}^{\boldsymbol{\tau}\text{-shift}}_I$. Thus we can apply Lemma 11.10 with $\mathcal{Q}=\mathcal{P}^{\text{big}}_{L,0}$ and $\mathcal{S}=\{L\}$ to obtain (11.25).

Next, we observe that the collections $\mathcal{P}^{\text{big}}_{L,0}$ are *mutually orthogonal*, namely

$$\mathcal{P}^{\text{big}}_{L,0} \subset \mathcal{C}_L \times \mathcal{C}^{\boldsymbol{\tau}\text{-shift}}_L\,, \quad \sum_{L\in\mathcal{L}} \mathbf{1}_{\mathcal{C}_L} \leq 1 \quad \text{and} \quad \sum_{L\in\mathcal{L}} \mathbf{1}_{\mathcal{C}^{\boldsymbol{\tau}\text{-shift}}_L} \leq \boldsymbol{\tau}\,.$$

Thus the orthogonality Lemma 11.11 shows that

$$\mathfrak{N}^{A,\mathcal{Q}^{\text{big}}_0}_{\text{stop},1,\triangle} \leq \sqrt{n\boldsymbol{\tau}}\,\sup_{L\in\mathcal{L}} \mathfrak{N}^{A,\mathcal{P}^{\text{big}}_{L,0}}_{\text{stop},1,\triangle} \leq \sqrt{n\boldsymbol{\tau}}\,C\,\mathcal{S}^{\alpha,A}_{\text{size}}(\mathcal{P})\,.$$

Now we turn to the collection

$$\mathcal{Q}^{\text{big}}_1 = \bigcup_{t\geq 1}\bigcup_{L\in\mathcal{L}} \mathcal{P}_{L,t} = \bigcup_{t\geq 1} \mathcal{P}^{\text{big}}_t\,; \quad \mathcal{P}^{\text{big}}_t \equiv \bigcup_{L\in\mathcal{L}} \mathcal{P}_{L,t}\,, \quad t\geq 0\,.$$

We claim that

$$\mathfrak{N}^{A,\mathcal{P}^{\text{big}}_t}_{\text{stop},1,\triangle} \leq C\,\rho^{-t/2}\,\mathcal{S}^{\alpha,A}_{\text{size}}(\mathcal{P}), \quad t\geq 1\,. \tag{11.26}$$

Note that with this claim established, we have

$$\mathfrak{N}_{\mathrm{stop},1,\triangle}^{A,\mathcal{P}^{\mathrm{big}}} \leq \mathfrak{N}_{\mathrm{stop},1,\triangle}^{A,\mathcal{Q}_0^{\mathrm{big}}} + \mathfrak{N}_{\mathrm{stop},1,\triangle}^{A,\mathcal{Q}_1^{\mathrm{big}}} \leq \mathfrak{N}_{\mathrm{stop},1,\triangle}^{A,\mathcal{Q}_0^{\mathrm{big}}} + \sum_{t=1}^{\infty} \mathfrak{N}_{\mathrm{stop},1,\triangle}^{A,\mathcal{P}_t^{\mathrm{big}}} \leq C_{\boldsymbol{\rho}}\, \mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{P}),$$

which proves (11.16) if we apply the orthogonal Lemma 11.11 to the set of collections $\{\mathcal{P}_{L,0}^{\mathrm{small}}\}_{L\in\mathcal{L}}$, which is mutually orthogonal since $\mathcal{P}_{L,0}^{\mathrm{small}} \subset \mathcal{C}_L' \times \mathcal{C}_L^{\boldsymbol{\tau}\text{-shift}}$. With this the proof of Proposition 11.8 is now complete since $\rho = 1+\varepsilon$. Thus it remains only to show that (11.26) holds.

The cases $1 \leq t \leq \mathbf{r}+1$ can be handled with relative ease since decay in t is not needed there. Indeed, $\mathcal{P}_{L,t}$ $\boldsymbol{\tau}$-straddles the collection $\mathfrak{C}_{\mathcal{L}}(L)$ of $\mathcal{L}$-children of L, and so the straddling lemma applies to give

$$\mathfrak{N}_{\mathrm{stop},1,\triangle}^{A,\mathcal{P}_{L,t}} \leq C\, \mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{P}_{L,t}) \leq C\, \mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{P}),$$

and then the orthogonality Lemma 11.11 applies to give

$$\mathfrak{N}_{\mathrm{stop},1,\triangle}^{A,\mathcal{P}_t^{\mathrm{big}}} \leq \sqrt{n\boldsymbol{\tau}} \sup_{L\in\mathcal{L}} \mathfrak{N}_{\mathrm{stop},1,\triangle}^{A,\mathcal{P}_{L,t}} \leq C\sqrt{n\boldsymbol{\tau}}\, \mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{P}),$$

since $\{\mathcal{P}_{L,t}\}_{L\in\mathcal{L}}$ is mutually orthogonal as $\mathcal{P}_{L,t} \subset \mathcal{C}_L \times \mathcal{C}_{L'}^{\boldsymbol{\tau}\text{-shift}}$ with $L \in \mathcal{G}_d$ and $L' \in \mathcal{G}_{d+t}$ for depth $d = d(L)$.

Now we consider the case $t \geq \mathbf{r}+2$, where it is essential to obtain decay in t. We again apply Lemma 11.10 to $\mathcal{P}_{L,t}$ with $\mathcal{S} = \mathfrak{C}_{\mathcal{L}}(L)$, but this time we must use the stronger localized bounds $\mathcal{S}_{\mathrm{size}}^{\alpha,A;S}$ with an S-hole, that give

$$\mathfrak{N}_{\mathrm{stop},1,\triangle}^{A,\mathcal{P}_{L,t}} \leq C \sup_{S\in\mathfrak{C}_{\mathcal{L}}(L)} \mathcal{S}_{\mathrm{size}}^{\alpha,A;S}(\mathcal{P}_{L,t}), \quad t \geq 0. \tag{11.27}$$

Fix $L \in \mathcal{G}_d$. Now we note that if $J \in \Pi_2^{L,\boldsymbol{\tau}\text{-deep}}\mathcal{P}_{L,t}$ then J belongs to the $\boldsymbol{\tau}$-shifted corona $\mathcal{C}_{L^{d+t}}^{\boldsymbol{\tau}\text{-shift}}$ for some cube $L^{d+t} \in \mathcal{G}_{d+t}$. Then $\pi^{\boldsymbol{\tau}}J$ is $\boldsymbol{\tau}$ levels above J, hence in the corona $\mathcal{C}_{L^{d+t}}$. This cube L^{d+t} lies in some child $S \in \mathcal{S} = \mathfrak{C}_{\mathcal{L}}(L)$. So fix $S \in \mathcal{S}$ and a cube $L^{d+t} \in \mathcal{G}_{d+t}$ that is contained in S with $t \geq \mathbf{r}+2$. Now the cubes K that arise in the supremum defining $\mathcal{S}_{\mathrm{size}}^{\alpha,A;S}(\mathcal{P}_{L,t})$ in (11.21) belong to either $\mathcal{N}_{\boldsymbol{\rho}-\boldsymbol{\tau}}(S)$ or $\mathcal{W}^{\mathrm{good}}(S)$. We will consider these two cases separately.

So first suppose that $K \in \mathcal{N}_{\boldsymbol{\rho}-1-\boldsymbol{\tau}}(S)$. A simple induction on levels yields

$$\begin{aligned}
\omega_{\mathcal{P}_{L,t}}(\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(K)) &= \sum_{\substack{J\in\Pi_2^{S,\boldsymbol{\tau}\text{-deep}}\mathcal{P}_{L,t}\\ J\subset K}} \|\triangle_J^{\omega}\mathbf{x}\|_{L^2(\omega)}^2 \\
&\leq \omega_{\mathcal{P}}\Big(\bigcup_{L^{d+t}\in\mathcal{G}_{d+t}:\,L^{d+t}\subset K} \mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(L^{d+t})\Big) \\
&\leq \frac{1}{\rho}\,\omega_{\mathcal{P}}\Big(\bigcup_{L^{d+t-1}\in\mathcal{G}_{d+t-1}:\,L^{d+t-1}\subset K} \mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(L^{d+t-1})\Big) \\
&\vdots \\
&\lesssim \boldsymbol{\rho}^{-(t-\boldsymbol{\rho}-\boldsymbol{\tau})}\,\omega_{\mathcal{P}}(\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(K)), \quad t \geq \boldsymbol{\rho}-1-\boldsymbol{\tau}+2.
\end{aligned}$$

Thus we have

$$\frac{1}{|K|_\sigma}\Big(\frac{\mathrm{P}^\alpha(K,\mathbf{1}_{A\setminus S}\sigma)}{|K|^{1/n}}\Big)^2\omega_{\mathcal{P}_{L,t}}(\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(K))$$
$$\lesssim \boldsymbol{\rho}^{-t}\frac{1}{|K|_\sigma}\Big(\frac{\mathrm{P}^\alpha(K,\mathbf{1}_{A\setminus S}\sigma)}{|K|^{1/n}}\Big)^2\omega_{\mathcal{P}}(\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(K))\lesssim \boldsymbol{\rho}^{-t}\,\mathcal{S}_{\text{size}}^{\alpha,A}(\mathcal{P})^2\,.$$

Now suppose that $K\in\mathcal{W}^{\text{good}}(S)$ and that $J\in\Pi_2^{S,\boldsymbol{\tau}\text{-deep}}\mathcal{P}_{L,t}$ and $J\subset K$. There is a unique cube $L^{d+\mathbf{r}+1}\in\mathcal{G}_{d+\mathbf{r}+1}$ such that $J\subset L^{d+\mathbf{r}+1}\subset S$. Now $L^{d+\mathbf{r}+1}$ is good so $L^{d+\mathbf{r}+1}\Subset_{\mathbf{r}} S$. Thus in particular $3L^{d+\mathbf{r}+1}\subset S$ so that $L^{d+\mathbf{r}+1}\subset K$. The above simple induction applies here to give

$$\sum_{\substack{J\in\Pi_2^{S,\boldsymbol{\tau}\text{-deep}}\mathcal{P}_{L,t}\\ J\subset L^{d+\mathbf{r}+1}}}\|\,\triangle_J^\omega\,\mathbf{x}\|^2_{L^2(\omega)}\le\omega_{\mathcal{P}}\Big(\bigcup_{L^{d+t}\in\mathcal{G}_{d+t}:\,L^{m-t}\subset L^{d+\mathbf{r}+1}}\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(L^{d+t})\Big)$$
$$\lesssim\boldsymbol{\rho}^{-(t-1-\mathbf{r})}\,\omega_{\mathcal{P}}\left(\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(L^{d+\mathbf{r}+1})\right),\quad t\ge\mathbf{r}+2\,.$$

Thus we have

$$\Big(\frac{\mathrm{P}^\alpha(K,\mathbf{1}_{A\setminus S}\sigma)}{|K|^{1/n}}\Big)^2\sum_{\substack{J\in\Pi_2^{K,\boldsymbol{\tau}\text{-deep}}\mathcal{P}_{L,t}\\ J\subset K}}\|\,\triangle_J^\omega\,\mathbf{x}\|^2_{L^2(\omega)}$$
$$\le C\Big(\frac{\mathrm{P}^\alpha(K,\mathbf{1}_{A\setminus S}\sigma)}{|K|^{1/n}}\Big)^2\boldsymbol{\rho}^{-(t-1-\mathbf{r})}\sum_{\substack{L^{d+\mathbf{r}+1}\in\mathcal{G}_{d+\mathbf{r}+1}\\ L^{d+\mathbf{r}+1}\subset K}}\omega_{\mathcal{P}}(\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(L^{d+\mathbf{r}+1}))$$
$$\le C\,\boldsymbol{\rho}^{-(t-1-\mathbf{r})}\Big(\frac{\mathrm{P}^\alpha(K,\mathbf{1}_{A\setminus S}\sigma)}{|K|^{1/n}}\Big)^2\omega_{\mathcal{P}}\left(\mathbf{T}^{\boldsymbol{\tau}\text{-deep}}(K)\right)\le C\,\boldsymbol{\rho}^{-(t-1-\mathbf{r})}\mathcal{S}_{\text{size}}^{\alpha,A}(\mathcal{P})^2.$$

So altogether we conclude that

$$\sup_{S\in\mathfrak{C}_{\mathcal{L}}(L)}\mathcal{S}_{\text{size}}^{\alpha,A;S}(\mathcal{P}_{L,t})^2$$
$$=\sup_{S\in\mathfrak{C}_{\mathcal{L}}(L)}\ \sup_{K\in\mathcal{N}_{\boldsymbol{\rho}-\boldsymbol{\tau}}(S)\cup\mathcal{W}^{\text{good}}(S)}\frac{1}{|K|_\sigma}\Big(\frac{\mathrm{P}^\alpha(K,\mathbf{1}_{A\setminus K}\sigma)}{|K|^{1/n}}\Big)^2\sum_{\substack{J\in\Pi_2^{K,\boldsymbol{\tau}\text{-deep}}\mathcal{P}_{L,t}\\ J\subset K}}\|\mathsf{P}_J^\omega\mathbf{x}\|^2_{L^2(\omega)}$$
$$\le C_{\mathbf{r},\boldsymbol{\tau},\boldsymbol{\rho}}\,\boldsymbol{\rho}^{-t}\mathcal{S}_{\text{size}}^{\alpha,A}(\mathcal{P})^2\,,$$

and combined with (11.27) this gives (11.26). As we pointed out above, this completes the proof of Proposition 11.8, hence of Proposition 11.4, and finally of Theorem 2.6. □

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