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Non-critical dimensions for critical problems involving fractional Laplacians

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Abstract. We study the Brezis–Nirenberg effect in two families of non-compact boundary value problems involving Dirichlet-Laplacian of arbitrary real order $m \in (0, n/2)$.

Dedicated to Haim Brezis in occasion of his 70th birthday

1. Introduction

Nonlocal differential operators are commonly used to model diffusion processes in presence of long range interactions, when pointwise defined operators, such as the Laplacian and standard polyharmonic operators, are totally inadequate. Rather, one is lead to introduce differential operators in a nonlocal way, usually via global integration. In this context, a widely used operator is the fractional Laplacian $(-\Delta)^m$, for real $m > 0$.

In recent years a lot of effort have been indeed spent in developing appropriate mathematical techniques, suitable to handle nonlocal differential operators. One of the main question is to understand how concentration phenomena may occur in presence of non-local terms.

In the present paper we study some model noncompact Dirichlet's problems in which both the leading differential operator and the perturbing term might have a nonlocal nature.

Let m and s be two given real numbers, with $0 \leq s < m < n/2$. Let $\Omega \subset \mathbb{R}^n$ be a bounded and smooth domain in $\mathbb{R}^n$ and put

$$2^*_m = \frac{2n}{n-2m}.$$

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We study equations

$$(1.1)\qquad (-\Delta)^m u = \lambda(-\Delta)^s u + |u|^{2^*_m - 2} u \quad \text{in } \Omega,$$

$$(1.2)\qquad (-\Delta)^m u = \lambda|x|^{-2s} u + |u|^{2^*_m - 2} u \quad \text{in } \Omega,$$

under suitably defined Dirichlet boundary conditions. In dealing with equation (1.2) we always assume that Ω contains the origin. For the definition of fractional Dirichlet–Laplace operators $(-\Delta)^m$, $(-\Delta)^s$ and for the variational approach to (1.1) and (1.2) we refer to the next section.

The celebrated paper [4] by Brezis and Nirenberg was the inspiration for a fruitful line of research about the effect of lower order perturbations in noncompact variational problems. They took as model the case $n > 2$, $m = 1$, $s = 0$, that is,

$$(1.3)\qquad -\Delta u = \lambda u + |u|^{4/(n-2)} u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega\,.$$

Brezis and Nirenberg pointed out a remarkable phenomenon that appears for positive values of the parameter λ: they proved existence of a nontrivial solution for any small $\lambda > 0$ if $n \geq 4$; in contrast, in the lowest dimension $n = 3$ non-existence phenomena for sufficiently small $\lambda > 0$ can be observed. For this reason, the dimension $n = 3$ has been named *critical* for problem (1.3) (compare with [14], [9]).

Note that the Brezis–Nirenberg effect is a nonlinear analog of the so-called zero-energy resonance for the Schrödinger operators (see, e.g., [19] and [20], pp. 287–288).

Clearly, as larger s is, as stronger the effects of the lower order perturbations are expected in equations (1.1), (1.2). We are interested in the following question: *Given $m < n/2$, how large must be s in order to have the existence of a ground state solution, for any arbitrarily small $\lambda > 0$?* In case of an affirmative answer, we say that n is *not* a critical dimension.

We present our main result, that holds for any dimension $n \geq 1$ (see Theorem 4.2 in Section 4 for a more precise statement).

Theorem. *If $s \geq 2m - n/2$, then n is not a critical dimension for the Dirichlet boundary value problems associated to equations* (1.1) *and* (1.2)*.*

We point out some particular cases that are included in this result.

- If m is an integer and $s = m-1$, then at most the lowest dimension $n = 2m+1$ is critical.
- For any $n > 2m$, there always exist lower order perturbations of the type $|x|^{-2s}u$ and of the type $(-\Delta)^s u$ such that n is not a critical dimension.
- If $m < 1/4$, then no dimension is critical, for any choice of $s \in [0, m)$.

After [4], a large number of papers have been focussed on studying the effect of linear perturbations in noncompact variational problems of the type (1.1). Most of these papers deal with $s = 0$, when the problems (1.1) and (1.2) coincide. Moreover, as far as we know, all of them consider either polyharmonic case $2 \leq m \in \mathbb{N}$, see for instance [14], [7], [3], [11], [8], or the case $m \in (0, 1)$, see [15], [16] and [2]. We cite also [5], where equation (1.1) is studied in case $m = 2$, $s = 1$. Thus, our Theorem 4.2 covers all earlier existence results.

Finally, we mention [1] (see also [17]), where equation (1.1) for the so-called Navier-Laplacian is studied in case $m \in (0,1)$, $s = 0$. For a comparison between the Dirichlet and Navier Laplacians we refer to [13].

The paper is organized as follows. After introducing some notation and preliminary facts in Section 2, we provide the main estimates in Section 3. In Section 4 we prove Theorem 4.2 and point out an existence result for the case $s < 2m - n/2$.

2. Preliminaries

The fractional Laplacian $(-\Delta)^m u$ of a function $u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ is defined via the Fourier transform

$$\mathcal{F}[u](\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x}\, u(x)\, dx$$

by the identity

$$\mathcal{F}\left[(-\Delta)^m u\right](\xi) = |\xi|^{2m} \mathcal{F}[u](\xi). \tag{2.1}$$

In particular, Parseval's formula gives

$$\int_{\mathbb{R}^n} (-\Delta)^m u \cdot u\, dx = \int_{\mathbb{R}^n} |(-\Delta)^{m/2} u|^2\, dx = \int_{\mathbb{R}^n} |\xi|^{2m}\, |\mathcal{F}[u]|^2\, d\xi\,.$$

For future convenience we recall that if $m \in (0,1)$ then

$$(-\Delta)^m u(x) = C_{n,m} \cdot p.v. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2m}}\, dy, \tag{2.2}$$

see for instance Chapter 2, Section 3 in [10]. Here *p.v.* stands for the principal value of the integral, while

$$C_{n,m} = 2^{2m+n/2}\, \frac{\Gamma(m+n/2)}{\Gamma(-m)}.$$

We also recall the well-known Sobolev inequality

$$\int_{\mathbb{R}^n} |(-\Delta)^{m/2} u|^2\, dx \ge \mathcal{S}_m \Big(\int_{\mathbb{R}^n} |u|^{2^*_m}\, dx \Big)^{2/2^*_m}, \tag{2.3}$$

that holds for any $u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $m < n/2$, see for example [18], 2.8.1/15.

Let $\mathcal{D}^m(\mathbb{R}^n)$ be the Hilbert space obtained by completing $\mathcal{C}_0^\infty(\mathbb{R}^n)$ with respect to the Gagliardo norm

$$\|u\|_m^2 = \int_{\mathbb{R}^n} |(-\Delta)^{m/2} u|^2\, dx. \tag{2.4}$$

Thanks to (2.3), the space $\mathcal{D}^m(\mathbb{R}^n)$ is continuously embedded into $L^{2^*_m}(\mathbb{R}^n)$. The *best Sobolev constant* $\mathcal{S}_m$ was explicitly computed in [6]. Moreover, it has been proved in [6] that $\mathcal{S}_m$ is attained in $\mathcal{D}^m(\mathbb{R}^n)$ by a unique family of functions, all of them being obtained from

$$\phi(x) = (1+|x|^2)^{(2m-n)/2} \tag{2.5}$$

by translations, dilations in $\mathbb{R}^n$ and multiplication by constants.

Dilations play a crucial role in the problems under consideration. Notice that for any $\omega \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $R>0$ it turns out that

$$\int_{\mathbb{R}^n} |\xi|^{2m}|\mathcal{F}[\omega](\xi)|^2\,d\xi = R^{n-2m}\int_{\mathbb{R}^n} |\xi|^{2m}|\mathcal{F}[\omega(R\cdot)](\xi)|^2\,d\xi \tag{2.6}$$
$$\int_{\mathbb{R}^n} |\omega|^{2^*_m}\,dx = R^n\int_{\mathbb{R}^n} |\omega(R\cdot)|^{2^*_m}\,dx\,.$$

Finally, we point out that the Hardy inequality

$$\int_{\mathbb{R}^n} |(-\Delta)^{m/2}u|^2\,dx \ge \mathcal{H}_m \int_{\mathbb{R}^n} |x|^{-2m}|u|^2\,dx \tag{2.7}$$

holds for any function $u \in \mathcal{D}^m(\mathbb{R}^n)$. The *best Hardy constant* $\mathcal{H}_m$ was explicitly computed in [12].

The natural ambient space to study the Dirichlet boundary value problems for (1.1) and (1.2) is

$$\widetilde{H}^m(\Omega) = \{u \in \mathcal{D}^m(\mathbb{R}^n)\ :\ \operatorname{supp} u \subset \overline{\Omega}\},$$

endowed with the norm $\|u\|_m$. By Theorem 4.3.2/1 in [18], for $m+1/2 \notin \mathbb{N}$ this space coincides with $H_0^m(\Omega)$ (that is the closure of $\mathcal{C}_0^\infty(\Omega)$ in $H^m(\Omega)$), while for $m+1/2 \in \mathbb{N}$ one has $\widetilde{H}^m(\Omega) \subsetneq H_0^m(\Omega)$. Moreover, $\mathcal{C}_0^\infty(\Omega)$ is dense in $\widetilde{H}^m(\Omega)$. Clearly, if m is an integer then $\widetilde{H}^m(\Omega)$ is the standard Sobolev space of functions $u \in H^m(\Omega)$ such that $D^\alpha u = 0$ for every multiindex $\alpha \in \mathbb{N}^n$ with $0 \le |\alpha| < m$.

We agree that $(-\Delta)^0 u = u$, $\widetilde{H}^0(\Omega) = L^2(\Omega)$, since (2.4) reduces to the standard L^2 norm in case $m=0$.

We define (weak) solutions of the Dirichlet problems for (1.1), (1.2) as suitably normalized critical points of the functionals

$$\mathcal{R}^\Omega_{\lambda,m,s}[u] = \frac{\displaystyle\int_{\mathbb{R}^n} |(-\Delta)^{m/2}u|^2\,dx - \lambda\int_{\mathbb{R}^n} |(-\Delta)^{s/2}u|^2\,dx}{\displaystyle\Big(\int_\Omega |u|^{2^*_m}dx\Big)^{2/2^*_m}} \tag{2.8}$$

$$\widetilde{\mathcal{R}}^\Omega_{\lambda,m,s}[u] = \frac{\displaystyle\int_{\mathbb{R}^n} |(-\Delta)^{m/2}u|^2\,dx - \lambda\int_\Omega |x|^{-2s}|u|^2\,dx}{\displaystyle\Big(\int_\Omega |u|^{2^*_m}dx\Big)^{2/2^*_m}}\,, \tag{2.9}$$

respectively. It is easy to see that both functionals (2.8), (2.9) are well defined on $\widetilde{H}^m(\Omega)\setminus\{0\}$.

We conclude this preliminary section with some embedding results.

Proposition 2.1. *Let m, s be given, with $0 \le s < m < n/2$.*

i) *The space $\widetilde{H}^m(\Omega)$ is compactly embedded into $\widetilde{H}^s(\Omega)$. In particular the infima*

$$\Lambda_1(m,s) := \inf_{\substack{u\in\widetilde{H}^m(\Omega)\\ u\neq 0}} \frac{\|u\|_m^2}{\|u\|_s^2}\ , \qquad \widetilde{\Lambda}_1(m,s) := \inf_{\substack{u\in\widetilde{H}^m(\Omega)\\ u\neq 0}} \frac{\|u\|_m^2}{\||x|^{-s}u\|_0^2} \tag{2.10}$$

are positive and achieved.

ii) $$\inf_{\substack{u\in\widetilde{H}^m(\Omega)\\ u\neq 0}} \frac{\|u\|_m^2}{\|u\|_{L^{2^*_m}}^2} = \mathcal{S}_m.$$

Statement i) is well known for $\Lambda_1(m,s)$ and follows from (2.7) for $\widetilde{\Lambda}_1(m,s)$. To check ii), use the inclusion $\widetilde{H}^m(\Omega) \hookrightarrow \mathcal{D}^m(\mathbb{R}^n)$ and a rescaling argument. Clearly, the Sobolev constant $\mathcal{S}_m$ is never achieved on $\widetilde{H}^m(\Omega)$.

3. Main estimates

Let ϕ be the extremal of the Sobolev inequality (2.3) given by (2.5). In particular, it holds that

$$M := \int_{\mathbb{R}^n} |(-\Delta)^{m/2}\phi|^2\, dx = \mathcal{S}_m \Big(\int_{\mathbb{R}^n} |\phi|^{2^*_m}\, dx\Big)^{2/2^*_m}. \tag{3.1}$$

Fix $\delta > 0$ and a cutoff function $\varphi \in \mathcal{C}^\infty_0(\Omega)$, such that $\varphi \equiv 1$ on the ball $\{|x| < \delta\}$ and $\varphi \equiv 0$ outside $\{|x| < 2\delta\}$. If δ is sufficiently small, the function

$$u_\varepsilon(x) := \varepsilon^{2m-n}\,\varphi(x)\,\phi\left(\frac{x}{\varepsilon}\right) = \varphi(x)\,\left(\varepsilon^2 + |x|^2\right)^{(2m-n)/2}$$

has compact support in Ω. Next we define

$$A^\varepsilon_m := \int_{\mathbb{R}^n} |(-\Delta)^{m/2}u_\varepsilon|^2 dx \qquad A^\varepsilon_s := \int_{\mathbb{R}^n} |(-\Delta)^{s/2}u_\varepsilon|^2 dx$$

$$\widetilde{A}^\varepsilon_s := \int_\Omega |x|^{-2s}|u_\varepsilon|^2 dx \qquad B^\varepsilon := \int_\Omega |u_\varepsilon|^{2^*_m} dx$$

and we denote by c any universal positive constant.

Lemma 3.1. *It holds that*

$$A^\varepsilon_m \le \varepsilon^{2m-n}\left(M + c\,\varepsilon^{n-2m}\right) \tag{3.2}$$

$$A^\varepsilon_s, \widetilde{A}^\varepsilon_s \ge c\,\varepsilon^{4m-n-2s} \qquad \text{if} \quad s > 2m - \frac{n}{2} \tag{3.3}$$

$$A^\varepsilon_s, \widetilde{A}^\varepsilon_s \ge c\,|\log\varepsilon| \qquad \text{if} \quad s = 2m - \frac{n}{2} \tag{3.4}$$

$$B^\varepsilon \ge \varepsilon^{-n}\big((M\mathcal{S}_m^{-1})^{2^*_m/2} - c\,\varepsilon^n\big)\,. \tag{3.5}$$

Proof of (3.2). First of all, from (2.6) we get

$$A_m^\varepsilon = \varepsilon^{2m-n} \int_{\mathbb{R}^n} |\xi|^{2m} \left|\mathcal{F}\left[\varphi(\varepsilon\,\cdot)\phi\right]\right|^2 d\xi. \tag{3.6}$$

Thus

$$\Gamma_m^\varepsilon := \varepsilon^{n-2m} A_m^\varepsilon - M = \int_{\mathbb{R}^n} |\xi|^{2m} \left|\mathcal{F}\left[\varphi(\varepsilon\,\cdot)\phi\right]\right|^2 d\xi - \int_{\mathbb{R}^n} |\xi|^{2m} \left|\mathcal{F}[\phi]\right|^2 d\xi.$$

We need to prove that

$$|\Gamma_m^\varepsilon| \le c\,\varepsilon^{n-2m}. \tag{3.7}$$

If $m \in \mathbb{N}$, the proof of (3.7) has been carried out in [4], [8]. Here we limit ourselves to the more difficult case, namely, when m is not an integer. See [15] and [2] for related computations in case $m \in (0,1)$. We denote by $k := \lfloor m \rfloor \ge 0$ the integer part of m, so that $m - k \in (0,1)$. Then

$$\begin{aligned} \Gamma_m^\varepsilon &= \int_{\mathbb{R}^n} |\xi|^{2k} \mathcal{F}[U_-] \cdot |\xi|^{2(m-k)} \overline{\mathcal{F}[U_+]}\, d\xi \\ &= C_{n,m-k} \cdot \int_{\mathbb{R}^n} (-\Delta)^k U_-(x) \cdot p.v. \int_{\mathbb{R}^n} \underbrace{\frac{U_+(x) - U_+(y)}{|x-y|^{n+2(m-k)}}}_{\Psi(x,y)}\, dy\, dx, \end{aligned}$$

where $U_\pm = \varphi(\varepsilon\,\cdot)\phi \pm \phi$, compare with (2.2).

We split the interior integral as follows:

$$p.v. \int_{\mathbb{R}^n} \Psi\, dy = \underbrace{p.v. \int_{|y-x|\le|x|/2} \Psi\, dy}_{I_1} + \underbrace{\int_{\substack{|y-x|\ge|x|/2\\ |y|\le|x|}} \Psi\, dy}_{I_2} + \underbrace{\int_{\substack{|y-x|\ge|x|/2\\ |y|\ge|x|}} \Psi\, dy}_{I_3}.$$

We claim that $|I_j| \le c\,|x|^{2k-n}$ for $j = 1,2,3$. Indeed, the Lagrange formula gives

$$\begin{aligned} |I_1| &\le \max_{|y-x|\le|x|/2} |D^2 U_+(y)| \cdot \int_{|z|\le|x|/2} \frac{dz}{|z|^{n+2(m-k)-2}} \\ &\le c\,|x|^{-(n-2m+2)} \cdot |x|^{2-2(m-k)} = c\,|x|^{2k-n}. \end{aligned}$$

As concerns the last two integrals, we estimate

$$|I_2| \le \int_{\substack{|y-x|\ge|x|/2\\ |y|\le|x|}} \frac{c|y|^{-(n-2m)}}{|x-y|^{n+2(m-k)}}\, dy \le |x|^{-(n+2(m-k))} \cdot c\,|x|^{2m} = c\,|x|^{2k-n},$$

and finally,

$$\begin{aligned} |I_3| &\le \int_{\substack{|y-x|\ge|x|/2\\ |y|\ge|x|}} \frac{c|x|^{-(n-2m)}}{|x-y|^{n+2(m-k)}}\, dy \le c\,|x|^{-(n-2m)} \cdot \int_{|z|\ge|x|/2} \frac{dz}{|z|^{n+2(m-k)}} \\ &\le c\,|x|^{-(n-2m)} \cdot |x|^{-2(m-k)} = c\,|x|^{2k-n}, \end{aligned}$$

and the claim follows. Now, since

$$|(-\Delta)^k U_-(x)| \le \frac{c}{|x|^{n-2(m-k)}}\,\chi_{\{|x|\ge\delta/\varepsilon\}} + \frac{c\,\varepsilon^{2k}}{|x|^{n-2m}}\,\chi_{\{\delta/\varepsilon\le|x|\le 2\delta/\varepsilon\}},$$

we obtain

$$|\Gamma^\varepsilon_m| \le c\int_{|x|\ge\delta/\varepsilon}\frac{dx}{|x|^{2n-2m}} + c\int_{\delta/\varepsilon\le|x|\le 2\delta/\varepsilon}\frac{\varepsilon^{2k}\,dx}{|x|^{2n-2(m+k)}} \le c\,\varepsilon^{n-2m},$$

that completes the proof of (3.7) and of (3.2).

Proof of (3.3) *and* (3.4). We use the Hardy inequality (2.7) to get

$$\begin{aligned} A^\varepsilon_s \ge c\,\widetilde{A}^\varepsilon_s &\ge c\,\varepsilon^{4m-2s-n}\int_{\mathbb{R}^n}|x|^{-2s}\,|\varphi(\varepsilon\,\cdot)\,\phi|^2\,dx \\ &\ge c\,\varepsilon^{4m-2s-n}\int_{|x|<\delta/\varepsilon}\frac{dx}{|x|^{2s}(1+|x|^2)^{n-2m}}\,. \end{aligned}$$

The last integral converges as $\varepsilon\to 0$ if $s>2m-n/2$, and diverges with speed $|\log\varepsilon|$ if $s=2m-n/2$.

Proof of (3.5). For ε small enough we estimate by below

$$\begin{aligned} \int_{\mathbb{R}^n}|u_\varepsilon|^{2^*_m} &= \varepsilon^{-n}\int_{\mathbb{R}^n}|\varphi(\varepsilon\,\cdot)\phi|^{2^*_m}\,dx = \varepsilon^{-n}\Big(\int_{\mathbb{R}^n}|\phi|^{2^*_m}\,dx - \int_{|x|>\delta/\varepsilon}|\varphi(\varepsilon\,\cdot)\phi|^{2^*_m}\,dx\Big) \\ &\ge \varepsilon^{-n}\Big((M\mathcal{S}_m^{-1})^{2^*_m/2} - c\int_{|x|>\delta/\varepsilon}|x|^{-2n}\,dx\Big) = \varepsilon^{-n}((M\mathcal{S}_m^{-1})^{2^*_m/2} - c\varepsilon^n), \end{aligned}$$

and Lemma 3.1 is completely proved. □

4. Two noncompact minimization problems

In this section we deal with the minimization problems

$$\mathcal{S}^\Omega_\lambda(m,s) = \inf_{\substack{u\in\widetilde{H}^m(\Omega)\\ u\ne 0}}\mathcal{R}^\Omega_{\lambda,m,s}[u]; \qquad \widetilde{\mathcal{S}}^\Omega_\lambda(m,s) = \inf_{\substack{u\in\widetilde{H}^m(\Omega)\\ u\ne 0}}\widetilde{\mathcal{R}}^\Omega_{\lambda,m,s}[u]\,,$$

where the functionals $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ are introduced in (2.8) and (2.9), respectively.

Lemma 4.1. *The following facts hold for any* $\lambda\in\mathbb{R}$ *:*

i) $\mathcal{S}^\Omega_\lambda(m,s)\le\mathcal{S}_m$*;*

ii) *If* $\lambda\le 0$ *then* $\mathcal{S}^\Omega_\lambda(m,s)=\mathcal{S}_m$ *and it is not achieved;*

iii) *If* $0<\mathcal{S}^\Omega_\lambda(m,s)<\mathcal{S}_m$*, then* $\mathcal{S}^\Omega_\lambda(m,s)$ *is achieved.*

The same statements hold for $\widetilde{\mathcal{S}}^\Omega_\lambda(m,s)$ *instead of* $\mathcal{S}^\Omega_\lambda(m,s)$*.*

Proof. The proof is nowadays standard, and is essentially due to Brezis and Nirenberg [4]. We sketch it for the infimum $\mathcal{S}^\Omega_\lambda(m,s)$, for the convenience of the reader.

Fix $\varepsilon > 0$ and take $u \in \mathcal{C}_0^\infty(\mathbb{R}^n) \setminus \{0\}$ such that

$$(\mathcal{S}_m + \varepsilon)\Big(\int_{\mathbb{R}^n} |u|^{2^*_m} dx\Big)^{2/2^*_m} \ge \int_{\mathbb{R}^n} |(-\Delta)^{m/2} u|^2 \, dx. \tag{4.1}$$

Let $R > 0$ be large enough, so that $u_R(\cdot) := u(R\cdot) \in \mathcal{C}_0^\infty(\Omega)$. Using (2.6) we get

$$\mathcal{S}_\lambda^\Omega(m,s) \le \frac{\|u\|_m^2 - \lambda R^{2(s-m)} \|u\|_s^2}{\|u\|^2_{L^{2^*_m}}} \le (\mathcal{S}_m + \varepsilon)\big(1 + cR^{2(s-m)}\big),$$

where c depends only on u and λ. Letting $R \to \infty$ we get $\mathcal{S}_\lambda^\Omega(m,s) \le (\mathcal{S}_m + \varepsilon)$ for any $\varepsilon > 0$, and i) is proved.

Next, if $\lambda \le 0$ then clearly $\mathcal{S}_\lambda^\Omega(m,s) = \mathcal{S}_m$. If $\lambda = 0$ then $\mathcal{S}_m$ is not achieved. The more it is not achieved for $\lambda < 0$, and ii) holds.

Finally, to prove iii) take a minimizing sequence u_h. It is convenient to normalize u_h with respect to the $L^{2^*_m}$-norm, so that

$$\int_{\mathbb{R}^n} |(-\Delta)^{m/2} u_h|^2 \, dx - \lambda \int_{\mathbb{R}^n} |(-\Delta)^{s/2} u_h|^2 \, dx = \mathcal{S}_\lambda^\Omega(m,s) + o(1).$$

We can assume that $u_h \to u$ weakly in $\widetilde{H}^m(\Omega)$ and strongly in $\widetilde{H}^s(\Omega)$ by Proposition 2.1. Since

$$\begin{aligned}\lambda \int_{\mathbb{R}^n} |(-\Delta)^{s/2} u|^2 \, dx &= \lambda \int_{\mathbb{R}^n} |(-\Delta)^{s/2} u_h|^2 \, dx + o(1) \\ &= \int_{\mathbb{R}^n} |(-\Delta)^{m/2} u_h|^2 \, dx - \mathcal{S}_\lambda^\Omega(m,s) + o(1) \ge (\mathcal{S}_m - \mathcal{S}_\lambda^\Omega(m,s)) + o(1),\end{aligned}$$

then $u \ne 0$. By the Brezis–Lieb lemma we get

$$1 = \|u_h\|^{2^*_m}_{L^{2^*_m}} = \|u_h - u\|^{2^*_m}_{L^{2^*_m}} + \|u\|^{2^*_m}_{L^{2^*_m}} + o(1).$$

Thus

$$\begin{aligned}\mathcal{S}_\lambda^\Omega(m,s) &= \|u_h\|_m^2 - \lambda \|u_h\|_s^2 + o(1) \\ &= \big(\|u_h - u\|_m^2 + \|u\|_m^2\big) - \lambda\big(\|u_h - u\|_s^2 + \|u\|_s^2\big) + o(1) \\ &= \frac{\big(\|u_h - u\|_m^2 - \lambda \|u_h - u\|_s^2\big) + \big(\|u\|_m^2 - \lambda \|u\|_s^2\big)}{\big(\|u_h - u\|^{2^*_m}_{L^{2^*_m}} + \|u\|^{2^*_m}_{L^{2^*_m}}\big)^{2/2^*_m}} + o(1) \\ &\ge \mathcal{S}_\lambda^\Omega(m,s) \cdot \frac{\xi_h^2 + 1}{(\xi_h^{2^*_m} + 1)^{2/2^*_m}} + o(1),\end{aligned}$$

where we have set

$$\xi_h := \frac{\|u_h - u\|_{L^{2^*_m}}}{\|u\|_{L^{2^*_m}}}.$$

Since $2^*_m > 2$, this implies that $\xi_h \to 0$, that is, $u_h \to u$ in $L^{2^*_m}$ and hence u achieves $\mathcal{S}_\lambda^\Omega(m,s)$. □

We are in position to prove our existence result, that includes the theorem already stated in the introduction.

Theorem 4.2. *Assume $s \geq 2m - n/2$.*

i) *If $0 < \lambda < \Lambda_1(m,s)$ then $\mathcal{S}_\lambda^\Omega(m,s)$ is achieved and (1.1) has a nontrivial solution in $\widetilde{H}^m(\Omega)$.*

ii) *If $0 < \lambda < \widetilde{\Lambda}_1(m,s)$ then $\widetilde{\mathcal{S}}_\lambda^\Omega(m,s)$ is achieved and (1.2) has a nontrivial solution in $\widetilde{H}^m(\Omega)$.*

Proof. Since $0 < \lambda < \Lambda_1(m,s)$ then $\mathcal{S}_\lambda^\Omega(m,s)$ is positive, by Proposition 2.1. The main estimates in Lemma 3.1 readily imply $\mathcal{S}_\lambda^\Omega(m,s) < \mathcal{S}_m$. By Lemma 4.1, $\mathcal{S}_\lambda^\Omega(m,s)$ is achieved by a nontrivial $u \in \widetilde{H}^m(\Omega)$, that solves (1.1) after multiplication by a suitable constant. Thus i) is proved. For ii), argue in the same way. □

In the case $s < 2m - n/2$ the situation is more complicated. We limit ourselves to point out the next simple existence result.

Theorem 4.3. *Assume $s < 2m - n/2$.*

i) *There exists $\lambda^* \in [0, \Lambda_1(m,s))$ such that the infimum $\mathcal{S}_\lambda^\Omega(m,s)$ is attained for any $\lambda \in (\lambda^*, \Lambda_1(m,s))$, and hence (1.1) has a nontrivial solution.*

ii) *There exists $\widetilde{\lambda}^* \in [0, \widetilde{\Lambda}_1(m,s))$ such that the infimum $\widetilde{\mathcal{S}}_\lambda^\Omega(m,s)$ is attained for any $\lambda \in (\widetilde{\lambda}^*, \widetilde{\Lambda}_1(m,s))$, and hence (1.2) has a nontrivial solution.*

Proof. Use Proposition 2.1 to find $\varphi_1 \in \widetilde{H}^m(\Omega)$, $\varphi_1 \neq 0$, such that

$$\int_{\mathbb{R}^n} |(-\Delta)^{m/2}\varphi_1|^2\,dx = \Lambda_1(m,s)\int_{\mathbb{R}^n} |(-\Delta)^{s/2}\varphi_1|^2\,dx\,.$$

Then test $\mathcal{S}_\lambda^\Omega(m,s)$ with φ_1 to get the strict inequality $\mathcal{S}_\lambda^\Omega(m,s) < \mathcal{S}_m$. Then i) follows by Proposition 2.1 and Lemma 4.1. For (1.2) argue similarly. □

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