



Fine properties of Newtonian functions and the Sobolev capacity on metric measure spaces

Lukáš Malý

Abstract. Newtonian spaces generalize first-order Sobolev spaces to abstract metric measure spaces. In this paper, we study regularity of Newtonian functions based on quasi-Banach function lattices. Their (weak) quasi-continuity is established, assuming density of continuous functions. The corresponding Sobolev capacity is shown to be an outer capacity. Assuming sufficiently high integrability of upper gradients, Newtonian functions are shown to be (essentially) bounded and (Hölder) continuous. Particular focus is put on the borderline case when the degree of integrability equals the “dimension of the measure”. If Lipschitz functions are dense in a Newtonian space on a proper metric space, then locally Lipschitz functions are proven dense in the corresponding Newtonian space on open subsets, where no hypotheses (besides being open) are put on these sets.

1. Introduction

First-order analysis in metric measure spaces requires a generalization of Sobolev spaces as the notion of a (distributional) gradient relies on the linear structure of \mathbb{R}^n . The Newtonian approach makes use of the so-called upper gradients and weak upper gradients, which were originally introduced by Heinonen and Koskela [20] and Koskela and MacManus [27], respectively. Shanmugalingam [37], [38] established the foundations for the Newtonian spaces $N^{1,p}$, based on the L^p norm of a function and its (weak) upper gradient and hence corresponding to the classical Sobolev spaces $W^{1,p}$, cf. Björn and Björn [5] or Heinonen, Koskela, Shanmugalingam and Tyson [21]. Various authors have developed the elements of the Newtonian theory based on function norms other than L^p in the past two decades, see e.g. [11], [13], [17], [28], [42]. So far, foundations of the theory in utmost generality were obtained by Malý in [31], [32], where complete quasi-normed lattices of measurable functions were considered as the base function spaces. The question of when

Mathematics Subject Classification (2010): Primary 46E35; Secondary 28A12, 30L99, 46E30.

Keywords: Newtonian space, Sobolev-type space, metric measure space, Banach function lattice, Sobolev capacity, quasi-continuity, outer capacity, locally Lipschitz function, continuity, doubling measure, Poincaré inequality.

Newtonian functions can be regularized using Lipschitz truncations was the focus of Malý [33]. The current paper goes further and studies regularity properties of Newtonian functions and of the corresponding Sobolev capacity.

One of the main points of interest is the so-called *quasi-continuity*, which can be understood as a Luzin-type condition, where a set of arbitrarily small capacity can be found for each Newtonian function so that its restriction to the complement of that set is continuous. Existence of quasi-continuous representatives was first shown by Deny [12] for functions of the unweighted Sobolev space $W^{1,2}(\mathbb{R}^n, dx)$. An analogous result in Sobolev spaces $W^{1,p}(\mathbb{R}^n, dx)$ is given in Federer and Ziemer [15], see also Malý and Ziemer [30], and the situation in weighted Sobolev spaces is discussed in Heinonen, Kilpeläinen, Martio [19]. In metric spaces, Shanmugalingam [38] showed that Newtonian functions in $N^{1,p}$ have quasi-continuous representatives if the metric space is endowed with a doubling measure and supports a p -Poincaré inequality (see Definition 6.1 below). The hypotheses were weakened in Björn, Björn and Shanmugalingam [8], where density of continuous functions was proven sufficient to obtain existence of quasi-continuous representatives of $N^{1,p}$ functions. The current paper provides an analogous result for the Newtonian space N^1X built upon an arbitrary quasi-Banach function lattice X .

In [8], still assuming the density of continuous functions, all $N^{1,p}$ functions were further proven to be quasi-continuous on condition that the metric space is *proper* (i.e., if all bounded closed sets are compact). In order to show a similar property of all N^1X functions in proper metric spaces, the quasi-Banach function lattice X needs to possess the *Vitali–Carathéodory property* (i.e., the quasi-norm of a function can be approximated by the quasi-norms of its lower semicontinuous majorants). Björn, Björn and Malý [7] give counterexamples that show that this property is vital.

Since the Vitali–Carathéodory property is crucial for the presented results, we will look into the question of when a general quasi-Banach function lattice X possesses it. Vitali [43] proved that the L^1 norm of a measurable function on \mathbb{R}^n can be approximated by the L^1 norms of its lower semicontinuous majorants. His result can be easily generalized to $L^p(\mathbb{R}^n)$ with $0 < p < \infty$. We will show that it suffices that X contains all simple functions (with bounded support) and these have absolutely continuous quasi-norm. Moreover, counterexamples of X violating either of these two conditions and hence lacking the Vitali–Carathéodory property are provided.

Quasi-continuity of Newtonian functions in N^1X is closely connected with regularity of the Sobolev capacity C_X . Namely, under the assumption that continuous functions are dense in the Newtonian space N^1X , all Newtonian functions are quasi-continuous if and only if C_X (or an equivalent capacity in case X is merely quasi-normed) is an outer capacity. Actually, the density of continuous functions need not be assumed in the forward implication.

Furthermore, quasi-continuity can be applied to show that locally Lipschitz functions are dense in a Newtonian space on any open subset of a metric space provided that locally Lipschitz functions are dense in the Newtonian space on the

entire metric space. The noteworthy part of this claim is that the open subset as a metric subspace need not support any Poincaré inequality and the restriction of the measure need not be doubling any more. In general, it is however impossible to obtain density of Lipschitz functions.

It was observed already by Morrey [35] in 1940 that the classical Sobolev functions in \mathbb{R}^n have (Hölder) continuous representatives if the degree of summability of the weak gradients is sufficiently high compared to the dimension. A similar result based on a p -Poincaré inequality was obtained by Hajlasz and Koskela [16] in metric spaces endowed with a doubling measure, after introducing an analogue of the dimension. Recently, Lu, Yang and Yuan [28] looked into Hölder continuity of Newtonian functions with gradients in Morrey spaces. As we are considering Newtonian spaces based on general function lattices, our tools suffice to study the borderline case when the degree of summability (in terms of a Banach function lattice quasi-norm) of upper gradients is essentially equal to the “dimension” of a doubling measure. We will establish conditions that guarantee that all Newtonian functions are essentially bounded and have continuous representatives (in equivalence classes given by equality up to sets of capacity zero). If the metric measure space is in addition locally compact, then all Newtonian functions are in fact continuous.

The structure of the paper is the following. Section 2 provides an overview of the used notation and preliminaries in the area of quasi-Banach function lattices and Newtonian spaces. In Section 3, we study the Sobolev capacity, still without the assumption of density of continuous functions. After that, in Section 4, we move on to quasi-continuity and its consequences for the Sobolev capacity and continuity of Newtonian functions. Density of locally Lipschitz functions on general open sets is shown in Section 5. A very short introduction to rearrangement-invariant spaces is provided in Section 6, whose main focus however lies in establishing sufficient conditions for Newtonian functions to be essentially bounded. In Section 7, existence of continuous representatives and continuity of all representatives is discussed. Several lemmata for calculus of (minimal) weak upper gradients are given in the appendix.

2. Preliminaries

We assume throughout the paper that $\mathcal{P} = (\mathcal{P}, d, \mu)$ is a metric measure space equipped with a metric d and a σ -finite Borel regular measure μ such that every ball in \mathcal{P} has finite positive measure. In our context, Borel regularity means that all Borel sets in \mathcal{P} are μ -measurable and for each μ -measurable set A there is a Borel set $D \supset A$ such that $\mu(D) = \mu(A)$. Since μ is Borel regular and \mathcal{P} can be decomposed into countably many (possibly overlapping) open sets of finite measure, it is outer regular, see Theorem 1.10 in Mattila [34].

The open ball centered at $x \in \mathcal{P}$ with radius $r > 0$ will be denoted by $B(x, r)$. Given a ball $B = B(x, r)$ and a scalar $\lambda > 0$, we let $\lambda B = B(x, \lambda r)$. We say that μ is a *doubling* measure, if there is a constant $c_{\text{dbl}} \geq 1$ such that $\mu(2B) \leq c_{\text{dbl}} \mu(B)$

for every ball B . We will assume that μ satisfies the doubling condition only in Sections 6 and 7, where essential boundedness and continuity of Newtonian functions are studied.

A metric space is *proper* if all its closed and bounded subsets are compact. A doubling metric measure space (and hence a metric space with a doubling measure) is proper if and only if it is complete, see Proposition 3.1 in Björn and Björn [5].

Let $\mathcal{M}(\mathcal{P}, \mu)$ denote the set of all extended real-valued μ -measurable functions on \mathcal{P} . The symbol $\text{Lip}_c(\Omega)$ stands for Lipschitz continuous functions with compact support in Ω . The set of extended real numbers, i.e., $\mathbb{R} \cup \{\pm\infty\}$, will be denoted by $\overline{\mathbb{R}}$. We will also use \mathbb{R}^+ , which denotes the set of positive real numbers, i.e., the interval $(0, \infty)$. The symbol \mathbb{N} will denote the set of positive integers, i.e., $\{1, 2, \dots\}$. We define the *integral mean* of a measurable function u over a set E of finite positive measure as

$$u_E := \int_E u \, d\mu = \frac{1}{\mu(E)} \int_E u \, d\mu,$$

whenever the integral on the right-hand side exists, not necessarily finite though. We write $E \Subset A$ if \overline{E} is a compact subset of A . The notation $L \lesssim R$ will be used to express that there exists a constant $c > 0$, perhaps dependent on other constants within the context, such that $L \leq cR$. If $L \lesssim R$ and simultaneously $R \lesssim L$, then we will simply write $L \approx R$ and say that the quantities L and R are *comparable*. The words *increasing* and *decreasing* will be used in their non-strict sense.

A linear space $X = X(\mathcal{P}, \mu)$ of equivalence classes of functions in $\mathcal{M}(\mathcal{P}, \mu)$ is a *quasi-Banach function lattice* over (\mathcal{P}, μ) equipped with the quasi-norm $\|\cdot\|_X$ if the following axioms hold:

(P0) $\|\cdot\|_X$ determines the set X , i.e., $X = \{u \in \mathcal{M}(\mathcal{P}, \mu) : \|u\|_X < \infty\}$;

(P1) $\|\cdot\|_X$ is a *quasi-norm*, i.e.,

- $\|u\|_X = 0$ if and only if $u = 0$ a.e.,
- $\|au\|_X = |a| \|u\|_X$ for every $a \in \mathbb{R}$ and $u \in \mathcal{M}(\mathcal{P}, \mu)$,
- there is a constant $c_\delta \geq 1$, the so-called *modulus of concavity*, such that $\|u + v\|_X \leq c_\delta (\|u\|_X + \|v\|_X)$ for all $u, v \in \mathcal{M}(\mathcal{P}, \mu)$;

(P2) $\|\cdot\|_X$ satisfies the *lattice property*, i.e., if $|u| \leq |v|$ a.e., then $\|u\|_X \leq \|v\|_X$;

(RF) $\|\cdot\|_X$ satisfies the *Riesz–Fischer property*, i.e., if $u_n \geq 0$ a.e. for all $n \in \mathbb{N}$, then $\|\sum_{n=1}^\infty u_n\|_X \leq \sum_{n=1}^\infty c_\delta^n \|u_n\|_X$, where $c_\delta \geq 1$ is the modulus of concavity. Note that the function $\sum_{n=1}^\infty u_n$ needs to be understood as a pointwise (a.e.) sum.

Observe that X contains only functions that are finite a.e., which follows from (P1) and (P2). In other words, if $\|u\|_X < \infty$, then $|u| < \infty$ a.e.

Throughout the paper, we will also assume that the quasi-norm $\|\cdot\|_X$ is *continuous*, i.e., if $\|u_n - u\|_X \rightarrow 0$ as $n \rightarrow \infty$, then $\|u_n\|_X \rightarrow \|u\|_X$. We do not lose any generality by this assumption as the Aoki–Rolewicz theorem (see Proposition H.2

in Benyamini and Lindenstrauss [4] or Theorem 1.2 in Maligranda [29]) implies that there is always an equivalent quasi-norm that is an r -norm, i.e., it satisfies

$$\|u + v\|^r \leq \|u\|^r + \|v\|^r,$$

where $r = 1/(1 + \log_2 c_a) \in (0, 1]$, which implies the continuity. The theorem’s proof shows that such an equivalent quasi-norm retains the lattice property. Moreover, $\|\cdot\|^r$ satisfies (RF) without any constants, i.e., $\|\sum_{n=1}^\infty u_n\|^r \leq \sum_{n=1}^\infty \|u_n\|^r$.

The Riesz–Fischer property is actually equivalent to the completeness of the quasi-normed space X , provided that the conditions (P0)–(P2) are satisfied and that the quasi-norm is continuous, see Theorem 1.1 in Maligranda [29].

If $c_a = 1$, then the functional $\|\cdot\|_X$ is a norm. We then drop the prefix *quasi* and hence call X a *Banach function lattice*.

A (quasi)Banach function lattice $X = X(\mathcal{P}, \mu)$ is a (quasi)Banach function space over (\mathcal{P}, μ) if the following axioms are satisfied as well:

(P3) $\|\cdot\|_X$ has the *Fatou property*, i.e., if $0 \leq u_n \nearrow u$ a.e., then $\|u_n\|_X \nearrow \|u\|_X$;

(P4) if a measurable set $E \subset \mathcal{P}$ has finite measure, then $\|\chi_E\|_X < \infty$;

(P5) for every measurable set $E \subset \mathcal{P}$ with $\mu(E) < \infty$ there is $C_E > 0$ such that $\int_E |u| d\mu \leq C_E \|u\|_X$ for every measurable function u .

Note that the Fatou property implies the Riesz–Fischer property. Axiom (P4) is equivalent to the condition that X contains all simple functions (with support of finite measure). Condition (P5) implies that $X \subset L^1_{loc}(\mathcal{P}, \mu)$.

In the further text, we will slightly deviate from this rather usual definition of (quasi)Banach function lattices and spaces. Namely, we will consider X to be a linear space of functions defined everywhere instead of equivalence classes defined a.e. Then, the functional $\|\cdot\|_X$ is really only a (quasi)seminorm. Unless explicitly stated otherwise, we will always assume that X is a quasi-Banach function lattice.

We will say that X is *continuously embedded* in Y_{loc} , denoted by $X \hookrightarrow Y_{loc}$, if for every ball $B \subset \mathcal{P}$ there is $c_{emb}(B) > 0$ such that $\|u\chi_B\|_Y \leq c_{emb}(B)\|u\chi_B\|_X$ whenever $u \in \mathcal{M}(\mathcal{P}, \mu)$. The global continuous embedding $X \hookrightarrow Y$ is defined in a similar fashion by letting $B = \mathcal{P}$. It is also worth pointing out that axioms (P4) and (P5) of a (quasi)Banach function space X are in general more restrictive than $L^\infty \hookrightarrow X_{loc}$ and $X \hookrightarrow L^1_{loc}$, respectively. Namely, the axioms express that the embedding inequality between function norms holds whenever B is a set of finite measure (possibly unbounded).

A measurable function u has *absolutely continuous quasi-norm in X* , if it satisfies that

(AC) $\|u\chi_{E_n}\|_X \rightarrow 0$ as $n \rightarrow \infty$ whenever $\{E_n\}_{n=1}^\infty$ is a decreasing sequence of measurable sets with $\mu(\bigcap_{n=1}^\infty E_n) = 0$.

The quasi-norm $\|\cdot\|_X$ is *absolutely continuous* if every $u \in X$ has absolutely continuous quasi-norm in X .

Observe that if $u \in \mathcal{M}(\mathcal{P}, \mu)$ has absolutely continuous quasi-norm in X and if μ is non-atomic, then $u \in X$. It follows from the dominated convergence theorem that the L^p norm is absolutely continuous for $p \in (0, \infty)$. On the other hand, $L^{p, \infty}$

(i.e., the weak- L^p spaces) and L^∞ lack this property apart from in a few exceptional cases. For example, if μ is purely atomic, $0 < \delta \leq \mu(A)$ for every atom $A \subset \mathcal{P}$, and $\mu(\mathcal{P}) < \infty$, then every quasi-Banach function lattice has absolutely continuous quasi-norm since the condition $\mu(\bigcap_{n=1}^\infty E_n) = 0$ implies that $\mu(E_n) = 0$ for all sufficiently large $n \in \mathbb{N}$. However, (purely) atomic measures lie outside of the main scope of our interest.

Next, we will show that the quasi-norm of a function in a quasi-Banach function lattice X can be approximated by the quasi-norms of its lower semicontinuous (lsc) majorants provided that all simple functions with bounded support satisfy (AC). Note that X may very well be a weak-type space (e.g., $L^{p,\infty}$ with $p < \infty$) since we do not need to assume that every function in X has absolutely continuous quasi-norm in X .

Proposition 2.1 (Vitali–Carathéodory theorem). *Let X be a quasi-Banach function lattice. Suppose that χ_B satisfies (AC) and lies in X whenever $B \subset \mathcal{P}$ is bounded and measurable. If $u: \mathcal{P} \rightarrow \overline{\mathbb{R}}$ is measurable, then*

$$(2.1) \quad \|u\|_X = \inf\{\|v\|_X : v \geq |u| \text{ on } \mathcal{P} \text{ and } v \in \text{lsc}(\mathcal{P})\}.$$

In particular, the hypotheses are fulfilled if X is a rearrangement-invariant quasi-Banach function space (see the definition in Section 6 below) whose fundamental function ϕ_X satisfies $\lim_{t \rightarrow 0^+} \phi_X(t) = 0$, which can also be expressed as $X \not\subset L^\infty$.

The theorem’s origin can be dated back to 1905, when Vitali [43] showed that every function $f \in L^1(\mathbb{R}^n, d\mu)$ coincides a.e. with a function of the Baire class 2. Namely, he showed that there exist sequences $\{u_k\}_{k=1}^\infty$ and $\{l_k\}_{k=1}^\infty$ of upper semicontinuous (usc) minorants and lsc majorants, respectively, such that $u_k \nearrow f$ and $l_k \searrow f$ everywhere in \mathbb{R}^n , and $\|u_k\|_{L^1} \rightarrow \|f\|_{L^1}$ and $\|l_k\|_{L^1} \rightarrow \|f\|_{L^1}$ as $k \rightarrow \infty$. In 1918, Carathéodory [10] has shown that usc minorants and lsc majorants with the same convergence properties exist for every $f \in \mathcal{M}(\mathbb{R}^n, d\mu)$, i.e., even if $f \notin L^1(\mathbb{R}^n, d\mu)$.

The lsc majorants in the proof below are constructed similarly as in the proof of Lemma 2.3 in E. & M. Järvenpää, K. & S. Rogovin, and Shanmugalingam [22], where merely $X = L^p$ with $p \in [1, \infty)$ was considered.

The hypotheses that χ_B lies in X and that χ_B has absolutely continuous (quasi)norm in X for every bounded set $B \subset \mathcal{P}$ are vital for the proposition. For example, take $u = \chi_{\{0\}}$ on \mathbb{R} . Then, $\|u\|_X = 0$ for every quasi-Banach function lattice X as $u = 0$ a.e. For $X = L^\infty(\mathbb{R})$, the norm of χ_B lacks the absolute continuity whenever $|B| > 0$. Then, $\|v\|_X \geq 1$ for every lsc majorant v of u . The norm $\|f\|_Y = \int_{\mathbb{R}} |f(t)|/t \, dt$ gives rise to a function space Y such that $L^\infty(\mathbb{R}) \not\subset Y_{\text{loc}}$. In particular, there are bounded measurable sets $B \subset \mathbb{R}$ with $\chi_B \notin Y$. We obtain that $\|v\|_Y = \infty$ for every lsc majorant v of u since $v > 1/2$ in some open neighborhood of zero.

Proof of Proposition 2.1. Let u be given. The lattice property (P2) yields that $\|u\|_X \leq \inf_v \|v\|_X$, where the infimum is taken over all lsc majorants v of u . Hence, if $\|u\|_X = \infty$, then the desired identity holds trivially. Suppose instead that

$\|u\|_X < \infty$. Without loss of generality, we may assume that u is non-negative. We will show that there is a sequence $\{v_j\}_{j=1}^\infty$ of lsc majorants of u such that $v_j \rightarrow u$ in X (i.e., $\|v_j - u\|_X \rightarrow 0$) as $j \rightarrow \infty$. Continuity of the (quasi)norm of X then yields that $\|v_j\|_X \rightarrow \|u\|_X$ as desired. Therefore, it suffices to find $v \in \text{lsc}(\mathcal{P})$ for every $\varepsilon > 0$ so that $v \geq u$ everywhere in \mathcal{P} and $\|v - u\|_X < \varepsilon$.

Fixing an arbitrary point $x_0 \in \mathcal{P}$, we can decompose \mathcal{P} as a union of an open ball and open spherical shells centered in x_0 , i.e., $\mathcal{P} = \bigcup_{k=1}^\infty \mathcal{P}_k$, where $\mathcal{P}_k = \{x \in \mathcal{P} : k - 2 < d(x, x_0) < k\}$.

Let $\varepsilon > 0$ be fixed. For each $k \in \mathbb{N}$, we will find an lsc function v_k that dominates u on \mathcal{P}_k while $\|(v_k - u)\chi_{\mathcal{P}_k}\|_X < \varepsilon / (2c_\Delta)^k$, where $c_\Delta \geq 1$ is the modulus of concavity of X (i.e., the constant in the triangle inequality in (P1)).

Fix $k \in \mathbb{N}$ and let $\delta = \varepsilon / c_\Delta^2 (2c_\Delta)^k (2 + \|\chi_{\mathcal{P}_k}\|_X)$. Let $E_\infty = \mathcal{P}_k \cap u^{-1}(\infty)$. Then, $\mu(E_\infty) = 0$. For every $n \in \mathbb{N}$, we define $E_n = \mathcal{P}_k \cap u^{-1}([(n - 1)\delta, n\delta])$. Due to the outer regularity of μ and the absolute continuity of the (quasi)norm of $\chi_{\mathcal{P}_k}$ in X , there are open sets U_n and V_n such that $E_n \subset U_n \subset \mathcal{P}_k$ with $\|\chi_{U_n \setminus E_n}\|_X < 1/n(2c_\Delta)^n$, and $E_\infty \subset V_n \subset \mathcal{P}_k$ with $\|\chi_{V_n}\|_X < \delta / (2c_\Delta)^n$. Define now $v_k : \mathcal{P} \rightarrow [0, \infty]$ by

$$v_k(x) = \sum_{n=1}^\infty (n\delta\chi_{U_n} + \chi_{V_n}).$$

Obviously, $v_k \in \text{lsc}(\mathcal{P}_k)$ and $v_k \geq u$ on \mathcal{P}_k . Then, we can estimate

$$\begin{aligned} v_k(x) - u(x) &\leq \sum_{n=1}^\infty (n\delta\chi_{U_n}(x) + \chi_{V_n}(x) - (n - 1)\delta\chi_{E_n}(x)) \\ &\leq \sum_{n=1}^\infty (n\delta\chi_{U_n \setminus E_n}(x) + \delta\chi_{E_n}(x) + \chi_{V_n}(x)) \\ &= \sum_{n=1}^\infty n\delta\chi_{U_n \setminus E_n}(x) + \delta\chi_{\mathcal{P}_k}(x) + \sum_{n=1}^\infty \chi_{V_n}(x). \end{aligned}$$

The triangle inequality and the Riesz–Fischer property give that

$$\begin{aligned} (2.2) \quad \|(v_k - u)\chi_{\mathcal{P}_k}\|_X &\leq c_\Delta^2 \left(\delta \sum_{n=1}^\infty n c_\Delta^n \|\chi_{U_n \setminus E_n}\|_X + \delta \|\chi_{\mathcal{P}_k}\|_X + \sum_{n=1}^\infty c_\Delta^n \|\chi_{V_n}\|_X \right) \\ &< \delta c_\Delta^2 (2 + \|\chi_{\mathcal{P}_k}\|_X) = \frac{\varepsilon}{(2c_\Delta)^k}. \end{aligned}$$

Let now $v(x) = \max_{k \in \mathbb{N}} v_k(x)$ for all $x \in \mathcal{P}$. Note that for each $x \in \mathcal{P}$, there may be at most two values of k such that $v_k(x) > 0$ and in that case they are consecutive. Define thus $\mathcal{P}'_k = \{x \in \mathcal{P}_k : v(x) = v_k(x) > v_{k+1}(x)\}$. Then, the sets \mathcal{P}'_k , $k \in \mathbb{N}$, are pairwise disjoint and $v(x) = u(x) = 0$ whenever $x \in \mathcal{P} \setminus \bigcup_{k=1}^\infty \mathcal{P}'_k$, which together with (2.2) leads to the estimate

$$\|v - u\|_X = \left\| \sum_{k=1}^\infty (v - u)\chi_{\mathcal{P}'_k} \right\|_X = \left\| \sum_{k=1}^\infty (v_k - u)\chi_{\mathcal{P}'_k} \right\|_X \leq \sum_{k=1}^\infty c_\Delta^k \|(v_k - u)\chi_{\mathcal{P}_k}\|_X < \varepsilon. \quad \square$$

By a *curve* in \mathcal{P} we will mean a non-constant continuous mapping $\gamma: I \rightarrow \mathcal{P}$ with finite total variation (i.e., length of $\gamma(I)$), where $I \subset \mathbb{R}$ is a compact interval. Thus, a curve can be (and we will always assume that all curves are) parametrized by arc length ds , see e.g. Section 7.1 in Heinonen [18]. Note that every curve is Lipschitz continuous with respect to its arc length parametrization. The family of all non-constant rectifiable curves in \mathcal{P} will be denoted by $\Gamma(\mathcal{P})$. By abuse of notation, the image of a curve γ will also be denoted by γ .

A statement holds for Mod_X -a.e. curve if the family of exceptional curves Γ_e , for which the statement fails, has *zero X -modulus*, i.e., if there is a Borel function $\rho \in X$ such that $\int_\gamma \rho ds = \infty$ for every curve $\gamma \in \Gamma_e$ (see Proposition 4.8 in [31]).

Definition 2.2. Let $u: \mathcal{P} \rightarrow \overline{\mathbb{R}}$. Then, a Borel function $g: \mathcal{P} \rightarrow [0, \infty]$ is an *upper gradient* of u if

$$(2.3) \quad |u(\gamma(0)) - u(\gamma(l_\gamma))| \leq \int_\gamma g ds$$

for every curve $\gamma: [0, l_\gamma] \rightarrow \mathcal{P}$. To make the notation easier, we are using the convention that $|(\pm\infty) - (\pm\infty)| = \infty$. If we allow g to be a measurable function and (2.3) to hold only for Mod_X -a.e. curve $\gamma: [0, l_\gamma] \rightarrow \mathcal{P}$, then g is an *X -weak upper gradient*.

Observe that the (X -weak) upper gradients are by no means given uniquely. Indeed, if we have a function u with an (X -weak) upper gradient g , then $g + h$ is another (X -weak) upper gradient of u whenever $h \geq 0$ is a Borel (measurable) function.

Definition 2.3. We say that function $u \in \mathcal{M}(\mathcal{P}, \mu)$ belongs to the *Dirichlet space* DX if it has an upper gradient $g \in X$. Let

$$(2.4) \quad \|u\|_{N^1X} = \|u\|_X + \inf_g \|g\|_X,$$

where the infimum is taken over all upper gradients g of u . The *Newtonian space* based on X is the space

$$N^1X = N^1X(\mathcal{P}, \mu) := \{u \in \mathcal{M}(\mathcal{P}, \mu) : \|u\|_{N^1X} < \infty\} = X \cap DX.$$

Given a measurable set $\Theta \subset \mathcal{P}$, we define

$$N^1_0X(\Theta) = \{u|_\Theta : u \in N^1X \text{ and } u = 0 \text{ in } \mathcal{P} \setminus \Theta\}.$$

Note that we may define DX via X -weak upper gradients and take the infimum over all X -weak upper gradients g of u in (2.4) without changing the value of the Newtonian quasi-norm, see Corollary 5.7 in [31]. Let us also point out that we assume that functions are defined everywhere, and not just up to equivalence classes μ -almost everywhere. This is essential for the notion of upper gradients since they are defined by a pointwise inequality.

It has been shown in [32] that the infimum in (2.4) is attained for functions in N^1X by a *minimal X -weak upper gradient*. Such an X -weak upper gradient is minimal both normwise and pointwise (a.e.) among all (X -weak) upper gradients in X , whence it is given uniquely up to equality a.e. The minimal X -weak upper gradient of a function $u \in N^1X$ will be denoted by $g_u \in X$.

The functional $\|\cdot\|_{N^1X}$ is a quasi-seminorm on N^1X and a quasi-norm on $\tilde{N}^1X := N^1X/\sim$, where the equivalence relation $u \sim v$ is given by $\|u - v\|_{N^1X} = 0$. The modulus of concavity for N^1X (i.e., the constant in the triangle inequality) is equal to c_Δ , the modulus of concavity for X . Furthermore, the Newtonian space \tilde{N}^1X is complete and thus a quasi-Banach space, see Theorem 7.1 in [31].

3. Sobolev capacity

When working with first-order analysis, it is the Sobolev capacity that provides a set function that distinguishes which sets do not carry any information about a Newtonian function and thus are negligible. In this section, we will show a certain rigidity property of the capacity. Furthermore, if \mathcal{P} is locally compact and if the Vitali–Carathéodory theorem holds, then we will obtain that the capacity is outer regular on sets of zero capacity.

Definition 3.1. The (*Sobolev*) X -*capacity* of a set $E \subset \mathcal{P}$ is defined as

$$C_X(E) = \inf\{\|u\|_{N^1X} : u \geq 1 \text{ on } E\}.$$

If X is r -normed for some $r \in (0, \infty)$, then we define the (*Sobolev*) X, r -*capacity* of a set $E \subset \mathcal{P}$ by

$$\tilde{C}_{X,r}(E) = \inf\{(\|u\|_X^r + \|g\|_X^r)^{1/r} : u \geq 1 \text{ on } E \text{ and } g \text{ is an upper gradient of } u\}.$$

The function lattice X will be implicitly assumed to be r -normed whenever the capacity $\tilde{C}_{X,r}$ is used. We say that a property of points in \mathcal{P} holds C_X -*quasi-everywhere* (C_X -*q.e.*) if the set of exceptional points has X -capacity zero.

Observe that the capacities C_X and $\tilde{C}_{X,r}$ are equivalent, viz.,

$$\min\{1, 2^{1/r-1}\}C_X(E) \leq \tilde{C}_{X,r}(E) \leq \max\{1, 2^{1/r-1}\}C_X(E) \quad \text{for every } E \subset \mathcal{P}.$$

Therefore, it is of no importance whether the notion of quasi-everywhere is defined using C_X or $\tilde{C}_{X,r}$. The capacities $C_X(E)$ and $\tilde{C}_{X,r}(E)$ may be equivalently defined considering only functions u such that $\chi_E \leq u \leq 1$, see Proposition 3.2 in [31]. If X is normed, then $C_X = \tilde{C}_{X,1}$. Despite the dependence on X , we will often write simply *capacity* and *q.e.* whenever there is no risk of confusion of the base function space.

A capacity C is an *outer capacity*, if $C(E) = \inf_G C(G)$, where the infimum is taken over all open sets $G \supset E$. Based on the quasi-continuity of Newtonian functions, we will show in Proposition 4.8 below that $\tilde{C}_{X,r}$ is an outer capacity.

It was established by Theorem 3.4 in [31] that C_X is σ -quasi-subadditive, i.e.,

$$C_X\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} c_{\Delta}^j C_X(E_j).$$

In particular, C_X is countably subadditive if $c_{\Delta} = 1$. Next, we will show that so is $\tilde{C}_{X,r}(\cdot)^r$ regardless of the value of $c_{\Delta} \geq 1$ or $r > 0$.

Lemma 3.2. *The function $\tilde{C}_{X,r}(\cdot)^r$ is σ -subadditive, i.e.,*

$$\tilde{C}_{X,r}\left(\bigcup_{j=1}^{\infty} E_j\right)^r \leq \sum_{j=1}^{\infty} \tilde{C}_{X,r}(E_j)^r$$

whenever $E_1, E_2, \dots \subset \mathcal{P}$.

Proof. If $\tilde{C}_{X,r}(E_j) = \infty$ for some $j \in \mathbb{N}$, then the wanted inequality holds trivially. Suppose therefore that $\tilde{C}_{X,r}(E_j) < \infty$ for every $j \in \mathbb{N}$. For each $E_j, j \in \mathbb{N}$, we can hence find $u_j \in N^1X$ with an upper gradient $g_j \in X$ such that $\chi_{E_j} \leq u_j \leq 1$, and $\|u_j\|_X^r + \|g_j\|_X^r < \tilde{C}_{X,r}(E_j)^r + 2^{-j}\varepsilon$. Let $u = \sup_{j \geq 1} u_j$ and $g = \sup_{j \geq 1} g_j$. Then, $\chi_{\bigcup_{j=1}^{\infty} E_j} \leq u \leq 1$, while g is an upper gradient of u by Lemma 3.3 in [31]. Hence,

$$\begin{aligned} \tilde{C}_{X,r}\left(\bigcup_{j=1}^{\infty} E_j\right)^r &\leq \|u\|_X^r + \|g\|_X^r = \left\| \sup_{j \geq 1} u_j \right\|_X^r + \left\| \sup_{j \geq 1} g_j \right\|_X^r \\ &\leq \left\| \sum_{j=1}^{\infty} u_j \right\|_X^r + \left\| \sum_{j=1}^{\infty} g_j \right\|_X^r \leq \sum_{j=1}^{\infty} (\|u_j\|_X^r + \|g_j\|_X^r) < \varepsilon + \sum_{j=1}^{\infty} \tilde{C}_{X,r}(E_j)^r. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ completes the proof. □

If $C_X(E) = 0$, then $\mu(E) = 0$. The converse is however not true in general. The natural equivalence classes in N^1X , where u and v are equivalent if $\|u - v\|_{N^1X} = 0$, are in fact given by equality q.e. as shown in Corollary 6.16 of [31].

The following proposition shows a certain rigidity property of the Sobolev capacity of an open set, which is an important hypothesis in Proposition 4.4 below. The usual Sobolev capacity in \mathbb{R}^n has this property trivially by definition, see Definition 2.35 in Heinonen, Kilpeläinen and Martio [19]. The idea of the claim and its proof originates in Proposition 5.22 of [5]. It is noteworthy that we do not need C_X (or $\tilde{C}_{X,r}$) to be an outer capacity to obtain this result.

Proposition 3.3. *Let $G \subset \mathcal{P}$ be open and suppose that $\mu(E) = 0$. Then,*

$$C_X(G) = C_X(G \setminus E) \quad \text{and} \quad \tilde{C}_{X,r}(G) = \tilde{C}_{X,r}(G \setminus E).$$

Proof. Obviously, $C_X(G) \geq C_X(G \setminus E)$ as the capacity is monotone.

The converse inequality holds trivially if $C_X(G \setminus E) = \infty$. Hence, suppose that $C_X(G \setminus E) < \infty$. Let $\varepsilon > 0$. Then, there is $u \in N^1X$ with an upper gradient $g \in X$ such that $\chi_{G \setminus E} \leq u \leq 1$ and $\|u\|_{N^1X} \leq \|u\|_X + \|g\|_X < C_X(G \setminus E) + \varepsilon$.

Let $v = \max\{\chi_G, u\}$. Then, $u = v$ outside of $G \cap E$, whose measure is zero. Hence, $\|u\|_X = \|v\|_X$. We will show that g is an X -weak upper gradient of v . Let $\gamma: [0, l_\gamma] \rightarrow \mathcal{P}$ be a curve such that $\lambda^1(\gamma^{-1}(G \cap E)) = 0$ and (2.2) is satisfied for all subcurves $\gamma' = \gamma|_I$, where $I \subset [0, l_\gamma]$ is a closed interval. By Lemma 4.9 and Corollary 5.9 in [31], Mod_X -a.e. curve γ satisfies these conditions.

If $\gamma(0) \in G \cap E$, then there is $\alpha \in (0, l_\gamma)$ such that $\gamma(\alpha) \in G \setminus E$ as $\gamma^{-1}(G)$ is open in $[0, l_\gamma]$ and $\lambda^1(\gamma^{-1}(G \cap E)) = 0$. If $\gamma(0) \notin G \cap E$, then we set $\alpha = 0$. We obtain that $v(\gamma(0)) = u(\gamma(\alpha))$.

Similarly, if $\gamma(l_\gamma) \in G \cap E$, then there is $\beta \in (\alpha, l_\gamma)$ such that $\gamma(\beta) \in G \setminus E$. We set $\beta = l_\gamma$ otherwise. Consequently, $v(\gamma(l_\gamma)) = u(\gamma(\beta))$. Therefore,

$$(3.1) \quad |v(\gamma(0)) - v(\gamma(l_\gamma))| = |u(\gamma(\alpha)) - u(\gamma(\beta))| \leq \int_{\gamma|_{[\alpha, \beta]}} g \, ds \leq \int_\gamma g \, ds.$$

Thus, g is an X -weak upper gradient of v as (3.1) holds for Mod_X -a.e. curve γ . Hence

$$C_X(G) \leq \|v\|_{N^1X} \leq \|v\|_X + \|g\|_X = \|u\|_X + \|g\|_X < C_X(G \setminus E) + \varepsilon.$$

Letting $\varepsilon \rightarrow 0$, we see that $C_X(G) \leq C_X(G \setminus E)$ as needed.

The equality for $\tilde{C}_{X,r}$ can be shown analogously. □

For an arbitrary set, adding a zero set (with respect to the capacity) does not change the capacity of the set even if C_X is not subadditive but merely quasi-subadditive as we are now about to see.

Lemma 3.4. *Let $E, F \subset \mathcal{P}$. Suppose that $C_X(F) = 0$. Then, $C_X(E \cup F) = C_X(E)$ and $\tilde{C}_{X,r}(E \cup F) = \tilde{C}_{X,r}(E)$.*

Proof. By monotonicity, $C_X(E \cup F) \geq C_X(E)$ and $\tilde{C}_{X,r}(E \cup F) \geq \tilde{C}_{X,r}(E)$. The converse inequality for $\tilde{C}_{X,r}$ follows from the σ -subadditivity of $\tilde{C}_{X,r}^*$.

If $C_X(E) = \infty$, then the converse inequality holds trivially. Suppose now that $C_X(E) < \infty$. Let $\varepsilon > 0$ and let $\chi_E \leq u \in N^1X$ be such that $\|u\|_{N^1X} \leq \|u\|_X + \|g\|_X < C_X(E) + \varepsilon$, where $g \in X$ is an upper gradient of u . Set $v = u + \chi_F$. Then, $v = u$ q.e., whence $\|v\|_X = \|u\|_X$ and g is a weak upper gradient of v by Corollary 5.11 in [31]. As $v \geq \chi_{E \cup F}$, we have $C_X(E \cup F) \leq \|v\|_{N^1X} \leq \|v\|_X + \|g\|_X < C_X(E) + \varepsilon$. Letting $\varepsilon \rightarrow 0$ shows that $C_X(E \cup F) \leq C_X(E)$. □

The following result generalizes Proposition 1.4 in [8] in a similar fashion as Proposition 4.7 in Björn, Björn, and Lehrbäck [6]. It shows that in locally compact (and hence in proper) metric measure spaces, the Sobolev capacity C_X (and hence also $\tilde{C}_{X,r}$) is an outer capacity at least for zero sets whenever lsc majorants provide good estimates of the function norm.

Proposition 3.5. *Assume that \mathcal{P} is locally compact and that X is a quasi-Banach function lattice possessing the Vitali–Carathéodory property, i.e., it satisfies (2.1). Let $E \subset \mathcal{P}$ with $C_X(E) = 0$. Then for every $\varepsilon > 0$, there is an open set $U \supset E$ with $C_X(U) < \varepsilon$.*

Proof. Suppose first that there is an open set $G \supset E$ such that \overline{G} is compact.

Let $\varepsilon > 0$. Since $C_X(E) = 0$, we have that $\chi_E = 0$ q.e. in \mathcal{P} and hence $\|\chi_E\|_{N^1X} = 0$ by Proposition 6.15 in [31]. Thus, there is an upper gradient $g \in X$ of χ_E such that $\|g\|_X < \varepsilon$. By (2.1), we can find $v, \rho \in X \cap \text{lsc}(\mathcal{P})$ that satisfy $v \geq \chi_E$ and $\rho \geq g$ everywhere in \mathcal{P} , while $\|v\|_X < \varepsilon$ and $\|\rho\|_X < \varepsilon$. Let $\tilde{v} = v\chi_G$. Since G is open and contains E , we have that $\chi_E \leq \tilde{v} \in X \cap \text{lsc}(\mathcal{P})$ and $\|\tilde{v}\|_X < \varepsilon$ as well.

Let $V = \{x \in \mathcal{P} : \tilde{v}(x) > 1/2\}$. Then, $E \subset V \subset G$ and V is open. Furthermore, $\|\chi_V\|_X \leq 2\|\tilde{v}\|_X < 2\varepsilon$. Let

$$u(x) = \min\left\{1, \inf_{\gamma} \int_{\gamma} (\rho + 1) ds\right\},$$

where the infimum is taken over all (including constant) curves connecting x to the closed set $\mathcal{P} \setminus V$. Then, $u|_{\overline{G}} \in \text{lsc}(\overline{G})$ by Lemma 3.3 in Björn, Björn and Shanmugalingam [8] since \overline{G} , being compact, is a proper metric space and $\rho + 1$ is bounded away from zero. Consequently, $u \in \text{lsc}(\mathcal{P})$ as $u \equiv 0$ on $\mathcal{P} \setminus V \supset \mathcal{P} \setminus G$.

Now, let $U = \{x \in \mathcal{P} : u(x) > 1/2\}$. The set U is open due to the semicontinuity of u . We can show that $u = 1$ on E , whence $E \subset U$. Indeed, let γ be a curve connecting arbitrary $x := \gamma(0) \in E$ with $y := \gamma(l_\gamma) \in \mathcal{P} \setminus V$. Then, $\int_{\gamma} (\rho + 1) ds \geq |\chi_E(x) - \chi_E(y)| + l_\gamma > 1$ as ρ is an upper gradient of χ_E . Furthermore, $u \leq \chi_V$ and $(\rho + 1)\chi_V$ is an upper gradient of u due to Lemmata 3.1 and 3.2 in [8]. We can therefore estimate the capacity

$$\begin{aligned} C_X(U) &\leq 2\|u\|_{N^1X} \leq 2(\|\chi_V\|_X + \|(\rho + 1)\chi_V\|_X) \\ &\leq 2((1 + c_\Delta)\|\chi_V\|_X + c_\Delta\|\rho\chi_V\|_X) \leq 2((1 + c_\Delta)2\varepsilon + c_\Delta\varepsilon) \leq 10c_\Delta\varepsilon. \end{aligned}$$

If no open neighborhood of E has a compact closure, then we can apply separability and the local compactness of \mathcal{P} to write $E = \bigcup_{n=1}^\infty E_n$ so that for each $n \in \mathbb{N}$ there is an open set $G_n \supset E_n$ with a compact closure. In particular, $C_X(E_n) = 0$. By the previous part of the proof, we can find open sets $U_n \supset E_n$ with $C_X(U_n) < \varepsilon/(2c_\Delta)^n$. Let now $U = \bigcup_{n=1}^\infty U_n$. Then, U is open and

$$C_X(U) \leq \sum_{n=1}^\infty c_\Delta^n C_X(U_n) < \varepsilon$$

by the σ -quasi-subadditivity of C_X . □

Remark 3.6. In the previous claim, it in fact suffices to assume that there exists an open set $G \supset E$ that is locally compact instead of requiring that the entire space \mathcal{P} is locally compact. It is, however, currently unknown whether such local compactness is really necessary. On the other hand, the Vitali–Carathéodory property is crucial. In view of Proposition 2.1, it suffices that X contains bounded functions with bounded support and these have absolutely continuous norm in X . In [7], Björn, Björn and Malý have constructed a metric measure space \mathcal{P} and a function space $X = X(\mathcal{P})$ such that Propositions 2.1 and 3.5 fail.

4. Quasi-continuity and its consequences

In this section, we study when Newtonian functions possess a Luzin-type property, the so-called quasi-continuity, which then leads to the fact that the Sobolev capacity is an outer capacity. The Sobolev capacity defined via the Newtonian quasi-norm characterizes the equivalence classes well, but in general it need not be an outer capacity then. Outside of the Newtonian setting, it is customary to introduce the Sobolev capacity so that it is an outer capacity by definition, cf. Definition 2.35 in Heinonen, Kilpeläinen and Martio [19] or Section 3 in Kinnunen and Martio [26].

In Section 5, the quasi-continuity will help us establishing the density of compactly supported Lipschitz functions in $N_0^1 X(\Omega)$, where $\Omega \subset \mathcal{P}$ is open.

Definition 4.1. A function $u: \mathcal{P} \rightarrow \overline{\mathbb{R}}$ is *weakly quasi-continuous* if for every $\varepsilon > 0$ there is a set $E \subset \mathcal{P}$ with $C_X(E) < \varepsilon$ such that $u|_{\mathcal{P} \setminus E}$ is continuous. If the set E can be chosen open for every $\varepsilon > 0$, then u is *quasi-continuous*.

By continuity of an extended real-valued function, we mean that the function does not attain the values $\pm\infty$ and is in fact real-valued.

As the capacities C_X and $\tilde{C}_{X,r}$ are equivalent, it is insignificant whether the notion of (weak) quasi-continuity is defined using C_X or $\tilde{C}_{X,r}$.

Lemma 4.2. *Assume that for every ε and every $E \subset \mathcal{P}$ with $C_X(E) = 0$, there is an open set $U \supset E$ with $C_X(U) < \varepsilon$. If u is quasi-continuous, then every v that coincides with u q.e. is also quasi-continuous.*

In view of Proposition 3.5, the hypotheses of the lemma are satisfied if \mathcal{P} is locally compact and if X has the Vitali–Carathéodory property (2.1) (which, in particular, it does by Proposition 2.1 if X contains all bounded functions with bounded support and these have absolutely continuous quasi-norm in X).

Proof. Let $\varepsilon > 0$. Define $E = \{x \in \mathcal{P} : u(x) \neq v(x)\}$, so $C_X(E) = 0$. Thus, we can find an open set $U \supset E$ with $C_X(U) < \varepsilon/2c_\Delta$. Since u is quasi-continuous, there is an open set V with $C_X(V) < \varepsilon/2c_\Delta^2$ such that $u|_{\mathcal{P} \setminus V}$ is continuous. Let $G = U \cup V$. Then, $C_X(G) < \varepsilon$ and $v|_{\mathcal{P} \setminus G} = u|_{\mathcal{P} \setminus G}$ is continuous. Hence, v is quasi-continuous. \square

Next, we will see that if the Sobolev capacity is an outer capacity, then the distinction between weak quasi-continuity and quasi-continuity is not needed.

Proposition 4.3. *Suppose that C_X or $\tilde{C}_{X,r}$ is an outer capacity. Then, a function is quasi-continuous if and only if it is weakly quasi-continuous.*

Proof. Quasi-continuous functions are trivially weakly quasi-continuous.

Let u be weakly quasi-continuous and let $\varepsilon > 0$. Then, there is a set $E \subset \mathcal{P}$ with $C_X(E) < \varepsilon$ (resp. $\tilde{C}_{X,r}(E) < \varepsilon$) such that $u|_{\mathcal{P} \setminus E}$ is continuous. Since C_X (resp. $\tilde{C}_{X,r}$) is an outer capacity, there is an open set $G \supset E$ with $C_X(G) < \varepsilon$ (resp. $\tilde{C}_{X,r}(G) < \varepsilon$). Then, $u|_{\mathcal{P} \setminus G}$ is also continuous, whence u is quasi-continuous. \square

For functions that are absolutely continuous along Mod_X -a.e. curve, the equality a.e. implies equality on a larger set, namely, q.e. (see Proposition 6.12 in [31]). Similarly, we have the following result for quasi-continuous functions.

Proposition 4.4. *Suppose that both u and v are quasi-continuous. If $u = v$ a.e., then $u = v$ q.e.*

Sketch of proof. Kilpeläinen [25] has proven the claim for abstract outer capacities that satisfy the rigidity condition of Proposition 3.3 under the assumption that u and v are weakly quasi-continuous. His proof works verbatim if the hypotheses of the capacity being outer and the weak quasi-continuity of u and v are replaced by the hypothesis that u and v are quasi-continuous. \square

Since a Egorov-type convergence theorem (see Corollary 7.2 in [31]) holds in Newtonian spaces based on an arbitrary quasi-Banach function lattice X , we will obtain that Newtonian functions are weakly quasi-continuous provided that continuous functions are dense in N^1X .

Sufficient conditions for density of (Lipschitz) continuous functions in N^1X have been discussed in [33] using the connection between Hajlasz gradients, fractional sharp maximal functions, and (weak) upper gradients in doubling p -Poincaré spaces (see Definition 6.1 below). Roughly speaking, Lipschitz functions are dense in N^1X if a certain maximal operator of Hardy–Littlewood type satisfies weak norm estimates and the quasi-norm of X is absolutely continuous.

In Theorem 7.1 below, it will be shown that Newtonian functions have continuous representatives (with respect to equality q.e.) if \mathcal{P} supports a certain Poincaré inequality, μ is doubling and the quasi-norm of X is sufficiently restrictive in comparison with the “dimension of the measure”. In that case, the continuous functions are trivially dense in N^1X .

It was established by Ambrosio, Colombo and Di Marino [1] (cf. Ambrosio, Gigli and Savaré [2]) that Lipschitz functions are dense in $N^{1,p} := N^1L^p$ for $p \in (1, \infty)$ if \mathcal{P} is proper, endowed with a doubling metric. In particular, neither a Poincaré inequality, nor a doubling property of the measure is needed.

Proposition 4.5. *If continuous functions are dense in N^1X , then every function $u \in N^1X$ has a quasi-continuous representative $\tilde{u} = u$ q.e. Hence, u is weakly quasi-continuous.*

Proof. Let $u \in N^1X$ be approximated by a sequence $\{u_k\}_{k=1}^\infty \subset \mathcal{C}(\mathcal{P}) \cap N^1X$ so that $u_k \rightarrow u$ in N^1X as $k \rightarrow \infty$. By Corollary 7.2 in [31], there is $\tilde{u} \in N^1X$ such that $\tilde{u} = u$ q.e. and for every $\varepsilon > 0$ there exists an open set U_ε with $C_X(U_\varepsilon) < \varepsilon$ such that a subsequence $\{u_{k_j}\}_{j=1}^\infty$ converges uniformly to \tilde{u} on $\mathcal{P} \setminus U_\varepsilon$. Hence, $\tilde{u}|_{\mathcal{P} \setminus U_\varepsilon}$ is continuous and \tilde{u} is quasi-continuous. Writing $E = \{x \in \mathcal{P} : u(x) \neq \tilde{u}(x)\}$, we have $C_X(U_\varepsilon \cup E) < \varepsilon$ by Lemma 3.4 and $u|_{\mathcal{P} \setminus (U_\varepsilon \cup E)}$ is continuous whence u is weakly quasi-continuous. \square

In Proposition 3.5, we saw that C_X is an outer capacity for zero sets under certain hypotheses. As a consequence, we obtain that functions with a quasi-

continuous representative are in fact quasi-continuous by Lemma 4.2. Then, Newtonian functions are quasi-continuous by Proposition 4.5 provided that they can be approximated by continuous functions. Hence, we have the following result.

Corollary 4.6. *Assume that \mathcal{P} is locally compact and X is a quasi-Banach function lattice with the Vitali–Carathéodory property (2.1). In particular, we may assume that X contains characteristic functions of all bounded sets and these have absolutely continuous norm in X . If continuous functions are dense in N^1X , then every $u \in N^1X$ is quasi-continuous.*

The original idea of Proposition 4.5 and Corollary 4.6 for $X = L^p$ under considerably stronger assumptions can be traced back to Shanmugalingam [38], whose result was later generalized by Björn, Björn, and Shanmugalingam [8].

Proposition 4.7. *Suppose that there is a cover $\mathcal{P} = \bigcup_{k=1}^\infty \mathcal{P}_k$, where \mathcal{P}_k is open and that for every $k \in \mathbb{N}$, there is $\delta_k > 0$ such that $C_X(\{x\}) \geq \delta_k$ for each $x \in \mathcal{P}_k$. Then, weakly quasi-continuous functions are continuous.*

In particular, if all functions in N^1X are weakly quasi-continuous (which holds, e.g., if continuous functions are dense in N^1X), then $N^1X \subset \mathcal{C}(\mathcal{P})$.

Proof. Set $\varepsilon_k = \delta_k/2$ for every $k \in \mathbb{N}$. Let u be weakly quasi-continuous. Then, there is E_k with $C_X(E_k) < \varepsilon_k$ such that $u|_{\mathcal{P} \setminus E_k}$ is continuous. Consequently, $\mathcal{P}_k \cap E_k = \emptyset$ because every $x \in \mathcal{P}_k$ satisfies $C_X(\{x\}) \geq \delta_k > C_X(E_k)$. Thus, $u|_{\mathcal{P}_k} \in \mathcal{C}(\mathcal{P}_k)$. Since \mathcal{P}_k is open, u is continuous at every point of \mathcal{P}_k .

Finally, u is continuous everywhere in \mathcal{P} as $\mathcal{P} = \bigcup_{k=1}^\infty \mathcal{P}_k$. Hence, $u \in \mathcal{C}(\mathcal{P})$.

If continuous functions are dense in N^1X , then all functions in N^1X are weakly quasi-continuous by Proposition 4.5. □

So far, we have seen that C_X (or $\tilde{C}_{X,r}$) being an outer capacity on zero sets implies that Newtonian functions are quasi-continuous (under some additional assumptions). The next proposition shows that the converse is actually stronger. Namely, if Newtonian functions are quasi-continuous, then $\tilde{C}_{X,r}$ is an outer capacity on all sets (without any additional assumptions). An analogous result for $\tilde{C}_{L^p,p}$ with $p \in [1, \infty)$ was given in Corollary 1.3 of [8].

Proposition 4.8. *Assume that all functions in N^1X are quasi-continuous. Then, $\tilde{C}_{X,r}$ is an outer capacity, i.e., $\tilde{C}_{X,r}(E) = \inf \tilde{C}_{X,r}(G)$ for every $E \subset \mathcal{P}$, where the infimum is taken over all open sets $G \supset E$. Moreover, if X is normed, then C_X is an outer capacity.*

Sketch of proof. If X is normed, then $C_X = \tilde{C}_{X,r}$ with $r = 1$. Hence, it suffices to prove that $\tilde{C}_{X,r}$ is an outer capacity.

The proof of the outer regularity of $\tilde{C}_{X,r}$ goes through almost verbatim as the proof of Corollary 1.3 in [8]. The only modifications needed are: (i) to replace the capacity $C_p(\cdot)^{1/p}$ by $\tilde{C}_{X,r}(\cdot)^r$; and (ii) to replace the Newtonian norm $\|f\|_{N^{1,p}}$ by $\|f\|_X^r + \|g_f\|_X^r$, where $g_f \in X$ is a minimal X -weak upper gradient of $f \in N^1X$ (while f stands for either u, v , or w in the original proof). □

The following proposition quantifies the difference between a.e. and q.e. equivalence classes in Newtonian spaces. Namely, N^1X contains only the “good” representatives of the functions that lie in an a.e. equivalence class of a Newtonian function.

Proposition 4.9. *Assume that \mathcal{P} is locally compact and that continuous functions are dense in N^1X . Suppose further that X has the Vitali–Carathéodory property (2.1). In particular, it suffices to assume that χ_B satisfies (AC) and lies in X whenever $B \subset \mathcal{P}$ is bounded and measurable. Let $u: \mathcal{P} \rightarrow \overline{\mathbb{R}}$ be such that $u = v$ a.e. in \mathcal{P} for some function $v \in N^1X$. Then, the following are equivalent:*

- (a) $u \in N^1X$;
- (b) $u \circ \gamma$ is absolutely continuous for Mod_X -a.e. curve γ ;
- (c) u is weakly quasi-continuous;
- (d) u is quasi-continuous.

Proof. Proposition 2.1 gives (2.1) if $\chi_B \in X$ satisfies (AC) whenever $B \subset \mathcal{P}$ is bounded.

The equivalence (a) \Leftrightarrow (b) was established by Proposition 6.18 in [31], without any assumptions on \mathcal{P} .

The implication (a) \Rightarrow (c) is shown in Proposition 4.5.

The equivalence (c) \Leftrightarrow (d) follows by Proposition 4.3, whose hypotheses are satisfied due to Corollary 4.6 and Proposition 4.8.

In order to show that (d) \Rightarrow (a), assume that u is quasi-continuous. It follows from (a) \Rightarrow (d) that v is quasi-continuous since $v \in N^1X$. Then, $u = v$ q.e. by Proposition 4.4. Therefore, $\|u - v\|_{N^1X} = 0$ by Proposition 6.15 in [31], which yields that $u \in N^1X$. \square

5. Density of locally Lipschitz functions

Let $\Theta \subset \mathcal{P}$ be measurable. Then, it can be naturally considered as a metric measure space on its own, where both the metric and the measure are inherited from \mathcal{P} (by restriction). Given a (quasi)Banach function lattice $X = X(\mathcal{P}, \mu)$, we define the restricted space $X(\Theta)$ by its (quasi)norm $\|u\|_{X(\Theta)} := \|\tilde{u}\chi_\Theta\|_X$, where \tilde{u} is any extension of a function $u: \Theta \rightarrow \overline{\mathbb{R}}$ to \mathcal{P} . The Newtonian space $N^1X(\Theta)$ is then defined in accordance with Definition 2.3 as the set of those functions in $X(\Theta)$ that have an upper gradient in $X(\Theta)$. Observe that $N^1X(\Theta) \supset \{u|_\Theta : u \in N^1X(\mathcal{P})\}$ and the inclusion may be strict unless Θ is an extension domain (see e.g. Björn and Shanmugalingam [9] and references therein for some results on $N^{1,p}$ -extension domains).

The aim of this section is to prove that locally Lipschitz functions are dense in $N^1X(\Omega)$, where $\Omega \subset \mathcal{P}$ is open, provided that (locally) Lipschitz functions are dense in $N^1X(\mathcal{P})$. Note that we will not pose any assumptions on Ω besides being open. We will however need \mathcal{P} to be proper and X to have absolutely continuous norm such that $L^\infty(\Omega) \hookrightarrow X_{\text{loc}}(\Omega)$. Here, we generalize the results of [8], where X was just L^p .

It has been shown in Section 3 of [33] that Newtonian functions can be approximated by their truncations if the (quasi)norm of X is absolutely continuous. We can extend this result if all Newtonian functions are quasi-continuous. Namely, Newtonian functions that vanish outside of a measurable set Θ can be approximated by bounded functions whose support is a bounded subset of Θ . The case $X = L^p$ with an open Θ was discussed in [8], where the fundamental idea came from [39].

By Definition 2.3, $N_0^1X(\Theta)$ consists of those functions in $N^1X(\Theta)$ that can be extended by zero yielding a function in $N^1X(\mathcal{P})$. Therefore, we may (and will) consider every function in $N_0^1X(\Theta)$ to be also an element of $N^1X(\mathcal{P})$ that vanishes outside of Θ .

Lemma 5.1. *Let X be a quasi-Banach function lattice with absolutely continuous quasi-norm. Assume that all functions in N^1X are quasi-continuous. Then, every function in $N_0^1X(\Theta)$ can be approximated in N^1X by bounded functions with bounded support lying in Θ .*

Proof. Let $u \in N_0^1X(\Theta)$ with an X -weak upper gradient $g \in X$. It has been shown in Corollary 3.4 of [33] that the truncations of u get arbitrarily close to u in N^1X . Therefore, we may assume that u is bounded.

Next, we will show that we may assume that u has bounded support. Let us fix $x_0 \in \mathcal{P}$ and write $\psi_n(x) = (1 - \text{dist}(x, B(x_0, n)))^+$ for $n \in \mathbb{N}$. We want to prove that $u\psi_n \rightarrow u$ in N^1X as $n \rightarrow \infty$. Since $\tilde{g} = \chi_{A(x_0, n, n+1)}$ is an upper gradient of $1 - \psi_n(x) = \min\{1, \text{dist}(x, B(x_0, n))\}$, where $A(x_0, n, n+1)$ is the closed annulus $\{x \in \mathcal{P} : n \leq d(x, x_0) \leq n+1\}$, the product rule (Theorem A.1) yields that the function $g_n = (1 - \psi_n)g + \chi_{A(x_0, n, n+1)}u$ is an X -weak upper gradient of $(1 - \psi_n)u$. Moreover, $g_n \leq (u + g)\chi_{\mathcal{P} \setminus B(x_0, n)}$. Hence,

$$\begin{aligned} \|u - u\psi_n\|_{N^1X} &\leq \|u - u\psi_n\|_X + \|g_n\|_X \\ &\leq \|u\chi_{\mathcal{P} \setminus B(x_0, n)}\|_X + \|(u + g)\chi_{\mathcal{P} \setminus B(x_0, n)}\|_X \rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

due to the absolute continuity of the quasi-norm of X . Therefore, we do not lose any generality if we suppose that u has bounded support.

Since u is quasi-continuous, there are open sets $U_k \subset \mathcal{P}$, $k \in \mathbb{N}$, such that $u|_{\mathcal{P} \setminus U_k}$ is continuous while $C_X(U_k) \rightarrow 0$ as $k \rightarrow \infty$. Thus, there exist functions $w_k \in N^1X$ such that $\chi_{U_k} \leq w_k \leq 1$ and $\|w_k\|_{N^1X} \rightarrow 0$ as $k \rightarrow \infty$. By Corollary 7.2 in [31], we may assume that $w_k \rightarrow 0$ q.e., passing to a subsequence if necessary. The sets $G_k := U_k \cup \{x \in \mathcal{P} \setminus U_k : u(x) < 1/k\}$ are open in \mathcal{P} , whence $\mathcal{P} \setminus G_k \subset \Theta$ is closed. Let

$$\eta_k(t) = \begin{cases} 0 & \text{for } |t| < 1/k, \\ 2(|t| - 1/k) \text{sgn } t & \text{for } 1/k \leq |t| \leq 2/k, \\ t & \text{for } |t| > 2/k. \end{cases}$$

By the chain rule (Theorem A.2), we obtain that $g_{\eta_k \circ u - u} = g_u \chi_{\{0 < |u| < 2/k\}}$ a.e. as the function $t \mapsto \eta_k(t) - t$ is 1-Lipschitz and supported in $[-2/k, 2/k]$. The

absolute continuity of the norm of X now yields

$$\|\eta_k \circ u - u\|_{N^1X} \leq \|u\chi_{\{0 < |u| < 2/k\}}\|_X + \|g_u\chi_{\{0 < |u| < 2/k\}}\|_X \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

since $\bigcap_{k=1}^\infty \{x \in \mathcal{P} : 0 < |u(x)| < 2/k\} = \emptyset$. As η_k is 2-Lipschitz, we can estimate $g_{\eta_k \circ u} \leq 2g_u$ a.e. Let $u_k = (1 - w_k)(\eta_k \circ u)$. Then, u_k is supported within $\mathcal{P} \setminus G_k$ and the product rule gives that

$$\begin{aligned} \|u_k - \eta_k \circ u\|_{N^1X} &= \|(\eta_k \circ u)w_k\|_{N^1X} \\ &\leq \|(\eta_k \circ u)w_k\|_X + \|(\eta_k \circ u)g_{w_k} + g_{\eta_k \circ u}w_k\|_X \\ &\leq \|\eta_k \circ u\|_{L^\infty}(\|w_k\|_X + c_\Delta \|g_{w_k}\|_X) + 2c_\Delta \|g_u w_k\|_X. \end{aligned}$$

For an arbitrary $\varepsilon > 0$, we obtain that $\|g_u w_k\|_X \leq c_\Delta(\varepsilon \|g_u\|_X + \|g_u\chi_{E_k(\varepsilon)}\|_X)$, where $E_k(\varepsilon) = \{x \in \mathcal{P} : w_k(x) > \varepsilon\}$. Then, $\limsup_{k \rightarrow \infty} \|g_u w_k\|_X \leq c_\Delta \varepsilon \|g_u\|_X$ by the absolute continuity of the quasi-norm of X since $\mu(\bigcap_{k=1}^\infty E_k(\varepsilon)) = 0$. Letting $\varepsilon \rightarrow 0$, we see that $\|g_u w_k\|_X \rightarrow 0$ as $k \rightarrow \infty$. The choice of w_k ensures that $\|w_k\|_X + \|g_{w_k}\|_X = \|w_k\|_{N^1X} \rightarrow 0$ as $k \rightarrow \infty$. Thus, $\|u_k - \eta_k \circ u\|_{N^1X} \rightarrow 0$ as $k \rightarrow \infty$. \square

We can go even further if \mathcal{P} is a proper metric measure space. If $\Omega \subset \mathcal{P}$ is open and if locally Lipschitz functions are dense in N^1X , then the Newtonian functions that vanish outside of Ω can be approximated (in the norm of N^1X) by Lipschitz functions that are compactly supported within Ω .

Proposition 5.2. *Suppose that \mathcal{P} is proper and $\Omega \subset \mathcal{P}$ is open. Assume that X is a quasi-Banach function lattice that has absolutely continuous quasi-norm. Assume further that $L^\infty(\Omega) \hookrightarrow X_{\text{loc}}(\Omega)$ (which in particular holds if X satisfies (P4)). If locally Lipschitz functions are dense in N^1X , then $\text{Lip}_c(\Omega) = \overline{N_0^1X(\Omega)}$.*

Proof. Since bounded functions with compact support in Ω are contained in X by the embedding $L^\infty(\Omega) \hookrightarrow X_{\text{loc}}(\Omega)$, we immediately obtain that $\text{Lip}_c(\Omega) \subset \overline{N_0^1X(\Omega)}$. Now, we will show that $N_0^1X(\Omega)$ is a closed subset of N^1X . Let $\{u_k\}_{k=1}^\infty$ be a Cauchy sequence in $N_0^1X(\Omega)$. Then, there exists a function $u \in N^1X$ such that $\|u_k - u\|_{N^1X} \rightarrow 0$ as $k \rightarrow \infty$ since N^1X is complete by Theorem 7.1 in [31]. By passing to a subsequence if needed, we have that $u_k \rightarrow u$ pointwise q.e. in \mathcal{P} by Corollary 7.2 in [31]. Hence, $u = 0$ q.e. in $\mathcal{P} \setminus \Omega$. Let $\tilde{u} = u\chi_\Omega$. Then, $\tilde{u} = u$ q.e. in \mathcal{P} . Therefore, $\|\tilde{u} - u\|_{N^1X} = 0$ by Proposition 6.15 in [31] and hence $\|u_k - \tilde{u}\|_{N^1X} \rightarrow 0$ as $k \rightarrow \infty$, where $\tilde{u} \in N_0^1X$. Consequently, $N_0^1X(\Omega) = \overline{N_0^1X(\Omega)}$.

Let now $u \in N_0^1X(\Omega)$, $u \not\equiv 0$. Due to Corollary 4.6 and Lemma 5.1, we may assume that u is bounded and has bounded support within Ω . In fact, $\text{spt } u$ is compact since \mathcal{P} is proper. Let $\varepsilon > 0$ be arbitrary. There exists a locally Lipschitz function $v \in N^1X$ such that $\|u - v\|_{N^1X} < \varepsilon$. Next, we define $\eta(x) = (1 - 2 \text{dist}(x, \text{spt } u)/\delta)^+$, where $\delta = \min\{1, \text{dist}(\text{spt } u, \mathcal{P} \setminus \Omega)\}$. The support of η is compact in Ω , and $g_\eta \leq 2/\delta$ since η is $2/\delta$ -Lipschitz. Moreover, $\chi_{\text{spt } u} \leq \eta \leq 1$. Consequently, $v\eta \in \text{Lip}_c(\Omega)$ and the product rule (Theorem A.1) gives $g_{v(1-\eta)} \leq |v|g_\eta + g_v$. Since $1 - \eta = 0$ on $\text{spt } u$, Corollary A.4 yields that $g_{v(1-\eta)} = 0$ a.e.

on $\text{spt } u$. Corollary A.4 further implies that $g_{u-v} = g_{-v} = g_v$ a.e. outside of $\text{spt } u$ because $u - v = -v$ there. Thus,

$$\begin{aligned} \|v - v\eta\|_{N^1X} &\leq \|v\chi_{\mathcal{P}\setminus\text{spt } u}\|_X + \|(|v|g_\eta + g_v)\chi_{\mathcal{P}\setminus\text{spt } u}\|_X \\ &\leq \|v\chi_{\mathcal{P}\setminus\text{spt } u}\|_X + c_\Delta (\|v\chi_{\mathcal{P}\setminus\text{spt } u}\|_X \|g_\eta\|_{L^\infty} + \|g_v\chi_{\mathcal{P}\setminus\text{spt } u}\|_X) \\ &\leq \left(1 + \frac{2c_\Delta}{\delta}\right) \|v\chi_{\mathcal{P}\setminus\text{spt } u}\|_X + c_\Delta \|g_v\chi_{\mathcal{P}\setminus\text{spt } u}\|_X \\ &\leq \left(1 + \frac{2c_\Delta}{\delta}\right) (\|(u - v)\chi_{\mathcal{P}\setminus\text{spt } u}\|_X + \|g_{u-v}\chi_{\mathcal{P}\setminus\text{spt } u}\|_X). \end{aligned}$$

Therefore,

$$\|v - v\eta\|_{N^1X} \leq \left(1 + \frac{2c_\Delta}{\delta}\right) \|u - v\|_{N^1X} < \left(1 + \frac{2c_\Delta}{\delta}\right) \varepsilon.$$

The triangle inequality in N^1X now yields that

$$\|u - v\eta\|_{N^1X} \leq c_\Delta (\|u - v\|_{N^1X} + \|v - v\eta\|_{N^1X}) < 2c_\Delta \left(1 + \frac{c_\Delta}{\delta}\right) \varepsilon,$$

completing the proof of the inclusion $N_0^1X(\Omega) \subset \overline{\text{Lip}_c(\Omega)}$. □

Finally, if we consider the space of Newtonian functions on an open subset Ω of a proper metric space \mathcal{P} , then the density of locally Lipschitz functions in $N^1X(\mathcal{P})$, implies the density in $N^1X(\Omega)$. What makes this claim interesting is that we do not impose any other conditions on Ω . In particular, it has been shown earlier that Lipschitz functions are dense in $N^1X(\mathcal{P})$ if \mathcal{P} supports a p -Poincaré inequality and the maximal operator M_p has certain bounds, but here we do not assume that Ω (as a metric subspace of \mathcal{P}) is a p -Poincaré space nor that $\mu|_\Omega$ is doubling. On the other hand, we merely obtain density of locally Lipschitz functions, which is however not unexpected in view of Examples 5.8–5.11 in [5].

Example 5.3 (Example 5.4 in [8]). Let \mathcal{P} be the slit disc $B(0, 1) \setminus (-1, 0] \subset \mathbb{C} = \mathbb{R}^2$. Then, $f(z) = \max\{0, 2|z| - 1\} \arg z$ belongs to $N^{1,p}(\mathcal{P}) \setminus \overline{\text{Lip}(\mathcal{P})}$ for all $p \geq 1$.

Theorem 5.4. *Under the hypotheses of Proposition 5.2, locally Lipschitz functions are dense in $N^1X(\Omega)$.*

Proof. Let $u \in N^1X(\Omega)$ and let $\varepsilon > 0$ be arbitrary. Since \mathcal{P} is proper, we may find an increasing sequence of open sets $\emptyset = \Omega_0 \neq \Omega_1 \in \Omega_2 \in \dots \in \Omega$ so that $\Omega = \bigcup_{j=1}^\infty \Omega_j$. Let $\eta_0 = 0$ in Ω . For each $j = 1, 2, \dots$, choose $\eta_j \in \text{Lip}_c(\Omega_{j+1})$ such that $\chi_{\Omega_j} \leq \eta_j \leq 1$ and then define $u_j = u\eta_j(1 - \eta_{j-1})$, which gives that $u_j \in N_0^1X(\Omega_{j+1} \setminus \overline{\Omega_{j-1}})$. It is easily seen by induction that $\sum_{j=1}^k u_j = u\eta_k$ for every $k = 1, 2, \dots$, which leads to equality $u = \sum_{j=1}^\infty u_j$ everywhere in Ω .

By Proposition 5.2, there exists $v_j \in \text{Lip}_c(\Omega_{j+1} \setminus \overline{\Omega_{j-1}})$ such that $\|u_j - v_j\|_{N^1X} \leq (2c_\Delta)^{-j} \varepsilon$ for every $j = 1, 2, \dots$. Let $v = \sum_{j=1}^\infty v_j$. For every $x \in \Omega$, there is a

neighborhood $U \ni x$ such that at most three terms in this sum are non-zero in U , whence v is locally Lipschitz in Ω . The triangle inequality now yields that

$$\|u - v\|_{N^1X(\Omega)} \leq \sum_{j=1}^{\infty} c_{\delta}^j \|u_j - v_j\|_{N^1X} \leq \varepsilon. \quad \square$$

6. Boundedness of Newtonian functions

In the setting of Sobolev spaces $W^{1,p}(\mathbb{R}^n)$, it is well known that Sobolev functions are essentially bounded and have continuous representatives if $p > n$. A finer distinction of function spaces is however needed for $p = n$, e.g., certain Zygmund or Lorentz norms can be used to ensure the boundedness. A similar result can be obtained for Newtonian functions if we introduce the notion of dimension of a doubling measure μ . Therefore, we will assume that μ satisfies the *doubling condition* in this (as well as in the next) section.

In \mathbb{R}^n with the Lebesgue measure λ^n , we have $\lambda^n(2B) = 2^n \lambda^n(B)$ for every ball $B \subset \mathbb{R}^n$. The doubling condition gives $\mu(2B) \leq c_{\text{dbl}} \mu(B) = 2^{\log_2 c_{\text{dbl}}} \mu(B)$. Even though $\log_2 c_{\text{dbl}}$ can play the role of the dimension, it need not be sharp for the results about boundedness and continuity of Newtonian functions. It can be easily shown (see, e.g., Lemma 3.3 in [5]) that for every $s \geq \log_2 c_{\text{dbl}}$ there is $c_s > 0$ such that

$$(6.1) \quad \frac{\mu(B(y, r))}{\mu(B(x, R))} \geq c_s \left(\frac{r}{R}\right)^s$$

whenever $0 < r \leq R$, $x \in \mathcal{P}$, and $y \in B(x, R)$. Considering a simple example of weighted \mathbb{R}^n with a non-constant weight $w \in L^\infty(\mathbb{R}^n)$ such that $1/w \in L^\infty(\mathbb{R}^n)$, we see that (6.1) holds with $s = n < \log_2 c_{\text{dbl}}$ and $c_s = 1/\|w\|_\infty \|1/w\|_\infty$. Therefore, the dimension will be replaced by $s \leq \log_2 c_{\text{dbl}}$, preferably as small as possible, such that (6.1) is satisfied. Note however that the set of admissible exponents s may be open, see e.g. Example 3.1 in Björn, Björn and Lehrbäck [6]. It is insignificant for the notion of dimension whether we require that (6.1) holds for all $y \in B(x, R)$ or only for $y = x$ (i.e., only for concentric balls) since

$$c_{\text{dbl}}^{-1} \frac{\mu(B(y, r))}{\mu(B(y, R))} \leq \frac{\mu(B(y, r))}{\mu(B(x, R))} = \frac{\mu(B(y, r))}{\mu(B(y, R))} \frac{\mu(B(y, R))}{\mu(B(x, R))} \leq c_{\text{dbl}} \frac{\mu(B(y, r))}{\mu(B(y, R))}.$$

If \mathcal{P} is connected, then there are $c_\sigma > 0$ and $0 < \sigma \leq s$ such that

$$(6.2) \quad \frac{\mu(B(y, r))}{\mu(B(x, R))} \leq c_\sigma \left(\frac{r}{R}\right)^\sigma$$

whenever $0 < r \leq R < 2 \text{diam } \mathcal{P}$, $x \in \mathcal{P}$, and $y \in B(x, R)$, see Corollary 3.8 in [5]. Similarly as above, if (6.2) holds with some σ , then it holds with all $\sigma' \leq \sigma$. The set of admissible exponents in (6.2) may be open, see Example 3.1 in [6]. Moreover, it may happen that $\sigma < s$ even if both σ and s are the best possible exponents

(provided that these exist). The metric measure space is called *Ahlfors Q -regular* if both (6.1) and (6.2) are satisfied with $\sigma = s =: Q$. However, the Ahlfors regularity is a very restrictive condition that fails even in weighted \mathbb{R}^n , unless the weight is bounded away both from zero and from infinity, see e.g. Example 3.5 in [5].

We will show that all Newtonian functions are locally essentially bounded (and have continuous representatives, which will be shown in the next section) provided that the function lattice X is continuously embedded into L^p_{loc} for some $p > s$ or into $L^s(\log L)^{1+\varepsilon}_{loc}$ in the borderline case $p = s \geq 1$, where s is the “dimension of the measure” given by (6.1), under the assumption that \mathcal{P} supports a p -Poincaré inequality. If a slightly stronger Poincaré inequality is assumed, then we will show that the embedding $X \hookrightarrow L^{s,1}_{loc}$ suffices to obtain local essential boundedness of functions in N^1X (and hence so does $X \hookrightarrow L^s(\log L)^{1-1/s}$).

Definition 6.1. We say that \mathcal{P} supports a *p -Poincaré inequality* with $p \in [1, \infty)$ if there exist constants $c_{PI} > 0$ and $\lambda \geq 1$ such that

$$(6.3) \quad \int_B |u - u_B| d\mu \leq c_{PI} \operatorname{diam}(B) \left(\int_{\lambda B} g^p d\mu \right)^{1/p}$$

for all balls $B \subset \mathcal{P}$, for all $u \in L^1_{loc}(\mathcal{P})$ and all upper gradients g of u .

This inequality is sometimes called a *weak p -Poincaré inequality* since we allow for $\lambda > 1$. Moreover, \mathcal{P} supports a p -Poincaré inequality if and only if (6.3) holds for all measurable functions u and all p -weak upper gradients g of u , where the left-hand side is interpreted as ∞ whenever u_B is not defined or $u_B = \pm\infty$. We can also equivalently require that (6.3) holds for all $u \in L^\infty(\mathcal{P})$ and all (p -weak) upper gradients g of u . These characterizations were shown in Proposition 4.13 of [5]. If $X \hookrightarrow L^p_{loc}$, i.e., if $\|f\chi_B\|_{L^p} \leq c_{\text{emb}}(B)\|f\chi_B\|_X$ for all balls $B \subset \mathcal{P}$ and $f \in X$, then we may also require validity of the inequality for all X -weak upper gradients g of u , which follows by Lemma 5.6 in [31].

If \mathcal{P} supports a p -Poincaré inequality for some $p \in [1, \infty)$, then it also supports a q -Poincaré inequality whenever $q \in [p, \infty)$ due to the Hölder inequality. It also follows that \mathcal{P} is connected (see, e.g., page 25 of Shanmugalingam [37]), whence μ , being doubling, is non-atomic.

Both Zygmund and Lorentz spaces, which have been mentioned earlier, belong to a wide class of function spaces, the so-called *r.i. spaces*, i.e., Banach function spaces that are *rearrangement-invariant*. Thus, they satisfy not only (P0)–(P5) with the modulus of concavity $c_\Delta = 1$, but also

(RI) if u and v are *equimeasurable*, i.e.,

$$\mu(\{x \in \mathcal{P} : u(x) > t\}) = \mu(\{x \in \mathcal{P} : v(x) > t\}) \quad \text{for all } t \geq 0,$$

then $\|u\|_X = \|v\|_X$.

For a detailed treatise on r.i. spaces, see Bennett and Sharpley [3].

For $f \in \mathcal{M}(\mathcal{P}, \mu)$, we define its *distribution function* μ_f and the *decreasing rearrangement* f^* by

$$\begin{aligned} \mu_f(t) &= \mu(\{x \in \mathcal{P} : |f(x)| > t\}), \quad t \in [0, \infty), \\ f^*(t) &= \inf\{\tau \geq 0 : \mu_f(\tau) \leq t\}, \quad t \in [0, \infty). \end{aligned}$$

The Cavalieri principle implies that $\|f\|_{L^1(\mathcal{P}, \mu)} = \|\mu_f\|_{L^1(\mathbb{R}^+, \lambda^1)} = \|f^*\|_{L^1(\mathbb{R}^+, \lambda^1)}$.

We define the *fundamental function* of a rearrangement-invariant quasi-Banach function lattice X as $\phi_X(t) = \|\chi_{E_t}\|_X$, where $E_t \subset \mathcal{P}$ is an arbitrary measurable set with $\mu(E_t) = \min\{t, \mu(\mathcal{P})\}$, $t > 0$. Note that different spaces may very well have the same fundamental function, which is the case, e.g., of the Lebesgue L^p and the Lorentz $L^{p,q}$ spaces as $\phi_{L^p}(t) = \phi_{L^{p,q}}(t) = t^{1/p}$ for $t < \mu(\mathcal{P})$ whenever $p \in [1, \infty)$ and $q \in [1, \infty]$. For another example, the Orlicz space L^Ψ based on an N -function Ψ has the fundamental function $\phi_{L^\Psi}(t) = 1/\Psi^{-1}(1/t)$.

We also have the continuous embedding of X into the weak- X space for every r.i. space X , which can be expressed by the inequality $\sup_{t>0} u^*(t)\phi_X(t) \leq \|u\|_X$.

In the next proposition, we will see that a p -Poincaré inequality gives not only an integral but also a supramal estimate for the oscillation of a Newtonian function, provided that p is sufficiently large.

Proposition 6.2. *Suppose that \mathcal{P} supports a p -Poincaré inequality with $p > s$ such that (6.1) is satisfied. Suppose further that $X \hookrightarrow L^p_{\text{loc}}$. Let $B_0 \subset \mathcal{P}$ be a fixed ball of radius $R > 0$. Then, there is a constant $c_{B_0} > 0$ such that for every ball $B \subset B_0$ of radius $r \in (0, R)$, we have*

$$(6.4) \quad C_X\text{-ess sup}_{x \in B} |u(x) - u_B| \lesssim \frac{c_{\text{emb}}(2\lambda B)r}{\mu(2\lambda B)^{1/p}} \|g\chi_{2\lambda B}\|_X \leq c_{B_0} r^{1-s/p} \|g\chi_{2\lambda B}\|_X$$

whenever $g \in X$ is an X -weak upper gradient of $u \in N^1X$. Moreover, we can estimate $c_{B_0} \approx c_{\text{emb}}(2\lambda B_0)(R^s/\mu(2\lambda B_0))^{1/p}$.

Note that we need to assume that $r \lesssim R$ for the second inequality in (6.4) as it may happen that $r \gg R$ even if $B \subset B_0$.

Proof. Let $B_0 = B(y, R) \subset \mathcal{P}$ and $B = B(z, r) \subset B_0$ be arbitrary balls with $r < R$. Then, $C_X\text{-ess sup}_{x \in B} |u(x) - u_B| = \text{ess sup}_{x \in B} |u(x) - u_B|$ by Corollary 6.13 in [31] since $|u - u_B| \in DX$.

By Proposition 4.27 in [5] and by the embedding $X \hookrightarrow L^p_{\text{loc}}$, we obtain that

$$\text{ess sup}_{x \in B} |u(x) - u_B| \lesssim r \frac{\|g\chi_{2\lambda B}\|_{L^p}}{\mu(2\lambda B)^{1/p}} \leq \frac{c_{\text{emb}}(2\lambda B)r}{\mu(2\lambda B)^{1/p}} \|g\chi_{2\lambda B}\|_X.$$

By applying (6.1), we see that $c_s(2\lambda r)^s/\mu(2\lambda B) \leq (2\lambda R)^s/\mu(2\lambda B_0)$. Hence,

$$\frac{r}{\mu(2\lambda B)^{1/p}} = r^{1-s/p} \left(\frac{r^s}{\mu(2\lambda B)} \right)^{1/p} \leq r^{1-s/p} \left(\frac{R^s}{c_s \mu(2\lambda B_0)} \right)^{1/p}.$$

Moreover, $c_{\text{emb}}(2\lambda B) \leq c_{\text{emb}}(2\lambda B_0)$, which yields the desired estimate

$$\frac{c_{\text{emb}}(2\lambda B)r}{\mu(2\lambda B)^{1/p}} \|g\chi_{2\lambda B}\|_X \leq c_{B_0} r^{1-s/p} \|g\chi_{2\lambda B}\|_X. \quad \square$$

Corollary 6.3. *Assume that \mathcal{P} supports a p -Poincaré inequality with $p > s$ such that (6.1) is satisfied. Let $\Omega \subset \mathcal{P}$ be a fixed open set. Assume further that X is an r.i. space with fundamental function ϕ and that*

$$(6.5) \quad c_\phi(\Omega) := \sup_{0 < t < \mu(\Omega)} \phi(t) \left(\int_0^t \frac{d\tau}{\phi(\tau)^p} \right)^{1/p} < \infty.$$

Then, for every ball B of radius $r > 0$ such that $2\lambda B \subset \Omega$, we have

$$C_{X\text{-ess sup}} |u(x) - u_B| \lesssim c_\phi(\Omega) r \frac{\|g\chi_{2\lambda B}\|_X}{\phi(\mu(2\lambda B))},$$

whenever $g \in X$ is an X -weak upper gradient of $u \in N^1X$.

Proof. In the proof of Proposition 6.2, we have seen that

$$C_{X\text{-ess sup}} |u(x) - u_B| \lesssim r \left(\int_{2\lambda B} g^p d\mu \right)^{1/p}.$$

By the Cavalieri principle and by the embedding $X \hookrightarrow \text{weak-}X$, it follows that

$$\begin{aligned} \left(\int_{2\lambda B} g^p d\mu \right)^{1/p} &= \left(\int_0^{\mu(2\lambda B)} \left(\frac{(g\chi_{2\lambda B})^*(t)\phi(t)}{\phi(t)} \right)^p dt \right)^{1/p} \\ &\leq \phi(\mu(2\lambda B)) \left(\int_0^{\mu(2\lambda B)} \frac{dt}{\phi(t)^p} \right)^{1/p} \sup_{0 < \tau < \mu(2\lambda B)} \frac{(g\chi_{2\lambda B})^*(\tau)\phi(\tau)}{\phi(\mu(2\lambda B))} \\ &\leq c_\phi(\Omega) \frac{\|g\chi_{2\lambda B}\|_X}{\phi(\mu(2\lambda B))}. \quad \square \end{aligned}$$

The previous proposition and corollary can be refined in the critical case $p = s$; we will however need to assume that X is embedded into the Zygmund space $L^s(\log L)^\alpha$ for some $\alpha > 1$. That result will be further improved under somewhat stronger assumptions on \mathcal{P} .

Definition 6.4. For $p \geq 1$ and $\alpha \geq 0$, the Zygmund space $L^p(\log L)^\alpha(E)$, where $E \subset \mathcal{P}$ is measurable and $\mu(E) < \infty$, consists of measurable functions $u: E \rightarrow \overline{\mathbb{R}}$ such that

$$\|u\|_{L^p(\log L)^\alpha(E)}^p = \int_0^{\mu(E)} \left(u^*(t) \left(1 + \log \frac{\mu(E)}{t} \right)^\alpha \right)^p dt < \infty.$$

As an alternative, which is well-defined even if $\mu(E) = \infty$, we may use

$$(6.6) \quad \|u\|_{L^p(\log L)^\alpha(E)}^p = \int_0^{\mu(E)} \left(u^*(t) \left(1 + \log^+ \frac{1}{t} \right)^\alpha \right)^p dt,$$

where \log^+ denotes the positive part of \log .

It is easy to see that $\|u\|_{L^p(\log L)^\alpha(E)} \approx \|u\|_{L^p(\log L)^\alpha(E)}$ where the constants depend on $\mu(E) < \infty$. Obviously, $L^p(\log L)^0 = L^p$. Furthermore, the Zygmund spaces are classical Lorentz spaces and they coincide with certain Orlicz classes whenever $\alpha \geq 1/p$. It is customary to drop the respective exponent in the notation $L^p(\log L)^\alpha(E)$ if $p = 1$ or $\alpha = 1$.

Proposition 6.5. *Suppose that \mathcal{P} supports an s -Poincaré inequality with s given by (6.1). Assume that $X \hookrightarrow L^s(\log L)_{\text{loc}}^\alpha$ for some $\alpha \geq 1$ if $s = 1$ and for some $\alpha > 1$ if $s > 1$. Let $B_0 \subset \mathcal{P}$ be a fixed ball of radius $R > 0$. Then, there is a constant $c_{B_0} > 0$ such that for every ball $B \subset B_0$ of radius $r \in (0, R)$, we have*

$$(6.7) \quad C_X\text{-ess sup}_{x \in B} |u(x) - u_B| \lesssim \frac{c_{\text{emb}}(2\lambda B)r}{\mu(2\lambda B)^{1/s}} \|g\chi_{2\lambda B}\|_X \leq c_{B_0} \|g\chi_{2\lambda B}\|_X$$

whenever $g \in X$ is an X -weak upper gradient of $u \in N^1X$. Moreover, we can estimate $c_{B_0} \approx c_{\text{emb}}(2\lambda B_0)R/\mu(2\lambda B_0)^{1/s}$.

Proof. Let $B_0 = B(y, R) \subset \mathcal{P}$ and $B = B(z, r) \subset B_0$ be arbitrary balls. It follows from Corollary 6.13 in [31] that

$$C_X\text{-ess sup}_{x \in B} |u(x) - u_B| = \text{ess sup}_{x \in B} |u(x) - u_B|$$

as $|u - u_B| \in DX$.

Let $x \in B$ be a Lebesgue point of u and set $\tilde{B} = B(x, r)$ and $\tilde{B}_n = B(x, 2^{-n}r)$ for $n = 0, 1, 2, \dots$. Then,

$$u(x) = \lim_{n \rightarrow \infty} u_{\tilde{B}_n} = u_{\tilde{B}} + \sum_{n=0}^{\infty} (u_{\tilde{B}_{n+1}} - u_{\tilde{B}_n}).$$

Applying the triangle inequality, the doubling condition, and the s -Poincaré inequality (where an X -weak upper gradient $g \in X$ of u may be used in light of the embedding $X \hookrightarrow L^s(\log L)_{\text{loc}}^\alpha \hookrightarrow L^s_{\text{loc}}$) yields

$$\begin{aligned} |u(x) - u_{\tilde{B}}| &\leq \sum_{n=0}^{\infty} |u_{\tilde{B}_{n+1}} - u_{\tilde{B}_n}| \leq \sum_{n=0}^{\infty} \int_{\tilde{B}_{n+1}} |u - u_{\tilde{B}_n}| d\mu \\ &\lesssim \sum_{n=0}^{\infty} \int_{\tilde{B}_n} |u - u_{\tilde{B}_n}| d\mu \lesssim r \sum_{n=0}^{\infty} 2^{-n} \left(\int_{\lambda \tilde{B}_n} g^s d\mu \right)^{1/s}. \end{aligned}$$

We have $c_s \mu(\lambda \tilde{B}) \leq 2^{ns} \mu(\lambda \tilde{B}_n)$ by (6.1), whence

$$\sum_{n=0}^{\infty} 2^{-n} \left(\int_{\lambda \tilde{B}_n} g^s d\mu \right)^{1/s} \lesssim \frac{1}{\mu(\lambda \tilde{B})^{1/s}} \sum_{n=0}^{\infty} \left(\int_{\lambda \tilde{B}_n} g^s d\mu \right)^{1/s}.$$

Let us, for the sake of brevity, write $\tilde{g} = g\chi_{\lambda \tilde{B}}$. Since \mathcal{P} is connected due to the Poincaré inequality, we have that $\mu(\lambda \tilde{B}_n) \leq c_\sigma 2^{-n\sigma} \mu(\lambda \tilde{B})$ for some $0 < \sigma \leq s$

and $c_\sigma \geq 1$ by (6.2). Let $A = c_\sigma \mu(\lambda \tilde{B})$. Applying the Hardy–Littlewood inequality and replacing \tilde{g} by its decreasing rearrangement, we obtain that

$$\sum_{n=0}^\infty \left(\int_{\lambda \tilde{B}_n} g^s \, d\mu \right)^{1/s} \leq \sum_{n=0}^\infty \left(\int_0^{\mu(\lambda \tilde{B}_n)} \tilde{g}^*(t)^s \, dt \right)^{1/s} \leq \sum_{n=0}^\infty \left(\int_0^{2^{-n\sigma} A} \tilde{g}^*(t)^s \, dt \right)^{1/s}.$$

If $s = 1$, then splitting the integration domain dyadically gives that

$$\sum_{n=0}^\infty \int_0^{2^{-n\sigma} A} \tilde{g}^*(t) \, dt = \sum_{n=0}^\infty \sum_{j=n}^\infty \int_{2^{-(j+1)\sigma} A}^{2^{-j\sigma} A} \tilde{g}^*(t) \, dt = \sum_{j=0}^\infty (j+1) \int_{2^{-(j+1)\sigma} A}^{2^{-j\sigma} A} \tilde{g}^*(t) \, dt.$$

If $s > 1$, then the Hölder inequality for series with $s' = s/(s - 1)$ yields that

$$\begin{aligned} (6.8) \quad \sum_{n=0}^\infty \left(\int_0^{2^{-n\sigma} A} \tilde{g}^*(t)^s \, dt \right)^{1/s} &= \sum_{n=0}^\infty \frac{(n+1)^{\alpha/s'}}{(n+1)^{\alpha/s'}} \left(\int_0^{2^{-n\sigma} A} \tilde{g}^*(t)^s \, dt \right)^{1/s} \\ &\leq \left(\sum_{n=0}^\infty (n+1)^{\alpha s/s'} \int_0^{2^{-n\sigma} A} \tilde{g}^*(t)^s \, dt \right)^{1/s} \left(\sum_{n=0}^\infty \frac{1}{(n+1)^\alpha} \right)^{1/s'}, \end{aligned}$$

where the latter series converges since $\alpha > 1$. Next,

$$\begin{aligned} \sum_{n=0}^\infty (n+1)^{\alpha(s-1)} \int_0^{2^{-n\sigma} A} \tilde{g}^*(t)^s \, dt &= \sum_{n=0}^\infty \sum_{j=n}^\infty (n+1)^{\alpha(s-1)} \int_{2^{-(j+1)\sigma} A}^{2^{-j\sigma} A} \tilde{g}^*(t)^s \, dt \\ &= \sum_{j=0}^\infty \sum_{n=0}^j (n+1)^{\alpha(s-1)} \int_{2^{-(j+1)\sigma} A}^{2^{-j\sigma} A} \tilde{g}^*(t)^s \, dt \lesssim \sum_{j=0}^\infty (j+1)^{\alpha s} \int_{2^{-(j+1)\sigma} A}^{2^{-j\sigma} A} \tilde{g}^*(t)^s \, dt. \end{aligned}$$

We have thus shown for all $s \geq 1$ that

$$\sum_{n=0}^\infty \left(\int_0^{2^{-n\sigma} A} \tilde{g}^*(t)^s \, dt \right)^{1/s} \lesssim \sum_{j=0}^\infty (j+1)^{\alpha s} \int_{2^{-(j+1)\sigma} A}^{2^{-j\sigma} A} \tilde{g}^*(t)^s \, dt.$$

We can estimate $(j+1)^{\alpha s} \lesssim (1 + \log A/t)^{\alpha s}$ for $t \in (2^{-(j+1)\sigma} A, 2^{-j\sigma} A)$. Therefore,

$$\sum_{j=0}^\infty (j+1)^{\alpha s} \int_{2^{-(j+1)\sigma} A}^{2^{-j\sigma} A} \tilde{g}^*(t)^s \, dt \lesssim \sum_{j=0}^\infty \int_{2^{-(j+1)\sigma} A}^{2^{-j\sigma} A} \tilde{g}^*(t)^s \left(1 + \log \frac{A}{t} \right)^{\alpha s} \, dt.$$

The decreasing rearrangement \tilde{g}^* is supported in $[0, \mu(\lambda \tilde{B})]$ and hence

$$\begin{aligned} \int_0^A \tilde{g}^*(t)^s \left(1 + \log \frac{A}{t} \right)^{\alpha s} \, dt &= \int_0^{\mu(\lambda \tilde{B})} \tilde{g}^*(t)^s \left(1 + \log \frac{c_\sigma \mu(\lambda \tilde{B})}{t} \right)^{\alpha s} \, dt \\ &\leq (1 + \log c_\sigma)^{\alpha s} \int_0^{\mu(\lambda \tilde{B})} \tilde{g}^*(t)^s \left(1 + \log \frac{\mu(\lambda \tilde{B})}{t} \right)^{\alpha s} \, dt \approx \|g\chi_{\lambda \tilde{B}}\|_{L^s(\log L)^\alpha(\lambda \tilde{B})}^s. \end{aligned}$$

Putting all the estimates together, we obtain

$$|u(x) - u_{\tilde{B}}| \lesssim r \frac{\|g\chi_{\lambda\tilde{B}}\|_{L^s(\log L)^\alpha(\lambda\tilde{B})}}{\mu(\lambda\tilde{B})^{1/s}}.$$

The triangle and the s -Poincaré inequality provide us with the estimate

$$\begin{aligned} |u_{\tilde{B}} - u_B| &\leq |u_{\tilde{B}} - u_{2B}| + |u_{2B} - u_B| \leq \int_{\tilde{B}} |u - u_{2B}| d\mu + \int_B |u - u_{2B}| d\mu \\ &\lesssim \int_{2B} |u - u_{2B}| d\mu \lesssim r \left(\int_{2\lambda B} g^s d\mu \right)^{1/s} \lesssim r \frac{\|g\chi_{2\lambda B}\|_{L^s(\log L)^\alpha(2\lambda B)}}{\mu(2\lambda B)^{1/s}}. \end{aligned}$$

Altogether, we see that

$$\begin{aligned} |u(x) - u_B| &\leq |u(x) - u_{\tilde{B}}| + |u_{\tilde{B}} - u_B| \lesssim r \frac{\|g\chi_{2\lambda B}\|_{L^s(\log L)^\alpha(2\lambda B)}}{\mu(2\lambda B)^{1/s}} \\ &\leq \frac{c_{\text{emb}}(2\lambda B)r}{\mu(2\lambda B)^{1/s}} \|g\chi_{2\lambda B}\|_X \leq c_{B_0} \|g\chi_{2\lambda B}\|_X, \end{aligned}$$

where (6.1) has been applied to estimate $r/\mu(2\lambda B)^{1/s} \leq R/(c_s\mu(2\lambda B_0))^{1/s}$. The Lebesgue differentiation theorem, which holds true since μ is doubling (see Section 1 of Heinonen [18]), yields that a.e. $x \in B$ is a Lebesgue point of u , whence this inequality holds for a.e. $x \in B$ and thus for the essential supremum over B . \square

The technique used in the previous proof for $s > 1$ seems to work even if we assume that $X \hookrightarrow L^s \log L(\log \log L)^\beta$ with $\beta > 1/s'$. The main difference in the proof would be to replace $(n + 1)^{\alpha/s'}$ in (6.8) by $(n + 1)^{1/s'} \log(n + 2)^\beta$. It also appears to be possible to iterate the logarithm several times raised to suitable powers. Nevertheless, we refrain from properly formulating and proving this claim.

It follows by Talenti [41] that Zygmund–Sobolev functions in \mathbb{R}^n , $n \geq 2$, are bounded for every $\alpha > 1/n' := 1 - 1/n$, which indicates that 1 is not the optimal borderline value for the exponent α in the previous proposition. On the other hand, $1/n'$ is sharp, which can be seen by considering $u(x) = \sqrt{\log \log(e^2/|x|)}$ for $x \in B := B(0, 1) \subset \mathbb{R}^n$, $n \geq 2$. Apparently, u is unbounded even though $u \in N^1 L^n(\log L)^\alpha(B) \subset W^1 L^n(\log L)^\alpha(B)$ for every $\alpha \in [0, 1/n']$.

We will show in Corollary 6.9 below that we can in fact obtain the local essential boundedness of Zygmund–Newtonian functions for all $\alpha > 1/n'$ as a special case of a more general result if \mathcal{P} supports a stronger Poincaré inequality.

Definition 6.6. The Lorentz space $L^{p,1}(\mathcal{P})$ for $1 \leq p < \infty$ is the Banach function space that consists of the measurable functions $u: \mathcal{P} \rightarrow \overline{\mathbb{R}}$ such that

$$\|u\|_{L^{p,1}(\mathcal{P})} := \frac{1}{p} \int_0^\infty u^*(t) t^{1/p-1} dt < \infty.$$

The following proposition shows that in the borderline case it suffices that the $(X$ -weak) upper gradient lies in the Lorentz $L^{s,1}$ space. A similar claim was proven

by Romanov [36] under a highly restrictive assumption that μ is s -Ahlfors regular, i.e., both (6.1) and (6.2) hold with $s = \sigma$. His paper served as an inspiration to use Abel’s partial summation formula in the proof below.

The price we have to pay is that a stronger Poincaré inequality is needed. Actually, assuming that \mathcal{P} supports an s -Poincaré inequality is enough if \mathcal{P} is complete. By Keith and Zhong [24], the Poincaré inequality is a self-improving property in that case and hence \mathcal{P} supports a p -Poincaré inequality with some $p < s$.

Proposition 6.7. *Suppose that \mathcal{P} supports a p -Poincaré inequality with $1 \leq p < s$ such that (6.1) is satisfied. Assume that $X \hookrightarrow L_{\text{loc}}^{s,1}$. Let $B_0 \subset \mathcal{P}$ be a fixed ball of radius $R > 0$. Then, there is a constant $c_{B_0} > 0$ such that for every ball $B \subset B_0$ of radius $r \in (0, R)$, we have*

$$(6.9) \quad C_X\text{-ess sup}_{x \in B} |u(x) - u_B| \lesssim \frac{c_{\text{emb}}(2\lambda B)r}{\mu(2\lambda B)^{1/s}} \|g\chi_{2\lambda B}\|_X \leq c_{B_0} \|g\chi_{2\lambda B}\|_X$$

whenever $g \in X$ is an X -weak upper gradient of $u \in N^1X$. Moreover, we can estimate $c_{B_0} \approx c_{\text{emb}}(2\lambda B_0)R/\mu(2\lambda B_0)^{1/s}$.

Proof. Let $B_0 = B(y, R) \subset \mathcal{P}$ and $B = B(z, r) \subset B_0$ be arbitrary balls with $r < R$. Then,

$$C_X\text{-ess sup}_{x \in B} |u(x) - u_B| = \text{ess sup}_{x \in B} |u(x) - u_B|$$

by Corollary 6.13 in [31] since $|u - u_B| \in DX$.

Let $x \in B$ be a Lebesgue point of u and set $\tilde{B} = B(x, r)$. Similarly as in the proof of Proposition 6.5, we obtain that

$$(6.10) \quad |u(x) - u_{\tilde{B}}| \lesssim r \sum_{n=0}^{\infty} 2^{-n} \left(\int_{2^{-n}\lambda\tilde{B}} g^p d\mu \right)^{1/p}.$$

Let us, for the sake of brevity, write $\tilde{g} = g\chi_{\lambda\tilde{B}}$. The embedding $L^{p,1} \hookrightarrow L^p$, whose norm is 1, and the doubling condition give that

$$(6.11) \quad \left(\int_{2^{-n}\lambda\tilde{B}} g^p d\mu \right)^{1/p} \leq \frac{\|g\chi_{2^{-n}\lambda\tilde{B}}\|_{L^{p,1}(\mathcal{P})}}{\mu(2^{-n}\lambda\tilde{B})^{1/p}} \lesssim \frac{\|\tilde{g}^*\chi_{(0,\mu(2^{-n}\lambda\tilde{B}))}\|_{L^{p,1}(\mathbb{R}^+)}}{\mu(2^{-n}\tilde{B})^{1/p}}.$$

Let $I_n = \|\tilde{g}^*\chi_{(0,\mu(2^{-n}\lambda\tilde{B}))}\|_{L^{p,1}(\mathbb{R}^+)}$. Summation by parts allows us to write

$$(6.12) \quad \sum_{n=0}^N \frac{2^{-n}}{\mu(2^{-n}\tilde{B})^{1/p}} I_n = \left(\sum_{n=0}^{N-1} \sum_{k=0}^n \frac{2^{-k}}{\mu(2^{-k}\tilde{B})^{1/p}} (I_n - I_{n+1}) \right) + \sum_{k=0}^N \frac{2^{-k}}{\mu(2^{-k}\tilde{B})^{1/p}} I_N$$

for every $N > 0$. Inequality (6.1) yields that $2^{-k} = 2^{-k}r/r \lesssim (\mu(2^{-k}\tilde{B})/\mu(\tilde{B}))^{1/s}$ and that $\mu(2^{-N}\lambda\tilde{B}) \approx \mu(2^{-N}\tilde{B})$. Recall also that $1 \leq p < s$. Then,

$$\begin{aligned}
 \sum_{k=0}^N \frac{2^{-k}}{\mu(2^{-k}\tilde{B})^{1/p}} I_N &\lesssim \sum_{k=0}^N \frac{\mu(2^{-k}\tilde{B})^{1/s-1/p}}{\mu(\tilde{B})^{1/s}} \int_0^{\mu(2^{-N}\lambda\tilde{B})} \tilde{g}^*(t)t^{1/p-1} dt \\
 &\lesssim \frac{1}{\mu(\tilde{B})^{1/s}} \sum_{k=0}^N \left(\frac{\mu(2^{-N}\tilde{B})}{\mu(2^{-k}\tilde{B})}\right)^{1/p-1/s} \int_0^{\mu(2^{-N}\lambda\tilde{B})} \tilde{g}^*(t)t^{1/s-1} dt \\
 (6.13) \quad &\lesssim \frac{\|\tilde{g}^*\chi_{(0,\mu(2^{-N}\lambda\tilde{B}))}\|_{L^{s,1}(\mathbb{R}^+)}}{\mu(\tilde{B})^{1/s}} \sum_{k=0}^N (2^{(k-N)\sigma})^{1/p-1/s},
 \end{aligned}$$

where the last inequality with some $\sigma \in (0, s]$ follows from (6.2). Due to the absolute continuity of the $L^{s,1}$ norm, we see that the last sum in (6.12) tends to zero as $N \rightarrow \infty$. Therefore,

$$(6.14) \quad \sum_{n=0}^{\infty} \frac{2^{-n}}{\mu(2^{-n}\tilde{B})^{1/p}} I_n = \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{2^{-k}}{\mu(2^{-k}\tilde{B})^{1/p}} (I_n - I_{n+1}).$$

Similarly as in (6.13), we can apply (6.1) and (6.2) to estimate

$$\begin{aligned}
 &\sum_{k=0}^n \frac{2^{-k}}{\mu(2^{-k}\tilde{B})^{1/p}} (I_n - I_{n+1}) \\
 &\lesssim \frac{1}{\mu(\tilde{B})^{1/s}} \sum_{k=0}^n \left(\frac{\mu(2^{-n}\tilde{B})}{\mu(2^{-k}\tilde{B})}\right)^{1/p-1/s} \int_{\mu(2^{-n-1}\lambda\tilde{B})}^{\mu(2^{-n}\lambda\tilde{B})} \tilde{g}^*(t)t^{1/s-1} dt \\
 &\lesssim \frac{1}{\mu(\tilde{B})^{1/s}} \sum_{k=0}^{\infty} 2^{-k\sigma(1/p-1/s)} \int_{\mu(2^{-n-1}\lambda\tilde{B})}^{\mu(2^{-n}\lambda\tilde{B})} \tilde{g}^*(t)t^{1/s-1} dt.
 \end{aligned}$$

Inserting this estimate into (6.14) yields that

$$(6.15) \quad \sum_{n=0}^{\infty} \frac{2^{-n}}{\mu(2^{-n}\tilde{B})^{1/p}} I_n \lesssim \frac{1}{\mu(\tilde{B})^{1/s}} \int_0^{\mu(\lambda\tilde{B})} \tilde{g}^*(t)t^{1/s-1} dt = \frac{\|\tilde{g}^*\|_{L^{s,1}(\mathbb{R}^+)}}{\mu(\tilde{B})^{1/s}}$$

Combining (6.10), (6.11), and (6.15) results in

$$|u(x) - u_{\tilde{B}}| \lesssim \frac{r}{\mu(\tilde{B})^{1/s}} \|\tilde{g}\|_{L^{s,1}(\mathcal{P})} \approx \frac{r}{\mu(2\lambda\tilde{B})^{1/s}} \|g\chi_{\lambda\tilde{B}}\|_{L^{s,1}(\mathcal{P})}.$$

The triangle, the p -Poincaré, and the Hölder inequality, as well as the embedding $L^{s,1} \hookrightarrow L^s$ provide us with the estimate

$$\begin{aligned}
 |u_{\tilde{B}} - u_B| &\leq |u_{\tilde{B}} - u_{2B}| + |u_{2B} - u_B| \lesssim \int_{2B} |u - u_{2B}| d\mu \\
 &\lesssim r \left(\int_{2\lambda B} g^p d\mu\right)^{1/p} \leq r \left(\int_{2\lambda B} g^s d\mu\right)^{1/s} \lesssim r \frac{\|g\chi_{2\lambda B}\|_{L^{s,1}(\mathcal{P})}}{\mu(2\lambda B)^{1/s}}.
 \end{aligned}$$

Altogether, we see that

$$\begin{aligned} |u(x) - u_B| &\leq |u(x) - u_{\bar{B}}| + |u_{\bar{B}} - u_B| \\ &\lesssim r \frac{\|g\chi_{2\lambda B}\|_{L^{s,1}(\mathcal{P})}}{\mu(2\lambda B)^{1/s}} \leq \frac{c_{\text{emb}}(2\lambda B)r}{\mu(2\lambda B)^{1/s}} \|g\chi_{2\lambda B}\|_X \leq c_{B_0} \|g\chi_{2\lambda B}\|_X, \end{aligned}$$

where (6.1) has been applied to estimate $r/\mu(2\lambda B)^{1/s} \leq R/(c_s\mu(2\lambda B_0))^{1/s}$. The Lebesgue differentiation theorem (see Section 1 of Heinonen [18]) now yields that a.e. $x \in B$ is a Lebesgue point of u , whence this inequality holds for a.e. $x \in B$ and thus for the essential supremum over B . \square

Edmunds, Kerman and Pick [14] have discussed the optimal Sobolev embeddings $W^m X(\Omega) \hookrightarrow Y(\Omega)$ for bounded domains $\Omega \subset \mathbb{R}^n$. Theorem 6.5 in [14] yields that $X = L^{n,1}(\Omega)$ is the largest r.i. space such that $W^1 X(\Omega) \hookrightarrow L^\infty(\Omega)$. Therefore, the result we have just obtained is sharp when considering r.i. spaces as the base function spaces X that $N^1 X$ are built upon.

As mentioned earlier, we also recover an analogue of the result of Talenti [41] on local essential boundedness of Newtonian functions based on the Zygmund space $X = L^s(\log L)^\alpha$ with $\alpha > 1/s' = 1 - 1/s$. Namely, it follows from the embedding $L^s(\log L)^\alpha_{\text{loc}} \hookrightarrow L^{s,1}_{\text{loc}}$, which is shown next.

Lemma 6.8. *Let $1 < p < \infty$ and suppose that $E \subset \mathcal{P}$ is a measurable set of finite positive measure. Then, $L^p(\log L)^\alpha(E) \hookrightarrow L^{p,1}(E)$ if and only if $\alpha > 1 - 1/p$.*

Proof. Let $a = \mu(E)$. We want to show that

$$\int_0^a f(t)t^{1/p-1} dt \lesssim \left(\int_0^a f(t)^p \left(1 + \log \frac{a}{t}\right)^{\alpha p} dt \right)^{1/p}$$

for every non-negative decreasing $f \in \mathcal{M}(\mathbb{R}^+, \lambda^1)$. According to Proposition 1 in Stepanov [40], this inequality holds true if and only if

$$b := \int_0^a \left(\frac{\int_0^t \tau^{1/p-1} d\tau}{\int_0^t (1 + \log(a/\tau))^{\alpha p} d\tau} \right)^{1/(p-1)} t^{1/p-1} dt < \infty.$$

We have the rough estimate $\int_0^t (1 + \log(a/\tau))^{\alpha p} d\tau \geq t(1 + \log(a/t))^{\alpha p}$. Conversely,

$$\begin{aligned} \int_0^t \left(1 + \log \frac{a}{\tau}\right)^{\alpha p} d\tau &= \sum_{n=0}^\infty \int_{t/2^{n+1}}^{t/2^n} \left(1 + \log \frac{a}{\tau}\right)^{\alpha p} d\tau \\ &\leq \sum_{n=0}^\infty \frac{t}{2^{n+1}} \left(1 + \log \frac{2^{n+1}a}{t}\right)^{\alpha p} \leq \sum_{n=0}^\infty t \left(1 + \log \frac{a}{t}\right)^{\alpha p} \frac{(1 + \log 2^{n+1})^{\alpha p}}{2^{n+1}}. \end{aligned}$$

Thus, $\int_0^t (1 + \log(a/\tau))^{\alpha p} d\tau \approx t(1 + \log(a/t))^{\alpha p}$ and we obtain that

$$b \approx \int_0^a \left(\frac{t^{1/p}}{t(1 + \log(a/t))^{\alpha p}} \right)^{1/(p-1)} t^{1/p-1} dt \approx \int_0^a \frac{dt}{t(1 + \log(a/t))^{\alpha p/(p-1)}}.$$

The integral on the right-hand side converges if and only if $\alpha > (p - 1)/p$. \square

Corollary 6.9. *Assume that \mathcal{P} supports a p -Poincaré inequality with $1 \leq p < s$ such that (6.1) is satisfied. Suppose also that $X \hookrightarrow L^s(\log L)_{\text{loc}}^\alpha$ with some $\alpha > 1 - 1/s$. Let $B_0 \subset \mathcal{P}$ be a fixed ball of radius $R > 0$. Then, there is a constant $c_{B_0} > 0$ such that for every ball $B \subset B_0$ of radius $r \in (0, R)$, we have*

$$C_X\text{-ess sup}_{x \in B} |u(x) - u_B| \lesssim \frac{c_{\text{emb}}(2\lambda B)r}{\mu(2\lambda B)^{1/s}} \|g\chi_{2\lambda B}\|_X \leq c_{B_0} \|g\chi_{2\lambda B}\|_X$$

whenever $g \in X$ is an X -weak upper gradient of $u \in N^1X$. Moreover, we can estimate $c_{B_0} \approx c_{\text{emb}}(2\lambda B_0)R/\mu(2\lambda B_0)^{1/s}$.

Observe that similarly as in Proposition 6.7, it suffices to assume that \mathcal{P} supports an s -Poincaré inequality if \mathcal{P} is complete.

Proof. By Lemma 6.8, we see that $X \hookrightarrow L_{\text{loc}}^{s,1}$. Then, the desired claim follows directly from Proposition 6.7. □

7. Continuity of Newtonian functions

In Sobolev spaces in \mathbb{R}^n , one may deduce that there exist continuous representatives if the L^∞ norm of a function is (locally) controlled by the quasi-norm of its gradient. A similar result, yet somewhat stronger, can be obtained in Newtonian spaces as well. Namely, it suffices in the Newtonian case to redefine a function on a set of capacity zero to obtain the continuous representative. If the metric measure space is in addition locally compact, then all representatives are continuous.

We assume in this (as well as in the previous) section that μ is a *doubling measure*. Moreover, μ is non-atomic since \mathcal{P} is connected, which follows from the Poincaré inequalities we will assume.

The following theorem refines Theorem 5.1 of Hajlasz and Koskela [16], where a p -Poincaré inequality was used to show that there exist $(1 - s/p)$ -Hölder continuous representatives (with equality a.e.) whenever the upper gradient lies in L^p and $p > s$. The case when the degree of summability of an upper gradient is essentially equal to s needs to be discussed using a finer scale of function spaces.

In \mathbb{R}^n , Kauhanen, Koskela, and Malý [23] have shown that it suffices that the gradient lies in the Lorentz space $L_{\text{loc}}^{n,1}$ to conclude that there are continuous representatives. Romanov [36] extended this result to Sobolev-type spaces on complete metric measure spaces with an Ahlfors s -regular measure (i.e., $\mu(B(x, r)) \approx r^s$ for $r < 2 \text{diam } \mathcal{P}$). The Ahlfors s -regularity of the measure is a very strong requirement that fails even in $(\mathbb{R}^n, w(x) dx)$ unless $w(x) \approx 1$. Besides, he did not work with Newtonian spaces as such, but solely with Poincaré inequalities. His result can be recovered as a special case of (d) in the following theorem.

Theorem 7.1. *Suppose that \mathcal{P} supports a p -Poincaré inequality and let s be given by (6.1). Suppose that one of the following sets of assumptions is satisfied:*

- (a) $p > s$ and $X \hookrightarrow L_{\text{loc}}^p$;
- (b) $1 = p = s$ and $X \hookrightarrow L^s(\log L)_{\text{loc}}^\alpha$ for some $\alpha \geq 1$;

- (c) $1 < p = s$ and $X \hookrightarrow L^s(\log L)^\alpha_{\text{loc}}$ for some $\alpha > 1$;
- (d) $1 \leq p < s$ and $X \hookrightarrow L^{s,1}_{\text{loc}}$.

Then, for every function $u \in N^1X$, there is $v \in N^1X \cap \mathcal{C}(\mathcal{P})$ such that $u = v$ C_X -quasi-everywhere. Moreover, v is locally $(1 - s/p)$ -Hölder continuous in the case (a).

Proof. Let us exhaust $\mathcal{P} = \bigcup_{n=1}^\infty B_n$, where $B_n = B(x_0, n)$ for an arbitrary point $x_0 \in \mathcal{P}$. Next, we will find an exceptional set E with $C_X(E) = 0$, where we lack control over the oscillation of $u \in N^1X$. Let $D = \{z_i \in \mathcal{P} : i \in \mathbb{N}\}$ be a dense subset of \mathcal{P} and let

$$E = \bigcup_{i=1}^\infty \bigcup_{q \in \mathbb{Q}^+} \left\{ x \in B(z_i, q) : |u(x) - u_{B(z_i, q)}| > C_X\text{-ess sup}_{w \in B(z_i, q)} |u(w) - u_{B(z_i, q)}| \right\}.$$

Since E is a countable union of sets of capacity zero, it has capacity zero as well.

Let us now fix a ball $\tilde{B} := 4B_n = B_{4n}$ for an arbitrary $n \in \mathbb{N}$. For every pair of points $x, y \in B_n \setminus E$, we can find $z \in D$ and $r \in \mathbb{Q}^+$ such that $d(x, y)/2 \leq r \leq 2d(x, y)$ and $x, y \in B := B(z, r) \subset \tilde{B}$.

In the case (a), we use (6.4) to obtain that

$$\begin{aligned} |u(x) - u(y)| &\leq 2C_X\text{-ess sup}_{w \in B} |u(w) - u_B| \leq c_{\tilde{B}} r^{1-s/p} \|g_u \chi_{2\lambda B}\|_X \\ &\leq c_{\tilde{B}} (2d(x, y))^{1-s/p} \|g_u\|_X. \end{aligned}$$

Therefore, $u|_{B_n \setminus E}$ is $(1 - s/p)$ -Hölder continuous and hence uniformly continuous. Since $C_X(E \cap B_n) = 0$, every point of $E \cap B_n$ is an accumulation point of $B_n \setminus E$, whence there is a unique continuation $v_n \in \mathcal{C}(B_n)$ of $u|_{B_n \setminus E}$. Moreover, v_n retains the Hölder continuity.

Let us now focus on the remaining three cases. Respective to the assumptions, let $Y = Y(\mathcal{P})$ be either the Zygmund space $L^s(\log L)^\alpha(\mathcal{P})$ with norm given by (6.6), or the Lorentz space $L^{s,1}(\mathcal{P})$. Since $|u - u_B| \in DX \subset DY_{\text{loc}}$, we obtain from Corollary 6.13 in [31] that

$$C_X\text{-ess sup}_{w \in B} |u(w) - u_B| = \text{ess sup}_{w \in B} |u(w) - u_B| = C_Y\text{-ess sup}_{w \in B} |u(w) - u_B|.$$

Applying (6.7) or (6.9) for the function space Y , we can find $c_{\tilde{B}} > 0$ such that

$$(7.1) \quad |u(x) - u(y)| \leq 2C_Y\text{-ess sup}_{w \in B} |u(w) - u_B| \leq c_{\tilde{B}} \|g_u \chi_{2\lambda B}\|_Y.$$

Now, let $E_B \subset 2\lambda\tilde{B}$ be a “super-level set” of $g_u|_{2\lambda\tilde{B}}$ such that $\mu(E_B) = \mu(2\lambda B)$. More precisely, E_B is chosen as an arbitrary measurable subset of $2\lambda\tilde{B}$ so that $\inf_{E_B} g_u \geq \sup_{2\lambda\tilde{B} \setminus E_B} g_u$. Abbreviating $g_u|_{2\lambda\tilde{B}}$ as \tilde{g}_u and $\mu(2\lambda B)$ as τ , we have

$$\{x \in 2\lambda\tilde{B} : \tilde{g}_u(x) > \tilde{g}_u^*(\tau)\} \subset E_B \subset \{x \in 2\lambda\tilde{B} : \tilde{g}_u(x) \geq \tilde{g}_u^*(\tau)\},$$

where \tilde{g}_u^* is the decreasing rearrangement of \tilde{g}_u . Consequently, we see that the set $E_B \subset 2\lambda\tilde{B}$ does not depend on the particular choice of B within \tilde{B} , but merely on

the measure of $2\lambda B$. Since Y is rearrangement-invariant, it holds that $\|g_u\chi_F\|_Y \leq \|g_u\chi_{E_B}\|_Y$ whenever $F \subset 2\lambda\tilde{B}$ with $\mu(F) \leq \mu(E_B)$. In particular, (7.1) leads to the inequality

$$|u(x) - u(y)| \leq c_{\tilde{B}} \|g_u\chi_{E_B}\|_Y.$$

Given an $\varepsilon > 0$, we can find $a > 0$ such that $c_{\tilde{B}} \|g_u\chi_{E_B}\|_Y < \varepsilon$ whenever $\mu(E_B) < a$ since Y has absolutely continuous norm. As \mathcal{P} is connected, we have $\mu(2\lambda B) \leq C_{2\lambda\tilde{B}} d(x, y)^\sigma$, where $\sigma \in (0, s]$ is from (6.2). If $d(x, y) < (a/C_{2\lambda\tilde{B}})^{1/\sigma}$, then $\mu(E_B) = \mu(2\lambda B) < a$. Thus, $c_{\tilde{B}} \|g_u\chi_{E_B}\|_Y < \varepsilon$ and hence $|u(x) - u(y)| < \varepsilon$.

This way, we have just shown uniform continuity of $u|_{B_n \setminus E}$. Thus, there is a unique continuation $v_n \in \mathcal{C}(B_n)$ of $u|_{B_n \setminus E}$.

We have thus proven that in all the cases (a)–(d) there is a unique continuous extension v_n of $u|_{B_n \setminus E}$ for every ball B_n , $n \in \mathbb{N}$. Now, we define v on \mathcal{P} by setting $v(x) = v_n(x)$ whenever $x \in B_n$. Then, $v \in \mathcal{C}(\mathcal{P})$ and $v = u$ outside of E , i.e., C_X -quasi-everywhere. Furthermore, $v \in C_{\text{loc}}^{0,1-s/p}(\mathcal{P})$ in the case (a). □

Several qualitative properties of the Sobolev capacity have been discussed in Section 4, where one of the crucial assumptions was density of continuous functions. Now, we have shown that under certain hypotheses, all Newtonian functions have continuous representatives, whence the continuous functions are dense. Thus, we may formulate the following corollary.

Corollary 7.2. *Suppose that \mathcal{P} is locally compact and X has the Vitali–Carathéodory property (2.1). In particular, it suffices to assume that χ_B satisfies (AC) for every bounded set $B \subset \mathcal{P}$. Suppose further that the hypotheses of Theorem 7.1 are satisfied. Then, $\tilde{C}_{X,r}$ is an outer capacity if X is r -normed. In particular, C_X is an outer capacity if X is normed.*

Proof. Proposition 2.1 yields that X has the Vitali–Carathéodory property under the given assumption on χ_B for bounded sets $B \subset \mathcal{P}$. (Recall that $\chi_B \in X$ due to (AC) as μ is non-atomic.) By Theorem 7.1, every Newtonian function has a continuous representative. Hence, continuous functions are dense in N^1X . By Corollary 4.6, every Newtonian function is quasi-continuous. Finally, it follows from Proposition 4.8 that $\tilde{C}_{X,r}$ and C_X are outer capacities. □

We are ready to apply Propositions 6.2, 6.5, and 6.7 to find a lower bound for the Sobolev capacity of a subset of a ball, in terms of the measure and radius of the ball, if we know beforehand that the set has non-zero capacity.

Proposition 7.3. *Suppose that \mathcal{P} supports a p -Poincaré inequality and let s be given by (6.1). Suppose that one of the following sets of assumptions is satisfied:*

- (a) $1 \leq s < p \leq q$ and $X \hookrightarrow L_{\text{loc}}^q$;
- (b) $1 = p = s =: q$ and $X \hookrightarrow L^s(\log L)_{\text{loc}}^\alpha$ for some $\alpha \geq 1$;
- (c) $1 < p = s =: q$ and $X \hookrightarrow L^s(\log L)_{\text{loc}}^\alpha$ for some $\alpha > 1$;
- (d) $1 \leq p < s =: q$ and $X \hookrightarrow L_{\text{loc}}^{s,1}$.

Let $B \subset \mathcal{P}$ be a ball with radius $r > 0$. Then, for every $E \subset B$ with $C_X(E) > 0$, we can estimate

$$C_X(E) \gtrsim \frac{\mu(2\lambda B)^{1/q}}{c_{\text{emb}}(2\lambda B)(r + 1)}.$$

In particular, this estimate holds if $\mu(E) > 0$.

Proof. Let $u \in N^1X$ be such that $\chi_E \leq u \leq 1$ everywhere in \mathcal{P} and let $g_u \in X$ be a minimal X -weak upper gradient of u . Propositions 6.2, 6.5, and 6.7 and the Hölder inequality then yield

$$\begin{aligned} 1 &\leq |1 - u_B| + u_B = C_X\text{-ess sup}_{x \in B} |u(x) - u_B| + \int_B u \, d\mu \\ &\lesssim \frac{c_{\text{emb}}(2\lambda B)r}{\mu(2\lambda B)^{1/q}} \|g_u \chi_{2\lambda B}\|_X + \frac{\|u \chi_B\|_{L^q}}{\mu(B)^{1/q}} \lesssim \frac{c_{\text{emb}}(2\lambda B)(r + 1)\|u\|_{N^1X}}{\mu(2\lambda B)^{1/q}}. \end{aligned}$$

Taking infimum over all such functions $u \in N^1X$, we obtain

$$C_X(E) = \inf \|u\|_{N^1X} \gtrsim \frac{\mu(2\lambda B)^{1/q}}{c_{\text{emb}}(2\lambda B)(r + 1)}. \quad \square$$

If X is an r.i. space, then it is possible to find an estimate of the capacity expressed using the fundamental function of X , provided that the integral means in L^p can be suitably rescaled to the norm means in X .

Corollary 7.4. *Assume that \mathcal{P} supports a p -Poincaré inequality with $p > s$ such that (6.1) is satisfied. Let $B \subset \mathcal{P}$ be a ball with radius $r > 0$. Suppose further that X is an r.i. space with fundamental function ϕ such that $c_\phi(\mathcal{P}) < \infty$, where c_ϕ is defined by (6.5), see Corollary 6.3. Then, for every $E \subset B$ with $C_X(E) > 0$, we can estimate*

$$C_X(E) \gtrsim \frac{\phi(\mu(2\lambda B))}{r + 1}.$$

In particular, this estimate holds if $\mu(E) > 0$.

Proof. In principle, it was shown in the proof of Corollary 6.3 that

$$c_{\text{emb}}(2\lambda B) \leq c_\phi(\mathcal{P}) \frac{\mu(2\lambda B)^{1/p}}{\phi(\mu(2\lambda B))}.$$

The desired result follows from Proposition 7.3. □

Recall that the natural equivalence classes in N^1X are given by equality outside of sets of capacity zero. Therefore, in order to be able to prove that all Newtonian functions in a locally compact doubling Poincaré space are continuous if the summability of the upper gradients (in terms of $\|\cdot\|_X$) is sufficiently high, we need to show that singletons have positive capacity. To that end, we will apply the outer regularity of the capacity on zero sets.

Proposition 7.5. *Assume that \mathcal{P} is locally compact and supports a p -Poincaré inequality and let s be given by (6.1). Suppose that one of the following sets of assumptions is satisfied:*

- (a) $1 \leq s < p \leq q$ and $X \hookrightarrow L^q_{\text{loc}}$;

- (b) $1 = p = s =: q$ and $X \hookrightarrow L^s(\log L)_{\text{loc}}^\alpha$ for some $\alpha \geq 1$;
- (c) $1 < p = s =: q$ and $X \hookrightarrow L^s(\log L)_{\text{loc}}^\alpha$ for some $\alpha > 1$;
- (d) $1 \leq p < s =: q$ and $X \hookrightarrow L_{\text{loc}}^{s,1}$.

Then, for every ball $B \subset \mathcal{P}$ of radius $r > 0$ and every $x \in B$, we have $C_X(\{x\}) \gtrsim \mu(B)^{1/q}/(r + 1) > 0$.

Proof. Let $B \subset \mathcal{P}$ be fixed. Let $Y = Y(B)$ be either $L^q(B)$, or $L^s(\log L)^\alpha(B)$, or $L^{s,1}(B)$ as in the proposition’s hypotheses. Then, Y has an absolutely continuous norm and satisfies (P4), whence Y has the Vitali–Carathéodory property by Proposition 2.1. Let $x \in B$ and suppose for a moment that $C_Y(\{x\}) = 0$. Then, Proposition 3.5 yields that $C_Y(\{x\}) = \inf_{G \ni x} C_Y(G)$, where G is open. Such a set G has positive measure and hence $C_Y(G) > 0$. Thus, we can estimate $C_Y(G) \geq c(B) > 0$ whenever $G \subset B$ by Proposition 7.3. Hence, $C_Y(\{x\}) \geq c(B) > 0$, which contradicts the assumption $C_Y(\{x\}) = 0$. Therefore, $0 < C_Y(\{x\}) \lesssim C_X(\{x\})$ and the claimed estimate follows from Proposition 7.3. \square

In view of the previous proposition, we see that the Newtonian functions considered in Propositions 6.2, 6.5, and 6.7 are not only C_X -essentially bounded on all balls, but bounded everywhere in the respective balls, provided that \mathcal{P} is locally compact.

Similarly, we will next show that Newtonian functions not only have continuous representatives, but in fact are continuous. Thus, the claim is stronger than its analogue for Sobolev functions in \mathbb{R}^n .

Theorem 7.6. *Under the assumptions of Proposition 7.5, every $u \in N^1X$ is continuous.*

Proof. Let $u \in N^1X$. By Theorem 7.1, there is $v \in N^1X \cap \mathcal{C}(\mathcal{P})$ such that $u = v$ q.e. In other words, the set $E = \{x \in \mathcal{P} : u(x) \neq v(x)\}$ has zero capacity. According to Proposition 7.5, E cannot contain any single point $x \in \mathcal{P}$ as it would have positive capacity then. Therefore, $u = v$ everywhere in \mathcal{P} , whence $u \in \mathcal{C}(\mathcal{P})$. \square

Appendix. Calculus for weak upper gradients

Throughout the paper, several tools for working with weak upper gradients have been needed. Note that none of the results in this section requires the measure to be doubling.

Theorem A.1 (Product rule). *Let $u, v : \mathcal{P} \rightarrow \overline{\mathbb{R}}$ be measurable. Assume that there are measurable $g, h \geq 0$ such that $u \circ \gamma, v \circ \gamma \in \text{AC}([0, l_\gamma])$ with*

$$|(u \circ \gamma)'(t)| \leq g(\gamma(t)) \quad \text{and} \quad |(v \circ \gamma)'(t)| \leq h(\gamma(t)) \quad \text{for a.e. } t \in (0, l_\gamma)$$

for Mod $_X$ -a.e. curve $\gamma : [0, l_\gamma] \rightarrow \mathcal{P}$. In particular, it suffices to assume that $g, h \in X$ are X -weak upper gradients of $u, v \in DX$, respectively. Then, $|u|h + |v|g$ is an X -weak upper gradient of uv .

The theorem can be proven essentially the same way as Proposition 6.10 in [31] or Theorem 2.15 in [5], whence the proof is omitted.

Observe that $|u|h + |v|g$ need not be a minimal X -weak upper gradient of uv even if g and h are minimal X -weak upper gradients of u and v , respectively. For example, suppose that $X = L^p(0, 1)$ with $p \geq 1$ and let $u(t) = g(t) = e^t$ and $v(t) = h(t) = e^{-t}$ for $t \in (0, 1)$. Then, $uv \equiv 1$, whence 0 is an upper gradient of uv , but $|u|h + |v|g \equiv 2$.

Theorem A.2 (Chain rule). *Suppose that $g_u \in X$ is a minimal X -weak upper gradient of $u \in N^1X$. Let $\phi : \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}$ be locally Lipschitz on \mathbb{R} . Then, $|\phi' \circ u|g_u$ is an X -weak upper gradient of $\phi \circ u$, where $\phi' \circ u := 0$ wherever undefined. Moreover, if $|\phi' \circ u|g_u \in X$, then it is a minimal X -weak upper gradient of $\phi \circ u$.*

The chain rule (without the minimality claim) was shown in Proposition 6.11 in [31]. To prove that $|\phi' \circ u|g_u$ is a minimal X -weak upper gradient of $\phi \circ u$, one can follow the argument of Theorem 2.16 (see also Errata) in [5] applying the background theory established in [31] and [32].

The following claims show that minimal weak upper gradients depend only on the local behavior of Newtonian functions.

Proposition A.3. *Let $g_u, g_v \in X$ be minimal X -weak upper gradients of $u, v \in DX$, respectively. Then, $g := g_u \chi_{\{u > v\}} + g_v \chi_{\{u \leq v\}}$ is a minimal X -weak upper gradient of $w := \max\{u, v\}$.*

Corollary A.4. *Let $g_u, g_v \in X$ be minimal X -weak upper gradients of $u, v \in DX$, respectively. Then, $g_u = g_v$ a.e. on $E := \{x \in \mathcal{P} : u(x) = v(x)\}$. In particular, we obtain that $g_u = 0$ a.e. on $\{x \in \mathcal{P} : u(x) = c\}$ for any constant $c \in \mathbb{R}$.*

Proofs can be led along the same lines as the proofs of Corollaries 2.20 and 2.21 in [5], while using the results of [31] and [32].

References

- [1] AMBROSIO, L., COLOMBO, M. AND DI MARINO, S.: Sobolev spaces in metric measure spaces: reflexivity and lower semicontinuity of slope. In *Variational methods for evolving objects (Sapporo, 2012)*, 1–58. Adv. Stud. Pure Math. 67, Mathematical Soc. of Japan, Tokyo, 2015.
- [2] AMBROSIO, L., GIGLI, N. AND SAVARÉ, G.: Density of Lipschitz functions and equivalence of weak gradients in metric measure spaces. *Rev. Mat. Iberoam.* **29** (2013), no. 3, 969–996.
- [3] BENNETT, C. AND SHARPLEY, R.: *Interpolation of operators*. Pure and Applied Mathematics 129, Academic Press, Boston, MA, 1988.
- [4] BENYAMINI, Y. AND LINDENSTRAUSS, J.: *Geometric nonlinear functional analysis. Vol. 1*. AMS Colloquium Publications 48, American Mathematical Society, Providence, RI, 2000.
- [5] BJÖRN, A. AND BJÖRN, J.: *Nonlinear potential theory on metric spaces*. EMS Tracts in Mathematics 17, European Mathematical Society, Zürich, 2011.

- [6] BJÖRN, A., BJÖRN, J. AND LEHRBÄCK, J.: Sharp capacity estimates for annuli in weighted \mathbb{R}^n and in metric spaces. Preprint, [arXiv: 1312.1668](https://arxiv.org/abs/1312.1668), 2013
- [7] BJÖRN, A., BJÖRN, J. AND MALÝ, L.: Nonquasicontinuous newtonian functions. In preparation.
- [8] BJÖRN, A., BJÖRN, J. AND SHANMUGALINGAM, N.: Quasicontinuity of Newton–Sobolev functions and density of Lipschitz functions on metric spaces. *Houston J. Math.* **34** (2008), 1197–1211.
- [9] BJÖRN, J. AND SHANMUGALINGAM, N.: Poincaré inequalities, uniform domains and extension properties for Newton–Sobolev functions in metric spaces. *J. Math. Anal. Appl.* **332** (2007), 190–208.
- [10] CARATHÉODORY, C.: *Vorlesungen über reelle Funktionen*. Teubner, Leipzig, 1918.
- [11] COSTEA, Ș. AND MIRANDA, M.: Newtonian Lorentz metric spaces. *Illinois J. Math.* **56** (2012), 579–616.
- [12] DENY, J.: Les potentiels d’énergie finie. *Acta Math.* **82** (1950), 107–183.
- [13] DURAND-CARTEGENA, E.: *Some topics in Lipschitz analysis on metric spaces*. Ph.D. Thesis, Complutense University of Madrid, Madrid, 2011.
- [14] EDMUNDS, D. E., KERMAN, R. AND PICK, L.: Optimal Sobolev imbeddings involving rearrangement-invariant quasinorms. *J. Funct. Anal.* **170** (2000), 307–355.
- [15] FEDERER, H. AND ZIEMER, W.: The Lebesgue set of a function whose distribution derivatives are p -th power summable. *Indiana Univ. Math. J.* **22** (1972/73), 139–158.
- [16] HAJLASZ, P. AND KOSKELA, P.: Sobolev met Poincaré. *Mem. Amer. Math. Soc.* **145** (2000), no. 688, x+101 pp.
- [17] HARJULEHTO, P., HÄSTÖ, P. AND PERE, M.: Variable exponent Sobolev spaces on metric measure spaces. *Funct. Approx. Comment. Math.* **36** (2006), 79–94.
- [18] HEINONEN, J.: *Lectures on analysis on metric spaces*. Universitext, Springer, New York, 2001.
- [19] HEINONEN, J., KILPELÄINEN, T. AND MARTIO, O.: *Nonlinear potential theory of degenerate elliptic equations*. Second edition. Dover Publications, Mineola, NY, 2006.
- [20] HEINONEN, J. AND KOSKELA, P.: From local to global in quasiconformal structures. *Proc. Nat. Acad. Sci. USA* **93** (1996), 554–556.
- [21] HEINONEN, J., KOSKELA, P., SHANMUGALINGAM, N. AND TYSON, J.: *Sobolev spaces on metric measure spaces: an approach based on upper gradients*. New Math. Monogr. 67, Cambridge University Press, Cambridge, 2015.
- [22] JÄRVENPÄÄ, E., JÄRVENPÄÄ, M., ROGOVIN, K., ROGOVIN, S. AND SHANMUGALINGAM, N.: Measurability of equivalence classes and MEC_p -property in metric spaces. *Rev. Mat. Iberoam.* **23** (2007), no. 3, 811–830.
- [23] KAUFMAN, J., KOSKELA, P. AND MALÝ, J.: On functions with derivatives in a Lorentz space. *Manuscripta Math.* **100** (1999), 87–101.
- [24] KEITH, S. AND ZHONG, X.: The Poincaré inequality is an open ended condition. *Ann. of Math. (2)* **167** (2008), 575–599.
- [25] KILPELÄINEN, T.: A remark on the uniqueness of quasi continuous functions. *Ann. Acad. Sci. Fenn. Math.* **23** (1998), 261–262.
- [26] KINNUNEN, J. AND MARTIO, O.: The Sobolev capacity on metric spaces. *Ann. Acad. Sci. Fenn. Math.* **21** (1996), 367–382.

- [27] KOSKELA, P. AND MACMANUS, P.: Quasiconformal mappings and Sobolev spaces. *Studia Math.* **131** (1998), 1–17.
- [28] LU, Y., YANG, D. AND YUAN, W.: Morrey–Sobolev spaces on metric measure spaces. *Potential Anal.* **41** (2014), 215–243.
- [29] MALIGRANDA, L.: Type, cotype and convexity properties of quasi-Banach spaces. In *Proceedings of the International Symposium on Banach and Function Spaces (Kitakyushu, 2003)*, 83–120. Yokohama Publ., Yokohama, 2004.
- [30] MALÝ, J. AND ZIEMER, W. P.: *Fine regularity of solutions of elliptic partial differential equations*. Mathematical Surveys and Monographs 51, American Mathematical Society, Providence, RI, 1997.
- [31] MALÝ, L.: Newtonian spaces based on quasi-Banach function lattices. To appear in *Math. Scand.* ArXiv:1210.1442v2.
- [32] MALÝ, L.: Minimal weak upper gradients in Newtonian spaces based on quasi-Banach function lattices. *Ann. Acad. Sci. Fenn. Math.* **38** (2013), 727–745.
- [33] MALÝ, L.: Regularization of Newtonian functions on metric spaces via weak boundedness of maximal operators. To appear in *J. Anal. Math.*
- [34] MATTILA, P.: *Geometry of sets and measures in Euclidean spaces*. Cambridge Studies in Advanced Mathematics 44, Cambridge University Press, Cambridge, 1995.
- [35] MORREY, C. B., JR.: Functions of several variables and absolute continuity, II. *Duke Math. J.* **6** (1940), 187–215.
- [36] ROMANOV, A. S.: On the absolute continuity of Sobolev-type functions on metric spaces. *Sibirsk. Mat. Zh.* **49** (2008), 1147–1156. English translation in *Sib. Math. J.* **49** (2008), 911–918.
- [37] SHANMUGALINGAM, N.: *Newtonian spaces: an extension of Sobolev spaces to metric measure spaces*. Ph.D. Thesis, University of Michigan, Ann Arbor, MI, 1999.
- [38] SHANMUGALINGAM, N.: Newtonian spaces: An extension of Sobolev spaces to metric measure spaces. *Rev. Mat. Iberoamericana* **16** (2000), no. 2, 243–279.
- [39] SHANMUGALINGAM, N.: Harmonic functions on metric spaces. *Illinois J. Math.* **45** (2001), 1021–1050.
- [40] STEPANOV, V. D.: The weighted Hardy’s inequality for nonincreasing functions. *Trans. Amer. Math. Soc.* **338** (1993), 173–186.
- [41] TALENTI, G.: An embedding theorem. In *Partial differential equations and the calculus of variations, Vol. II*, 919–924. Progr. Nonlinear Differential Equations Appl. 2, Birkhäuser, Boston, MA, 1989.
- [42] TUOMINEN, H.: Orlicz–Sobolev spaces on metric measure spaces. *Ann. Acad. Sci. Fenn. Math. Diss.* **135** (2004), 86 pp.
- [43] VITALI, G.: Una proprietà delle funzioni misurabili. *R. Ist. Lombardo Sci. Lett. Rend.* **38** (1905), 599–603.

Received April 29, 2014; revised August 8, 2014.

LUKÁŠ MALÝ: Department of Mathematics and Statistics, P.O. Box 35 (MaD), 400 14 University of Jyväskylä, Finland. *Current address*: Department of Mathematical Sciences, University of Cincinnati, P.O. Box 210025, Cincinnati, OH 45221-0025, USA.
E-mail: lukas.maly@matfyz.cz