



Boundary Harnack estimates in slit domains and applications to thin free boundary problems

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Abstract. We provide a higher order boundary Harnack inequality for harmonic functions in slit domains. As a corollary we obtain the C^∞ regularity of the free boundary in the Signorini problem near non-degenerate points.

1. Introduction

In our recent work [10] we proved a higher order boundary Harnack estimate for harmonic functions vanishing on a part of the boundary of a domain Ω in \mathbb{R}^n . We recall briefly the main result.

Theorem 1.1. *Let v and $u > 0$ be two harmonic functions in $\Omega \subset \mathbb{R}^n$ that vanish continuously on some portion of the boundary $\Gamma \subset \partial\Omega$ and let $k \geq 1$.*

If $\Gamma \in C^{k,\alpha}$, then $v/u \in C^{k,\alpha}$ up to the boundary in a neighborhood of Γ .

By classical Schauder estimates both functions u and v are of class $C^{k,\alpha}$ up to the boundary. In general, the quotient of two $C^{k,\alpha}$ functions that vanish on the boundary is only of class $C^{k-1,\alpha}$ in a neighborhood of the boundary. The theorem states that the quotient of two harmonic functions is in fact one derivative better than what is expected from boundary Schauder estimates.

Theorem 1.1 is well known as the *boundary Harnack theorem* in the case $k = 0$ (see [14], [4], [12]). An easy application of Theorem 1.1 gives C^∞ regularity for $C^{1,\alpha}$ free boundaries in the classical obstacle problem, see [10].

In this paper we obtain the corresponding theorems in the case of slit domains and the thin obstacle problem. A *slit domain* is a domain in \mathbb{R}^{n+1} from which we remove an n -dimensional set $\mathcal{P} \subset \{x_{n+1} = 0\}$ (slit), with $C^{k,\alpha}$ boundary in \mathbb{R}^n , $\Gamma := \partial_{\mathbb{R}^n} \mathcal{P}$, $k \geq 1$.

In [9] we investigated the higher regularity of the free boundary for the *thin one-phase* free boundary problem (see [5], [6], [7], [8]). In particular we developed

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a Schauder type estimate in slit domains, see Theorem 2.4 in the next section for the precise statement. The Schauder estimate states that if u is even in x_{n+1} , and it is harmonic in the slit domain $B_1 \setminus \mathcal{P}$, and u vanishes continuously on the slit \mathcal{P} then

$$\Gamma \in C^{k,\alpha} \implies \frac{u}{U_0} \in C^{k-1,\alpha}(x_1, \dots, x_n, r), \quad k \geq 2.$$

Here

$$U_0 := \frac{1}{\sqrt{2}} \sqrt{d+r}$$

where d represents the signed distance to Γ in \mathbb{R}^n and r denotes the distance to Γ in \mathbb{R}^{n+1} . The above statement says that the quotient u/U_0 is essentially a $C^{k-1,\alpha}$ function in the variables (x_1, \dots, x_n, r) .

The explicit function U_0 is an approximation of a harmonic function which vanishes on \mathcal{P} and plays the same role as the distance function in smooth (non-slit) domains. For example when $\Gamma = \{(x_n, x_{n+1}) = (0, 0)\}$ is a “straight” boundary then U_0 is indeed harmonic.

In this paper we obtain the boundary Harnack estimate in slit domains, see Theorem 2.3 in the next section. We show that if u and $U > 0$ are even, harmonic functions vanishing on \mathcal{P} then

$$\Gamma \in C^{k,\alpha} \implies \frac{u}{U} \in C^{k,\alpha}(x_1, \dots, x_n, r), \quad k \geq 1.$$

We also provide the Schauder estimates in slit domains with $C^{1,\alpha}$ boundary, which were not completed in [8].

The proofs of the Schauder estimates and the boundary Harnack estimates are essentially the same. We approximate u in a sequence of concentric balls B_{ρ^l} by functions $U_0 P$ or $U P$ with $P(x, r)$ a polynomial in x and r . The gain of one extra derivative comes from the fact that while U is harmonic, the explicit function U_0 only approximates a harmonic function up to an error, thus $U P$ provides a better approximation than $U_0 P$.

Signorini problem. As an application of the boundary Harnack estimates in slit domains we obtain C^∞ regularity of the free boundary near regular points in the Signorini problem, also known as the thin obstacle problem. It consists in minimizing

$$(1.1) \quad \min_{u \in \mathcal{A}} \int_{B_1} |\nabla u|^2 dX,$$

with $B_1 \subset \mathbb{R}^{n+1}$, $X = (x, x_{n+1}) \in \mathbb{R}^{n+1}$, and \mathcal{A} the convex set

$$\mathcal{A} := \{u \in H^1(B_1), \quad u = \varphi \quad \text{on } \partial B_1, \quad u(x, 0) \geq 0\}.$$

There is considerable literature on the regularity properties of the solution (see [11], [3], [16]). In particular, the minimizer u is Lipschitz in B_1 , and is harmonic in the slit domain $B_1 \setminus \mathcal{P}$ with $\mathcal{P} := \{u = 0\} \cap \{x_{n+1} = 0\}$. Athanasopoulos and Caffarelli obtained in [1] the optimal regularity of the solution on the free

boundary $\Gamma := \partial_{\mathbb{R}^n} \mathcal{P} \cap B_1$, i.e., the solution u is pointwise $C^{1,1/2}$ at all points on Γ . A point $X_0 \in \Gamma$ is called a *singular point* of the free boundary if

$$u(X) = o(|X - X_0|^{3/2}).$$

Otherwise, X_0 is called a *regular point* of the free boundary.

Concerning the regularity of the free boundary Γ , Athanasopoulos, Caffarelli and Salsa showed in [2] that if X_0 is a regular point, then Γ is given locally by the graph of a $C^{1,\alpha}$ function g in some direction, say the e_n direction (see also [13] concerning the singular set). Moreover, the derivatives u_i , $i = 1, 2, \dots, n$, are harmonic in the slit domain $B_1 \setminus \mathcal{P}$ and vanish continuously on \mathcal{P} , and $u_n > 0$ in a neighborhood of X_0 . We remark that we may assume that u is even in the x_{n+1} variable since the even part of u solves the obstacle problem with the same free boundary.

Now we can apply the boundary Harnack estimate, Theorem 2.3, and improve the $C^{1,\alpha}$ regularity of Γ to $\Gamma \in C^\infty$. Indeed, since $\Gamma \in C^{1,\alpha}$, we obtain that u_i/u_n is $C^{1,\alpha}$ when restricted to Γ in a neighborhood of X_0 . On the other hand, u_i/u_n restricted to Γ represents the derivative g_i . In conclusion $g \in C^{2,\alpha}$, hence $\Gamma \in C^{2,\alpha}$. We iterate this indefinitely and obtain $\Gamma \in C^\infty$.

Theorem 1.2. *Let X_0 be a regular point of the free boundary Γ of a solution u to the Signorini problem (1.1). Then $\Gamma \in C^\infty$ in a neighborhood of X_0 .*

We remark that analyticity of the free boundary Γ near regular points was obtained by Koch, Petrosyan and Shi in [15] at the same time this paper was completed. They used a different method based on partial Legendre transformation.

The paper is organized as follows. In Section 2 we introduce some notation and state our main result, the boundary Harnack estimate Theorem 2.3. In Section 3 we prove this theorem in the case $\Gamma \in C^{k,\alpha}$ with $k \geq 2$. In Section 4 we obtain both the Schauder and the boundary Harnack estimates when $\Gamma \in C^{1,\alpha}$. Finally, in Section 5 we collect the proofs of some technical lemmas used in our proofs.

2. Notation and statement of main results

2.1. Notation

Let Γ be a $C^{k+1,\alpha}$ surface in \mathbb{R}^n , $k \geq 0$. Assume for simplicity that Γ is given by the graph of a function g of $n - 1$ variables,

$$(2.1) \quad \Gamma := \{(x', g(x'))\}, \quad g : B'_1 \subset \mathbb{R}^{n-1} \rightarrow \mathbb{R},$$

satisfying

$$g(0) = 0, \quad \nabla_{x'} g(0) = 0, \quad \|g\|_{C^{k+1,\alpha}(B'_1)} \leq 1.$$

Let \mathcal{P} denote the n dimensional slit in \mathbb{R}^{n+1} given by

$$\mathcal{P} := \{X = (x, x_{n+1}) \in B_1 \mid x_{n+1} = 0, \quad x_n \leq g(x')\}.$$

Notice that in the n dimensional ball $B'_1 \times \{0\}$ we have $\partial_{\mathbb{R}^n} \mathcal{P} = \Gamma$.

Given a point $X = (x, x_{n+1})$ we denote by d the signed distance in \mathbb{R}^n from x to Γ with $d > 0$ above Γ (in the e_n direction). Let

$$r := \sqrt{x_{n+1}^2 + d^2}$$

be the distance in \mathbb{R}^{n+1} from X to Γ . We have

$$(2.2) \quad \nabla_x r = \frac{d}{r} \nu, \quad \nu = \nabla_x d,$$

where $\nu(x)$ represents the unit normal in \mathbb{R}^n to the parallel surface to Γ passing through x .

We recall the definition of the class $C_{xr}^{k,\alpha}$ introduced in [9]. We denote by

$$P(x, r) = a_{\mu m} x^\mu r^m, \quad \deg P = k,$$

a polynomial of degree k in the (x, r) variables, and we use throughout the paper the summation convention over repeated indices. Also

$$x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}, \quad |\mu| = \mu_1 + \cdots + \mu_n, \quad \mu_i \geq 0.$$

Sometimes we think that $a_{\mu m}$ are defined for all indices (μ, m) , by extending them to be 0. We also denote

$$\|P\| := \max |a_{\mu m}|.$$

Definition 2.1. We say that a function $f: B_1 \subset \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ is *pointwise* $C^{k,\alpha}$ in the (x, r) -variables at $0 \in \Gamma$ and write $f \in C_{xr}^{k,\alpha}(0)$ if there exists a (tangent) polynomial $P_0(x, r)$ of degree k such that

$$f(X) = P_0(x, r) + O(|X|^{k+\alpha}).$$

We define $\|f\|_{C_{xr}^{k,\alpha}(0)}$ as the smallest constant M such that

$$\|P_0\| \leq M \quad \text{and} \quad |f(X) - P_0(x, r)| \leq M|X|^{k+\alpha},$$

for all X in the domain of definition.

Similarly, we may write the definition for f to be pointwise $C_{xr}^{k,\alpha}$ at some other point $Z \in \Gamma$.

Definition 2.2. Let $K \subset \Gamma$. We say that $f \in C_{xr}^{k,\alpha}(K)$ if there exists a constant M such that $f \in C_{xr}^{k,\alpha}(Z)$ for all $Z \in K$ and $\|f\|_{C_{xr}^{k,\alpha}(Z)} \leq M$ for all $Z \in K$.

The smallest M in the definition above is denoted by $\|f\|_{C_{x,r}^{k,\alpha}(K)}$.

We remark that if $f \in C_{xr}^{k,\alpha}(\Gamma)$ and $\Gamma \in C^{k,\alpha}$ then the restriction of f on Γ is a $C^{k,\alpha}$ function.

Finally, let $\theta \in (-\pi, \pi]$ be the angle between the segment of length r from X to Γ and the x -hyperplane and define

$$(2.3) \quad U_0(X) := r^{1/2} \cos \frac{\theta}{2} = \frac{1}{\sqrt{2}} \sqrt{d+r}.$$

2.2. Main results

Let $u \in C(B_1)$ be even in the x_{n+1} coordinate, with $\|u\|_{L^\infty} \leq 1$, and

$$(2.4) \quad \begin{cases} \Delta u = \frac{U_0}{r} f & \text{in } B_1 \setminus \mathcal{P}, \\ u = 0 & \text{on } \mathcal{P}. \end{cases}$$

Also, let U be a positive harmonic function, even in x_{n+1} , which is normalized such that $U(\frac{1}{2}e_n) = 1$. Precisely we assume that $U > 0$ solves the problem above with $f = 0$, i.e.,

$$(2.5) \quad \begin{cases} \Delta U = 0 & \text{in } B_1 \setminus \mathcal{P}, \\ U = 0 & \text{on } \mathcal{P}, \end{cases} \quad \text{and } U(\frac{1}{2}e_n) = 1.$$

Notice that by a version of the Hopf lemma in slit domains (see Lemma 7.1 in [7]) and the boundary Harnack inequality we have

$$(2.6) \quad cU_0 \leq U \leq CU_0$$

for some universal $c, C > 0$.

Our main result reads as follows.

Theorem 2.3 (Boundary Harnack in slit domains). *Let $\Gamma \in C^{k+1,\alpha}$ satisfy (2.1), and let u and U satisfy (2.4), (2.5) with $k \geq 0$, and*

$$f \in C_{xr}^{k,\alpha}(\Gamma \cap B_1), \quad \|f\|_{C_{xr}^{k,\alpha}(\Gamma \cap B_1)} \leq 1.$$

Then,

$$(2.7) \quad \left\| \frac{u}{U} \right\|_{C_{xr}^{k+1,\alpha}(\Gamma \cap B_{1/2})} \leq C,$$

with C depending only on n, k and α .

The boundary Harnack Theorem 2.3 complements the Schauder type estimates obtained in [9] that we state below.

Theorem 2.4 (Schauder estimates in slit domains). *Let $\Gamma \in C^{k+1,\alpha}$ with $k \geq 1$ satisfy (2.1), and assume u solves (2.4) with*

$$f \in C_{xr}^{k-1,\alpha}(\Gamma \cap B_1), \quad \|f\|_{C_{xr}^{k-1,\alpha}(\Gamma \cap B_1)} \leq 1.$$

Then,

$$(2.8) \quad \left\| \frac{u}{U_0} \right\|_{C_{xr}^{k,\alpha}(\Gamma \cap B_{1/2})} \leq C$$

and

$$(2.9) \quad \left\| \frac{\nabla_x u}{(U_0/r)} \right\|_{C_{xr}^{k,\alpha}(\Gamma \cap B_{1/2})} \leq C,$$

with C a constant depending only on n, k and α .

In Section 4 we also prove the Schauder estimates when $\Gamma \in C^{1,\alpha}$.

Theorem 2.5. *Let $\Gamma \in C^{1,\alpha}$ satisfy (2.1) with $k = 0$, and assume u solves (2.4) with*

$$|f| \leq r^{\alpha-3/2}.$$

Then,

$$\left\| \frac{u}{U_0} \right\|_{C_{x_r}^\alpha(\Gamma \cap B_{1/2})} \leq C,$$

with C a constant depending only on n and α .

Throughout the paper we denote by c, C various positive constants that depend only on n, k and α and we refer to them as universal constants.

3. The case $\Gamma \in C^{k+1,\alpha}$ with $k \geq 1$

In this section we prove Theorem 2.3 in the case $k \geq 1$. We follow the same strategy as in the proof of the Schauder estimates from [9]. The difference is that now we approximate u by functions $U(X)P(x, r)$ instead of $U_0(X)P(x, r)$ as in [9].

It suffices to prove the following slightly stronger pointwise estimate for u/U .

Proposition 3.1. *Let $\Gamma \in C^{k+1,\alpha}$, satisfy (2.1) with $k \geq 1$, and let U be as in (2.5). Assume that $u \in C(B_1)$ is even and vanishes on \mathcal{P} , $\|u\|_{L^\infty} \leq 1$, and*

$$\Delta u(X) = \frac{U_0}{r} R(x, r) + F(X) \quad \text{in } B_1 \setminus \mathcal{P},$$

with

$$|F(X)| \leq r^{-1/2} |X|^{k+\alpha} \quad \text{and } R(x, r) \text{ a polynomial of degree } k \text{ with } \|R\| \leq 1.$$

There exists a polynomial $P(x, r)$ of degree $k + 1$ with coefficients bounded by C such that

$$\left| \frac{u}{U} - P \right| \leq C |X|^{k+1+\alpha},$$

with C depending on k, α, n .

Now Theorem 2.3 follows at the origin (and therefore at all points in $\Gamma \cap B_{1/2}$) since

$$f(X) = R(x, r) + h(X), \quad \deg R = k, \quad h(X) = O(|X|^{k+\alpha}),$$

and $F := \frac{U_0}{r} h(X)$ satisfies the bound above.

Before we proceed with the proof of Proposition 3.1 we first need to express U in terms of U_0 . This is given by the Schauder estimates Theorem 2.4 applied to U a solution of (2.5). Thus U satisfies the following expansion at $0 \in \Gamma$:

$$(3.1) \quad U(X) = U_0(X) (P_0(x, r) + O(|X|^{k+\alpha})),$$

for some polynomial $P_0(x, r)$ of degree k . The derivatives U_i are in fact obtained by differentiating formally this expansion in the x_i direction (see Subsection 5.2 in [9]). It is easy to check that

$$(3.2) \quad \nabla_x U_0 = \frac{U_0}{2r} \nu.$$

Thus, using (2.2)–(3.2) we have

$$(3.3) \quad \nabla_x U = \frac{U_0}{r} \left[\frac{1}{2} P_0 \nu + r \partial_x P_0 + (D_r P_0) d\nu + O(|X|^{k+\alpha}) \right],$$

where $D_r P_0$ represents the formal derivative of P_0 with respect to the r variable. Since $\nu, d \in C_x^{k,\alpha}$ we obtain

$$U_i = \frac{U_0}{r} (P_0^i(x, r) + O(|X|^{k+\alpha})), \quad \deg P_0^i = k,$$

for some polynomial P_0^i .

In the next lemma we state that this expansion holds also for the radial derivative $\partial_r U := \nabla r \cdot \nabla U$. Its proof can be found in Section 5.

Lemma 3.2. *Let Γ and U be as above. Then*

$$\left| \partial_r U - \frac{U_0}{r} P_0^r \right| \leq C \frac{U_0}{r} |X|^{k+\alpha},$$

with $\deg P_0^r = k$ and $\frac{U_0}{r} P_0^r$ is obtained by formally differentiating $U_0 P_0$ in the r -direction, i.e.,

$$(3.4) \quad \partial_r U = \frac{U_0}{r} \left[\frac{1}{2} P_0 + \nabla_x P_0 \cdot (d\nu) + r (D_r P_0) + O(|X|^{k+\alpha}) \right].$$

We also recall the following theorem from [9], which deals with the case when Γ is straight.

Theorem 3.3. *Assume $\Gamma = \{x_n = 0\}$ and $u \in C(B_1)$ is even, $\|u\| \leq 1$ and satisfies*

$$\Delta u = 0 \quad \text{in } B_1 \setminus \mathcal{P}, \quad u = 0 \quad \text{on } \mathcal{P}.$$

For any $m \geq 0$, there exists a polynomial $P_0(x, r)$ of degree m such that $U_0 P_0$ is harmonic in $B_1 \setminus \mathcal{P}$ and

$$|u - U_0 P_0| \leq K |X|^{m+1} U_0,$$

for some constant K depending on m and n .

We now proceed to prove Proposition 3.1. After careful computations are carried on, the proof follows from similar arguments as in Proposition 5.1 in [9].

After performing an initial dilation we may assume that:

$$(3.5) \quad \|g\|_{C^{k+1,\alpha}} \leq \delta, \quad |R| \leq \delta, \quad |F| \leq \delta r^{-1/2} |X|^{k+\alpha}.$$

Moreover, after multiplying U by a constant (see (2.6) and (3.1)) we may also assume

$$(3.6) \quad U = U_0(1 + \delta Q_0 + \delta O(|X|^{k+\alpha})), \quad \deg Q_0 = k, \quad \|Q_0\| \leq 1,$$

with the 0-th order term of Q_0 being 0. The constant δ will be made precise later.

Next we define the notion of ‘‘approximating polynomial’’ which plays a crucial role in our analysis.

Let $\kappa(x)$ be the mean curvature of the parallel surface to Γ passing through x and $\nu(x)$ denote the normal to this parallel surface. Thus,

$$\kappa(x) = -\Delta d \in C_x^{k-1,\alpha}, \quad \nu(x) = \nabla d \in C_x^{k,\alpha}.$$

Then, one easily computes that ($m \geq 0$)

$$(3.7) \quad \Delta r^m = m r^{m-2}(m - \kappa d).$$

Now, let $|\mu| + m \leq k + 1$ and let \bar{i} denote the multi-index with 1 on the i -th position and zeros elsewhere. Using (3.7), (2.2), and the fact that U is harmonic in $B_1 \setminus \mathcal{P}$ we obtain

$$\begin{aligned} \Delta(x^\mu r^m U) &= U \Delta(x^\mu r^m) + 2 \nabla(x^\mu r^m) \cdot \nabla U \\ &= U \left(r^m \mu_i (\mu_i - 1) x^{\mu - 2\bar{i}} + m x^\mu r^{m-2} (m - \kappa d) + 2m r^{m-1} \frac{d}{r} \nu \cdot \nabla_x x^\mu \right) \\ &\quad + 2 \left(r^m \nabla x^\mu \cdot \nabla_x U + m x^\mu r^{m-1} \partial_r U \right) \\ &= \frac{U}{r} \text{I} + 2 \text{II}. \end{aligned}$$

By Taylor expansion at 0, we write (see (3.5) and recall $\nabla_x g(0) = 0$)

$$(3.8) \quad \nu^i = \delta_n^i + \dots, \quad \kappa = \kappa(0) + \dots, \quad d = x_n + \dots$$

We arrange the terms in I by the degree up to order k and group the remaining ones in a remainder. Precisely,

$$\text{I} = m(m + 2\mu_n) x^\mu r^{m-1} + \mu_i (\mu_i - 1) x^{\mu - 2\bar{i}} r^{m+1} + b_{\sigma l}^{\mu m} x^\sigma r^l + \delta O(|X|^{k+\alpha}),$$

with

$$b_{\sigma l}^{\mu m} \neq 0 \quad \text{only if} \quad |\mu| + m - 1 < |\sigma| + l \leq k.$$

Notice that the monomials $b_{\sigma l}^{\mu m} x^\sigma r^l$ have strictly higher degree than the first terms and together with the remainder can be thought as lower order terms. Also they are linear combinations of coefficients of the tangent polynomials at 0 for $d\kappa(x)$, $d\nu^i$ thus,

$$|b_{\sigma l}^{\mu m}| \leq C \delta.$$

Notice that $b_{\sigma l}^{\mu m}$ vanish in the flat case $\Gamma = \{x_n = 0\}$.

To estimate II we use (3.3)–(3.4) and obtain

$$\text{II} = \frac{U_0}{r} \left[\frac{1}{2} r^m \mu_n x^{\mu - \bar{n}} + \frac{1}{2} m x^\mu r^{m-1} + p_{\sigma l}^{\mu m} x^\sigma r^l + \delta O(|X|^{k+\alpha}) \right],$$

where the coefficients $p_{\sigma l}^{\mu m}$ have the same properties as the $b_{\sigma l}^{\mu m}$.

Thus, using (3.6) we conclude that

$$\Delta(x^\mu r^m U) = \frac{U_0}{r} \left[m(m+1+2\mu_n) x^\mu r^{m-1} + r^m \mu_n x^{\mu-\bar{n}} + \mu_i(\mu_i-1) x^{\mu-2\bar{i}} r^{m+1} + c_{\sigma l}^{\mu m} x^\sigma r^l + \delta O(|X|^{k+\alpha}) \right],$$

with

$$c_{\sigma l}^{\mu m} \neq 0 \quad \text{only if} \quad |\mu| + m - 1 < |\sigma| + l \leq k,$$

and

$$(3.9) \quad |c_{\sigma l}^{\mu m}| \leq C \delta.$$

If $P = a_{\mu m} x^\mu r^m$ is a polynomial of degree $k+1$, then

$$(3.10) \quad \Delta(UP) = \frac{U_0}{r} (A_{\sigma l} x^\sigma r^l + \delta O(|X|^{k+\alpha})), \quad |\sigma| + l \leq k,$$

with

$$(3.11) \quad A_{\sigma l} = (l+1)(l+2+2\sigma_n) a_{\sigma, l+1} + (\sigma_n+1) a_{\sigma+\bar{n}, l} + (\sigma_i+1)(\sigma_i+2) a_{\sigma+2\bar{i}, l-1} + c_{\sigma l}^{\mu m} a_{\mu m}.$$

From (3.11) we see that $a_{\sigma, l+1}$ (whose coefficient is different than 0) can be expressed in terms of $A_{\sigma l}$ and a linear combination of $a_{\mu m}$ with $\mu+m < |\sigma|+l+1$ plus a linear combination of $a_{\mu m}$ with $\mu+m = |\sigma|+l+1$ and $m < l+1$. This shows that the coefficients $a_{\mu m}$ are uniquely determined from the linear system (3.11) once $A_{\sigma l}$ and $a_{\mu 0}$ are given.

Definition 3.4. We say that P is approximating for u/U at 0 if $A_{\sigma l}$ coincide with the coefficients of R .

Remark 3.5. Here we point out the difference between approximating using U and U_0 . If we want to obtain an expansion as in (3.10) for $\Delta(U_0 P)$ then we need to require $\Gamma \in C^{k+2, \alpha}$ in order to deal with the terms $x^\mu r^m \Delta U_0$.

The following improvement of flatness lemma is the key ingredient in the proof of Proposition 3.1.

Lemma 3.6. *There exist universal constants ρ, δ depending only on k, α and n , such that if P with $\|P\| \leq 1$ is an approximating polynomial for u/U in B_λ , that is P is approximating for u/U at 0 and*

$$\|u - UP\|_{L^\infty(B_\lambda)} \leq \lambda^{k+3/2+\alpha},$$

for some $\lambda > 0$, then there exists an approximating polynomial \bar{P} for u/U at 0 such that, in $B_{\rho\lambda}$,

$$\|u - U\bar{P}\|_{L^\infty(B_{\rho\lambda})} \leq (\rho\lambda)^{k+3/2+\alpha} \quad \text{and} \quad \|\bar{P} - P\|_{L^\infty(B_\lambda)} \leq C\lambda^{k+1+\alpha}.$$

Proof. Set

$$u - UP =: \lambda^{k+3/2+\alpha} \tilde{u}\left(\frac{X}{\lambda}\right).$$

Thus, since P is approximating,

$$(3.12) \quad \Delta \tilde{u}\left(\frac{X}{\lambda}\right) = \lambda^{1/2-k-\alpha} \left(F(X) - \delta \frac{U_0}{r} O(|X|^{k+\alpha}) \right) =: \tilde{F}\left(\frac{X}{\lambda}\right).$$

Using the hypothesis on u and F we find

$$|\tilde{u}(X)| \leq 1, \quad |\Delta \tilde{u}(X)| \leq C \delta r^{-1/2} \quad \text{in } B_1.$$

Denote by $\tilde{\Gamma}, \tilde{\mathcal{P}}, \tilde{U}_0, \tilde{U}$ the rescalings of Γ, \mathcal{P}, U_0 and \tilde{U} from B_λ to B_1 , i.e.,

$$\tilde{\Gamma} := \frac{1}{\lambda} \Gamma, \quad \tilde{\mathcal{P}} := \frac{1}{\lambda} \mathcal{P}, \quad \tilde{U}_0(X) := \lambda^{-1/2} U_0(\lambda X), \quad \tilde{U}(X) := \lambda^{-1/2} U(\lambda X).$$

We decompose \tilde{u} as

$$\tilde{u} = \tilde{u}_0 + \tilde{v},$$

with

$$\begin{cases} \Delta \tilde{u}_0 = 0 & \text{in } B_1 \setminus \tilde{\mathcal{P}}, \\ \tilde{u}_0 = \tilde{u} & \text{on } \partial B_1 \cup \tilde{\mathcal{P}}, \end{cases} \quad \text{and} \quad \begin{cases} |\Delta \tilde{v}| \leq C \delta r^{-1/2} & \text{in } B_1 \setminus \tilde{\mathcal{P}}, \\ \tilde{v} = 0 & \text{on } \partial B_1 \cup \tilde{\mathcal{P}}. \end{cases}$$

Using barriers it follows that (see (5.6) in [9] or Lemma 5.2 in Section 5)

$$(3.13) \quad \|\tilde{v}\|_{L^\infty(B_1)} \leq C \delta \tilde{U}_0.$$

To estimate \tilde{u}_0 we observe that \tilde{u}_0 is a harmonic function in $B_1 \setminus \tilde{\mathcal{P}}, |\tilde{u}_0| \leq 1$ and as $\delta \rightarrow 0$, $\tilde{\Gamma}$ converges in the $C^{k+1,\alpha}$ norm to the hyperplane $\{x_n = 0\}$. Moreover, \tilde{u}_0 is uniformly Hölder continuous in $B_{1/2}$. By compactness, if δ is sufficiently small, \tilde{u}_0 can be approximated in $B_{1/2}$ by a solution of the Laplace problem with $\Gamma = \{x_n = 0\}$. Thus by Theorem 3.3, and the fact that $\tilde{U} \rightarrow \tilde{U}_0$ uniformly as $\delta \rightarrow 0$ (see (3.6)) we deduce that

$$(3.14) \quad \|\tilde{u}_0 - \tilde{U}Q\|_{L^\infty(B_\rho)} \leq C \rho^{k+2+1/2}, \quad \deg Q = k + 1,$$

with $\|Q\| \leq C$. Since U_0Q is harmonic we also get that the coefficients of Q satisfy (see (3.11))

$$(3.15) \quad (l+1)(l+2+2\sigma_n) q_{\sigma,l+1} + (\sigma_n+1) q_{\sigma+\bar{n},l} + (\sigma_i+1)(\sigma_i+2) q_{\sigma+2\bar{i},l-1} = 0,$$

with bounded $q_{\mu m}$.

Using also (3.13) we find

$$\|\tilde{u} - \tilde{U}Q\|_{L^\infty(B_\rho)} \leq C \rho^{k+5/2} + C \delta \leq \frac{1}{2} \rho^{k+3/2+\alpha}$$

provided that we choose first ρ and then δ , universal, sufficiently small.

Writing this inequality in terms of the original function u , we find

$$\left| u - U \left(P + \lambda^{k+1+\alpha} Q \left(\frac{X}{\lambda} \right) \right) \right| \leq \frac{1}{2} (\lambda \rho)^{3/2+\alpha} \quad \text{in } B_{\rho\lambda}.$$

However $P(X) + \lambda^{k+1+\alpha} Q(X/\lambda)$ is not an approximating polynomial and therefore we need to perturb Q by a small amount.

Precisely, we need to modify Q into \bar{Q} such that $\bar{Q}(x/\lambda, r/\lambda)$ is approximating for $R \equiv 0$. Thus its coefficients solve the system (3.11) with $A_{\sigma l} = 0$ and rescaled $c_{\sigma l}^{\mu m}$, i.e.,

$$(3.16) \quad \begin{aligned} (l+1)(l+2+2\sigma_n)\bar{q}_{\sigma, l+1} + (\sigma_n+1)\bar{q}_{\sigma+\bar{n}, l} \\ + (\sigma_i+1)(\sigma_i+2)\bar{q}_{\sigma+2\bar{i}, l-1} + \bar{c}_{\sigma l}^{\mu m} \bar{q}_{\mu m} = 0, \end{aligned}$$

with

$$\bar{c}_{\sigma l}^{\mu m} := \lambda^{|\sigma|+l+1-|\mu|-m} c_{\sigma l}^{\mu m}, \quad \text{hence} \quad |\bar{c}_{\sigma l}^{\mu m}| \leq |c_{\sigma l}^{\mu m}| \leq C\delta.$$

After subtracting (3.16) from (3.15) we see that the coefficients of $Q - \bar{Q}$ solve the linear system (3.16) with right hand side $A_{\sigma l} = \bar{c}_{\sigma l}^{\mu m} q_{\mu m}$, hence $|A_{\sigma l}| \leq C\delta$. As we mentioned before Definition 3.4, this system is uniquely solvable after choosing $\bar{q}_{\mu 0} - q_{\mu 0} = 0$ and we find

$$\|\bar{Q} - Q\| \leq C\delta.$$

This concludes the proof. □

Remark 3.7. The classical boundary Harnack inequality implies that $|\tilde{u}_0| \leq C\tilde{U}_0$ which together with (3.13) gives

$$|\tilde{u}| \leq C\tilde{U}_0 \quad \text{in } B_{1/2}.$$

This shows that the hypothesis of Proposition 3.1 can be improved to

$$|u - UP| \leq CU_0 \lambda^{k+1+\alpha} \quad \text{in } B_{\lambda/2}.$$

We can now conclude the proof of Proposition 3.1.

After multiplying u by a small constant, we see that the hypotheses of the lemma are satisfied for some initial λ_0 small. Indeed, the coefficients of R become sufficiently small and, by (3.11), we can choose an initial approximating polynomial P_{λ_0} with $\|P_{\lambda_0}\| \leq 1/2$. Now we may iterate the lemma for all $\lambda = \lambda_0 \rho^m$ and conclude that there exists a limiting approximating polynomial P_0 , $\|P_0\| \leq 1$, such that

$$|u - UP_0| \leq C|X|^{k+3/2+\alpha} \quad \text{in } B_1.$$

Moreover, in view of the remark above, the right-hand side can be replaced by $CU_0|X|^{k+1+\alpha}$, or equivalently by $CU|X|^{k+1+\alpha}$, and the proposition is proved.

4. $C^{1,\alpha}$ boundaries

We start by proving the Schauder estimate Theorem 2.5. For the reader’s convenience we state it again.

Theorem 4.1. *Let u be a solution to*

$$|\Delta u| \leq r^{\alpha-3/2} \quad \text{in } B_1 \setminus \mathcal{P}, \quad u = 0 \quad \text{on } \mathcal{P}$$

with $\|u\|_{L^\infty} \leq 1$ and $\|\Gamma\|_{C^{1,\alpha}} \leq 1$. Then,

$$\left\| \frac{u}{U_0} \right\|_{C_{x^r}^\alpha(\Gamma \cap B_{1/2})} \leq C$$

with $C > 0$ depending on n and α .

At the origin, the theorem states that there exists a constant a , $|a| \leq C$ such that

$$(4.1) \quad |u - aU_0| \leq C |X|^\alpha U_0.$$

It turns out that if u is harmonic then we can differentiate formally the inequality above.

Lemma 4.2. *Assume that u is harmonic, satisfies the hypotheses of Theorem 4.1, and the expansion (4.1) holds. Then, for a.e. $X \in B_{1/2}$ we have*

$$|\nabla u - \nabla(aU_0)| \leq C |X|^\alpha r^{-1/2}, \quad |\nabla_x u - \nabla_x(aU_0)| \leq C |X|^\alpha \frac{U_0}{r}.$$

We prove the Schauder estimates and the boundary Harnack in slit domains with $C^{1,\alpha}$ boundary using the same strategy as in the case of $C^{k+1,\alpha}$ domains with $k \geq 1$. However, due to the lack of regularity of r, d and U_0 the “test” functions in the proof for $k \geq 1$ must be slightly modified. We achieve this by working with “regularizations” of the functions r, U_0 that we denote by \bar{r}, \bar{U}_0 . Their main properties are given in the next lemma. Notice that r, U_0 are differentiable a.e.

Lemma 4.3. *Let $\|\Gamma\|_{C^{1,\alpha}} \leq \delta$. There exist smooth functions \bar{r}, \bar{U}_0 such that*

$$\begin{aligned} \left| \frac{\bar{r}}{r} - 1 \right| &\leq C\delta r^\alpha, & \left| \frac{\bar{U}_0}{U_0} - 1 \right| &\leq C\delta r^\alpha, \\ |\nabla \bar{r} - \nabla r| &\leq C\delta r^\alpha, & |\partial_{x_{n+1}} \bar{r} - \partial_{x_{n+1}} r| &\leq C\delta r^{\alpha-1/2} U_0, \\ \left| \Delta \bar{r} - \frac{1}{r} \right| &\leq C\delta r^{\alpha-1}, & |\Delta \bar{U}_0| &\leq C\delta r^{\alpha-3/2}, \end{aligned}$$

with C universal.

The proof of Lemma 4.3 is postponed until Section 5.

As usually, Theorem 4.1 follows from the next improvement of flatness lemma.

Lemma 4.4. *Let*

$$|\Delta u| \leq \delta r^{\alpha-3/2} \quad \text{in } B_1 \setminus \mathcal{P}, \quad u = 0 \quad \text{on } \mathcal{P}, \quad \|\Gamma\|_{C^{1,\alpha}} \leq \delta.$$

Assume that there exists a constant a , $|a| \leq 1$ such that for some $\lambda > 0$

$$\|u - aU_0\|_{L^\infty(B_\lambda)} \leq \lambda^{\alpha+1/2}.$$

Then there exists $\rho > 0$ such that

$$\|u - bU_0\|_{L^\infty(B_{\rho\lambda})} \leq (\rho\lambda)^{\alpha+1/2},$$

with $|a - b| \leq C\lambda^\alpha$, as long as δ is sufficiently small.

Proof. From Lemma 4.3 we can replace U_0 by \bar{U}_0 and assume that

$$|u - a\bar{U}_0| \leq 2\lambda^{\alpha+1/2} \quad \text{in } B_\lambda.$$

Set

$$u - a\bar{U}_0 = 2\lambda^{\alpha+1/2} \tilde{u} \left(\frac{X}{\lambda} \right).$$

Then, using the bound for $\Delta \bar{U}_0$, we obtain that

$$|\tilde{u}| \leq 1, \quad |\Delta \tilde{u}| \leq C\delta r^{\alpha-3/2} \quad \text{in } B_1.$$

We now write

$$\tilde{u} = \tilde{u}_1 + \tilde{u}_2,$$

with

$$\Delta \tilde{u}_1 = \Delta \tilde{u} \quad \text{in } B_1 \setminus \mathcal{P}, \quad \tilde{u}_1 = 0 \quad \text{on } \partial B_1 \cup \tilde{\mathcal{P}}$$

and

$$\Delta \tilde{u}_2 = 0 \quad \text{in } B_1 \setminus \mathcal{P}, \quad \tilde{u}_2 = \tilde{u} \quad \text{on } \partial B_1 \cup \tilde{\mathcal{P}}.$$

By Lemma 5.2 in Section 5 we have $\|\tilde{u}_1\|_{L^\infty} \leq C\delta\tilde{U}_0$, hence \tilde{u}_1 converges to 0 uniformly as $\delta \rightarrow 0$.

To estimate \tilde{u}_2 we argue by compactness, as in the case $k \geq 1$. If δ is sufficiently small universal, \tilde{u}_2 can be approximated in $B_{1/2}$ by a solution of the Laplace problem with $\Gamma = \{x_n = 0\}$. Thus by Theorem 3.3, we deduce that

$$\|\tilde{u}_2 - b\tilde{U}_0\|_{L^\infty(B_\rho)} \leq C\rho^{1+1/2},$$

for some constant b , $|b| \leq C$. Thus,

$$\|\tilde{u} - b\tilde{U}_0\|_{L^\infty(B_\rho)} \leq C\rho^{1+1/2} + o(\delta) \leq \frac{1}{4}\rho^{1/2+\alpha},$$

provided we choose first ρ then δ sufficiently small.

Writing this inequality in terms of the original function u we obtain (for δ small enough),

$$|u - a\bar{U}_0 - 2b\lambda^\alpha U_0| \leq \frac{1}{2}(\lambda\rho)^{\alpha+1/2} \quad \text{in } B_{\rho\lambda}.$$

Then, by Lemma 4.3, we obtain

$$|u - (a + 2b\lambda^\alpha)U_0| \leq (\lambda\rho)^{\alpha+1/2}, \quad \text{in } B_{\rho\lambda}$$

as desired. □

For the remaining of this section we prove our main Theorem 2.3, for $k = 0$. For clarity of exposition we write below the statements of Proposition 3.1 and Lemma 3.6 leading to it, in the case $k = 0$.

Proposition 4.5. *Let $\Gamma \in C^{1,\alpha}$, $\|\Gamma\|_{C^{1,\alpha}} \leq 1$, and U be as in (2.5). Assume that $u \in C(B_1)$ is even and vanishes on \mathcal{P} , $\|u\|_{L^\infty} \leq 1$, and*

$$\Delta u(X) = q \frac{U_0}{r} + F(X) \quad \text{in } B_1 \setminus \mathcal{P},$$

with

$$|F(X)| \leq r^{-1/2} |X|^\alpha \quad \text{and } |q| \leq 1.$$

There exists a polynomial $P(x, r)$ of degree 1 with coefficients bounded by C such that

$$\left| \frac{u}{U} - P \right| \leq C |X|^{1+\alpha},$$

with C depending on α, n .

As usually, after performing an initial dilation we may assume that:

$$(4.2) \quad \|g\|_{C^{1,\alpha}} \leq \delta, \quad |q| \leq \delta, \quad |F| \leq \delta r^{-1/2} |X|^\alpha.$$

Proposition 4.5 will follow as in the case $k \geq 1$, after we extend the definition of approximating polynomial to this case i.e. now P is a polynomial in (x, \bar{r}) , rather than (x, r) .

Let $P(x, \bar{r})$ be a polynomial of degree one in x and \bar{r} ,

$$P(x, \bar{r}) = a_0 + \sum_{i=1}^n a_i x_i + a_{n+1} \bar{r}.$$

We claim that

$$(4.3) \quad \Delta(UP) = \frac{U_0}{r} [a_n + 2a_{n+1} + O(\delta |X|^\alpha)].$$

Definition 4.6. We say that P is approximating for u/U at 0 if

$$a_n + 2a_{n+1} = q.$$

The proof of the claim is postponed until later. Now, with this definition the proof of Proposition 4.5 is a consequence of the next lemma, whose proof is identical to the case $k \geq 1$.

Lemma 4.7. *There exist universal constants ρ and δ , depending only on α and n , such that if $P_0(x, \bar{r})$ with $\|P_0\| \leq 1$ is an approximating polynomial for u/U in B_λ , that is P is approximating for u/U at 0 and*

$$|u - UP|_{L^\infty(B_\lambda)} \leq \lambda^{3/2+\alpha},$$

for some $\lambda > 0$, then there exists an approximating polynomial $P_1(x, \bar{r})$ for u/U at 0 such that, in $B_{\rho\lambda}$,

$$|u - UP_1|_{L^\infty(B_{\rho\lambda})} \leq (\rho\lambda)^{3/2+\alpha} \quad \text{and} \quad \|P_1 - P_0\|_{L^\infty(B_\lambda)} \leq C\lambda^{1+\alpha}.$$

Notice that, in view of Lemma 4.3,

$$|P(x, r) - P(x, \bar{r})| \leq C\delta \|P\|r^{1+\alpha},$$

and Proposition 4.5 follows as before.

It remains to prove formula (4.3). We compute that

$$(4.4) \quad \Delta(U P) = a_{n+1} U \Delta \bar{r} + 2a_i U_i + 2a_{n+1} \nabla \bar{r} \cdot \nabla U.$$

Now we use Theorem 4.1, Lemma 4.2 and estimate $U, \nabla U$ in terms of $U_0, \nabla U_0$ together with the estimates for $\nabla \bar{r}, \Delta \bar{r}$ from Lemma 4.3. From Theorem 4.1, Lemma 4.2, we see that after multiplication by a constant and a dilation we may suppose that the function U satisfies

$$U = U_0(1 + O(\delta|X|^\alpha)),$$

$$\nabla_x U = \nabla_x U_0 + O\left(\delta \frac{U_0}{r} |X|^\alpha\right), \quad \partial_{x_{n+1}} U = \partial_{x_{n+1}} U_0 + O(\delta r^{-1/2} |X|^\alpha).$$

Since

$$\Delta \bar{r} = \frac{1}{r} + O(\delta r^{\alpha-1}),$$

$$\nabla_x \bar{r} = \nabla_x r + O(\delta r^\alpha), \quad \partial_{x_{n+1}} \bar{r} = \partial_{x_{n+1}} r + O(\delta U_0 r^{\alpha-1/2}),$$

and also

$$\begin{aligned} \nabla_x U_0 &= \frac{U_0}{2r} \nabla_x d, \quad \nabla_x d = e_n + O(\delta|X|^\alpha), \\ |\nabla U_0| &\leq C r^{-1/2}, \quad \nabla r \cdot \nabla U_0 = \frac{U_0}{2r}, \quad |\partial_{x_{n+1}} r| \leq C U_0 r^{-1/2}, \end{aligned}$$

we easily obtain (4.3) from (4.4).

5. Proof of some technical lemmas

In this section we collect the proofs of several technical lemmas. We start with the approximation Lemma 4.3. For the reader's convenience we state it again.

Lemma 5.1. *Let $\|\Gamma\|_{C^{1,\alpha}} \leq 1$. There exist smooth functions \bar{r}, \bar{U}_0 such that*

$$\begin{aligned} \left| \frac{\bar{r}}{r} - 1 \right| &\leq C r^\alpha, & \left| \frac{\bar{U}_0}{U_0} - 1 \right| &\leq C r^\alpha, \\ |\nabla \bar{r} - \nabla r| &\leq C r^\alpha, & |\partial_{x_{n+1}} \bar{r} - \partial_{x_{n+1}} r| &\leq C r^{\alpha-1/2} U_0, \\ \left| \Delta \bar{r} - \frac{1}{r} \right| &\leq C r^{\alpha-1}, & |\Delta \bar{U}_0| &\leq C r^{\alpha-3/2}, \end{aligned}$$

with C universal. Moreover if we assume $\|\Gamma\|_{C^{1,\alpha}} \leq \delta$, small, then the constant C above is replaced by $C\delta$.

Proof. The idea is to smooth out the signed distance function d to Γ and then smooth out r and U_0 using the formulas

$$r = \sqrt{d^2 + x_{n+1}^2} \quad \text{and} \quad U_0 = \frac{1}{\sqrt{2}}\sqrt{r + d}.$$

We divide the proof in three steps. Whenever we write $\nabla d, \nabla r, \nabla U_0$ we assume we are at a point where these locally Lipschitz functions are differentiable.

Step 1. We start by constructing \bar{d} by smoothing the signed distance d to Γ in dyadic tubular neighborhoods, and then we glue them together. First, define the open tubular neighborhood of Γ ,

$$D_\lambda = \{x \in \mathbb{R}^n : |d| < \lambda\}, \quad \lambda \text{ small.}$$

We set

$$d_\lambda := d * \rho_\lambda, \quad \rho_\lambda = \lambda^{-n} \rho\left(\frac{x}{\lambda}\right),$$

with ρ a symmetric kernel supported in $B_{1/10}$.

We claim that

$$(5.1) \quad |d_\lambda - d| \leq C\lambda^{1+\alpha}, \quad |\nabla d_\lambda - \nabla d| \leq C\lambda^\alpha, \quad |D^2 d_\lambda| \leq C\lambda^{\alpha-1} \quad \text{in } D_{4\lambda}.$$

We check our claim at a point x_0 on the x_n -axis. Since $\|\Gamma\|_{C^{1,\alpha}} \leq 1$, we have

$$|d - x_n| \leq C\lambda^{1+\alpha}, \quad \text{in } B_{4\lambda},$$

and we remark that if $\|\Gamma\|_{C^{1,\alpha}} \leq \delta$ then we may replace C by $C\delta$.

Thus,

$$d = x_n + \lambda^{1+\alpha}v, \quad |v| \leq C,$$

and using that $x_n * \rho_\lambda = x_n$ we get

$$d_\lambda = d * \rho_\lambda = x_n + \lambda^{1+\alpha} (v * \rho_\lambda).$$

This gives,

$$\nabla d_\lambda = e_n + \lambda^{1+\alpha} (v * \nabla \rho_\lambda), \quad D^2 d_\lambda = \lambda^{1+\alpha} (v * D^2 \rho_\lambda)$$

and the claim (5.1) follows by using that

$$\int \lambda |\nabla \rho_\lambda| dx \leq C, \quad \int \lambda^2 |D^2 \rho_\lambda| dx \leq C, \quad |\nabla d(x_0) - e_n| \leq C\lambda^\alpha.$$

The function d_λ approximates d up to an error $\lambda^{1+\alpha}$ in D_λ . Next we interpolate between various d_λ with $\lambda = \lambda_k = 4^{-k}$ in the annular sets $\mathcal{A}_\lambda := \{\lambda < d < 4\lambda\}$.

We define \bar{d} to coincide with d_λ in $\mathcal{A}_\lambda \cap D_{2\lambda}$ and with $d_{4\lambda}$ in $\mathcal{A}_\lambda \setminus D_{3\lambda}$. Precisely, let

$$\bar{d} = \varphi d_\lambda + (1 - \varphi) d_{4\lambda},$$

with φ a cutoff function

$$(5.2) \quad \varphi = 0 \quad \text{if } d > 3\lambda, \quad \varphi = 1 \quad \text{if } d < 2\lambda.$$

We set

$$\varphi = h\left(\frac{d_\lambda}{\lambda}\right),$$

with

$$h(t) = 1 \quad \text{if } t \leq 2 + 1/4, \quad h(t) = 0 \quad \text{if } t \geq 2 + 3/4$$

and h smooth in between. Thus,

$$(5.3) \quad |\nabla\varphi| \leq C\lambda^{-1}, \quad |D^2\varphi| \leq C\lambda^{-2}.$$

Then, \bar{d} satisfies

$$(5.4) \quad |\bar{d} - d| \leq C\lambda^{1+\alpha}, \quad |\nabla\bar{d} - \nabla d| \leq C\lambda^\alpha, \quad |D^2\bar{d}| \leq C\lambda^{\alpha-1} \quad \text{in } \mathcal{A}_\lambda.$$

This follows immediately after computing

$$(5.5) \quad \begin{aligned} \nabla\bar{d} &= \varphi\nabla d_\lambda + (1 - \varphi)\nabla d_{4\lambda} + (d_\lambda - d_{4\lambda})\nabla\varphi, \\ D^2\bar{d} &= \varphi D^2 d_\lambda + (1 - \varphi)D^2 d_{4\lambda} + 2(\nabla d_\lambda - \nabla d_{4\lambda}) \otimes \nabla\varphi + (d_\lambda - d_{4\lambda})D^2\varphi, \end{aligned}$$

and then using (5.1) and (5.3).

Step 2. We construct \bar{r} in a similar fashion as \bar{d} . We first construct approximations r_λ in dyadic annular regions \mathcal{R}_λ in \mathbb{R}^{n+1} , and then we “glue” them together. We define

$$(5.6) \quad r_\lambda := \sqrt{d_\lambda^2 + x_{n+1}^2}, \quad \text{in } \mathcal{R}_\lambda = \{\lambda/2 < r < 4\lambda\}.$$

with d_λ as in Step 1. Then r , r_λ and λ are all comparable to each other in \mathcal{R}_λ and we claim that

$$(5.7) \quad |r_\lambda - r| \leq C\lambda^{1+\alpha}, \quad |\Delta r_\lambda - 1/r| \leq C\lambda^{\alpha-1}$$

$$(5.8) \quad |\nabla r_\lambda - \nabla r| \leq C\lambda^\alpha, \quad \|D^2 r_\lambda\| \leq C/\lambda.$$

Indeed, using the first inequality in (5.1) we obtain

$$|r_\lambda^2 - r^2| = |d_\lambda^2 - d^2| \leq C\lambda^{2+\alpha}.$$

This gives the first inequality in (5.7), hence

$$(5.9) \quad \left| \frac{r_\lambda}{r} - 1 \right| \leq C\lambda^\alpha.$$

Since

$$(5.10) \quad \nabla r_\lambda = \frac{1}{r_\lambda}(d_\lambda \nabla_x d_\lambda, x_{n+1})$$

the inequalities in (5.8) follow easily from the estimates for d_λ (see (5.1), (5.9)).

From (5.8) and (5.1) we obtain

$$|\nabla r_\lambda| - 1 = O(\lambda^\alpha), \quad |\nabla d_\lambda| - 1 = O(\lambda^\alpha).$$

Then the identity

$$r_\lambda \Delta r_\lambda + |\nabla r_\lambda|^2 = \frac{1}{2} \Delta r_\lambda^2 = \frac{1}{2} \Delta (d_\lambda^2 + x_{n+1}^2) = d_\lambda \Delta d_\lambda + |\nabla d_\lambda|^2 + 1$$

implies

$$r_\lambda \Delta r_\lambda = 1 + O(\lambda^\alpha),$$

which gives the second inequality in (5.7) and our claim is proved.

Next we glue various r_λ 's with $\lambda = \lambda_k = 4^{-k}$. In the regions $\{\lambda_k < r < 4\lambda_k\}$ we define

$$\bar{r} = \varphi r_\lambda + (1 - \varphi)r_{4\lambda}, \quad \text{with } \varphi := h(r_\lambda/\lambda), \quad h \text{ as above.}$$

Notice that φ satisfies (5.3), and (5.8) holds with d replaced by r . This shows that \bar{r} still satisfies in this region

$$(5.11) \quad |\bar{r} - r| \leq Cr^{1+\alpha}, \quad |\nabla \bar{r} - \nabla r| \leq Cr^\alpha, \quad \left| \Delta \bar{r} - \frac{1}{r} \right| \leq Cr^{\alpha-1}.$$

Moreover (5.10) and $|\partial_{x_{n+1}} \varphi| \leq C|x_{n+1}|/\lambda^2$ imply

$$|\partial_{x_{n+1}} \bar{r} - \partial_{x_{n+1}} r| \leq C \frac{|x_{n+1}|}{r} \lambda^\alpha \leq C \frac{U_0}{r^{1/2}} \lambda^\alpha.$$

Step 3. We construct \bar{U}_0 . As before,

$$(U_0)_\lambda = \frac{1}{\sqrt{2}}(d_\lambda + r_\lambda)^{1/2}, \quad \text{in } \mathcal{R}_\lambda.$$

Below we show that $(U_0)_\lambda$ satisfies the following inequalities:

$$(5.12) \quad \left| \frac{(U_0)_\lambda}{U_0} - 1 \right| \leq C \lambda^\alpha, \quad |\nabla (U_0)_\lambda - \nabla U_0| \leq C \lambda^{\alpha-1/2},$$

$$(5.13) \quad |\Delta (U_0)_\lambda| \leq C \lambda^{\alpha-3/2}.$$

Then the function \bar{U}_0 is obtained as in Step 2 by interpolating the various $(U_0)_\lambda$'s.

We check the inequalities above separately in the regions

$$\mathcal{R}_\lambda^1 := \mathcal{R}_\lambda \cap \{d > -r/2\} \quad \text{and} \quad \mathcal{R}_\lambda^2 := \mathcal{R}_\lambda \cap \{d < -r/2\},$$

depending whether or not we are closer to the set \mathcal{P} where U_0 and $(U_0)_\lambda$ vanish.

Case 1. In the region \mathcal{R}_λ^1 we know that U_0 , $(U_0)_\lambda$ and $\lambda^{1/2}$ are comparable to each other and

$$(U_0)_\lambda = U_0 \left(\frac{r_\lambda + d_\lambda}{r + d} \right)^{1/2}.$$

From Steps 1 and 2 we have

$$\frac{r_\lambda + d_\lambda}{r + d} = 1 + O(\lambda^\alpha), \quad \nabla \frac{r_\lambda + d_\lambda}{r + d} = O(\lambda^{\alpha-1}),$$

and we easily obtain the two inequalities in (5.12). In particular we find

$$|\nabla(U_0)_\lambda| = |\nabla U_0| + O(\lambda^{\alpha-1/2}) = \frac{1}{2} r^{-1/2} + O(\lambda^{\alpha-1/2}).$$

Now (5.13) follows from the identity (see also (5.1), (5.7))

$$(U_0)_\lambda \Delta(U_0)_\lambda + |\nabla(U_0)_\lambda|^2 = \frac{1}{4} \Delta(d_\lambda + r_\lambda).$$

Case 2. In the region \mathcal{R}_λ^2 we know that U_0 , $(U_0)_\lambda$ and $|x_{n+1}| \lambda^{-1/2}$ are comparable to each other since

$$(U_0)_\lambda = |x_{n+1}| (r_\lambda - d_\lambda)^{-1/2} = U_0 \left(\frac{r_\lambda - d_\lambda}{r - d} \right)^{-1/2},$$

and (5.12) is obtained as above. In order to prove (5.13) we write (assume $x_{n+1} > 0$)

$$\begin{aligned} \Delta(U_0)_\lambda &= 2\partial_{x_{n+1}}(r_\lambda - d_\lambda)^{-1/2} + x_{n+1} \Delta(r_\lambda - d_\lambda)^{-1/2} \\ (5.14) \quad &= \frac{x_{n+1}}{2} (r_\lambda - d_\lambda)^{-3/2} \left(-2 \frac{1}{r_\lambda} - \Delta(r_\lambda - d_\lambda) + \frac{3}{2} \frac{|\nabla(r_\lambda - d_\lambda)|^2}{r_\lambda - d_\lambda} \right), \end{aligned}$$

and we used $\partial_{x_{n+1}} r_\lambda = x_{n+1}/r_\lambda$ and $\partial_{x_{n+1}} d_\lambda = 0$. Since

$$|\nabla(r_\lambda - d_\lambda)|^2 = r_\lambda^{-2} ((r_\lambda - d_\lambda)^2 |\nabla d_\lambda|^2 + x_{n+1}^2) = r_\lambda^{-2} (2r_\lambda(r_\lambda - d_\lambda) + O(\lambda^{2+\alpha}))$$

we find that the quantity in the parenthesis in (5.14) is $O(\lambda^{\alpha-1})$ and our claim is proved. \square

In the next lemma we obtain an L^∞ bound for solutions to the Laplace equation with right hand side that degenerates near Γ .

Lemma 5.2 (Barrier). *Assume $\|\Gamma\|_{C^{1,\alpha}} \leq \delta$ with $\alpha \in (0, 1/2)$, and*

$$|\Delta u| \leq r^{\alpha-3/2} \quad \text{in } B_1 \setminus \mathcal{P}, \quad u = 0 \quad \text{on } \mathcal{P} \cup \partial B_1.$$

Then

$$|u| \leq C U_0 \quad \text{in } B_1,$$

with C and δ depending on n and α .

Proof. We construct an upper barrier for u . Let

$$V := \bar{U}_0 - \bar{U}_0^{1+2\alpha},$$

and notice that $V \geq 0$ in B_1 .

We compute

$$\Delta V = \Delta \bar{U}_0 - (1 + 2\alpha)\bar{U}_0^{2\alpha-1}(\bar{U}_0 \Delta \bar{U}_0 + 2\alpha|\nabla \bar{U}_0|^2).$$

By Lemma 5.1 we have

$$|\bar{U}_0| \leq r^{1/2}, \quad |\Delta \bar{U}_0| \leq C\delta r^{\alpha-3/2}, \quad |\nabla \bar{U}_0| \geq cr^{-1/2},$$

thus we obtain

$$\Delta V \leq -cr^{\alpha-3/2}.$$

Since $V \geq 0$ and $\Delta(CV) \leq \Delta u$, we apply maximum principle in $B_1 \setminus \mathcal{P}$ and obtain

$$u \leq CV \leq C\bar{U}_0 \quad \text{in } B_1. \quad \square$$

Remark 5.3. L^∞ bounds for u hold also for more degenerate right hand side. If for some small $\gamma > 0$,

$$|\Delta u| \leq r^{\gamma-2}, \quad u = 0 \quad \text{on } \mathcal{P} \cup \partial B_1,$$

then

$$|u| \leq Cr^\gamma.$$

Indeed, in this case we can use $V = \bar{U}_0^{2\gamma}$ as an upper barrier.

Proof of Lemma 4.2. We have

$$|u - aU_0| \leq Cr^\alpha U_0 \leq Cr^{1/2+\alpha}.$$

The idea is to replace U_0 with an appropriate function U_0^* which measures the distances d and r to a straight boundary instead of Γ . Notice that u and U_0^* are both harmonic.

We pick a point X_0 at distance λ from Γ . Assume for simplicity of notation that the closest point to X_0 on Γ is the origin 0, thus X_0 belongs to the 2D plane $\{x' = 0\}$. Let U_0^* denote the 1-dimensional solution with respect to the straight boundary $L := \{x_n = 0\}$, i.e.,

$$U_0^* := U_{0,L} = \frac{1}{\sqrt{2}}\sqrt{x_n + r^*}, \quad r^* := \sqrt{x_n^2 + x_{n+1}^2}.$$

Notice that U_0^* and r^* coincide with U_0 and r at the point X_0 . Moreover, if d , r , and U_0 are differentiable at X_0 , then

$$\nabla d = \nabla x_n = e_n, \quad \nabla U_0 = \nabla U_0^*, \quad \nabla r = \nabla r^* \quad \text{at } X_0.$$

In the conical region

$$\mathcal{C} := \{|x'| < r^*\} \cap \{\lambda/2 < r^* < 2\lambda\}$$

we use that $\|\Gamma\|_{C^{1,\alpha}} \leq 1$ and obtain (as in (5.12))

$$|U_0^* - U_0| \leq C\lambda^{1/2+\alpha},$$

thus

$$|u - aU_0^*| \leq C\lambda^{1/2+\alpha} \quad \text{in } \mathcal{C}.$$

Since $u - aU_0^*$ is harmonic and vanishes on \mathcal{P} we apply gradient estimates and obtain

$$|\nabla u - a \nabla U_0^*| \leq C \lambda^{\alpha-1/2}, \quad |\nabla_x u - a \nabla_x U_0^*| \leq C \lambda^{\alpha-1} U_0^* \quad \text{at } X_0,$$

and we replace U_0^* by U_0 in the inequalities above.

In conclusion, at an arbitrary point $X \in B_{1/2}$ where U_0 is differentiable we have

$$|\nabla u - a_{\pi(X)} \nabla U_0| \leq C r^{\alpha-1/2} \quad \text{and} \quad |\nabla_x u - a_{\pi(X)} \nabla_x U_0| \leq C r^{\alpha-1} U_0,$$

where $\pi(X)$ is the projection of X onto Γ and $a_{\pi(X)}$ represents the corresponding constant for the expansion of u at $\pi(X)$. The lemma is proved since

$$\begin{aligned} |a_{\pi(X)} - a| &\leq C |\pi(X)|^\alpha \leq C |X|^\alpha, \quad r \leq |X|, \\ |\nabla U_0| &\leq C r^{-1/2}, \quad \nabla_x U_0 = \frac{U_0}{2r} \nabla_x d, \quad |\nabla_x U_0| \leq C \frac{U_0}{r}. \end{aligned}$$

□

Finally we conclude with the proof of Lemma 3.2.

Proof of Lemma 3.2. By Lemma 5.5 in [9], we see that

$$\begin{aligned} U_i &= \partial_{x_i}(U_0 P_0) + O\left(\frac{U_0}{r} |X|^{k+\alpha}\right) \quad i = 1, 2, \dots, n, \\ U_{n+1} &= \partial_{x_{n+1}}(U_0 P_0) + O(|X|^{k-1/2+\alpha}) \end{aligned}$$

in the cone $\mathcal{C}_0 = \{|(x_n, x_{n+1})| > |x'|\}$. Since $|\partial_{x_{n+1}} r| \leq r^{-1/2} U_0$, we find

$$\partial_r U = \partial_r(U_0 P_0) + O\left(\frac{U_0}{r} |X|^{k+\alpha}\right) \quad \text{in } \mathcal{C}_0,$$

and the conclusion of the lemma follows as in [9], by writing the equality above for all corresponding cones \mathcal{C}_Z , $Z \in \Gamma$. □

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