



Bi-Lipschitz pieces between manifolds

Guy C. David

Abstract. A well-known class of questions asks the following: if X and Y are metric measure spaces and $f: X \rightarrow Y$ is a Lipschitz mapping whose image has positive measure, then must f have large pieces on which it is bi-Lipschitz? Building on methods of David and Semmes, we answer this question in the affirmative for Lipschitz mappings between certain types of Ahlfors s -regular, topological d -manifolds. In general, these manifolds need not be bi-Lipschitz embeddable in any Euclidean space. To prove the result, we use some facts on the Gromov–Hausdorff convergence of manifolds and a topological theorem of Bonk and Kleiner. This also yields a new proof of the uniform rectifiability of some metric manifolds.

1. Introduction

There are nowadays many different theorems of the following general form: let (X, d, μ) and (Y, ρ, ν) be metric measure spaces (satisfying some assumptions), and let $f: X \rightarrow Y$ be a Lipschitz map whose image has positive ν -measure. Then f must be bi-Lipschitz on a subset of large measure, in a quantitative way.

This class of theorems is not true in general, and later on in Section 8 we will mention some interesting cases where it fails. However, there are a number of situations in which results of this form can be proven.

The idea started with [7], where David¹ examined the case in which (X, d, μ) is Ahlfors d -regular and Y is \mathbb{R}^d with the standard metric and Lebesgue measure. David showed that if, in addition to these assumptions, f satisfies a certain technical condition that we will discuss below, then it is quantitatively bi-Lipschitz on a set of large measure. By verifying his technical condition, David then applied this theorem to show that if an L -Lipschitz map f from the unit cube $[0, 1]^d$ into \mathbb{R}^d has an image of Lebesgue measure at least $\delta > 0$, then f is M -bi-Lipschitz on a set of Lebesgue measure θ in the cube, where θ and M depend only on L and δ .

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¹The Guy David mentioned here and in the references is a professor at Université Paris-Sud and has no relation to the author of this paper, who is a postdoc at NYU. The author wishes to apologize for any confusion generated by this amusing coincidence.

Quite different methods were then invented by Jones [17] and David [8] to show the result in the case $X = [0, 1]^d$ and $Y = \mathbb{R}^D$ equipped with d -dimensional Hausdorff measure, where $D \geq d$. In 2009, Schul [26] showed the result in the case where $X = [0, 1]^d$ and Y is an arbitrary metric space, again equipped with d -dimensional Hausdorff measure. In addition, Meyerson [23] used techniques of Jones and David to show the result when X and Y are Carnot groups.

Here we do not use the later methods of [17], [8], and [26], but rather the original method of David, which required verifying a certain technical condition on the Lipschitz map and the spaces in question. Originally, this applied only in the case $Y = \mathbb{R}^d$, but later Semmes [30] generalized David's theorem to the case of arbitrary target spaces Y .

In this paper, we apply this theorem of Semmes and adapt David's original argument to show the "Lipschitz implies bi-Lipschitz" result for Lipschitz maps between certain types of abstract manifolds. Our main result is the following theorem.

Theorem 1.1. *Let X and Y be Ahlfors s -regular, linearly locally contractible, complete, oriented, topological d -manifolds, for $s > 0$, $d \in \mathbb{N}$. Suppose in addition that Y has d -manifold weak tangents.*

Suppose I_0 is a dyadic 0-cube in X and $z: I_0 \rightarrow Y$ is a Lipschitz map. Then for every $\epsilon > 0$, there are measurable subsets $E_1, \dots, E_l \subset I_0$, such that $z|_{E_i}$ is M -bi-Lipschitz, and

$$\left| z \left(I_0 \setminus \bigcup_{i=1}^l E_i \right) \right| < \epsilon |I_0|.$$

Here l and M depend only on ϵ , the Lipschitz constant of z , the data of X , and the space Y .

All the relevant definitions will be given in Subsections 1.1 and 1.2 below.

In fact, in Theorem 1.1, the condition that Y is Ahlfors s -regular can be relaxed to the condition that Y is doubling and satisfies the upper mass bound

$$\mathcal{H}^s(\overline{B}_Y(x, r)) \leq C_0 r^s.$$

It is only this half of the Ahlfors regularity of Y that is used in the proof.

For more information regarding the dependence of Theorem 1.1 on the space Y (and not just its data), see Remark 1.10.

Note that Theorem 1.1 implies in particular the type of result mentioned at the beginning of this paper: if the image of z has positive measure in Y , then z is bi-Lipschitz on a set of definite size in I_0 . This stronger conclusion, in which the domain of the mapping admits a decomposition into pieces on which the mapping is bi-Lipschitz and a "garbage" piece of small image, is typical and appears in the works [17], [8], [30], [26], and [23] mentioned above.

1.1. Definitions

Recall that if (X, d) and (Y, ρ) are metric spaces, then a map $f: X \rightarrow Y$ is called *Lipschitz* if there is a constant C such that

$$\rho(f(x), f(y)) \leq C d(x, y).$$

The map f is called *bi-Lipschitz*, if there is a constant C such that

$$C^{-1}d(x, y) \leq \rho(f(x), f(y)) \leq C d(x, y).$$

If we wish to emphasize the constant, we will call such mappings C -Lipschitz or C -bi-Lipschitz.

The following definition makes sense for general measures, but, following [30], we will consider only Hausdorff measure \mathcal{H}^s .

Definition 1.2. A metric space (X, d) is *Ahlfors s -regular* if there is a constant C_0 such that for all $x \in X$ and $r \leq \text{diam } X$, we have

$$C_0^{-1}r^s \leq \mathcal{H}^s(\overline{B}(x, r)) \leq C_0 r^s.$$

From now on, whenever we speak of a measure we will speak of s -dimensional Hausdorff measure in an Ahlfors s -regular space. To simplify notation, we therefore always write $|A|$ or, to avoid confusion, $|A|_X$ for the s -dimensional Hausdorff measure of a set A in a space X .

Definition 1.3. A metric space (X, d) is called *linearly locally contractible* if there are constants $L, r_0 > 0$ such that every open ball $B \subset X$ of radius $r < r_0$ is contractible inside a ball with the same center of radius Lr . We may abbreviate the condition as LLC or (L, r_0) -LLC to emphasize the constants.

Remark 1.4. In some contexts, the abbreviation LLC refers to the weaker condition of “linear local connectivity”. We do not use this condition in this paper.

The class of source and target spaces we consider in this paper are complete, oriented topological d -manifolds that are Ahlfors s -regular and LLC. If X is such a space, the phrase “the data of X ” refers to the collection of constants associated to X : the dimensions d and s , the constant C_0 appearing in the Ahlfors regularity of X , and the constants L and r_0 appearing in the LLC property of X .

There is also an additional constraint on the class of target spaces for which our theorem applies. This requires the notion of convergence of a sequence of pointed metric spaces, which we introduce in Definition 2.3 below.

Definition 1.5. We say a complete metric space (Y, ρ) has *d -manifold weak tangents* if the following holds: Whenever r_i is a sequence of positive real numbers that is bounded above, p_i are points in Y , and $(Y, \frac{1}{r_i}\rho, p_i)$ converges (as in Definition 2.3) to a space $(Y_\infty, \rho_\infty, p_\infty)$, then Y_∞ is a topological d -manifold.

Remark 1.6. Note that Definition 1.5 includes the assumption that Y itself is a topological d -manifold, by taking $r_i = 1$ and $p_i = p$ for all i .

Remark 1.7. While Definitions 1.2 and 1.3 are rather standard, Definition 1.5 is more unusual, and somewhat restrictive. Here are some examples of spaces that satisfy it:

- The simplest example is \mathbb{R}^d , for $d \geq 1$. Indeed, if $Y = \mathbb{R}^d$, then all the pointed metric spaces $(Y, \frac{1}{r_i}\rho, p_i)$ are isometric to $(\mathbb{R}^d, |\cdot|, 0)$ by rescaling and translating. Therefore, the limiting space of this sequence is also \mathbb{R}^d , which is a topological d -manifold.
- For the same reasons, every Carnot group G , equipped with its Carnot–Carathéodory metric, has d -manifold weak tangents, where d is the topological dimension of G .
- If X is a compact, doubling metric space with d -manifold weak tangents, and Y is quasisymmetric to X , then Y has d -manifold weak tangents. This follows, e.g., from [19], Lemmas 2.4.3 and 2.4.7. (For the definition and properties of quasisymmetric mappings, see [13].)
- Similarly, if G is a topologically d -dimensional Carnot group, and Y is quasisymmetric to G , then Y has d -manifold weak tangents (even if Y has larger Hausdorff dimension than G). This includes all “snowflaked” Carnot groups, i.e., metric spaces of the form (G, ρ^α) , where $0 < \alpha \leq 1$ and (G, ρ) is a Carnot group.
- The Cartesian product of two spaces (X, d_X) , (Y, d_Y) with n - and m -manifold weak tangents, respectively, (equipped, e.g., with the metric $d((x, y), (x', y')) = d_X(x, x') + d_Y(y, y')$) has $(n + m)$ -manifold weak tangents.
- Any complete, doubling, linearly locally contractible topological 2-manifold has 2-manifold weak tangents. Indeed, by Proposition 2.19 below, every weak tangent of such a space is a homology 2-manifold (see Definition 2.18), and the only homology 2-manifolds are topological 2-manifolds. (See [4], Theorem V.16.32.)
- Suppose a compact metric space Z has the property that every triple of points can be blown up to a uniformly separated triple by a uniformly quasi-Möbius map. (This condition was studied by Bonk and Kleiner in [2] and is satisfied by boundaries of hyperbolic groups equipped with their visual metrics.) Then Z has d -manifold weak tangents if and only if Z is itself a topological d -manifold. This follows from [2], Lemma 5.3. (Note that the definition of a weak tangent given in [2] is different than ours, in that it requires the sequence of scales $1/r_i$ tend to infinity. However, the proof of Lemma 5.3 in [2] works the same way without this restriction.)

1.2. Dyadic cubes

If X is a complete metric space that is Ahlfors s -regular with constant C_0 , we can equip X with a type of “dyadic decomposition”. The formulation in [30], Section 2.3, is the easiest to apply here. It says that there exists $j_0 \in \mathbb{N} \cup \{\infty\}$ (with $2^{j_0} \leq \text{diam } X < 2^{j_0+1}$ if X is bounded) such that for each $j < j_0$, there exists a partition Δ_j of X into measurable subsets $Q \in \Delta_j$ such that

- $Q \cap Q' = \emptyset$ if $Q, Q' \in \Delta_j$ and $Q \neq Q'$.
- If $j \leq k < j_0$ and $Q \in \Delta_j, Q' \in \Delta_k$, then either $Q \subseteq Q'$ or $Q \cap Q' = \emptyset$.
- $C_0^{-1} 2^j \leq \text{diam } Q \leq C_0 2^j$ and $C_0^{-1} 2^{sj} \leq |Q| \leq C_0 2^{sj}$.
- For every $j < j_0$, $Q \in \Delta_j$, and $\tau > 0$, we have

$$\begin{aligned} |\{x \in Q : \text{dist}(x, X \setminus Q) \leq \tau 2^j\}| &\leq C_0 \tau^{1/C_0} |Q|, \\ |\{x \in X \setminus Q : \text{dist}(x, Q) \leq \tau 2^j\}| &\leq C_0 \tau^{1/C_0} |Q|. \end{aligned}$$

We call an element of Δ_j a (dyadic) cube, or a (dyadic) j -cube if we wish to emphasize the scale.

Note that these dyadic cubes are not necessarily closed or open, but merely measurable. They are also disjoint, and do not merely have disjoint interiors. In \mathbb{R}^d , one should think of these as analogous to “half-open” cubes of the form

$$[n_1 2^j, (n_1 + 1) 2^j) \times [n_2 2^j, (n_2 + 1) 2^j) \times \cdots \times [n_d 2^j, (n_d + 1) 2^j),$$

where $n_i \in \mathbb{Z}$.

It follows from the third and fourth conditions that for every $j < j_0$ and $Q \in \Delta_j$, there exists $x \in Q$ such that

$$B(x, c_0 2^j) \subseteq Q \subseteq B(x, C_0 2^j).$$

All the constants in the cube decomposition depend only on s and the Ahlfors-regularity constant of the space, and so we have denoted the larger cube constant also by C_0 .

We remark here that the full force of Ahlfors regularity is not required to construct dyadic cubes with nice properties in metric measure spaces: a construction of Christ [6] provides similar cubes in all doubling metric measure spaces.

1.3. Background and results

In [7], condition (9), David introduced the following condition for a Lipschitz map defined on a dyadic cube in an Ahlfors-regular space. Though David gave the condition for maps into \mathbb{R}^d , in [30], Condition 9.1, Semmes re-formulated David’s condition for arbitrary target spaces. This is the formulation we give here. Recall that $|\cdot|$ denotes s -dimensional Hausdorff measure.

Definition 1.8. Let (X, d) be an Ahlfors s -regular metric space with a system of dyadic cubes as above. Let (Y, ρ) be a metric space. Let I_0 be a 0-cube in X , and $z: I_0 \rightarrow Y$ be a Lipschitz map. We will say that z satisfies *David’s condition* on I_0 if the following holds:

For every $\lambda, \gamma > 0$, there exist $\Lambda, \eta > 0$ such that, for every $x \in I_0$ and $j < j_0$, if T is the union of all j -cubes intersecting $B(x, \Lambda 2^j)$, and if $T \subseteq I_0$ and $|z(T)| \geq \gamma |T|$, then either:

- (i) $z(T) \supseteq B(z(x), \lambda 2^j)$, or

(ii) there is a j -cube $R \subset T$ such that

$$\frac{|z(R)|}{|R|} \geq (1 + 2\eta) \frac{|z(T)|}{|T|}.$$

As in Theorem 1.1, it is convenient to phrase David's condition for 0-cubes because that is how we will use it, although it makes sense for cubes of all sizes. Note that, given the definition of our cubes in Subsection 1.2, a space may contain no 0-cubes at all, but one can always create some by rescaling the space and relabeling the levels of the cubes.

In essence, David's condition says the following. At every location and scale within I_0 , if the map z does not collapse the measure of a ball too much, then one of two things must happen: either (i) the image of this ball contains a ball of definite size (centered at the image of its center), or (ii) some sub-cube of this ball is expanded by a larger factor than the ball itself. The upshot of (i) is that the map z does not "fold" at this location and scale.

To take a concrete example, suppose $I_0 = [0, 1]^2 \subset \mathbb{R}^2$ and z is the map

$$z(x, y) = \left(|x - \frac{1}{2}|, y \right),$$

which folds the square in half along its central vertical axis. If T is well away from the folding line $\{x = 1/2\}$, then z essentially acts isometrically on T and so condition (i) of David's condition holds. If T is centered on the folding line, then $|z(T)|/|T| = 1/2$ and (i) fails, but some sub-square R of T to the left or right of the folding line satisfies $|z(R)|/|R| = 1$, so (ii) holds.

Theorem 10.1 of [30], which is a generalization of Theorem 1 of [7], says the following.

Theorem 1.9 ([30], Theorem 10.1). *Let (X, d) be an Ahlfors s -regular metric space with a system of dyadic cubes as above. Let (Y, ρ) be an arbitrary metric space. Let I_0 be a 0-cube in X , and $z: I_0 \rightarrow Y$ be a Lipschitz map. Suppose that z satisfies David's condition on I_0 .*

Then for every $\epsilon > 0$, there are measurable subsets $E_1, \dots, E_l \subset I_0$, such that $z|_{E_i}$ is M -bi-Lipschitz, and

$$\left| z \left(I_0 \setminus \bigcup_{i=1}^l E_i \right) \right| < \epsilon |I_0|.$$

The constants l and M depend only on ϵ , the constants associated to the Ahlfors-regularity of the space X , the Lipschitz constant of z , and the numbers Λ and η from David's condition (for $\lambda = 1$ and γ depending only on ϵ and the Lipschitz constant of z .)

We will apply Theorem 1.9 and a modification of the proof of Theorem 2 of [7] to prove Theorem 1.1.

On the other hand, the fact that X and Y have the same topological dimension d is crucial in the setting of Theorem 1.1. In Proposition 8.1 below, we will give a counterexample to Theorem 1.1 in which X and Y satisfy all the assumptions of the theorem, except that they are manifolds of different topological dimensions.

A few further remarks on the statement of Theorem 1.1 are in order.

Remark 1.10. That Theorem 1.1 gives dependence of constants on the space Y (and not just its data) is a consequence of our compactness style of proof. However, the proof of Theorem 1.1 can be modified slightly to reduce the dependence on Y in the following manner. Let Y be a complete, oriented d -manifold that is LLC, Ahlfors s -regular, and has d -manifold weak tangents. Suppose that Y' is LLC, Ahlfors s -regular, and is η -quasisymmetric to Y , by a quasisymmetry that maps balls in Y' of radius 1 to sets of uniformly bounded diameter. Then Theorem 1.1 holds for maps $z: X \rightarrow Y'$ with constants depending only on the space Y , the data of Y' , and the quasisymmetry function η (as well as the data of z and X).

In particular, if $\xi \geq \xi_0 > 0$, then the theorem holds for target space $Y' = (Y, \xi\rho)$ with l, M depending only on Y and ξ_0 (as well as on ϵ and the data of X and z), and not on ξ itself. That is because this rescaling is quasisymmetric (with $\eta(t) = t$) and does not alter the data of Y , other than changing the contractibility radius r_0 to r_0/ξ_0 .

Remark 1.11. We have phrased Theorem 1.1 for 0-cubes to parallel Theorem 2 of [7]. However, it is easy to see that the following statement also holds:

Suppose $j_1 \leq j_0$, Q_0 is a dyadic j -cube in X , $j \leq j_1$, and $z: Q_0 \rightarrow Y$ is Lipschitz. Then the conclusion of Theorem 1.1 holds for z on Q_0 , i.e., for every $\epsilon > 0$, there are measurable subsets $E_1, \dots, E_l \subset Q_0$, such that $z|_{E_i}$ is M -bi-Lipschitz, and

$$\left| z \left(I_0 \setminus \bigcup_{i=1}^l E_i \right) \right| < \epsilon |I_0|.$$

Here l and M depend only on ϵ , the Lipschitz constant of z , j_1 , the space Y , and the data of X .

Indeed, if Q_0 is an j -cube for $j \leq j_1$, one need only apply Theorem 1.1 to the rescaled spaces $(X, 2^{-j}d)$ and $(Y, 2^{-j}\rho)$, and the same Lipschitz map z , relabeling the cubes so that Q_0 is a 0-cube. The rescaled spaces $(X, 2^{-j}d)$ and $(Y, 2^{-j}\rho)$ have the same data as X and Y , except that their LLC radii r_0 must be replaced by $2^{-j_1}r_0$. So we can apply Theorem 1.1 and Remark 1.10 to obtain this result.

David proved Theorem 1.1 in the case $X = Y = \mathbb{R}^d$ (see [7], Theorem 2). In doing so, he used a compactness argument to verify a modified version of what we have called David's condition. The general idea is the following: Consider a sequence of counterexample maps z_k , which in the case of \mathbb{R}^d may all be defined on the unit cube, that fail both conditions of Definition 1.8 with increasingly worse constants as $k \rightarrow \infty$. Extract a sub-limit z , and by a careful argument show that z has constant Jacobian. Because z is in addition Lipschitz, it is a quasi-regular mapping, and a theorem of Reshetnyak implies that it is an open mapping. A degree argument then shows that, for k large, the image of the maps z_k must contain a fixed size ball around $z_k(0)$, with radius independent of k . For k large, this contradicts the assumption that the maps z_k fail the first condition of Definition 1.8.

In our setting, we follow a similar approach. The compactness argument of [7] is modified to be a Gromov–Hausdorff compactness argument; to make the degree theory work in this setting we require some results on the Gromov–Hausdorff limits of locally contractible manifolds: see Section 2 below. In addition, the theory of quasi-regular mappings and the result of Reshetnyak are not available to us. They are replaced by a topological theorem of Bonk and Kleiner (Theorem 2.29 below) on mappings of bounded multiplicity.

A completely different method for verifying David’s condition in some situations is a type of detailed homotopy argument, as in [11], Chapter 9. This approach allows for much weaker topological assumptions on X , but it seems to rely on having $s = d$, $Y = \mathbb{R}^d$, and X embedded in some Euclidean space.

Even under the assumptions $s = d$ and $Y = \mathbb{R}^d$, Theorem 1.1 appears to be new if X is not a subset of some Euclidean space. This observation has a consequence for the geometry of the space X . The following concept has many definitions, but the one we give is most natural for abstract metric spaces.

Definition 1.12. An Ahlfors d -regular space X is called *uniformly rectifiable* if there exist constants $\alpha > 1$ and $0 < \beta \leq 1$ such that for every open ball B in X , there is a subset $E \subset B$ with $|E| \geq \beta|B|$ and an α -bi-Lipschitz map $f: E \rightarrow \mathbb{R}^d$.

We will call X *locally uniformly rectifiable* if for every $r > 0$, there exist constants α and β , depending on r , such that for every open ball B in X of radius less than r , there is a subset $E \subset B$ with $|E| \geq \beta|B|$ and an α -bi-Lipschitz map $f: E \rightarrow \mathbb{R}^d$.

We can apply Theorem 1.1 and a theorem of Semmes [27] to show that some abstract manifolds are uniformly rectifiable. Note that in this case we require that the Ahlfors regularity dimension and the topological dimension of X coincide. Snowflaked metric spaces such as $(\mathbb{R}^n, |\cdot|^{1/2})$ provide counterexamples in the absence of this assumption.

Theorem 1.13. *An Ahlfors d -regular, LLC, complete, oriented topological d -manifold is locally uniformly rectifiable. The local uniform rectifiability constants α and β depend on the scale r and otherwise only on the data of the space.*

In particular, a compact, Ahlfors d -regular, LLC, oriented topological d -manifold is uniformly rectifiable, with constants depending only on d, C_0, L , and $r_0/\text{diam}(X)$.

If X admits a bi-Lipschitz embedding into some Euclidean space, then Theorem 1.13 follows from work of David and Semmes in [11]. However, examples of Semmes [28] and Laakso [21] show that such an embedding need not always exist.

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2. The main tools

In this section, we introduce the main concepts and results used in the proof of Theorem 1.1.

2.1. Convergence of metric spaces

We will use the notion of convergence of “mapping packages”, a version of Gromov–Hausdorff convergence, that is described in Chapter 8 of [10]. All material in this sub-section is from that source. A brief exposition of this material is also given in [18].

While the notation in this set-up is a bit more cumbersome than for other definitions of Gromov–Hausdorff convergence, the detailed results of [10] make it very flexible for discussing simultaneous convergence of metric spaces and mappings.

Definition 2.1. We say that a sequence $\{F_j\}$ of non-empty closed subsets of some Euclidean space \mathbb{R}^N converges to a non-empty closed set $F \subseteq \mathbb{R}^N$ if, for all $R > 0$,

$$\lim_{j \rightarrow \infty} \sup_{x \in F_j \cap B(0,R)} \text{dist}(x, F) = 0$$

and

$$\lim_{j \rightarrow \infty} \sup_{y \in F \cap B(0,R)} \text{dist}(y, F_j) = 0.$$

This convergence is stable under taking products, in the sense that if $\{F_j\}$ converges to F in \mathbb{R}^N and $\{G_j\}$ converges to G in \mathbb{R}^M , then $\{F_j \times G_j\}$ converges to $F \times G$ in \mathbb{R}^{N+M} .

Definition 2.2. Suppose $\{F_j\}$ is a sequence of closed sets converging to a closed set F in \mathbb{R}^N as in the previous definition. Let Y be a metric space and $\phi_j : F_j \rightarrow Y$, $\phi : F \rightarrow Y$ be mappings. We say that $\{\phi_j\}$ converges to ϕ if for each sequence $\{x_j\}$ in \mathbb{R}^N such that $x_j \in F_j$ for all j and $x_j \rightarrow x \in F$, we have that

$$\lim_{j \rightarrow \infty} \phi_j(x_j) = \phi(x).$$

A *pointed metric space* is a triple (X, d, p) , where (X, d) is a metric space and p is a point in X . All metric spaces that we consider are complete and doubling.

Definition 2.3. A sequence of pointed metric spaces $\{(X_j, d_j, p_j)\}$ converges to a pointed metric space (X, d, p) if the following conditions hold. There exist $\alpha \in (0, 1]$, $N \in \mathbb{N}$, and L -bi-Lipschitz embeddings $f_j : (X_j, d_j^\alpha) \rightarrow \mathbb{R}^N$, $f : (X, d^\alpha) \rightarrow \mathbb{R}^N$, with $f_j(p_j) = f(p) = 0$ for all j . Furthermore, we require that $f_j(X_j)$ converge to $f(X)$ in the sense of Definition 2.1, and that the real-valued functions $d_j(f_j^{-1}(x), f_j^{-1}(y))$ defined on $f_j(X_j) \times f_j(X_j)$ converge to $d(f^{-1}(x), f^{-1}(y))$ on $f(X) \times f(X)$ in the sense of Definition 2.2.

We only use Definition 2.3 when the metric spaces $\{(X_j, d_j)\}$ and (X, d) are uniformly doubling. In that case, the embeddings f_j and f can always be found, by Assouad’s embedding theorem (see [13], Theorem 12.2).

Definition 2.4. A mapping package consists of a pair of pointed metric spaces (M, d_M, p) and (N, d_N, q) as well as a mapping $g : M \rightarrow N$ such that $g(p) = q$.

Definition 2.5. A sequence of mapping packages $\{((X_j, d_j, p_j), (Y_j, \rho_j, q_j), h_j)\}$ is said to converge to another mapping package $((X, d, p), (Y, \rho, q), h)$ if the following conditions hold. The sequences $\{(X_j, d_j, p_j)\}$ and $\{(Y_j, \rho_j, q_j)\}$ converge to (X, d, p) and (Y, ρ, q) , respectively, in the sense of Definition 2.3. Furthermore, the maps $g_j \circ h_j \circ f_j^{-1}$ converge to $g \circ h \circ f^{-1}$ in the sense of Definition 2.2, where f_j, g_j, f, g are the embeddings of Definition 2.3.

The following proposition is a special case of Lemma 8.22 of [10].

Proposition 2.6. *Let $\{((X_j, d_j, p_j), (Y_j, \rho_j, q_j), h_j)\}$ be a sequence of mapping packages, in which all the metric spaces are complete and uniformly doubling, and in which the maps h_j are uniformly Lipschitz and satisfy $h_j(p_j) = q_j$. Then there exists a mapping package $((X, d, p), (Y, \rho, q), h)$ that is the limit of a subsequence of $\{((X_j, d_j, p_j), (Y_j, \rho_j, q_j), h_j)\}$.*

We will now describe some consequences of the convergence of a sequence of mapping packages, which are Lemmas 8.11 and 8.19 of [10].

Proposition 2.7. *Suppose a sequence of pointed metric spaces $\{(X_k, d_k, p_k)\}$ converges to the pointed metric space (X, d, p) , in the sense of Definition 2.3.*

Then there exist (not necessarily continuous) mappings $\phi_k : X \rightarrow X_k$ and $\psi_k : X_k \rightarrow X$ such that:

- *For all k , $\phi_k(p) = p_k$ and $\psi_k(p_k) = p$.*
- *For all $R > 0$,*

$$\limsup_{k \rightarrow \infty} \{d_X(\psi_k(\phi_k(x)), x) : x \in B_X(p, R)\} = 0$$

and

$$\limsup_{k \rightarrow \infty} \{d_{X_k}(\phi_k(\psi_k(x)), x) : x \in B_{X_k}(p_k, R)\} = 0.$$

- *For all $R > 0$,*

$$\limsup_{k \rightarrow \infty} \{|d_{X_k}(\phi_k(x), \phi_k(y)) - d_X(x, y)| : x, y \in B_X(p, R)\} = 0$$

and

$$\limsup_{k \rightarrow \infty} \{|d_X(\psi_k(x), \psi_k(y)) - d_{X_k}(x, y)| : x, y \in B_{X_k}(p, R)\} = 0.$$

Proposition 2.8. *Suppose a sequence of mapping packages*

$$\{((X_k, d_k, p_k), (Y_k, \rho_k, q_k), h_k)\}$$

converges to a mapping package

$$((X, d, p), (Y, \rho, q), h),$$

where the mappings h_k are uniformly Lipschitz and satisfy $h_k(p_k) = q_k$.

Then there exist (not necessarily continuous) mappings $\phi_k: X \rightarrow X_k$ and $\psi_k: X_k \rightarrow X$ satisfying exactly the conditions of Proposition 2.7, and mappings $\sigma_k: Y \rightarrow Y_k$ and $\tau_k: Y_k \rightarrow Y$ satisfying the analogous properties of Proposition 2.7, such that in addition we have the following: for all $x \in X$,

$$\lim_{k \rightarrow \infty} \tau_k(h_k(\phi_k(x))) = h(x),$$

and this convergence is uniform on bounded subsets of X .

We will be interested in mapping packages in which the mappings h_k are defined only on subsets of the source spaces X_k . For this, we need the following fact, which is Lemma 8.32 of [10].

Lemma 2.9. *Suppose that $\{(X_k, d_k, p_k)\}$ is a sequence of pointed metric spaces that converges to the pointed metric space $\{(X, d, p)\}$ in the sense of Definition 2.3. Let $\{F_k\}$ be a sequence of nonempty closed sets with $F_k \subset X_k$ for each k . Suppose that*

$$\sup_k d_k(F_k, p_k) < \infty.$$

Then we can pass to a subsequence to get convergence to a nonempty closed subset F of X .

We make one final remark in this Subsection, which is Lemma 8.29 of [10].

Lemma 2.10. *Let the pointed metric spaces (X_j, d_j, p_j) converge to (X, d, p) in the sense of Definition 2.3. Suppose that (X_j, d_j) are Ahlfors s -regular, with Ahlfors regularity constant uniformly bounded (see Definition 1.2). Then (X, d) is Ahlfors s -regular, with constant controlled by the Ahlfors regularity constants of the spaces (X_j, d_j) .*

2.2. Convergence of LLC spaces

Here we state some results that apply to the convergence of metric spaces (in the sense of the previous section) when those metric spaces also happen to be linearly locally contractible. The main goals are to show that a convergent sequence of uniformly LLC spaces has an LLC limit (essentially a result of Borsuk [3]), and to describe a result that improves Proposition 2.8 in this context.

The following basic fact about LLC spaces will be used a number of times.

Lemma 2.11. *Let X be a (L, r_0) -LLC space. Fix $x \in X$ and $r < r_0$. Then there is a connected open set U satisfying*

$$B(x, r/(2L)) \subset U \subset B(x, r).$$

Proof. Consider a point $y \in X$ and radius $0 < r < r_0$. Let $H : B(y, r/(2L)) \times [0, 1] \rightarrow B(x, r/2)$ be a homotopy contracting $B(y, r/(2L))$ to a point. Define

$$E(y, r) = H(B(y, r/(2L)) \times [0, 1]).$$

Then $E(y, r)$ is a connected subset of $B(y, r/2)$ containing $B(y, r/(2L))$.

Let $E_0 = E(x, r)$. For $i \in \mathbb{N}$, inductively define sets

$$E_i = \bigcup_{y \in E_{i-1}} E(y, 2^{-i}r).$$

By induction, each set E_i is connected. In addition, for each i we have the relation

$$(2.1) \quad E_i \subset \text{int } E_{i+1}.$$

Now let

$$U = \bigcup_{i=1}^{\infty} E_i.$$

Then, as the union of connected sets that all contain the point x , U is connected. In addition, by (2.1) U is open: if $x \in U$, then, for some i ,

$$x \in E_i \subset \text{int } E_{i+1} \subset \text{int } U.$$

Finally, if $y \in E_i \subset U$, then

$$d(x, y) \leq (2^{-(i+1)} + 2^{-i} + \dots + 2^{-1})r < r.$$

Thus, U is a connected open set, $U \subset B(x, r)$, and $U \supseteq E_0 \supseteq B(x, r/(2L))$. □

The following is our main lemma about convergence of uniformly LLC sets.

Lemma 2.12. *Let F_k be a sequence of closed sets in some Euclidean space \mathbb{R}^N that are each (L, r_0) -LLC (as spaces equipped with the induced Euclidean metric). Suppose that $F_k \rightarrow F$ in the sense of Definition 2.1. Then F is LLC, with constants depending only on L and r_0 .*

In the case of compact sets converging in the usual Hausdorff metric, Lemma 2.12 is due to Borsuk [3]. A similar localized version of the result was noted in [14]. Here we provide a proof, following the method of Borsuk.

The proof is somewhat technical, though the main idea is not difficult: For subsets of Euclidean space, the LLC property for a set E implies the existence of a retraction to E , from an open neighborhood of E of fixed size, that moves points by an amount proportional to their distance from E . We use the existence of these retractions on the limiting sets F_k to construct a retraction onto the limit F . This retraction can then be used to show that F is LLC. Because the convergence is local, there are some minor technical complications.

Proof of Lemma 2.12. Let $B_R = B(0, R) \subset \mathbb{R}^N$. For a set $E \subseteq \mathbb{R}^N$, let $U_\epsilon(E)$ denote the open ϵ -neighborhood of E .

We note first that the LLC property implies that there exist constants $c < 1$ and $C = c^{-1} > 1$ such that each F_k admits a continuous retraction $r_k : U_c(F_k) \rightarrow F_k$ satisfying

$$(2.2) \quad |r_k(x) - x| \leq C \text{dist}(x, F_k)$$

for $x \in U_c(F_k)$. The proof of this can be found in Section 13 of [3] (and does not require compactness of the sets).

Fix a ball $B = B(p, r) \cap F$ for $p \in F$ and $r < r_0/4$. Fix $R > \max\{4Lr, 12C\}$ large enough so that $\overline{B} \subset B_R$.

By passing to a subsequence, we may without loss of generality assume that, for all k ,

$$\begin{aligned} \sup\{\text{dist}(x, F_k) : x \in F \cap B_{10R}\} &< c/4, \\ \sup\{\text{dist}(x, F) : x \in F_k \cap B_{10R}\} &< c/4. \end{aligned}$$

It follows that

$$U := \bigcap_{k=1}^{\infty} U_c(F_k)$$

contains a $c/2$ -neighborhood of $B_{9R} \cap F$ as well as of $\bigcup_{k=1}^{\infty} (B_{9R} \cap F_k)$.

For $k \in \mathbb{N}$, fix decreasing sequences

$$(2.3) \quad \eta_k = c^2 4^{-k},$$

$$(2.4) \quad \eta'_k = c^2 4^{-k}/3.$$

We may now pass to a further subsequence of our sets on which we assume that

$$(2.5) \quad \sup\{\text{dist}(x, F_k) : x \in F \cap B_{9R}\} < \eta'_k/8,$$

$$(2.6) \quad \sup\{\text{dist}(x, F) : x \in F_k \cap B_{9R}\} < \eta'_k/8.$$

Let $U_k = U_{\eta_k}(F_k)$ and $V_k = U_{\eta'_k}(F_k)$. Then, if $x \in U_{k+1} \cap B_{7R}$, we have, by (2.3), (2.4), (2.5), and (2.6), that

$$\text{dist}(x, F_k \cap B_{8R}) < \eta'_k.$$

Therefore, for every $R' \leq 7R$,

$$(2.7) \quad (U_{k+1} \cap B_{R'}) \subset (V_k \cap B_{R'}) \subset (\overline{V}_k \cap B_{R'}) \subset (U_k \cap B_{R'}) \subset (U \cap B_{R'}).$$

We will now inductively construct a new sequence of retractions $s_k : U \cap B_{5R} \rightarrow F_k$ by modifying the maps r_k .

Let $s_1 = r_1$. Suppose that s_k has already been defined and in addition satisfies $s_k = r_k$ on $V_k \cap B_{5R}$. Let $f : U \rightarrow \mathbb{R}$ be a continuous function that is 0 on $U \setminus U_{k+1}$ and 1 on V_{k+1} . For $x \in U \cap B_{5R}$, define

$$s_{k+1}(x) = r_{k+1}((1 - f(x))s_k(x) + f(x)x).$$

We first check that s_{k+1} is well-defined, i.e., that for $x \in U \cap B_{5R}$, the point $(1 - f(x))s_k(x) + f(x)x$ is in U . If $x \in U \setminus U_{k+1}$, then $(1 - f(x))s_k(x) + f(x)x = s_k(x) \in F_k \in U$, so s_{k+1} is well-defined. In the case $x \in U_{k+1}$, we have by (2.7) that $x \in V_k$. By our inductive assumption that $s_k = r_k$ on $V_k \cap B_{5R}$, we get

$$|x - s_k(x)| = |x - r_k(x)| \leq C \eta'_k < c.$$

Thus, every point on the line segment from x to $s_k(x)$ is in the c -neighborhood of F_k and so is in U .

That s_{k+1} is the identity on points of F_{k+1} follows from the fact that, by definition, $s_{k+1} = r_{k+1}$ on F_{k+1} .

We now make the following claim: if $x \in U \cap B_{5R}$ and $s_k(x) \in B_{6R}$, then

$$(2.8) \quad |s_{k+1}(x) - s_k(x)| < 3C4^{-k}.$$

To prove this, we consider three cases.

(i) The case $x \in V_{k+1}$.

In this case, using (2.2) and the definitions of s_k, s_{k+1} , we get

$$\begin{aligned} |s_{k+1}(x) - s_k(x)| &= |r_{k+1}(x) - r_k(x)| \leq |r_{k+1}(x) - x| + |x - r_k(x)| \\ &\leq C(\eta'_{k+1} + \eta'_k) < 3C4^{-k}. \end{aligned}$$

(ii) The case $x \in U \setminus U_{k+1}$.

In this case, $s_{k+1}(x) = r_{k+1}(s_k(x))$. By assumption, $s_k(x) \in F_k \cap B_{6R}$ and therefore $\text{dist}(s_k(x), F_{k+1}) < \eta'_k/4$ by (2.6). Therefore, by (2.2),

$$|s_{k+1}(x) - s_k(x)| = |r_{k+1}(s_k(x)) - s_k(x)| \leq C\eta'_k/4 < 3C4^{-k}.$$

(iii) The case $x \in U_{k+1} \setminus V_{k+1}$.

Note that $x \in U_{k+1} \cap B_{5R} \subset V_k \cap B_{5R}$, so $s_k(x) = r_k(x)$. Let

$$y = (1 - f(x))s_k(x) + f(x)x,$$

which is on the line segment L joining x to $s_k(x) = r_k(x)$. The diameter of L is therefore $|x - r_k(x)| \leq C\eta'_k$, by (2.2) and the fact that $x \in V_k$.

In addition, because $x \in U_{k+1}$, we have $\text{dist}(x, F_{k+1}) < \eta_k$.

From these calculations, it follows that

$$\text{dist}(y, F_{k+1}) \leq \text{dist}(x, F_{k+1}) + \text{diam}(L) \leq \eta_k + C\eta'_k,$$

and therefore, by (2.2), that

$$\begin{aligned} |s_{k+1}(x) - x| &= |r_{k+1}(y) - x| \leq |r_{k+1}(y) - y| + |y - x| \\ &\leq C(\eta_k + C\eta'_k) + C\eta'_k \leq 2C4^{-k}. \end{aligned}$$

From this, we see that

$$|s_{k+1}(x) - s_k(x)| \leq |s_{k+1}(x) - x| + |x - r_k(x)| < 2C4^{-k} + \eta_k < 3C4^{-k}.$$

This concludes the proof of the claim that $|s_{k+1}(x) - s_k(x)| < 3C4^{-k}$ if $x \in U \cap B_{5R}$ and $s_k(x) \in B_{6R}$.

Now note that

$$|s_1(x) - x| = |r_1(x) - x| \leq Cc = 1.$$

Therefore $r_1(x) \in B_{5.5R}$. Because $\sum_{k=0}^{\infty} (3C4^{-k}) \leq 6C < R/2$, it follows from the above claim that $s_k(x) \in B_{6R}$ for all k , and therefore that

$$|s_{k+1}(x) - s_k(x)| < 3C4^{-k}$$

for all k .

It follows immediately from this and from (2.2) that $s_k|_{U \cap B_{5R}}$ converge uniformly to a map

$$s : U \cap B_{5R} \rightarrow F \cap B_{6R}$$

that is the identity on $F \cap B_{5R}$.

Note that the map s is indeed the identity on F : if $x \in F \cap B_{5R}$, then by (2.5) and the definition of s_k we see that $s_k(x) = r_k(x)$. It follows that

$$|s(x) - x| = \lim_{k \rightarrow \infty} |s_k(x) - x| = \lim_{k \rightarrow \infty} |r_k(x) - x| \leq C \lim_{k \rightarrow \infty} \text{dist}(x, F_k) = 0.$$

To finish the proof of the lemma, recall our fixed ball $B = B(p, r) \cap F$ in $F \cap B_R$. The map s , when restricted to $F \cap \overline{B}_{4R}$, is the identity. Therefore, for every positive number $\eta < r$ sufficiently small, there is a neighborhood $V \subset (U \cap B_{5R})$ of $F \cap \overline{B}_{4R}$ such that

$$x \in V \Rightarrow |s(x) - x| < \eta.$$

We may now choose k large so that $|s_k(x) - s(x)| < \eta$ for all $x \in U \cap B_{5R}$ (by uniform convergence) and in addition so that

$$F_k \cap B_{3R} \subset V.$$

Now we contract B in the following manner. First, consider the homotopy

$$h(x, t) = (1 - t)x + ts_k(x)$$

for $x \in B$ and $t \in [0, 1]$. Because $|s_k(x) - x| = |s_k(x) - s(x)| < \eta$, we have $h(B \times [0, 1]) \subset B_{3R}$. In addition, h deforms B onto a set $E \subset F_k \cap B_{3R}$ of diameter no more than $2r + 2\eta$. By our choices of r and η , $2r + 2\eta < 4r < r_0$, and therefore E is contractible inside a set $E' \subset F_k \cap B_{3R}$ of diameter $L(2r + 2\eta)$.

Let $g : B \times [0, 1] \rightarrow E' \subset (F_k \cap B_{3R})$ denote the homotopy of B onto a point that first deforms by h and then by the contraction in F_k . Then $s \circ g$ is a contraction of B to a point within the set $s(E') \subset F$, which has diameter no more than $L(2r + 2\eta) + 2\eta$.

In summary, if we recall that $\eta < r$, we have shown that the ball $B = B(p, r) \cap F$ is contractible within the ball $B' = B(p, (4L + 2)r) \cap F$ whenever $r < r_0/4$. This completes the proof. \square

Lemma 2.13. *Suppose the pointed metric spaces (X_k, d_k, v_k) are (L, r_0) -LLC and converge to the pointed metric space (X, d, v) in the sense of Definition 2.3. Then (X, d) is LLC, with constants depending only on L and r_0 .*

Proof. This follows immediately from Lemma 2.12 and Definition 2.3, as the “snowflake” transformations of Definition 2.3 distort the LLC constants in a quantitative way. \square

To conclude this section, we give two lemmas which improve Propositions 2.7 and 2.8 in the setting of LLC spaces. They say that if a sequence of mapping packages converges, then the “almost-isometries” ϕ_k and ψ_k between the limiting spaces and the limit space can be taken to be continuous.

Definition 2.14. For $\eta > 0$, we say that continuous maps $f, g: M \rightarrow N$ between metric spaces are η -homotopic if they are homotopic by a homotopy $H: M \times [0, 1] \rightarrow N$ such that, for all $x \in M$ and $t \in [0, 1]$, we have

$$d_N(f(x), H(x, t)) < \eta.$$

Note in particular that if f and g are η -homotopic, then $d_N(f(x), g(x)) < \eta$ for all x .

The following is an immediate consequence of Proposition 2.8 above, combined with Propositions 5.4 and 5.8 of [27]. (See also [24], Section 3, for a cleaner statement in the compact case.) Note that all our spaces are Ahlfors s -regular and thus have topological dimension bounded above by s , so those results apply.

Lemma 2.15. *Suppose the pointed metric spaces (X_k, d_k, v_k) are (L, r_0) -LLC, uniformly Ahlfors s -regular, and converge to the pointed metric space (X, d, v) in the sense of Definition 2.3.*

Fix a point $x \in X$ and radius $R > 0$. Then there exist continuous mappings $f_k: \overline{B}_X(x, R) \rightarrow X_k$ and $g_k: \overline{B}_{X_k}(f_k(x), R) \rightarrow X$ satisfying the following conditions:

(i) *They almost preserve distances, in the sense that*

$$\lim_{k \rightarrow \infty} \sup\{|d_{X_k}(f_k(p), f_k(q)) - d_X(p, q)| : p, q \in B_X(x, R)\} = 0$$

and

$$\lim_{k \rightarrow \infty} \sup\{|d_X(g_k(p), g_k(q)) - d_{X_k}(p, q)| : p, q \in B_{X_k}(f_k(x), R)\} = 0.$$

(ii) *For every $0 < r < R$, we have*

$$\lim_{k \rightarrow \infty} \inf\{\eta : g_k \circ f_k|_{\overline{B}(x, r)} \text{ is } \eta\text{-homotopic to the inclusion of } \overline{B}(x, r) \text{ into } B(x, R)\} = 0$$

and

$$\lim_{k \rightarrow \infty} \inf\{\eta : f_k \circ g_k|_{\overline{B}(f_k(x), r)} \text{ is } \eta\text{-homotopic to the inclusion of } \overline{B}(f_k(x), r) \text{ into } B(f_k(x), R)\} = 0.$$

(iii) *If x is the basepoint $v \in X$, then in addition we have*

$$\lim_{k \rightarrow \infty} d_k(f_k(v), v_k) = 0.$$

Proof. Take $\eta > 0$. We will find, for all k sufficiently large, continuous mappings f_k and g_k as above that preserve distances up to additive error η and such that $f_k \circ g_k$ and $g_k \circ f_k$ are η -homotopic to the appropriate inclusion maps.

Fix numbers η'', η' sufficiently small, with $0 < \eta'' < \eta' < \eta$. They will depend only on η and the (uniform) data of the spaces $X, \{X_k\}$.

By Proposition 2.7, there is an index $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$, the maps $\phi_k : X \rightarrow X_k$ and $\psi_k : X_k \rightarrow X$ preserve distances up to additive error η'' on $\overline{B}(x, R)$ and $\overline{B}(f_k(x), R)$, respectively. By [27], Proposition 5.4, if η'' was chosen sufficiently small compared to η' , then there exist continuous maps $f_k : \overline{B}(x, R) \rightarrow X_k$ and $g_k : \overline{B}(f_k(x), R) \rightarrow X$ such that

$$(2.9) \quad d_k(f_k(z), \phi_k(z)) < \eta' \quad \text{and} \quad d(g_k(y), \psi_k(y)) < \eta'$$

on their respective domains. Part (i) of the lemma follows immediately from this by taking $\eta' < \eta/10$. Part (iii) also follows, because $\phi_k(v) = v_k$.

Now fix $0 < r < R$. By Proposition 2.7 we may also assume that, for all $k \geq k_0$, we have

$$d(\phi_k(\psi_k(x)), x) < \eta' \quad \text{and} \quad d(\psi_k(\phi_k(x)), x) < \eta',$$

in addition to the properties above.

If η' was chosen sufficiently small, then $f_k(\overline{B}(x, r)) \subset B(f_k(x), R)$ and so the composition $g_k \circ f_k$ is defined on $\overline{B}(x, r)$. Similarly, the composition $f_k \circ g_k$ is defined on $\overline{B}(f_k(x), r)$. By choosing $\eta'' < \eta'/10$ and using equation (2.9) and the properties of ϕ_k and ψ_k , we also see that

$$d(f_k(g_k(x)), x) < 2\eta' \quad \text{and} \quad d(g_k(f_k(x)), x) < 2\eta'.$$

Therefore, if η' was chosen sufficiently small, depending on η and the data of the spaces $X, \{X_k\}$, Proposition 5.8 of [27] implies that

$$g_k \circ f_k|_{\overline{B}(x,r)} \quad \text{and} \quad f_k \circ g_k|_{\overline{B}(f_k(x),r)}$$

are η -homotopic to the inclusions

$$\overline{B}(x, r) \rightarrow B(x, R) \quad \text{and} \quad \overline{B}(f_k(x), r) \rightarrow B(f_k(x), R).$$

This proves part (ii) of the lemma. □

Propositions 5.4 and 5.8 of [27], on which the proof of the previous lemma is based, are important consequences of the linear local contractibility of the spaces X and $\{X_k\}$. Roughly speaking, they say that if a mapping into an LLC space is “roughly continuous” (as the maps ϕ_k and ψ_k are), then it is close to a continuous mapping, and if two continuous mappings into an LLC space are close, then they are η -homotopic. The proofs of these facts use polyhedral approximations of the source space and an induction on the skeleta of the polyhedra. We encourage the reader to look at Semmes’ paper [27] or Petersen’s work [24] for the details.

The following additional fact is an immediate consequence of Proposition 2.8 and equation (2.9) above.

Lemma 2.16. *Suppose we have convergence of a sequence of mapping packages*

$$((X_k, d_k, p_k), (Y_k, \rho_k, q_k), h_k) \rightarrow ((X, d, p), (Y, \rho, q), h)$$

in the sense of Definition 2.5. Suppose that all the spaces involved are uniformly Ahlfors s -regular and uniformly LLC, and that the mappings $\{h_k\}$ and h are uniformly C -Lipschitz and satisfy $h_k(p_k) = q_k, h(p) = q$.

Then for all $R > 0$, there exist continuous mappings

$$f_k : \overline{B}_X(p, R) \rightarrow X_k \quad \text{and} \quad g_k : \overline{B}_{X_k}(p_k, R) \rightarrow X$$

satisfying exactly the conditions of Lemma 2.15, and continuous mappings

$$\tilde{f}_k : \overline{B}_Y(q, R) \rightarrow Y_k \quad \text{and} \quad \tilde{g}_k : \overline{B}_{Y_k}(q_k, R) \rightarrow Y$$

satisfying the analogous properties of Lemma 2.15, such that in addition we have that for all $x \in X$,

$$\lim_{k \rightarrow \infty} \tilde{g}_k(h_k(f_k(x))) = h(x)$$

uniformly on $\overline{B}_X(p, R/2C)$.

2.3. Convergence of manifolds

Here we state some facts on the convergence of metric spaces that are LLC topological manifolds. Our main goal is to give a proof of Proposition 2.19 below, which says that the limit of a sequence of uniformly Ahlfors regular, uniformly LLC, topological d -manifolds is a homology d -manifold (see Definition 2.18). This result essentially goes back to Begle [1] (see also [12]) and appears to be well-known, but we did not find a modern proof in the literature in the generality necessary here.

Below, H_* denotes singular homology with integer coefficients.

Lemma 2.17. *Let M be an (L, r_0) -LLC oriented topological d -manifold. Let $v \in M$ and let $K_1 \subset K_2$ be compact sets satisfying $v \in K_1 \subset B(v, r) \subset B(v, 2Lr) \subset K_2 \subset B(v, r_0)$. Then the following facts hold.*

- (i) *The map $j_* : H_p(M, M \setminus K_2) \rightarrow H_p(M, M \setminus K_1)$, induced by inclusion, is trivial if $p \neq d$.*
- (ii) *The map $i_* : H_d(M, M \setminus K_2) \rightarrow H_d(M, M \setminus \{v\}) \cong \mathbb{Z}$, induced by inclusion, is surjective.*
- (iii) *With this notation, we also have $\ker i_* \subseteq \ker j_*$ in the top degree d .*

Proof. By use of the natural duality isomorphisms ([31], Theorem 6.2.17) we obtain the following commutative diagram. Here \overline{H} denotes Čech cohomology, and all maps in the diagram are the natural maps induced by inclusion.

$$(2.10) \quad \begin{array}{ccccc} H_p(M, M \setminus K_2) & \xrightarrow{j_*} & H_p(M, M \setminus K_1) & \xrightarrow{k_*} & H_p(M, M \setminus \{v\}) \\ \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\ \overline{H}^{d-p}(K_2) & \xrightarrow{j^*} & \overline{H}^{d-p}(K_1) & \xrightarrow{k^*} & \overline{H}^{d-p}(\{v\}) \end{array}$$

If $p \neq d$, then j^* is trivial because K_1 is contractible in K_2 , which proves (i).

Now let $p = d$. The map $i^* = k^*j^* : \overline{H}^0(K_2) \rightarrow \overline{H}^0(\{v\})$ is surjective, as $v \in K_2$, which proves (ii).

Finally, by Lemma 2.11, K_1 is entirely contained in a connected component E of K_2 . Therefore, every connected component E' of K_2 that does not contain $\{v\}$ is in fact disjoint from K_1 . It follows that if $i^*\phi = k^*j^*\phi$ is trivial in $\overline{H}^0(\{v\})$, then $j^*\phi$ is already trivial in $\overline{H}^0(K_1)$. This proves claim (iii). \square

We now set up some definitions for the main result of this sub-section. A *Euclidean neighborhood retract* (ENR) is a space X which, for every $N \in \mathbb{N}$ and every topological embedding $e : X \rightarrow \mathbb{R}^N$, has the property that $e(X)$ is a retract of some open neighborhood of $e(X)$ in \mathbb{R}^N . Every LLC space with finite topological dimension is a Euclidean neighborhood retract (see [16], Theorem V.7.1).

Definition 2.18. A space M that is an ENR and that satisfies the condition

$$H_*(M, M \setminus \{x\}) = H_*(\mathbb{R}^d, \mathbb{R}^d \setminus \{0\}),$$

for all $x \in M$, is called a *homology d -manifold*.

Proposition 2.19. *Suppose the metric spaces (X_k, d_k) are uniformly Ahlfors s -regular, (L, r_0) -LLC, oriented topological d -manifolds, $v_k \in X_k$, and the sequence of pointed metric spaces (X_k, d_k, v_k) converges to (X, d, v) in the sense of Definition 2.3. Then (X, d) is an LLC homology d -manifold.*

Proof. The fact that (X, d) is LLC is Lemma 2.13 above. As this statement is quantitative, we will denote the LLC constants of (X, d) also by (L, r_0) .

The fact that X is a homology d -manifold can be proven by the methods of Begle [1], again as remarked in [14]. For convenience, we provide a proof using the tools introduced in this section.

We know that X is Ahlfors s -regular, and therefore it has finite Hausdorff dimension and thus finite topological dimension. Because X is also LLC, it is an ENR, as noted above. It now suffices to show that for every $x \in X$, the local integer (singular) homology groups $H_p(X, X \setminus \{x\})$ are isomorphic to \mathbb{Z} if $p = d$ and trivial otherwise.

To set up the proof we need some notation.

Let $L' = 4L$. Fix an integer $p \geq 0$, a point $x \in X$, and a radius $R > 0$. In addition, for each $k \in \mathbb{N}$, fix continuous maps

$$f_k : B_X(x, R) \rightarrow X_k \quad \text{and} \quad g_k : B_{X_k}(f_k(x), R) \rightarrow X,$$

as in Lemma 2.15. These maps have the property that, up to arbitrarily small additive error (decreasing to zero with k), they preserve distances and are inverses of each other.

For $n \in \mathbb{N}$, let

$$F_n = H_p(X, X \setminus \overline{B}(x, (L')^{-n}r_0))$$

and

$$G_n^k = H_p(X_k, X_k \setminus \overline{B}(f_k(x), (L')^{-n}r_0))$$

(Of course these groups depend on p , but we will make it clear from context which value of p we take.)

Note that for $m \geq n$ there are natural maps $(i_{n,m})_* : F_n \rightarrow F_m$ and $(j_{n,m}^k)_* : G_n^k \rightarrow G_m^k$ induced by inclusion.

Claim 2.20. *We have the direct limits*

$$F_\infty := \varinjlim F_n \cong H_p(X, X \setminus \{x\})$$

and

$$G_\infty^k := \varinjlim G_n^k \cong H_p(X_k, X_k \setminus \{f_k(x)\}) \cong \begin{cases} \mathbb{Z}, & \text{if } p = d, \\ 0, & \text{if } p \neq d. \end{cases}$$

Proof of Claim 2.20. We will show the first direct limit; the proof of the second is identical. The proof follows from standard properties of direct limits and singular homology. There are natural maps $\phi_n : F_n \rightarrow H_p(X, X \setminus \{x\})$ induced by inclusion. To show that $F_\infty \cong H_p(X, X \setminus \{x\})$, we must show two things (see, e.g., [22], Proposition A.4):

1. For every $a \in H_p(X, X \setminus \{x\})$, there exists $n \in \mathbb{N}$ and $b \in F_n$ such that $\phi_n(b) = a$.
2. If $b \in F_n$ and $\phi_n(b) = 0$, then $(i_{n,m})_*(b) = 0$ for some $m \geq n$.

To show (1), consider $a \in H_p(X, X \setminus \{x\})$. By excision and the fact that singular homology has compact support (see [31], 4.8.11), $a = j_*(c)$, where $c \in H_p(X, X \setminus U)$ for some open set U containing x , and

$$j_* : H_p(X, X \setminus U) \rightarrow H_p(X, X \setminus \{x\})$$

is the mapping induced by inclusion.

We now choose $n \in \mathbb{N}$ large enough so that $\overline{B}(x, (L')^{-n}r_0) \subset U$. There is a mapping

$$k_* : H_p(X, X \setminus U) \rightarrow H_p(X, X \setminus \overline{B}(x, (L')^{-n}r_0))$$

induced by inclusion.

Because all mappings are induced by inclusion, we have $\phi_n k_* = j_*$. Thus, if we let $b = k_*(c) \in H_p(X, X \setminus \overline{B}(x, (L')^{-n}r_0))$, we see that $\phi_n(b) = \phi_n k_*(c) = j_*(c) = a$. This proves part (1) of Claim 2.20.

To show part (2), suppose that $b \in F_n$ is such that $\phi_n(b) = 0 \in H_p(X, X \setminus \{x\})$. As before, using the fact that singular homology has compact support, we can write $b = l_*(c)$, where $c \in H_p(X, X \setminus U)$ for some open set U containing x , and

$$l_* : H_p(X, X \setminus U) \rightarrow F_n$$

is the mapping induced by inclusion.

By excision and [31], Theorem 4.8.13, we see that $i_*(c) = 0 \in H_p(X, X \setminus V)$, where $V \subset U$ is an open set containing x and

$$i_* : H_p(X, X \setminus U) \rightarrow H_p(X, X \setminus V)$$

is the mapping induced by inclusion.

We now choose $m \in \mathbb{N}$ large enough so that $\overline{B}(x, (L')^{-n}r_0) \subset V$. Let

$$h_* : H_p(X, X \setminus V) \rightarrow F_m$$

be induced by inclusion. Again because all mappings are compatible, we have

$$(i_{n,m})_*(b) = (i_{n,m})_*l_*(c) = h_*i_*(c) = h_*(0) = 0 \in F_m.$$

This completes the proof of Claim 2.20. □

Let $(i_n)_* : F_n \rightarrow F_\infty$ and $(j_n^k)_* : G_n^k \rightarrow G_\infty^k$ denote the natural inclusion maps.

The excision property of homology and the properties of f_k and g_k allow us to conclude the following: For all $n_0 \in \mathbb{N}$, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and $n \leq n_0$, there are group homomorphisms $a_n^k : F_n \rightarrow G_{n+1}^k$ and $b_n^k : G_n^k \rightarrow F_{n+1}$ that commute with the inclusion maps above, and that satisfy

$$b_{n+1}^k a_n^k = i_{n,n+2} \quad \text{and} \quad a_{n+1}^k b_n^k = j_{n,n+2}^k.$$

Indeed, a_n^k and b_n^k are simply the maps on homology induced by f_k and g_k , and so these properties follow from Lemma 2.15. The fact that a_n^k maps into G_{n+1}^k if $n \leq n_0$ k is sufficiently large follows from the fact that f_k preserves distances up to a small additive error, by Lemma 2.15.

In summary, for each n_0 there exists a k so that we have the following commutative diagram, in which the diagonal arrows do not exist past column n_0 :

$$(2.11) \quad \begin{array}{ccccccc} F_1 & \xrightarrow{i_{1,2}} & F_2 & \xrightarrow{i_{2,3}} & F_3 & \xrightarrow{i_{3,4}} & \cdots & \xrightarrow{i_{n_0-1,n_0}} & F_{n_0} & \xrightarrow{i_{n_0,n_0+1}} & \cdots & F_\infty \\ & \searrow a_1^k & \nearrow & \searrow a_2^k & \nearrow & \searrow a_3^k & \nearrow & \searrow a_{n_0-1}^k & \nearrow & & & \\ & b_1^k & & b_2^k & & b_3^k & & b_{n_0-1}^k & & & & \\ G_1^k & \xrightarrow{j_{1,2}^k} & G_2^k & \xrightarrow{j_{2,3}^k} & G_3^k & \xrightarrow{j_{3,4}^k} & \cdots & \xrightarrow{j_{n_0-1,n_0}^k} & G_{n_0}^k & \xrightarrow{j_{n_0,n_0+1}^k} & \cdots & G_\infty^k \end{array}$$

Note that Lemma 2.17 translates to the following information in this setting:

- (i) If $p \neq d$, then for each k and for each $m > n$, the map $j_{n,m}^k : G_n^k \rightarrow G_m^k$ is trivial.
- (ii) If $p = d$, then for each k, n the map $j_n^k : G_n^k \rightarrow G_\infty^k$ is surjective.
- (iii) If $p = d$, then for each k, n , we have $\ker j_n^k \subseteq \ker j_{n,n+1}^k$.

We wish to show that F_∞ is isomorphic to \mathbb{Z} if $p = d$ and is trivial if $p \neq d$, just as each of the spaces G_∞^k are.

Consider first the case $p \neq d$. By (i), we have that for all k and for all $m > n$, the maps $j_{n,m}^k$ are trivial. It follows by the diagram that the maps $i_{n,n+3}$ are all trivial (as they factor through $j_{n,n+1}^k$ for some k) and therefore that F_∞ is trivial when $p \neq d$.

Now we consider the case $p = d$.

Claim 2.21. *In degree $p = d$, $i_2 : F_2 \rightarrow F_\infty$ is surjective.*

Proof of Claim 2.21. This is just diagram-chasing. We will freely use the three properties of the diagram (2.11) described above, and we encourage the reader to simply trace the proof in that diagram.

Fix $x \in F_\infty$. Then $x = i_m(x_m)$ for some $m \in \mathbb{N}$, by the definition of the direct limit. Fix k large so that diagram (2.11) has diagonal arrows $a_l := a_l^k$ and $b_l := b_l^k$ up to $l = m + 3$. (We will suppress all superscripts k in the proof of this claim.)

Let $y_{m+1} = a_m(x_m) \in G_{m+1}$. Then some $y_1 \in G_1$ satisfies

$$j_1(y_1) = j_{m+1}(y_{m+1}),$$

by (ii), and so

$$j_{m+1}(y_{m+1}) = j_{m+1} j_{1,m+1}(y_1).$$

It follows, by (iii), that

$$j_{m+1,m+2}(y_{m+1}) = j_{1,m+2}(y_1).$$

Denote this element by $y_{m+2} \in G_{m+2}$.

Let $x_{m+3} = b_{m+2}(y_{m+2}) \in F_{m+3}$. We have

$$i_{2,m+3} b_1(y_1) = b_{m+2} j_{1,m+2}(y_1) = b_{m+2}(y_{m+2}) = x_{m+3}.$$

In addition,

$$\begin{aligned} x_{m+3} &= b_{m+2}(y_{m+2}) = b_{m+2} j_{m+1,m+2}(y_{m+1}) \\ &= b_{m+2} j_{m+1,m+2} a_m(x_m) = i_{m,m+3}(x_m) \end{aligned}$$

It follows that $i_{m+3}(x_{m+3}) = i_m(x_m) = x$, and so

$$i_2 b_1(y_1) = i_{m+3} i_{2,m+3} b_1(y_1) = i_{m+3}(x_{m+3}) = x,$$

and so i_2 is surjective. □

The following claim is also proven by a similar diagram chase.

Claim 2.22. *In dimension $p = d$, $\ker i_n \subset \ker i_{n,n+3}$.*

Proof of Claim 2.22. Suppose that for some $x_n \in F_n$, $i_n(x_n) = 0$. Then for some $m \geq n$, $i_{n,m}(x_n) = 0$. As in the previous claim, we now fix k large so that diagram (2.11) has diagonal arrows $a_l := a_l^k$ and $b_l := b_l^k$ up to column $l = m$. We then see that

$$j_{n+1,m+1} a_n(x_n) = a_m i_{n,m}(x_n) = a_m(0) = 0.$$

By (ii) above, it follows that $j_{n+1,n+2} a_n(x_n) = 0$. Thus,

$$i_{n,n+3}(x_n) = b_{n+2} j_{n+1,n+2} a_n(x_n) = b_{n+2}(0) = 0.$$

This completes the proof of Claim 2.22. □

Now fix k so that the diagonal arrows in diagram (2.11) exist up to $n = 10$. Let $G_\infty = G_\infty^k \cong \mathbb{Z}$. (Now that k is fixed we will again suppress the superscripts k .) We now define homomorphisms $\psi_n : F_n \rightarrow G_\infty \cong \mathbb{Z}$ by

$$\psi_n(x_n) = j_3 a_2 i_2^{-1} i_n(x_n).$$

Note that i_2 is surjective but not necessarily injective; nonetheless we have the following fact:

Claim 2.23. *The maps ψ_n are well-defined homomorphisms (i.e., independent of the choice of $i_2^{-1} i_n(x_n)$), and are compatible, in the sense that $\psi_m i_{n,m}(x_n) = \psi_n(x_n)$ for $m \geq n$.*

Proof of Claim 2.23. Suppose first that $i_2(x_2) = i_2(x'_2)$ for some $x_2, x'_2 \in F_2$. To show that ψ_n is well-defined we must show that

$$j_3 a_2(x_2) = j_3 a_2(x'_2).$$

By Claim 2.22, $i_{2,3}(x_2) = i_{2,3}(x'_2)$. Thus,

$$j_3 a_2(x_2) = j_4 j_{3,4}(x_2) = j_4 a_3 i_{2,3}(x_2) = j_4 a_3 i_{2,3}(x'_2) = j_3 a_2(x'_2).$$

This shows that ψ_n is well-defined. That ψ_n is a homomorphism is clear.

To see that $\psi_m(i_{n,m}(x_n)) = \psi_n(x_n)$, we note that if $i_2(x_2) = i_n(x_n)$, then $i_2(x_2) = i_m i_{n,m}(x_n)$. Thus,

$$\psi_m(i_{n,m}(x_n)) = j_3 a_2(x_2) = \psi_n(x_n). \quad \square$$

It follows that the maps ψ_n induce a homomorphism $h : F_\infty \rightarrow G_\infty \cong \mathbb{Z}$ satisfying $h \circ i_n = \psi_n$ for all n . We will show that h is injective and surjective.

Suppose $h(x) = 0$ for $x \in F_\infty$. By Claim 2.21, we can write $x = i_2 x_2$, for $x_2 \in F_2$. Therefore,

$$0 = h i_2(x_2) = \psi_2(x_2) = j_3 a_2(x_2).$$

Because $\ker j_3 \subseteq \ker j_{3,4}$, we have

$$j_{3,4} a_2(x_2) = 0 \in G_4.$$

It follows that

$$i_{2,5}(x_2) = b_4 j_{3,4} a_2(x_2) = 0 \in F_5$$

and therefore that $x = i_5 i_{2,5}(x_2) = 0 \in F_\infty$. This shows that h is injective.

To show that h is surjective, it suffices to show that $\psi_2 = j_3 a_2$ is surjective. Consider $y \in G_\infty$. Because j_1 is surjective, $y = j_1(y_1)$ for some $y_1 \in G_1$. Letting $x_2 = b_1(y_1)$, we see that

$$\psi_2(x_2) = j_3 a_2(x_2) = j_3 a_2 b_1(y_1) = j_1(y_1) = y.$$

This shows that h is surjective and is therefore an isomorphism $F_\infty \rightarrow G_\infty$.

We have therefore shown that $F_\infty \cong H_p(X, X \setminus \{x\})$ is isomorphic to $G_\infty \cong H_p(X_k, X_k \setminus \{x_k\})$, which completes the proof of Proposition 2.19. \square

2.4. Some basic degree theory

In this section, we give a degree-type lemma for close mapping packages. The idea here is quite simple, though the notation is cumbersome: If the limit of a suitable sequence of mappings is a homeomorphism, then sufficiently close limiting mappings should have images which contain a ball of fixed radius.

We now fix a set-up and some notation.

Let $\{(X_k, p_k)\}$ and $\{(Y_k, q_k)\}$ be two sequences of pointed metric spaces converging to (X, p) and (Y, q) , respectively. Suppose that all the spaces are uniformly Ahlfors s -regular, (L, r_0) -LLC, homology d -manifolds, and furthermore that $\{Y_k\}$ and Y are topological d -manifolds.

For some fixed $R > 0$, let $F_k = \overline{B}(p_k, R)$ and assume also that the sequence of pointed metric spaces $\{(F_k, p_k)\}$ converge to the pointed metric space (F, p) , where $F \subset X$ and $F \supset B(p, R)$ in X .

Finally, assume that $w_k : F_k \rightarrow Y_k$ are uniformly C -Lipschitz and that we have convergence of the sequence of mapping packages:

$$\{(F_k, p_k), (Y_k, q_k), w_k\} \rightarrow \{(F, p), (Y, q), w\}$$

By Lemma 2.15, there are continuous mappings

$$\begin{aligned} f_k &: F \rightarrow X_k, \\ g_k &: F_k \rightarrow X, \\ \tilde{f}_k &: \overline{B}_Y(q, 3CR) \rightarrow Y_k, \\ \tilde{g}_k &: \overline{B}_{Y_k}(q_k, 3CR) \rightarrow Y, \end{aligned}$$

that almost preserve distances and are almost inverses, up to additive error that decreases to zero with k .

Fix an open set $A \subseteq F$ such that the set $w(A)$ lies within a single chart of Y homeomorphic to an open subset of \mathbb{R}^d .

Lemma 2.24. *Suppose that, for some $r, r' \in (0, r_0)$, the ball $B_X(z, 4Lr)$ is contained in A , the map $w|_A$ is a homeomorphism, and $w(B_X(z, r)) \supseteq B_Y(w(z), r')$. Then for all k sufficiently large, $w_k(B_{X_k}(f_k(z), 2r)) \supseteq B_{Y_k}(w_k(f_k(z)), r'/2)$.*

Proof. First of all, it is clear that in the proof we may assume without loss of generality that Y is an orientable d -manifold, because all arguments can be carried out in the orientable chart of Y containing $w(A)$. This will allow us to apply Lemma 2.17 to subsets of $w(A)$.

Now let $0 < \eta < r'/(100L)$. For all k sufficiently large, the maps $f_k, g_k, \tilde{f}_k, \tilde{g}_k$ preserve distances up to additive error η . In addition, again by Lemma 2.15, we may assume that, for all k large and for all $r < 2CR$, the map

$$\tilde{g}_k \circ \tilde{f}_k|_{B(p,r)}$$

is η -homotopic to the inclusion map of $B(p, r)$ into $B(p, R)$.

Fix $k \in \mathbb{N}$ large enough for this to hold; from now on, this k will be fixed, so we drop the subscript and denote the above maps by $f, g, \tilde{f}, \tilde{g}$. By Lemma 2.16 and Proposition 5.8 in [27], we can also arrange that the maps $\tilde{f} \circ w$ and $w_k \circ f$, when restricted to A , are η -homotopic on F .

Let $B = \overline{B}_X(z, r)$. Fix $y \in B_{Y_k}(w_k(f_k(z)), r'/2)$.

First, because w is a homeomorphism on A , the induced mapping on relative homology,

$$w_* : H_d(A, A \setminus B) \rightarrow H_d(Y, Y \setminus w(B))$$

is an isomorphism. (Here we use excision for singular homology, see [31], Corollary 4.6.5.) Note that $H_d(Y, Y \setminus w(B))$ is non-trivial, by duality (e.g. [31], Theorem 6.2.17).

Let $V = B_Y(q, 2CR)$. The map \tilde{f} induces a non-trivial map

$$\tilde{f}_* : H_d(V, V \setminus w(B)) \rightarrow H_d(Y_k, Y_k \setminus B''),$$

where $B'' = \overline{B}(y, r'')$, $r'' = r'/(10L)$. Indeed, if this map were trivial, then the map

$$\tilde{g}_* \tilde{f}_* : H_d(V, V \setminus w(B)) \rightarrow H_d(Y, Y \setminus \{a\}),$$

factoring as it does through the previous map, would be trivial for some $a \in w(B)$. But this map on homology is the same as that induced by inclusion, so this cannot be the case by the duality argument of Lemma 2.17 (ii).

It follows that the map

$$(\tilde{f} \circ w)_* = \tilde{f}_* w_* : H_d(A, A \setminus B) \rightarrow H_d(Y_k, Y_k \setminus B'')$$

is non-trivial.

Because $\tilde{f} \circ w$ and $w_k \circ f$ are η -homotopic, the map

$$(w_k \circ f)_* : H_d(A, A \setminus B) \rightarrow H_d(Y_k, Y_k \setminus B'')$$

is non-trivial.

This implies that

$$(w_k \circ f)_* : H_d(A, A \setminus B) \rightarrow H_d(Y_k, Y_k \setminus \{y\})$$

is non-trivial. Indeed, if not, then by Lemma 2.17 (iii),

$$(w_k \circ f)_* : H_d(A, A \setminus B) \rightarrow H_d(Y_k, Y_k \setminus B'')$$

would be trivial, but we just showed that it is not.

So we have shown that

$$(w_k \circ f)_* : H_d(A, A \setminus B) \rightarrow H_d(Y_k, Y_k \setminus \{y\})$$

is non-trivial. It follows from this that $y \in (w_k \circ f)(B)$, otherwise this map would factor through the trivial $H_d(Y_k \setminus \{y\}, Y_k \setminus \{y\})$.

Because $f(B) \subset B(f_k(z), 2r)$, we get that

$$y \in w_k(f(B)) \subset w_k(B(f_k(z), 2r)). \quad \square$$

Later on, it will be convenient to work with a cohomological notion of local degree, which we introduce now. The following material is taken from [15]. For proofs, see [25].

Let H_c^* denote the Alexander–Spanier cohomology groups with compact supports and coefficients in \mathbb{Z} . (For the definition and properties of Alexander–Spanier cohomology, see [22].) The following definition is taken from [15], I.1.

Definition 2.25. A locally compact, Hausdorff, connected, and locally connected space M is called a *generalized d -manifold* if

- $H_c^p(U) = 0$ whenever $U \subseteq M$ is open and $p \geq d + 1$;
- for every $x \in M$ and every open neighborhood U of x , there is another open neighborhood V of x contained in U such that

$$H_c^p(V) = \begin{cases} \mathbb{Z} & \text{if } p = d \\ 0 & \text{if } p = d - 1 \end{cases}$$

and the standard homomorphism $H_c^n(W) \rightarrow H_c^n(V)$ is surjective whenever W is an open neighborhood of x contained in V ;

- X has finite topological dimension.

A generalized d -manifold X is called *orientable* if $H_c^d(X) = \mathbb{Z}$, and *oriented* if we fix a choice of generator in $H_c^d(X)$.

Remark 2.26. Any homology d -manifold is a generalized d -manifold, as noted in [15], Example 1.4 (c).

A generalized d -manifold X is said to be *oriented* if $H_c^d(X) = \mathbb{Z}$. In this case we can simultaneously orient all connected open subsets U of X via the isomorphism between $H_c^d(U)$ and $H_c^d(X)$.

We will not use any sophisticated facts about cohomology below, but only the following object and its basic properties: Let X and Y be oriented generalized d -manifolds, and let $f: X \rightarrow Y$ be continuous. For any relatively compact domain D in X , and for every $y \in Y \setminus f(\partial D)$, we can associate an integer called the local degree $\mu(y, D, f)$. In the following lemma, we collect the only properties of μ we will need.

Lemma 2.27. *For continuous maps f and g between oriented generalized d -manifolds X and Y , and a relatively compact domain $D \subseteq X$, the local degree μ has the following properties.*

- The function $y \rightarrow \mu(y, D, f)$ is constant on each connected component of $Y \setminus f(\partial D)$.
- If $y \notin f(\overline{D})$, then $\mu(y, D, f) = 0$.
- If $f: D \rightarrow f(D)$ is a homeomorphism, then $\mu(y, D, f) = \pm 1$ for each $y \in f(D)$.
- If $y \in Y \setminus f(\partial D)$ and if $f^{-1}(y) \subset D'$, where D' is a domain contained in D such that $y \in Y \setminus f(\partial D')$, then

$$\mu(y, D, f) = \mu(y, D', f).$$

Proof. These facts can all be found in [15], 2.3 or [25], II.2. □

2.5. The Bonk–Kleiner theorem on mappings of bounded multiplicity

This material is taken from [2].

Definition 2.28. A map f between spaces X and Y is of *bounded multiplicity* if there is a constant $N \in \mathbb{N}$ such that $\#f^{-1}(y) \leq N$ for all $y \in Y$.

The following result of Bonk and Kleiner provides a partial substitute, in our setting, for Reshetnyak’s theorem on quasi-regular mappings. (See the discussion of David’s proof in Subsection 1.3.)

Theorem 2.29 ([2], Theorem 3.4). *Suppose X is a compact metric space, every non-empty open subset of X has topological dimension at least d , and $f: X \rightarrow \mathbb{R}^d$ is a continuous map of bounded multiplicity. Then there is an open subset $V \subseteq f(X)$ with $\bar{V} = f(X)$ such that $U = f^{-1}(V)$ is dense in X and $f|_U : U \rightarrow V$ is a covering map.*

3. Warm-up: getting bi-Lipschitz weak tangents

In this section, we prove a result that is much weaker than Theorem 1.1, but whose proof illustrates some of the techniques used in the proof of Theorem 1.1. Nothing in this section is needed in the proof of Theorem 1.1, so a reader who is solely interested in that proof can skip this section without missing anything needed later in the paper.

Let (X, d) and (Y, ρ) be metric spaces and $f: X \rightarrow Y$ be Lipschitz. Define a *weak tangent of f* to be a mapping package

$$((X_\infty, d_\infty, x_\infty), (Y_\infty, \rho_\infty, y_\infty), f_\infty)$$

for which there is a sequence of positive real numbers λ_n , bounded above, and a sequence $x_n \in X$ such that, in the sense of Definition 2.5, we have

$$((X, \lambda_n^{-1}d, x_n), (Y, \lambda_n^{-1}\rho, f(x_n)), f) \rightarrow ((X_\infty, d_\infty, x_\infty), (Y_\infty, \rho_\infty, y_\infty), f_\infty)$$

as $n \rightarrow \infty$.

Note that the spaces X_∞ and Y_∞ here are “weak tangents” of the spaces X and Y , as in Definition 1.5.

We will say that f has a bi-Lipschitz weak tangent at x if, for one of its weak tangent mapping packages, the mapping f_∞ which arises is bi-Lipschitz.

Suppose that f is Lipschitz and that X and Y are doubling metric spaces. Suppose also that X is equipped with a doubling measure, and that x is a point of density of a set $E \subset X$ such that $f|_E$ is bi-Lipschitz. Consider any sequence $\lambda_n \rightarrow 0$. Then every weak tangent of f along the sequence of scales $\{\lambda_n\}$ and the sequence of points $\{x_n = x\}$ yields a mapping f_∞ that is bi-Lipschitz. This is a standard fact, and its proof is very similar to that given in Proposition 8.1 below.

Thus, a mapping having a positive-measure set on which it is bi-Lipschitz is a much stronger condition than a map merely having a bi-Lipschitz weak tangent.

In the setting of Theorem 1.1, one can give a simpler argument which shows that the mapping has a bi-Lipschitz weak tangent. This argument is really contained in [2], though our context is slightly different.

In the proof, we will need one definition that we have not yet introduced, coming from Chapter 12 of [10]. (This will not be used in the proof of the main Theorem 1.1.)

Definition 3.1. A Lipschitz mapping $f: M \rightarrow N$ between two metric spaces is said to be *David–Semmes regular* if there is a constant $C > 0$ such that, for every ball $B \subseteq N$ of radius r , the set $f^{-1}(B)$ can be covered by at most C balls of radius Cr .

In particular, David–Semmes regular maps always have bounded multiplicity.

Proposition 3.2. *Let X and Y be Ahlfors s -regular, linearly locally contractible, complete, oriented, topological d -manifolds, for $s, d \geq 1$. Suppose in addition that Y has d -manifold weak tangents.*

Suppose that $f: X \rightarrow Y$ has $|f(X)| > 0$. Then, for some $x \in X$, f has a bi-Lipschitz weak tangent.

Proof. The first step is to apply Proposition 12.8 of [10]. This says that, for some $x \in X$, we can find a weak tangent

$$((X_\infty, d_\infty, x_\infty), (Y_\infty, \rho_\infty, y_\infty), f_\infty)$$

of f such that f_∞ is a David–Semmes regular map. In particular, this means that f_∞ is a mapping of bounded multiplicity, in the sense of Definition 2.28.

The next step is to examine the spaces X_∞ and Y_∞ . By Proposition 2.19 and the assumption that Y has d -manifold weak tangents, we see that X_∞ is a homology d -manifold and Y_∞ is a topological d -manifold. We also have, by Proposition 2.13, that Y_∞ is (L, r_0) -LLC, for some constants L and r_0 .

We would now like to apply Theorem 2.29 to f_∞ . Fix a small open ball $B \subset X_\infty$. We can choose B so small that $f_\infty(\overline{B})$ lies in a set $V \subset Y_\infty$ which is homeomorphic to an open set in \mathbb{R}^d , and which has diameter less than the contractibility radius r_0 of Y_∞ .

Let $K = \overline{B}$. Then every open subset of K contains an open subset of the homology d -manifold X_∞ and thus has topological dimension at least d . Because we also know that f_∞ has bounded multiplicity on K , we can apply Theorem 2.29.

In particular, we obtain an open set $U \subset K \subset X$ such that f_∞ , when restricted to U , is a homeomorphism. Let $V' \subset f_\infty(U)$ be a small open set such that

$$\text{dist}(V', Y_\infty \setminus f_\infty(U)) > L \text{diam } V'$$

and let $U' = f_\infty^{-1}(V') \cap U$.

We claim that f_∞ is in fact bi-Lipschitz on U' . We already know it to be Lipschitz, so it suffices to establish the other bound. Fix $x, y \in U'$ and consider $f(x), f(y) \in V' \subset Y_\infty$. Let $r = \rho_\infty(f(x), f(y))$; note that $r < r_0$ by our assumptions.

First of all, there is a compact connected set $S \subset f(U)$ containing $f(x)$ and $f(y)$ such that $\text{diam } S \leq Lr$. Indeed, by our assumptions, the compact set $\overline{B}(f(x), r)$ is contractible within $\overline{B}(f(x), Lr)$. If H is the homotopy realizing this contractibility, then

$$S = H(\overline{B}(f(x), r) \times [0, 1])$$

contains $f(x)$ and $f(y)$ and is compact, connected, and contained in $\overline{B}(f(x), Lr) \subset f_\infty(U)$.

Now consider $E = f_\infty^{-1}(S) \cap U$. Because f_∞ is a homeomorphism on U , we have that E is a compact, connected set in U that contains x and y . Because f_∞ is David–Semmes regular, E is contained in the union of C balls of radius CLr in X_∞ . It follows that $\text{diam } E \leq 2C^2Lr$.

Thus,

$$d_\infty(x, y) \leq \text{diam } E \leq 2C^2Lr = 2C^2L\rho_\infty(f(x), f(y)),$$

and so f_∞ is bi-Lipschitz on U' .

To complete the proof of the Proposition, we take another weak tangent of f_∞ along a sequence of scales $\{\lambda_n\}$ tending to zero and a fixed base-point sequence $\{x_n = x \in U'\}$. This yields a weak tangent of f_∞ which is globally bi-Lipschitz. That this is also a weak tangent of f itself is a standard fact (see [10], Lemma 9.22). \square

4. Setting up the proof of Theorem 1.1

We first introduce the following notation:

$$\tilde{B}_n(x, r) = \bigcup \{Q \in \Delta_n : Q \cap B(x, r) \neq \emptyset\}$$

By Theorem 1.9, to prove Theorem 1.1 it suffices to show the following proposition, which is just a restatement of David’s condition, formulated in Definition 1.8.

Proposition 4.1. *Let $d \in \mathbb{N}$ and $s > 0$. Suppose (Y, ρ) is LLC, Ahlfors s -regular, and has d -manifold weak tangents. For all C_0, L, r_0, M and for all $\lambda, \gamma \geq 0$, there exist $\Lambda, \eta > 0$ such that the following holds.*

Let X be a complete, oriented, topological d -manifold which is Ahlfors s -regular with constant C_0 and (L, r_0) -LLC. Let I_0 be a 0-cube and $z: I_0 \rightarrow Y$ an M -Lipschitz map. If $x \in X, n \in \mathbb{Z}$, and $T = \tilde{B}_n(x, \Lambda 2^n) \subseteq I_0$ satisfies $|z(T)|/|T| \geq \gamma$, then one of the following holds:

- (i) $z(T) \supseteq B(z(x), \lambda 2^n)$, or
- (ii) there is an n -cube $R \subset T$ such that

$$\frac{|z(R)|}{|R|} \geq (1 + 2\eta) \frac{|z(T)|}{|T|}.$$

We emphasize that in Proposition 4.1 the constants Λ and η depend only on the “input” constants λ and γ , as well as the “data” d, s, C_0, L, r_0, M , and the space Y .

We will actually prove the following similar statement, which implies Proposition 4.1. (This is analogous to Lemma 4 of [7].) For $r > 0$, define n_r to be the largest integer n such that

$$(4.1) \quad 10 C_0 2^n \leq r.$$

Proposition 4.2. *Let $d \in \mathbb{N}$ and $s > 0$. Suppose (Y, ρ) is LLC, Ahlfors s -regular, and has d -manifold weak tangents. For all C_0, L, r_0 and for all $\gamma > 0$, there exist $\tau, \sigma > 0$ such that the following holds:*

Let X be a complete, oriented, topological d -manifold which is Ahlfors s -regular with constant C_0 and (L, r_0) -LLC. If $v \in X$, $0 < r \leq C_0$, $T = \tilde{B}_{n_r}(v, r)$, and $z: T \rightarrow Y$ is 1-Lipschitz satisfying $|z(T)|/|T| \geq \gamma$, then one of the following holds:

- (i) $z(T) \supseteq B(z(v), \tau r)$, or
- (ii) *there is a dyadic cube $R \subset T$ of diameter at least τr such that*

$$\frac{|z(R)|}{|R|} \geq (1 + \sigma) \frac{|z(T)|}{|T|}.$$

As before, the constants τ and σ in Proposition 4.2 depend only on d, s, C_0, L, r_0 , and γ , as well as the space Y .

Lemma 4.3. *Proposition 4.2 implies Proposition 4.1.*

Proof. Suppose that Proposition 4.2 is true but that Proposition 4.1 fails. The failure of Proposition 4.1, first of all, implies the existence of dimensions $d \in \mathbb{N}$, $s > 0$, and a space (Y, ρ) . It also implies that for some data C_0, L, r_0, M , some constants $\lambda, \gamma > 0$ and every $\Lambda, \eta > 0$, there exists an Ahlfors s -regular, LLC, complete oriented topological d -manifold X (with data given by C_0, L, r_0), a 0-cube $I_0 \subset X$, and

$$T = \tilde{B}_n(x, \Lambda 2^n) \subset I_0,$$

as well as an M -Lipschitz $z: T \rightarrow Y$ with $|z(T)|/|T| \geq \gamma$ such that

- $z(T) \not\supseteq B(z(x), \lambda 2^n)$, and
- for every n -cube $R \subset T$,

$$\frac{|z(R)|}{|R|} \leq (1 + 2\eta) \frac{|z(T)|}{|T|}.$$

In the proof, our goal is to choose Λ large enough and η small enough to reach a contradiction.

We now reduce to the 1-Lipschitz case by letting $\tilde{z}: T \rightarrow (Y, \frac{1}{M}\rho)$. Then $\tilde{z}: T \rightarrow (Y, \frac{1}{M}\rho)$ satisfies

- $|\tilde{z}(T)|/|T| \geq \tilde{\gamma} = \gamma/M^s$
- $\tilde{z}(T) \not\supseteq B_{\frac{1}{M}Y}(z(x), \lambda 2^n/M)$, and

- for every n -cube $R \subset T$,

$$\frac{|\tilde{z}(R)|}{|R|} \leq (1 + 2\eta) \frac{|\tilde{z}(T)|}{|T|}.$$

Let $T' = \tilde{B}_{n_r}(x, r)$ for $r = \Lambda 2^n / 10$. Note that, as $T \subset I_0$, we have $\text{diam } T \leq \text{diam } I_0$ and so $r \leq C_0$.

Note also that $T' \subseteq T$, by a simple triangle inequality argument. On the other hand, as we may choose $\Lambda > C_0$, we have

$$B(x, \Lambda 2^n / 10) \subset T' \subset T \subset B(x, (\Lambda + C_0) 2^n) \subset B(x, 2\Lambda 2^n),$$

and so the relative measure $|T'|/|T|$ is bounded below by a constant depending only on s and C_0 .

If $\Lambda > 200C_0$, then T' , and therefore also $T \setminus T'$, is a disjoint union of n -cubes. Indeed, in this case $n_r \geq n$, and T is a disjoint union of n_r -cubes, each of which is a disjoint union of n -cubes.

It follows from the second property of \tilde{z} above that

$$\frac{|\tilde{z}(T')|}{|T'|} \leq (1 + 2\eta) \frac{|\tilde{z}(T)|}{|T|} \quad \text{and} \quad \frac{|\tilde{z}(T \setminus T')|}{|T \setminus T'|} \leq (1 + 2\eta) \frac{|\tilde{z}(T)|}{|T|}.$$

Therefore,

$$\begin{aligned} |\tilde{z}(T')| &\geq |\tilde{z}(T)| - |\tilde{z}(T \setminus T')| \geq |\tilde{z}(T)| - (1 + 2\eta) \frac{|\tilde{z}(T)|}{|T|} |T \setminus T'| \\ &= (|T| - (1 + 2\eta) |T \setminus T'|) \frac{|\tilde{z}(T)|}{|T|} = ((1 + 2\eta) |T'| - 2\eta |T|) \frac{|\tilde{z}(T)|}{|T|} \\ &\geq ((1 + 2\eta) |T'| - 2\eta C |T'|) \frac{|\tilde{z}(T)|}{|T|} \geq (1 - C' \eta) |T'| \frac{|\tilde{z}(T)|}{|T|} \geq \frac{\tilde{\gamma}}{3} |T'| \end{aligned}$$

if η is small depending on γ . (Here C and C' depend only on the Ahlfors regularity constants s and C_0 .)

Now, apply Proposition 4.2 to $\tilde{z}: T' \rightarrow (Y, \frac{1}{M}\rho)$ with γ as $\tilde{\gamma}/3$. We obtain τ and σ . Note that τ and σ depend only on the data d, s, C_0, L, r_0, M , the space Y , and the constant γ .

If $\Lambda > \max\{\frac{10\lambda}{M\tau}, \frac{10C_0}{\tau}\}$ and η is sufficiently small relative to σ , we get that either

- $\tilde{z}(T) \supseteq \tilde{z}(T') \supseteq B_{\frac{1}{M}Y}(\tilde{z}(x), \tau \Lambda 2^n / 10) \supset B_{\frac{1}{M}Y}(\tilde{z}(x), \lambda 2^n / M)$, or
- there is a dyadic cube $R \subset T'$ of diameter at least $\tau \Lambda 2^n$ such that

$$\begin{aligned} \frac{|\tilde{z}(R)|}{|R|} &\geq (1 + \sigma) \frac{|\tilde{z}(T')|}{|T'|} \\ &\geq (1 + \sigma)(1 - C' \eta) \frac{|\tilde{z}(T)|}{|T|} \geq (1 + 3\eta) \frac{|\tilde{z}(T)|}{|T|}. \end{aligned}$$

In the first case we contradict the assumption that the first conclusion in Proposition 4.1 fails. In the second case, note that R is a cube at scale larger than n (because $\tau\Lambda 2^n > C_0 2^n$) and therefore a disjoint union of n -cubes. At least one of those n -cubes R' must then also satisfy

$$\frac{|\tilde{z}(R')|}{|R'|} \geq (1 + 3\eta) \frac{|\tilde{z}(T)|}{|T|},$$

which contradicts the assumption that the second conclusion of Proposition 4.1 fails. □

5. Proof of Proposition 4.2

We will use the notation of the previous section; recall especially the definition of n_r from (4.1).

Suppose now that Proposition 4.2 is false. Then there exists constants $d, s, C_0, L, r_0, \gamma$, and a space (Y, ρ) that is LLC, Ahlfors s -regular and has d -manifold weak tangents, such that the following holds:

For every $k \in \mathbb{N}$, there are spaces Z_k that are Ahlfors s -regular with constant C_0 and that are (L, r_0) -LLC, complete oriented topological d -manifolds. In addition, there are radii $0 < r_k \leq C_0$, subsets $T_k = \tilde{B}_{n_{r_k}}(v_k, r_k) \subset Z_k$ and 1-Lipschitz maps $z_k : Z_k \rightarrow Y$ satisfying $|z_k(T_k)|_Y \geq \gamma|T_k|_{Z_k}$ and such that:

- (i) $z_k(T_k) \not\supseteq B(z_k(v_k), \frac{1}{k}r_k)$, and
- (ii) for every dyadic cube $R \subseteq T_k$ of diameter at least r_k/k , we have

$$\frac{|z_k(R)|}{|R|} \leq \left(1 + \frac{1}{k}\right) \frac{|z_k(T_k)|}{|T_k|}.$$

Let X_k be the metric space

$$\left(Z_k, \frac{1}{r_k}d_{Z_k}\right).$$

Let $S_k \subset X_k$ denote the corresponding rescaled version of T_k . Then

$$B(v_k, 1) \subseteq S_k \subseteq B(v_k, 2) \quad \text{and} \quad C_0 \leq |S_k|_{X_k} \leq 2^s C_0.$$

Note that S_k has a dyadic cube decomposition given by the rescaled versions of cubes in T_k . The following additional technical fact about this decomposition of S_k is obvious but useful.

Lemma 5.1. *For every $0 < r < 1/20$ and every $k \in \mathbb{N}$, the set S_k can be written as a disjoint union of measurable sets R_j satisfying*

- $(2C_0^2)^{-1}r \leq \text{diam } R_j \leq r$, and
- $(2C_0^2)^{-s}r^s \leq |R_j|_{X_k} \leq r^s$

Proof. Choose n such that

$$C_0 2^n \leq r r_k \leq 2 C_0 2^n .$$

If $r < 1/20$, then

$$2 C_0 2^n \leq 2r r_k < r_k/10 \leq 2 C_0 2^{n r_k}$$

and so $n \leq n_{r_k}$.

Therefore, we can write T_k as a disjoint union of dyadic cubes in Δ_n . The rescaled versions of these cubes in S_k are now immediately seen to satisfy the required properties. \square

For each k , we also consider the rescaled target spaces

$$Y_k = (Y, \rho_k) = \left(Y, \left(\frac{\gamma |T_k|}{|z_k(T_k)|} \right)^{1/s} \frac{1}{r_k} \rho \right) .$$

Let $w_k : S_k \rightarrow Y_k$ be the map z_k (making the natural identification between points of Z_k and points of X_k). Then each w_k is Lipschitz with constant

$$\left(\frac{\gamma |T_k|}{|z_k(T_k)|} \right)^{1/s} \leq 1 .$$

In addition, the maps w_k satisfy

$$|w_k(S_k)| = \gamma |S_k|$$

for all k . (The extra rescaling factor $(\gamma |T_k| / |z_k(T_k)|)^{1/s}$ in the target Y is to ensure this last convenient fact.)

Finally, the two important properties of z_k pass to w_k in the following way:

$$(5.1) \quad w_k(S_k) \not\supseteq B\left(w_k(v_k), \frac{1}{k}\right)$$

and for every dyadic cube $R \subseteq S_k$ of diameter at least $1/k$, we have

$$(5.2) \quad \frac{|w_k(R)|}{|R|} \leq \left(1 + \frac{1}{k}\right) \frac{|w_k(S_k)|}{|S_k|} = \left(1 + \frac{1}{k}\right) \gamma .$$

Let $F_k = \overline{B(v_k, 1/2)} \subset S_k \subset X_k$. We may now consider the following sequence of mapping packages (see Definition 2.4):

$$\left\{ \left((F_k, d_{X_k}, v_k), \left(Y, \frac{1}{r_k} \rho_k, w_k(v_k) \right), w_k \right) \right\} .$$

Note that all the spaces in the above mapping packages are complete and uniformly doubling, and the mappings w_k are uniformly 1-Lipschitz. By applying Proposition 2.6, we obtain a subsequence of this mapping package that converges to a limit $\{(F, d, v), (M, d', q), w\}$. In addition, by Lemma 2.9 we may assume that along this subsequence we also have the convergence of the sequence of ambient source spaces (X_k, d_{X_k}, v_k) to a space (X, d, v) that contains F as a subset. (We continue to index this sequence by the original parameter k .)

The following diagram may be useful for keeping track of this convergence. The dotted arrows represent convergence of spaces in the sense of Definition 2.3.

$$(5.3) \quad \begin{array}{ccccc} X_k & \supset & S_k & \supset & F_k \xrightarrow{w_k} Y_k \\ \vdots \downarrow & & & & \vdots \downarrow \\ X & & & \supset & F \xrightarrow{w} M \\ & & & & \vdots \downarrow \end{array}$$

We now know, by Proposition 2.19, that the space X is an LLC, Ahlfors s -regular, homology d -manifold. In addition, by Lemmas 2.10 and 2.13 and the assumption that Y has d -manifold weak tangents, the space M is an Ahlfors s -regular, LLC, topological d -manifold. Finally, it is clear that the set $F \subset X$ contains the open ball $B(v, 1/2)$.

The space X is a generalized d -manifold (see Definition 2.25), so we may now fix an open subset of $B(v, 1/2) \subset F$ which has H_c^d isomorphic to \mathbb{Z} , i.e., is itself an oriented generalized d -manifold. We will only work in this oriented subset of X from now on.

Let A be a small open ball in X (of diameter smaller than half the contractibility radius of X) centered at v and compactly contained in this oriented open subset. Because M is a manifold and w is Lipschitz, by making A small enough, we may assume that $w(\overline{A})$ lies in a single chart of M . Let $K = \overline{A}$, which is compact.

We now investigate the limit map w .

Lemma 5.2. *The map $w|_K$ is of bounded multiplicity on K . In other words, there exists $N \in \mathbb{N}$ such that for every $x \in M$, there are at most N points in $w^{-1}(x) \cap K$.*

Proof. We will show that there exists N such that for all $r < 1/20$ and every $y \in M$, $w^{-1}(B(y, r)) \cap K$ is contained in the union of N balls of radius r in X . This clearly suffices to prove the lemma. (This essentially shows the stronger statement that w is a David–Semmes regular mapping, as in Definition 3.1, but we do not need this here.)

Recall from Propositions 2.7 and 2.8 that there are “almost-isometries” $\phi_k : F \rightarrow F_k \subset X_k$ and $\sigma_k : Y \rightarrow Y_k$, which, on some fixed ball, preserve distances up to an additive error that tends to zero as k approaches infinity. In addition, it follows immediately from those propositions that

$$\lim_{k \rightarrow \infty} \rho_k(w_k(\phi_k(x)), \sigma_k(w(x))) = 0$$

locally uniformly on $F \subset X$.

Fix a ball $B(y, r)$ in M . Let $E = w^{-1}(B(y, r)) \cap K$. Let $E_k = \phi_k(E) \subseteq X_k$. Note that if k is sufficiently large, we have both that $E_k \subset S_k$ and $w_k(E_k) \subset B(\sigma_k(y), 2r)$. By Lemma 5.1 we may write S_k as a disjoint union of cubes Q , each satisfying

$$(2C_0^2)^{-1}r \leq \text{diam } Q \leq r \quad \text{and} \quad (2C_0^2)^{-d}r^s \leq |Q| \leq r^s.$$

We will call these cubes “ r -sized”.

Let \mathcal{Q} denote the collection of r -sized cubes in S_k that intersect E_k , and let $N_k = \#\{Q \in \mathcal{Q}\}$. Because w_k is 1-Lipschitz on S_k ,

$$w_k(Q) \subset B(\sigma_k(y), (2 + 2C_0)r) \subset Y_k$$

for all $Q \in \mathcal{Q}$.

Therefore, dividing S_k into those r -sized cubes that are in \mathcal{Q} and those that are not (and taking all Hausdorff measures with respect to X_k and Y_k) we see that

$$\begin{aligned} \gamma|S_k| &= |w_k(S_k)| \leq \left| \bigcup_{Q \in \mathcal{Q}} w_k(Q) \right| + \left| \bigcup_{Q \notin \mathcal{Q}} w_k(Q) \right| \\ &\leq |B(\sigma_k(y), (2 + 2C_0)r)| + \gamma(1 + 1/k) \sum_{Q \subset S_k, Q \in \Delta_{n_r} \setminus \mathcal{Q}} |Q| \\ &= |B(\sigma_k(y), (2 + 2C_0)r)| + \gamma(1 + 1/k) \left(|S_k| - \sum_{Q \in \mathcal{Q}} |Q| \right) \\ &\leq C_1 r^s + \gamma(1 + 1/k)(|S_k| - N_k C_2 r^s) \end{aligned}$$

where C_1 depends only on C_0 and the Ahlfors-regularity constant of Y , and $C_2 = (2C_0^2)^{-s}$.

Rearranging this inequality yields

$$N_k \leq \frac{C_1 r^s + \frac{1}{k}|S_k|}{\gamma(1 + \frac{1}{k})C_2 r^s}.$$

Because the measures $|S_k|$ are uniformly bounded, we see that for all k sufficiently large (depending on r , but that is fine), we have

$$N_k \leq \frac{2C_1}{C_2 \gamma}.$$

Since each cube in \mathcal{Q} is contained in a ball of radius $2C_0r$ in X_k , and each X_k is doubling with constant depending only on C_0 and d , we get that E_k is contained in a union of N balls of radius r , where N depends only on s, C_0 and γ . (This holds for all k sufficiently large.)

It immediately follows that the same holds for E (with a possibly larger N) by using the distance-preserving properties of ψ_k and ϕ_k for k large, and the fact that X is doubling. This proves the lemma. □

Remark 5.3. In the proof of Lemma 5.2, we used the fact that w is a limit of mappings w_k , each of which does not multiply the measure of cubes of size at least $1/k$ by much more than the factor γ . The proof would be a bit simpler if we knew that w itself does not expand the measure of any cube by more than a factor γ , because then the computations above could all be carried out in the limit $w: X \rightarrow M$, rather than in the limiting objects $w_k: X_k \rightarrow Y_k$. However, it is not clear (and not necessary for us) that this “non-expanding” property of the maps w_k passes directly to the limit map w , especially since there is no obvious relationship between cubes in X_k and cubes in X . The same issue arises in Lemma 5.4 below. For an interesting and somewhat related example, see Theorem2 in [5].

Note now that the set K is a compact set that is the closure of an open set in the homology d -manifold X . It follows that every relatively open subset of K contains an open subset of X and thus has topological dimension at least d (see [15], Remark 1.3(b)). Recall our assumption that $w(K)$ lies in a single chart of M . As w has bounded multiplicity on K , we can apply Theorem 2.29 to obtain a dense open subset V in $w(K)$ such that $U = w^{-1}(V) \cap K$ is dense in K and $w|_U$ is a covering map.

Lemma 5.4. *Every point in V has exactly one pre-image in K under w .*

Proof. In other words, what we must show is that if $x \in U$ and $x' \in K$ with $x' \neq x$, then $w(x) \neq w(x')$. Suppose to the contrary that $w(x) = w(x') = y \in V$. As $x \in U$ and w is a covering map when restricted to U , we obtain a ball $B(x, r) \subset U$ such that $w|_{B(x, r)}$ is a homeomorphism and $w(B(x, r))$ contains a ball $B(y, r') \subset M$. Without loss of generality, we may take $r < d(x, x')/10C_0$ and $r < 1/20$.

Recall the the continuous “almost isometries” $f_k : K \rightarrow X$ from Lemma 2.15. By Lemma 2.24, for all k sufficiently large, we obtain $x_k = f_k(x) \in S_k$ such that $w_k(B(x_k, 2r))$ contains the ball $B(y_k, r'/2) \subset Y_k$, where $y_k = w_k(x_k)$. Also let $x'_k = f_k(x')$. For all k large, we have $\rho_k(w_k(x_k), w_k(x'_k)) < r/10$, because $w(x) = w(x')$.

Let $r_1 = \min\{r, r'\}$. By Lemma 5.1, we may write S_k as the disjoint union of sets Q such that

$$(2C_0^2)^{-1}r_1/10 \leq \text{diam } Q \leq r_1/10$$

and

$$(2C_0^2)^{-d}(r_1/10)^s \leq |Q| \leq (r_1/10)^s.$$

One of these sets Q contains the point x'_k ; let Q_0 denote that set. In addition, let T be the union of all these sets Q that intersect $B(x_k, 2r)$. Note that Q_0 is not in T by our choice of r and r_1 . Then

$$w_k(Q_0) \subset B(y_k, r'/2) \subset w_k(T).$$

We now sum over all the sets Q in S_k as above that are not Q_0 . Because $w_k(Q_0) \subseteq \bigcup_{Q \neq Q_0} w_k(Q)$, we have that

$$\gamma|S_k| = |w_k(S_k)| \leq \sum_{Q \neq Q_0} |w_k(Q)| \leq \gamma(1 + 1/k) \sum_{Q \neq Q_0} |Q| \leq \gamma(1 + 1/k)(|S_k| - C_3r_1^s)$$

where $C_3 = (2C_0^2)^{-s}$.

Rearranging and recalling that $|S_k| \leq C_02^s = C_4$, we get

$$\gamma C_3r_1^s \leq \frac{\gamma}{k}(C_4 - C_3r_1^s),$$

which is a contradiction for k large. □

Lemma 5.5. *The map $w|_A : A \rightarrow M$ is an open mapping.*

Proof. We use the notion of local degree defined in Subsection 2.4, which we may apply to the oriented generalized d -manifold containing A .

Suppose w is not an open mapping on A . Then there is a point $x \in A$ and an open set $G \subseteq A$ containing x such that $y = w(x)$ is not an interior point of $w(G)$. Since w has bounded multiplicity, we can find a closed ball in G containing x and no other pre-images of y . Let B be a connected open subset of this ball containing x . Then $\overline{B} \cap w^{-1}(y) = \{x\}$.

We now claim that the local degree $\mu(y, B, w)$ is 0. Suppose to the contrary that $\mu(y, B, w) \neq 0$. Choose a small connected neighborhood N of y that does not intersect the compact set $w(\partial B)$. Then $\mu(y', B, w) \neq 0$ for all $y' \in N$. It follows (by Lemma 2.27) that $N \subseteq w(B)$, which contradicts our assumption that $y \notin \text{int}(w(G))$. So $\mu(y, B, w) = 0$.

On the other hand, we can choose $x' \in B \cap U$ so that $y' = w(x') \in V$ is arbitrarily close to y . By Lemma 5.4, x' is the only pre-image of y' . As before, choose a small connected neighborhood $B' \subset B$ around x' so that $w|_{B'}$ is a homeomorphism and $\partial B'$ avoids the (finitely many) pre-images of y . Remember that B' contains the only pre-image x' of y' in B . It follows from Lemma 2.27 that

$$\mu(y', B, w) = \mu(y', B', w) = \pm 1.$$

Now, if y' is sufficiently close to y , then y' is in the same connected component of $M \setminus w(\partial B)$ as y . Because the local degree is locally constant (Lemma 2.27), we see that

$$\mu(y, B, w) = \mu(y', B, w).$$

But the left-hand side is 0 while the right-hand side is not. This completes the proof that w is an open mapping. \square

From the previous two lemmas it immediately follows that w is a homeomorphism on A . Indeed, we only need show it is injective. Suppose $w(x) = w(x')$. Choose small disjoint balls B and B' containing x and x' , respectively. Then $w(B) \cap w(B')$ is an open set in $w(A)$ and therefore contains a point of V . This contradicts Lemma 5.4.

Because w is a homeomorphism, there are radii $r, r' > 0$ such that $w(B(v, r)) \supseteq B(w(v), r')$. It follows by Lemmas 2.24 and 2.15 that for all k sufficiently large,

$$w_k(S_k) \supseteq w_k(B(f_k(v), 2r)) \supseteq B(w_k(f_k(v)), r'/2) \supseteq B(w_k(v_k), r'/3).$$

This contradicts property (5.1) of w_k if k is large enough.

This completes the proof of Proposition 4.2 and thus of Theorem 1.1.

6. Proof of Theorem 1.13

Let X be an Ahlfors d -regular, LLC, oriented topological d -manifold. (We re-emphasize the fact that here the Ahlfors regularity dimension and the topological dimension of X must coincide.) We will apply Theorem 1.1 (in the case $Y = \mathbb{R}^d$) to a class of maps on X provided by a theorem of Semmes. These are given in the following result, which is a slightly weakened version of Theorem 1.29(a) of [27].

Theorem 6.1 ([27], Theorem 1.29 (a)). *Let B be an open ball in X of radius $r > 0$. Then there is a surjective Lipschitz map f from X onto the standard d -dimensional unit sphere \mathbb{S}^d with Lipschitz constant $\leq Cr^{-1}$ that is constant on $X \setminus B$. The constant C depends only on the data of X .*

Remark 6.2. In Theorem 6.1, it makes no difference whether one endows \mathbb{S}^d with the standard Riemannian metric of diameter π or with the “chordal” metric arising from writing $\mathbb{S}^d = \{x \in \mathbb{R}^{d+1} : |x| = 1\}$ and letting $d(x, y) = |x - y|$. These metrics are bi-Lipschitz equivalent. For convenience, we will use the latter.

Proof of Theorem 1.13. As above, write $\mathbb{S}^d = \{x \in \mathbb{R}^{d+1} : |x| = 1\}$. Consider the projection p from \mathbb{S}^d onto the first d coordinates in \mathbb{R}^{d+1} . Then p is 1-Lipschitz and $|p(\mathbb{S}^d)| = \sigma_d$, the d -dimensional Hausdorff measure of the unit ball in \mathbb{R}^d .

Therefore, by post-composing the maps of Theorem 6.1 with p , we see that for every ball $B(x, r) \subseteq X$ there is a Cr^{-1} -Lipschitz map $g_B : B \rightarrow \mathbb{R}^d$ with $|g_B(B)| = \sigma_d$.

To show X is locally uniformly rectifiable, we must show that for all $R > 0$ there exists constants α, β such that for every ball B of radius at most R , there is a set $E \subseteq B$ and a map $f : E \rightarrow \mathbb{R}^d$ such that $|E| \geq \beta|B|$ and f is α -bi-Lipschitz.

Fix a ball $B = B(x, r)$, where $r < R$. Let n be such that $C_02^n < r \leq C_02^{n+1}$. Then B contains a dyadic cube $Q \in \Delta_n$.

As $c_02^n \geq \frac{c_0}{2C_0}r$, Q contains a ball B' of radius $\frac{c_0}{2C_0}r$. Let $g = g_{B'}$ be a map as above associated to B' . Then g is Lipschitz with Lipschitz constant bounded by $\frac{2CC_0}{c_0r}$.

Therefore, the map $h = \frac{c_0r}{2CC_0}g$ is 1-Lipschitz and $|h(B')| \geq c_5r^d$, for $c_5 = \sigma_d(c_0/2CC_0)^d$.

Thus, $|h(Q)| \geq \delta|Q|$ for some constant δ depending only on the data of X . By choosing $\epsilon > 0$ sufficiently small in Theorem 1.1 (see Remark 1.11) we get that h is α -bi-Lipschitz on a set $E \subset Q \subset B$ of measure at least $\theta|Q| \geq \beta|B|$, where α and β depend only on R and the data of X . This proves Theorem 1.13. \square

7. Consequences of Theorem 1.13

It is now possible to derive many corollaries which result immediately from applying deep theorems of David and Semmes on uniformly rectifiable sets to the conclusion of Theorem 1.13. We state two geometric examples below.

First of all, Theorem 1.13, in combination with a result of Semmes in [29], provides a quasisymmetric embedding result for suitable compact metric manifolds. For the definition and basic properties of quasisymmetric homeomorphisms, see [13].

Corollary 7.1. *Let X be an Ahlfors d -regular, LLC, compact, oriented topological d -manifold. Then X is quasisymmetrically equivalent to a space X' that is also an Ahlfors d -regular, LLC, compact, oriented topological d -manifold and that is a subset of some \mathbb{R}^N .*

Both the dimension N and the quasisymmetry function of the homeomorphism between X and X' depend only on the data of X .

Proof. By Theorem 1.13, the space X is uniformly rectifiable. Proposition 2.10 of [29], combined with equation (3.27) in that paper, shows that X can be quasisymmetrically deformed by a weight (with quantitative control) so that the resulting space admits a bi-Lipschitz embedding into some \mathbb{R}^N .

Both the deformation and the bi-Lipschitz embedding quantitatively preserve the Ahlfors s -regularity of X . For the former, this is explained in the discussion following the proof of Lemma 4.4 in [29]; the latter is a general fact about bi-Lipschitz mappings.

Thus, if we let X' be the image of the deformed X under the bi-Lipschitz embedding, then X' is Ahlfors s -regular. Because it is quasisymmetrically homeomorphic to X , it is also a compact, LLC, oriented topological d -manifold. \square

Remark 7.2. Every doubling metric space quasisymmetrically embeds in some Euclidean space by Assouad’s theorem (see [13], Theorem 12.2), but in general this embedding first “snowflakes” the metric, increasing the Hausdorff dimension and destroying the rectifiability properties of the space. Corollary 7.1 is false if one replaces “quasisymmetrically” by “bi-Lipschitz”, as examples of Semmes [28] and Laakso [21] show.

Once there is a nice embedding of the abstract metric space X as a uniformly rectifiable subset of Euclidean space, all the theory of these sets developed by David and Semmes can be applied. Here we merely mention one further example, which says that the image of the embedding in Corollary 7.1 can be taken to lie in a particularly nice subset of \mathbb{R}^N .

Recall the definition of David–Semmes regular maps, introduced in Definition 3.1. We define the following class of subsets of Euclidean space.

Definition 7.3. Let E be an Ahlfors d -regular subset of \mathbb{R}^n . We say that E is *quasisymmetrically d -regular* if $E = g(f(\mathbb{R}^d))$, where $f: \mathbb{R}^d \rightarrow Y$ is a quasisymmetric homeomorphism of \mathbb{R}^d onto an Ahlfors d -regular space Y , and $g: Y \rightarrow \mathbb{R}^N$ is a David–Semmes regular mapping.

A set that is quasisymmetrically d -regular admits a bounded-multiplicity parametrization by \mathbb{R}^d in a controlled way.

The following corollary follows from a weakened version of the implication (C6) \Rightarrow (C7) in the main result of [9]. (The full version of the result should discuss deformations by A_1 -weights, which we have not mentioned.)

Corollary 7.4. *Let X be an Ahlfors d -regular, LLC, compact, oriented topological d -manifold. Let X' be a quasisymmetrically equivalent subset of \mathbb{R}^N provided by Corollary 7.1. Assume $N \geq 2d$. Then X' is contained in a quasisymmetrically d -regular set $E \subset \mathbb{R}^N$.*

Proof. This follows from Corollary 7.1, Theorem 1.13, and the main result of [9] (specifically, the implication (C6) \Rightarrow (C7)). \square

In general, it is not possible to find good (e.g. quasisymmetric or bi-Lipschitz) parametrizations of metric spaces such as those in Corollary 7.4 by standard spaces such as \mathbb{S}^d or \mathbb{R}^d . Corollary 7.4 provides a weaker form of “parametrization”, in that it yields a mapping onto but not into the space, and that is bounded-multiplicity rather than injective.

8. Counterexamples

To conclude, we wish to briefly describe some counterexamples regarding the class of “Lipschitz implies bi-Lipschitz” theorems mentioned at the beginning of this paper. By this we mean the class of theorems that say that if $f: X \rightarrow Y$ is a Lipschitz mapping with positive-measure image, then f is bi-Lipschitz on a set of positive measure, quantitatively. None of these counterexamples are new, but they are scattered in a few different places in the literature and it may be convenient to collect them in one place. The first two can be found in Meyerson’s paper [23], the third is due to David and Semmes [10], and the fourth is an example of Laakso [20].

The first counterexample shows that, in the setting of Theorem 1.1, the requirement that the two spaces have the same topological dimension is necessary. This proposition is proven by Meyerson in [23], Theorem 4.1. Here we give a slightly different argument.

Proposition 8.1. *There is an Ahlfors 2-regular, linearly locally contractible, complete oriented topological 1-manifold X and a Lipschitz map $f: X \rightarrow \mathbb{R}^2$ with positive measure image that is not bi-Lipschitz on any subset of positive measure.*

Proof. The metric space X will be the “snowflaked” space $(\mathbb{R}, |\cdot|^{1/2})$, equipped with two-dimensional Hausdorff measure (which is the same as one-dimensional Hausdorff measure on $(\mathbb{R}, |\cdot|)$). It is clear that X satisfies all the required properties.

It is well-known (see, e.g., [32], Theorem 7.3.1) that there is a space-filling curve $f: (\mathbb{R}, |\cdot|) \rightarrow \mathbb{R}^2$ that is Hölder continuous with exponent $1/2$ and whose image contains the unit square in \mathbb{R}^2 . Therefore, when considered as a mapping $f: X \rightarrow \mathbb{R}^2$, f is Lipschitz, and it has positive-measure image.

However, no Lipschitz map from X to \mathbb{R}^2 can be bi-Lipschitz on a set of positive measure. Indeed, suppose that f is bi-Lipschitz on a set of positive measure E in X , with $f(0) = 0$. Let $E' = f(E) \subseteq \mathbb{R}^2$. Without loss of generality, we may assume that E is compact, that $0 \in \mathbb{R}$ is a point of density of E in X , and that $f(0) = 0 \in \mathbb{R}^2$ is a point of density of E' in \mathbb{R}^2 . (We can always find such points.)

We now consider the sequences of mapping packages

$$(8.1) \quad \left\{ \left(\left(E, \frac{1}{n} d_X, 0 \right), \left(E', \frac{1}{n} |\cdot|, 0 \right), f \right) \right\}.$$

Because $0 \in X$ is a point of density of E and $0 \in \mathbb{R}^2$ is a point of density of E' , we have by [10], Lemmas 9.12 and 9.13, that, in the sense of pointed metric spaces,

$$\left(E, \frac{1}{n} d_X, 0 \right) \rightarrow (X, d_X, 0) \quad \text{and} \quad \left(E', \frac{1}{n} |\cdot|, 0 \right) \rightarrow (\mathbb{R}^2, |\cdot|, 0).$$

Therefore, some subsequence of the sequence of mapping packages in (8.1) converges to a mapping package

$$((X, d_X, 0), (\mathbb{R}^2, |\cdot|, 0), g).$$

The mapping g is bi-Lipschitz, because $f|_E$ is bi-Lipschitz. In addition, the map g is surjective. We may see this by passing to another subsequence along which the sequence of inverse mapping packages

$$\left\{ \left(\left(E', \frac{1}{n} |\cdot|, 0 \right), \left(E, \frac{1}{n} d_X, 0 \right), (f|_E)^{-1} \right) \right\}$$

converges to a mapping package

$$((\mathbb{R}^2, |\cdot|, 0), (X, d_X, 0), h).$$

It is then easy to see that $g(h(y)) = y$ for all $y \in \mathbb{R}^2$ and therefore that g is surjective.

So g is a bi-Lipschitz homeomorphism of X onto \mathbb{R}^2 . But this is impossible, as X is homeomorphic to \mathbb{R} . □

The two spaces in Proposition 8.1 satisfy all the conditions of Theorem 1.1, except that they are manifolds of different topological dimensions.

For the remaining three counterexamples that we mention here, we merely indicate the statements and refer the reader to the original sources for the proofs.

The second example is Theorem 4.2 of [23]. Let us first note that, as a consequence of Theorem 1.1, we know the following: Let X and Y be spaces as in Theorem 1.1. Let $U \subset X$ be an open set, and let $f: U \rightarrow Y$ be Lipschitz and satisfy $|f(U)| > 0$. Then there is a countable collection of measurable sets $E_i \subset U$ such that $f|_{E_i}$ is bi-Lipschitz for each i and $|f(U \setminus \cup E_i)| = 0$. (Here the sets E_i are not necessarily disjoint.) On the other hand, we have the following fact:

Proposition 8.2 ([23], Theorem 4.2). *There is a doubling, LLC, complete, oriented topological 2-manifold X of Hausdorff dimension 2, an open set $U \subset X$, and a Lipschitz map $f: U \rightarrow \mathbb{R}^2$ that cannot be represented in the above manner. In other words, there is no countable collection of measurable sets $E_i \subset U$ such that $f|_{E_i}$ is bi-Lipschitz for each i and $|f(U \setminus \cup E_i)| = 0$.*

In particular, the conclusion of Theorem 1.1 does not hold for this choice of X and $Y = \mathbb{R}^2$. In this result, the space X can be chosen to be the sub-Riemannian manifold known as the Grushin plane. The source and target spaces in Proposition 8.2 satisfy all the conditions of Theorem 1.1, except that the source X is not Ahlfors 2-regular. The idea behind Proposition 8.2 is to reduce to Proposition 8.1, because the Grushin plane X contains a bi-Lipschitz equivalent copy of the snowflaked line $(\mathbb{R}, |\cdot|^{1/2})$ as a positive-measure subset.

If one completely relaxes the strong topological conditions imposed in Theorem 1.1, then one can find Lipschitz mappings between metric spaces with large images but no bi-Lipschitz pieces, even in the presence of very strong analytic assumptions on the spaces and mappings.

Proposition 8.3 ([10], Proposition 14.5). *There is a compact, Ahlfors regular metric space X and a Lipschitz mapping $f: X \rightarrow X$ which is not bi-Lipschitz on any positive-measure subset. Furthermore, the mapping f can be taken to be a homeomorphism which is in addition David–Semmes regular and preserves measure, in the sense that $|f(K)| = |K|$ for all compact $K \subseteq X$.*

The space X in Proposition 8.3 is a totally disconnected Cantor set. See Chapter 14 of [10] for the proof and some other related constructions.

In both the positive result Theorem 1.1 and the counterexample Propositions 8.1 and 8.3, the spaces in question may have no “good calculus”, i.e., they may have no rectifiable curves and therefore no Poincaré inequality. (For the definition of Poincaré inequalities on metric measure spaces, see [13].) It is not known to what extent this type of calculus is helpful in proving “Lipschitz implies bi-Lipschitz” theorems, but in closing we wish to note the following theorem of Laakso [20], which shows that Ahlfors regular spaces with Poincaré inequalities may still fail to have such results.

Proposition 8.4 ([20]). *There exists an Ahlfors regular space X admitting a Poincaré inequality and a Lipschitz map $f: X \rightarrow X$ with positive-measure image such that there is no positive-measure subset of X on which f is bi-Lipschitz.*

In fact, in Laakso’s example the mapping f does not even have any bi-Lipschitz tangents, in the sense of Section 3.

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GUY C. DAVID: Courant Institute of Mathematical Sciences, New York University,
251 Mercer Street, New York, NY 10012, USA.

E-mail: guydavid@math.nyu.edu