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Monodromy representations of completed coverings

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Abstract. In this paper we consider completed coverings that are branched coverings in the sense of Fox. For completed coverings between PL manifolds we give a characterization of the existence of a monodromy representation and the existence of a locally compact monodromy representation. These results stem from a characterization of the discreteness of a completed normal covering. We also show that a completed covering admitting a monodromy representation is discrete and that the image of the branch set is closed.

1. Introduction

By the classical theory of covering spaces, a covering map $f\colon X \to Z$ between manifolds is a factor of a normal (or regular) covering; there exists a normal covering $p\colon Y \to X$ so that $q = f \circ p\colon Y \to Z$ is a normal covering and the deck-transformation group of q is isomorphic to the monodromy group of f,

$$\begin{array}{ccc} & Y & \\ {}^{p}\swarrow & & \searrow^{q} \\ X & \xrightarrow{\;f\;} & Z. \end{array}$$

In this case, the monodromy group G of f acts on Y and there exists a subgroup $H \subset G$ for which $Y/G \approx Z$ and $Y/H \approx X$. The normal coverings $p\colon Y \to X$ and $q\colon Y \to Z$ are orbit maps. In this article we are interested in ramifications of this construction for discrete open mappings $f\colon X \to Z$ between manifolds.

By Černavskiĭ [2] and Väisälä [18] a discrete and open mapping $f\colon X \to Z$ between manifolds is almost a local homeomorphism in the following sense. Let $B_f \subset X$ be the branch set i.e. the set of points in X, where f is not a local homeomorphism. By the theorem of Černavskiĭ and Väisälä the topological codimension

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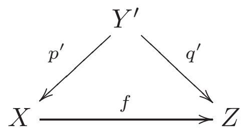
of B_f is at least 2. In fact, if we set $Z' := Z \setminus f(B_f)$ and $X' := X \setminus f^{-1}(f(B_f))$, then $X' \subset X$ is a dense and connected subset of X and $Z' \subset Z$ is a dense and connected subset of Z and $g := f|X' \colon X' \to Z'$ is a local homeomorphism.

Suppose now in addition that $Z' \subset Z$ is open and g is a covering. By the classical argument above there exist an open manifold Y' and a commutative diagram of discrete and open mappings

(1.1)



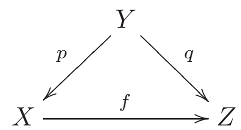
where $p' \colon Y' \to X'$ and $q' \colon Y' \to Z'$ are normal coverings, the deck-transformation group of the covering $q' \colon Y' \to Z'$ is isomorphic to the monodromy group of g and $q' = f \circ p'$. It becomes a question, whether there exists a space $Y \supset Y'$ so that p' and q' extend to discrete orbit maps $p \colon Y \to X$ and $q \colon Y \to Z$ satisfying $q = f \circ p$. We formalize this question as follows.

Question 1. *Suppose $f \colon X \to Z$ is an open discrete mapping between manifolds so that $Z \setminus f(B_f) \subset Z$ is open and*

$$f|\big(X \setminus f^{-1}(f(B_f))\big) \colon \big(X \setminus f^{-1}(f(B_f))\big) \to \big(Z \setminus f(B_f)\big)$$

is a covering. Does there exist a locally connected Hausdorff space Y, an embedding $\iota \colon Y' \to Y$ and orbit maps $p \colon Y \to X$ and $q \colon Y \to Z$ so that

(1.2)



is a commutative diagram of discrete and open mappings satisfying $p' = p \circ \iota$ and $q' = q \circ \iota$, where p' and q' are as in (1.1)? Further, we require that $\iota(Y') \subset Y$ is a dense subset and $Y \setminus \iota(Y')$ does not locally separate Y.

This question stems from an article of Berstein and Edmonds [1], where they show that for open and discrete mappings between compact manifolds the answer to Question 1 is positive and the orbit maps p and q are induced by the action of the monodromy group of f on Y. They use this construction to give degree estimates for simplicial maps between compact manifolds; see also Pankka–Souto [15] for another application.

In this article we extend the construction of Berstein and Edmonds introduced in the proof of Proposition 2.2 in [1] for completed coverings (Definition 3.7) that are branched coverings in the sense of Fox [7]. Completed coverings form a subclass of open and surjective mappings from a Hausdorff space onto a manifold. Examples of completed coverings between manifolds are discrete and open mappings between compact manifolds (see [3]), proper quasiregular mappings (see [16]) and surjective open and discrete simplicial mappings between PL manifolds (see [11]). The class of completed coverings also include mappings outside these classes of mappings (see [5] and [12]). The definition of a completed covering is technical and it comes from the theory of complete spreads in Fox [7]. We only mention here, that the class of completed coverings is sufficiently large and natural in our setting.

Edmonds' results on orbit maps with finite multiplicity [6] ensures that the answer to Question 1 is known for completed coverings that have finite multiplicity: by Theorem 4.1 in Edmonds [6], completed normal coverings with finite multiplicity are discrete orbit maps. Based on this, the argument of Berstein and Edmonds [1] relies on the observations that a discrete and open mapping $f\colon X \to Z$ between compact manifolds is a completed covering and has a finite monodromy group. For completed coverings, the finiteness of the monodromy group is a sufficient condition for the argument of Berstein and Edmonds. Thus, by the finiteness of the monodromy group, the answer to Question 1 is positive for every completed covering $f\colon X \to Z$ between manifolds that has finite multiplicity.

On the other hand, for Question 1 to have a positive answer for a mapping $f\colon X \to Z$ with infinite multiplicity, it is not enough to assume that f is a completed covering. The reason for this is that the monodromy group of f is in this case infinite, and a completed normal covering $p\colon Y \to Z$ with infinite multiplicity is not necessary discrete; see Example 10.6 in Montesinos [14]. To answer Question 1, our first main theorem is the following.

Theorem 1.1. *A completed normal covering $p\colon Y \to Z$ from a Hausdorff space Y onto a manifold Z is an orbit map if and only if p is a discrete map.*

For this reason in the heart of Question 1 is the characterization of discrete completed normal coverings. In this direction we obtain the following results.

Theorem 1.2. *A completed normal covering $p\colon Y \to Z$ from a Hausdorff space Y onto a PL manifold Z is discrete if and only if p is stable.*

Stability of completed coverings is defined in Section 4.3. A completed normal covering $p\colon Y \to Z$ from a Hausdorff space Y onto a manifold Z is discrete if it has locally finite multiplicity; see Theorem 9.14 in [14]. We prove the following.

Theorem 1.3. *Let $p\colon Y \to Z$ be a discrete completed normal covering from a Hausdorff space Y onto a PL manifold Z. Then Y is locally compact if and only if p has locally finite multiplicity.*

Regularity and existence of monodromy representations. We begin with a result showing the naturality of completed coverings. We show that if the space Y in diagram (1.2) is locally compact and locally connected, then all the maps in the diagram are completed coverings.

Theorem 1.4. *Suppose $f\colon X \to Z$ is a discrete and open mapping between manifolds, $X' := X \setminus f^{-1}(f(B_f))$ and $Z' := Z \setminus f(B_f)$ for the branch set B_f of f.*

Suppose $Z' \subset Z$ is open and $f|X' \colon X' \to Z'$ is a covering. Let

$$\begin{array}{ccc} & Y' & \\ {}^{p'}\swarrow & & \searrow^{q'} \\ X & \xrightarrow{\quad f \quad} & Z \end{array}$$

be a commutative diagram of discrete and open mappings so that $p' \colon Y' \to X'$ and $q' \colon Y' \to Z'$ are normal coverings. Suppose Y is a locally compact and locally connected Hausdorff space so that there exist an embedding $\iota \colon Y' \to Y$ and orbit maps $p \colon Y \to X$ and $q \colon Y \to Z$ for which

$$\begin{array}{ccc} & Y & \\ {}^{p}\swarrow & & \searrow^{q} \\ X & \xrightarrow{\quad f \quad} & Z \end{array}$$

is a commutative diagram of discrete and open mappings satisfying $p' = p \circ \iota$ and $q' = q \circ \iota$, and so that $\iota(Y') \subset Y$ is dense and $Y \setminus \iota(Y')$ does not locally separate Y.

Then $f \colon X \to Z$, $p \colon Y \to X$ and $q \colon Y \to Z$ are completed coverings.

Let X and Z be manifolds and $f \colon X \to Z$ a completed covering. We say that a triple (Y, p, q) is *a monodromy representation* of f if Y is a locally connected Hausdorff space and the monodromy group G of f has an action on Y and a subgroup $H \subset G$ so that $Y/G \approx Z$, $Y/H \approx X$ and the associated orbit maps $p \colon Y \to X$ and $q \colon Y \to Z$ are completed coverings satisfying $q = p \circ f$. We call a monodromy representation (Y, p, q) *locally compact* if Y is locally compact. We present the following four regularity results for monodromy representations.

Theorem 1.5. *Let $f \colon X \to Z$ be a completed covering between manifolds. Suppose (Y, p, q) is a monodromy representation of f. Then f, p and q are discrete maps.*

Theorem 1.6. *Let $f \colon X \to Z$ be a completed covering between manifolds. Suppose (Y, p, q) is a monodromy representation of f and $B_f \subset X$, $B_p \subset Y$ and $B_q \subset Y$ the respective branch sets. Then $f(B_f) \subset Z$, $p(B_p) \subset X$ and $q(B_q) \subset Z$ are closed sets.*

Theorem 1.7. *Let $f \colon X \to Z$ be a completed covering between PL manifolds. Suppose (Y, p, q) is a monodromy representation of f. Then f, p and q are stable.*

Theorem 1.8. *Let $f \colon X \to Z$ be a completed covering between PL manifolds. Suppose (Y, p, q) is a locally compact monodromy representation of f. Then for every polyhedral path-metric d_s on Z there exists a path-metric d_s^* on Y, so that*

(a) *the topology induced by d_s^* coincides with the topology of Y,*

(b) *(Y, d_s^*) is a locally proper metric space,*

(c) *$q \colon (Y, d_s^*) \to (Z, d_s)$ is a 1-Lipschitz map, and*

(d) *the action of the monodromy group of f in the monodromy representation (Y, p, q) is by isometries on (Y, d_s^*).*

We note that the path metric d_s^* in Theorem 1.8 is a pullback of the path metric d_s in Z. Thus the map $q\colon (Y, d_s^*) \to (Z, d_s)$ is a 1-BLD map. Further, if $f\colon (X, e_s) \to (Z, d_s)$ is a L-BLD map for a path metric e_s on X, then $p\colon (Y, d_s^*) \to (X, e_s)$ is a L-BLD map. A mapping $f\colon X \to Z$ between length manifolds is L-BLD for $L \geq 1$ if f is discrete and open and satisfies

$$\frac{1}{L}\,\ell(\gamma) \leq \ell(f \circ \gamma) \leq L\,\ell(\gamma)$$

for every path γ in X, where $\ell(\cdot)$ is the length of a path. We refer to [10] for a detailed discussion of BLD-maps.

For the existence of a monodromy representations, we have the following characterization. Together with Theorem 1.5, this answers to Question 1 in the context of completed coverings between PL manifolds.

Theorem 1.9. *A completed covering $f\colon X \to Z$ between PL manifolds has a monodromy representation (Y, p, q) if and only if f is stable.*

As a corollary of Theorem 1.9 open and surjective simplicial mappings between PL manifolds have monodromy representations. Another corollary is the following.

Corollary 1.10. *Let $f\colon X \to Z$ be a L-BLD mapping between PL 2-manifolds. Then f has a monodromy representation if and only if $f(B_f) \subset Z$ is a discrete set.*

Indeed, if $f(B_f) \subset Z$ is a discrete set, then the BLD-mapping f is a completed covering by Luisto [12], and further f is a stable completed covering. Thus f has a monodromy representation by Theorem 1.9.

If $f(B_f) \subset Z$ is not a discrete set, then $f(B_f) \subset Z$ it not a closed set by Stoilow's theorem. Thus, by Theorem 1.6, f has no monodromy representation.

We characterize the existence of locally compact monodromy representations as follows. Local monodromy groups are defined in Definition 5.4.

Theorem 1.11. *A completed covering $f\colon X \to Z$ between PL manifolds has a locally compact monodromy representation (Y, p, q) if and only if f is stable and f has a finite local monodromy group at each point of Z.*

We show that not every monodromy representation is locally compact. However, our results combined with previous results by Fox, Edmonds and Montesinos have an easy corollary in the positive direction. For the following statement we say that a covering $g\colon X' \to Z'$ is *virtually normal* if $g_*(\pi(X', x_0)) \subset \pi(Z', z_0)$ contains a finite index subgroup which is normal in $\pi(Z', z_0)$.

Corollary 1.12. *Let $f\colon X \to Z$ be a completed virtually normal covering between PL manifolds. Then f has a locally compact monodromy representation (Y, p, q), where p has finite multiplicity.*

Proof. By Fox [7] we find a completed normal covering $p\colon Y \to X$ so that p has finite multiplicity and $q := f \circ p$ is a completed normal covering. Since p has finite multiplicity, p is an orbit map by Theorem 4.1 in Edmonds [6] and q has locally finite multiplicity. Since q has locally finite multiplicity, the map q is a discrete map by Theorem 9.14 in Montesinos [14]. Thus q is an orbit map by Theorem 1.1 and (Y, p, q) a monodromy representation of f. Since $q\colon Y \to Z$ is a completed normal covering onto a PL manifold that is an orbit map and that has locally finite multiplicity, the space Y is by Theorem 1.3 locally compact. Thus the monodromy representation (Y, p, q) is locally compact. □

This article is organized as follows. In Sections 2.1 and 2.2 we discuss monodromy of covering maps. In Sections 3.1 and 3.2 we introduce completed coverings. In Section 3.3 we study the relation between orbit maps and completed normal coverings and prove Theorem 1.4. In Section 4.1 we prove Theorem 1.5 by showing that completed normal coverings are orbit maps if and only if they are discrete maps. In Section 4.2 we prove Theorem 1.6. In Section 4.3 we prove Theorems 1.7 and 1.9 by showing that a completed normal covering onto a PL manifold is discrete if and only if it is stable. In Sections 5.2 and 5.3 we study completed normal coverings having locally finite multiplicity and prove Theorems 1.3, 1.8 and 1.11.

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2. Preliminaries

2.1. Normal coverings

In this section we recall some facts on normal coverings and fix some notation related to coverings; we refer to Chapter 1 in Hatcher [9] for a detailed discussion.

Let Y be a topological space. A *path* is a continuous map $\alpha\colon [0,1] \to Y$. The *inverse path* $\alpha^{\leftarrow}\colon [0,1] \to Y$ of a path $\alpha\colon [0,1] \to Y$ is the path $t \mapsto \alpha(1-t)$. The *path composition* $\alpha\beta\colon [0,1] \to Y$ of paths $\alpha\colon [0,1] \to Y$ and $\beta\colon [0,1] \to Y$ satisfying $\alpha(1) = \beta(0)$ is the path $t \mapsto \alpha(2t)$ for $t \in [0, 1/2]$ and $t \mapsto \beta(2(t-1/2))$ for $t \in [1/2, 1]$. For $y_0, y_1 \in Y$ a path $\alpha\colon [0,1] \to Y$ satisfying $\alpha(0) = y_0$ and $\alpha(1) = y_1$ is denoted $\alpha\colon y_0 \curvearrowright y_1$. Given $y_0 \in Y$, we denote by $\pi(Y, y_0)$ the fundamental group of Y (at y_0).

A map $p\colon X \to Y$ between path-connected spaces X and Y is a covering map if for every $y \in Y$ there exists an open neighbourhood U of y such that the preimage $p^{-1}(U) \subset X$ is a union of pairwise disjoint open sets homeomorphic to U. A space X is called *a cover* of Y if there is a covering map from X to Y. Let $p\colon X \to Y$ be a covering map, $y_0 \in Y$ and $x_0 \in p^{-1}\{y_0\}$. A path $\widetilde{\alpha}\colon [0,1] \to X$ is called a *lift* of $\alpha\colon [0,1] \to Y$ in p if $\alpha = p \circ \widetilde{\alpha}$. For every path $\alpha\colon [0,1] \to Y$ satisfying $\alpha(0) = \alpha(1) = y_0$ there exists a unique lift $\widetilde{\alpha}$ satisfying $\widetilde{\alpha}(0) = x_0$. We denote this lift by $\widetilde{\alpha}_{x_0}$. Given a loop $\alpha\colon (S^1, e_0) \to (Y, y_0)$ we denote by $\widetilde{\alpha}$ the lift of the path $[0,1] \to Y, t \mapsto \alpha(\cos(2\pi t), \sin(2\pi t))$.

The *deck-transformation group* $\mathcal{T}(p)$ of a map $p\colon X \to Y$ is the group of all homeomorphisms $\tau\colon X \to X$ satisfying $p(\tau(x)) = p(x)$ for all $x \in X$. Let $\tau_1\colon X \to X$ and $\tau_2\colon X \to X$ be deck-transformations of a covering $p\colon X \to Y$. Then $\tau_1 = \tau_2$ if and only if there exists $x \in X$ so that $\tau_1(x) = \tau_2(x)$.

A covering $p\colon X \to Y$ is *normal* if the subgroup $p_*(\pi(X, x_0)) \subset \pi(Y, y_0)$ is normal for $y_0 \in Y$ and $x_0 \in p^{-1}\{y_0\}$. The following statements are equivalent:

(a) p is a normal covering,

(b) for every $y \in Y$ and pair of points $x_1, x_2 \in p^{-1}\{y\}$ there exists a unique deck transformation $\tau\colon X \to X$ satisfying $\tau(x_1) = x_2$,

(c) $X/\mathcal{T}(p) \approx Y$, and

(d) $p_*(\pi(X, x_0)) = p_*(\pi(X, x_1))$ for all $x_1 \in p^{-1}\{p(x_0)\}$.

Moreover, $\mathcal{T}(p) \cong \pi(Y, y_0)/p_*(\pi(X, x_0))$ if and only if p is a normal covering.

For every manifold Y, base point $y_0 \in Y$ and normal subgroup $N \subset \pi(Y, y_0)$, there exists a normal covering $p\colon X \to Y$ so that $p_*(\pi(X, x_0)) = N$ for every $x_0 \in p^{-1}\{y_0\}$. Moreover, the cover X is a manifold. Thus $\mathcal{T}(p)$ is countable, since $p^{-1}\{y_0\}$ is countable.

We also recall the following properties of covering maps. Let $p\colon (X, x_0) \to (Y, y_0)$ be a normal covering and $\widetilde{\alpha}_{x_0}\colon [0,1] \to X$ a lift of a loop $\alpha\colon (S^1, e_0) \to (Y, y_0)$. Then the point $\widetilde{\alpha}_{x_0}(1) \in p^{-1}\{y_0\}$ depends only on the class of α in $\pi(Y, y_0)/p_*(\pi(X, x_0))$. For any pair of points $y_1 \in Y$ and $x_1 \in p^{-1}\{y_1\}$ and a path $\beta\colon y_1 \curvearrowright y_0$, the lift $(\widetilde{\beta\alpha\beta^{\leftarrow}})_{x_1}$ is a loop if and only if the lift $\widetilde{\alpha}_{\widetilde{\beta}_{x_1}(1)}$ is a loop. Further, $(\widetilde{\beta\alpha\beta^{\leftarrow}})_{x_1}(1) = (\widetilde{\beta\gamma\beta^{\leftarrow}})_{x_1}(1)$ for a loop $\gamma\colon (S^1, e_0) \to (Y, y_0)$ if and only if $\widetilde{\alpha}_{\widetilde{\beta}_{x_1}}(1) = \widetilde{\gamma}_{\widetilde{\beta}_{x_1}}(1)$.

2.2. Monodromy

In this section we recall some facts on the monodromy of maps.

Let $f\colon X \to Y$ be a covering, $y_0 \in Y$. For every $[\alpha] \in \pi(Y, y_0)$ we define a map $m_{[\alpha]}\colon f^{-1}\{y_0\} \to f^{-1}\{y_0\}$ by setting $m_{[\alpha]}(x) = \widetilde{\alpha}_x(1)$ for every $x \in f^{-1}\{y_0\}$. For every $[\alpha] \in \pi(Y, y_0)$, the map $m_{[\alpha]}$ is a bijection by the uniqueness of lifts. The *monodromy* of f is the homomorphism

$$\sigma_f\colon \pi(Y, y_0) \to \mathrm{Sym}(f^{-1}\{y_0\}),\ [\alpha] \mapsto m_{[\alpha]}.$$

We call the quotient $\pi(Y, y_0)/\mathrm{Ker}(\sigma_f)$ the monodromy group of f.

Let $p\colon (Y, y_0) \to (X, x_0)$ be a normal covering, $f\colon (X, x_0) \to (Z, z_0)$ a covering and $q := f \circ p\colon (Y, y_0) \to (Z, z_0)$. Then q is a covering and $q_*(\pi(Y, y_0))$ is a normal subgroup of $f_*(\pi(Z, z_0))$. We note that, q is a normal covering if and only if $q_*(\pi(Y, y_0))$ is a normal subgroup of $\pi(Z, z_0)$. The question whether the covering q is a normal is related to the monodromy of the covering f in the following way.

Proposition 2.1. *Let X be a manifold, $f\colon (X, x_0) \to (Z, z_0)$ a covering and $N \subset \pi(Z, z_0)$ a normal subgroup. Then there exists a normal covering $p\colon (Y, y_0) \to (X, x_0)$ satisfying $(f \circ p)_*(\pi(Y, y_0)) = N$ if and only if $N \subset \mathrm{Ker}(\sigma_f)$.*

In order to prove Proposition 2.1, we prove first Lemmas 2.2 and 2.3.

Lemma 2.2. *Let* $f\colon (X,x_0) \to (Y,y_0)$ *be a covering map. Then* $\mathrm{Ker}(\sigma_f)$ *is a normal subgroup of* $\pi(Y,y_0)$ *satisfying* $\mathrm{Ker}(\sigma_f) \subset f_*(\pi(X,x_0))$.

Proof. Clearly, $\mathrm{Ker}(\sigma_f) \subset \pi(Y,y_0)$ is a normal subgroup. Let $\alpha\colon (S^1,e_0) \to (Y,y_0)$ be a loop satisfying $[\alpha] \in \mathrm{Ker}(\sigma_f)$ and let $\widetilde{\alpha}_{x_0}\colon [0,1] \to X$ be the lift of α in f beginning at x_0. Since

$$\widetilde{\alpha}_{x_0}(1) = m_{[\alpha]}(x_0) = \mathrm{id}(x_0) = x_0 = \widetilde{\alpha}_{x_0}(0),$$

$\widetilde{\alpha}_{x_0}$ is a loop. Thus $[\alpha] = [f \circ \widetilde{\alpha}_{x_0}] \in f_*(\pi(X,x_0))$. □

Lemma 2.3. *Let* $f\colon (X,x_0) \to (Y,y_0)$ *be a covering and* $N \subset f_*(\pi(X,x_0))$ *a normal subgroup of* $\pi(Y,y_0)$. *Then* $N \subset \mathrm{Ker}(\sigma_f)$.

Proof. Let $\alpha\colon (S^1,e_0) \to (Y,y_0)$ be a loop satisfying $[\alpha] \in N$. We need to show that for every $x \in f^{-1}\{y_0\}$ the lift $\widetilde{\alpha}_x$ of α in f is a loop. For this let $x \in f^{-1}\{y_0\}$ and let $\beta\colon x_0 \curvearrowright x$ be a path. Then

$$[f \circ \beta\widetilde{\alpha}_x\beta^{\leftarrow}] = [f \circ \beta][\alpha][f \circ \beta]^{-1} \in N \subset f_*(\pi(X,x_0)),$$

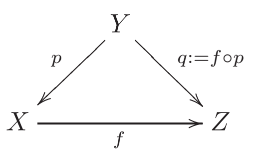
since N is a normal subgroup of $\pi(Y,y_0)$. Thus $\widetilde{\alpha}_x$ is a loop and $N \subset \mathrm{Ker}(\sigma_f)$ is a normal subgroup. □

Proof of Proposition 2.1. Let $p\colon (Y,y_0) \to (X,x_0)$ be a normal covering. Suppose first that $(f \circ p)_*(\pi(Y,y_0)) = N$. Then $N \subset \mathrm{Ker}(\sigma_f)$ by Lemma 2.3.

Suppose now that $N \subset \mathrm{Ker}(\sigma_f)$. Then, by Lemma 2.2, $N \subset f_*(\pi(X,x_0))$. Since f_* is a monomorphism, the pre-image $\tilde{N}$ of N under f_* is a normal subgroup of $\pi(X,x_0)$ and isomorphic to N. Hence there exists a normal covering $p\colon (Y,y_0) \to (X,x_0)$ satisfying $p_*(\pi(Y,y_0)) = \tilde{N} \cong N$. In particular, $(f \circ p)_*(\pi(Y,y_0)) = f_*(\tilde{N}) = N$. □

The following observation is due to Berstein and Edmonds [1].

Proposition 2.4. *Let*



be a diagram of covering maps so that $q_*(\pi(Y,y_0)) = \mathrm{Ker}(\sigma_f)$ *for* $z_0 \in Z$ *and* $y_0 \in q^{-1}\{z_0\}$. *Then the deck-transformation group* $\mathcal{T}(q)$ *of the covering* $q\colon Y \to Z$ *is finite if and only if the set* $f^{-1}\{z_0\} \subset X$ *is finite.*

Proof. Suppose that the group $\mathcal{T}(q)$ is finite and $\#\mathcal{T}(q) = m$. Since q is a normal covering, $\#(q^{-1}\{z_0\}) = m$. Hence $\#(f^{-1}\{z_0\}) \le m$, since $q = f \circ p$.

Suppose that $f^{-1}\{z_0\} \subset X$ is finite. Then the finiteness of the symmetric group $\mathrm{Sym}(f^{-1}\{z_0\})$ implies that

$$\mathcal{T}(q) \cong \pi(Z, z_0)/\mathrm{Ker}(\sigma_f) \cong \mathrm{Im}(\sigma_f) \subset \mathrm{Sym}(f^{-1}\{y_0\})$$

is a finite group. □

Next we define monodromy for a broader class of maps including completed coverings defined in the following section. We say that an open dense subset $Z' \subset Z$ is *large* if $Z \setminus Z'$ does not locally separate Z.

Lemma 2.5. *Let $f: X \to Z$ be a continuous and open map. Suppose that for $i \in \{1, 2\}$ there exist large subsets $X_i \subset X$ and $Z_i \subset Z$ so that $g_i := f|X_i : X_i \to Z_i$ is a covering. Let $z_0 \in Z_1 \cap Z_2$ and let $\sigma_{g_i} : \pi(Z_i, z_0) \to \mathrm{Sym}(f^{-1}\{z_0\})$ be the monodromy for $i \in \{1, 2\}$. Then $\mathrm{Im}(\sigma_{g_1}) = \mathrm{Im}(\sigma_{g_2})$.*

Proof. Let $g := f|X_1 \cap X_2 : X_1 \cap X_2 \to Z_1 \cap Z_2$. Then g is a covering. It is sufficient to show that $\mathrm{Im}(\sigma_{g_1}) = \mathrm{Im}(\sigma_g)$. Let $\iota_* : \pi(Z_1 \cap Z_2, z_0) \to \pi(Z_1, z_0)$ be the homomorphism induced by the inclusion $\iota: Z_1 \cap Z_2 \hookrightarrow Z_1$. Then ι_* is surjective, since $Z_1 \cap Z_2 \subset Z_1$ is dense and $Z_1 \setminus (Z_1 \cap Z_2)$ does not locally separate Z_1. Thus $\mathrm{Im}(\sigma_{g_1}) = \mathrm{Im}(\sigma_g)$. □

Let $f: X \to Z$ be an open and continuous map so that $g := f|X' : X' \to Z'$ is a covering for large subsets $X' \subset X$ and $Z' \subset Z$. Then we say that the monodromy $\sigma_g : \pi(Z', z_0) \to \mathrm{Sym}(f^{-1}\{z_0\})$ is a *monodromy σ_f of f*. We call the monodromy group $\pi(Z', z_0)/\mathrm{Ker}(\sigma_g)$ of g a *monodromy group of f*.

Remark 2.6. The isomorphism class of the monodromy group of f is independent of the choice of $z_0 \in Z'$, and by Lemma 2.5, it is independent of the choice of monodromy.

3. Spreads and completed coverings

3.1. Spreads

In this section we discuss the theory of spreads; see Fox [7] and Montesinos [14] for a more detailed discussion.

Definition 3.1. A mapping $g: Y \to Z$ between locally connected Hausdorff spaces is called a *spread* if the components of the inverse images of the open sets of Z form a basis of Y.

Clearly spreads are continuous and a composition of spreads is a spread. A map $g: Y \to Z$ between topological manifolds is a spread if and only if g is light; for every $z \in Z$ the set $g^{-1}\{z\}$ is totally disconnected, see Corollary 4.7 in [14]. In particular, if Z is a manifold and $g: Y \to Z$ is an open mapping which is a covering to its image, then g is a spread.

The definition of a complete spread is formulated by Fox [7]: a spread $g\colon Y \to Z$ is *complete* if for every point z of Z and every open neighbourhood W of z there is selected a component V_W of $g^{-1}(W)$ in such a way that $V_{W_1} \subset V_{W_2}$ whenever $W_1 \subset W_2$, then $\cap_W V_W \neq \varnothing$. This definition can equivalently be written as follows.

For every $z \in Z$ we denote by $\mathcal{N}_Z(z)$ the set of open connected neighbourhoods of z. A function $\Theta\colon \mathcal{N}_Z(z) \to \mathrm{Top}(Y)$ is a *selection function* of a spread $g\colon Y \to Z$ if $\Theta(W)$ is a component of $g^{-1}(W)$ for every $W \in \mathcal{N}_Z(z)$ and $\Theta(W_1) \subset \Theta(W_2)$ whenever $W_1 \subset W_2$.

Definition 3.2. A spread $g\colon Y \to Z$ is *complete* if for all $z \in Z$ every selection function $\Theta\colon \mathcal{N}_Z(z) \to \mathrm{Top}(Y)$ of g satisfies

$$\bigcap_{W \in \mathcal{N}_Z(z)} \Theta(W) \neq \varnothing.$$

Remark 3.3. Let $g\colon Y \to Z$ be a complete spread. Since Y is Hausdorff, the set $\bigcap_{W \in \mathcal{N}_Z(z)} \Theta(W) \subset Y$ is a point for every $z \in Z$ and every selection function $\Theta\colon \mathcal{N}_Z(z) \to \mathrm{Top}(Y)$. Moreover, for every $z \in Z$ and $y \in g^{-1}\{z\}$ there exists a selection function $\Theta\colon \mathcal{N}_Z(z) \to \mathrm{Top}(Y)$ of g for which $\bigcap_{W \in \mathcal{N}_Z(z)} \Theta(W) = \{y\}$.

Lemma 3.4. *Let $g\colon Y \to Z$ be a spread. Suppose that for every $z \in Z$ the following conditions are satisfied:*

(1) *$g(V) = W$ for every $W \in \mathcal{N}_Z(z)$ and every component V of $g^{-1}(W)$, and*

(2) *there exists $W_0 \in \mathcal{N}_Z(z)$ so that for every component V of $g^{-1}(W_0)$ the set $V \cap g^{-1}\{y\}$ is finite.*

Then g is a complete spread.

Proof. Let $z \in Z$ and let $\Theta\colon \mathcal{N}_Z(z) \to \mathrm{Top}(Y)$ be a selection function for g. We need to show that $\bigcap_{W \in \mathcal{N}_Z(z)} \Theta(W) \neq \varnothing$.

Suppose $\bigcap_{W \in \mathcal{N}_Z(z)} \Theta(W) = \varnothing$. By condition (2), the set $g^{-1}\{z\} \cap \Theta(W_0)$ is finite for some $W_0 \in \mathcal{N}_Z(z)$. Let $y_1, \ldots, y_k \in Y$ be such that

$$\{y_1, \ldots, y_k\} = g^{-1}\{z\} \cap \Theta(W_0).$$

Since $\bigcap_{W \in \mathcal{N}_Z(z)} \Theta(W) = \varnothing$, we find for every $j = 0, \ldots, k$ a set $W_j \in \mathcal{N}_Z(z)$ for which $y_j \notin \Theta(W_j)$. Let $W' = W_0 \cap \cdots \cap W_k$. Then $\Theta(W') \cap g^{-1}\{z\} = \varnothing$. Thus $z \notin \bigcap_{W \in \mathcal{N}_Z(z)} g(\Theta(W)) \subset g(\Theta(W'))$.

On the other hand, $z \in \bigcap_{W \in \mathcal{N}_Z(z)} g(\Theta(W))$ as a consequence of condition (1). This is a contradiction and we conclude that $\bigcap_{W \in \mathcal{N}_Z(z)} \Theta(W) \neq \varnothing$. □

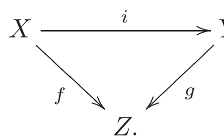
For completion of presentation we include the following proposition well known for the experts.

Proposition 3.5. *Let $f\colon X \to Z$ be a discrete and open mapping between compact manifolds. Then f is a complete spread.*

Proof. By Corollary 4.7 in [14] the map f is a spread; see also proof of Theorem 3.14. We prove that f is complete by showing that conditions (1) and (2) in Lemma 3.4 hold. Since f is an open and discrete map from a compact manifold X, the map f has finite multiplicity. Thus condition (2) holds since f is a spread. We then suppose that condition (1) does not hold. Then there exist $x \in X$ and $V \in \mathcal{N}_Z(f(x))$ so that the y-component U of $f^{-1}(V)$ satisfies $f(U) \subsetneq V$. Since V is connected there is a point $z \in \partial f(U) \cap V$. Since X is compact $\overline{U} \subset X$ is compact. Thus $z \in \overline{f(U)} \subset f(\overline{U})$ and there exists a point $x' \in \overline{U} \cap p^{-1}\{z\}$. This is a contradiction since $x' \in (\overline{U} \setminus U) \cap p^{-1}(V)$ and U is a component of $p^{-1}(V)$. We conclude that (1) holds and that f is a complete spread. $\square$

We recall that an open dense subset $Z' \subset Z$ is large if $Z \setminus Z'$ does not locally separate Z.

Definition 3.6. Let X, Y and Z be Hausdorff-spaces. A complete spread $g\colon Y \to Z$ is a *completion* of a spread $f\colon X \to Z$ if there is an embedding $i\colon X \to Y$ such that $i(X) \subset Y$ is large and



Two completions $g_1\colon Y_1 \to Z$ and $g_2\colon Y_2 \to Z$ of a spread $f\colon X \to Z$ are *equivalent* if there exists a homeomorphism $j\colon Y_1 \to Y_2$ such that $g_2 \circ j = g_1$ and $j(x) = x$ for all $x \in X$. In Sections 2 and 3 in Fox [7] it is shown that every spread has a completion and it is unique up to equivalence, see also Corollaries 2.8 and 7.4 in Montesinos [14].

3.2. Completed coverings

Let Y be a Hausdorff space and Z a manifold. Suppose $f\colon Y \to Z$ is a complete spread so that there are large subsets $Y' \subset Y$ and $Z' \subset Z$ for which $f' := f|Y'\colon Y' \to Z'$ is a covering. Then f is the completion of $f'\colon Y' \to Z$, and since $Z' \subset Z$ is large, f is an open surjection, see Teorema 2.1 in [4].

Definition 3.7. Let Y be a locally connected Hausdorff space and Z a manifold. A map $f\colon Y \to Z$ is a *completed covering* if there are large subsets $Y' \subset Y$ and $Z' \subset Z$ so that $f|Y'\colon Y' \to Z'$ is a covering and f is the completion of $f'|Y'\colon Y' \to Z$.

The results in [11] cover the following example.

Example 3.8. Let $f\colon X \to Z$ be an open and discrete simplicial map from a PL n-manifold X with triangulation K onto a PL n-manifold Z with triangulation L. Let K^{n-2} be the $(n-2)$-skeleton of K, L^{n-2} the $(n-2)$-skeleton of L, $X' = |K| \setminus |K^{n-2}|$ and $Z' = |L| \setminus |L^{n-2}|$. Then $f|X'\colon X' \to Z'$ is a covering and f is the completion of $f|X'\colon X' \to Z$.

Next we prove some basic properties of completed coverings. The following two properties will be used in the proof of Proposition 4.14 and Lemma 5.6.

Lemma 3.9. *Let $g\colon Y' \to Z'$ be a normal covering onto a large subset $Z' \subset Z$ of a manifold Z and $f\colon Y \to Z$ the completion of $g\colon Y' \to Z$. Then $f^{-1}(Z') = Y'$.*

Proof. Suppose there exists a point $y \in Y \setminus Y'$ so that $z := f(y) \in Z'$. Let $\Theta\colon \mathcal{N}_Z(z) \to \mathrm{Top}(Y)$ be a selection function of f satisfying

$$\bigcap_{W \in \mathcal{N}_Z(z)} \Theta(W) = \{y\}.$$

Since $Z' \subset Z$ is open, we have $\mathcal{N}_{Z'}(z) \subset \mathcal{N}_Z(z)$. Moreover, since $Y' \subset Y$ is large, the set $\Theta(W) \cap Y'$ is a component of $g^{-1}(W)$ for every $W \in \mathcal{N}_{Z'}(z)$. We define $\Theta'\colon \mathcal{N}_{Z'}(z) \to \mathrm{Top}(Y')$ by $\Theta'(W) = \Theta(W) \cap Y'$. Then Θ' is a selection function for $g\colon Y' \to Z'$. Since g is a covering, it is also a complete spread. Thus there exists $y' \in Y'$ so that

$$\bigcap_{W \in \mathcal{N}_{Z'}(z)} \Theta'(W) = \{y'\}.$$

This is a contradiction, since

$$y' \in \bigcap_{W \in \mathcal{N}_Z(z)} \Theta(W) = \{y\} \subset Y \setminus Y'. \qquad \square$$

The following lemma will also be used in the proof of Lemmas 3.12, 4.5 and 5.14, and Propositions 4.15 and 5.18.

Lemma 3.10. *Let $g\colon Y' \to Z'$ be a covering onto a large subset $Z' \subset Z$ of a manifold Z and $f\colon Y \to Z$ the completion of $g\colon Y' \to Z$. Let $U \subset Z$ be a connected and open set, and let V be a component of $f^{-1}(U)$. Then $f(V) = U$.*

Proof. Since $Z' \subset Z$ is large, the set $U \cap Z'$ is path-connected. Furthermore, since $Y' \subset Y$ is dense, we may fix a component W of $g^{-1}(U \cap Z')$ contained in $V \cap Y'$. Every path into $U \cap Z'$ has a total lift into W, since g is a covering map. Thus $U \cap Z' = \mathrm{Im}(g|W)$. In particular, $g|W\colon W \to Z' \cap U$ is a covering onto a large subset of U.

Since W is a connected and open set and f is a complete spread, there exists such a connected subset $V' \subset Y$ that $f|V'$ is the completion of the spread $g|W\colon W \to U$. Since $U \cap Z' \subset U$ is large, the map $f|V'\colon V' \to U$ is a surjective spread by Teorema 2.1 in [4]. Since V' is connected and $W \subset V$, we have $V' \subset V$. Thus $U \subset f(V') \subset f(V) \subset U$. $\square$

As final observations we show that compositions of completed coverings are completed coverings and that completed coverings factor to completed coverings under certain assumptions.

Lemma 3.11. *Let $f_1\colon X \to Y$ and $f_2\colon Y \to Z$ be completed coverings. Then $f_2 \circ f_1\colon X \to Z$ is a completed covering.*

Proof. Let $X' \subset X$, $Y' \subset Y$ and $Z' \subset Z$ be large subsets so that $g_1 := f_1|X' \colon X' \to Y'$ and $g_2 := f_1|Y' \colon Y' \to Z'$ are coverings. Then f_1 is the completion of $g_1 \colon X' \to Y$ and f_2 is the completion of $g_2 \colon Y' \to Z$ since f_1 and f_2 are complete spreads. Since $Z' \subset Z$ is large $f_2 \circ f_1 \colon X' \to Z$ is a spread. Moreover, since $X' \subset X$ is large, it is sufficient to show that $f_2 \circ f_1 \colon X \to Z$ is complete.

If there are no selection functions for $f_2 \circ f_1$, then $f_2 \circ f_1$ is trivially a complete spread. Then suppose $z \in Z$ and $\Theta \colon \mathcal{N}_Z(z) \to \mathrm{Top}(X)$ is a selection function for $f_2 \circ f_1$. We need to show that $\bigcap_{W \in \mathcal{N}_Z(z)} \Theta(W) \neq \varnothing$.

Since Θ is a selection function, there are for every $W \in \mathcal{N}_Z(z)$ components U_W of $f_2^{-1}(W)$ and V_{U_W} of $f_2^{-1}(U_W)$ so that $\Theta(W) = V_{U_W}$ and $U_{W_1} \subset U_{W_2}$ whenever $W_1 \subset W_2$. We define $\Theta_2 \colon \mathcal{N}_Z(z) \to \mathrm{Top}(Y)$ by $W \mapsto U_W$. Then Θ_2 is a selection function of f_2.

Since f_2 is a complete spread, there exists a point $y \in f_2^{-1}\{z\}$ satisfying $\bigcap_{W \in \mathcal{N}_Z(z)} \Theta_2(W) = \{y\}$. Thus there exists a selection function $\Theta_1 \colon \mathcal{N}_Y(y) \to \mathrm{Top}(X)$ satisfying $\Theta_1(U_W) = V_{U_W}$ for every $W \in \mathcal{N}_Z(z)$. Since f_1 is complete, there exists a point $x \in f_1^{-1}\{y\}$ satisfying $\bigcap_{U \in \mathcal{N}_Y(y)} \Theta_1(U) = \{x\}$. Now

$$\{x\} = \bigcap_{U \in \mathcal{N}_Y(y)} \Theta_1(U) \subseteq \bigcap_{W \in \mathcal{N}_Z(z)} V_{U_W} = \bigcap_{W \in \mathcal{N}_Z(z)} \Theta(W).$$

Thus

$$\bigcap_{W \in \mathcal{N}_Z(z)} \Theta(W) \neq \varnothing,$$

and we conclude that $f_2 \circ f_1$ is a complete spread. □

Lemma 3.12. *Let $f_1 \colon X \to Y$ be a completed covering and $f_2 \colon Y \to Z$ a spread between manifolds and suppose $Y'' \subset Y$ and $Z'' \subset Z$ are large subsets so that the map $f_2|Y'' \colon Y'' \to Z''$ is a covering. Then f_2 is a completed covering if $f := f_2 \circ f_1 \colon X \to Z$ is a completed covering.*

Proof. Let $X' \subset X$, $Y' \subset Y$ and $Z' \subset Z$ be large subsets so that the maps $g_1 := f_1|X' \colon X' \to Y'$ and $f_2|Y' \colon Y' \to Z'$ are coverings. Then f_1 is the completion of $g_1 \colon X' \to Y$, $g := f|X' \colon X' \to Z'$ is a covering and f is the completion of $g \colon X' \to Z$. Since f_2 is a spread and $Y' \subset Y$ is large, it is sufficient to show that $f_2 \colon Y \to Z$ is complete.

Suppose there are no selection functions for f_2. Then f_2 is trivially complete. Suppose then that $z \in Z$ and $\Theta \colon \mathcal{N}_Z(z) \to \mathrm{Top}(Y)$ is a selection function for f_2. Since f_1 is a completed covering, there exists (as a consequence of Lemma 3.10) a selection function $\Theta' \colon \mathcal{N}_Z(z) \to \mathrm{Top}(X)$ so that $\Theta'(W)$ is a component of $f_1^{-1}(\Theta(W))$ for every $W \in \mathcal{N}_Z(z)$. Since f is complete, there exists a point $x \in \bigcap_{W \in \mathcal{N}_Z(z)} \Theta'(W)$. Now,

$$f_1(x) \in f\Big(\bigcap_{W \in \mathcal{N}_Z(z)} \Theta'(W)\Big) \subseteq \bigcap_{W \in \mathcal{N}_Z(z)} f(\Theta'(W)) \subseteq \bigcap_{W \in \mathcal{N}_Z(z)} \Theta(W).$$

Thus f_2 is complete and we conclude that f_2 is a completed covering. □

3.3. Orbit maps

In this section we prove Theorem 1.4 in the introduction.

An open continuous map $p\colon Y \to Z$ is an *orbit map* if there exist a subgroup $G \subset \operatorname{Homeo}(Y)$ and a homeomorphism $\phi\colon Y/G \to Z$ so that the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\quad p \quad} & Z\,, \\ & {}_{\pi}\searrow \qquad \nearrow_{\phi} & \\ & Y/G & \end{array}$$

commutes for the canonical map $\pi\colon Y \to Y/G$

Lemma 3.13. *Let $f\colon Y \to Z$ be an orbit map. Let $U \subset Z$ be a connected and open set and let V_1 and V_2 be components of $f^{-1}(U)$ satisfying $f(V_1) = f(V_2) = U$. Then there exists a deck-transformation $\tau\colon Y \to Y$ satisfying $\tau(V_1) = V_2$.*

Proof. Let $z \in U$ and fix points $y_1 \in V_1 \cap f^{-1}\{z\}$ and $y_2 \in V_2 \cap f^{-1}\{z\}$. Since f is an orbit map there is a deck-transformation $\tau\colon Y \to Y$ for which $\tau(y_1) = y_2$. Since $\tau(f^{-1}(U)) = f^{-1}(U)$ and V_1 and V_2 are components of $f^{-1}(U)$, we have $\tau(V_1) \subset V_2$ and $\tau^{-1}(V_2) \subset V_1$. Thus $\tau(V_1) = V_2$. □

Theorem 3.14. *Let Y be a locally compact and locally connected Hausdorff space, let Z be a manifold and let $p\colon Y \to Z$ be a discrete orbit map. Suppose there are large subsets $Y' \subset Y$ and $Z' \subset Z$ so that $g\colon Y' \to Z'$ is a covering. Then p is a completed covering.*

Proof. Suppose $U \subset Y$ is open. To show that p is a spread, it is sufficient to show that for every $y \in U$ there exists a neighbourhood $V \subset p(U)$ of $p(y)$ for which the y-component D of $p^{-1}(V)$ is contained in U. Fix $y \in U$. Since Y is locally compact and p discrete there exists a neighbourhood $W \subset U$ of y, so that $\overline{W}$ is compact and $\partial W \cap p^{-1}\{p(y)\} = \varnothing$. Since p is open and continuous the set $p(W) \setminus p(\partial W) \subset Z$ is open. Since Z is locally connected the $p(y)$-component V of $p(W) \setminus p(\partial W)$ is open and since Y is locally connected the y-component D of $f^{-1}(V)$ is open. Since $D \cap \partial W = \varnothing$, we get $D \subset W \subset U$. We conclude that p is a spread.

Since $Y' \subset Y$ and $Z' \subset Z$ are large and $p|Y'\colon Y' \to Z'$ is a covering, to show that p is a completed covering it is sufficient to show that the spread p is complete.

For this let $z \in Z$. By Lemma 3.4 it is sufficient to show that $p(U) = W$ for every $W \in \mathcal{N}_Z(z)$ and U a component of $p^{-1}(W)$, and that there is a connected neighbourhood W_0 of z so that for every component V_0 of $p^{-1}(W_0)$ the set $V_0 \cap p^{-1}\{z\}$ is finite. Towards this let $W \in \mathcal{N}_Z(z)$. Since p is an orbit map p is necessarily onto. Hence there is a component U_1 of $p^{-1}(W)$ for which $z \in p(U_1)$. Let U_2 be a component of $p^{-1}(W)$. Since p is an orbit map the deck-transformation group $\mathcal{T}(p)$ acts transitively on fibers $p^{-1}\{z'\}$ for every $z' \in Z$. Thus Lemma 3.13 implies that $p(U_1) = p(U_2)$ or $p(U_1) \cap p(U_2) = \varnothing$. This implies that both $p(U_1)$ and $W \setminus p(U_1)$ are open sets. Since W is connected, this is only possible if $W \setminus p(U_1) = \varnothing$. Thus $p(U) = W$ for every component U of $p^{-1}(W)$.

Let then $y \in p^{-1}\{z\}$. Since Y is locally compact and locally connected there is a neighbourhood U_0 of y so that $p^{-1}\{z\} \cap U_0$ is finite and U_0 is the y-component of $p^{-1}(p(U_0))$. Further by Lemma 3.13, $\#(p^{-1}\{z\} \cap V_0) = \#(p^{-1}\{z\} \cap U_0)$ for every component V_0 of $p^{-1}(p(U_0))$. Thus p is a complete spread by Lemma 3.4. $\square$

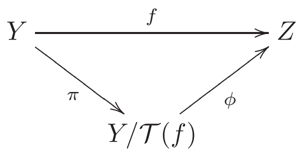
Proof of Theorem 1.4. Let $f\colon X \to Z$ be a mapping and $X' \subset X$ and $Z' \subset Z$ large subsets as in the statement. Similarly, let Y be a space and $p\colon Y \to X$ and $q\colon Y \to Z$ be orbit maps as in the statement. By assumption, there exists a large subset $Y' \subset Y$ for which the maps $p|Y'\colon Y' \to X'$ and $q|Y'\colon Y' \to X'$ are normal coverings. Thus, by Theorem 3.14, the maps p and q are completed coverings. Since $q = f \circ p$ and p and q are completed coverings, we have by Lemma 3.12, that the map f is a completed covering. $\square$

4. Existence of monodromy representations

4.1. Completed normal coverings and orbit maps

In this section we will prove Theorems 1.1 and 1.5 in the introduction.

Theorem 4.1. *Let $g\colon Y' \to Z'$ be a normal covering onto a large subset $Z' \subset Z$ of a manifold Z, $f\colon Y \to Z$ the completion of $g\colon Y' \to Z$ and $\mathcal{T}(f)$ the deck-transformation group of f. Then for the canonical maps π and ϕ*



the map ϕ is a homeomorphism if and only if f is discrete. In particular, f is an orbit map if and only if f is discrete.

Remark 4.2. Let $g\colon Y' \to Z'$ be a normal covering onto a large subset $Z' \subset Z$ of a manifold Z and $f\colon Y \to Z$ the completion of $g\colon Y' \to Z$. Then the deck transformations of g are restrictions of deck transformations of f by Lemma 3.9, that is, $\mathcal{T}(g) = \{\tau|Y'\colon Y' \to Y' : \tau \in \mathcal{T}(f)\}$, and each $\tau \in \mathcal{T}(f)$ is the completion of $\tau|Y'\colon Y' \to Y$, since $Y' \subset Y$ is large. In particular, the homomorphism

$$\varphi\colon \mathcal{T}(f) \to \mathcal{T}(g), \tau \mapsto \tau|Y',$$

is an isomorphism by the uniqueness of completions, see Corollary 7.8 in [14].

Proof of Theorem 4.1. We suppose first that f is discrete. Since ϕ is open and continuous, it is sufficient to show that it is bijective; we need to show that for every $z \in Z$ and $y_1, y_2 \in f^{-1}\{z\}$ there exists $\tau \in \mathcal{T}(f)$ so that $\tau(y_1) = y_2$.

Since f is discrete, there exists $W \in \mathcal{N}_Z(z)$ so that the y_i-components V_i of $f^{-1}(W)$ satisfy $f^{-1}\{z\} \cap V_i = \{y_i\}$ for $i \in \{1, 2\}$. Since g is a normal covering and $Y' \subset Y$ is large, there exists $\tau'\colon Y' \to Y'$ in $\mathcal{T}(g)$ for which $\tau'(Y' \cap V_1) = Y' \cap V_2$. The completion $\tau \in \mathcal{T}(f)$ of $\tau'\colon Y' \to Y$ satisfies $\tau(y_1) = y_2$.

We now suppose that ϕ is a homeomorphism. Then $\mathcal{T}(f)$ acts transitively on $f^{-1}\{z\}$ for every $z \in Z$. Arguing towards contradiction we assume f is not discrete. Then there exists a point $y \in Y$ so that $f^{-1}\{f(y)\} \cap U$ is an infinite set for every neighbourhood U of y. Let $z := f(y)$. Since $g\colon Y' \to Z'$ is a covering between manifolds, $\mathcal{T}(g)$ is countable. Since $\mathcal{T}(g) \cong \mathcal{T}(f)$, for a contradiction it is sufficient to show that $f^{-1}\{z\}$ is uncountable.

Towards a contradiction we suppose that the set $f^{-1}\{z\}$ is countable and let $f^{-1}\{z\} = \{y_0, y_1, \ldots\}$ be an enumeration of $f^{-1}\{z\}$. Then for every $k \in \mathbb{N}$ and neighbourhood U of y_k the set $f^{-1}\{z\} \cap U$ is infinite, since f is an orbit map.

We fix $W_0 \in \mathcal{N}_Z(z)$ and let U_0 be the y_0-component of $f^{-1}(W_0)$. Then we fix a point $y_1' := y_k \in \{y_1, y_2, \ldots\} \cap U_0$ for which $y_i \notin U_0, i < k$, and let $W_1 \in \mathcal{N}_Z(z)$ be such that the y_1'-component U_1 of $f^{-1}(W_1)$ satisfies $U_0 \supset U_1$ but $y_0 \notin U_1$. We proceed in this fashion to construct sequences $W_0 \supset W_1 \supset \cdots$ and $U_0 \supset U_1 \supset \cdots$ so that $\bigcap_{n=0}^{\infty} W_n = \{z\}$.

Since f is a complete spread, there exists a point y_∞ so that $\bigcap_{n=0}^{\infty} U_n = \{y_\infty\}$. However, by the construction, $\bigcap_{n=0}^{\infty} U_n \cap f^{-1}\{z\}$ is empty. This is a contradiction. Hence $f^{-1}\{z\}$ is uncountable and ϕ is not injective. Since ϕ is bijective, this is a contradiction and we conclude that f is discrete. □

Theorems 1.1 and 1.5 in the introduction are now easy corollaries of Theorem 4.1.

Proof of Theorem 1.1. Let $p\colon Y \to Z$ be a completed normal covering. Then there are large subsets $Y' \subset Y$ and $Z' \subset Z$ so that $p|Y'\colon Y' \to Z'$ is a normal covering and p is the completion of $p|Y'\colon Y' \to Z$. By Theorem 4.1, p is an orbit map if and only if p is a discrete map. □

Proof of Theorem 1.5. Let $f\colon X \to Z$ be a completed covering. Suppose f has a monodromy representation (Y, p, q). We need to show that f, p and q are discrete maps. Since p and q are completed coverings and orbit maps, p and q are completed normal coverings. Thus p and q are discrete maps by Theorem 4.1. Since p is an open surjection and $q = p \circ f$, the discreteness of f follows from the discreteness of q. □

4.2. Image of the branch set of a completed covering

In this section we prove Theorem 1.6 in the introduction. We recall that the branch set B_f of a completed covering $f\colon X \to Z$ is the set of points where f fails to be a local homeomorphism. We say that a completed covering $f\colon X \to Z$ is *uniformly discrete* if every $z \in Z$ has a neighbourhood U so that $f^{-1}\{z\} \cap D$ is a point for every component D of $f^{-1}(U)$.

Proposition 4.3. *Let $p\colon Y \to Z$ be a discrete completed normal covering. Then $p(B_p) \subset Z$ is closed for the branch set B_p of p.*

Proof. Let $U := Y \setminus p^{-1}(p(B_p))$. Since p is an orbit map by Theorem 4.1, we have $p^{-1}(p(B_p)) = B_p$ and $p(B_p) = Z \setminus p(U)$. Now $U \subset Y$ is open, since $B_p \subset Y$ is closed. Thus $p(B_p) = Z \setminus p(U)$ is closed, since p is an open map. □

Proposition 4.4. *Let $f\colon X \to Z$ be a completed covering between manifolds. Suppose there are uniformly discrete completed normal coverings $p\colon Y \to X$ and $q\colon Y \to Z$ satisfying $q := f \circ p$. Then $f(B_f) \subset Z$ is closed.*

For proving Proposition 4.4 we need the following lemma.

Lemma 4.5. *Let $p\colon Y \to Z$ be a uniformly discrete completed normal covering. Let $z \in Z$ and $U, V \in \mathcal{N}_Z(z)$ be such that $U \subset V$ and $p^{-1}\{z\} \cap D$ is a point for every component D of $p^{-1}(V)$. Let D be a component of $p^{-1}(V)$ and C a component of $p^{-1}(U)$ so that $C \subset D$. Then the groups*

$$H_D := \{\tau \in \mathcal{T}(p) : \tau(D) = D\} \quad \textit{and} \quad H_C := \{\tau \in \mathcal{T}(p) : \tau(C) = C\}$$

satisfy $H_D = H_C$.

Proof. Let $Y' \subset Y$ and $Z' \subset Z$ be large subsets so that $g := p|Y'\colon Y' \to Z'$ is a normal covering. Then $D \cap Y'$ is a component of $V \cap Z'$ and $C \cap Y'$ is a component of $U \cap Z'$. By Theorem 4.1 p is an orbit map, and in particular, the map $\mathcal{T}(p) \to \mathcal{T}(g), \tau \mapsto \tau|Y'$ is an isomorphism by Corollary 7.8 in [14]. Thus it is sufficient to show that $H'_D = H'_C$ for the groups

$$H'_D := \{\tau \in \mathcal{T}(g) : \tau(D \cap Y') = D \cap Y'\}$$

and

$$H'_C := \{\tau \in \mathcal{T}(g) : \tau(C \cap Y') = C \cap Y'\}.$$

Let $z' \in Z' \cap U$. Since $p^{-1}\{z\} \cap D$ is a point, we have $p^{-1}(U) \cap D = C$ as a consequence of Lemma 3.10. Thus $g^{-1}\{z'\} \cap (D \cap Y') = g^{-1}\{z'\} \cap (C \cap Y')$. Let $y' \in g^{-1}\{z'\} \cap (C \cap Y')$ and let $\tau_y\colon Y' \to Y'$ be for every $y \in p^{-1}\{z'\}$ the unique deck-transformation of g that satisfies $\tau_y(y') = y$. Then

$$\begin{aligned} H'_D &= \{\tau_y : y \in g^{-1}\{z'\} \cap (D \cap Y')\} \\ &= \{\tau_y : y \in g^{-1}\{z'\} \cap (C \cap Y')\} = H'_C. \end{aligned}$$

Thus $H_D = H_C$. □

Proof of Proposition 4.4. Let $z \in Z \setminus f(B_f)$ and $U \in \mathcal{N}_Z(z)$ be a neighbourhood of z so that the set $q^{-1}\{z\} \cap D$ is a point for every component D of $q^{-1}(U)$. We note that, since $p\colon Y \to X$ is onto X, $U \subset Z \setminus f(B_f)$ if $f|p(D)$ is an injection for every component D of $q^{-1}(U)$.

Let D be a component of $q^{-1}(U)$ and

$$H_D := \{\tau \in \mathcal{T}(q) : \tau(D) = D\}.$$

By Theorem 4.1, q is an orbit map. Thus $f|p(D)$ is an injection if $H_D \subset \mathcal{T}(p)$.

Let $y \in q^{-1}\{z\} \cap D$ and $x := p(y)$. Since $x \in f^{-1}(Z \setminus f(B_f))$ there exists $U_0 \in \mathcal{N}_Z(z)$ so that $f|V$ is an injection for the x-component V of $f^{-1}(U_0)$ and so that $U_0 \subset U$. Let C be the y-component of $q^{-1}(U_0)$. Then $f|p(C) = f|V$ by Lemma 3.10. Thus

$$H_C := \{\tau \in \mathcal{T}(q) : \tau(C) = C\} \subset \mathcal{T}(p),$$

since $f|p(C)$ is an injection.

By Lemma 4.5, $H_D = H_C$, since $C \subset D$. Thus $H_D = H_C \subset \mathcal{T}(p)$. Thus $f|p(D)$ is an injection. Thus $U \subset Z \setminus f(B_f)$ and $f(B_f) \subset Z$ is closed. □

Towards proving Theorem 1.6 we make the following observation.

Lemma 4.6. *Let $p\colon Y \to Z$ be a completed normal covering. Then p is a discrete map if and only if p is a uniformly discrete map.*

Proof. Suppose p is discrete. Then, by Theorem 4.1, the map p is an orbit map. Let $y \in Y$ and $z := p(y)$. Then, since p is both a discrete map and a spread, there exists a neighbourhood V of y so that V is the y-component of $p^{-1}(p(V))$ and $V \cap p^{-1}\{p(y)\} = \{y\}$. Let U be a component of $p^{-1}(p(V))$. By Lemma 3.13 there is a deck-transformation $\tau\colon Y \to Y$ of p so that $\tau(V) = U$. Thus

$$p^{-1}\{p(y)\} \cap U = \{\tau(y)\}$$

and p is uniformly discrete.

Suppose now that every $z \in Z$ has a neighbourhood W such that $p^{-1}\{z\} \cap V$ is a point for each component V of $p^{-1}(W)$. Then p is discrete, since p is a spread. □

Proof of Theorem 1.6. Let $f\colon X \to Z$ be a completed covering between manifolds and (Y, p, q) a monodromy representation of f. As a consequence of Theorem 4.1 and Lemma 4.6 the maps $p\colon Y \to X$ and $q\colon Y \to Z$ are uniformly discrete completed normal coverings. By Proposition 4.3, $p(B_p) \subset X$ and $q(B_q) \subset Z$ are closed. By Proposition 4.4, $f(B_f) \subset Z$ is closed, since $q = f \circ p$. □

4.3. Discrete completed normal coverings

In this section we prove Theorems 1.7 and 1.9 in the introduction. Let $g\colon Y' \to Z'$ be a normal covering onto a large subset $Z' \subset Z$ of a PL manifold Z, $p\colon Y \to Z$ the completion of $g\colon Y' \to Z$, $z_0 \in Z'$ and $y_0 \in g^{-1}\{z_0\}$. In general p is not discrete, see Example 10.6. in [14], but if p has locally finite multiplicity, then p is discrete, see Theorem 9.14. in [14]. The discreteness of the map p depends on the properties of the group $\pi(Z', z_0)/g_*(\pi(Y', y_0))$ and the manifold Z. To capture this relation we define stability of the covering g with respect to Z and show that $p\colon Y \to Z$ is discrete if and only if g is stable with respect to Z.

Let $\mathcal{N}_Z(z; z_0)$ be the set of open and connected neighbourhoods $U \subset Z$ of z satisfying $z_0 \in U$. We denote by $\iota_{U,Z'}$ the inclusion $U \hookrightarrow Z'$.

Definition 4.7. Let Z be a PL manifold, $Z' \subset Z$ a large subset, $z_0 \in Z'$ and $N \subset \pi(Z', z_0)$ a normal subgroup. A subset $U \in \mathcal{N}_Z(z; z_0)$ is a (Z', N)-*stable* neighbourhood of z if

$$\mathrm{Im}(\pi \circ (\iota_{V \cap Z', Z'})_*) = \mathrm{Im}(\pi \circ (\iota_{U \cap Z', Z'})_*)$$

for every $V \in \mathcal{N}_Z(z; z_0)$ contained in U;

$$\begin{array}{ccc} \pi(V \cap Z', z_0) & & \pi(U \cap Z', z_0) \\ & \searrow^{(\iota_{V\cap Z', Z'})_*} \quad {}^{(\iota_{U\cap Z', Z'})_*}\swarrow & \\ & \pi(Z', z_0) & \\ & \downarrow \pi & \\ & \pi(Z', z_0)/N, & \end{array}$$

where $\pi\colon \pi(Z', z_0) \to \pi(Z', z_0)/N$ is the canonical quotient map.

Definition 4.8. Let Z be a PL manifold and $Z' \subset Z$ a large subset, $z_0 \in Z'$ and $N \subset \pi(Z', z_0)$ a normal subgroup. The manifold Z is (Z', N)-*stable* if every point $z \in Z$ has a (Z', N)-stable neighbourhood.

Remark 4.9. Let Z be a PL manifold, $Z' \subset Z$ a large subset, $z_0 \in Z'$ and N' and N normal subgroups of $\pi(Z', z_0)$ so that $N' \subset N$. Then Z is (Z', N)-stable if Z is (Z', N')-stable. Indeed, if $U \in \mathcal{N}_Z(z; z_0)$ is a (Z', N')-stable neighbourhood of z, then U is a (Z', N)-stable neighbourhood of z.

Definition 4.10. Let Z be a PL manifold and $g\colon Y' \to Z'$ a covering onto a subset $Z' \subset Z$. Then g is *stable* with respect to Z, if $Z' \subset Z$ is large and Z is $(Z', \mathrm{Ker}(\sigma_g))$-stable.

We say that a completed covering $f\colon Y \to Z$ onto a PL manifold Z is *stable* if there is a large subset $Y' \subset Y$ so that $g := f|Y'\colon Y' \to f(Y')$ is a covering and g is stable with respect to Z. We say that a completed normal covering $p\colon Y \to Z$ onto a PL manifold Z is *stable* if there is a large subset $Y' \subset Y$ so that $h := p|Y'\colon Y' \to p(Y')$ is a normal covering and h is stable with respect to Z.

Theorem 4.11. *Let $p\colon Y \to Z$ be a completed normal covering onto a PL manifold Z, $Y' \subset Y$ and $Z' \subset Z$ large subsets so that $g := p|Y'\colon Y' \to Z'$ is a normal covering, $z_0 \in Z'$ and $y_0 \in g^{-1}\{z_0\}$. Then the following conditions are equivalent:*

(a) *$p\colon Y \to Z$ is discrete,*

(b) *$p\colon Y \to Z$ is uniformly discrete and*

(c) *$g\colon Y' \to g(Y')$ is stable with respect to Z.*

The proof consists of three parts. Lemma 4.6 proves $(a) \Leftrightarrow (b)$, Proposition 4.14 proves $(c) \Rightarrow (b)$ and Proposition 4.15 proves $(b) \Rightarrow (c)$.

Next we are going to state two lemmas for proving Propositions 4.14 and 4.15.

Lemma 4.12. *Let Z be a manifold, $Z' \subset Z$ large and U and V simply-connected open subsets of Z so that $U \subset Z'$ and the set $U \cap V$ is a non-empty path-connected set. Let $z_0 \in U$ and $z_1 \in U \cap V \cap Z'$. Then for every path $\gamma\colon z_0 \curvearrowright z_1$ in $(U \cup V) \cap Z'$ and every loop $\alpha\colon (S_1, e_0) \to ((U \cup V) \cap Z', z_0)$ there exists a loop $\beta\colon (S^1, e_0) \to (V \cap Z', z_1)$ so that $[\alpha] = [\gamma\beta\gamma^{\leftarrow}]$ in the group $\pi((U \cup V) \cap Z', z_0)$.*

Proof. We denote $W := U \cup V$. Let $\gamma\colon z_0 \curvearrowright z_1$ be a path in $W \cap Z'$ and $\alpha\colon (S^1, e_0) \to (W \cap Z', z_0)$ a loop. Since $U \subset Z'$, we have $W \cap Z' = U \cup (V \cap Z')$. Moreover, the sets U, $V \cap Z'$ and $U \cap (V \cap Z')$ are path-connected open subsets of $W \cap Z'$, since $Z' \subset Z$ is large. Since U is simply-connected, the homomorphism $\iota_*\colon \pi(V \cap Z', z_1) \to \pi(W \cap Z', z_1)$ induced by the inclusion $\iota\colon V \cap Z' \to W \cap Z'$ is, by the Seifert–van Kampen theorem, an epimorphism, see Theorem 4.1 in [13]. Hence there exists a loop $\beta\colon (S^1, e_0) \to (V \cap Z', z_1)$ so that $[\beta] = [\gamma^{\leftarrow}\alpha\gamma]$ in $\pi(W \cap Z', z_1)$. Now $[\alpha] = [\gamma\beta\gamma^{\leftarrow}]$ in $\pi(W \cap Z', z_0)$. □

For the terminology of simplicial structures we refer to Chapter 2 in Hatcher [9]. We denote by $\sigma(v_0, \ldots, v_n)$ the simplex in $\mathbb{R}^k$ spanned by vertices $v_0, \ldots, v_n \in \mathbb{R}^k$. The open simplex $\mathrm{Int}(\sigma) := \sigma \setminus \partial(\sigma)$ we call the simplicial interior of σ. For a simplicial complex K consisting of simplices in $\mathbb{R}^k$ the polyhedron $|K|$ is a subset of $\mathbb{R}^k$. For $z \in |K|$ we call $\mathrm{St}(z) := \bigcup_{\sigma \in K; z \in \sigma} \mathrm{Int}(\sigma)$ the star neighbourhood of z (with respect to K).

A *triangulation* of a manifold Z is a simplicial complex T satisfying $Z \approx |T|$. We say that a manifold is a *PL manifold* if it has a triangulation. If Z is a PL manifold and T a triangulation of Z with simplices in $\mathbb{R}^k$, then we assume from now on that

$$Z = |T| \subset \mathbb{R}^k.$$

Let K be a simplicial complex. We consider a basis of topology of $|K|$ related to star neighbourhoods of points in $|K|$. Let $x_0 \in |K|$, $t \in (0, 1]$ and let $\sigma = \sigma(v_0, \ldots, v_m) \in K$ be a simplex containing x_0. Let $\tau(\sigma, x_0, t)$ be the simplex spanned by vertices $v_j' = x_0 + t(v_j - x_0)$, $j \in \{0, \ldots, m\}$. The *t-star* of x_0 is the set

$$\mathrm{St}_t(x_0) = \bigcup_{\{\sigma \in K : x_0 \in \sigma\}} \mathrm{Int}(\tau(\sigma, x_0, t)),$$

where $\mathrm{Int}(\tau(\sigma, x_0, t))$ is the simplicial interior of $\tau(\sigma, x_0, t)$.

The collection $(\mathrm{St}_{1/n}(x_0))_{n \in \mathbb{N}}$ is a neighbourhood basis of the point $x_0 \in |K|$. For every $x_0 \in |K|$, there exists a contraction

$$H\colon \mathrm{St}(x_0) \times [0, 1] \to \mathrm{St}(x_0)$$

of the open star $\mathrm{St}(x_0)$ to x_0 for which $H(\mathrm{St}(x_0) \times \{1/n\}) = \mathrm{St}_{1/n}(x_0)$ for every $n \in \mathbb{N}$ and $H(\{x\} \times [0, 1]) \subset [x, x_0]$ for all $x \in \mathrm{St}(x_0)$. The collection $(\mathrm{St}_{1/n}(x_0))_{n \in \mathbb{N}}$ also has the following technical property.

Lemma 4.13. *Let Z be a PL manifold, $Z' \subset Z$ a large subset and T a triangulation of Z having a vertex $z_0 \in Z'$. Let $z \in Z$ be a point and $\mathrm{St}_{1/m}(z)$ the $1/m$-star of z for $m > 1$. Suppose $X \subset Z$ is an open connected subset satisfying $z_0 \in X$ and $\mathrm{St}_{1/m}(z) \subset X$. Then there exists a simply-connected open set $U \subset X \cap Z'$ for which $z_0 \in U$ and for which the set $U \cap \mathrm{St}_{1/m}(z)$ is non-empty and path-connected.*

Proof. Suppose $z_0 \in \overline{\mathrm{St}_{1/m}(z)}$. Since z_0 is a vertex of T and $m > 1$, we have $z = z_0$. Thus $z = z_0 \in \mathrm{St}_{1/m}(z) \cap Z'$. Since $\mathrm{St}_{1/m}(z) \cap Z' \subset Z$ is open, there exists a simply-connected neighbourhood $U \subset \mathrm{St}_{1/m}(z) \cap Z'$ of z. This proves the claim in the case $z_0 \in \overline{\mathrm{St}_{1/m}(z)}$.

Suppose then that $z_0 \notin \overline{\mathrm{St}_{1/m}(z)}$. Since $Z' \subset Z$ is large and $\mathrm{St}_{1/m}(z) \subsetneq X$, there exists a point $z_1 \in \partial(\mathrm{St}_{1/m}(z)) \cap X \cap Z'$, that is a simplicial interior point of an n-simplex $\sigma \in T$ and a face f of $\tau(\sigma, z, 1/m)$. Since $|T| \subset \mathbb{R}^k$ for some $k \in \mathbb{N}$, the Euclidean metric of $\mathbb{R}^k$ induces a metric on $|T|$. Let

$$r_1 := \min\{\mathrm{dist}(z_1, f) \mid f \text{ is a face of } \tau(\sigma, z, 1/m) \text{ and } z_1 \notin f\}$$

and

$$r_2 := \mathrm{dist}(z_1, Z \setminus (Z' \cap X)).$$

Then $r := \min\{r_1, r_2\} > 0$.

We fix $k \in \mathbb{N}$ and a 1-subcomplex L in the k^{th} barycentric subdivision $\mathrm{Bd}^k(T)$ of T so that $|L| \subset |T|$ is a simple closed curve in $X \cap Z' \cap (Z \setminus \overline{\mathrm{St}_{1/m}(z)})$ from z_0 to $\mathrm{Int}(\sigma)$ satisfying $\mathrm{dist}(z_1, |L| \cap \sigma) < r/4$.

We fix $l \geq k + 2$ large enough so that for

$$W := \bigcup_{z' \in |L|} \mathrm{St}^l(z'),$$

where $\mathrm{St}^l(z')$ is for every $z' \in |L|$ the open star of z' in $\mathrm{Bd}^l(T)$, we have

$$\overline{W} \subset X \cap Z' \cap (Z \setminus \overline{\mathrm{St}_{1/m}(z)}).$$

Then $|L|$ is a strong deformation retract of W since $l \geq k + 2$; see regular neighbourhoods in Section 2 in [17]. In particular, W is simply-connected since $|L|$ is simply-connected.

Let $K < \mathrm{Bd}^l(T)$ be the $(n-1)$-subcomplex satisfying $\partial W = |K|$ and let S be the collection of $(n-1)$-simplices $\tau \in K$ for which

$$\mathrm{dist}(z_1, \overline{W} \cap \sigma) = \mathrm{dist}(z_1, \tau).$$

Since z_1 is an interior point of f and $\mathrm{dist}(z_1, \overline{W} \cap \sigma) < r_1$, there exists $\tau \in S$ and $z_2 \in \mathrm{Int}(\tau)$ for which the line segment $[z_1, z_2] \subset \mathrm{Int}(\sigma)$ has length at most $r/2$ and

$$[z_1, z_2] \cap (\mathrm{St}_{1/m}(z) \cup W) = \{z_1, z_2\}.$$

Now, for all $t < r/2$ the t-neighbourhood $W_t \subset \mathrm{Int}(\sigma)$ of $[z_1, z_2]$ satisfies $W_t \subset X \cap Z'$ and the intersections $W \cap W_t$ and $\mathrm{St}_{1/m}(z) \cap W_t$ are non-empty. Since $z_2 \in \mathrm{Int}(\tau)$, the intersections $W \cap W_{t_0}$ and $\mathrm{St}_{1/m}(z) \cap W_{t_0}$ are non-empty intersections of two convex sets for some $t_0 < r/2$. Thus $W \cap W_{t_0}$ and $\mathrm{St}_{1/m}(z) \cap W_{t_0}$ are path-connected. Since the sets W_{t_0} and W are simply-connected and the set $W \cap W_{t_0}$ path-connected, the set $U = W \cup W_{t_0} \subset Z'$ is simply-connected by the Seifert–van Kampen theorem. Further, $z_0 \in W \subset U$. Since $\mathrm{St}_{1/m}(z) \cap W_{t_0}$ is path-connected and $U \cap \mathrm{St}_{1/m}(z) = \mathrm{St}_{1/m}(z) \cap W_{t_0}$, this proves the claim. □

Proposition 4.14. *Let $p\colon Y \to Z$ be a completed normal covering onto a PL manifold Z and $Y' \subset Y$ and $Z' \subset Z$ large subsets so that $g := p|Y' \colon Y' \to Z'$ is a normal covering. If g is stable with respect to Z, then $p\colon Y \to Z$ is a uniformly discrete mapping.*

Proof. Let $z_0 \in Z'$ and $y_0 \in g^{-1}\{z_0\}$. Let T be a triangulation of Z so that z_0 is a vertex of T. Let $z \in Z$. Since g is stable with respect to Z the manifold Z is $(Z', g_*(\pi(Y', y_0))$-stable. Let V be a $(Z', g_*(\pi(Y', y_0)))$-stable neighbourhood of z and D a component of $p^{-1}(V)$. It suffices to show that $p^{-1}\{z\} \cap D$ is a point. By Lemma 3.10, we have $p(D) = V$, and hence $p^{-1}\{z\} \cap D \neq \varnothing$.

We suppose towards a contradiction that there are distinct points d_1 and d_2 in $p^{-1}\{z\} \cap D$. Using points d_1 and d_2, we construct a loop $\alpha_1\colon (S^1, e_0) \to (V \cap Z', z_0)$ and an open set $V_1 \in \mathcal{N}_Z(z; z_0)$ so that $V_1 \subset V$ and

$$[\alpha_1]g_*(\pi(Y', y_0)) \notin \operatorname{Im}(\pi \circ \iota_{Z' \cap V_1, Z'})).$$

This contradicts the $(Z', g_*(\pi(Y', y_0)))$-stability of Z and proves the claim.

We begin by choosing the neighbourhood $V_1 \subset V$ of z. For all $n \in \mathbb{N}$ let $\mathrm{St}_{1/n}(z)$ be the $1/n$-star of z in the triangulation T. Then $(\mathrm{St}_{1/n}(z))_{n\in\mathbb{N}}$ is a neighbourhood basis for z. Since $p\colon Y \to Z$ is a spread and Y a Hausdorff space, there exists $n \geq 2$ so that $\mathrm{St}_{1/n}(z) \subset V$ and the d_i-components C_i, $i \in \{1, 2\}$, of $p^{-1}(\mathrm{St}_{1/n}(z))$ are disjoint. By Lemma 4.13, there is such an open simply-connected set $W \subset V \cap Z'$ containing z_0 that the sets $W \cup \mathrm{St}_{1/n}(z)$ and $W \cap \mathrm{St}_{1/n}(z)$ are path-connected. Hence

$$V_1 = W \cup \mathrm{St}_{1/n}(z) \subset V$$

is an open connected neighbourhood of z containing z_0. The configuration is illustrated in Figure 1.

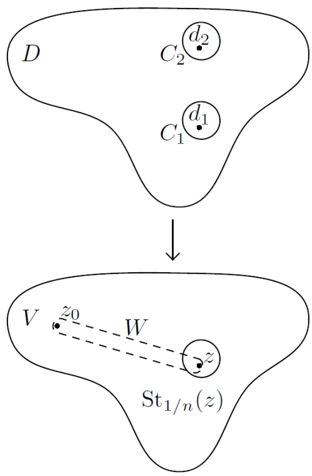


Figure 1. Neighborhood V of z and a component D of $p^{-1}(V)$.

We continue by choosing a loop $\alpha_1\colon (S^1, e_0) \to (V \cap Z', z_0)$. Fix $z_1 \in W \cap \mathrm{St}_{1/n}(z) \cap Z'$. Since W is a connected set containing points z_0 and z_1, there is a path $\gamma\colon z_0 \curvearrowright z_1$ in W. Since $\mathrm{St}_{1/n}(z)$ is connected, Lemma 3.10 implies that there are points $c_i \in p^{-1}\{z_1\} \cap C_i$ for $i \in \{1,2\}$. By Lemma 3.9, $p^{-1}(Z') = Y'$. Thus $c_1, c_2 \in D \cap Y'$. Since $Y' \subset Y$ is large, the set $Y \setminus Y'$ does not locally separate Y and the set $D \cap Y'$ is path-connected. Let $\beta\colon [0,1] \to D \cap Y'$ be a path $\beta\colon c_1 \curvearrowright c_2$. Then $g \circ \beta\colon [0,1] \to V \cap Z'$ is a loop at z_1. We set $\alpha_1 = \gamma(g \circ \beta)\gamma^{\leftarrow}$.

Suppose there is a loop $\alpha_2\colon (S^1, e_0) \to (V_1 \cap Z', z_0)$ satisfying

$$[\alpha_2] \in [\gamma(g \circ \beta)\gamma^{\leftarrow}]g_*(\pi(Y', y_0)).$$

Since $W \subset Z'$ and γ is a path in $W \subset V_1 \cap Z'$, there is by Lemma 4.12 a loop $\alpha_3\colon (S^1, e_0) \to (\mathrm{St}_{1/n}(z) \cap Z', z_1)$ satisfying

$$[\alpha_2] = [\gamma\alpha_3\gamma^{\leftarrow}].$$

Next we change the base point of Y as follows. Since $g\colon Y' \to Z'$ is a normal covering and $W \subset Z'$, the path $\gamma^{\leftarrow}\colon [0,1] \to W$ has a lift $\widetilde{\gamma^{\leftarrow}}_{c_1}\colon [0,1] \to Y'$. We set $y_1 := \widetilde{\gamma^{\leftarrow}}_{c_1}(1)$. Since $g\colon Y' \to Z'$ is a normal covering, we have

$$g_*(\pi(Y', y_1)) = g_*(\pi(Y', y_0)).$$

Hence

$$[\gamma\alpha_3\gamma^{\leftarrow}] \in [\gamma(g \circ \beta)\gamma^{\leftarrow}]g_*(\pi(Y', y_0)) = [\gamma(g \circ \beta)\gamma^{\leftarrow}]g_*(\pi(Y', y_1)).$$

Thus the lifts $\widetilde{\gamma\alpha_3\gamma^{\leftarrow}}_{y_1}$ and $\gamma\widetilde{(g \circ \beta)}\gamma^{\leftarrow}{}_{y_1}$ have a common endpoint. By the uniqueness of lifts,

$$\widetilde{\alpha_3}_{c_1}(1) = \widetilde{(g \circ \beta)}_{c_1}(1) = \beta(1) = c_2.$$

On the other hand, since α_3 is a path in $\mathrm{St}_{1/n}(z)$, the lift $\widetilde{\alpha_3}_{c_1}$ is a path in $p^{-1}(\mathrm{St}_{1/n}(z))$. Since C_1 is a component of $p^{-1}(\mathrm{St}_{1/n}(z))$, this implies that $\widetilde{\alpha_3}_{c_1}[0,1] \subset C_1$. Thus $c_2 = \widetilde{\alpha_3}_{c_1}(1) \in C_1$, which is a contradiction. Hence $p^{-1}\{p(y)\} \cap D$ is a point. □

Proposition 4.15. *Let $p\colon Y \to Z$ be a completed normal covering onto a PL manifold Z, $Y' \subset Y$ and $Z' \subset Z$ large subsets so that $g := p|Y'\colon Y' \to Z'$ is a normal covering. If p is uniformly discrete, then g is stable with respect to Z.*

Proof. Let $z_0 \in Z'$ and $y_0 \in g^{-1}\{z_0\}$. Let T be a triangulation of Z so that z_0 is a vertex of T. We prove the statement by constructing for every $z \in Z$ a $(Z', g_*(\pi(Y', y_0))$-stable neighbourhood of z.

Suppose $z = z_0$. Since the map p is uniformly discrete and $z_0 \in Z'$, there exists a simply-connected neighbourhood $V \subset Z'$ of z so that $p^{-1}\{z\} \cap D$ is a point for each component D of $p^{-1}(V)$. Now V is a $(Z', g_*(\pi(Y', y_0))$-stable neighbourhood of z, since $V \subset Z'$ and V is simply-connected.

Let then $z \in Z \setminus \{z_0\}$. For all $n \in \mathbb{N}$ let $\mathrm{St}_{1/n}(z)$ be the $1/n$-star of z with respect to the triangulation T. Since $(\mathrm{St}_{1/n}(z))_{n\in\mathbb{N}}$ is a neighbourhood basis of z

and p is uniformly discrete, there exists $n \geq 2$ so that $p^{-1}\{z\} \cap D$ is a point for each component D of $p^{-1}(\mathrm{St}_{1/n}(z))$. By Lemma 4.13, there is an open simply-connected set $U \subset Z'$ containing z_0 so that $U \cup \mathrm{St}_{1/n}(z)$ and $U \cap \mathrm{St}_{1/n}(z)$ are path-connected. Let $V := U \cup \mathrm{St}_{1/n}(z)$.

Towards showing that V is a $(Z', g_*(\pi(Y', y_0))$-stable neighbourhood of z, let $\alpha\colon (S^1, e_0) \to (V \cap Z', z_0)$ be a loop and let $V_1 \in \mathcal{N}_Z(z; z_0)$ be an open set contained in V. It is sufficient to show that there exists a loop $\beta\colon (S^1, e_0) \to (V_1 \cap Z', z_0)$ satisfying $[\beta] g_*(\pi(Y', y_0)) = [\alpha] g_*(\pi(Y', y_0))$.

We choose first a suitable representative from the class $[\alpha]$ as follows. Let W be the z-component of $\mathrm{St}_{1/n}(z) \cap V_1$. Since $Z' \subset Z$ is large, there exist a point $z_1 \in W \cap Z'$ and a path $\gamma\colon z_0 \curvearrowright z_1$ in $V_1 \cap Z'$. By Lemma 4.12, there exists a loop $\alpha_1\colon (S^1, e_0) \to (\mathrm{St}_{1/n}(z) \cap Z', z_1)$ for which $[\alpha] = [\gamma \alpha_1 \gamma^{\leftarrow}]$ in $\pi(V \cap Z', z_0)$.

Denote the endpoint of the lift $\widetilde{\gamma}_{y_0}$ in g by y_1 and the endpoint of the lift $\widetilde{\alpha_1}_{y_1}$ in g by y_2. Let D_1 be the y_1-component of $p^{-1}(\mathrm{St}_{1/n}(z))$. We proceed by showing that there exists a loop $\alpha_2\colon (S^1, e_0) \to (W \cap Z', z_1)$ so that the lift $\widetilde{\alpha_2}_{y_1}$ in g that begins at y_1 ends at y_2.

Let $C_1 \subset D_1$ be the y_1-component of $p^{-1}(W)$. Since W is a connected neighbourhood of z, every component of $p^{-1}(W)$ intersects the set $p^{-1}\{z\}$ by Lemma 3.10. Since $W \subset \mathrm{St}_{1/n}(z)$ and the set $p^{-1}\{z\} \cap D_1$ is a point, this implies

$$p^{-1}(W) \cap D_1 = C_1.$$

Hence the points y_1 and y_2 belong to $C_1 \cap Y'$. Since $Y \setminus Y'$ does not locally separate Y, there is a path $\beta\colon y_1 \curvearrowright y_2$ in $C_1 \cap Y'$. Let $\alpha_2 := g \circ \beta$.

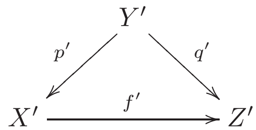
Since the lifts $\widetilde{\gamma \alpha_1 \gamma^{\leftarrow}}_{y_0}$ and $\widetilde{\gamma \alpha_2 \gamma^{\leftarrow}}_{y_0}$ have a common endpoint,

$$[\gamma \alpha_2 \gamma^{\leftarrow}] \in [\gamma \alpha_1 \gamma^{\leftarrow}] g_*(\pi(Y', y_0)) = [\alpha] g_*(\pi(Y', y_0)).$$

Since $\gamma \alpha_2 \gamma^{\leftarrow}$ is a loop in $V_1 \cap Z'$ at z_0, V is a $(Z', g_*(\pi(Y', y_0)))$-stable neighbourhood of the point z. Since this holds for every $z \in Z$ the manifold Z is $(Z', g_*(\pi(Y', y_0)))$-stable. □

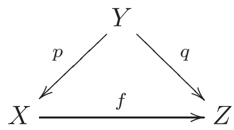
This concludes the proof of Theorem 4.11. After the following two consequences of Theorem 4.11, we will be ready to prove Theorems 1.2, 1.7 and 1.9 in the introduction.

Corollary 4.16. *Let $f\colon X \to Z$ be a completed covering between PL manifolds, $X' \subset X$ and $Z' \subset Z$ large subsets and*



a commutative diagram of coverings so that p' and q' are normal coverings.

Let $z_0 \in Z'$, $y_0 \in q'^{-1}\{z_0\}$ *and* $N = q'_*(\pi(Y', y_0))$. *Let*



be a commutative diagram of open mappings so that p *is the completion of* $p' \colon Y' \to X$ *and* q *is the completion of* $q' \colon Y' \to Z$. *Then, the maps* p *and* q *are orbit maps if and only if* Z *is* (Z', N)*-stable.*

Let $f \colon X \to Z$ be a completed covering between manifolds and $q \colon Y \to Z$ a completed covering that is an orbit map. We say that q is *natural of* f if there are large subsets $X' \subset X$, $Y' \subset Y$ and $Z' \subset Z$ so that $g := f|X' \colon X' \to Z'$ and $h := q|Y' \colon Y' \to Z'$ are coverings satisfying $\mathrm{Ker}(\sigma_g) = \mathrm{Ker}(\sigma_h)$. Recall that the monodromy group of f is defined as the monodromy group of g.

Theorem 4.17. *Let* $f \colon X \to Z$ *be a stable completed covering between PL manifolds. Then* f *has a monodromy representation* (Y, p, q), *where* q *is natural of* f.

Proof. Let $X' \subset X$ and $Z' \subset Z$ be large subsets so that $g := f|X' \colon X' \to Z'$ is a covering and Z is $(Z', \mathrm{Ker}(\sigma_g))$-stable. By Theorem 2.1 there is a normal covering $p' \colon Y' \to X'$ so that $q' := g \circ p'$ is a normal covering satisfying $q'_*(\pi(Y', y_0)) = \mathrm{Ker}(\sigma_g)$. Thus $\mathrm{Ker}(\sigma_{q'}) = q'_*(\pi(Y', y_0)) = \mathrm{Ker}(\sigma_g)$ and the deck-transformation group $\mathcal{T}(q')$ is isomorphic to the monodromy group of g.

Let $p \colon Y \to X$ be the completion of $p' \colon Y' \to X$ and $q := f \circ p \colon Y \to Z$. Then q is the completion of $q' \colon Y' \to Z$ by Proposition 3.11 and q is natural of f. Since $\mathcal{T}(q')$ is isomorphic to the monodromy group of g, it is sufficient to show that p and q are orbit maps and $\mathcal{T}(q) \cong \mathcal{T}(q')$. Since Z is $(Z', \mathrm{Ker}(\sigma_{q'}))$-stable, q is discrete by Theorem 4.11. Thus q is an orbit map by Theorem 4.1, and $\mathcal{T}(q) \cong \mathcal{T}(q')$ by Remark 4.2. Since $q = f \circ p$ is discrete, p is discrete. Thus p is an orbit map by Theorem 4.1. □

Proof of Theorem 1.2. Let $p \colon Y \to Z$ be a completed normal covering onto a PL manifold Z. Let $Y' \subset Y$ and $Z' \subset Z$ be large subsets so that $g := p|Y' \colon Y' \to Z'$ is a normal covering. Since p is discrete, $g \colon Y' \to Z$ is stable by Theorem 4.11. Thus p is stable.

Suppose then that p is a stable completed normal covering. Then there are such large subsets $Y'' \subset Y$ and $Z'' \subset Z$ that $h := p|Y'' \colon Y'' \to Z''$ is a stable normal covering with respect to Z. Thus the map p is discrete by Theorem 4.11. □

Proof of Theorem 1.7. Suppose $f \colon X \to Z$ is a completed covering between PL manifolds and (Y, p, q) is a monodromy representation of f. By Corollary 4.16, the maps p and q are stable completed normal coverings. Since q is a stable, as a consequence of Theorem 2.1 and Remark 4.9, also f is stable. □

Proof of Theorem 1.9. By Theorem 1.7 and Theorem 4.17, a completed covering $f \colon X \to Z$ between PL manifolds is a stable completed covering if and only if f has a monodromy representation (Y, p, q). □

We end this section with a corollary of Theorem 4.16. We recall that for a simplicial n-complex K, K^{n-2} is the codimension 2 skeleton of K.

Corollary 4.18. *Let $f\colon X \to Z$ be a surjective, open and discrete simplicial map from a PL n-manifold X with triangulation K onto a PL n-manifold Z with triangulation L. Let also $z_0 \in |L| \setminus |L^{n-2}|$ and $x_0 \in f^{-1}\{z_0\}$. Then $g := f||K| \setminus |K^{n-2}|\colon |K| \setminus |K^{n-2}| \to |L| \setminus |L^{n-2}|$ is a covering and for every normal subgroup $N \subset \pi(|L| \setminus |L^{n-2}|, z_0)$ contained in $\mathrm{Ker}(\sigma_g)$ there is a commutative diagram*

$$\begin{array}{ccc} & Y & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} \\ X & \xrightarrow{\quad f \quad} & Z \end{array}$$

of completed coverings so that p and q are orbit maps and the deck-transformation groups $\mathcal{T}(p)$ and $\mathcal{T}(q)$ satisfy

$$\mathcal{T}(p) \cong \pi(|K| \setminus |K^{n-2}|, x_0)/N \quad \textit{and} \quad \mathcal{T}(q) \cong \pi(|L| \setminus |L^{n-2}|, z_0)/N.$$

Proof. Since $f\colon X \to Z$ is an open discrete simplicial map the branch set of f is contained in $|K^{n-2}|$. In particular $g := f||K| \setminus |K^{n-2}|\colon |K| \setminus |K^{n-2}| \to |L| \setminus |L^{n-2}|$ is a covering and f is the completion of $f\colon |K| \setminus |K^{n-2}| \to Z$. Clearly, Z is $(|L| \setminus |L^{n-2}|, \{e\})$-stable for the trivial group $\{e\} \subset \pi(|L| \setminus |L^{n-2}|, z_0)$. Further, Z is $(|L| \setminus |L^{n-2}|, N)$-stable by Remark 4.9. We conclude the statement from Theorem 2.1 and Theorem 4.16. □

Remark 4.19. If the map $p\colon Y \to X$ in Corollary 4.18 has locally finite multiplicity, then the space Y is a locally finite simplicial complex and p and q are simplicial maps, see Section 6 in [7]. However, in the next section we show that p need not to have locally finite multiplicity.

5. Existence of locally compact monodromy representations

5.1. Uniformly bounded local multiplicities

In this section we show the existence of a monodromy representation that is not locally compact. Let $f\colon X \to Z$ be a completed covering and $z \in Z$. We say that f has *uniformly bounded local multiplicities* in $f^{-1}\{z\} \subset X$ if there exist a neighbourhood $U \in \mathcal{N}_Z(z)$ so that every point $z' \in U$ satisfies

$$\sup\{\#(f^{-1}\{z'\} \cap D) : D \text{ is a component of } f^{-1}(U)\} < \infty.$$

Lemma 5.1. *Suppose $f\colon X \to Z$ is a completed covering and (Y, p, q) is a locally compact monodromy representation of f. Then f has uniformly bounded local multiplicities in $f^{-1}\{z\} \subset X$ for every $z \in Z$.*

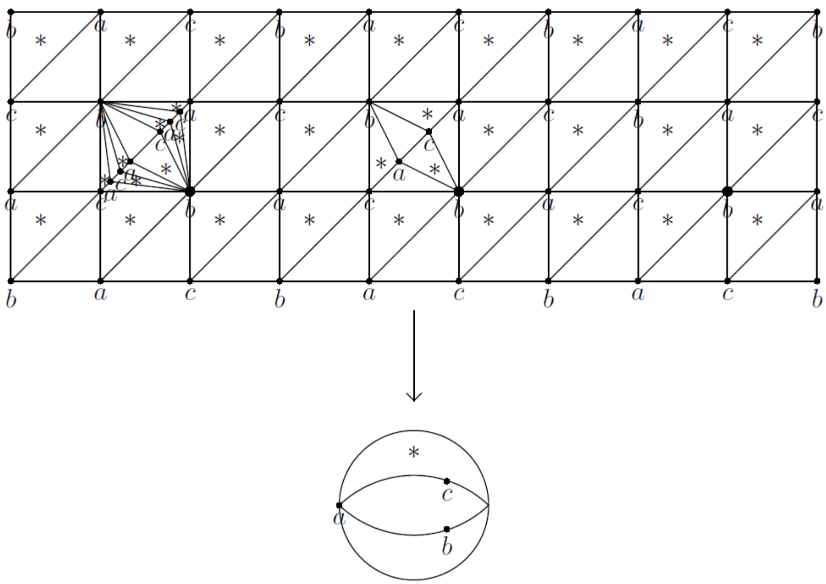


FIGURE 2. A simplicial completed covering from $\mathbb{R}^2$ to S^2.

Proof. Let $z \in Z$ and $y \in q^{-1}\{z\}$. Since $q: Y \to Z$ is an open and discrete map, the map q has locally finite multiplicity, since Y is locally compact. Thus there exists such $U \in \mathcal{N}_Z(z)$ that $q|D_y$ has finite multiplicity for the y-component D_y of $q^{-1}(U)$. Let $z' \in U$ and $k_{z'} := \#(q^{-1}\{z'\} \cap D_y)$. Since q is an orbit map,

$$\sup\{\#(q^{-1}\{z'\} \cap D) : D \text{ is a component of } q^{-1}(U)\} = k_{z'}.$$

Hence

$$\sup\{\#(f^{-1}\{z'\} \cap E) : E \text{ is a component of } f^{-1}(U)\} \leq k_{z'} < \infty,$$

since $q = f \circ p$. Thus f has uniformly bounded local multiplicities in $f^{-1}\{z\}$. □

Remark 5.2. Let $f: X \to Z$ be a completed covering between manifolds. Suppose there exists $z \in Z$ so that f does not have uniformly bounded local multiplicities in $f^{-1}\{z\} \subset X$. Then, as a consequence of Lemma 5.1, f is not an orbit map.

Proposition 5.3. *There exists a simplicial completed covering $f: \mathbb{R}^2 \to S^2$ onto the 2-sphere S^2 having a monodromy representation, which is not locally compact.*

Proof. Let $f: \mathbb{R}^2 \to S^2$ be a simplicial completed covering for which $f(B_f) \subset S^2$ is finite and there exists a point $z \in f(B_f)$ so that f does not have uniformly bounded local multiplicities in $f^{-1}\{z\} \subset \mathbb{R}^2$; see Figure 2 and take $z = b$.

Since $f(B_f)$ is finite, f is stable. Thus f has a monodromy representation (Y, p, q) by Theorem 4.17. By Proposition 5.1, (Y, p, q) is not locally compact. □

5.2. Completed normal coverings with locally finite multiplicity

Let Z be a PL manifold and $f\colon X \to Z$ a stable completed covering. Then there are large subsets $Y' \subset Y$ and $Z' \subset Z$ so that $g := p|Y'\colon Y' \to Z'$ is a covering and Z is $(Z', \mathrm{Ker}(\sigma_g))$-stable.

Fix $z_0 \in Z'$. Let $z \in Z$ and $U \in \mathcal{N}_Z(z; z_0)$ be a $(Z', \mathrm{Ker}(\sigma_g))$-stable neighbourhood of z. Let $\pi\colon \pi(Z', z_0) \to \pi(Z', z_0)/\mathrm{Ker}(\sigma_g)$ be the canonical map.

Definition 5.4. A group $H \subset \pi(Z', z_0)/\mathrm{Ker}(\sigma_g)$ is a *local monodromy group* of $f\colon X \to Z$ at z if there exists a $(Z', \mathrm{Ker}(\sigma_g))$-stable neighbourhood $U \in \mathcal{N}_Z(z; z_0)$ of z so that $H = \mathrm{Im}(\pi \circ (\iota_{Z' \cap U, Z'})_*)$ for the inclusion $\iota_{Z' \cap U, Z'}$.

For a $(Z', \mathrm{Ker}(\sigma_g))$-stable neighbourhood $U \in \mathcal{N}_Z(z; z_0)$ we denote

$$\mathrm{Mono}_f(U; Z') := \mathrm{Im}(\pi \circ (\iota_{Z' \cap U, Z'})_*).$$

Remark 5.5. If $U, V \in \mathcal{N}_Z(z; z_0)$ are $(Z', \mathrm{Ker}(\sigma_g))$-stable neighbourhoods of $z \in Z$, then

$$\mathrm{Mono}_f(V; Z') = \mathrm{Mono}_f(U; Z'),$$

if $V \subset U$.

Lemma 5.6. *Let Z be a PL manifold and $p\colon Y \to Z$ a stable completed normal covering. Let $Y' \subset Y$ and $Z' \subset Z$ be such large subsets that $g := p|Y'\colon Y' \to Z'$ is a normal covering and Z is $(Z', \mathrm{Ker}(\sigma_g))$-stable. Let $y \in Y$, $z := p(y)$ and V be a $(Z', \mathrm{Ker}(\sigma_g)))$-stable neighbourhood of the point z. Let Z have a triangulation T having a vertex $z_0 \in Z'$ and let $n \geq 2$ be such that $\mathrm{St}_{1/n}(z) \subset V$. Let D_y be the y-component of $p^{-1}(\mathrm{St}_{1/n}(z))$. Then*

(a) $\#(p^{-1}\{z'\} \cap D_y) = \#\mathrm{Mono}_p(V; Z')$ *for* $z' \in \mathrm{St}_{1/n}(z) \cap Z'$ *and*

(b) $\#(p^{-1}\{z'\} \cap D_y) \leq \#\mathrm{Mono}_p(V; Z')$ *for* $z' \in \mathrm{St}_{1/n}(z)$.

Proof. We first prove claim (a). Let z' and z'' be points in $\mathrm{St}_{1/n}(z) \cap Z'$. Then $\#(p^{-1}\{z'\} \cap D_y) = \#(p^{-1}\{z''\} \cap D_y)$, since $\mathrm{St}_{1/n}(z) \cap Z'$ is connected and $D_y \cap Y'$ is a component of $g^{-1}(\mathrm{St}_{1/n}(z) \cap Z')$. Thus we only need to show that there is a point $z_1 \in \mathrm{St}_{1/n}(z) \cap Z'$ satisfying $\#(p^{-1}\{z_1\} \cap D_y) = \#\mathrm{Mono}_p(V)$.

By Lemma 4.13, there is an open set $U \subset V \cap Z'$ so that $U \cap \mathrm{St}_{1/n}(z)$ is path-connected and $V_n := U \cup \mathrm{St}_{1/n}(z) \in \mathcal{N}_Z(z; z_0)$. Since $V_n \subset V$, the set V_n is a $(Z', \mathrm{Ker}(\sigma_g))$-stable neighbourhood of z. Hence

$$\#\mathrm{Im}(\pi \circ (\iota_{V_n \cap Z', Z'})_*) = \#\mathrm{Im}(\pi \circ (\iota_{V \cap Z', Z'})_*).$$

Let $z_1 \in V_n \cap Z'$ and let $\beta\colon [0,1] \to V_n \cap Z'$ be a path $\beta\colon z_0 \curvearrowright z_1$. Let $y_0 \in p^{-1}\{z_0\}$ and $y_1 := \widetilde{\beta}_{y_0}(1)$. Let D_{y_1} be the y_1-component of $p^{-1}(\mathrm{St}_{1/n}(z))$. Since p is an orbit map, there is by Lemma 3.13 a deck-transformation $\tau \in \mathcal{T}(p)$ satisfying $\tau(D_y) = D_{y_1}$. Hence,

$$\#(p^{-1}\{z_1\} \cap D_y) = \#(p^{-1}\{z_1\} \cap D_{y_1}).$$

Since g is a normal covering, we have $\mathrm{Ker}(\sigma_g) = \mathrm{Im}(g_*)$. By Lemma 4.12 there exist for every loop $\gamma\colon (S^1, e_0) \to (V_n \cap Z', z_0)$ a loop $\alpha\colon (S^1, e_0) \to (\mathrm{St}_{1/n}(z) \cap Z', z_1)$ satisfying $[\gamma] = [\beta\alpha\beta^{\leftarrow}]$ in $V_n \cap Z'$. By Lemma 3.9, $g^{-1}(Z') = Y'$. Hence,

$$\#(p^{-1}\{z_1\} \cap D_{y_1}) = \#\mathrm{Im}(\pi \circ (\iota_{V_n \cap Z', Z'})_*).$$

We conclude that

$$\begin{aligned}\#(p^{-1}\{z_1\} \cap D_y) &= \#(p^{-1}\{z_1\} \cap D_{y_1}) = \#\mathrm{Im}(\pi \circ (\iota_{V_n \cap Z', Z'})_*)\\ &= \#\mathrm{Im}(\pi \circ (\iota_{V \cap Z', Z'})_*) = \#\mathrm{Mono}_p(V).\end{aligned}$$

We then prove claim (b). Since $p^{-1}\{z'\}$ is a countable set for every $z' \in Z$, the statements holds trivially if $\#\mathrm{Mono}_p(V) = \infty$. Suppose $\#\mathrm{Mono}_p(V) = k$ for $k \in \mathbb{N}$.

Towards a contradiction suppose that $z' \in \mathrm{St}_{1/n}(z) \setminus Z'$ and that there exist $k+1$ points $y_0, \ldots, y_k$ in $p^{-1}\{z'\} \cap D_y$. Let $U \subset \mathrm{St}_{1/n}(z)$ be an open connected set so that the y_i-components $D'_{y_i} \subset D$ of $p^{-1}(U)$ are pairwise disjoint. By Lemma 3.10, $p(D'_{y_i}) = U$ for every $i \in \{0, \ldots, k\}$. Since $Z' \subset Z$ is large, there exists a point $z'_1 \in U \cap Z'$ for which $\#(p^{-1}\{z'\} \cap D_y) \geq k$. This is a contradiction with (a). Hence $\#(p^{-1}\{z'\} \cap D_y) \leq k$ for all $z' \in \mathrm{St}_{1/n}(z)$. □

Theorem 5.7. *Let Z be a PL manifold and $p\colon Y \to Z$ a discrete completed normal covering. Then p has locally finite multiplicity if and only if p has a finite local monodromy group at each point of Z.*

Proof. The map p is an orbit map by Theorem 4.1. Thus there is by Theorem 4.11 large subsets $Y' \subset Y$ and $Z' \subset Z$ so that $g := p|Y'\colon Y' \to Z'$ is a normal covering and Z is $(Z', \mathrm{Ker}(\sigma_g))$-stable. Fix a triangulation T of Z having a vertex $z_0 \in Z'$.

Suppose first that p has locally finite multiplicity. Let $z \in Z$ and $y \in p^{-1}\{z\}$. Let D_n be the y-component of $p^{-1}(\mathrm{St}_{1/n}(z))$ for every $n \geq 1$. Then $(D_n)_{n\in\mathbb{N}}$ is a neighbourhood basis of y, since p is a spread.

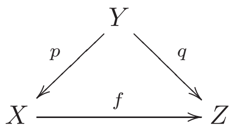
Let $V \in \mathcal{N}_Z(z; z_0)$ be a $(Z', \mathrm{Ker}(\sigma_g))$-stable neighbourhood of z. Since p has locally finite multiplicity, there exists $n \geq 2$ so that $\mathrm{St}_{1/n}(z) \subset V$ and $\#(p^{-1}\{z'\} \cap D_n) < \infty$ for every $z' \in \mathrm{St}_{1/n}(z)$. Let $z_1 \in \mathrm{St}_{1/n}(z) \cap Z'$. By Lemma 5.6,

$$\#\mathrm{Mono}_p(V) = \#(p^{-1}\{z_1\} \cap D_n) < \infty.$$

Suppose then that p has a finite local monodromy group at each $z \in Z$. Let $y \in Y$ and $z := p(y)$. Let $V \in \mathcal{N}_Z(z; z_0)$ be a $(Z', \mathrm{Ker}(\sigma_g))$-stable neighbourhood of the point z so that $\#\mathrm{Mono}_p(V) < \infty$. Let $n \geq 2$ be such that $\mathrm{St}_{1/n}(z) \subset V$ and let D_y be the y-component of $p^{-1}(\mathrm{St}_{1/n}(z))$. By Lemma 5.6, $p|D_y$ has finite multiplicity. Thus p has locally finite multiplicity. □

Next we show regularity results for local monodromy.

Theorem 5.8. *Let $f\colon X \to Z$ be a stable completed covering between PL manifolds. Let*



be a commutative diagram of completed coverings so that p and q are orbit maps and q is natural of f. Then q has locally finite multiplicity if and only if every $z \in Z$ has a finite local monodromy group of f at z.

Proof. By naturality of q we may fix large subsets $X' \subset X$, $Y' \subset Y$ and $Z' \subset Z$ for which $g := f|X'\colon X' \to Z'$ and $h := q|Y'\colon Y' \to Z'$ are coverings satisfying $\mathrm{Ker}(\sigma_g) = \mathrm{Ker}(\sigma_h)$.

Suppose first that f has a finite local monodromy group of f at each point of Z. Let $z \in Z$. Then there exists a $(Z', \mathrm{Ker}(\sigma_g))$-stable neighbourhood $V \in \mathcal{N}_Z(z; z_0)$ of z so that $\#\mathrm{Mono}(V) < \infty$. Since $\mathrm{Ker}(\sigma_g) = \mathrm{Ker}(\sigma_h)$, V is a $(Z', \mathrm{Ker}(\sigma_h))$-stable neighbourhood of z. Since $\#\mathrm{Mono}(V) < \infty$, q has a finite local monodromy group at z. Thus q has locally finite multiplicity by Theorem 5.7.

Suppose then that q has locally finite multiplicity. Let $z \in Z$. By Theorem 5.7, q has a finite local monodromy group at z. Since $\mathrm{Ker}(\sigma_g) = \mathrm{Ker}(\sigma_h)$, the finite local monodromy group of q at z is a finite local monodromy group of f at z. □

As a direct consequence of Theorem 5.8 we obtain an analogy of Theorem 1.9.

Corollary 5.9. *Let $f\colon X \to Z$ be a stable completed covering between PL manifolds, so that f has a finite local monodromy group at each $z \in Z$. Then there exists a monodromy representation (Y, p, q) of f, where q has locally finite multiplicity.*

We conclude this section by defining the homotopical index $\mathcal{H}(y, p)$ of y in p for a completed normal covering $p\colon Y \to Z$ onto a PL manifold Z and $y \in Y$. By Lemmas 5.6 and 5.7 we may define as follows.

Definition 5.10. Let Z be a PL manifold, $p\colon Y \to Z$ a completed normal covering with locally finite multiplicity and $y \in Y$. Let $Y' \subset Y$ and $Z' \subset Z$ be large subsets so that $g := p|Y'\colon Y' \to Z'$ is a normal covering. Let T be a triangulation of Z having a vertex $z_0 \in Z'$. Let $n \geq 2$ be such that $\mathrm{St}_{1/n}(p(y))$ is contained in a $(Z', \mathrm{Ker}(\sigma_g))$-stable neighbourhood of $p(y)$. Let D be the y-component of $\mathrm{St}_{1/n}(p(y))$ and $z' \in \mathrm{St}_{1/n}(p(y)) \cap Z'$. Then

$$\mathcal{H}(p, y) := \#(p^{-1}\{z'\} \cap D)$$

is the *homotopical index* of y in p.

When $p\colon Y \to Z$, $y \in Y$ and $D \subset Y$ are as in Definition 5.10 we get the following upper bound for the homotopical indices of points in D.

Lemma 5.11. *Let $y' \in D$ and $\mathcal{H}(y', p)$ be the homotopical index of y' in p. Then $\mathcal{H}(y', p) \leq \mathcal{H}(y, p)$.*

Proof. Let $y' \in D$. Let $m \geq 2$ be such that $\mathrm{St}_{1/m}(p(y')) \subset \mathrm{St}_{1/n}(p(y))$ and such that $\mathrm{St}_{1/m}(p(y'))$ is contained in a $(Z', \mathrm{Ker}(\sigma_g))$-stable neighbourhood of $p(y')$. Let $C \subset D$ be the y'-component of $\mathrm{St}_{1/m}(p(y))$ and $z' \in \mathrm{St}_{1/m}(p(y')) \cap \mathrm{St}_{1/n}(p(y)) \cap Z'$. Then by Lemma 5.6,

$$\mathcal{H}(p, y') = \#(p^{-1}\{z'\} \cap C) \leq \#(p^{-1}\{z'\} \cap D) = \mathcal{H}(p, y),$$

since $C \subset D$. □

5.3. Metrization of monodromy representations

In this section we consider PL manifolds as length manifolds and prove Theorem 1.8 and Theorem 1.11 in the introduction.

Let $|K| \subset \mathbb{R}^n$ be the polyhedron of a simplicial complex K and

$$\ell(\gamma) := \sup \left\{ \sum_{i=1}^{k-1} |\gamma(t_i) - \gamma(t_{i+1})| : t_1, \ldots, t_k \in [0,1], t_1 < \cdots < t_k, k \in \mathbb{N} \right\}$$

the *length* of γ for every path γ in $|K|$. A path γ in $|K|$ is called *rectifiable* if the length $\ell(\gamma)$ is finite. Since $|K| \subset \mathbb{R}^n$ is a polyhedron for $z_1, z_2 \in |K|$ there exists a rectifiable path $\gamma\colon z_1 \curvearrowright z_2$. In particular, the formula

$$d_s(z_1, z_2) = \inf_\gamma \{\ell(\gamma) \mid \gamma\colon z_1 \curvearrowright z_2\}$$

defines a path-metric on $|K|$. For a detailed study of path length structures see Section 1 in [8].

We note that the path-metric d_s coincides with the metric of $\mathbb{R}^n$ when restricted to any simplex $\sigma \in K$. By local finiteness of K the metric topology induced on $|K|$ by d_s coincides with the relative topology of $|K|$ as a subset of $\mathbb{R}^n$.

For a PL manifold Z, there exist a simplicial complex K and an embedding $\iota\colon Z \hookrightarrow \mathbb{R}^n$ satisfying $\iota(Z) = |K|$. For the next theorem we assume $Z = (|K|, d_s)$. For the next theorem we also assume the following: if $p\colon Y \to Z$ is a completed normal covering and there are fixed large subsets $Y' \subset Y$ and $Z' \subset Z$ so that $p|Y'\colon Y' \to Z'$ is a normal covering, then K has a vertex $z_0 \in Z'$.

Theorem 5.12. *Let Z be a PL manifold, $p\colon Y \to Z$ a completed normal covering with locally finite multiplicity. Then there exists a path metric d_s^* on Y so that*

(a) *the topology induced by d_s^* on Y coincide with the original topology of Y,*

(b) *(Y, d_s^*) is a locally proper metric space,*

(c) *$p\colon (Y, d_s^*) \to (Z, d_s)$ is a 1-Lipschitz map and*

(d) *every deck-transformation $\tau \in \mathcal{T}(p)$ is an isometry with respect to d_s^*.*

The proof consists of Proposition 5.17 and Proposition 5.19. We obtain metric d_s^* in Theorem 5.12 by lifting paths. In Lemma 5.15 we show an analogy of the result II.3.4 in [16] concerning local path-lifting for open discrete maps between manifolds. To obtain this result we first prove Lemmas 5.13 and 5.14.

In Lemmas 5.13, 5.14 and 5.15 we assume that Z is a PL manifold and $p\colon Y \to Z$ is a completed normal covering with locally finite multiplicity. Since p has locally finite multiplicity, it is discrete by Theorem 9.14 in [14]. We recall also that p is an orbit map by Theorem 4.1, and that there are by Theorem 4.11 large subsets $Y' \subset Y$ and $Z' \subset Z$ so that $g := p|Y'\colon Y' \to Z'$ is a normal covering and Z is $(Z', \mathrm{Ker}(\sigma_g))$-stable.

Lemma 5.13. *Let $z \in Z$ and $y \in p^{-1}\{z\}$. Let $m \geq 2$ be such that $\mathrm{St}_{1/m}(z)$ is contained in a $(Z', \mathrm{Ker}(\sigma_g))$-stable neighbourhood of z and let D be the y-component of $p^{-1}(\mathrm{St}_{1/m}(z))$. Let $\gamma\colon [0,1] \to \mathrm{St}_{1/m}(z)$ be a path and let $\gamma'\colon [0,1] \to D$ be a function satisfying $p \circ \gamma' = \gamma$. Then γ' is continuous at every $t \in [0,1]$ satisfying $\mathcal{H}(\gamma'(t); p) = \mathcal{H}(y; p)$.*

Proof. Let $\mathrm{St}_{1/n}(\gamma(t))$ be the $1/n$-star of $\gamma(t)$ and C_n the $\gamma'(t)$ component of $p^{-1}(\mathrm{St}_{1/n}(\gamma(t)))$ for every $n \geq 2$. Let $n_0 \geq 2$ be such that $\mathrm{St}_{1/n_0}(\gamma(t))$ is contained in $\mathrm{St}_{1/m}(z)$ and in a $(Z', \mathrm{Ker}(\sigma_p))$-stable neighbourhood of $\gamma(t)$. Since $\mathcal{H}(\gamma'(t); p) = \mathcal{H}(y; p)$, we have by Lemma 5.6, $p^{-1}(\mathrm{St}_{1/n}(\gamma(t))) \cap D = C_n$ for every $n \geq n_0$. Since γ is continuous, there exists for every $n \geq n_0$ such $\epsilon_n > 0$ that $\gamma(t - \epsilon_n, t + \epsilon_n) \subset \mathrm{St}_{1/n}(\gamma(t))$. Now for every $n \geq n_0$ we have $\gamma'(t - \epsilon_n, t + \epsilon_n) \subset C_n$. Since $(C_n)_{n \geq n_0}$ is a neighbourhood basis at $\gamma'(t)$ the function γ' is continuous at t. □

Lemma 5.14. *Let $h\colon (0,1) \to Z$ be a continuous map. Suppose there exists for every $t \in (0,1)$ such $\epsilon > 0$ that for every $y \in p^{-1}\{h(t)\}$ there exists a continuous map $h'\colon (t - \epsilon, t + \epsilon) \to Y$ satisfying $h'(t) = y$ and $p \circ h' = h|(t - \epsilon, t + \epsilon)$. Then there exists for every $y \in p^{-1}\{h(1/2)\}$ a continuous map $\widetilde{h}\colon (0,1) \to Y$ satisfying $\widetilde{h}(1/2) = y$ and $p \circ \widetilde{h} = h$.*

Proof. Let $y \in p^{-1}\{h(1/2)\}$. By Zorn's lemma there exists a maximal connected set $I \subset (0,1), 1/2 \in I$, for which there exists a continuous map $h'\colon I \to Y$ satisfying $p \circ h'(1/2) = y$ and $p \circ h' = h|I$. By the existence of local lifts I is an open interval $(a,b) \subset [0,1]$. We need to show that $a = 0$ and $b = 1$.

Fix a map $h'\colon I \to Y$ satisfying $h'(1/2) = y$ and $p \circ h' = h|I$. Suppose $b \neq 1$. Let $m \geq 2$ be such that $\mathrm{St}_{1/m}(h(b))$ is contained in a $(Z', \mathrm{Ker}(\sigma_g))$-stable neighbourhood of $h(b)$. Let $\delta \in (0, (b-a)/2)$ be such that $h(b - 2\delta, b) \subset \mathrm{St}_{1/m}(h(b))$. Let D be the component of $p^{-1}(\mathrm{St}_{1/m}(h(b)))$ that contains $h'(b - 2\delta, b)$ and $y' \in D \cap p^{-1}\{h(b)\}$. Let $h''\colon (a, b] \to Y$ be the extension of h' defined by $h''(b) = y'$.

By Lemma 5.13, $h''|[b-\delta, b]\colon [b-\delta, b] \to D$ is continuous. Thus h'' is continuous. This is a contradiction with the maximality of $I = (a,b) \subset [0,1]$. Thus $b = 1$. By a similar argument $a = 1$. □

Lemma 5.15. *Let $z \in Z$ and let $m \geq 2$ be such that $\mathrm{St}_{1/m}(z)$ is contained in a $(Z', \mathrm{Ker}(\sigma_g))$-stable neighbourhood of z. Let $\gamma\colon [0,1] \to \mathrm{St}_{1/m}(z)$ be a path satisfying $\gamma(0) = z$ and $y \in p^{-1}\{z\}$. Then γ has a lift $\widetilde{\gamma}\colon [0,1] \to Y$ satisfying $\gamma(0) = y$.*

Proof. Let $k := \mathcal{H}(y, p)$. We prove the existence of the lift $\widetilde{\gamma}_y$ by induction on k. Let D be the y-component of $p^{-1}(\mathrm{St}_{1/m}(z))$. By Lemma 3.10, $p(D) = \mathrm{St}_{1/m}(z)$.

Suppose $k = 1$. Then $p|D\colon D \to \mathrm{St}_{1/m}(z)$ is a homeomorphism. Thus γ has a lift $\widetilde{\gamma}$ satisfying $\widetilde{\gamma}(0) = y$.

Suppose the statement holds for all $j \le k-1$. Let

$$F := \{z' \in \mathrm{St}_{1/m}(z) : \mathcal{H}(y',p) = \mathcal{H}(y,p) \text{ for every } y' \in p^{-1}\{z'\}\}$$

and

$$U := \{z' \in \mathrm{St}_{1/m}(z) : \mathcal{H}(y',p) < \mathcal{H}(y,p) \text{ for every } y' \in p^{-1}\{z'\}\}.$$

By Lemma 5.6 and Lemma 5.11, $U = \mathrm{St}_{1/m}(z) \setminus F$, $U \subset \mathrm{St}_{1/m}(z)$ is open and $p^{-1}\{z'\} \cap D$ is a point for every $z' \in F$.

Now $\gamma^{-1}(U) \subset [0,1]$ is a countable union $\bigcup_{n\in N} I_n$ of disjoint open intervals $I_n \subset [0,1]$. We use Lemma 5.14 to show that there exists for every $n \in N$ a continuous map $\widetilde{\gamma|I_n}\colon I_n \to D$ satisfying $p \circ \widetilde{\gamma|I_n} = \gamma|I_n$.

Let $n \in N$ and $t \in I_n$. Then there exists such $p \ge 2$ that $\mathrm{St}_{1/p}(\gamma(t))$ is contained in a $(Z', \mathrm{Ker}(\sigma_g))$-stable neighbourhood of $\gamma(t)$ and such $\epsilon > 0$ that $(t-\epsilon, t+\epsilon) \subset I_n$ and $\gamma(t-\epsilon, t+\epsilon) \subset \mathrm{St}_{1/p}(\gamma(t))$. Let $h := I_n|(t-\epsilon, t+\epsilon)$.

Let $y' \in p^{-1}(\gamma(t))$. Since $\gamma(t) \in U$, we have $\mathcal{H}(y',p) < k$. Thus, by the induction hypotheses there are continuous maps $h_1'\colon (t-\epsilon, t] \to Y$ and $h_2'\colon [t, t+\epsilon) \to Y$ satisfying $h_1'(t) = y' = h_2'(t)$, $p \circ h_1' = h|(t-\epsilon, t]$ and $p \circ h_2' = h|[t, t+\epsilon)$. Hence there exists a continuous map $h'\colon (t-\epsilon, t+\epsilon) \to Y$ satisfying $h'(t) = y'$ and $p \circ h' = h$.

Let $t_0 \in I_n$ be the center of I_n and $y'' \in p^{-1}\{\gamma(t_0)\} \cap D$. By Lemma 5.14, there exists a continuous map $\widetilde{\gamma|I_n}\colon I_n \to Y$ satisfying $p \circ \widetilde{\gamma|I_n} = \gamma|I_n$ and $\widetilde{\gamma|I_n}(t_0) = y'' \in D$. Since D is a component of $\mathrm{St}_{1/m}(z)$, $\widetilde{\gamma|I_n}\colon I_n \to D$.

Let then $J := \gamma^{-1}(F)$. Since $p^{-1}\{z'\} \cap D$ is a point for every $z' \in F$, there exists a unique function $\widetilde{\gamma|J}\colon J \to D$ satisfying $p \circ \widetilde{\gamma|J} = \gamma|J$.

Let $\widetilde{\gamma}\colon [0,1] \to D$ be the unique function satisfying $\widetilde{\gamma}|I_n = \widetilde{\gamma|I_n}$ for every $n \in N$ and $\widetilde{\gamma}|J = \widetilde{\gamma|J}$. Then $p \circ \widetilde{\gamma} = \gamma$. Since $[0,1] \setminus J$ is open, the function $\widetilde{\gamma}$ is continuous at every $t \in [0,1] \setminus J$. By Lemma 5.13, $\widetilde{\gamma}$ is continuous at every $t \in J$. Thus $\widetilde{\gamma}$ is continuous. Since $\gamma(0) = z \in F$, $\widetilde{\gamma}(0) = y$. □

After the following lemma we are ready to define the pullback metric d_s^* on Y.

Lemma 5.16. *Let Z be a PL manifold and $p\colon Y \to Z$ a completed normal covering with locally finite multiplicity. Then for every pair of points y_1 and y_2 in Y there exists a path $\gamma\colon y_1 \curvearrowright y_2$ so that $p \circ \gamma$ is rectifiable.*

Proof. Let $Y' \subset Y$ and $Z' \subset Z$ be large subsets so that $g := p|Y'\colon Y' \to Z'$ is a normal covering. By Lemma 5.15, there exist points $z_1, z_2 \in Z'$ and rectifiable paths $\alpha_1\colon p(y_1) \curvearrowright z_1$ and $\alpha_2\colon p(y_2) \curvearrowright z_2$ having lifts $(\widetilde{\alpha_1})_{y_1}$ and $(\widetilde{\alpha_2})_{y_2}$ by p. We denote $y_1' = (\widetilde{\alpha}_i)_{y_1}(1)$ and $y_2' = (\widetilde{\alpha}_i)_{y_2}(1)$.

Since $g\colon Y' \to Z'$ is a covering between open manifolds, there exists a path $\beta\colon y_1' \curvearrowright y_2'$ so that $\ell(p \circ \beta) < \infty$. Now $\gamma := (\widetilde{\alpha_1})_{y_1}\beta(\widetilde{\alpha_2})_{y_1}^{\leftarrow}\colon y_1 \curvearrowright y_2$ satisfies

$$\ell(p \circ \gamma) \le \ell(\alpha_1) + \ell(p \circ \beta) + \ell(\alpha_2^{\leftarrow}) < \infty.$$ □

Let Z be a PL manifold and $p\colon Y \to Z$ a completed normal covering that has locally finite multiplicity and d_s the path-metric of Z. We call $d_s^*\colon Y \times Y \to \mathbb{R}_+$ defined by

$$d_s^*(y_1, y_2) = \inf\{\ell(p \circ \gamma) \mid p \circ \gamma,\ \gamma\colon y_1 \curvearrowright y_2\}$$

the *pullback* of the path-metric d_s by p.

Corollary 5.17. *Let Z be a PL manifold and $p\colon Y \to Z$ a completed normal covering that has locally finite multiplicity, d_s the path-metric of Z and d_s^* the pullback of d_s by p. Then d_s^* is a metric, every $\tau \in \mathcal{T}(p)$ is an isometry with respect to d_s^* and $p\colon (Y, d_s^*) \to (Y, d_s)$ is a 1-Lipschitz map.*

Proof. The map p is discrete by Theorem 9.14 in [14]. Thus d_s^* separates points of Y and d_s^* is a metric by Lemma 5.16; see Section 1 in [8]. By the definition of d_s^*, every $\tau \in \mathcal{T}(p)$ is an isometry with respect to d_s^* and $p\colon (Y, d_s^*) \to (Y, d_s)$ is a 1-Lipschitz map. □

In the following proposition we show that the topology induced by d_s^* on Y coincides with the original topology of Y.

Proposition 5.18. *Let Z be a PL manifold and $p\colon Y \to Z$ a completed normal covering with locally finite multiplicity. Let d_s be the path-metric on Z and $\mathcal{T}$ the topology of Y. Let d_s^* be the pullback of the path-metric d_s of Z by p and $\mathcal{T}_{d_s^*}$ the topology induced on Y by the metric d_s^*. Then $\mathcal{T} = \mathcal{T}_{d_s^*}$.*

Proof. We first show that $\mathcal{T} \subset \mathcal{T}_{d_s^*}$. Since p is a spread, it is sufficient to show that $U \in \mathcal{T}_{d_s^*}$ for every open connected subset $V \subset Z$ and component U of $p^{-1}(V)$. Let $V \subset Z$ be an open and connected set, and let $U \subset Y$ be a component of $p^{-1}(V)$. Fix $y \in U$. Since $p\colon (Y, d_s^*) \to (Y, d_s)$ is a 1-Lipschitz map, there exists $r_y \in (0,1)$ so that $p(B(y, r_y)) \subset B(y, r_y) \subset V$. Now for every point $y' \in B(y, r_y)$ there exists by the definition of d_s^* a path $\gamma\colon y \curvearrowright y'$ in $p^{-1}(B(y, r_y)) \subset p^{-1}(V)$. Thus $B(y, r_y) \subset U$, since U is the y-component of $p^{-1}(V)$. We conclude $U \in \mathcal{T}_{d_s^*}$.

We then show that $\mathcal{T}_{d_s^*} \subset \mathcal{T}$. Let $U \in \mathcal{T}_{d_s^*}$. The map p is discrete by Theorem 9.14 in [14], uniformly discrete by Theorem 4.11 and an orbit map by Theorem 4.1. Thus, as a consequence of Lemma 5.15, there exists for every $y \in U$ a radius $r_y \in (0,1)$ that satisfies the following conditions:

(a) for the y-component U_y of $p^{-1}(B(p(y), r_y))$, $U_y \cap p^{-1}\{p(y)\} = \{y\}$,

(b) every path in $B(p(y), r_y)$ beginning at $p(y)$ has a total lift into Y beginning at y,

(c) $[z, p(y)] \subset B(p(y), r_y)$ for every $z \in B(p(y), r_y)$ and

(d) $B(y, r_y) \subset U$.

Since p is a spread, $U_y \in \mathcal{T}$ for every $y \in U$. It suffices to show that $U_y \subset B(y, r_y)$ for every $y \in U$, since then

$$U = \bigcup_{y \in U} B(y, r_y) = \bigcup_{y \in U} U_y \in \mathcal{T}.$$

Let $y_1 \in U_y$. Then $p(y_1) \in B(p(y), r_y)$ and there exists a path $\gamma\colon p(y) \curvearrowright p(y_1)$ in $B(p(y), r_y)$ satisfying $\ell(\gamma) < r_y$.

Let $\widetilde{\gamma}_y$ be a lift of γ in U_y beginning at y. Then

$$d_s^*(y, \widetilde{\gamma}_y(1)) \le \ell(p \circ \widetilde{\gamma}_y) = \ell(\gamma) < r_y.$$

Hence $\widetilde{\gamma}_y(1) \in B(y, r_y)$.

Since p is an orbit map there is a deck-transformation $\tau \in \mathcal{T}(p)$ satisfying $\tau(\widetilde{\gamma}(1)) = y_1$. Since y_1 and $\widetilde{\gamma}_y(1)$ belong to U_y, we have $\tau(U_y) = U_y$ by Lemma 3.10. Hence $\tau(y) = y$, since $U_y \cap p^{-1}\{p(y)\} = \{y\}$. Thus

$$d_s^*(y_1, y) = d_s^*(\tau(\widetilde{\gamma}_y(1)), \tau(y)) = d_s^*(\widetilde{\gamma}_y(1), y) < r_y,$$

since τ is an isometry with respect to d_s^*. Thus $y_1 \in B(y, r_y)$. Thus $U_y \subset B(y, r_y)$ and we conclude that $\mathcal{T}_{d_s^*} \subset \mathcal{T}$. □

In the following proposition we show that the topology induced by d_s^* on Y is locally proper.

Proposition 5.19. *Let $p\colon Y \to Z$ be a completed normal covering onto a PL manifold Z that has locally finite multiplicity. Let d_s^* be the pullback of the path-metric d_s of Z by p. Then Y is a locally proper metric space with respect to d_s^*.*

Proof. Fix $y \in Y$. Since p has locally finite multiplicity, Lemma 5.15 implies that there exists $r \in (0,1)$ that satisfies $p(\overline{B(y,r)}) = \overline{B(p(y), r)}$ and satisfies for the y-component D of $p^{-1}(B(p(y), 2r))$ the following conditions:

(a) $p^{-1}\{p(y)\} \cap D = \{y\}$

(b) $p|D$ has finite multiplicity and

(c) $\overline{B(y,r)} \subset D$.

We prove the claim by showing that $\overline{B(y,r)}$ is compact.

Let $\mathcal{U}$ be an open cover of $\overline{B(y,r)}$. Now p is an open map and the set $p^{-1}\{z\} \cap \overline{B(y,r)}$ is finite for every $z \in p(\overline{B(y,r)})$. Thus there exists for every $z \in p(\overline{B(y,r)})$ a radius $r_z \in (0,1)$ so that for every $y' \in p^{-1}\{z\} \cap \overline{B(y,r)}$ there exists such $U \in \mathcal{U}$ that $B(y', r_z) \subset U$ and $p(B(y', r_z)) = B(z, r_z)$.

Since $p(\overline{B(y,r)}) = \overline{B(p(y), r)} \subset Z$ is compact, we may fix points $z_1, \ldots, z_k \in p(\overline{B(y,r)})$ for which $\{B(z_i, r_{z_i}) : 1 \le i \le k\}$ is an open cover of $p(\overline{B(y,r)})$. The set

$$\{B(y', r_{z_i}) \mid y' \in p^{-1}\{z_i\} \cap \overline{B(y,r)}, i \in \{1, \ldots, k\}\}$$

is now finite. Further, for every $i \in \{1, \ldots, k\}$ and $y' \in p^{-1}\{z_i\} \cap \overline{B(y,r)}$ there exists $U \in \mathcal{U}$ so that $B(y', r_{z_i}) \subset U$. Thus it suffices to show that

$$\overline{B(y,r)} \subset V := \bigcup \{B(y', r_{z_i}) : y' \in p^{-1}\{z_i\} \cap \overline{B(y,r)}, i \in \{1, \ldots, k\}\}.$$

Let $y' \in \overline{B(y,r)}$. Fix $i \in \{1, \ldots, k\}$ satisfying $p(y') \subset B(z_i, r_{z_i})$ and $e \in p^{-1}\{z_i\} \cap \overline{B(y,r)}$. Then there exists a point $y'' \in p^{-1}\{p(y')\} \cap B(e, r_{z_i})$, since

$p(B(e, r_{z_i})) = B(z_i, r_{z_i})$. Let $\tau\colon (Y, d_s^*) \to (Y, d_s^*)$ be a deck-transformation isometry satisfying $\tau(y'') = y'$. Then $y' \in B(\tau(e), r_{z_i})$ and $\tau(e) \in p^{-1}\{z_i\}$. Since y'' and y' belong to D, $\tau(D) = D$ by Lemma 3.13. Thus $\tau(y) = y$, since $p^{-1}\{p(y)\} \cap D = \{y\}$. Hence

$$d_s^*(\tau(e), y) = d_s^*(\tau(e), \tau(y)) = d_s^*(e, y) \leq r.$$

Thus $\tau(e) \in p^{-1}\{z_i\} \cap \overline{B(y, r)}$ and $y' \in B(\tau(e), r_{z_i}) \subset V$. We conclude that $\mathcal{U}$ has a finite subcover. Thus $\overline{B}(y, r) \subset Y$ is compact and (Y, d_s^*) is a locally proper metric space. □

This concludes the proof of Theorem 5.12, and we are ready for the proofs of Theorems 1.3, 1.8 and 1.11 in the introduction.

We say that a path-metric d'_s on Z is a *polyhedral path-metric* on Z, if there exists a simplicial complex K, a polyhedron $|K| \subset \mathbb{R}^n$ and an embedding $\iota\colon Z \to \mathbb{R}^n$ satisfying $\iota(Z) = |K|$, such that d'_s is the pullback of d_s by ι for the path-metric d_s of $|K|$.

We recall that a completed normal covering $p\colon Y \to Z$ is an open map and the domain Y is by definition a Hausdorff space. Thus p has locally finite multiplicity, if p is discrete and Y is locally compact.

Proof of Theorem 1.3. Let Z be a PL manifold and let $p\colon Y \to Z$ be a discrete completed normal covering. If Y is locally compact, then p has locally finite multiplicity. Suppose that p has locally finite multiplicity. By Theorem 5.12, there is a metric d_s^* on Y so that the topology induced by d_s^* coincides with the original topology of Y and (Y, d_s^*) is a locally proper metric space. Thus Y is locally compact. □

Proof of Theorem 1.8. Let $f\colon X \to Z$ be a completed covering between PL manifolds, (Y, p, q) a locally compact monodromy representation of f and d_s a polyhedral path-metric of Z. By Theorem 4.1, the orbit map q is discrete. Thus q has locally finite multiplicity, since Y is locally compact. Since q has locally finite multiplicity, by Theorem 5.12, the pullback d_s^* of d_s is a path-metric on Y satisfying conditions (a)–(d) in Theorem 4.1. □

Proof of Theorem 1.11. For the statement we need to show that a completed covering $f\colon X \to Z$ between PL manifolds has a locally compact monodromy representation (Y, p, q) if and only if f is stable and f has a finite local monodromy group at each $z \in Z$.

Suppose first that f is stable and that f has a finite local monodromy group at each $z \in Z$. By Corollary 5.9 f has a monodromy representation (Y, p, q), where q has locally finite multiplicity. By Theorem 5.12, there is a metric d_s^* on Y so that the topology induced by d_s^* coincides with the original topology of Y and (Y, d_s^*) is a locally proper metric space. Thus Y is locally compact and (Y, p, q) a locally compact monodromy representation of f.

Suppose then that f has a locally compact monodromy representation (Y, p, q). Then the map q is discrete by Theorem 4.1. Thus q has locally finite multiplicity. By Theorem 4.11, q is a stable completed normal covering. Thus, by Theorem 5.7, q has a finite local monodromy group at each $z \in Z$, since q has locally finite multiplicity.

Fix $z \in Z$ and let H be a finite local monodromy group of q at z. By Theorem 2.1 and Remark 4.9 there is a quotient of H that is a local monodromy group of f at z. Since every quotient of a finite group is finite, f has a finite local monodromy group at z. □

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