Rev. Mat. Iberoam. **32** (2016), no. 2, 649–750
DOI 10.4171/RMI/897

Fitting a Sobolev function to data II

Charles Fefferman, Arie Israel, and Garving Luli

Abstract. In this paper and two companion papers, we produce efficient algorithms to solve the following interpolation problem. Let $m \geq 1$ and $p > n \geq 1$. Given a finite set $E \subset \mathbb{R}^n$ and a function $f : E \to \mathbb{R}$, compute an extension F of f belonging to the Sobolev space $W^{m,p}(\mathbb{R}^n)$ with norm having the smallest possible order of magnitude; secondly, compute the order of magnitude of the norm of F. The combined running time of our algorithms is at most $CN \log N$, where N denotes the cardinality of E, and C depends only on m, n, and p.

Introduction

Continuing from [1], we interpolate data by a function $F : \mathbb{R}^n \to \mathbb{R}$ whose Sobolev norm has the least possible order of magnitude. More precisely, let $m \geq 1$ and $p > n \geq 1$. Given a function $f : E \to \mathbb{R}$ with $E \subset \mathbb{R}^n$ finite, we compute a function $F \in W^{m,p}(\mathbb{R}^n)$ such that $F = f$ on E, and $\|F\|_{W^{m,p}} \leq C\|\widetilde{F}\|_{W^{m,p}}$ for any competing function $\widetilde{F} \in W^{m,p}(\mathbb{R}^n)$ such that $\widetilde{F} = f$ on E. Here, C depends only on m, n, and p.

Our computations consist of efficient algorithms to be implemented on an (idealized) von Neumann computer. We study two distinct models of computation. In the first model ("infinite precision"), we assume that our computer deals with exact real numbers, without roundoff error. Our second, more realistic model of computation assumes that our machine handles only S-bit machine numbers, for some fixed, large S.

In our previous paper [1], we stated our main results for infinite precision, and developed technical tools to be used in the proof of those results. Here, we complete the proof of our results for infinite precision. Issues arising from the finite-precision model of computation will be addressed in [2].

Mathematics Subject Classification (2010): Primary 65D17; Secondary 65D05.
Keywords: Algorithm, interpolation, Sobolev spaces.

1. Proof of the main technical results

1.1. Review of notation

Let $\mathcal{M}$ denote the collection of all multiindices $\alpha = (\alpha_1, \ldots, \alpha_n)$ of order $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq m - 1$. Let $\mathcal{A} \subset \mathcal{M}$ be a multiindex set. We refer the reader to Section 3 of [1] for a description of the main technical results for $\mathcal{A}$. We will prove the main technical results by induction on $\mathcal{A}$.

We recall the relevant order relation $<$ on multiindex sets $\mathcal{A} \subset \mathcal{M}$ (see Section 2.6 of [1]). We first define a total order relation on $\mathcal{M}$ as follows: given distinct $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in \mathcal{M}$, let $k \in \{1, \ldots, n\}$ be the maximal index such that $\alpha_1 + \cdots + \alpha_k \neq \beta_1 + \cdots + \beta_k$. Then $\alpha < \beta$ if $\alpha_1 + \cdots + \alpha_k < \beta_1 + \cdots + \beta_k$, and $\alpha > \beta$ otherwise. We next define a total order relation $<$ on $2^{\mathcal{M}}$. Given distinct subsets $\mathcal{A}, \mathcal{B} \subset \mathcal{M}$, pick the minimal element $\alpha \in \mathcal{A} \Delta \mathcal{B}$ (with respect to the order relation defined above). Then $\mathcal{A} < \mathcal{B}$ if $\alpha \in \mathcal{A}$, and $\mathcal{B} < \mathcal{A}$ otherwise. ($\mathcal{A}\Delta\mathcal{B}$ denotes the symmetric difference of $\mathcal{A}$ and $\mathcal{B}$.) Note that $\mathcal{M}$ is minimal and the empty set $\emptyset$ is maximal with respect to this order relation on $2^{\mathcal{M}}$.

We recall the notion of *monotonic* multiindex sets (see Section 2.6 of [1]). A set $\mathcal{A} \subset \mathcal{M}$ is called monotonic if for every $\alpha \in \mathcal{A}$ and $\gamma \in \mathcal{M}$ with $|\gamma| \leq m - 1 - |\alpha|$, we have $\alpha + \gamma \in \mathcal{A}$. One of the key properties of monotonic multiindex sets is the following: let $x \in \mathbb{R}^n$, and let P be a polynomial on $\mathbb{R}^n$ of degree at most $m - 1$. Then

$$\mathcal{A} \text{ monotonic}, \quad \partial^\alpha P(x) = 0 \ \ \forall \alpha \in \mathcal{A} \quad \Longrightarrow \quad \partial^\alpha P \equiv 0 \quad \text{on } \mathbb{R}^n \ \ \forall \alpha \in \mathcal{A}. \tag{1.1}$$

Fix a finite subset $E \subset \frac{1}{32} Q^\circ$, where Q° denotes the unit cube $[0, 1)^n$. Let $N := \#(E)$. We assume that $N \geq 2$.

Given a multiindex set $\mathcal{A} \subset \mathcal{M}$ and $\epsilon > 0$, we say that a cube $Q \subset \mathbb{R}^n$ is tagged with $(\mathcal{A}, \epsilon)$ if either $\#(E \cap Q) \leq 1$ or there exists $\mathcal{A}' \leq \mathcal{A}$ such that

$$\sigma(Q) \text{ has an } (\mathcal{A}', x_Q, \epsilon, \delta_Q)\text{-basis} \quad (\text{recall: } x_Q = \text{the center of } Q).$$

We refer the reader to Section 2.7.1 of [1] for the definition of a basis for a convex set of polynomials.

1.2. Starting the induction

We first establish the base case of the induction. This corresponds to proving the main technical results for $\mathcal{A} = \mathcal{M}$.

Let $CZ(\mathcal{M})$ be the collection of maximal dyadic cubes $Q \subset Q^\circ$ such that $\#(E \cap 3Q) \leq 1$.

Using one time-work at most $CN \log N$ in space CN, we produce a $CZ(\mathcal{M})$-ORACLE that answers queries as follows.

- A query consists of a point $\underline{x} \in Q^\circ$.
- The response to the query $\underline{x}$ is a list of all the cubes $Q \in CZ(\mathcal{M})$ such that $\underline{x} \in \frac{65}{64} Q$.
- The work and storage required to answer a query are at most $C \log N$.

We simply apply the PLAIN VANILLA CZ-ORACLE from Section 4.6.3 of [1]; see Remark 47 from the same section.

Since $\#(E) \geq 2$ and $E \subset Q^\circ$, the collection $\mathrm{CZ}(\mathcal{M})$ does not contain the cube Q°. Therefore, each $Q \in \mathrm{CZ}(\mathcal{M})$ is a strict subcube of Q°, hence Q has a dyadic parent $Q^+ \subset Q^\circ$ such that $\#(3Q^+ \cap E) \geq 2$ (because Q is maximal), and so in particular

$$\#(9Q \cap E) \geq 2 \quad \text{for all } Q \in \mathrm{CZ}(\mathcal{M}). \tag{1.2}$$

Recall that we say two dyadic cubes Q and Q' *touch* either if $Q = Q'$, or if Q is disjoint from Q' but the boundaries ∂Q and $\partial Q'$ have a nonempty intersection; we write $Q \leftrightarrow Q'$ to indicate that Q touches Q'.

Lemma 1. *If* $Q, Q' \in \mathrm{CZ}(\mathcal{M})$ *and* $Q \leftrightarrow Q'$, *then* $\frac{1}{2}\delta_Q \leq \delta_{Q'} \leq 2\delta_Q$.

Proof. We proceed by contradiction. Suppose that $Q \leftrightarrow Q'$ and $\delta_Q \leq \frac{1}{4}\delta_{Q'}$ for some $Q, Q' \in \mathrm{CZ}(\mathcal{M})$. Then $3Q^+ \subset 3Q'$, and hence $\#(E \cap 3Q^+) \leq \#(E \cap 3Q') \leq 1$. However, this contradicts that $\#(3Q^+ \cap E) \geq 2$, completing the proof of the lemma. □

Lemma 2. *There exists* $\epsilon_1 > 0$, *depending only on* m, n, *and* p, *such that* $9Q$ *is not tagged with* $(\mathcal{M}, \epsilon_1)$ *for any* $Q \in \mathrm{CZ}(\mathcal{M})$.

Proof. Assume that $\epsilon_1 \in (0, 1)$ is less than a small enough universal constant.

Let $Q \in \mathrm{CZ}(\mathcal{M})$. Thanks to (1.2), it suffices to show that $\sigma(9Q)$ does not have an $(\mathcal{M}, x_Q, \epsilon_1, \delta_{9Q})$-basis.

We argue by contradiction. Suppose that $(P_\alpha)_{\alpha \in \mathcal{M}}$ is an $(\mathcal{M}, x_Q, \epsilon_1, \delta_{9Q})$-basis for $\sigma(9Q)$. Therefore, $P_0(x_Q) = 1$, and $\partial^\alpha P_0(x_Q) = 0$ for $\alpha \in \mathcal{M}$, $\alpha \neq 0$. In other words, $P_0 \equiv 1$.

Moreover, there exists $\varphi_0 \in \mathbb{X}$ such that $\varphi_0 = 0$ on $E \cap 9Q$ and

$$\|\varphi_0\|_{\mathbb{X}(9Q)} + \delta_{9Q}^{-m}\|\varphi_0 - P_0\|_{L^p(9Q)} \leq \epsilon_1 \delta_{9Q}^{n/p-m}.$$

We know that $\#(E \cap 9Q) \geq 2$. Fix $x \in E \cap 9Q$. We apply the Sobolev-type estimate stated in Lemma 10 of [1] to obtain

$$\delta_Q^{n/p-m} \cdot |\varphi_0(x) - P_0(x)| \leq C\big(\|\varphi_0\|_{\mathbb{X}(9Q)} + \delta_Q^{-m}\|\varphi_0 - P_0\|_{L^p(9Q)}\big) \leq C'\epsilon_1 \delta_{9Q}^{n/p-m}.$$

But $\varphi_0(x) = 0$, and thus $|P_0(x)| \leq C''\epsilon_1$. However, if we take $\epsilon_1 < 1/C''$, then this inequality contradicts the fact that $P_0 \equiv 1$. □

Recall that $\#(3Q \cap E) \leq 1$ for each $Q \in \mathrm{CZ}(\mathcal{M})$. This implies the next result.

Lemma 3. *If* $Q \in \mathrm{CZ}(\mathcal{M})$ *then* $3Q$ *is tagged with* $(\mathcal{M}, 1/2)$.

We refer the reader to Section 3 of [1] for a statement of conditions **(CZ1)**–**(CZ5)**, which are a component of the main technical results. We now establish **(CZ1)**–**(CZ5)** for $\mathrm{CZ}(\mathcal{M})$. Indeed, **(CZ1)**, **(CZ2)**, and **(CZ4)** follow from

Lemmas 1, 2, and 3, respectively. Furthermore, **(CZ3)** and **(CZ5)** are vacuously true because we are treating the base case $\mathcal{A} = \mathcal{M}$.

We next describe the algorithms COMPUTE MAIN-CUBES, COMPUTE FUNCTIONALS, and COMPUTE EXTENSION OPERATORS from the main technical results.

We set $\mathrm{CZ}_{\mathrm{main}}(\mathcal{M}) := \{Q \in \mathrm{CZ}(\mathcal{M}) : (65/64)Q \cap E \neq \emptyset\}$. For $Q \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{M})$, we denote the unique point in $E \cap \frac{65}{64}Q$ by $x(Q)$. (Recall that $\#(E \cap 3Q) \leq 1$ for $Q \in \mathrm{CZ}(\mathcal{M})$.)

Next, we define an extension operator and a linear functional associated to each $Q \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{M})$.

- We define a linear map $T_{(Q,\mathcal{M})} : \mathbb{X}(\frac{65}{64}Q \cap E) \oplus \mathcal{P} \to \mathbb{X}$ given by

$$T_{(Q,\mathcal{M})}(f, P) = P + f(x(Q)) - P(x(Q)). \tag{1.3}$$

- We define a list $\Xi(Q, \mathcal{M}) = \{\xi_Q\}$, where

$$\xi_Q(f, P) = \big(f(x(Q)) - P(x(Q))\big) \cdot \delta_Q^{n/p-m}. \tag{1.4}$$

- We take the list of assist functionals $\Omega(Q, \mathcal{M})$ to be empty.

The functional ξ_Q and linear map $T_{(Q,\mathcal{M})}$ obviously have $\Omega(Q, \mathcal{M})$-assisted bounded depth (i.e., bounded depth).

ALGORITHM: COMPUTE MAIN-CUBES AND COMPUTE EXTENSION OPERATOR

We compute a list of the cubes in $\mathrm{CZ}_{\mathrm{main}}(\mathcal{M})$. For each $Q \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{M})$, we compute a short form description of the bounded depth functional

$$\xi_Q : \mathbb{X}\Big(\frac{65}{64}Q \cap E\Big) \oplus \mathcal{P} \to \mathbb{R}.$$

We give a query algorithm, which requires work at most $C \log N$ to answer a query. A query consists of a cube $\underline{Q} \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{M})$ and point $\underline{x} \in Q^\circ$. The response to the query $(\underline{Q}, \underline{x})$ is a short form description of the linear map

$$(f, P) \mapsto J_{\underline{x}} T_{(\underline{Q},\mathcal{M})}(f, P).$$

These computations require one-time work at most $CN \log N$ in space CN.

Explanation. We compute a list of all the cubes $Q \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{M})$ and a list of associated points $x(Q) \in E \cap \frac{65}{64}Q$. This computation requires work at most $CN \log N$ in space CN; see the algorithm FIND MAIN-CUBES in Section 4.6.4 of [1].

For each Q in $\mathrm{CZ}_{\mathrm{main}}(\mathcal{M})$, we compute the linear functional

$$\xi_Q(f, P) = \big\{f(x(Q)) - P(x(Q))\big\} \cdot \delta_Q^{n/p-m}.$$

There are at most CN such functionals, and each one is computed using work and storage at most C.

Given a query $(\underline{Q}, \underline{x}) \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{M}) \times Q^\circ$, by a binary search we determine the position of $\underline{Q}$ in the list $\mathrm{CZ}_{\mathrm{main}}(\mathcal{M})$. We next compute the linear map

$$(f, P) \mapsto J_{\underline{x}} T_{(\underline{Q},\mathcal{M})}(f, P) = P + f(x(\underline{Q})) - P(x(\underline{Q})).$$

This requires work at most $C \log N$ per query. □

We finish by giving a proof of the remaining properties **(E1)**–**(E3)** in the main technical results for $\mathcal{M}$. (See Section 3 of [1].)

Lemma 4. *There exists* $C \geq 1$*, depending only on* m, n*, and* p*, such that for each* $Q \in CZ_{\mathrm{main}}(\mathcal{M})$*, the following properties hold.*

- $T_{(Q,\mathcal{M})}(f,P) = f$ *on* $\frac{65}{64}Q \cap E$.
- $\|T_{(Q,\mathcal{M})}(f,P)\|_{\mathbb{X}(\frac{65}{64}Q)} + \delta_Q^{-m}\|T_{(Q,\mathcal{M})}(f,P) - P\|_{L^p(\frac{65}{64}Q)} \leq C \cdot |\xi_Q(f,P)|$.
- $C^{-1} \cdot \|(f,P)\|_{\frac{65}{64}Q} \leq |\xi_Q(f,P)| \leq C \cdot \|(f,P)\|_{\frac{65}{64}Q}$.

Proof. Note that $E \cap \frac{65}{64}Q = \{x(Q)\}$ and $T_{(Q,\mathcal{M})}(f,P)(x(Q)) = f(x(Q))$ for each $Q \in CZ_{\mathrm{main}}(\mathcal{M})$. This implies the first bullet point.

Recall that $T_{(Q,\mathcal{M})}(f,P) \in \mathcal{P}$, hence $\|T_{(Q,\mathcal{M})}(f,P)\|_{\mathbb{X}(\frac{65}{64}Q)} = 0$. Moreover,

$$\begin{aligned}\delta_Q^{-m}\|T_{(Q,\mathcal{M})}(f,P) - P\|_{L^p(\frac{65}{64}Q)} &= \delta_Q^{-m}\|f(x(Q)) - P(x(Q))\|_{L^p(\frac{65}{64}Q)} \\ &\leq C\delta_Q^{-m+n/p}|f(x(Q)) - P(x(Q))| = C|\xi_Q(f,P)|.\end{aligned}$$

This implies the second bullet point.

From the first and second bullet points we have

$$\begin{aligned}\|(f,P)\|_{\frac{65}{64}Q} &\leq \|T_{(Q,\mathcal{M})}(f,P)\|_{\mathbb{X}(\frac{65}{64}Q)} + (\delta_{\frac{65}{64}Q})^{-m}\|T_{(Q,\mathcal{M})}(f,P) - P\|_{L^p(\frac{65}{64}Q)} \\ &\leq C\,|\xi_Q(f,P)|.\end{aligned}$$

Let $F \in \mathbb{X}$ satisfy $F = f$ on $\frac{65}{64}Q \cap E$. Then the Sobolev inequality implies that

$$\begin{aligned}\delta_Q^{n/p-m}|(f-P)(x(Q))| &= \delta_Q^{n/p-m}|(F-P)(x(Q))| \\ &\leq C \cdot \big(\|F\|_{\mathbb{X}(\frac{65}{64}Q)} + \delta_Q^{-m}\|F-P\|_{L^p(\frac{65}{64}Q)}\big).\end{aligned}$$

Taking the infimum over such F, we obtain the estimate $|\xi_Q(f,P)| \leq C\|(f,P)\|_{\frac{65}{64}Q}$. Thus we obtain the third bullet point, and this completes the proof. □

This completes the proof of the main technical results for $\mathcal{M}$, which was the base case of our proof by induction. We start to prove the induction step in the next section.

1.3. The induction step

Fix a set of multiindices $\mathcal{A} \subset \mathcal{M}$ with

$$\mathcal{A} \neq \mathcal{M}. \tag{1.5}$$

We assume by induction that we have already carried out the main technical results for each $\mathcal{A}' < \mathcal{A}$. Our goal is to find suitable constants $\mathfrak{a}(\mathcal{A})$, $\epsilon_1(\mathcal{A})$, $\epsilon_2(\mathcal{A})$, $c_*(\mathcal{A})$, and $S(\mathcal{A})$, and to carry out the main technical results for $\mathcal{A}$.

Let $\mathcal{A}^-$ be the maximal mutiindex set that is smaller than $\mathcal{A}$ with respect to the order relation $<$ on $2^{\mathcal{M}}$. By our induction hypothesis, we have already carried out the main technical results for $\mathcal{A}^-$. We have thus produced the following (see Section 3 in [1]):

- A decomposition $\mathrm{CZ}(\mathcal{A}^-)$ of Q° into dyadic cubes, with the following properties.
  - If $\frac{65}{64}Q \cap \frac{65}{64}Q' \neq \emptyset$ with $Q, Q' \in \mathrm{CZ}(\mathcal{A}^-)$, then $Q \leftrightarrow Q'$ and $\frac{1}{2}\delta_{Q'} \leq \delta_Q \leq 2\delta_{Q'}$.
  - The collection of cubes $\{\frac{65}{64}Q\colon Q \in \mathrm{CZ}(\mathcal{A}^-)\}$ has *bounded overlap*, meaning that there exists a constant $C = C(n)$ such that, for each $Q \in \mathrm{CZ}(\mathcal{A}^-)$ there are at most C cubes $Q' \in \mathrm{CZ}(\mathcal{A}^-)$ with $(65/64)Q \cap (65/64)Q' \neq \emptyset$.
  - $\mathrm{CZ}(\mathcal{M})$ refines $\mathrm{CZ}(\mathcal{A}^-)$; thus, from (1.2) we conclude that

    (1.6) ("E is nearby") $\#(E \cap 9Q) \geq 2$ for each $Q \in \mathrm{CZ}(\mathcal{A}^-)$.
- An oracle that accepts a query point $\underline{x} \in Q^\circ$ and responds with a list of all cubes $Q \in \mathrm{CZ}(\mathcal{A}^-)$ such that $\underline{x} \in \frac{65}{64}Q$.
- A list $\mathrm{CZ}_{\mathrm{main}}(\mathcal{A}^-)$ consisting of all the $Q \in \mathrm{CZ}(\mathcal{A}^-)$ such that $\frac{65}{64}Q \cap E \neq \emptyset$.
- For each $Q \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A}^-)$, a list of assists $\Omega(Q, \mathcal{A}^-) \subset [\mathbb{X}(E)]^*$.
- For each $Q \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A}^-)$, a list of $\Omega(Q, \mathcal{A}^-)$-assisted bounded depth linear functionals $\Xi(Q, \mathcal{A}^-) \subset \left[\mathbb{X}(\frac{65}{64}Q \cap E) \oplus \mathcal{P}\right]^*$ written in short form, as well as a linear extension operator

  $$T_{(Q,\mathcal{A}^-)} : \mathbb{X}\Big(\frac{65}{64}Q \cap E\Big) \oplus \mathcal{P} \to \mathbb{X},$$

  which we "compute" in the sense that (after one-time work) we can answer queries: In response to a query $\underline{x} \in Q^\circ$ we return a short form description of the $\Omega(Q, \mathcal{A}^-)$-assisted bounded depth linear map

  $$(f, P) \mapsto J_{\underline{x}} T_{(Q,\mathcal{A}^-)}(f, P).$$

These objects and algorithms have good properties as part of the induction assumption on $\mathcal{A}^-$. We listed some of these properties just above. The remaining properties are mentioned later, as required.

We denote

(1.7) $a = a(\mathcal{A}^-)$, the geometric constant from the main technical results for $\mathcal{A}^-$.

ALGORITHM: APPROXIMATE OLD TRACE NORM

For each $Q \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A}^-)$, we compute linear functionals $\xi_1^Q, \ldots, \xi_D^Q$ on $\mathcal{P}$, such that

$$\sum_{\xi \in \Xi(Q,\mathcal{A}^-)} |\xi(0, P)|^p \quad \text{and} \quad \sum_{i=1}^{D} |\xi_i^Q(P)|^p \quad (P \in \mathcal{P}) \tag{1.8}$$

differ by at most a factor of C. We carry this out using work and storage $\leq CN$.

Explanation. For each ξ in the list $\Xi(Q, \mathcal{A}^-)$, we compute the map $P \mapsto \xi(0, P)$ using work and storage at most C, by examining the short form description of $(f, P) \mapsto \xi(f, P)$ that has been computed. Applying COMPRESS NORMS (see Section 2.8 of [1]), we compute linear functionals $\xi_1^Q, \cdots, \xi_D^Q$ such that (1.8) holds, using work and storage at most $C \cdot \#[\Xi(Q, \mathcal{A}^-)]$. By the inductive hypothesis, we know that the sum of $\#[\Xi(Q, \mathcal{A}^-)]$ over all $Q \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A}^-)$ is bounded by CN, hence the work and storage guarantees are met. □

1.3.1. The non-monotonic case. Here, we assume that $\mathcal{A} \subset \mathcal{M}$ is not monotonic and prove the main technical results for $\mathcal{A}$.

We define $\mathrm{CZ}(\mathcal{A}) = \mathrm{CZ}(\mathcal{A}^-)$ and

$$\epsilon_2(\mathcal{A}) = \epsilon_2(\mathcal{A}^-), \quad c_*(\mathcal{A}) = c_*(\mathcal{A}^-), \quad a(\mathcal{A}) = a(\mathcal{A}^-), \quad \text{and } S(\mathcal{A}) = S(\mathcal{A}^-).$$

The constant $\epsilon_1(\mathcal{A})$ is chosen later in this section.

We define

$$\Omega(Q, \mathcal{A}) := \Omega(Q, \mathcal{A}^-), \quad \Xi(Q, \mathcal{A}) := \Xi(Q, \mathcal{A}^-) \text{ and}$$
$$T_{(Q,\mathcal{A})} := T_{(Q,\mathcal{A}^-)} \text{ for each } Q \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A}) = \mathrm{CZ}_{\mathrm{main}}(\mathcal{A}^-).$$

The properties of $\Omega(Q, \mathcal{A})$, $\Xi(Q, \mathcal{A})$ and $T_{(Q,\mathcal{A})}$ asserted in the main technical results for $\mathcal{A}$ are immediate from the corresponding properties of $\Omega(Q, \mathcal{A}^-)$, $\Xi(Q, \mathcal{A}^-)$ and $T_{(Q,\mathcal{A}^-)}$ asserted in the main technical results for $\mathcal{A}^-$.

Next, we prove properties **(CZ1)**–**(CZ5)** for the label $\mathcal{A}$. We refer the reader to Section 3 of [1] for the statements of **(CZ1)**–**(CZ5)**, which are a component of the main technical results.

Note that **(CZ1)** for $\mathcal{A}$ follows from **(CZ1)** for $\mathcal{A}^-$. Also, note that **(CZ5)** for $\mathcal{A}$ holds because $\mathrm{CZ}(\mathcal{A}) = \mathrm{CZ}(\mathcal{A}^-)$.

Note that **(CZ3)** for $\mathcal{A}$ holds vacuously: There do not exist cubes $Q \in \mathrm{CZ}(\mathcal{A})$, $Q' \in \mathrm{CZ}(\mathcal{A}^-)$ which satisfy the hypotheses of **(CZ3)**. This follows because $\mathrm{CZ}(\mathcal{A}) = \mathrm{CZ}(\mathcal{A}^-)$.

We need not check **(CZ4)**, since $\mathcal{A} \neq \mathcal{M}$; see (1.5).

It remains to prove **(CZ2)** for $\mathcal{A}$, which we accomplish in the next lemma. We determine $\epsilon_1(\mathcal{A}) = \epsilon_1$ in the lemma below.

Lemma 5. *There exists a universal constant $\epsilon_1 > 0$ such that the following holds. Suppose that $Q \in \mathrm{CZ}(\mathcal{A})$ and $\delta_Q \leq c_*(\mathcal{A})$. Then $S(\mathcal{A})Q$ is not tagged with $(\mathcal{A}, \epsilon_1)$.*

Proof. We assume that $\epsilon_1 > 0$ is less than a small enough universal constant.

Let $Q \in \mathrm{CZ}(\mathcal{A})$ satisfy $\delta_Q \leq c_*(\mathcal{A})$. Assume for the sake of contradiction that $S(\mathcal{A})Q$ is tagged with $(\mathcal{A}, \epsilon_1)$.

If $\#(S(\mathcal{A})Q \cap E) \leq 1$ then $S(\mathcal{A}^-)Q = S(\mathcal{A})Q$ is tagged with $(\mathcal{A}^-, \epsilon_1(\mathcal{A}^-))$. However, this contradicts the induction hypothesis. Hence, we may assume from now on that $\#(S(\mathcal{A})Q \cap E) \geq 2$. Thus,

$$\sigma(S(\mathcal{A})Q) \text{ has an } (\widetilde{\mathcal{A}}, \varkappa_Q, \epsilon_1, \delta_{S(\mathcal{A})Q})\text{-basis for some } \widetilde{\mathcal{A}} \leq \mathcal{A}.$$

Hence, Lemma 25 of [1] implies that there exists $\kappa \in [\kappa_1, \kappa_2]$ such that

$$\sigma(S(\mathcal{A})Q) \text{ has an } (\mathcal{A}', x_Q, \epsilon_1^\kappa, \delta_{S(\mathcal{A})Q}, \Lambda)\text{-basis,}$$
$$\text{with } \mathcal{A}' \le \mathcal{A} \text{ and } \epsilon_1^\kappa \Lambda^{100D} \le \epsilon_1^{\kappa/2}.$$

Here, $\kappa_1, \kappa_2 > 0$ are universal constants. (For the definition of an $(\mathcal{A}, x, \epsilon, \delta, \Lambda)$-basis, we refer the reader to Section 2.7.1 of [1].)

Suppose for the moment that $\mathcal{A}' < \mathcal{A}$. Then $S(\mathcal{A})Q$ is tagged with $(\mathcal{A}^-, \epsilon_1^\kappa)$. Note that $\epsilon_1^\kappa \le \epsilon_1^{\kappa_1} \le \epsilon_1(\mathcal{A}^-)$, for small enough ϵ_1. Thus, $S(\mathcal{A}^-)Q = S(\mathcal{A})Q$ is tagged with $(\mathcal{A}^-, \epsilon_1(\mathcal{A}^-))$. However, this contradicts the induction hypothesis. Hence,

$$\sigma(S(\mathcal{A})Q) \text{ has an } (\mathcal{A}, x_Q, \epsilon_1^\kappa, \delta_{S(\mathcal{A})Q}, \Lambda)\text{-basis.}$$

Thus, there exists $(P_\alpha)_{\alpha\in\mathcal{A}}$ with

$$P_\alpha \in \epsilon_1^\kappa \cdot (\delta_{S(\mathcal{A})Q})^{n/p+|\alpha|-m}\sigma(S(\mathcal{A})Q) \qquad (\alpha \in \mathcal{A}) \tag{1.9}$$

- $\partial^\beta P_\alpha(x_Q) = \delta_{\beta\alpha}$ $\qquad (\beta, \alpha \in \mathcal{A})$
- $|\partial^\beta P_\alpha(x_Q)| \le \epsilon_1^\kappa \cdot (\delta_{S(\mathcal{A})Q})^{|\alpha|-|\beta|}$ $\qquad (\alpha \in \mathcal{A}, \beta \in \mathcal{M}, \beta > \alpha)$
- $|\partial^\beta P_\alpha(x_Q)| \le \Lambda \cdot (\delta_{S(\mathcal{A})Q})^{|\alpha|-|\beta|}$ $\qquad (\alpha \in \mathcal{A}, \beta \in \mathcal{M})$.

We are assuming that $\mathcal{A}$ is not monotonic. Thus we can pick $\alpha_0 \in \mathcal{A}$ and $\gamma \in \mathcal{M}$ such that $\alpha_0 + \gamma \in \mathcal{M} \setminus \mathcal{A}$. We define

$$\overline{\alpha} = \alpha_0 + \gamma \quad \text{and} \quad \overline{\mathcal{A}} = \mathcal{A} \cup \{\overline{\alpha}\}.$$

Note that $\overline{\mathcal{A}}\Delta\mathcal{A} = \{\overline{\alpha}\}$ with $\overline{\alpha} \in \overline{\mathcal{A}}$. Consequently, $\overline{\mathcal{A}} < \mathcal{A}$.

We define $P_{\overline{\alpha}} = P_{\alpha_0} \odot_{x_Q} q$, where $q(y) = \frac{\alpha_0!}{\overline{\alpha}!}(y - x_Q)^\gamma$. That is,

$$P_{\overline{\alpha}}(y) = \frac{\alpha_0!}{\overline{\alpha}!} \sum_{|\omega|\le m-1-|\gamma|} \frac{1}{\omega!}\partial^\omega P_{\alpha_0}(x_Q)(y - x_Q)^{\omega+\gamma}.$$

Note that $P_{\overline{\alpha}} = q \cdot P_{\alpha_0}^{\text{main}}$, where

$$P_{\alpha_0}^{\text{main}} = \sum_{|\omega|\le m-1-|\gamma|} \frac{1}{\omega!}\partial^\omega P_{\alpha_0}(x_Q)(y - x_Q)^{\omega},$$

and that

$$R_{\alpha_0} := P_{\alpha_0} - P_{\alpha_0}^{\text{main}} = \sum_{|\omega|> m-1-|\gamma|} \frac{1}{\omega!}\partial^\omega P_{\alpha_0}(x_Q)(y - x_Q)^{\omega}.$$

In the above sum for R_{α_0}, since $|\omega| > m - 1 - |\gamma| \ge |\alpha_0|$ we have $\omega > \alpha_0$, and so

$$|\partial^\omega P_{\alpha_0}(x_Q)| \le C\,\epsilon_1^\kappa\,\delta_Q^{|\alpha_0|-|\omega|}.$$

Consequently,

$$\|R_{\alpha_0}\|_{L^p(S(\mathcal{A})Q)} \le C'\,\epsilon_1^\kappa\,\delta_Q^{n/p+|\alpha_0|}.$$

The bullet point properties of P_{α_0} now yield the following properties of $P_{\overline{\alpha}}$.

- $\partial^{\overline{\alpha}} P_{\overline{\alpha}}(x_Q) = 1$,
- $|\partial^\beta P_{\overline{\alpha}}(x_Q)| \le C\,\epsilon_1^\kappa\,\delta_Q^{|\overline{\alpha}|-|\beta|} \quad (\beta \in \mathcal{M}, \beta > \overline{\alpha})$,
- $|\partial^\beta P_{\overline{\alpha}}(x_Q)| \le C\,\Lambda\,\delta_Q^{|\overline{\alpha}|-|\beta|} \quad (\beta \in \mathcal{M})$.

We now show that

- $P_{\overline{\alpha}} \in C\,\epsilon_1^\kappa \cdot (\delta_Q)^{n/p+|\overline{\alpha}|-m} \cdot \sigma(S(\mathcal{A})Q)$.

To start, (1.9) implies that there exists $\varphi \in \mathbb{X}$ with $\varphi = 0$ on $S(\mathcal{A})Q \cap E$ and

$$\|\varphi\|_{\mathbb{X}(S(\mathcal{A})Q)} + \delta_Q^{-m}\|\varphi - P_{\alpha_0}\|_{L^p(S(\mathcal{A})Q)} \le C\,\epsilon_1^\kappa \cdot (\delta_Q)^{n/p+|\alpha_0|-m}.$$

We recall that $P_{\alpha_0}^{\text{main}} = P_{\alpha_0} - R_{\alpha_0}$. Thus, using the estimate $\|R_{\alpha_0}\|_{L^p(S(\mathcal{A})Q)} \le C\,\epsilon_1^\kappa\,\delta_Q^{n/p+|\alpha_0|}$, we learn that

$$\|\varphi - P_{\alpha_0}^{\text{main}}\|_{\mathbb{X}(S(\mathcal{A})Q)} + \delta_Q^{-m}\|\varphi - P_{\alpha_0}^{\text{main}}\|_{L^p(S(\mathcal{A})Q)} \le C\,\epsilon_1^\kappa \cdot (\delta_Q)^{n/p+|\alpha_0|-m}. \tag{1.10}$$

By definition of q, we have $\|\partial^\omega q\|_{L^\infty(S(\mathcal{A})Q)} \le C\delta_Q^{|\gamma|-|\omega|}$ for any multiindex ω. By the Leibniz rule, we obtain the estimate

$$\|gh\|_{\mathbb{X}(Q')} \lesssim \sum_{|\omega|+|\beta|=m} \|\partial^\omega g\|_{L^\infty(Q')}\|\partial^\beta h\|_{L^p(Q')} \quad \text{for } g, h \in \mathbb{X}(Q').$$

Using the previous estimates, we have

$$\begin{aligned}
&\|q \cdot (\varphi - P_{\alpha_0}^{\text{main}})\|_{\mathbb{X}(S(\mathcal{A})Q)} + \delta_Q^{-m}\|q \cdot (\varphi - P_{\alpha_0}^{\text{main}})\|_{L^p(S(\mathcal{A})Q)} \\
&\quad \le C \sum_{|\beta|\le m} (\delta_Q)^{|\beta|+|\gamma|-m}\|\partial^\beta(\varphi - P_{\alpha_0}^{\text{main}})\|_{L^p(S(\mathcal{A})Q)} \\
&\quad \le C \cdot (\delta_Q)^{|\gamma|}\big(\|\varphi - P_{\alpha_0}^{\text{main}}\|_{\mathbb{X}(S(\mathcal{A})Q)} + \delta_Q^{-m}\|\varphi - P_{\alpha_0}^{\text{main}}\|_{L^p(S(\mathcal{A})Q)}\big) \\
&\quad \le C\,\epsilon_1^\kappa \cdot (\delta_Q)^{|\gamma|}(\delta_Q)^{n/p+|\alpha_0|-m} \quad \text{(by (1.10))} \\
&\quad = C\,\epsilon_1^\kappa \cdot (\delta_Q)^{n/p+|\overline{\alpha}|-m}.
\end{aligned}$$

In the second inequality above, we used the Sobolev-type estimate stated in Proposition 9 of [1]. Note that $q \cdot P_{\alpha_0}^{\text{main}} \in \mathcal{P}$, hence the above estimate yields

$$\|q \cdot \varphi\|_{\mathbb{X}(S(\mathcal{A})Q)} + \delta_Q^{-m}\|q \cdot \varphi - P_{\overline{\alpha}}\|_{L^p(S(\mathcal{A})Q)} \le C\,\epsilon_1^\kappa(\delta_Q)^{n/p+|\overline{\alpha}|-m}.$$

Since $q \cdot \varphi = 0$ on $S(\mathcal{A})Q \cap E$, we have shown that $P_{\overline{\alpha}} \in C\,\epsilon_1^\kappa \cdot (\delta_Q)^{n/p+|\overline{\alpha}|-m} \cdot \sigma(S(\mathcal{A})Q)$. This proves all the bullet point properties of $P_{\overline{\alpha}}$.

The bullet point properties of the P_α ($\alpha \in \overline{\mathcal{A}}$) imply that $(\partial^\beta P_\alpha(x_Q))_{\alpha,\beta\in\overline{\mathcal{A}}}$ is $(C\,\epsilon_1^\kappa, C\Lambda, \delta_Q)$-near triangular. Inverting the matrix $(\partial^\beta P_\alpha(x_Q))_{\alpha,\beta\in\overline{\mathcal{A}}}$, we obtain a matrix $(M_{\alpha\omega})_{\alpha,\omega\in\overline{\mathcal{A}}}$ such that

$$\sum_{\alpha\in\overline{\mathcal{A}}} \partial^\beta P_\alpha(x_Q) M_{\alpha\omega} = \delta_{\beta\omega} \quad (\beta, \omega \in \overline{\mathcal{A}})$$

and

$$|M_{\alpha\omega} - \delta_{\alpha\omega}| \leq \begin{cases} C\,\epsilon_1^{\kappa}\Lambda^D \cdot \delta_Q^{|\omega|-|\alpha|}, & \text{if } \alpha,\omega \in \overline{\mathcal{A}},\ \alpha \geq \omega, \\ C\Lambda^D \cdot \delta_Q^{|\omega|-|\alpha|}, & \text{if } \alpha,\omega \in \overline{\mathcal{A}}. \end{cases}$$

Set $P_\omega^\# = \sum_{\alpha\in\overline{\mathcal{A}}} P_\alpha M_{\alpha\omega}$. The bullet point properties of $(P_\alpha)_{\alpha\in\overline{\mathcal{A}}}$ imply that

- $P_\omega^\# \in C\,\epsilon_1^{\kappa} \cdot \Lambda^{2D} \cdot \delta_Q^{n/p+|\omega|-m}\sigma(S(\mathcal{A})Q) \quad (\omega \in \overline{\mathcal{A}})$,
- $\partial^\beta P_\omega^\#(x_Q) = \delta_{\beta\omega} \quad (\beta,\omega \in \overline{\mathcal{A}})$.

For $\omega \in \overline{\mathcal{A}}$ and $\beta \in \mathcal{M}$ with $\beta > \omega$, we write

$$\partial^\beta P_\omega^\#(x_Q) = \sum_{\alpha<\beta} \partial^\beta P_\alpha(x_Q)M_{\alpha\omega} + \sum_{\alpha\geq\beta} \partial^\beta P_\alpha(x_Q)M_{\alpha\omega}. \tag{1.11}$$

An arbitrary term in the first sum in (1.11) is $\leq \left[C\,\epsilon_1^{\kappa}\,\delta_Q^{|\alpha|-|\beta|}\right] \cdot \left[C\Lambda^D\delta_Q^{|\omega|-|\alpha|}\right]$. Hence, this sum is at most $C'\epsilon_1^{\kappa}\Lambda^D\delta_Q^{|\omega|-|\beta|}$.

If $\alpha \geq \beta$, then $\alpha > \omega$, since $\beta > \omega$. Thus, an arbitrary term in the second sum in (1.11) is $\leq \left[C\Lambda\delta_Q^{|\alpha|-|\beta|}\right] \cdot \left[C\,\epsilon_1^{\kappa}\Lambda^D \cdot \delta_Q^{|\omega|-|\alpha|}\right]$. Hence, this sum is at most $C'\epsilon_1^{\kappa}\Lambda^{D+1}\delta_Q^{|\omega|-|\beta|}$.

Thus,

- $|\partial^\beta P_\omega^\#(x_Q)| \leq C\,\epsilon_1^{\kappa}\Lambda^{2D}\delta_Q^{|\omega|-|\beta|} \quad (\beta \in \mathcal{M}, \omega \in \overline{\mathcal{A}}, \beta > \omega)$.

According to the bullet point properties of $(P_\omega^\#)_{\omega\in\overline{\mathcal{A}}}$, we see that $\sigma(S(\mathcal{A})Q)$ has an $(\overline{\mathcal{A}}, x_Q, C\epsilon_1^{\kappa}\Lambda^{2D}, \delta_Q)$-basis, hence $\sigma(S(\mathcal{A})Q)$ has an $(\overline{\mathcal{A}}, x_Q, C'\epsilon_1^{\kappa}\Lambda^{2D}, \delta_{S(\mathcal{A})Q})$-basis. (See Remark 18 in [1].)

For small enough ϵ_1 we have $C'\epsilon_1^{\kappa}\Lambda^{2D} \leq C'\epsilon_1^{\kappa/2} \leq \epsilon_1^{\kappa_1/4} \leq \epsilon_1(\mathcal{A}^-)$, hence

$$\sigma(S(\mathcal{A})Q) \text{ has an } (\overline{\mathcal{A}}, x_Q, \epsilon_1(\mathcal{A}^-), \delta_{S(\mathcal{A})Q})\text{-basis.}$$

Hence, $S(\mathcal{A})Q$ is tagged with $(\mathcal{A}^-, \epsilon_1(\mathcal{A}^-))$. However, since $\delta_Q \leq c_*(\mathcal{A}^-)$ and $S(\mathcal{A}) = S(\mathcal{A}^-)$, this contradicts the induction hypothesis.

This completes the contradiction, and with it, the proof of the lemma. □

We have thus proven the main technical results for $\mathcal{A}$ in the non-monotonic case.

1.3.2. The monotonic case. From this point onward, we assume that $\mathcal{A}$ is monotonic. We drop this assumption when we prove our main theorem in Section 2. We will now begin the task of carrying out the induction step by proving the main technical results for $\mathcal{A}$. (See Section 3 in [1].)

We begin by treating a preliminary case.

Lemma 6. *Suppose that $\delta_Q \geq 1/4$ for all $Q \in CZ(\mathcal{A}^-)$. Then the main technical results for $\mathcal{A}^-$ imply the main technical results for $\mathcal{A}$.*

Proof. We take $CZ(\mathcal{A})$ to equal $CZ(\mathcal{A}^-)$. The other objects and algorithms in the main technical results for $\mathcal{A}$ are copies of the corresponding objects and algorithms in the main technical results for $\mathcal{A}^-$. □

By making at most C calls to the $CZ(\mathcal{A}^-)$-ORACLE, we can check whether the hypothesis of Lemma 6 holds. This takes one-time work at most $C \log N$. In the sequel, we assume that we are in the case that

$$\delta_Q \le 1/8 \quad \text{for some } Q \in CZ(\mathcal{A}^-). \tag{1.12}$$

Recall that the decomposition $CZ(\mathcal{A}^-)$ has the following properties:

- $CZ(\mathcal{A}^-)$ is a finite partition of $Q^\circ = [0,1)^n$ into pairwise disjoint dyadic subcubes.
- If $Q, Q' \in CZ(\mathcal{A}^-)$ and $Q \leftrightarrow Q'$ then $\delta_Q/\delta_{Q'} \in \{1/2, 1, 2\}$.
- If $Q \in CZ(\mathcal{A}^-)$ then $\#(9Q \cap E) \ge 2$. (See (1.6).)

Lemma 7. *If* $Q \in CZ(\mathcal{A}^-)$ *and* $\operatorname{dist}(Q, \mathbb{R}^n \setminus Q^\circ) = 0$ *then* $\delta_Q \in \{1/2, 1/4, 1/8\}$.

Proof. Let $Q \in CZ(\mathcal{A}^-)$ with $\operatorname{dist}(Q, \mathbb{R}^n \setminus Q^\circ) = 0$.

Recall that $\delta_Q \ne 1$, because $CZ(\mathcal{A}^-) \ne \{Q^\circ\}$ (see (1.12)).

We need to show that $\delta_Q \ge 1/8$. For the sake of contradiction assume that $\delta_Q \le 1/16$. Then since $\operatorname{dist}(Q, \mathbb{R}^n \setminus Q^\circ) = 0$, we have $9Q \subset \mathbb{R}^n \setminus \frac{1}{10}Q^\circ$, hence $9Q \subset \mathbb{R}^n \setminus E$. But $\#(E \cap 9Q) \ge 2$, according to the above bullet points. This contradiction completes the proof of Lemma 7. □

We now pass from the decomposition $CZ(\mathcal{A}^-)$ of Q° to a decomposition $\overline{CZ}(\mathcal{A}^-)$ of $\mathbb{R}^n$.

Proposition 1. *There exists a decomposition* $\overline{CZ}(\mathcal{A}^-)$ *of* $\mathbb{R}^n$ *into pairwise disjoint dyadic cubes, with the following properties:*

(a) $CZ(\mathcal{A}^-) \subset \overline{CZ}(\mathcal{A}^-)$.

(b) *If* $Q, Q' \in \overline{CZ}(\mathcal{A}^-)$ *and* $Q \leftrightarrow Q'$ *then* $\frac{1}{8}\delta_{Q'} \le \delta_Q \le 8\delta_{Q'}$ (*"good geometry"*). *Moreover, the collection of cubes* $\{\frac{65}{64}Q : Q \in \overline{CZ}(\mathcal{A}^-)\}$ *has bounded overlap* (*each cube intersects a bounded number of other cubes*).

(c) *If* $Q \in \overline{CZ}(\mathcal{A}^-) \setminus CZ(\mathcal{A}^-)$, *then* $\frac{65}{64}Q \cap E = \emptyset$.

(d) *If* $Q \in \overline{CZ}(\mathcal{A}^-) \setminus CZ(\mathcal{A}^-)$, *then* $100Q$ *intersects cubes in* $CZ(\mathcal{A}^-)$ *with side-length less than* δ_Q.

(e) *If* $Q \in \overline{CZ}(\mathcal{A}^-) \setminus CZ(\mathcal{A}^-)$, *then* $\delta_Q \ge 1$.

(f) *If* $Q \in \overline{CZ}(\mathcal{A}^-)$ *then* $\#(9Q \cap E) \ge 2$.

We produce a $\overline{CZ}(\mathcal{A}^-)$-ORACLE. *The* $\overline{CZ}(\mathcal{A}^-)$-ORACLE *accepts a query consisting of a point* $\underline{x} \in \mathbb{R}^n$. *The response to a query* $\underline{x}$ *is the list of cubes* $Q \in \overline{CZ}(\mathcal{A}^-)$ *such that* $\underline{x} \in \frac{65}{64}Q$. *The work and storage required to answer a query are at most* $C \log N$.

Proof. Let $\mathcal{Q}$ consist of the maximal dyadic cubes $Q \subset \mathbb{R}^n$ satisfying the condition $[\delta_Q \leq 1 \text{ or } 0 \notin 2Q]$. A dyadic cube $Q \subset \mathbb{R}^n$ belongs to $\mathcal{Q}$ if and only if

$$\delta_Q = 1 \text{ or } 0 \notin 2Q, \tag{1.13}$$

and

$$\delta_{Q^+} \geq 2 \text{ and } 0 \in 2Q^+. \tag{1.14}$$

Here, as usual, Q^+ denotes the parent of a dyadic cube Q.

For any $x \in \mathbb{R}^n$, there exists a dyadic cube $\overline{Q}$ containing x such that $\delta_{\overline{Q}} \geq 2$ and $0 \in 2\overline{Q}$. Hence, each $x \in \mathbb{R}^n$ is contained in some cube $Q \in \mathcal{Q}$. Hence, $\mathcal{Q}$ partitions $\mathbb{R}^n$ into pairwise disjoint dyadic cubes.

Note that the cube $Q^\circ = [0,1)^n$ belongs to $\mathcal{Q}$.

We now establish good geometry of $\mathcal{Q}$ (with constant $1/4$). We prove that if $Q, Q' \in \mathcal{Q}$ and $Q \leftrightarrow Q'$ then $\frac{1}{4}\delta_{Q'} \leq \delta_Q \leq 4\delta_{Q'}$.

Assume for the sake of contradiction that there exist cubes $Q, Q' \in \mathcal{Q}$ with $\delta_Q \leq \frac{1}{8}\delta_{Q'}$ and $Q \leftrightarrow Q'$. By (1.14), we have $\delta_{Q^+} \geq 2$ and $0 \in 2Q^+$. Moreover, note that $2Q^+ \subset 2Q'$ (since $Q \leftrightarrow Q'$ and $\delta_Q \leq \frac{1}{8}\delta_{Q'}$, hence $Q^+ \leftrightarrow Q'$ and $\delta_{Q^+} \leq \frac{1}{4}\delta_{Q'}$). Hence, $0 \in 2Q'$. Moreover, $\delta_{Q'} \geq 4\delta_{Q^+} \geq 8$. However, since $Q' \in \mathcal{Q}$, the analogue of (1.13) with Q replaced by Q' must hold. This yields a contradiction. This completes the proof that the cubes in $\mathcal{Q}$ have good geometry.

We define the collection $\overline{\mathrm{CZ}}(\mathcal{A}^-)$ to consist of all the cubes $Q \in \mathcal{Q}$ except for $Q = Q^\circ$, together with all the cubes $Q \in \mathrm{CZ}(\mathcal{A}^-)$. Since $\mathcal{Q}$ partitions $\mathbb{R}^n$ and $\mathrm{CZ}(\mathcal{A}^-)$ partitions Q°, we see that $\overline{\mathrm{CZ}}(\mathcal{A}^-)$ partitions $\mathbb{R}^n$ into pairwise disjoint dyadic cubes. Moreover, property (a) clearly holds.

If $Q \in \mathcal{Q}$, $Q' \in \mathrm{CZ}(\mathcal{A}^-)$, and $Q \leftrightarrow Q'$, then both Q and Q' touch the boundary of Q°.

We prove the claim that $\mathcal{Q}$ contains all 4^n of the dyadic cubes $Q \subset [-2,2)^n$ with $\delta_Q = 1$. Indeed, we have $Q^+ \subset [-2,2)^n$, $\delta_{Q^+} = 2$ and $0 \in 2Q^+$ for any such Q. Hence, each Q satisfies (1.13) and (1.14), which implies that Q belongs to $\mathcal{Q}$. This proves our claim. Hence, in particular, any $Q \in \mathcal{Q}$ that intersects the boundary of $Q^\circ = [0,1)^n$ must satisfy $\delta_Q = 1$.

Moreover, by Lemma 7, any $Q' \in \mathrm{CZ}(\mathcal{A}^-)$ that intersects the boundary of Q° must satisfy $\delta_{Q'} \in \{1/2, 1/4, 1/8\}$.

Hence, the previous two statements imply that for any $Q \in \mathcal{Q}$ and $Q' \in \mathrm{CZ}(\mathcal{A}^-)$ with $Q \leftrightarrow Q'$ we have $\frac{1}{8}\delta_Q \leq \delta_{Q'} \leq \delta_Q$.

Finally, for $Q, Q' \in \mathrm{CZ}(\mathcal{A}^-)$ with $Q \leftrightarrow Q'$, we have $\frac{1}{2}\delta_Q \leq \delta_{Q'} \leq 2\delta_Q$, by good geometry of the cubes in $\mathrm{CZ}(\mathcal{A}^-)$.

Recall that the cubes in $\mathcal{Q}$ satisfy good geometry (with constant $1/4$).

Thus, combining the previous three statements, for any $Q, Q' \in \overline{\mathrm{CZ}}(\mathcal{A}^-)$ with $Q \leftrightarrow Q'$, we have $\frac{1}{8}\delta_{Q'} \leq \delta_Q \leq 8\delta_{Q'}$. Thus, the collection $\overline{\mathrm{CZ}}(\mathcal{A}^-)$ satisfies the hypothesis of Lemma 46 from [1] with $\gamma = 1/8$. Hence, for $Q, Q' \in \overline{\mathrm{CZ}}(\mathcal{A}^-)$ with $\frac{65}{64}Q \cap \frac{65}{64}Q' \neq \emptyset$, we have $Q \leftrightarrow Q'$. It follows that the collection $\{\frac{65}{64}Q : Q \in \overline{\mathrm{CZ}}(\mathcal{A}^-)\}$ has bounded overlap. This completes the proof of property (b).

From (1.14), each $Q \in \overline{\mathrm{CZ}}(\mathcal{A}^-) \setminus \mathrm{CZ}(\mathcal{A}^-)$ satisfies $\delta_Q \geq 1$. This proves property (e).

We now prove property (c). Let $Q \in \overline{CZ}(\mathcal{A}^-) \setminus CZ(\mathcal{A}^-)$. Then $Q \in \mathcal{Q}$ and $Q \subset \mathbb{R}^n \setminus Q^\circ$. According to property (e), there are only two cases to consider

- If $\delta_Q = 1$, then $\frac{65}{64}Q$ cannot intersect $\frac{1}{32}Q^\circ$ (because $Q \cap Q^\circ = \emptyset$ and $\delta_Q = \delta_{Q^\circ} = 1$). Since $E \subset \frac{1}{32}Q^\circ$, we conclude that $\frac{65}{64}Q \cap E = \emptyset$.

- If $\delta_Q \geq 2$, then $0 \notin 2Q$ thanks to (1.13). Assume for the sake of contradiction that $\frac{65}{64}Q \cap \frac{1}{32}Q^\circ \neq \emptyset$. Since Q and Q° are disjoint, we conclude that $\frac{1}{64}\delta_Q \geq \frac{1}{4} \implies \delta_Q \geq 16$. Hence, $0 \in 2Q$ (since $\frac{65}{64}Q \cap \frac{1}{32}Q^\circ \neq \emptyset$ and $\delta_Q \geq 16$, and $Q^\circ = [0,1)^n$). Hence, we derive a contradiction. Thus, $\frac{65}{64}Q$ cannot intersect $\frac{1}{32}Q^\circ$. Since $E \subset \frac{1}{32}Q^\circ$, we conclude that $\frac{65}{64}Q \cap E = \emptyset$.

This completes the proof of property (c).

Property (d) is easy to prove. Let $Q \in \mathcal{Q}$. Then $0 \in 2Q^+$ thanks to (1.14). Hence, $0 \in 6Q$ (since $2Q^+ \subset 6Q$). Since $\delta_Q \geq 1$, this implies $Q^\circ \subset 9Q$ (recall that $Q^\circ = [0,1)^n$). Together with (1.12), this implies property (d).

We now prove property (f). Let $Q \in \overline{CZ}(\mathcal{A}^-)$ be given.
If $Q \in CZ(\mathcal{A}^-)$ then $\#(9Q \cap E) \geq 2$, thanks to (1.6).
If $Q \in \mathcal{Q}$, then $9Q \supset Q^\circ$, hence $\#(9Q \cap E) = \#(E) \geq 2$.
This concludes the proof of property (f).

We prepare to describe the construction of the $\overline{CZ}(\mathcal{A}^-)$-ORACLE.

We can determine whether a dyadic cube $Q \subset \mathbb{R}^n$ belongs to $\overline{CZ}(\mathcal{A}^-)$ using work and storage at most $C \log N$. We explain the procedure below.

Let $Q \subset \mathbb{R}^n$ be given.

First, suppose that $Q \subset Q^\circ$. Then $Q \in \overline{CZ}(\mathcal{A}^-)$ if and only if $Q \in CZ(\mathcal{A}^-)$. We can determine whether $Q \in CZ(\mathcal{A}^-)$ by using the $CZ(\mathcal{A}^-)$-ORACLE to produce a list of all the cubes $Q' \in CZ(\mathcal{A}^-)$ satisfying $x_Q \in \frac{65}{64}Q'$. (Recall, x_Q denotes the center of Q.) Then $Q \in CZ(\mathcal{A}^-)$ if and only if Q belongs to the aforementioned list. Thus, in this case, we can determine whether $Q \in \overline{CZ}(\mathcal{A}^-)$ using work at most $C \log N$.

Next, suppose that $Q^\circ \subsetneq Q$. Then Q can never belong to $\overline{CZ}(\mathcal{A}^-)$.

Lastly, suppose that $Q \subset \mathbb{R}^n \setminus Q^\circ$. Then $Q \in \overline{CZ}(\mathcal{A}^-)$ if and only if $Q \in \mathcal{Q}$. Recall from (1.13) and (1.14) that $Q \in \mathcal{Q}$ if and only if [$\delta_Q = 1$ or $0 \notin 2Q$] and [$\delta_{Q^+} \geq 2$ and $0 \in 2Q^+$]. We can check each of these conditions using at most C computer operations. Thus, in this case we can determine whether $Q \in \overline{CZ}(\mathcal{A}^-)$ using work at most C.

Hence, we can determine whether a given cube belongs to $\overline{CZ}(\mathcal{A}^-)$ using work at most $C \log N$.

We next explain how to compute the unique cube $Q_{\underline{x}} \in \overline{CZ}(\mathcal{A}^-)$ containing $\underline{x}$. It will then not be difficult to produce a list of the cubes $Q \in \overline{CZ}(\mathcal{A}^-)$ satisfying $\underline{x} \in \frac{65}{64}Q$. We describe this step at the very end.

We check whether or not $\underline{x} \in Q^\circ$. We split into cases depending on the result.

First, suppose that $\underline{x} \in Q^\circ$. We then compute the cube $\overline{Q} \in CZ(\mathcal{A}^-)$ containing $\underline{x}$ using the $CZ(\mathcal{A}^-)$-ORACLE. We set $Q_{\underline{x}} = \overline{Q}$.

Now suppose that $\underline{x} \in \mathbb{R}^n \setminus Q^\circ$.

Let Q be the unique cube in $\mathcal{Q} \setminus \{Q^\circ\}$ containing $\underline{x}$. We will explain how to compute Q.

We compute the dyadic cube $\widetilde{Q} \subset \mathbb{R}^n$ such that $\delta_{\widetilde{Q}} = 1$ and $\underline{x} \in \widetilde{Q}$.

We test to see whether $0 \in 2\widetilde{Q}$. We can do that using at most C computer operations.

If $0 \in 2\widetilde{Q}$ then $\widetilde{Q}$ is a maximal dyadic cube satisfying the condition [$\delta_{\widetilde{Q}} \leq 1$ or $0 \notin 2\widetilde{Q}$]. Hence, in that case, $\widetilde{Q}$ is the unique cube in $\mathcal{Q}$ containing $\underline{x}$. We set $Q_{\underline{x}} = \widetilde{Q}$.

Now suppose that $0 \notin 2\widetilde{Q}$. Thus, $\widetilde{Q}$ satisfies (1.13). Since Q and $\widetilde{Q}$ are intersecting dyadic cubes (they both contain $\underline{x}$), and since Q is maximal with respect to the property (1.13), we conclude that $\widetilde{Q} \subset Q$.

Assume that $\widetilde{Q} = Q$. Then $0 \notin 2Q$, by assumption. On the other hand, suppose that $\widetilde{Q} \subsetneq Q$. Then $\delta_Q > 1$ (since $\delta_{\widetilde{Q}} = 1$). Since Q satisfies (1.13), we conclude that $0 \notin 2Q$.

Thus, in the case where $0 \notin 2\widetilde{Q}$, we know that $0 \notin 2Q$. Since $\underline{x} \in Q$ this shows that $|\underline{x}| \geq \frac{1}{4}\delta_Q$. Moreover, since Q satisfies (1.14) we know that $0 \in 9Q$. Hence,

$$\frac{1}{4}\delta_Q \leq |\underline{x}| \leq 9\,\delta_Q \tag{1.15}$$

for the unique cube $Q \in \mathcal{Q}$ containing $\underline{x}$.

There are no more than C dyadic cubes $Q \subset \mathbb{R}^n$ satisfying (1.15) with $\underline{x} \in Q$; moreover, it takes work at most C to list all these cubes. We examine each cube and test to see whether it belongs to $\overline{CZ}(\mathcal{A}^-)$. We set aside the unique cube Q that passes the test. We set $Q_{\underline{x}} = Q$.

We have just explained how to compute the cube $Q_{\underline{x}} \in \overline{CZ}(\mathcal{A}^-)$ containing a given point $\underline{x} \in \mathbb{R}^n$. The work requires is at most $C \log N$. We now explain how to construct the $\overline{CZ}(\mathcal{A}^-)$-ORACLE.

Suppose that $\overline{Q} \in \overline{CZ}(\mathcal{A}^-)$ satisfies $\underline{x} \in \frac{65}{64}\overline{Q}$. Then

$$\overline{Q} \leftrightarrow Q_{\underline{x}} \quad \text{and} \quad \frac{1}{8}\delta_{Q_{\underline{x}}} \leq \delta_{\overline{Q}} \leq 8\,\delta_{Q_{\underline{x}}}. \tag{1.16}$$

This follows from condition (b) in Proposition 1 and an application of Lemma 46 of [1] (with $\gamma = 1/8$).

We produce a list of all the dyadic cubes $\overline{Q}$ that satisfy both (1.16) and $\underline{x} \in \frac{65}{64}\overline{Q}$. There are at most C such cubes and it takes work at most C to list them all. We examine each $\overline{Q}$ to see whether it belongs to $\overline{CZ}(\mathcal{A}^-)$. We return a list of all those cubes that belong to $\overline{CZ}(\mathcal{A}^-)$.

This completes the description of the $\overline{CZ}(\mathcal{A}^-)$-ORACLE and finishes the proof of the proposition. □

1.3.3. Keystone cubes. We define integer constants

$$\begin{cases} S_0 := S(\mathcal{A}^-), \\ S_1 := \text{the smallest integer greater than } 100, 10^5 \cdot S_0, \text{ and } 2 \cdot [c_*(\mathcal{A}^-)]^{-1}, \\ S_2 := \text{the smallest odd integer greater than } 10^5 S_1. \end{cases} \tag{1.17}$$

We let $\epsilon > 0$ be a small parameter. We assume in what follows that

(1.18) $\epsilon > 0$ is less than a small enough universal constant.

We eventually fix ϵ to be a universal constant, but only much later in the proof. We will take $\epsilon_2(\mathcal{A}) = \epsilon^\kappa$ and $\epsilon_1(\mathcal{A}) = \epsilon^{1/\kappa}$ for a small universal constant κ. The discussion of the final choice of the numerical constants relevant to the Main Results for $\mathcal{A}$ occurs in Section 1.8.3. See also (1.180).

We next define the keystone cubes associated to the decomposition $\overline{\mathrm{CZ}}(\mathcal{A}^-)$. We will prove a few basic properties of the keystone cubes and introduce the relevant algorithms.

Definition 1. A cube $Q^\# \in \overline{\mathrm{CZ}}(\mathcal{A}^-)$ is keystone if and only if $\delta_Q \geq \delta_{Q^\#}$ for every $Q \in \overline{\mathrm{CZ}}(\mathcal{A}^-)$ that meets $S_2 Q^\#$.

Lemma 8. *The collection* $\{S_1 Q^\# : Q^\# \in \overline{\mathrm{CZ}}(\mathcal{A}^-)$ *keystone*$\}$ *has bounded overlap. Moreover, each keystone cube* $Q^\# \in \overline{\mathrm{CZ}}(\mathcal{A}^-)$ *belongs to* $\mathrm{CZ}(\mathcal{A}^-)$.

Proof. Suppose that $Q_1^\#, Q_2^\#$ are keystone cubes such that $S_1 Q_1^\# \cap S_1 Q_2^\# \neq \emptyset$ and $\delta_{Q_1^\#} \leq \delta_{Q_2^\#}$. Then $Q_1^\# \cap S_2 Q_2^\# \neq \emptyset$, since $S_2 \geq 10^5 S_1$. Therefore, $\delta_{Q_1^\#} \geq \delta_{Q_2^\#}$, by definition of the keystone cubes.

Consequently, $\delta_{Q_1^\#} = \delta_{Q_2^\#}$ whenever $S_1 Q_1^\# \cap S_1 Q_2^\# \neq \emptyset$. Thus, no more than C cubes $S_1 Q_2^\#$ can intersect any given cube $S_1 Q_1^\#$. This implies the first conclusion of Lemma 8.

Finally, observe that no cube in $\overline{\mathrm{CZ}}(\mathcal{A}^-) \setminus \mathrm{CZ}(\mathcal{A}^-)$ can be keystone, thanks to condition (d) in Proposition 1 and the fact that $S_2 \geq 100$. This completes the proof of the lemma. □

The above definition of keystone cubes agrees with the definition of keystone cubes in Section 4.5 of [1], if we let $K = S_2$ and let A be a large universal constant in Section 4.5 of [1]. The MAIN KEYSTONE CUBE ALGORITHM from Section 4.5 of [1] says the following: given $Q \in \mathrm{CZ}(\mathcal{A}^-)$, we can compute a keystone cube $\mathcal{K}(Q) \in \mathrm{CZ}(\mathcal{A}^-)$ such that there exists a sequence $\mathcal{S} = (Q_1, Q_2, \ldots, Q_{\underline{L}})$ of $\overline{\mathrm{CZ}}(\mathcal{A}^-)$ cubes such that

$$Q = Q_1 \leftrightarrow Q_2 \leftrightarrow \cdots \leftrightarrow Q_{\underline{L}} = \mathcal{K}(Q),$$

and with

$$\delta_{Q_\ell} \leq C \cdot (1-c)^{\ell-\ell'} \delta_{Q_{\ell'}}, \quad \text{for } 1 \leq \ell' \leq \ell \leq \underline{L}.$$

We do not compute the sequence $\mathcal{S}$, we just claim its existence.

We now modify the sequence $\mathcal{S}$ to consist only of cubes from $\mathrm{CZ}(\mathcal{A}^-)$ while retaining the important properties of $\mathcal{S}$.

We first discuss the case in dimension $n = 1$. We let $\mathcal{S}'$ denote the sequence formed by omitting from $\mathcal{S}$ all the cubes that belong to $\overline{\mathrm{CZ}}(\mathcal{A}^-) \setminus \mathrm{CZ}(\mathcal{A}^-)$. Recall that all the cubes in $\mathrm{CZ}(\mathcal{A}^-)$ are contained in $Q^\circ = [0,1)$ and all the cubes in $\overline{\mathrm{CZ}}(\mathcal{A}^-)$ are contained in $\mathbb{R} \setminus [0,1)$. Consider a maximal subsequence $Q_{k_1}, \ldots, Q_{k_2}$ of cubes in $\mathcal{S}$ that belong to $\overline{\mathrm{CZ}}(\mathcal{A}^-)$. Then, by connectedness,

each Q_k ($k_1 \le k \le k_2$) is contained in either $[1,\infty)$ or $(-\infty,0)$. Assume for sake of definiteness that each Q_k is contained in $[1,\infty)$. Then Q_{k_1-1} and Q_{k_2+1} are the same cube in $\mathrm{CZ}(\mathcal{A}^-)$, namely the unique cube in $\mathrm{CZ}(\mathcal{A}^-)$ that meets the endpoint $x = 1$. (This is because the sequence must exit and reenter $[0,1)$ using the same cube that borders the endpoint $x = 1$.) Thus we can remove the aforementioned subsequence from $\mathcal{S}$ and obtain a connected path of cubes. The resulting sequence is exponentially decreasing with the same constants C and c above. The same argument shows that we can remove every maximal subsequence of $\mathcal{S}$ consisting of cubes in $\overline{\mathrm{CZ}}(\mathcal{A}^-) \setminus \mathrm{CZ}(\mathcal{A}^-)$.

We now handle the case when the dimension n is at least 2.

Suppose that some of the cubes in $\mathcal{S}$ belong to $\overline{\mathrm{CZ}}(\mathcal{A}^-)$. Let Q_{k_1} and Q_{k_2} denote the first and last cubes in the sequence $\mathcal{S}$ belonging to $\overline{\mathrm{CZ}}(\mathcal{A}^-) \setminus \mathrm{CZ}(\mathcal{A}^-)$. Let $\mathcal{S}_{\mathrm{sub}} = (Q_{k_1}, \dots, Q_{k_2})$ denote the corresponding subsequence of $\mathcal{S}$.

We know that $Q_1 = Q$ and $Q_{\underline{L}} = \mathcal{K}(Q)$ both belong to $\mathrm{CZ}(\mathcal{A}^-)$. Hence, $1 < k_1 \le k_2 < \underline{L}$.

Note that both Q_{k_1-1} and Q_{k_2-1} intersect the boundary of Q° and belong to $\mathrm{CZ}(\mathcal{A}^-)$.

We join Q_{k_1-1} and Q_{k_2+1} with a sequence $\mathcal{S}'_{\mathrm{sub}} = (\widetilde{Q}_{k_1}, \dots, \widetilde{Q}_{k_3})$ with the following properties.

- The cubes $\widetilde{Q}_k \in \mathrm{CZ}(\mathcal{A}^-)$ intersect the boundary of Q°.
- $\widetilde{Q}_{k_1} \leftrightarrow Q_{k_1-1}$, $\widetilde{Q}_{k_3} \leftrightarrow Q_{k_2+1}$, and
$$\widetilde{Q}_k \leftrightarrow \widetilde{Q}_{k+1} \quad \text{for } k_1 \le k \le k_3 - 1.$$
- $k_3 - k_1$ is bounded by a universal constant.
- Each $\widetilde{Q}_k$ has sidelength between $1/2$ and $1/8$.

These properties can be arranged due to Lemma 7.

We replace the subsequence $\mathcal{S}_{\mathrm{sub}}$ with the sequence $\mathcal{S}'_{\mathrm{sub}}$ in $\mathcal{S}$. We obtain a sequence $\mathcal{S}' = (\widetilde{Q}_1, \widetilde{Q}_2, \dots, \widetilde{Q}_L)$ of cubes in $\mathrm{CZ}(\mathcal{A}^-)$ such that $\widetilde{Q}_1 = Q$ and $\widetilde{Q}_L = \mathcal{K}(Q)$; moreover,

$$\widetilde{Q}_\ell \leftrightarrow \widetilde{Q}_{\ell+1} \ \ (1 \le \ell \le L-1) \ \text{ and } \ \delta_{\widetilde{Q}_\ell} \le C' \cdot (1-c')^{\ell-\ell'} \delta_{\widetilde{Q}_{\ell'}} \ \ (1 \le \ell' \le \ell \le L).$$

Indeed, the fact that $\mathcal{S}'$ satisfies the exponentially decreasing property follows directly from the construction: We removed a subsequence of connected cubes in $\mathcal{S}$ and replaced it with a subsequence of bounded length consisting of cubes of size $\in \{1/2, 1/4, 1/8\}$. This has no effect on the fact that the sidelengths are exponentially decreasing in the sense of the above estimate.

Hence, the sequence $\mathcal{S}'$ joining Q and $\mathcal{K}(Q)$ is exponentially decreasing.

We never actually compute the sequences $\mathcal{S}$ or $\mathcal{S}'$, we just claim their existence.

Using the above analysis, the MAIN KEYSTONE CUBE ALGORITHM and the algorithm LIST ALL KEYSTONE CUBES from Section 4.5 of [1], we obtain the following result.

ALGORITHM: KEYSTONE-ORACLE

After one-time work at most $CN\log N$ in space CN we produce the following outcomes:

- We list all the keystone cubes $Q^{\#}$ in $CZ(\mathcal{A}^-)$.
- We can answer queries: A query consists of a cube $Q \in CZ(\mathcal{A}^-)$, and the response to a query Q is a keystone cube $\mathcal{K}(Q)$ to which Q is connected by an exponentially decreasing path
$$Q = \widetilde{Q}_1 \leftrightarrow \widetilde{Q}_2 \leftrightarrow \cdots \leftrightarrow \widetilde{Q}_L = \mathcal{K}(Q)$$
with
$$\delta_{\widetilde{Q}_\ell} \le C\cdot(1-c)^{\ell-\ell'}\delta_{\widetilde{Q}_{\ell'}} \quad \text{for } 1 \le \ell' \le \ell \le L.$$
We guarantee that $\widetilde{Q}_\ell \in CZ(\mathcal{A}^-)$ and $\frac{65}{64}\widetilde{Q}_\ell \subset CQ$ for each ℓ. We guarantee that $S_1\mathcal{K}(Q) \subset CQ$; also that $\mathcal{K}(Q) = Q$ if Q is keystone. The work required to answer a query is at most $C\log N$.
- We list all $(Q', Q'') \in CZ(\mathcal{A}^-) \times CZ(\mathcal{A}^-)$ such that $Q' \leftrightarrow Q''$ and $\mathcal{K}(Q') \neq \mathcal{K}(Q'')$. Let $BD(\mathcal{A}^-)$ (the "border disputes") denote the set of all such pairs (Q', Q''). We guarantee that the cardinality of $BD(\mathcal{A}^-)$ is at most CN.

Remark 1. Let $\widetilde{Q}_1 \leftrightarrow \cdots \leftrightarrow \widetilde{Q}_L$ be as above. For fixed Q', we can have $Q' = \widetilde{Q}_\ell$ for at most C distinct ℓ. This is because the path $\widetilde{Q}_1 \leftrightarrow \widetilde{Q}_2 \leftrightarrow \cdots \leftrightarrow \widetilde{Q}_L$ is exponentially decreasing.

Remark 2. We do not attempt to compute the sequence of cubes $\widetilde{Q}_1, \ldots, \widetilde{Q}_{L-1}$, we only guarantee that this sequence exists, and we guarantee that we can compute the keystone cube $\mathcal{K}(Q) = \widetilde{Q}_L$ located at the end of the sequence.

1.4. An approximation to the sigma

We begin the proof of the main technical results for $\mathcal{A}$. We recall that $\mathcal{A} \subsetneq \mathcal{M}$ is a monotonic set.

In Sections 1.3.2 and 1.3.3, we have defined a dyadic decomposition $\overline{CZ}(\mathcal{A}^-)$ of $\mathbb{R}^n$ and a notion of keystone cubes in $\overline{CZ}(\mathcal{A}^-)$. We cannot compute all the cubes in $\overline{CZ}(\mathcal{A}^-)$ since there are infinitely many. Instead, we have access to a $\overline{CZ}(\mathcal{A}^-)$-ORACLE and a KEYSTONE-ORACLE.

The integer constants S_0, S_1 and S_2 relating to the keystone cubes are defined in (1.17).

According to the main technical results for $\mathcal{A}^-$ (see Section 3 in [1]), for each $Q \in CZ_{\text{main}}(\mathcal{A}^-)$ the functional

$$M_{(Q,\mathcal{A}^-)}(f,R) := \Big(\sum_{\xi\in\Xi(Q,\mathcal{A}^-)} |\xi(f,R)|^p\Big)^{1/p} \tag{1.19}$$

satisfies

$$c\|(f,R)\|_{(1+a)Q} \le M_{(Q,\mathcal{A}^-)}(f,R) \le C\|(f,R)\|_{\frac{65}{64}Q}. \tag{1.20}$$

Recall that a is the constant $a(\mathcal{A}^-)$ from the main technical results; see (1.7).

We set $\Xi(Q, \mathcal{A}^-) := \emptyset$ and $M_{(Q,\mathcal{A}^-)}(f, R) := 0$ for all $Q \in \overline{CZ}(\mathcal{A}^-) \backslash CZ_{main}(\mathcal{A}^-)$. By definition of the collection $CZ_{main}(\mathcal{A}^-)$ and by property (c) in Proposition 1, we have $\frac{65}{64}Q \cap E = \emptyset$. Thus, $\|(f, R)\|_{(1+a)Q} = 0$ for any $Q \in \overline{CZ}(\mathcal{A}^-) \backslash CZ_{main}(\mathcal{A}^-)$. Thus, we see that (1.20) holds for all $Q \in \overline{CZ}(\mathcal{A}^-)$.

1.4.1. Assigning jets to keystone cubes. Let $Q^\# \in CZ(\mathcal{A}^-)$ be a keystone cube. We define its associated CZ cubes to be the collection

$$\mathcal{I}(Q^\#) := \left\{Q \in \overline{CZ}(\mathcal{A}^-) : Q \cap S_0 Q^\# \neq \emptyset\right\}. \tag{1.21}$$

We note that the cubes in $\mathcal{I}(Q^\#)$ belong to $\overline{CZ}(\mathcal{A}^-)$ rather than $CZ(\mathcal{A}^-)$. Hence, the cubes in $\mathcal{I}(Q^\#)$ are contained in $\mathbb{R}^n$, and may not be contained in Q°.

Lemma 9. *Let $A \geq 1$ be given. Assume that $Q, \overline{Q} \in \overline{CZ}(\mathcal{A}^-)$ and $Q \cap A\overline{Q} \neq \emptyset$. Then*

$$\delta_Q \leq 10^3 A \delta_{\overline{Q}}, \text{ and} \tag{1.22}$$

$$\frac{65}{64}Q \subset 10^5 A \overline{Q}. \tag{1.23}$$

Proof. We first prove (1.22). Assume for the sake of contradiction that $\delta_Q > 10^3 A \delta_{\overline{Q}}$ for some $Q, \overline{Q} \in \overline{CZ}(\mathcal{A}^-)$ with $Q \cap A\overline{Q} \neq \emptyset$. Then $\frac{65}{64}Q \cap \overline{Q} \neq \emptyset$. However, this contradicts the good geometry of the cubes in $\overline{CZ}(\mathcal{A}^-)$ (see Proposition 1). This completes the proof of (1.22) by contradiction. Lastly, (1.23) follows from (1.22) and our assumption that $Q \cap A\overline{Q} \neq \emptyset$. □

By Lemma 9, the CZ cubes associated to a given $Q^\#$ satisfy the following property: for each $Q \in \mathcal{I}(Q^\#)$ we have

$$\delta_Q \leq 10^3 S_0 \cdot \delta_{Q^\#}, \text{ and} \tag{1.24}$$

$$\frac{65}{64}Q \subset S_1 Q^\#. \tag{1.25}$$

(Recall (1.17), which states that $S_1 \geq 10^5 S_0$.)

Remark 3. The definition of keystone cubes shows that $\delta_Q \geq \delta_{Q^\#}$ whenever $Q \in \mathcal{I}(Q^\#)$. Hence, (1.24) implies that the cardinality of $\mathcal{I}(Q^\#)$ is bounded by C for each keystone cube $Q^\#$.

If $Q \in \mathcal{I}(Q_1^\#)$ and $Q \in \mathcal{I}(Q_2^\#)$ then (1.25) implies that $S_1 Q_1^\# \cap S_1 Q_2^\# \supset \frac{65}{64}Q$. Recall that the cubes $S_1 Q^\#$ ($Q^\#$ keystone) have bounded overlap. (See Lemma 8.) Thus, each $Q \in \overline{CZ}(\mathcal{A}^-)$ belongs to $\mathcal{I}(Q^\#)$ for at most C distinct keystone cubes $Q^\#$.

ALGORITHM: MAKE NEW ASSISTS AND ASSIGN KEYSTONE JETS

For each keystone cube $Q^\#$, we compute a list of new assists $\Omega^{new}(Q^\#) \subset \left[\mathbb{X}(S_1 Q^\# \cap E)\right]^*$, written in short form, and we produce an $\Omega^{new}(Q^\#)$-assisted bounded depth linear map $R^\#_{Q^\#} : \mathbb{X}(S_1 Q^\# \cap E) \oplus \mathcal{P} \to \mathcal{P}$, written in short form. Furthermore, we guarantee that the following conditions are met.

- The sum of depth(ω) over all ω in $\Omega^{\text{new}}(Q^{\#})$, and over all keystone cubes $Q^{\#}$, is bounded by CN.

Given $(f, P) \in \mathbb{X}(S_1 Q^{\#} \cap E) \oplus \mathcal{P}$, denote $R^{\#} = R^{\#}_{Q^{\#}}(f, P)$.

- Then $\partial^{\alpha}\left(R^{\#} - P\right) \equiv 0$ for all $\alpha \in \mathcal{A}$. (This condition is natural because $\mathcal{A}$ is monotonic; see (1.1).)
- Let $R \in \mathcal{P}$, with $\partial^{\beta}(R - P) \equiv 0$ for all $\beta \in \mathcal{A}$. Then

$$\sum_{Q \in \mathcal{I}(Q^{\#})} \sum_{\xi \in \Xi(Q, \mathcal{A}^-)} |\xi(f, R^{\#})|^p \leq C \sum_{Q \in \mathcal{I}(Q^{\#})} \sum_{\xi \in \Xi(Q, \mathcal{A}^-)} |\xi(f, R)|^p. \tag{1.26}$$

To compute the assists $\Omega^{\text{new}}(Q^{\#})$ and the short form of the maps $R^{\#}_{Q^{\#}}$ (for all the keystone cubes $Q^{\#}$) requires work at most $CN \log N$, and storage at most CN.

Explanation. Given $P \in \mathcal{P}$, we define V_P to be the affine subspace consisting of all polynomials $R \in \mathcal{P}$ satisfying $\partial^{\alpha}(R - P) \equiv 0$ for all $\alpha \in \mathcal{A}$. We note that $R \in V_P \iff \partial^{\alpha}(R - P)(0) = 0$ for all $\alpha \in \mathcal{A}$, since $\mathcal{A}$ is monotonic.

We introduce coordinates on V_P, defined by

$$w = (w_1, \ldots, w_k) \in \mathbb{R}^k \implies R_w(x) = \sum_{\alpha \in \mathcal{A}} \frac{\partial^{\alpha} P(0)}{\alpha!} x^{\alpha} + \sum_{j=1}^{k} w_j \cdot \frac{x^{\alpha_j}}{\alpha_j!},$$

where we write $\mathcal{M} \setminus \mathcal{A} = \{\alpha_1, \ldots, \alpha_k\}$.

We consider the sum of the p-th powers of the functionals $\xi(f, R_w)$ over all $\xi \in \Xi(Q, \mathcal{A}^-)$ and $Q \in \mathcal{I}(Q^{\#})$. We want to minimize this expression with respect to $w \in \mathbb{R}^k$. We can approximately solve this minimization problem using the algorithm OPTIMIZE VIA MATRIX from Section 2.8 of [1]. We describe the process below.

For each $Q \in CZ_{\text{main}}(\mathcal{A}^-)$ with $Q \cap S_0 Q^{\#} \neq \emptyset$, we have $\delta_{Q^{\#}} \leq \delta_Q$ by definition of keystone cubes. Hence, from (1.24) we have

$$Q \cap S_0 Q^{\#} \neq \emptyset \quad \text{and} \quad \delta_{Q^{\#}} \leq \delta_Q \leq C \cdot \delta_{Q^{\#}} \quad \text{for a universal constant } C. \tag{1.27}$$

We list all the dyadic cubes Q that satisfy (1.27). There are at most C cubes in this list. We test each Q to see whether it belongs to $CZ_{\text{main}}(\mathcal{A}^-)$. Thus, we can compute the list

$$\mathfrak{L} = \left\{Q \in CZ_{\text{main}}(\mathcal{A}^-) : \ Q \cap S_0 Q^{\#} \neq \emptyset\right\}.$$

The list $\mathfrak{L}$ contains all cubes Q that participate in (1.26) for which $\Xi(Q, \mathcal{A}^-) \neq \emptyset$. (Recall that $\Xi(Q, \mathcal{A}^-) = \emptyset$ for $Q \in \overline{CZ}(\mathcal{A}^-) \setminus CZ_{\text{main}}(\mathcal{A}^-)$.)

We list all the functionals ξ appearing in $\Xi(Q, \mathcal{A}^-)$ for some $Q \in \mathfrak{L}$. From the main technical results for $\mathcal{A}^-$ (see Section 3 in [1]), each such ξ is given in the form

$$\xi(f, R_w) = \lambda(f) + \sum_{a=1}^{I} \mu_a \cdot \omega_a(f) + \check{\lambda}((\partial^{\alpha} P(0))_{\alpha \in \mathcal{A}}) + \sum_{j=1}^{k} \check{\mu}_j \cdot w_j,$$

where λ and $\check{\lambda}$ are linear functionals; $\omega_a \in \Omega(Q, \mathcal{A}^-)$ for some $Q \in \mathfrak{L}$; μ_a and $\check{\mu}_j$ are real coefficients; and $\mathrm{depth}(\lambda) = \mathcal{O}(1)$, $I = \mathcal{O}(1)$. In this discussion, we write $X = \mathcal{O}(Y)$ to indicate that $X \leq CY$ for a universal constant C.

Processing each functional ξ this way takes work $\mathcal{O}(1)$ per functional. Thus, with work $\mathcal{O}(L)$ (see below), we obtain a list of all the above ξ's, written as

$$\xi_\ell(f, R_w) = \lambda_\ell(f) + \sum_{a=1}^{I_\ell} \mu_{\ell a}\omega_{\ell a}(f) + \check{\lambda}_\ell((\partial^\alpha P(0))_{\alpha\in\mathcal{A}}) + \sum_{j=1}^{k} \check{\mu}_{\ell j} w_j \tag{1.28}$$
$$\text{for } \ell = 1, \dots, L; \text{ here, } L = \sum_{Q\in\mathcal{I}(Q^\#)} \#\big[\Xi(Q, \mathcal{A}^-)\big].$$

Here, each $I_\ell = \mathcal{O}(1)$, each λ_ℓ has bounded depth, and each $\omega_{\ell a}$ belongs to $\Omega(Q_{\ell a}, \mathcal{A}^-)$ for some $Q_{\ell a} \in \mathfrak{L}$. Of course the $Q_{\ell a}$ need not be distinct, and $k \leq \dim(\mathcal{P}) = D$.

Processing the functionals $w \mapsto \xi_\ell(f, R_w)$ in (1.28) with the algorithm OPTIMIZE VIA MATRIX (see Section 2.8 of [1]), we compute a matrix $(b_{j\ell})$ with the following property. The sum of the absolute values of the p-th powers of the functionals $\xi_\ell(f, R_w)$ $(1 \leq \ell \leq L)$ is essentially minimized for fixed f, $(\partial^\alpha P(0))_{\alpha\in\mathcal{A}}$ by setting $w = w^* = (w_1^*, \dots, w_k^*)$, where

$$\begin{aligned} w_j^* &= \sum_{\ell=1}^{L} b_{j\ell}\Big[\lambda_\ell(f) + \sum_{a=1}^{I_\ell} \mu_{\ell a}\omega_{\ell a}(f) + \check{\lambda}_\ell((\partial^\alpha P(0))_{\alpha\in\mathcal{A}})\Big] \\ &= \Big\{\sum_{\ell=1}^{L} b_{j\ell}\Big[\lambda_\ell(f) + \sum_{a=1}^{I_\ell} \mu_{\ell a}\omega_{\ell a}(f)\Big]\Big\} + \sum_{\ell=1}^{L} b_{j\ell}\cdot\check{\lambda}_\ell((\partial^\alpha P(0))_{\alpha\in\mathcal{A}}) \\ &\equiv \qquad \omega_j^{\mathrm{new}}(f) \qquad\qquad + \quad \lambda_j^{\mathrm{new}}((\partial^\alpha P(0))_{\alpha\in\mathcal{A}}). \end{aligned} \tag{1.29}$$

We have thus defined new assists ω_j^{new} and new functionals λ_j^{new}.

We may therefore set $R^\#_{Q^\#}(f, P) := R_{w^*}$ with $w_j^* = \omega_j^{\mathrm{new}}(f) + \lambda_j^{\mathrm{new}}((\partial^\alpha P(0))_{\alpha\in\mathcal{A}})$ $(1 \leq j \leq k)$ and we obtain the estimate (1.26). Note that $R^\#_{Q^\#}$ has assisted bounded depth, with assists ω_j^{new} $(j = 1, \dots, k)$. Indeed,

$$\partial^\alpha[R^\#_{Q^\#}(f, P)](0) = \begin{cases} \omega_j^{\mathrm{new}}(f) + \lambda_j^{\mathrm{new}}((\partial^\alpha P(0))_{\alpha\in\mathcal{A}}) & \text{if } \alpha = \alpha_j,\ j = 1, \dots, k \\ \partial^\alpha P(0) & \text{if } \alpha \in \mathcal{A}. \end{cases} \tag{1.30}$$

We can compute the new functionals λ_j^{new} $(1 \leq j \leq k)$ using the obvious method. This requires work

$$\mathcal{O}(L) = \mathcal{O}\Big(\sum_{Q\in\mathcal{I}(Q^\#)} \#\big[\Xi(Q, \mathcal{A}^-)\big]\Big).$$

We will now express the new assists ω_j^{new} in short form.

We write $\omega_j^{\mathrm{new}} = \omega_j^{\mathrm{new},1} + \omega_j^{\mathrm{new},2}$, where $\omega_j^{\mathrm{new},1}$ and $\omega_j^{\mathrm{new},2}$ are defined below (see (1.31) and (1.34)).

Each $\lambda_\ell(f)$ has bounded depth, so the functional

$$\omega_j^{\text{new},1} : f \mapsto \sum_{\ell=1}^{L} b_{j\ell} \cdot \lambda_\ell(f) \tag{1.31}$$

can be computed in short form using

$$\text{work} \quad \mathcal{O}(L \log L) = \mathcal{O}\Big(\log N \cdot \sum_{Q \in \mathcal{I}(Q^\#)} \#\left[\Xi(Q, \mathcal{A}^-)\right] \Big)$$
$$\text{and storage} \quad \mathcal{O}(L) = \mathcal{O}\Big(\sum_{Q \in \mathcal{I}(Q^\#)} \#\left[\Xi(Q, \mathcal{A}^-)\right] \Big).$$

This computation follows by a simple sorting procedure. We provide details below.

- Recall that λ_ℓ has bounded depth and is given in short form (without assists):

$$\lambda_\ell(f) = \sum_{x \in S_\ell} c_\ell(x) \cdot f(x), \quad \text{where } \#(S_\ell) \le C. \tag{1.32}$$

  Thus, we can express the functional (1.31) as

$$\omega_j^{\text{new},1} : f \mapsto \sum_{x \in S} d_j(x) \cdot f(x), \quad \text{where } S = \bigcup_{\ell=1}^{L} S_\ell \text{ and}$$
$$d_j(x) = \sum_{\ell=1}^{L} b_{j\ell} \cdot c_\ell(x) \quad \text{for } x \in S. \tag{1.33}$$

  We compute the weights $d_j(x)$ by sorting. More precisely, we sort the points of S. We make an array indexed by S. We initialize the array to have all zero entries. We loop over $\ell = 1, \dots, L$, and we loop over all the points $y \in S_\ell$. We determine the position of each y in the list S, and we add the number $b_{j\ell} \cdot c_\ell(y)$ at the relevant position in the array. This requires work at most $C \log(S) \le C \log L$ for a fixed pair (ℓ, y). Hence, the total work required is at most $CL \log L$, since the number of relevant pairs (ℓ, y) is $\sum_{\ell=1}^{L} \#(S_\ell) \le \sum_{\ell=1}^{L} C \le CL$. Similarly, the total storage is at most $C \sum_{\ell=1}^{L} \#(S_\ell) \le CL$.

Thus, we can compute the functional (1.31) using work $\mathcal{O}(L \log L)$ and storage $\mathcal{O}(L)$.

It remains to compute the functional

$$\omega_j^{\text{new},2} : f \mapsto \sum_{\ell=1}^{L} b_{j\ell} \sum_{a=1}^{I_\ell} \mu_{\ell a} \omega_{\ell a}(f) \tag{1.34}$$

in short form. (Recall, each $I_\ell = \mathcal{O}(1)$.)

We recall that each $\omega_{\ell a}$ belongs to $\bigcup_{Q \in \mathcal{I}(Q^\#)} \Omega(Q, \mathcal{A}^-)$.

We can express the functional (1.34) in the form

$$\omega_j^{\mathrm{new},2} : f \mapsto \sum_{\omega \in \bigcup_{Q \in \mathcal{I}(Q^\#)} \Omega(Q,\mathcal{A}^-)} q_{j\omega} \cdot \omega(f), \tag{1.35}$$

with work $\mathcal{O}(L \log L) = \mathcal{O}\Big(\log N \cdot \sum_{Q \in \mathcal{I}(Q^\#)} \#\big[\Xi(Q,\mathcal{A}^-)\big] \Big)$

and storage $\mathcal{O}(L) = \mathcal{O}\Big(\sum_{Q \in \mathcal{I}(Q^\#)} \#\big[\Xi(Q,\mathcal{A}^-)\big] \Big)$.

We can compute the relevant numbers $q_{j\omega}$ by sorting, since

$$q_{j\omega} = \sum_{(\ell,a):\omega_{\ell a}=\omega} b_{j\ell} \cdot \mu_{\ell a}. \tag{1.36}$$

Finally, once our functional is in the form (1.35), we can easily write it in short form

$$\omega_j^{\mathrm{new},2} : f \mapsto \sum_{x \in S} k_j(x) \cdot f(x) \tag{1.37}$$

with work $\mathcal{O}\Big(\log N \cdot \sum_{Q \in \mathcal{I}(Q^\#)} \sum_{\omega \in \Omega(Q,\mathcal{A}^-)} \mathrm{depth}(\omega) \Big)$

and storage $\mathcal{O}\Big(\sum_{Q \in \mathcal{I}(Q^\#)} \sum_{\omega \in \Omega(Q,\mathcal{A}^-))} \mathrm{depth}(\omega) \Big)$.

Again, we perform a sort to carry this out.

We compute $\omega_j^{\mathrm{new}} = \omega_j^{\mathrm{new},1} + \omega_j^{\mathrm{new},2}$ in short form by adding the short form expressions (1.33) and (1.37).

Altogether, we obtain the new assists ω_j^{new} and the new functionals λ_j^{new} for a given $Q^\#$ using work at most

$$C \cdot (\log N) \cdot \Big[\sum_{Q \in \mathcal{I}(Q^\#)} \Big\{ 1 + \#\big[\Xi(Q,\mathcal{A}^-)\big] + \sum_{\omega \in \Omega(Q,\mathcal{A}^-)} \mathrm{depth}(\omega) \Big\} \Big]$$

and storage at most

$$C \cdot \Big[\sum_{Q \in \mathcal{I}(Q^\#)} \Big\{ 1 + \#\big[\Xi(Q,\mathcal{A}^-)\big] + \sum_{\omega \in \Omega(Q,\mathcal{A}^-)} \mathrm{depth}(\omega) \Big\} \Big].$$

(Again, recall that $\Xi(Q,\mathcal{A}^-) = \Omega(Q,\mathcal{A}^-) = \emptyset$ for any $Q \in \overline{\mathrm{CZ}}(\mathcal{A}^-) \setminus \mathrm{CZ}_{\mathrm{main}}(\mathcal{A}^-)$.)

Each $Q \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A}^-)$ belongs to $\mathcal{I}(Q^\#)$ for at most C distinct $Q^\#$ (see Remark 3). Therefore, we can compute the new assists and the new functionals for

all the keystone cubes $Q^{\#}$ using work at most

$$C \cdot (\log N) \cdot \Big[\#\{ \text{Keystone Cubes } Q^{\#} \} + \sum_{Q \in CZ_{\text{main}}(\mathcal{A}^-)} \#\big[\Xi(Q, \mathcal{A}^-)\big] + \sum_{Q \in CZ_{\text{main}}(\mathcal{A}^-)} \sum_{\omega \in \Omega(Q, \mathcal{A}^-)} \text{depth}(\omega) \Big],$$

which is at most $CN \log N$ by the induction hypothesis and the statement of the KEYSTONE-ORACLE (which guarantees that the number of keystone cubes is bounded by CN). Similarly, we see that all the new assists can be computed using storage at most CN.

Finally, note that there are at most D new assists for each given keystone cube $Q^{\#} \in CZ(\mathcal{A}^-)$, and each such assist has depth at most $\#(S_1 Q^{\#} \cap E)$. By the bounded overlap of the cubes $S_1 Q^{\#}$ (see Lemma 8), we see that the sum of the depths of all the new assists is at most $C \cdot \#(E) = CN$.

This completes the explanation of the algorithm. □

Let $Q^{\#} \in CZ(\mathcal{A}^-)$ be a keystone cube. For each $(f, R) \in \mathbb{X}(S_1 Q^{\#} \cap E) \oplus \mathcal{P}$, we define

$$\big[M^{\#}_{Q^{\#}}(f, R)\big]^p := \sum_{Q \in \mathcal{I}(Q^{\#})} \sum_{\xi \in \Xi(Q, \mathcal{A}^-)} |\xi(f, R)|^p = \sum_{Q \in \mathcal{I}(Q^{\#})} \big[M_{(Q, \mathcal{A}^-)}(f, R)\big]^p. \tag{1.38}$$

The terms $\xi(f, R)$ appearing above are well-defined, since (1.25) states that $\frac{65}{64} Q \subset S_1 Q^{\#}$ for each $Q \in \mathcal{I}(Q^{\#})$. (Recall that the domain of each functional ξ in $\Xi(Q, \mathcal{A}^-)$ is $\mathbb{X}((65/64)Q \cap E) \oplus \mathcal{P}$.)

We now show that the "keystone functional" defined in (1.38) is comparable to the trace semi-norm near the given keystone cube.

Lemma 10. *Let $Q^{\#}$ be a keystone cube. Then*

$$c \cdot \|(f, R)\|_{S_0 Q^{\#}} \le M^{\#}_{Q^{\#}}(f, R) \le C \cdot \|(f, R)\|_{S_1 Q^{\#}}$$

for all $(f, R) \in \mathbb{X}(S_1 Q^{\#} \cap E) \oplus \mathcal{P}$.

Proof. From (1.25) we learn that $(1 + a)Q \subset (65/64)Q \subset S_1 Q^{\#}$ for any $Q \in \mathcal{I}(Q^{\#})$. (Recall that $a = a(\mathcal{A}^-) \le 1/64$; see (1.7).)

Let $(f, R) \in \mathbb{X}(S_1 Q^{\#} \cap E) \oplus \mathcal{P}$ be given.

For each $Q \in \mathcal{I}(Q^{\#})$, by definition of the seminorm $\|(\cdot, \cdot)\|_{(1+a)Q}$, we can choose $F_Q \in \mathbb{X}$ with $F_Q = f$ on $E \cap (1 + a)Q$ and

$$\|F_Q\|_{\mathbb{X}((1+a)Q)} + \delta^{-m}_{(1+a)Q} \|F_Q - R\|_{L^p((1+a)Q)} \le 2 \cdot \|(f, R)\|_{(1+a)Q}.$$

From the left-hand estimate in (1.20) we have $\|(f, R)\|_{(1+a)Q} \le C \cdot M_{(Q, \mathcal{A}^-)}(f, R)$. Thus, by definition (1.38) we have

$$\|F_Q\|_{\mathbb{X}((1+a)Q)} + \delta^{-m}_{Q} \|F_Q - R\|_{L^p((1+a)Q)} \le C \cdot M^{\#}_{Q^{\#}}(f, R) \quad \text{for } Q \in \mathcal{I}(Q^{\#}). \tag{1.39}$$

We will need to use a few results from Sections 4.6.4 and 4.6.5 in [1]. Consider:

- $CZ := \overline{CZ}(\mathcal{A}^-)$ and $\mathcal{Q} := \mathcal{I}(Q^\#)$.
- $\widehat{Q} := S_0 Q^\#$;
- $\overline{r} := a$, and $A := 10^3$.

We have already given a $\overline{CZ}(\mathcal{A}^-)$-ORACLE. We proved good geometry for $\overline{CZ}(\mathcal{A}^-)$, which is a dyadic decomposition of $\mathbb{R}^n$; see Proposition 1. This puts us in the setting of Section 4.6.4 of [1].

Now, note the following *covering condition*: every $x \in S_0 Q^\#$ belongs to some $Q \in \mathcal{I}(Q^\#)$. This is due to the definition of $\mathcal{I}(Q^\#)$ in (1.21) and because $\overline{CZ}(\mathcal{A}^-)$ is a decomposition of $\mathbb{R}^n$. We also point out the *size condition* stated in (1.24): for each $Q \in \mathcal{I}(Q^\#)$, we have $\delta_Q \le 10^3 \delta_{S_0 \widehat{Q}}$. These two conditions are equivalent to (4.53) and (4.54) in Section 4.6.5, with the choice of CZ, $\mathcal{Q}$, $\widehat{Q}$, $\overline{r}$, and A as in the above bullet points. This puts us in the setting of Section 4.6.5 of [1].

By Lemma 48 in [1], there exists a partition of unity $\theta_Q^{Q^\#} \in C^m(\mathbb{R}^n)$ ($Q \in \mathcal{I}(Q^\#)$) such that

$$\sum_{Q \in \mathcal{I}(Q^\#)} \theta_Q^{Q^\#} = 1 \quad \text{on } S_0 Q^\#, \tag{1.40}$$

where $\operatorname{supp} \theta_Q^{Q^\#} \subset (1+a)Q$ and $|\partial^\alpha \theta_Q^{Q^\#}(x)| \le C \cdot \delta_Q^{-|\alpha|}$ for $x \in (1+a)Q$, $|\alpha| \le m$.

Define

$$F := \sum_{Q \in \mathcal{I}(Q^\#)} F_Q \cdot \theta_Q^{Q^\#}.$$

Note that the cardinality of $\mathcal{I}(Q^\#)$ is at most C (see Remark 3). Thus, from (1.39), and Lemma 49 in [1], we have

$$\|F\|_{\mathbb{X}(S_0 Q^\#)} \le C \cdot M^\#_{Q^\#}(f, R).$$

Moreover, since $\operatorname{supp}(\theta_Q^{Q^\#}) \subset (1+a)Q$ and $\delta_{Q^\#} \le \delta_Q \le C\delta_{Q^\#}$ for all $Q \in \mathcal{I}(Q^\#)$, we have

$$\begin{aligned}
\delta_{Q^\#}^{-m} \|F - R\|_{L^p(S_0 Q^\#)} &= \delta_{Q^\#}^{-m} \Big\| \sum_{Q \in \mathcal{I}(Q^\#)} (F_Q - R) \cdot \theta_Q^{Q^\#} \Big\|_{L^p(S_0 Q^\#)} \qquad \text{(see (1.40))} \\
&\le C \sum_{Q \in \mathcal{I}(Q^\#)} \delta_Q^{-m} \|F_Q - R\|_{L^p((1+a)Q)} \|\theta_Q^{Q^\#}\|_{L^\infty((1+a)Q)} \\
&\le C \cdot M^\#_{Q^\#}(f, R) \qquad \text{(see (1.39))}.
\end{aligned}$$

Because $\operatorname{supp}(\theta_Q^{Q^\#}) \subset (1+a)Q$ and $F_Q = f$ on $E \cap (1+a)Q$ for all $Q \in \mathcal{I}(Q^\#)$, and from (1.40), we learn that $F = f$ on $E \cap S_0 Q^\#$. Hence, the above estimates imply that

$$\|(f, R)\|_{S_0 Q^\#} \le \|F\|_{\mathbb{X}(S_0 Q^\#)} + \delta_{S_0 Q^\#}^{-m} \|F - R\|_{L^p(S_0 Q^\#)} \lesssim M^\#_{Q^\#}(f, R).$$

This proves one inequality in the statement of the lemma.

Next, using the right-hand estimate in (1.20), we see that

$$\left[M^{\#}_{Q^{\#}}(f,R)\right]^p = \sum_{\substack{Q\in\mathcal{I}(Q^{\#})\\ Q\in CZ_{\mathrm{main}}(\mathcal{A}^-)}} \left[M_{(Q,\mathcal{A}^-)}(f,R)\right]^p \le C \sum_{Q\in\mathcal{I}(Q^{\#})} \|(f,R)\|^p_{\frac{65}{64}Q}.$$

Recall that $\mathcal{I}(Q^{\#})$ contains at most C cubes. Thus, by Lemma 14 in [1], where we use the fact that $\delta_{Q^{\#}} \le \delta_Q \le C\delta_{Q^{\#}}$ and $\frac{65}{64}Q \subset S_1 Q^{\#}$ for all $Q \in \mathcal{I}(Q^{\#})$, we learn that the right-hand side in the above estimate is bounded by $C\cdot\|(f,R)\|^p_{S_1Q^{\#}}$. This completes the proof of the lemma. □

The parameter $\epsilon > 0$ now makes its first appearance. Recall that ϵ is assumed to be less than a small enough universal constant. See (1.18).

Proposition 2. *Let $\widehat{Q}$ be a dyadic subcube of Q°, such that $3\widehat{Q}$ is tagged with $(\mathcal{A},\epsilon)$.*

Assume also that $Q^{\#} \in CZ(\mathcal{A}^-)$ is a keystone cube, and that $S_1Q^{\#} \subseteq \frac{65}{64}\widehat{Q}$.

Suppose that $H \in \mathbb{X}$ satisfies $H = f$ on $E \cap S_1Q^{\#}$ and $\partial^\alpha H(x_{Q^{\#}}) = \partial^\alpha P(x_{Q^{\#}})$ for all $\alpha \in \mathcal{A}$. Then

$$\delta^{-m}_{Q^{\#}}\|H - R^{\#}_{Q^{\#}}\|_{L^p(S_1Q^{\#})} \le C\cdot\|H\|_{\mathbb{X}(S_1Q^{\#})}. \tag{1.41}$$

Here, $C \ge 1$ is a universal constant; and $R^{\#}_{Q^{\#}} = R^{\#}_{Q^{\#}}(f,P)$.

(See the algorithm MAKE NEW ASSISTS AND ASSIGN KEYSTONE JETS.*)*

Proof. Recall that $S_0Q^{\#} \subset \frac{65}{64}\widehat{Q}$ and $3\widehat{Q}$ is tagged with $(\mathcal{A},\epsilon)$. Thus, Lemma 28 from [1] implies that $S_0Q^{\#}$ is tagged with $(\mathcal{A},\epsilon^\kappa)$ for some universal constant $\kappa > 0$.

Recall that $S_1 \ge 2[c_*(\mathcal{A}^-)]^{-1}$; see (1.17). Thus, since $S_1Q^{\#} \subset \frac{65}{64}\widehat{Q}$ we have

$$\delta_{Q^{\#}} \le S_1^{-1}\delta_{\frac{65}{64}\widehat{Q}} \le c_*(\mathcal{A}^-). \tag{1.42}$$

Hence, the induction hypothesis implies that

$$S_0Q^{\#} \text{ is not tagged with } (\mathcal{A}', \epsilon_1(\mathcal{A}^-)) \text{ for any } \mathcal{A}' < \mathcal{A}. \tag{1.43}$$

In particular, $S_0Q^{\#}$ is not tagged with $(\mathcal{A}',\epsilon^\kappa)$ for any $\mathcal{A}' < \mathcal{A}$.

Since $S_0Q^{\#}$ is tagged with $(\mathcal{A},\epsilon^\kappa)$ but not with $(\mathcal{A}',\epsilon^\kappa)$ for any $\mathcal{A}' < \mathcal{A}$, we see that $\sigma(S_0Q^{\#})$ has an $(\mathcal{A}, x_{Q^{\#}}, \epsilon^\kappa, \delta_{S_0Q^{\#}})$-basis. Thus there exist polynomials $(P_\alpha)_{\alpha\in\mathcal{A}}$ such that

$$P_\alpha \in \epsilon^\kappa\left[\delta_{S_0Q^{\#}}\right]^{|\alpha|+n/p-m}\cdot\sigma(S_0Q^{\#}) \qquad \text{for } \alpha\in\mathcal{A}, \tag{1.44}$$

$$\partial^\beta P_\alpha(x_{Q^{\#}}) = \delta_{\alpha\beta} \qquad \text{for } \beta,\alpha\in\mathcal{A}, \tag{1.45}$$

$$|\partial^\beta P_\alpha(x_{Q^{\#}})| \le \epsilon^\kappa\left[\delta_{S_0Q^{\#}}\right]^{|\alpha|-|\beta|} \qquad \text{for } \beta\in\mathcal{M},\ \alpha\in\mathcal{A},\ \beta>\alpha. \tag{1.46}$$

To start, we prove the following statement.

- Suppose that

$$R \in \sigma(S_0 Q^{\#}), \quad \text{and} \tag{1.47}$$

$$\partial^{\alpha} R(x_{Q^{\#}}) = 0 \quad \text{for all } \alpha \in \mathcal{A}. \tag{1.48}$$

Then, for some $W = W(m, n, p) \geq 0$ we have

$$|\partial^{\beta} R(x_{Q^{\#}})| \leq W \cdot \delta_{Q^{\#}}^{m-n/p-|\beta|} \quad \text{for } \beta \in \mathcal{M}. \tag{1.49}$$

For the sake of contradiction, suppose that (1.47) and (1.48) do not imply (1.49). Then, for some large constant $\widehat{W} \geq 0$, which will be determined later, there exists $R \in \sigma(S_0 Q^{\#})$ satisfying (1.48) and

$$\max_{\beta \in \mathcal{M}} |\partial^{\beta} R(x_{Q^{\#}})| \cdot (\delta_{Q^{\#}})^{n/p+|\beta|-m} = \widehat{W}. \tag{1.50}$$

For each integer $\ell \geq 0$, define the multi-index set

$$\Delta_\ell = \left\{\beta \in \mathcal{M} : |\partial^{\beta} R(x_{Q^{\#}})| \cdot (\delta_{Q^{\#}})^{n/p+|\beta|-m} \geq \widehat{W}^{(2^{-\ell})}\right\}.$$

Note that $\Delta_0 \neq \emptyset$ thanks to (1.50), and also $\Delta_\ell \subset \Delta_{\ell+1}$ for $\ell \geq 0$.

Since $\#\mathcal{M} = D$ and $\Delta_\ell \subset \mathcal{M}$ is an increasing sequence, there is an index $\ell_* \in \{0, \ldots, D\}$ such that $\Delta_{\ell_*} = \Delta_{\ell_*+1}$. Pick the maximal element $\overline{\alpha} \in \Delta_{\ell_*}$ (under the total order relation $<$ on $\mathcal{M}$). By definition of Δ_{ℓ_*}, we have

$$|\partial^{\overline{\alpha}} R(x_{Q^{\#}})| \cdot (\delta_{Q^{\#}})^{n/p+|\overline{\alpha}|-m} \geq \widehat{W}^{(2^{-\ell_*})}. \tag{1.51}$$

Now, if $\beta \in \mathcal{M}$ and $\beta > \overline{\alpha}$, then $\beta \notin \Delta_{\ell_*} = \Delta_{\ell_*+1}$, by maximality of $\overline{\alpha}$. Hence,

$$|\partial^{\beta} R(x_{Q^{\#}})| \cdot (\delta_{Q^{\#}})^{n/p+|\beta|-m} \leq \widehat{W}^{(2^{-\ell_*-1})} \quad \text{for each } \beta \in \mathcal{M} \text{ with } \beta > \overline{\alpha}. \tag{1.52}$$

We define Z and A by setting $\widehat{W} = Z^A$, with $A = 2^{\ell_*}$ and $0 \leq \ell_* \leq D$. Then (1.50)–(1.52) state that

$$\begin{aligned}
&|\partial^{\overline{\alpha}} R(x_{Q^{\#}})| \geq Z \cdot [\delta_{Q^{\#}}]^{m-n/p-|\overline{\alpha}|}, \\
&|\partial^{\beta} R(x_{Q^{\#}})| \leq Z^{1/2} \cdot [\delta_{Q^{\#}}]^{m-n/p-|\beta|} \quad \text{for } \beta \in \mathcal{M},\ \beta > \overline{\alpha}, \text{ and} \\
&|\partial^{\beta} R(x_{Q^{\#}})| \leq Z^{A} \cdot [\delta_{Q^{\#}}]^{m-n/p-|\beta|} \quad \text{for } \beta \in \mathcal{M}.
\end{aligned}$$

Define $\overline{P}_{\overline{\alpha}} := (\partial^{\overline{\alpha}} R(x_{Q^{\#}}))^{-1} \cdot R$. Then (1.48) implies that

$$\partial^{\beta} \overline{P}_{\overline{\alpha}}(x_{Q^{\#}}) = \delta_{\alpha\beta} \quad \text{for } \beta \in \mathcal{A} \cup \{\overline{\alpha}\}. \tag{1.53}$$

Since $R \in \sigma(S_0 Q^{\#})$,

$$\overline{P}_{\overline{\alpha}} \in Z^{-1} \left[\delta_{Q^{\#}}\right]^{|\overline{\alpha}|+n/p-m} \cdot \sigma(S_0 Q^{\#}), \tag{1.54}$$

and also

$$|\partial^\beta \overline{P}_{\overline{\alpha}}(x_{Q^\#})| \le Z^{-1/2} \cdot [\delta_{Q^\#}]^{|\overline{\alpha}|-|\beta|} \quad \text{for } \beta \in \mathcal{M},\ \beta > \overline{\alpha}, \text{ and} \tag{1.55}$$

$$|\partial^\beta \overline{P}_{\overline{\alpha}}(x_{Q^\#})| \le Z^A \cdot [\delta_{Q^\#}]^{|\overline{\alpha}|-|\beta|} \quad \text{for } \beta \in \mathcal{M}. \tag{1.56}$$

Set $\overline{\mathcal{A}} := \{\alpha \in \mathcal{A} : \alpha < \overline{\alpha}\} \cup \{\overline{\alpha}\}$. Since $\partial^\alpha R(x_{Q^\#}) = 0$ for all $\alpha \in \mathcal{A}$, we see that $\overline{\alpha} \notin \mathcal{A}$. Thus,

$$\overline{\mathcal{A}} < \mathcal{A}. \tag{1.57}$$

For $\alpha \in \mathcal{A}$ with $\alpha < \overline{\alpha}$, we set

$$\overline{P}_\alpha := P_\alpha - \partial^{\overline{\alpha}} P_\alpha(x_{Q^\#}) \overline{P}_{\overline{\alpha}}.$$

Note that

$$\overline{P}_\alpha \in C\epsilon^\kappa Z^A \left[\delta_{Q^\#}\right]^{|\alpha|+n/p-m} \cdot \sigma(S_0 Q^\#) \quad \text{(by (1.44), (1.46), (1.54))}, \tag{1.58}$$

$$\partial^\beta \overline{P}_\alpha(x_{Q^\#}) = \delta_{\alpha\beta} \quad \text{for } \beta \in \overline{\mathcal{A}} \text{ (by (1.45), (1.53))}, \text{ and} \tag{1.59}$$

$$|\partial^\beta \overline{P}_\alpha(x_{Q^\#})| \le C\epsilon^\kappa Z^A [\delta_{Q^\#}]^{|\alpha|-|\beta|} \quad \text{for } \beta \in \mathcal{M}, \beta > \alpha \text{ (by (1.46), (1.56))}. \tag{1.60}$$

Examining (1.53)–(1.60), we see that

$$(\overline{P}_\alpha)_{\alpha \in \overline{\mathcal{A}}} \text{ is an } (\overline{\mathcal{A}}, x_{Q^\#}, C \cdot (S_0)^m \max\{\epsilon^\kappa Z^A, Z^{-1/2}\}, \delta_{S_0 Q^\#})\text{-basis for } \sigma(S_0 Q^\#). \tag{1.61}$$

Recall that $Z = \widehat{W}^{1/A}$ with $A = 2^{\ell_*}$ and $0 \le \ell_* \le D$. We pick $\widehat{W}$ to be a large enough universal constant, and assume that ϵ is less than a small enough universal constant. Then (1.57) and (1.61) imply that $S_0 Q^\#$ is tagged with $(\mathcal{A}^-, \epsilon_1(\mathcal{A}^-))$. But this contradicts (1.43). This completes our proof that (1.49) holds whenever the polynomial R satisfies (1.47) and (1.48).

We now prove the main assertion in Proposition 2. Suppose that $H \in \mathbb{X}$ satisfies $H = f$ on $E \cap S_1 Q^\#$ and $\partial^\alpha H(x_{Q^\#}) = \partial^\alpha P(x_{Q^\#})$ for all $\alpha \in \mathcal{A}$. Then $\partial^\alpha(J_{x_{Q^\#}} H - P) \equiv 0$ for all $\alpha \in \mathcal{A}$. Here, we use the monotonicity of $\mathcal{A}$; see (1.1). Next, we apply the estimate (1.26) followed by Lemma 10, which gives

$$M^\#_{Q^\#}(f, R^\#_{Q^\#}) \le C \cdot M^\#_{Q^\#}(f, J_{x_{Q^\#}} H) \le C \cdot \|(f, J_{x_{Q^\#}} H)\|_{S_1 Q^\#} \le C \cdot \|H\|_{\mathbb{X}(S_1 Q^\#)},$$

which implies that

$$M^\#_{Q^\#}(0, R^\#_{Q^\#} - J_{x_{Q^\#}} H) \le C \cdot \|H\|_{\mathbb{X}(S_1 Q^\#)}.$$

Thus, Lemma 10 implies that $\|(0, R^\#_{Q^\#} - J_{x_{Q^\#}} H)\|_{S_0 Q^\#} \le C\|H\|_{\mathbb{X}(S_1 Q^\#)}$, hence

$$R^\#_{Q^\#} - J_{x_{Q^\#}} H \in C\|H\|_{\mathbb{X}(S_1 Q^\#)} \cdot \sigma(S_0 Q^\#).$$

By the defining properties of $R^\#_{Q^\#}$ (see the algorithm MAKE NEW ASSISTS AND ASSIGN KEYSTONE JETS), and by our assumption on $\partial^\alpha H(x_{Q^\#})$, we have

$$\partial^\alpha(R^\#_{Q^\#} - J_{x_{Q^\#}} H)(x_{Q^\#}) = \partial^\alpha(P - P)(x_{Q^\#}) = 0 \quad \text{for all } \alpha \in \mathcal{A}.$$

Thus, (1.49) shows that

$$|\partial^\beta(J_{x_{Q^\#}} H - R^\#_{Q^\#})(x_{Q^\#})| \le C \cdot (\delta_{Q^\#})^{m-n/p-|\beta|} \|H\|_{\mathbb{X}(S_1 Q^\#)} \quad \text{for all } \beta \in \mathcal{M}.$$

Hence, by the Sobolev inequality we have

$$\delta_{Q^\#}^{-m} \|H - R^\#_{Q^\#}\|_{L^p(S_1 Q^\#)} \leq C \cdot \|H\|_{\mathbb{X}(S_1 Q^\#)}.$$

That proves (1.41) and completes the proof of Proposition 2. □

1.4.2. Marked cubes. We summarize various objects that we have computed in previous sections of the paper. This is meant to serve as a reference for the reader.

- *The main cubes.* We compute the collection of cubes $Q \in CZ_{\text{main}}(\mathcal{A}^-)$, each marked with pointers to the following objects.
  - The list $\Omega(Q, \mathcal{A}^-)$ of assist functionals on $\mathbb{X}(\frac{65}{64}Q \cap E)$, expressed in short form.
  - The list $\Xi(Q, \mathcal{A}^-)$ of functionals on $\mathbb{X}(\frac{65}{64}Q \cap E) \oplus \mathcal{P}$, each of $\Omega(Q, \mathcal{A}^-)$-assisted bounded depth, expressed in short form in terms of assists $\Omega(Q, \mathcal{A}^-)$.
  - The list of functionals $\xi_1^Q, \ldots, \xi_D^Q$ on $\mathcal{P}$.

  (See the main technical results for $\mathcal{A}^-$ in Section 3 of [1], and the algorithm APPROXIMATE OLD TRACE NORM in Section 1.3.)
- *The keystone cubes.* We list all the keystone cubes $Q^\#$ for $CZ(\mathcal{A}^-)$, each marked with pointers to the following objects.
  - The list $\Omega^{\text{new}}(Q^\#)$ of new assist functionals on $\mathbb{X}(S_1 Q^\# \cap E)$, expressed in short form.
  - The linear map $R^\#_{Q^\#} : \mathbb{X}(S_1 Q^\# \cap E) \oplus \mathcal{P} \to \mathcal{P}$, which has $\Omega^{\text{new}}(Q^\#)$-assisted bounded depth, and is expressed in short form in terms of assists $\Omega^{\text{new}}(Q^\#)$.

  (See MAKE NEW ASSISTS AND ASSIGN KEYSTONE JETS in Section 1.4.1.)
- *The border-dispute pairs.* We list all the border-dispute pairs $(Q', Q'') \in BD(\mathcal{A}^-)$. (See the KEYSTONE-ORACLE in Section 1.3.3.)

We store these cubes in memory along with their markings.

1.4.3. Testing cubes. Let $\widehat{Q}$ be a dyadic subcube of Q°. Since $CZ(\mathcal{A}^-)$ is a dyadic decomposition of Q°, one and only one of the following alternatives holds.

(A) $\widehat{Q}$ is a disjoint union of cubes from $CZ(\mathcal{A}^-)$.

(B) $\widehat{Q}$ is strictly contained in one of the cubes of $CZ(\mathcal{A}^-)$.

Definition 2. Let $\widehat{Q} \subset Q^\circ$ be a dyadic cube. If alternative (A) holds, we call $\widehat{Q}$ a *testing cube.*

Let $0 < \lambda < 1$. We say that a testing cube $\widehat{Q}$ is λ-*simple* if $\delta_Q \geq \lambda \cdot \delta_{\widehat{Q}}$ for any $Q \in CZ(\mathcal{A}^-)$ with $Q \subset (65/64)\widehat{Q}$.

We introduce a geometric parameter

$$t_G \in \mathbb{R}, \text{ which is an integer power of two.} \tag{1.62}$$

We assume that $0 < t_G < c$, where c is a small enough constant determined by m, n and p.

We will later determine $a(\mathcal{A})$ to be an appropriate constant depending on t_G. For the main conditions satisfied by $a(\mathcal{A})$, see the fourth and fifth bullet points in Section 3 of [1]. Near the end of Section 1.4 we determine t_G to be a constant depending only on m, n, and p – but not yet.

We recall that $a = a(\mathcal{A}^-)$ is a fixed universal constant.

Lemma 11. *Let $\widehat{Q}$ be a testing cube. Assume that $t_G > 0$ is less than a small enough universal constant. The following properties hold.*

- *There exists a constant $a_{new} > 0$, depending only on t_G, m, n, p, such that the cube $(1 + a_{new})\widehat{Q}$ is contained in the union of the cubes $(1 + a/2)Q$ over all $Q \in CZ(\mathcal{A}^-)$ with $Q \subset (1 + t_G)\widehat{Q}$.*
- *If $Q \in CZ(\mathcal{A}^-)$ and $Q \subset (1 + 100t_G)\widehat{Q}$, then $\frac{65}{64}Q \subset \frac{65}{64}\widehat{Q}$.*

Proof. We assume a_{new} is less than a small enough constant determined by t_G, m, n, and p. We will later fix a_{new} to be a constant depending only on t_G, m, n, and p, but not yet.

Let $x \in (1 + a_{new})\widehat{Q}$ be given. We will produce a cube $Q \in CZ(\mathcal{A}^-)$ with $Q \subset (1 + t_G)\widehat{Q}$ such that $x \in (1 + a/2)Q$, thus proving the first bullet point.

Pick a point $x_{near} \in \widehat{Q}$ with $|x_{near} - x| \le a_{new}\delta_{\widehat{Q}}$. (Recall that we use the ℓ^∞ metric on $\mathbb{R}^n$.)

Since the cubes in $CZ(\mathcal{A}^-)$ partition Q°, one of the following cases must occur

Case 1. There exists $Q_1 \in CZ(\mathcal{A}^-)$ with $\delta_{Q_1} \le (t_G/40)\delta_{\widehat{Q}}$ such that $x \in Q_1$.

Because $x \in (1 + a_{new})\widehat{Q}$, we have $Q_1 \subset (1 + a_{new} + t_G/10)\widehat{Q}$ in Case 1. Therefore, $Q_1 \subset (1 + t_G)\widehat{Q}$. (Here, we assume that $a_{new} \le 9t_G/10$.)

Case 2. There exists $Q_2 \in CZ(\mathcal{A}^-)$ with $\delta_{Q_2} > (t_G/40)\delta_{\widehat{Q}}$ and $x \in Q_2$.

Because $\widehat{Q}$ is a testing cube, there exists $Q \in CZ(\mathcal{A}^-)$ such that $Q \subset \widehat{Q}$ and $x_{near} \in Q$. Moreover, note that

$$|x - x_{near}| \le a_{new}\delta_{\widehat{Q}} \le \frac{40a_{new}}{t_G}\delta_{Q_2} \le \frac{a}{8}\delta_{Q_2}.$$

(Here, we assume that $a_{new} \le at_G/320$.) The above estimate and the fact that $x \in Q_2$ imply that $x_{near} \in (1 + a)Q_2$. Since $x_{near} \in Q$, we have $\delta_{Q_2} \le 2\delta_Q$ by good geometry. Therefore, $|x - x_{near}| \le \frac{a}{4}\delta_Q$. Consequently, since $x_{near} \in Q$ we have $x \in (1 + a/2)Q$.

Case 3. $x \in \mathbb{R}^n \setminus Q^\circ$.

Because $\widehat{Q}$ is a testing cube, there exists $Q \in CZ(\mathcal{A}^-)$ with $Q \subset \widehat{Q}$ and $x_{near} \in Q$. Note that

$$\operatorname{dist}(Q, \mathbb{R}^n \setminus Q^\circ) \le |x - x_{near}| \le a_{new}\,\delta_{\widehat{Q}} \le a_{new}.$$

If $a_{\text{new}} < 10^{-3}$, then Lemma 7 implies that $\delta_Q \in \{1/2, 1/4, 1/8\}$, hence $|x - x_{\text{near}}| \le 8a_{\text{new}}\delta_Q$. Since $x_{\text{near}} \in Q$, we see that $x \in (1 + 100a_{\text{new}})Q \subset (1 + a/2)Q$.

Thus, in all cases we have produced some cube $Q' \in \mathrm{CZ}(\mathcal{A}^-)$ such that $Q' \subset (1 + t_G)\widehat{Q}$ and $x \in (1 + a/2)Q'$. Here, $x \in (1 + a_{\text{new}})\widehat{Q}$ is arbitrary. We now fix a_{new} to be a small enough constant depending on t_G, m, n, and p. This completes the proof of the first bullet point.

We now prove the second bullet point.

We assume we are given a cube $Q \in \mathrm{CZ}(\mathcal{A}^-)$ with $Q \subset (1 + 100t_G)\widehat{Q}$. Since $\widehat{Q}$ is a testing cube, either $Q \subset \widehat{Q}$ or $Q \subset (1 + 100t_G)\widehat{Q} \setminus \widehat{Q}$.

In the former case, clearly $\frac{65}{64}Q \subset \frac{65}{64}\widehat{Q}$.

In the latter case, we have $\delta_Q \le 50t_G\delta_{\widehat{Q}}$ and so $\frac{65}{64}Q \subset (1 + 1000t_G)\widehat{Q} \subset \frac{65}{64}\widehat{Q}$.

This proves the second bullet point and completes the proof of the lemma. □

1.4.4. Testing functionals. We recall that we have computed linear maps $R^{\#}_{Q^{\#}}$ associated to the keystone cubes $Q^{\#}$ in $\mathrm{CZ}(\mathcal{A}^-)$. See the algorithm MAKE NEW ASSISTS AND ASSIGN KEYSTONE JETS in Section 1.4.1.

We assume we are given a parameter t_G as in (1.62). We assume we are given a testing cube $\widehat{Q} \subset Q^\circ$. (See Definition 2.)

For each $Q \in \mathrm{CZ}(\mathcal{A}^-)$ with $Q \subseteq (1 + 100t_G)\widehat{Q}$, we define

$$R^{\widehat{Q}}_Q(f, P) := \begin{cases} P, & \delta_Q \ge t_G\delta_{\widehat{Q}}, \\ R^{\#}_{\mathcal{K}(Q)}(f, P), & \delta_Q < t_G\delta_{\widehat{Q}}, \end{cases} \tag{1.63}$$

for $(f, P) \in \mathbb{X}((65/64)\widehat{Q} \cap E) \oplus \mathcal{P})$. We guarantee that $S_1\mathcal{K}(Q) \subset CQ$ as in the KEYSTONE-ORACLE in Section 1.3.3. If $\delta_Q < t_G\delta_{\widehat{Q}}$, then $CQ \subset (1 + Ct_G)\widehat{Q}$. For small enough t_G, we conclude that

$$S_1\mathcal{K}(Q) \subset (65/64)\widehat{Q}. \tag{1.64}$$

This shows that the map $R^{\widehat{Q}}_Q$ is well-defined.

We define the "testing functional" $[M_{\widehat{Q}}(f, P)]^p$ to be the sum of the following terms.

$$\textbf{(I)} = \text{the sum of } \big[M_{(Q,\mathcal{A}^-)}(f, R^{\widehat{Q}}_Q(f, P))\big]^p = \sum_{\xi \in \Xi(Q,\mathcal{A}^-)} \big|\xi\big(f, R^{\widehat{Q}}_Q(f, P)\big)\big|^p \tag{1.65}$$

$$\text{over all } Q \in \mathrm{CZ}_{\text{main}}(\mathcal{A}^-) \text{ such that } Q \subset (1 + t_G)\widehat{Q}.$$

$$\textbf{(II)} = \text{the sum of } \sum_{\beta \in \mathcal{M}} \delta_{Q'}^{n-(m-|\beta|)p} \big|\partial^\beta \big[R^{\widehat{Q}}_{Q'}(f, P) - R^{\widehat{Q}}_{Q''}(f, P)\big](x_{Q'})\big|^p \tag{1.66}$$

$$\text{over all } (Q', Q'') \in \mathrm{BD}(\mathcal{A}^-) \text{ such that } Q' \subset (1+t_G)\widehat{Q},\ \delta_{Q'} < t_G\delta_{\widehat{Q}}.$$

$$\textbf{(III)} = \text{the sum of } \sum_{\beta \in \mathcal{M}} \delta_Q^{n-(m-|\beta|)p} \big|\partial^\beta \big[R^{\widehat{Q}}_Q(f, P) - P\big](x_Q)\big|^p \tag{1.67}$$

$$\text{over all } Q \in \mathrm{CZ}(\mathcal{A}^-) \text{ such that } Q \subset (1 + t_G)\widehat{Q},\ \delta_Q \ge t_G^2\delta_{\widehat{Q}}.$$

and

$$\text{(1.68)}\quad \textbf{(IV)} = \text{the sum of } \sum_{\beta\in\mathcal{M}} \delta_{\widehat{Q}}^{n-(m-|\beta|)p} \left|\partial^\beta \left[R^{\widehat{Q}}_{Q_{sp}}(f,P) - P\right](x_{\widehat{Q}})\right|^p$$
$$\text{for a single (arbitrarily chosen) } Q_{sp} \in \mathrm{CZ}(\mathcal{A}^-) \text{ contained in } \widehat{Q}.$$

(Note that $Q'' \subset (1+100t_G)\widehat{Q}$ in (1.66), thanks to the good geometry of cubes in $\mathrm{CZ}(\mathcal{A}^-)$; hence the sum **(II)** is well-defined.)

Thus we have defined a functional $M_{\widehat{Q}}(f,P)$. Although $M_{\widehat{Q}}(f,P)$ depends on the parameter t_G, we leave this dependence implicit in our notation for the sake of brevity.

For each testing cube $\widehat{Q}$, we define

$$\text{(1.69)}\qquad \overline{\overline{\sigma}}(\widehat{Q}) = \left\{P \in \mathcal{P} : M_{\widehat{Q}}(0,P) \le 1\right\}.$$

ALGORITHM: APPROXIMATE NEW TRACE NORM

Given a number $t_G > 0$ as in (1.62), we perform one-time work at most $C(t_G)N\log N$ in space $C(t_G)N$, after which we can answer queries.

A query consists of a testing cube $\widehat{Q}$.

The response to the query $\widehat{Q}$ is a list $\mu_1^{\widehat{Q}}, \dots, \mu_D^{\widehat{Q}}$ of linear functionals on $\mathcal{P}$ such that

$$\text{(1.70)}\qquad c\left[M_{\widehat{Q}}(0,P)\right]^p \le \sum_{i=1}^{D} |\mu_i^{\widehat{Q}}(P)|^p \le C\left[M_{\widehat{Q}}(0,P)\right]^p.$$

Define a quadratic form on $\mathcal{P}$ by

$$\text{(1.71)}\qquad q_{\widehat{Q}}(P) := \sum_{i=1}^{D} |\mu_i^{\widehat{Q}}(P)|^2.$$

This quadratic form satisfies

$$\text{(1.72)}\qquad c\left[M_{\widehat{Q}}(0,P)\right]^2 \le q_{\widehat{Q}}(P) \le C\left[M_{\widehat{Q}}(0,P)\right]^2.$$

In particular,

$$\text{(1.73)}\qquad \{q_{\widehat{Q}} \le c\} \subset \overline{\overline{\sigma}}(\widehat{Q}) \subset \{q_{\widehat{Q}} \le C\}.$$

The work required to answer a query is at most $C(t_G)\log N$.

Explanation. For each keystone cube $Q^\# \in \mathrm{CZ}(\mathcal{A}^-)$ and each $\beta \in \mathcal{M}$, we have stored a short form description of the $\Omega^{\mathrm{new}}(Q^\#)$-assisted bounded depth linear functional $(f,P) \mapsto \partial^\beta\left[R^\#_{Q^\#}(f,P)\right](0)$. This corresponds to an expansion

$$\partial^\beta\left[R^\#_{Q^\#}(f,P)\right](0) = \lambda_{(Q^\#,\beta)}(f) + \overline{\lambda}_{(Q^\#,\beta)}(P);$$

here, $\lambda_{(Q^\#,\beta)}(f)$ and $\overline{\lambda}_{(Q^\#,\beta)}(P)$ are linear functionals (with $\lambda_{(Q^\#,\beta)}$ given in short form in terms of some set of assists). We mark each keystone cube $Q^\#$ with the linear map

$$P \mapsto R^\#_{Q^\#}(0,P) = \sum_{\beta\in\mathcal{M}} \overline{\lambda}_{(Q^\#,\beta)}(P)\cdot\frac{1}{\beta!}x^\beta. \tag{1.74}$$

This requires work and storage at most C for each $Q^\#$. (We simply produce the functionals $\overline{\lambda}_{(Q^\#,\beta)} : \mathcal{P}\to\mathbb{R}$ for all $\beta\in\mathcal{M}$.) The number of keystone cubes is at most CN, hence this computation requires total work at most CN.

We now perform the marking procedure described below.

- For each cube $Q\in CZ_{\text{main}}(\mathcal{A}^-)$, we mark Q with the linear functionals

$$\xi_{(Q,i)}(P) := \xi_i^Q\big(R^\#_{\mathcal{K}(Q)}(0,P)\big)\quad (i=1,\dots,D).$$

  To compute these functionals we simply compose linear maps that were already computed. The functionals ξ_i^Q on $\mathcal{P}$ satisfy (1.8), and are computed using the algorithm APPROXIMATE OLD TRACE NORM. We produce the keystone cube $\mathcal{K}(Q)$ using the KEYSTONE-ORACLE. We locate the map $P\mapsto R^\#_{\mathcal{K}(Q)}(0,P)$ using a binary search.

  This requires work at most $C\log N$ for each given $Q\in CZ_{\text{main}}(\mathcal{A}^-)$. (The binary search requires work at most $C\log N$.)

- For each border-dispute pair $(Q',Q'')\in BD(\mathcal{A}^-)$, and $\beta\in\mathcal{M}$, we mark Q' with linear functionals

$$\xi_{(Q',Q'',\beta)}(P) := \delta_{Q'}^{n/p-m+|\beta|}\partial^\beta\left\{R^\#_{\mathcal{K}(Q')}(0,P) - R^\#_{\mathcal{K}(Q'')}(0,P)\right\}(x_{Q'}).$$

  The linear maps $P\mapsto R^\#_{\mathcal{K}(Q')}(0,P)$ and $P\mapsto R^\#_{\mathcal{K}(Q'')}(0,P)$ are computed using the KEYSTONE-ORACLE and a binary search, as in the previous bullet point.

  This requires work at most $C\log N$ for each given $(Q',Q'')\in BD(\mathcal{A}^-)$.

Each relevant cube is marked with at most $\mathcal{O}(1)$ functionals by the above bullet points. Since the number of cubes Q and Q' arising above is at most CN, the marking procedure requires work at most $CN\log N$ in space CN.

We perform the one-time work for the algorithm COMPUTE NORMS FROM MARKED CUBOIDS on the marked cubes Q, Q' arising above, which is at most $CN\log N$ work in space CN (see Section 4.1.5 of [1]). Again, we use the fact that each cube is marked by $\mathcal{O}(1)$ functionals. This concludes the one-time work for the present algorithm.

We now explain the query work. Suppose that $\widehat{Q}$ is a given testing cube (a query).

We partition $(1+t_G)\widehat{Q}$ into dyadic cubes $Q_1,\dots,Q_L\subset\mathbb{R}^n$ such that $\delta_{Q_\ell} = (t_G/4)\delta_{\widehat{Q}}$. Note that $L = L(t_G)$ is a constant determined by n and t_G. (Recall that $0<t_G<1$ is an integer power of 2; see (1.62).)

Note that

$$\begin{aligned}(1.75)\qquad &Q \in \mathrm{CZ}(\mathcal{A}^-),\ Q \subset (1+t_G)\widehat{Q},\ \text{and } \delta_Q \le (t_G/4)\delta_{\widehat{Q}} \iff\\ &Q \in \mathrm{CZ}(\mathcal{A}^-) \text{ and } Q \subset Q_\ell \quad \text{for some } \ell \in \{1,\dots,L\}.\end{aligned}$$

Next, we apply the query algorithm from COMPUTE NORMS FROM MARKED CUBOIDS (see Section 4.1.5 in [1]) with each cube Q_ℓ used as a query ($\ell = 1,\dots,L$). We obtain linear functionals $\mu_1^{Q_\ell},\dots,\mu_D^{Q_\ell}$ on $\mathcal{P}$ such that

$$(1.76)\qquad c\sum_{k=1}^{D}|\mu_k^{Q_\ell}(P)|^p \le \sum_{\substack{Q\in \mathrm{CZ}(\mathcal{A}^-),\ \xi\in\mathcal{P}^*\\ Q\subset Q_\ell\\ Q \text{ marked with } \xi}} |\xi(P)|^p \le C\sum_{k=1}^{D}|\mu_k^{Q_\ell}(P)|^p.$$

This requires work and storage at most $C\log N$ for each fixed ℓ, and total work and storage at most $C(t_G)\log N$. Summing the above estimate from $\ell = 1,\dots,L$ and using (1.75), we learn that

$$\begin{aligned}&\sum_{\ell=1}^{L}\sum_{k=1}^{D}|\mu_k^{Q_\ell}(P)|^p\\ &\quad\sim \sum\Big\{\sum_{i=1}^{D}|\xi_i^Q(R^{\#}_{\mathcal{K}(Q)}(0,P))|^p : Q\in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A}^-),\ Q\subset(1+t_G)\widehat{Q},\\ &\qquad\qquad \delta_Q \le \frac{t_G}{4}\delta_{\widehat{Q}}\Big\}.\\ &\qquad + \sum\big\{|\xi_{(Q',Q'',\beta)}(P)|^p :\ (Q',Q'')\in \mathrm{BD}(\mathcal{A}^-),\ Q'\subset(1+t_G)\widehat{Q},\\ &\qquad\qquad \delta_{Q'} \le \frac{t_G}{4}\delta_{\widehat{Q}},\ \beta\in\mathcal{M}\big\}\\ &\quad =: \mathfrak{S}_1+\mathfrak{S}_2.\end{aligned}\tag{1.77}$$

We now compute the functionals described below.

($\mathbf{F_1}$) $\boxed{\mu_k^{Q_\ell}(P)}$ for $1\le k\le D$, $1\le \ell\le L$.

($\mathbf{F_2}$) $\boxed{\xi_i^Q(R^{\#}_{\mathcal{K}(Q)}(0,P))}$ for $1\le i\le D$, $Q\in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A}^-)$, $Q\subset(1+t_G)\widehat{Q}$, $\delta_Q = \frac{t_G}{2}\delta_{\widehat{Q}}$.

($\mathbf{F_3}$) $\boxed{\xi_i^Q(P)}$ for $1\le i\le D$, $Q\in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A}^-)$, $Q\subset(1+t_G)\widehat{Q}$, $\delta_Q \ge t_G\delta_{\widehat{Q}}$.

($\mathbf{F_4}$) $\boxed{\xi_{(Q',Q'',\beta)}(P)}$ for $\beta\in\mathcal{M}$, $(Q',Q'')\in \mathrm{BD}(\mathcal{A}^-)$, $\delta_{Q'} = \frac{t_G}{2}\delta_{\widehat{Q}}$, $\delta_{Q''} \le \frac{t_G}{2}\delta_{\widehat{Q}}$.

$$(\mathbf{F_5}) \quad \boxed{\delta_Q^{n/p-m+|\beta|}\left\{\partial^\beta(R_Q^{\widehat{Q}}(0,P)-P)(x_Q)\right\}} \qquad \text{for } \beta\in\mathcal{M},\ Q\in CZ(\mathcal{A}^-),\ Q\subset(1+t_G)\widehat{Q},\ \delta_Q\geq t_G^2\delta_{\widehat{Q}}.$$

$$(\mathbf{F_6}) \quad \boxed{\delta_{\widehat{Q}}^{n/p-m+|\beta|}\left\{\partial^\beta(R_{Q_{sp}}^{\widehat{Q}}(0,P)-P)(x_{\widehat{Q}})\right\}} \qquad \text{for } \beta\in\mathcal{M}.$$

The number of functionals listed here is at most $C(t_G)$. To compute these functionals, we proceed as follows.

We have already produced the functionals in $(\mathbf{F_1})$ that satisfy (1.77).

We can compute the functionals arising in $(\mathbf{F_6})$. If $\delta_{Q_{sp}}\geq t_G\delta_{\widehat{Q}}$ then the functionals in $(\mathbf{F_6})$ vanish identically. If instead $\delta_{Q_{sp}}<t_G\delta_{\widehat{Q}}$ then the map $R_{Q_{sp}}^{\widehat{Q}}(0,P)=R^{\#}_{\mathcal{K}(Q_{sp})}(0,P)$ has been computed, and we easily produce the expression in $(\mathbf{F_6})$.

Next, we loop over all dyadic cubes $Q\subset(1+t_G)\widehat{Q}$ with $\delta_Q\geq t_G^2\delta_{\widehat{Q}}$. For each such Q, we do the following.

If $\delta_Q=\frac{t_G}{2}\delta_{\widehat{Q}}$ and $Q\in CZ_{main}(\mathcal{A}^-)$ then we compute the functional in $(\mathbf{F_2})$.

If $\delta_Q\geq t_G\delta_{\widehat{Q}}$ and $Q\in CZ_{main}(\mathcal{A}^-)$ then we compute the functional in $(\mathbf{F_3})$.

If $Q\in CZ(\mathcal{A}^-)$ then we can compute the functionals in $(\mathbf{F_5})$. These functionals are identically zero whenever $\delta_Q\geq t_G\delta_{\widehat{Q}}$. Otherwise, since we have already computed the map $R_Q^{\widehat{Q}}(0,P)=R^{\#}_{\mathcal{K}(Q)}(0,P)$, we can easily compute the expression in $(\mathbf{F_5})$. That concludes the loop over Q.

Finally, we loop over the dyadic cubes $Q'\subset(1+t_G)\widehat{Q}$ with $\delta_{Q'}=\frac{t_G}{2}\delta_{\widehat{Q}}$. If $Q'\in CZ(\mathcal{A}^-)$, then we loop over $Q''\in CZ(\mathcal{A}^-)$ such that $Q''\leftrightarrow Q'$. If $\delta_{Q''}\leq(t_G/2)\delta_{\widehat{Q}}$ and $\mathcal{K}(Q'')\neq\mathcal{K}(Q')$ then we compute the functionals arising in $(\mathbf{F_4})$. That concludes the loop over Q'.

Thus we have computed all the functionals arising in $(\mathbf{F_1})$-$(\mathbf{F_6})$. We define the functional $\big[X(P)\big]^p$ to be the sum of the p-th powers of all these functionals.

We will now show that $\big[X(P)\big]^p$ well approximates $\big[M_{\widehat{Q}}(0,P)\big]^p$.

The sum of the p-th powers of the functionals arising in $(\mathbf{F_1})$ is estimated in (1.77). We obtain from this the estimate

$$\begin{aligned}\big[X(P)\big]^p \sim &\sum\Big\{\sum_{i=1}^{D}|\xi_i^Q(R_Q^{\widehat{Q}}(0,P))|^p : Q\in CZ_{main}(\mathcal{A}^-),\ Q\subset(1+t_G)\widehat{Q}\Big\}\\ &+\sum\Big\{|\xi_{(Q',Q'',\beta)}(P)|^p :\ (Q',Q'')\in BD(\mathcal{A}^-),\ Q'\subset(1+t_G)\widehat{Q},\\ &\qquad\qquad \delta_{Q'}\leq\frac{t_G}{2}\delta_{\widehat{Q}},\ \delta_{Q''}\leq\frac{t_G}{2}\delta_{\widehat{Q}},\ \beta\in\mathcal{M}\Big\}\\ &+\mathfrak{S}_3+\mathfrak{S}_4. \qquad (1.78)\end{aligned}$$

Here, $\mathfrak{S}_3$ and $\mathfrak{S}_4$ are the terms **(III)** and **(IV)**, respectively, with f set to 0 (see (1.67) and (1.68)). Let us explain how we obtained this formula. The sum of the term $\mathfrak{S}_1$ in (1.77) and the sum of the p-th powers of all the functionals

in $(\mathbf{F_2})$ and $(\mathbf{F_3})$ is equal to the first line in (1.78). (Recall the definition of $R_Q^{\widehat{Q}}$ in (1.63).) The sum of the term $\mathfrak{S}_2$ in (1.77) and the sum of the p-th powers of all the functionals in $(\mathbf{F_4})$ is equal to the second line in (1.78). The sum of the p-th powers of all the functionals in $(\mathbf{F_5})$ and $(\mathbf{F_6})$ is equal to the third line in (1.78), i.e., the quantity $\mathfrak{S}_3 + \mathfrak{S}_4$.

The sum in the first line in (1.78) is comparable to the term **(I)** with $f \equiv 0$ (see (1.65)), thanks to the estimate (1.8). Note that the sum in the second line in (1.78) is equal to the term **(II)** with $f \equiv 0$ (see (1.66)) minus all the summands in **(II)** with $\delta_{Q'} = (t_G/2)\delta_{\widehat{Q}}$ and $\delta_{Q''} = t_G\delta_{\widehat{Q}}$. (Recall that by good geometry the sidelengths of Q' and Q'' can differ by at most a factor of 2.) However, these discarded summands appear also in the term **(III)**. Thus, $[X(P)]^p$ is comparable to the sum of the terms **(I)**, **(II)**, **(III)**, **(IV)** (with $f \equiv 0$). Thus, in summary, we have

$$c \cdot \left[M_{\widehat{Q}}(0,P)\right]^p \le \left[X(P)\right]^p \le C \cdot \left[M_{\widehat{Q}}(0,P)\right]^p$$

for universal constants $c > 0$ and $C \ge 1$.

Processing the functionals in $(\mathbf{F_1})$–$(\mathbf{F_6})$ using the algorithm COMPRESS NORMS (see Section 2.8 of [1]), we compute functionals $\mu_1^{\widehat{Q}}, \dots, \mu_D^{\widehat{Q}}$ on $\mathcal{P}$ such that

$$c \cdot \sum_{i=1}^{D} |\mu_i^{\widehat{Q}}(P)|^p \le [X(P)]^p \le C \cdot \sum_{i=1}^{D} |\mu_i^{\widehat{Q}}(P)|^p.$$

The previous two estimates imply the desired estimate (1.70).

The estimate (1.72), concerning the quadratic form $q_{\widehat{Q}}(P)$ defined in (1.71), follows because the ℓ_p and ℓ_2 norms on the space $\mathbb{R}^D$ are comparable up to a constant factor depending on D, which is, in turn, a universal constant. (Recall that $D = \dim(\mathcal{P})$ depends only on m and n.) The pair of inclusions in (1.73) follows directly from (1.72) and the definition of $\overline{\overline{\sigma}}(\widehat{Q})$ in (1.69).

This completes the description of the query work, which consists of at most $C(t_G)\log N$ computer operations.

This completes the explanation of the algorithm APPROXIMATE NEW TRACE NORM. □

1.4.5. Computing data associated to a testing cube. Let $\widehat{Q}$ be a testing cube (see Definition 2), and let $t_G > 0$ be as in (1.62).

The *supporting data for* $\widehat{Q}$ consists of the following:

(SD1) Pointers to the cubes $Q \in CZ_{\text{main}}(\mathcal{A}^-)$ with $Q \subset (1 + t_G)\widehat{Q}$.
(These are the cubes appearing in the sum **(I)**; see (1.65).)

(SD2) Pointers to the pairs $(Q', Q'') \in BD(\mathcal{A}^-)$ with $Q' \subset (1 + t_G)\widehat{Q}$ and $\delta_{Q'} < t_G\delta_{\widehat{Q}}$.
(These are the pairs of cubes appearing in the sum **(II)**; see (1.66).)

(SD3) Pointers to the cubes $Q \in CZ(\mathcal{A}^-)$ with $Q \subset (1 + t_G)\widehat{Q}$ and $\delta_Q \ge t_G^2\delta_{\widehat{Q}}$.
(These are the cubes appearing in the sum **(III)**; see (1.67).)

(SD4) A pointer to a cube $Q_{sp} \in CZ(\mathcal{A}^-)$ with $Q_{sp} \subset \widehat{Q}$.
(This cube appears in **(IV)**; see (1.68).)

(SD5) Pointers to the keystone cubes $Q^\#$ of $CZ(\mathcal{A}^-)$ with $S_1 Q^\# \subset (65/64)\widehat{Q}$.
(See (1.17) for the definition of S_1.)

We are given markings as in Section 1.4.2. Each cube $Q \in CZ_{main}(\mathcal{A}^-)$ is marked with pointers to the lists $\Omega(Q, \mathcal{A}^-)$ and $\Xi(Q, \mathcal{A}^-)$, and each keystone cube $Q^\# \in CZ(\mathcal{A}^-)$ is marked with a pointer to the list $\Omega^{new}(Q^\#)$. We define

$$\Omega(\widehat{Q}) := \Big[\bigcup \Big\{ \Omega(Q, \mathcal{A}^-) : Q \in CZ_{main}(\mathcal{A}^-), Q \subset (1 + t_G)\widehat{Q} \Big\} \Big]$$
$$\bigcup \Big[\bigcup \Big\{ \Omega^{new}(Q^\#) : Q^\# \in CZ(\mathcal{A}^-) \text{ keystone}, \ S_1 Q^\# \subset (65/64)\widehat{Q} \Big\} \Big]. \tag{1.79}$$

Using the supporting data for $\widehat{Q}$ and the above markings, we produce a list of all the functionals in $\Omega(\widehat{Q})$. To form the list (1.79), we examine all the relevant Q and $Q^\#$, and we copy each assist functional ω from $\Omega(Q, \mathcal{A}^-)$ or $\Omega^{new}(Q^\#)$ into a location in memory. The work and space required are bounded by the sum of the depths of all the ω that are copied. We make no attempt to remove duplicates in the list (1.79). See Section 2.1 in [1] for more details about our notation for unions of lists. We summarize the procedure in the following algorithm.

ALGORITHM: COMPUTE NEW ASSISTS

Given a testing cube $\widehat{Q}$, and given the supporting data for $\widehat{Q}$, we compute a list of all the functionals in $\Omega(\widehat{Q})$. We mark all the functionals that appear in the lists $\Omega(Q, \mathcal{A}^-)$ (for $Q \in CZ_{main}(\mathcal{A}^-)$, $Q \subset (1 + t_G)\widehat{Q}$ in the supporting data) and $\Omega^{new}(Q^\#)$ (for $Q^\#$ keystone, $S_1 Q^\# \subset (65/64)\widehat{Q}$ in the supporting data) with pointers to their position in the list $\Omega(\widehat{Q})$. This requires work at most

$$\mathfrak{W}_1(\widehat{Q}) = C \log N \cdot \Big[1 + \sum_{\substack{Q \in CZ_{main}(\mathcal{A}^-) \\ Q \subset (1+t_G)\widehat{Q}}} \sum_{\omega \in \Omega(Q, \mathcal{A}^-)} \text{depth}(\omega) + \sum_{\substack{\text{keystone } Q^\# \in CZ(\mathcal{A}^-) \\ S_1 Q^\# \subset \frac{65}{64}\widehat{Q}}} \sum_{\omega \in \Omega^{new}(Q^\#)} \text{depth}(\omega) \Big] \tag{1.80}$$

and storage at most

$$\mathfrak{S}_1(\widehat{Q}) = C \cdot \Big[1 + \sum_{\substack{Q \in CZ_{main}(\mathcal{A}^-) \\ Q \subset (1+t_G)\widehat{Q}}} \sum_{\omega \in \Omega(Q, \mathcal{A}^-)} \text{depth}(\omega) + \sum_{\substack{\text{keystone } Q^\# \in CZ(\mathcal{A}^-) \\ S_1 Q^\# \subset \frac{65}{64}\widehat{Q}}} \sum_{\omega \in \Omega^{new}(Q^\#)} \text{depth}(\omega) \Big]. \tag{1.81}$$

Remark 4. Let $\widehat{Q}$ be a testing cube, and let ξ be a linear functional that has $\Omega(Q, \mathcal{A}^-)$-assisted bounded depth for $Q \in CZ_{\text{main}}(\mathcal{A}^-)$, $Q \subset (1 + t_G)\widehat{Q}$ relevant to the supporting data for $\widehat{Q}$. Then ξ has $\Omega(\widehat{Q})$-assisted bounded depth, since $\Omega(Q, \mathcal{A}^-)$ is a sublist of $\Omega(\widehat{Q})$. If ξ is given in short form in terms of the assists $\Omega(Q, \mathcal{A}^-)$, then we can convert ξ into a short form in terms of the assists $\Omega(\widehat{Q})$. That is because we have marked each functional in $\Omega(Q, \mathcal{A}^-)$ with a pointer to its position in the list $\Omega(\widehat{Q})$. The conversion requires a constant amount of work once we have carried out the algorithm COMPUTE NEW ASSISTS for the given $\widehat{Q}$.

Similarly, let ξ be a linear functional that has $\Omega^{\text{new}}(Q^\#)$-assisted bounded depth, for some $Q^\#$ relevant to the supporting data for $\widehat{Q}$. Given a short form description of ξ in terms of the assists $\Omega^{\text{new}}(Q^\#)$, we can express ξ in short form in terms of the assists $\Omega(\widehat{Q})$ using a constant amount of work.

ALGORITHM: COMPUTE SUPPORTING MAP

We perform one-time work at most $CN \log N$ in space CN, after which we can answer queries as follows.

A query consists of a testing cube $\widehat{Q}$, its supporting data, and a cube $Q \in CZ(\mathcal{A}^-)$ with $Q \subset (1 + 100t_G)\widehat{Q}$.

The response to a query $(\widehat{Q}, Q)$ is a short form description of the linear map $R^{\widehat{Q}}_Q : \mathbb{X}((65/64)\widehat{Q} \cap E) \oplus \mathcal{P} \to \mathcal{P}$ in terms of the assists $\Omega(\widehat{Q})$ (see (1.63)).

The work and storage required to answer a query are at most $C \log N$.

(Here, we do not count the storage used to hold the supporting data for $\widehat{Q}$.)

Explanation. We simply use the definition in (1.63).

We first test to see whether $\delta_Q < t_G\delta_{\widehat{Q}}$ or $\delta_Q \geq t_G\delta_{\widehat{Q}}$.

In the first case when $\delta_Q \geq t_G\delta_{\widehat{Q}}$, we have $R^{\widehat{Q}}_Q(f, P) = P$, and we produce a short-form description of this map.

In the second case when $\delta_Q < t_G\delta_{\widehat{Q}}$, we compute the map $R^{\widehat{Q}}_Q = R^\#_{\mathcal{K}(Q)}$ as follows.

First, we compute the keystone cube $Q^\# = \mathcal{K}(Q)$ using the KEYSTONE-ORACLE. Recall that $S_1 Q^\# \subset \frac{65}{64}\widehat{Q}$ (see (1.64)). We locate $Q^\#$ in the list of pointers appearing in (SD5) using a binary search. We have already computed the $\Omega^{\text{new}}(Q^\#)$-assisted bounded depth linear map

$$R^\#_{Q^\#} : \mathbb{X}((65/64)\widehat{Q} \cap E) \oplus \mathcal{P} \to \mathcal{P}$$

in short form in terms of the assists $\Omega^{\text{new}}(Q^\#)$, as described in Section 1.4.2. Thanks to Remark 4 we can express $R^\#_{Q^\#}$ in short form in terms of the assists $\Omega(\widehat{Q})$.

Thus we have computed the desired expression for $R^{\widehat{Q}}_Q = R^\#_{Q^\#}$ in the second case.

That concludes the explanation of the algorithm. □

ALGORITHM: COMPUTE NEW ASSISTED FUNCTIONALS

Given a testing cube $\widehat{Q}$ and its supporting data, we produce a list $\Xi(\widehat{Q})$ consisting of $\Omega(\widehat{Q})$-assisted bounded depth functionals on $\mathbb{X}(\frac{65}{64}\widehat{Q} \cap E) \oplus \mathcal{P}$, with each functional written in short form, such that the following hold.

- $\big[M_{\widehat{Q}}(f,P)\big]^p = \sum_{\xi\in\Xi(\widehat{Q})} |\xi(f,P)|^p$ for each $(f,P)\in \mathbb{X}(\frac{65}{64}\widehat{Q}\cap E)\oplus\mathcal{P}$.
- Denote

(1.82) $$\mathfrak{N}(\widehat{Q}) := \#\big\{(Q',Q'')\in \mathrm{BD}(\mathcal{A}^-):\ Q'\subset (1+t_G)\widehat{Q},\ \delta_{Q'} < t_G\delta_{\widehat{Q}}\big\}.$$

We carry out the preceding computation using work at most

(1.83) $$\mathfrak{W}_2(\widehat{Q}) := C(t_G)\cdot\log N\cdot\Big[1+\mathfrak{N}(\widehat{Q}) + \sum_{\substack{Q\in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A}^-)\\ Q\subset(1+t_G)\widehat{Q}}} \#\big[\Xi(Q,\mathcal{A}^-)\big]\Big]$$

in space

(1.84) $$\mathfrak{S}_2(\widehat{Q}) := C(t_G)\cdot\Big[1+\mathfrak{N}(\widehat{Q}) + \sum_{\substack{Q\in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A}^-)\\ Q\subset(1+t_G)\widehat{Q}}} \#\big[\Xi(Q,\mathcal{A}^-)\big]\Big].$$

In particular, $\#\Xi(\widehat{Q})\le \mathfrak{S}_2(\widehat{Q})$.

(Again, we do not count the space used to hold the supporting data for $\widehat{Q}$.)

Explanation. We compute the list $\Xi(\widehat{Q})$ of all the functionals appearing in the sums **(I)**–**(IV)** in (1.65)–(1.68).

We loop over all the cubes $Q\in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A}^-)$ with $Q\subset(1+t_G)\widehat{Q}$ (as in (SD1)).

We form the functionals

(1.85) $$(f,P)\mapsto \xi(f,R_Q^{\widehat{Q}}(f,P)) \quad (\text{for } \xi\in\Xi(Q,\mathcal{A}^-)).$$

The linear maps $R_Q^{\widehat{Q}}$ are written in short form in terms of the assists $\Omega(\widehat{Q})$ (see the algorithm COMPUTE SUPPORTING MAP). The functionals $\xi\in\Xi(Q,\mathcal{A}^-)$ are written in short form in terms of assists $\Omega(Q,\mathcal{A}^-)$. We can write the functionals $\xi\in\Xi(Q,\mathcal{A}^-)$ in short form in terms of the assists $\Omega(\widehat{Q})$ (see Remark 4). Hence, we can express each functional in (1.85) in short form in terms of assists $\Omega(\widehat{Q})$. This requires work at most C for each ξ.

That concludes the loop on Q.

We now loop over all pairs $(Q',Q'')\in \mathrm{BD}(\mathcal{A}^-)$ with $Q'\subset(1+t_G)\widehat{Q}$ and $\delta_{Q'} < t_G\delta_{\widehat{Q}}$ (as in (SD2)). For each such pair, we compute the $\Omega(\widehat{Q})$-assisted bounded depth linear maps $R_{Q'}^{\widehat{Q}}$ and $R_{Q''}^{\widehat{Q}}$ in short form. We form the functionals

(1.86) $$\delta_{Q'}^{n/p-m+|\beta|}\big\{\partial^\beta(R_{Q'}^{\widehat{Q}}(f,P)-R_{Q''}^{\widehat{Q}}(f,P))(x_{Q'})\big\} \quad (\text{for } \beta\in\mathcal{M}).$$

That concludes the loop on (Q',Q'').

We loop over all the cubes $Q\in \mathrm{CZ}(\mathcal{A}^-)$ such that $Q\subset(1+t_G)\widehat{Q}$ and $\delta_Q\ge t_G^2\delta_{\widehat{Q}}$ (as in (SD3)). We form the functionals

(1.87) $$\delta_Q^{n/p-m+|\beta|}\big\{\partial^\beta(R_Q^{\widehat{Q}}(f,P)-P)(x_Q)\big\} \quad (\text{for } \beta\in\mathcal{M}).$$

That concludes the loop on Q.

We form the functionals

$$\delta_{\widehat{Q}}^{n/p-m+|\beta|}\big\{\partial^\beta(R^{\widehat{Q}}_{Q_{sp}}(f,P)-P)(x_{\widehat{Q}})\big\} \quad (\text{for } \beta\in\mathcal{M}). \tag{1.88}$$

Here, we use the cube Q_{sp} in (SD4).

Let $\Xi(\widehat{Q})$ denote the list of functionals arising in (1.85)–(1.88). All these functionals have $\Omega(\widehat{Q})$-assisted bounded depth and are expressed in short form in terms of assists $\Omega(\widehat{Q})$. Comparing with (1.65)–(1.68), we see that $[M_{\widehat{Q}}(f,P)]^p$ is equal to the sum of $|\xi(f,P)|^p$ over all $\xi\in\Xi(\widehat{Q})$. Clearly, the number of functionals in $\Xi(\widehat{Q})$ is bounded by

$$C(t_G)\cdot\Big[1+\mathfrak{N}(\widehat{Q})+\sum_{\substack{Q\in CZ_{main}(\mathcal{A}^-)\\ Q\subset(1+t_G)\widehat{Q}}}\#\big[\Xi(Q,\mathcal{A}^-)\big]\Big].$$

Since we perform work at most $C\log N$ (using storage at most C) to compute each functional, the total work and storage used by our algorithm are at most $\mathfrak{W}_2(\widehat{Q})$ and $\mathfrak{S}_2(\widehat{Q})$, respectively. □

The extension operator. Given a testing cube $\widehat{Q}$, the *covering cubes* for $\widehat{Q}$ are

$$\mathcal{I}_{cov}(\widehat{Q}) := \big\{Q\in CZ(\mathcal{A}^-) : Q\subset(1+t_G)\widehat{Q}\big\}. \tag{1.89}$$

We assume that

$$t_G \text{ satisfies the hypothesis of Lemma 11.} \tag{1.90}$$

We do not fix t_G just yet. Let $a_{new}=a_{new}(t_G)$ be as in Lemma 11. Thus,

(1.91) $(1+a_{new})\widehat{Q}$ is contained in the union of $(1+\frac{a}{2})Q$ as Q ranges over $\mathcal{I}_{cov}(\widehat{Q})$.

We will need several results from Sections 4.6.4 and 4.6.5 in [1]. Consider:

- $CZ=CZ(\mathcal{A}^-)$ and $\mathcal{Q}=\mathcal{I}_{cov}(\widehat{Q})$.
- The cube called $\widehat{Q}$ in Sections 4.6.4 and 4.6.5 is taken to be $(1+a_{new})\widehat{Q}$, with $\widehat{Q}$ as in the present section.
- $\overline{r}=a$, and $A=C$ for a large enough universal constant C.

We have already proven the good geometry of $CZ(\mathcal{A}^-)$ and exhibited a $CZ(\mathcal{A}^-)$-ORACLE. This is a part of the main technical results for $\mathcal{A}^-$, which can be found in Section 3 of [1]. This puts us in the setting of Section 4.6.4.

We point out the **covering condition** stated in (1.91), and the **size condition**: $\delta_Q\le C\cdot\delta_{\widehat{Q}}$ for all $Q\in\mathcal{I}_{cov}(\widehat{Q})$ (this follows by definition of $\mathcal{I}_{cov}(\widehat{Q})$). These two conditions are precisely the conditions (4.53) and (4.54) in Section 4.6.5, with the choice of CZ, $\mathcal{Q}$, $\widehat{Q}$, $\overline{r}$, and A as in the above bullet points. This puts us in the setting of Section 4.6.5 of [1].

By Lemma 48 of [1], there exist cutoff functions $\theta_Q^{\widehat{Q}} \in C^m(\mathbb{R}^n)$ such that

$$\sum_{Q \in \mathcal{I}_{cov}(\widehat{Q})} \theta_Q^{\widehat{Q}} = 1 \quad \text{on } (1 + a_{new})\widehat{Q}, \tag{1.92}$$

$$\text{supp}(\theta_Q^{\widehat{Q}}) \subset (1+a)Q \quad \text{and} \quad |\partial^\alpha \theta_Q^{\widehat{Q}}| \leq C \cdot \delta_Q^{-|\alpha|} \quad \text{for } |\alpha| \leq m, \text{ and} \tag{1.93}$$

$$\theta_Q^{\widehat{Q}} = 1 \text{ near } x_Q, \quad \text{and} \quad \theta_Q^{\widehat{Q}} = 0 \text{ near } x_{Q'} \text{ for each } Q' \in \mathcal{I}_{cov}(\widehat{Q}) \setminus \{Q\}. \tag{1.94}$$

Moreover, the cutoff functions may be computed with the following algorithm:

ALGORITHM: COMPUTE POU

After one-time work at most $CN \log N$ in space CN, we can answer queries as follows.

A query consists of a testing cube $\widehat{Q}$ and a point $\underline{x} \in Q^\circ$.

The response to the query $(\widehat{Q}, \underline{x})$ is a list of all the cubes $Q_1, \ldots, Q_L \in \mathcal{I}_{cov}(\widehat{Q})$ (with $Q_1, \ldots, Q_L$ all distinct) such that $\underline{x} \in \frac{65}{64} Q_\ell$, and the list of polynomials $J_{\underline{x}} \theta_{Q_1}^{\widehat{Q}}, \ldots, J_{\underline{x}} \theta_{Q_L}^{\widehat{Q}}$.

To answer a query requires work and storage at most $C \log N$.

Explanation. We list all the cubes $Q \in CZ(\mathcal{A}^-)$ for which $\underline{x} \in \frac{65}{64} Q$ using the $CZ(\mathcal{A}^-)$-ORACLE. We then discard any cubes that are not contained in $(1+t_G)\widehat{Q}$. The remaining cubes give the desired list $Q_1, \ldots, Q_L$.

We now compute the jet $J_{\underline{x}} \theta_{Q_\ell}^{\widehat{Q}}$ for each ℓ. In the proof of Lemma 48 in [1], we defined

$$\theta_{Q_\ell}^{\widehat{Q}} = \widetilde{\theta}_{Q_\ell} \cdot \big[\eta \circ \Psi\big]^{-1}, \quad \text{where } \Psi = \sum_{Q \in \mathcal{I}_{cov}(\widehat{Q})} \widetilde{\theta}_Q.$$

Applying the algorithm COMPUTE CUTOFF FUNCTION from Section 4.6.5 of [1], we compute the jet $J_{\underline{x}} \widetilde{\theta}_{Q_\ell}$ for each $\ell = 1, \ldots, L$. We can compute a formula for $\partial^\alpha J_{\underline{x}}(\theta_{Q_\ell}^{\widehat{Q}})(\underline{x})$ given a formula for the jet $J_{\underline{x}}(\eta \circ \Psi)$. Indeed, by the Leibniz rule, $\partial^\alpha J_{\underline{x}}(\theta_{Q_\ell}^{\widehat{Q}})(\underline{x})$ ($|\alpha| \leq m-1$) is given by a rational function of the derivatives $\partial^\beta J_{\underline{x}}(\widetilde{\theta}_{Q_\ell})(\underline{x})$ and $\partial^\beta J_{\underline{x}}(\eta \circ \Psi)(\underline{x})$ ($|\beta| \leq m-1$).

Since each $\widetilde{\theta}_{Q_\ell}$ is supported on $\frac{65}{64} Q_\ell$, we have

$$J_{\underline{x}} \Psi = \sum_{\ell=1}^{L} J_{\underline{x}} \widetilde{\theta}_{Q_\ell}.$$

In the proof of Lemma 48 in [1], we use a function $\eta : [0, \infty) \to \mathbb{R}$ defined by the conditions: $\eta(t) \geq 1/4$ for all $t \in [0, 1/2)$, and $\eta(t) = t$ for $t \in [1/2, \infty)$. Given $t_* \geq 0$ and $k \leq m$, we may assume that the number $\frac{d^k \eta}{dt^k}(t_*)$ can be computed using work and storage at most C. This is achieved by taking η to be a suitable spline function. Thus, the jet $J_{\underline{x}}(\eta \circ \Psi)$ can be computed using the chain rule.

Thus, we can compute the jets $J_{\underline{x}}(\theta_{Q_\ell}^{\widehat{Q}})$ using work and storage at most C once we know the list $Q_1, \ldots, Q_L$. □

Let $(f, P) \in \mathbb{X}(\frac{65}{64}\widehat{Q} \cap E) \oplus \mathcal{P}$ be given.

For ease of notation we write $R_Q^{\widehat{Q}} = R_Q^{\widehat{Q}}(f, P)$ for the polyomial defined in (1.63) (the dependence on (f, P) should be understood).

For each $Q \in \mathcal{I}_{\mathrm{cov}}(\widehat{Q})$ we define

$$F_Q^{\widehat{Q}} := \begin{cases} T_{(Q,\mathcal{A}^-)}(f, R_Q^{\widehat{Q}}), & \text{if } \frac{65}{64}Q \cap E \neq \emptyset, \\ R_Q^{\widehat{Q}}, & \text{if } \frac{65}{64}Q \cap E = \emptyset. \end{cases} \tag{1.95}$$

Note that the function $F_Q^{\widehat{Q}} \in \mathbb{X}$ is well-defined. According to the main technical results for $\mathcal{A}^-$ (see Section 3 in [1]) we have

$$F_Q^{\widehat{Q}} = f \quad \text{on } (1+a)Q \cap E. \tag{1.96}$$

$$\begin{aligned} \|F_Q^{\widehat{Q}}\|_{\mathbb{X}((1+a)Q)} + \delta_Q^{-m}\|F_Q^{\widehat{Q}} - R_Q^{\widehat{Q}}\|_{L^p((1+a)Q)} \\ \leq \begin{cases} CM_{(Q,\mathcal{A}^-)}(f, R_Q^{\widehat{Q}}), & \text{if } \frac{65}{64}Q \cap E \neq \emptyset, \\ 0, & \text{if } \frac{65}{64}Q \cap E = \emptyset. \end{cases} \end{aligned} \tag{1.97}$$

(Recall that $a = a(\mathcal{A}^-) \leq 1/64$; see (1.7).)

Finally, we define

$$T_{\widehat{Q}}(f, P) := \sum_{Q \in \mathcal{I}_{\mathrm{cov}}(\widehat{Q})} F_Q^{\widehat{Q}} \cdot \theta_Q^{\widehat{Q}} \in \mathbb{X}, \quad \text{with } \theta_Q^{\widehat{Q}} \text{ as in (1.92)–(1.94).} \tag{1.98}$$

ALGORITHM: COMPUTE NEW EXTENSION OPERATOR

We perform one-time work at most $CN \log N$ in space CN, after which we can answer queries.

A query consists of a testing cube $\widehat{Q}$, the supporting data for $\widehat{Q}$, and a point $\underline{x} \in Q^\circ$.

The response to the query $\underline{x}$ is a short form description of the $\Omega(\widehat{Q})$-assisted bounded depth linear map

$$(f, P) \mapsto J_{\underline{x}} T_{\widehat{Q}}(f, P).$$

To answer a query requires work at most $C \log N$.

Explanation. We compute a list of the cubes $Q_1, \ldots, Q_L \in \mathcal{I}_{\mathrm{cov}}(\widehat{Q})$ ($Q_1, \ldots, Q_L$ all distinct) such that $\underline{x} \in \frac{65}{64}Q_\ell$, and a list of the jets $J_{\underline{x}}\theta_{Q_1}^{\widehat{Q}}, \cdots, J_{\underline{x}}\theta_{Q_L}^{\widehat{Q}}$. See the algorithm COMPUTE POU. Recall that $L \leq C$.

Recall that $\mathrm{supp}(\theta_Q^{\widehat{Q}}) \subset \frac{65}{64}Q$. Therefore,

$$J_{\underline{x}} T_{\widehat{Q}}(f, P) = \sum_{\ell=1}^{L} J_{\underline{x}}\theta_{Q_\ell}^{\widehat{Q}} \odot_{\underline{x}} J_{\underline{x}} T_{(Q_\ell,\mathcal{A}^-)}(f, R_{Q_\ell}^{\widehat{Q}}(f, P)). \tag{1.99}$$

For each $\ell = 1, \dots, L$, we compute (see below) the map

$$(f, R) \mapsto J_{\underline{x}} T_{(Q_\ell, \mathcal{A}^-)}(f, R) \qquad ((f, R) \in \mathbb{X}((65/64)\widehat{Q} \cap E) \oplus \mathcal{P}). \tag{1.100}$$

We recall the definition (1.89) of $\mathcal{I}_{\mathrm{cov}}(\widehat{Q})$. Since $Q_\ell \in \mathcal{I}_{\mathrm{cov}}(\widehat{Q})$, we have $Q_\ell \subset (1 + t_G)\widehat{Q}$. Thus, Lemma 11 implies that $\frac{65}{64} Q_\ell \subset \frac{65}{64}\widehat{Q}$, hence the map (1.100) is well defined.

If $\frac{65}{64} Q_\ell \cap E = \emptyset$ then $J_{\underline{x}} T_{(Q_\ell, \mathcal{A}^-)}(f, R) = R$. Otherwise, if $\frac{65}{64} Q_\ell \cap E \neq \emptyset$, then we can compute the map (1.100) in short form in terms of the assists $\Omega(Q, \mathcal{A}^-)$, thanks to the main technical results for $\mathcal{A}^-$. We check whether $\frac{65}{64} Q_\ell \cap E \neq \emptyset$, by checking whether Q_ℓ appears in the list $CZ_{\mathrm{main}}(\mathcal{A}^-)$ using a binary search. We write each of the maps (1.100) in short form in terms of the assists $\Omega(\widehat{Q})$ (see Remark 4).

We compute a short form description of the $\Omega(\widehat{Q})$-assisted bounded depth map

$$R^{\widehat{Q}}_{Q_\ell} : \mathbb{X}\Big(\frac{65}{64}\widehat{Q} \cap E\Big) \oplus \mathcal{P} \to \mathcal{P} \quad \text{for } \ell = 1, \dots, L.$$

We use the algorithm COMPUTE SUPPORTING MAP (see Section 1.4.5).

Substituting $R = R^{\widehat{Q}}_{Q_\ell}(f, P)$ in the formula for each of the maps (1.100), we can express the map $(f, P) \mapsto J_{\underline{x}} T_{\widehat{Q}}(f, P)$ in short form in terms of the assists $\Omega(\widehat{Q})$ using (1.99).

The query work is clearly bounded by $C \log N$. □

1.4.6. The main estimates. We first prove a few properties of the extension operator $T_{\widehat{Q}}$ defined in (1.98).

Let $a_{\mathrm{new}} = a_{\mathrm{new}}(t_G)$ be as in Lemma 11.

Proposition 3. *Let $\widehat{Q}$ be a testing cube, and let $(f, P) \in \mathbb{X}(\frac{65}{64}\widehat{Q} \cap E) \oplus \mathcal{P}$. Then the following properties hold.*

- $T_{\widehat{Q}}(f, P) = f$ *on* $(1 + a_{\mathrm{new}})\widehat{Q} \cap E$.
- $\|T_{\widehat{Q}}(f, P)\|_{\mathbb{X}((1+a_{\mathrm{new}})\widehat{Q})} + \delta_{\widehat{Q}}^{-m} \|T_{\widehat{Q}}(f, P) - P\|_{L^p((1+a_{\mathrm{new}})\widehat{Q})} \leq C \cdot M_{\widehat{Q}}(f, P)$.

Here, the constant $C \geq 1$ depends only on m, n, and p.

Proof. The first bullet point follows from (1.92)–(1.94), (1.96) and (1.98). We now prove the second bullet point.

Recall that we defined the collection of cubes $\mathcal{I}_{\mathrm{cov}}(\widehat{Q})$ in (1.89).

For ease of notation, we set $\overline{a} = a_{\mathrm{new}}$ throughout the proof.

We apply Lemma 49 in [1] to the cube $(1 + \overline{a})\widehat{Q}$, the covering set $\mathcal{I}_{\mathrm{cov}}(\widehat{Q})$, the functions $F^{\widehat{Q}}_Q$, the polynomials $R^{\widehat{Q}}_Q$, and the partition of unity $\theta^{\widehat{Q}}_Q$ (defined for all

$Q \in \mathcal{I}_{\mathrm{cov}}(\widehat{Q})$). Thus, for $G := T_{\widehat{Q}}(f, P)$ we have

$$\|G\|^p_{\mathbb{X}((1+\overline{a})\widehat{Q})} \lesssim \sum_{Q \in \mathcal{I}_{\mathrm{cov}}(\widehat{Q})} \Big[\|F^{\widehat{Q}}_Q\|^p_{\mathbb{X}((1+a)Q)} + \delta_Q^{-mp} \|F^{\widehat{Q}}_Q - R^{\widehat{Q}}_Q\|^p_{L^p((1+a)Q)} \Big] + \sum_{\substack{Q', Q'' \in \mathcal{I}_{\mathrm{cov}}(\widehat{Q}) \\ Q' \leftrightarrow Q''}} \sum_{|\beta| \le m-1} \delta_{Q'}^{(|\beta|-m)p+n} \big|\partial^\beta (R^{\widehat{Q}}_{Q'} - R^{\widehat{Q}}_{Q''})(x_{Q'})\big|^p.$$

From (1.97), this implies that

$$\begin{aligned} \|G\|^p_{\mathbb{X}((1+\overline{a})\widehat{Q})} &\lesssim \sum_{\substack{Q \in \mathcal{I}_{\mathrm{cov}}(\widehat{Q}) \\ \frac{65}{64} Q \cap E \neq \emptyset}} \big[M_{(Q, \mathcal{A}^-)}(f, R^{\widehat{Q}}_Q)\big]^p \\ &\quad + \sum_{\substack{Q', Q'' \in \mathcal{I}_{\mathrm{cov}}(\widehat{Q}) \\ Q' \leftrightarrow Q''}} \sum_{|\beta| \le m-1} \delta_{Q'}^{(|\beta|-m)p+n} \big|\partial^\beta (R^{\widehat{Q}}_{Q'} - R^{\widehat{Q}}_{Q''})(x_{Q'})\big|^p \\ &= A_1(f, P) + A_2(f, P). \end{aligned} \tag{1.101}$$

The expression $A_1(f, P)$ is equal to the sum of the terms in (1.65).

We now analyze the expression $A_2(f, P)$. Suppose that $Q', Q'' \in \mathcal{I}_{\mathrm{cov}}(\widehat{Q})$ and $Q' \leftrightarrow Q''$. Then one of the following cases must occur.

(A) Both $\delta_{Q'}$ and $\delta_{Q''}$ are less than $t_G \cdot \delta_{\widehat{Q}}$, and $\mathcal{K}(Q') = \mathcal{K}(Q'')$;

(B) Both $\delta_{Q'}$ and $\delta_{Q''}$ are less than $t_G \cdot \delta_{\widehat{Q}}$, and $\mathcal{K}(Q') \neq \mathcal{K}(Q'')$;

(C) Both $\delta_{Q'}$ and $\delta_{Q''}$ are at least $t_G \cdot \delta_{\widehat{Q}}$;

(D) Exactly one of $\delta_{Q'}$ and $\delta_{Q''}$ is at least $t_G \cdot \delta_{\widehat{Q}}$.

If (A) or (C) occurs, then $R^{\widehat{Q}}_{Q'} = R^{\widehat{Q}}_{Q''}$, hence the summand in $A_2(f, P)$ vanishes identically; see (1.63).

On the other hand, suppose that (D) occurs. Since Q' and Q'' play symmetric roles in the summand from the second sum in (1.101) (switching Q' and Q'' does not change the order of magnitude of this term) we may assume that $\delta_{Q''} \ge t_G \cdot \delta_{\widehat{Q}}$. Hence, $R^{\widehat{Q}}_{Q''} = P$; see (1.63). Since $Q' \leftrightarrow Q''$, we also have $\delta_{Q'} \ge \frac{1}{2}\delta_{Q''} \ge \frac{t_G}{2}\delta_{\widehat{Q}} \ge t_G^2 \delta_{\widehat{Q}}$ by good geometry.

The previous three paragraphs imply the following estimate:

$$\begin{aligned} A_2(f, P) \lesssim \sum \big\{ &\text{terms in (1.66)} : Q', Q'' \subset (1 + t_G)\widehat{Q},\ \ \delta_{Q'}, \delta_{Q''} < t_G \cdot \delta_{\widehat{Q}}, \\ &\qquad Q' \leftrightarrow Q'',\ \ \mathcal{K}(Q') \neq \mathcal{K}(Q'') \big\} \\ + \sum \big\{ &\text{terms in (1.67)} : Q \subset (1 + t_G)\widehat{Q},\ \delta_Q \ge t_G^2 \cdot \delta_{\widehat{Q}} \big\}. \end{aligned}$$

Thus, we have shown that

$$\|G\|_{\mathbb{X}((1+\overline{a})\widehat{Q})} \le C \cdot M_{\widehat{Q}}(f, P). \tag{1.102}$$

We now estimate $\|G-P\|_{L^p((1+\overline{a})\widehat{Q})}$. Let Q_{sp} be as in (1.68). Observe that

$$\begin{aligned}\delta_{\widehat{Q}}^{-m}\|G-P\|_{L^p((1+\overline{a})\widehat{Q})} &\lesssim \delta_{\widehat{Q}}^{-m}\|G-J_{\chi_{Q_{sp}}}G\|_{L^p((1+\overline{a})\widehat{Q})}\\ &\quad+\delta_{\widehat{Q}}^{-m}\|J_{\chi_{Q_{sp}}}G-R^{\widehat{Q}}_{Q_{sp}}\|_{L^p((1+\overline{a})\widehat{Q})}\\ &\quad+\delta_{\widehat{Q}}^{-m}\|R^{\widehat{Q}}_{Q_{sp}}-P\|_{L^p((1+\overline{a})\widehat{Q})}.\end{aligned}$$

We estimate each of the three terms on the right-hand side above. To estimate the first term, we use the Sobolev inequality to write

$$\delta_{\widehat{Q}}^{-m}\|G-J_{\chi_{Q_{sp}}}G\|_{L^p((1+\overline{a})\widehat{Q})}\le C\|G\|_{\mathbb{X}((1+\overline{a})\widehat{Q})}.$$

To estimate the second term, we will need an elementary estimate about L^p norms of polynomials:

$$Q_2\subset CQ_1,\ \ P\in\mathcal{P}\implies \delta_{Q_1}^{-m}\|P\|_{L^p(Q_1)}\le C'\delta_{Q_2}^{-m}\|P\|_{L^p(Q_2)}. \tag{1.103}$$

This estimate is simply a consequence of Lemma 7 in [1]. Now, since $Q_{sp}\subset\widehat{Q}$ (see (1.68)), and since $J_{\chi_{Q_{sp}}}G=J_{\chi_{Q_{sp}}}F^{\widehat{Q}}_{Q_{sp}}$ (see (1.94)), we can write

$$\begin{aligned}\delta_{\widehat{Q}}^{-m}\|J_{\chi_{Q_{sp}}}G-R^{\widehat{Q}}_{Q_{sp}}\|_{L^p((1+\overline{a})\widehat{Q})} &= \delta_{\widehat{Q}}^{-m}\|J_{\chi_{Q_{sp}}}F^{\widehat{Q}}_{Q_{sp}}-R^{\widehat{Q}}_{Q_{sp}}\|_{L^p((1+\overline{a})\widehat{Q})}\\ &\lesssim \delta_{Q_{sp}}^{-m}\|J_{\chi_{Q_{sp}}}F^{\widehat{Q}}_{Q_{sp}}-R^{\widehat{Q}}_{Q_{sp}}\|_{L^p(Q_{sp})}\quad\text{(by (1.103))}\\ &\lesssim \|F^{\widehat{Q}}_{Q_{sp}}\|_{\mathbb{X}(Q_{sp})}+\delta_{Q_{sp}}^{-m}\|F^{\widehat{Q}}_{Q_{sp}}-R^{\widehat{Q}}_{Q_{sp}}\|_{L^p(Q_{sp})}.\end{aligned}$$

To prove the last inequality above, we have used the Sobolev-type inequality stated in Proposition 9 of [1]. According to (1.97), the previous estimate implies that

$$\delta_{\widehat{Q}}^{-m}\|J_{\chi_{Q_{sp}}}G-R^{\widehat{Q}}_{Q_{sp}}\|_{L^p((1+\overline{a})\widehat{Q})}\le C\cdot M_{\widehat{Q}}(f,P).$$

To estimate the third term, we write

$$\begin{aligned}&\delta_{\widehat{Q}}^{-m}\|R^{\widehat{Q}}_{Q_{sp}}-P\|_{L^p((1+\overline{a})\widehat{Q})}\\ &\quad\sim\sum_{|\beta|\le m-1}\delta_{\widehat{Q}}^{|\beta|+n/p-m}\left|\partial^\beta(R^{\widehat{Q}}_{Q_{sp}}-P)(x_{\widehat{Q}})\right|\quad\text{(see Lemma 7 in [1])}\\ &\quad\le C\cdot M_{\widehat{Q}}(f,P)\qquad\qquad\text{(see (1.68)).}\end{aligned}$$

We combine (1.102) with the above estimates to obtain

$$\|G\|_{\mathbb{X}((1+\overline{a})\widehat{Q})}+\delta_{\widehat{Q}}^{-m}\|G-P\|_{L^p((1+\overline{a})\widehat{Q})}\le C\cdot M_{\widehat{Q}}(f,P). \tag{1.104}$$

This completes the proof of Proposition 3. □

We next prove the following result.

Proposition 4. *Let $\widehat{Q}$ be a testing cube, and let $(f,P) \in \mathbb{X}(\frac{65}{64}\widehat{Q} \cap E) \oplus \mathcal{P}$. Then the following inequalities hold.*

(Unconditional inequality) $\|(f,P)\|_{(1+a_{new})\widehat{Q}} \le C \cdot M_{\widehat{Q}}(f,P)$.

(Conditional inequality) *If $3\widehat{Q}$ is tagged with $(\mathcal{A},\epsilon)$, then*

$$M_{\widehat{Q}}(f,P) \le C(t_G) \cdot (1/\epsilon) \cdot \|(f,P)\|_{\frac{65}{64}\widehat{Q}}.$$

The rest of Section 1.4.6 is devoted to the proof of Proposition 4. We set $\overline{a} = a_{new}$ for the remainder of the section for ease of notation.

The unconditional inequality in Proposition 4 follows easily from Proposition 3. Indeed, Proposition 3 states that $T_{\widehat{Q}}(f,P) = f$ on $(1+\overline{a})\widehat{Q} \cap E$. Hence, by definition of the trace norm,

$$\|(f,P)\|_{(1+\overline{a})\widehat{Q}} \le \|T_{\widehat{Q}}(f,P)\|_{\mathbb{X}((1+\overline{a})\widehat{Q})} + \delta_{\widehat{Q}}^{-m}\|T_{\widehat{Q}}(f,P) - P\|_{L^p((1+\overline{a})\widehat{Q})}.$$

Again thanks to Proposition 3, the right-hand side is bounded by $C \cdot M_{\widehat{Q}}(f,P)$, which proves the unconditional inequality.

We now begin the proof of the conditional inequality in Proposition 4. We assume that

$$3\widehat{Q} \text{ is tagged with } (\mathcal{A},\epsilon). \tag{1.105}$$

and

$$t_G \le \eta, \quad \text{where } \eta = \min\left\{c_*(\mathcal{A}^-), \left[100 \cdot S(\mathcal{A}^-)\right]^{-1}\right\} \tag{1.106}$$

Now, we consider two separate cases: either $\widehat{Q}$ is η-simple or $\widehat{Q}$ is not η-simple. For the definition of simple testing cubes, see Definition 2.

The conditional inequality is easy to prove in the former case.

Lemma 12. *Suppose that a testing cube $\widehat{Q}$ is η-simple with $\eta \ge t_G$. Then $M_{\widehat{Q}}(f,P) \le C(t_G) \cdot \|(f,P)\|_{\frac{65}{64}\widehat{Q}}$, where $C(t_G)$ depends only on m, n, p, and t_G.*

Proof. We examine the definition of $\left[M_{\widehat{Q}}(f,P)\right]^p$ as a sum of terms **(I)**–**(IV)** (see (1.65)–(1.68)).

Suppose that $Q \in CZ(\mathcal{A}^-)$ with $Q \subset (1+100t_G)\widehat{Q}$. Then $Q \subset \frac{65}{64}\widehat{Q}$ for small enough t_G. Our assumption that $\widehat{Q}$ is η-simple with $\eta \ge t_G$ implies that $\delta_Q \ge t_G\delta_{\widehat{Q}}$. Hence, from (1.63) we see that

$$Q \in CZ(\mathcal{A}^-),\ Q \subset (1+100t_G)\widehat{Q} \implies R_Q^{\widehat{Q}}(f,P) = P.$$

For every Q', Q'' as in (1.66), by good geometry of $CZ(\mathcal{A}^-)$ we have $Q', Q'' \subset (1+100t_G)\widehat{Q}$, hence $R_{Q'}^{\widehat{Q}} = R_{Q''}^{\widehat{Q}} = P$ in **(II)**. Similarly, for each Q in (1.67), we have $R_Q^{\widehat{Q}} = P$ in **(III)**. Similarly, $R_{Q_{sp}}^{\widehat{Q}} = P$ in **(IV)**. Hence, the terms **(II)**, **(III)**, and **(IV)**, all vanish, and thus $\left[M_{\widehat{Q}}(f,P)\right]^p =$ **(I)**.

We estimate the remaining term **(I)** (see (1.65)).

Let $Q \in CZ_{main}(\mathcal{A}^-)$ satisfy $Q \subset (1+t_G)\widehat{Q}$. We will bound each of the summands $\left[M_{(Q,\mathcal{A})}(f, R_Q^{\widehat{Q}})\right]^p$, which are relevant to the term **(I)**. As above, we have $R_Q^{\widehat{Q}}(f,P) = P$. Note that $\frac{65}{64}Q \subset \frac{65}{64}\widehat{Q}$ by Lemma 11. From the right-hand estimate in (1.20), Lemma 14 in [1], and the estimate $\delta_Q \geq t_G \delta_{\widehat{Q}}$ (which follows because $\widehat{Q}$ is η-simple with $\eta \geq t_G$), we have

$$M_{(Q,\mathcal{A}^-)}(f,P) \leq C \cdot \|(f,P)\|_{\frac{65}{64}Q} \leq C(t_G) \cdot \|(f,P)\|_{\frac{65}{64}\widehat{Q}}.$$

Therefore, each summand $\left[M_{(Q,\mathcal{A})}(f, R_Q^{\widehat{Q}})\right]^p$ relevant to **(I)** is bounded by $C(t_G)^p \cdot \|(f,P)\|^p_{\frac{65}{64}\widehat{Q}}$.

Since $\widehat{Q}$ is η-simple, we can have $Q \subset (1+t_G)\widehat{Q}$ for no more than $C(t_G)$ of the cubes $Q \in CZ(\mathcal{A}^-)$. Hence, no more than $C(t_G)$ many cubes Q arise in (1.65). Hence, by summing the estimates just obtained, we learn that $\left[M_{\widehat{Q}}(f,P)\right]^p \leq C(t_G) \cdot \|(f,P)\|^p_{\frac{65}{64}\widehat{Q}}$. This completes the proof of Lemma 12. □

If $\widehat{Q}$ is η-simple with $\eta = \min\{c_*(\mathcal{A}^-), [100S(\mathcal{A}^-)]^{-1}\}$, then the conditional inequality follows from Lemma 12. Here, note that the assumption (1.106) implies the hypotheses of Lemma 12.

Thus, in proving the conditional inequality, we may assume that

$$\widehat{Q} \text{ is } \textit{not} \ \eta\text{-simple, with } \eta = \min\{c_*(\mathcal{A}^-), \left[100S(\mathcal{A}^-)\right]^{-1}\}. \tag{1.107}$$

This is the latter, more difficult case in the dichotomy mentioned before. By definition, in this case, there exists a cube $\overline{Q} \in CZ(\mathcal{A}^-)$ with $\overline{Q} \subset \frac{65}{64}\widehat{Q}$ and $\delta_{\overline{Q}} \leq \eta \cdot \delta_{\widehat{Q}}$. Hence, we note that $S(\mathcal{A}^-)\overline{Q} \subset 3\widehat{Q}$. Moreover, we have $\delta_{\overline{Q}} \leq c_*(\mathcal{A}^-)\delta_{\widehat{Q}} \leq c_*(\mathcal{A}^-)$.

Thus, **(CZ2)** in the main technical results for $\mathcal{A}^-$ (see Section 3 in [1]) implies that

$$S(\mathcal{A}^-)\overline{Q} \text{ is not tagged with } (\mathcal{A}^-, \epsilon_1(\mathcal{A}^-)). \tag{1.108}$$

Hence, in particular, we have $\#(E \cap 3\widehat{Q}) \geq \#(E \cap S(\mathcal{A}^-)\overline{Q}) \geq 2$.

Now, from (1.105) we know that $3\widehat{Q}$ is tagged with $(\mathcal{A}, \epsilon)$. Hence, since $\#(E \cap 3\widehat{Q}) \geq 2$, we know that $\sigma(3\widehat{Q})$ has an $(\mathcal{A}', x_{\widehat{Q}}, \epsilon, \delta_{3\widehat{Q}})$-basis, for some $\mathcal{A}' \leq \mathcal{A}$.

We next apply Lemma 25 from [1] to the convex set $\sigma = \sigma(3\widehat{Q})$. Thus, we can guarantee that there exist numbers $\Lambda \geq 1$, and $\kappa_1 \leq \overline{\kappa} \leq \kappa_2$, and a multiindex set $\mathcal{A}'' \leq \mathcal{A}'$, such that

$$\sigma(3\widehat{Q}) \text{ has an } (\mathcal{A}'', x_{\widehat{Q}}, \epsilon^{\overline{\kappa}}, \delta_{3\widehat{Q}}, \Lambda)\text{-basis, where } \epsilon^{\overline{\kappa}} \cdot \Lambda^{100D} \leq \epsilon^{\overline{\kappa}/2}.$$

Here, $\kappa_1, \kappa_2 \in (0,1]$ are universal constants. Hence, $3\widehat{Q}$ is tagged with $(\mathcal{A}'', \epsilon^{\kappa_1/2})$, which implies that $S(\mathcal{A}^-)\overline{Q}$ is tagged with $(\mathcal{A}'', \epsilon^{\kappa'})$ for a universal constant $\kappa' > 0$. (Here, we use that $S(\mathcal{A}^-)\overline{Q} \subset 3\widehat{Q}$; see Lemma 28 from [1]). Comparing this statement and (1.108), we deduce that $\mathcal{A}'' = \mathcal{A}$.

In summary,

$$\sigma(3\widehat{Q}) \text{ has an } (\mathcal{A}, x_{\widehat{Q}}, \epsilon^{\overline{\kappa}}, \delta_{3\widehat{Q}}, \Lambda)\text{-basis, where } \epsilon^{\overline{\kappa}} \cdot \Lambda^{100D} \le \epsilon^{\overline{\kappa}/2}. \tag{1.109}$$

The assumptions (1.105)–(1.109) will be used in the remainder of this section. We finish the section by completing the proof of the conditional inequality in Proposition 4 and by deriving a useful corollary.

The next result represents a main step in the proof of the conditional inequality.

Proposition 5. *Assume that* (1.105)–(1.109) *hold. Then there exists an* $H \in \mathbb{X}$ *such that*

- $H = f$ *on* $E \cap \frac{65}{64}\widehat{Q}$.
- $\partial^\alpha H(x_Q) = \partial^\alpha P(x_Q)$ *for each* $\alpha \in \mathcal{A}$ *and* $Q \in \mathrm{CZ}(\mathcal{A}^-)$ *such that* $Q \subset \frac{65}{64}\widehat{Q}$.
- $\|H\|_{\mathbb{X}(\frac{65}{64}\widehat{Q})} + \delta_{\widehat{Q}}^{-m}\|H - P\|_{L^p(\frac{65}{64}\widehat{Q})} \le C\Lambda^{2D+1} \cdot \|(f,P)\|_{\frac{65}{64}\widehat{Q}}$.

Here, $C \ge 1$ *depends only on* m, n, *and* p.

Proof. We set

$$\mathcal{J}(\widehat{Q}) := \left\{ Q \in \overline{\mathrm{CZ}}(\mathcal{A}^-) : Q \cap \frac{65}{64}\widehat{Q} \neq \emptyset \right\}.$$

Recall that the cubes in $\{(65/64)Q : Q \in \mathcal{J}(\widehat{Q})\}$ have bounded overlap, and that the cubes in $\mathcal{J}(\widehat{Q})$ have good geometry, i.e.,

$$\text{If } Q, Q' \in \mathcal{J}(\widehat{Q}) \text{ and } Q \leftrightarrow Q' \text{ then } \frac{1}{16} \cdot \delta_Q \le \delta_{Q'} \le 16 \cdot \delta_Q. \tag{GG}$$

This follows from Proposition 1, since $\mathcal{J}(\widehat{Q}) \subset \overline{\mathrm{CZ}}(\mathcal{A}^-)$. We now prove that

$$\delta_Q \le C \cdot \delta_{\widehat{Q}} \quad \text{for each } Q \in \mathcal{J}(\widehat{Q}). \tag{1.110}$$

For the sake of contradiction, assume that $\delta_Q \ge 10^5 \delta_{\widehat{Q}}$ for some $Q \in \mathcal{J}(\widehat{Q})$. By definition of $\mathcal{J}(\widehat{Q})$, we have $Q \cap \frac{65}{64}\widehat{Q} \neq \emptyset$. Hence, since $\delta_Q \ge 10^5 \delta_{\widehat{Q}}$, we see that there exists $x \in \frac{65}{64}Q \cap \widehat{Q}$. Now, $\widehat{Q}$ is partitioned into cubes in $\mathrm{CZ}(\mathcal{A}^-)$, since $\widehat{Q} \subset Q^\circ$ is a testing cube. Thus, we can pick $Q_* \in \mathrm{CZ}(\mathcal{A}^-)$ with $x \in Q_*$ and $Q_* \subset \widehat{Q}$. Note that $x \in \frac{65}{64}Q \cap Q_*$. By good geometry of the cubes in $\overline{\mathrm{CZ}}(\mathcal{A}^-)$, we conclude that $\delta_Q \le 16 \cdot \delta_{Q_*}$. Hence, $\delta_Q \le 16\delta_{Q_*} \le 16\delta_{\widehat{Q}} < 10^5 \delta_{\widehat{Q}}$. This gives a contradiction and completes the proof of (1.110).

For each $Q \in \mathcal{J}(\widehat{Q})$ we select $y_Q \in Q \cap \frac{65}{64}\widehat{Q}$ such that

$$\text{if } Q \subset \frac{65}{64}\widehat{Q} \quad \text{then} \quad y_Q = x_Q \text{ (the center of } Q). \tag{1.111}$$

By definition of the trace seminorm $\|(\cdot,\cdot)\|_{\frac{65}{64}\widehat{Q}}$, there exists a function $F \in \mathbb{X}$ with

$$\|F\|_{\mathbb{X}(\frac{65}{64}\widehat{Q})} + \delta_{\frac{65}{64}\widehat{Q}}^{-m}\|F - P\|_{L^p(\frac{65}{64}\widehat{Q})} \le 2\,\|(f,P)\|_{\frac{65}{64}\widehat{Q}}, \text{ and} \tag{1.112}$$

$$F = f \quad \text{on} \quad \frac{65}{64}\widehat{Q} \cap E. \tag{1.113}$$

Part I: Defining local basis functions.

By (1.109), there exist $P_\alpha \in \mathcal{P}$ and $\varphi_\alpha \in \mathbb{X}$ such that

$$\|\varphi_\alpha\|_{\mathbb{X}(3\widehat{Q})} + \delta_{3\widehat{Q}}^{-m}\|\varphi_\alpha - P_\alpha\|_{L^p(3\widehat{Q})} \le \epsilon^{\overline{\kappa}}\delta_{3\widehat{Q}}^{|\alpha|+n/p-m} \quad (\alpha \in \mathcal{A}), \tag{1.114}$$

$$\varphi_\alpha = 0 \text{ on } E \cap 3\widehat{Q} \quad (\alpha \in \mathcal{A}), \tag{1.115}$$

$$|\partial^\beta P_\alpha(x_{\widehat{Q}}) - \delta_{\alpha\beta}| \le \begin{cases} \epsilon^{\overline{\kappa}}\delta_{3\widehat{Q}}^{|\alpha|-|\beta|}, & \text{if } \beta \ge \alpha \\ \Lambda\delta_{3\widehat{Q}}^{|\alpha|-|\beta|}, & \text{if } \beta < \alpha \end{cases} \quad (\alpha \in \mathcal{A}, \beta \in \mathcal{M}), \tag{1.116}$$

$$\partial^\beta P_\alpha(x_{\widehat{Q}}) = \delta_{\alpha\beta} \quad (\alpha, \beta \in \mathcal{A}). \tag{1.117}$$

Moreover, by (1.114) and (1.116),

$$\|\varphi_\alpha\|_{L^p(3\widehat{Q})} \le \|\varphi_\alpha - P_\alpha\|_{L^p(3\widehat{Q})} + \|P_\alpha\|_{L^p(3\widehat{Q})} \le C\epsilon^{\overline{\kappa}}\delta_{\widehat{Q}}^{|\alpha|+\frac{n}{p}} + C\Lambda\delta_{\widehat{Q}}^{|\alpha|+\frac{n}{p}}. \tag{1.118}$$

For each $\beta \in \mathcal{M}$, and each $Q \in \mathcal{J}(\widehat{Q})$, we have

$$\begin{aligned}|\partial^\beta\varphi_\alpha(y_Q) - \delta_{\alpha\beta}| \le{}& |\partial^\beta(\varphi_\alpha - P_\alpha)(y_Q)| \\ &+ |\partial^\beta P_\alpha(y_Q) - \partial^\beta P_\alpha(x_{\widehat{Q}})| + |\partial^\beta P_\alpha(x_{\widehat{Q}}) - \delta_{\alpha\beta}|.\end{aligned}$$

We shall estimate separately each of the three terms on the above right-hand side. For the first term, we use a Sobolev-type estimate that is stated in Lemma 10 of [1] to show that

$$\begin{aligned}\delta_{\widehat{Q}}^{|\beta|+n/p-m}|\partial^\beta(\varphi_\alpha - P_\alpha)(y_Q)| &\le C\big(\delta_{\widehat{Q}}^{-m}\|\varphi_\alpha - P_\alpha\|_{L^p(3\widehat{Q})} + \|\varphi_\alpha\|_{\mathbb{X}(3\widehat{Q})}\big) \\ &\le C\epsilon^{\overline{\kappa}}\delta_{\widehat{Q}}^{|\alpha|+n/p-m}.\end{aligned}$$

(Recall that $y_Q \in \frac{65}{64}\widehat{Q} \subset 3\widehat{Q}$.) For the second term, we use a Taylor expansion to write

$$\begin{aligned}|\partial^\beta P_\alpha(y_Q) - \partial^\beta P_\alpha(x_{\widehat{Q}})| &= \Big|\sum_{0<|\gamma|\le m-|\beta|-1} \frac{\partial^{\beta+\gamma}P_\alpha(x_{\widehat{Q}})}{\gamma!}\cdot(y_Q - x_{\widehat{Q}})^\gamma\Big| \\ &\le \begin{cases} C\epsilon^{\overline{\kappa}}\delta_{\widehat{Q}}^{|\alpha|-|\beta|}, & \text{if } \beta \ge \alpha, \\ C\Lambda\delta_{\widehat{Q}}^{|\alpha|-|\beta|}, & \text{if } \beta < \alpha \end{cases} \quad \text{(see (1.116)).}\end{aligned}$$

For the third term, we simply use (1.116). Thus, all together we have shown that

$$|\partial^\beta\varphi_\alpha(y_Q) - \delta_{\alpha\beta}| \le \begin{cases} C\epsilon^{\overline{\kappa}}\delta_{\widehat{Q}}^{|\alpha|-|\beta|}, & \beta \ge \alpha, \\ C\Lambda\delta_{\widehat{Q}}^{|\alpha|-|\beta|}, & \beta < \alpha \end{cases} \quad (\text{for } \alpha \in \mathcal{A}, \beta \in \mathcal{M}). \tag{1.119}$$

In particular, the matrix $(\partial^\beta\varphi_\alpha(y_Q))_{\alpha,\beta\in\mathcal{A}}$ is $(C\epsilon^{\overline{\kappa}}, C\Lambda, \delta_{\widehat{Q}})$-near triangular (with $\epsilon^{\overline{\kappa}}\Lambda^{100D} \le \epsilon^{\overline{\kappa}/2}$); hence, as claimed in Lemma 22 of [1], this matrix has an inverse $(A^Q_{\gamma\alpha})_{\gamma,\alpha\in\mathcal{A}}$ such that

$$\sum_{\alpha\in\mathcal{A}} A^Q_{\gamma\alpha}\cdot\partial^\beta\varphi_\alpha(y_Q) = \delta_{\beta\gamma} \quad (\text{for all } \beta, \gamma \in \mathcal{A}); \tag{1.120}$$

and

$$|A^Q_{\gamma\alpha} - \delta_{\gamma\alpha}| \le \begin{cases} C\epsilon^{\overline{\kappa}}\Lambda^D \delta_{\widehat{Q}}^{|\gamma|-|\alpha|}, & \text{if } \alpha \ge \gamma, \\ C\Lambda^D \delta_{\widehat{Q}}^{|\gamma|-|\alpha|}, & \text{if } \alpha < \gamma, \end{cases} \quad \text{(for all } \alpha, \gamma \in \mathcal{A}\text{)}. \tag{1.121}$$

We define

$$\varphi^Q_\alpha := \sum_{\beta \in \mathcal{A}} A^Q_{\alpha\beta} \varphi_\beta \quad \text{on } \mathbb{R}^n \quad \text{(for each } \alpha \in \mathcal{A}\text{)}. \tag{1.122}$$

For any $Q' \in \mathcal{J}(\widehat{Q})$, we can write

$$\varphi^{Q'}_\alpha = \sum_{\gamma \in \mathcal{A}} \omega^{QQ'}_{\alpha\gamma} \varphi^Q_\gamma, \quad \text{where } \omega^{QQ'} := A^{Q'} \cdot \left[A^Q\right]^{-1}. \tag{1.123}$$

For each $Q \in \mathcal{J}(\widehat{Q})$, we have

$$\|\varphi^Q_\alpha\|_{\mathbb{X}(3\widehat{Q})} \le C\epsilon^{\overline{\kappa}}\Lambda^D \delta_{\widehat{Q}}^{|\alpha|+n/p-m}. \tag{1.124}$$

$$\varphi^Q_\alpha = 0 \text{ on } E \cap 3\widehat{Q}. \tag{1.125}$$

$$\partial^\gamma \varphi^Q_\alpha(y_Q) = \delta_{\gamma\alpha} \qquad \text{for } \gamma \in \mathcal{A}. \tag{1.126}$$

$$|\partial^\gamma \varphi^Q_\alpha(y_Q)| \le C\epsilon^{\overline{\kappa}}\Lambda^{D+1} \delta_{\widehat{Q}}^{|\alpha|-|\gamma|} \qquad \text{for } \gamma \in \mathcal{M}, \gamma > \alpha. \tag{1.127}$$

$$|\partial^\gamma \varphi^Q_\alpha(y_Q)| \le C\Lambda^{D+1} \delta_{\widehat{Q}}^{|\alpha|-|\gamma|} \qquad \text{for } \gamma \in \mathcal{M}. \tag{1.128}$$

Here, (1.124), (1.125), (1.126) are immediate consequences of (1.114) and (1.121), (1.115), and (1.120), respectively. Moreover, (1.127) and (1.128) are both consequences of (1.119) and (1.121) (see Lemma 21 from [1]).

We now show that there exists $Z > 0$, depending only on m, n, and p, such that

$$|\partial^\gamma \varphi^Q_\alpha(y_Q)| \le Z\Lambda^{D+1} \delta_{S(\mathcal{A}^-)Q}^{|\alpha|-|\gamma|} \quad \text{for } \alpha \in \mathcal{A},\ \gamma \in \mathcal{M}. \tag{1.129}$$

For the sake of contradiction, assume that (1.129) fails to hold for some number $Z \ge 1$. We assume that Z exceeds a large enough constant determined by m, n, and p. We later take $Z = Z(m, n, p)$, but not yet. We assume that ϵ is less than a small enough constant determined by Z, m, n, and p.

If $\delta_Q \ge \min\{c_*(\mathcal{A}^-), 1/16, [100S(\mathcal{A}^-)]^{-1}\} \cdot \delta_{\widehat{Q}}$ then since also $\delta_Q \le C\delta_{\widehat{Q}}$ (see (1.110)), the estimate (1.129) follows from (1.128).

Alternatively, assume that

$$\delta_Q < \min\{c_*(\mathcal{A}^-), 1/16, [100S(\mathcal{A}^-)]^{-1}\} \cdot \delta_{\widehat{Q}}.$$

Thus, we have $S(\mathcal{A}^-)Q \subseteq 3\widehat{Q}$, since $Q \cap \frac{65}{64}\widehat{Q} \neq \emptyset$. Therefore, (1.125) implies that $\varphi^Q_\alpha = 0$ on $E \cap S(\mathcal{A}^-)Q$. Moreover, the Sobolev inequality and (1.124) imply that

$$\begin{aligned} &\|\varphi^Q_\alpha\|_{\mathbb{X}(S(\mathcal{A}^-)Q)} + \delta^{-m}_{S(\mathcal{A}^-)Q} \|\varphi^Q_\alpha - J_{y_Q}\varphi^Q_\alpha\|_{L^p(S(\mathcal{A}^-)Q)} \\ &\le C\|\varphi^Q_\alpha\|_{\mathbb{X}(S(\mathcal{A}^-)Q)} \le C\|\varphi^Q_\alpha\|_{\mathbb{X}(3\widehat{Q})} \le C\epsilon^{\overline{\kappa}}\Lambda^D \delta_{3\widehat{Q}}^{|\alpha|+n/p-m} \le C\epsilon^{\overline{\kappa}}\Lambda^D \delta_{S(\mathcal{A}^-)Q}^{|\alpha|+n/p-m}. \end{aligned}$$

(In the last inequality, we use that $\delta_{S(\mathcal{A}^-)Q} \le \delta_{3\widehat{Q}}$ and $|\alpha| + n/p - m < 0$.)

From the previous paragraph and (1.126), (1.127), we see that $(J_{y_Q}\varphi_\alpha^Q)_{\alpha\in\mathcal{A}}$ is an $(\mathcal{A}, y_Q, C\epsilon^{\overline{\kappa}}\Lambda^{D+1}, \delta_{S(\mathcal{A}^-)Q})$-basis for $\sigma(S(\mathcal{A}^-)Q)$.

Note that $C\epsilon^{\overline{\kappa}}\Lambda^{D+1} \le C\epsilon^{\overline{\kappa}/2} \le \epsilon^{\kappa_1/4}$ as long as ϵ is less than a small enough universal constant. Hence, $(J_{y_Q}\varphi_\alpha^Q)_{\alpha\in\mathcal{A}}$ is an $(\mathcal{A}, y_Q, \epsilon^{\kappa_1/4}, \delta_{S(\mathcal{A}^-)Q})$-basis for $\sigma(S(\mathcal{A}^-)Q)$.

We are assuming that (1.129) does not hold, hence

$$\max\left\{|\partial^\gamma\varphi_\alpha^Q(y_Q)|\delta_{S(\mathcal{A}^-)Q}^{|\gamma|-|\alpha|} : \alpha\in\mathcal{A},\ \gamma\in\mathcal{M}\right\} \ge Z.$$

If Z exceeds a large enough universal constant, and if $\epsilon^{\kappa_1/4} < Z^{-2}$, then from Lemma 24 in [1] we deduce that

$$\sigma(S(\mathcal{A}^-)Q) \text{ has an } (\mathcal{A}', y_Q, Z^{-\kappa}, \delta_{S(\mathcal{A}^-)Q})\text{-basis, with } \mathcal{A}' < \mathcal{A}.$$

Hence, $\sigma(S(\mathcal{A}^-)Q)$ has an $(\mathcal{A}'', x_Q, Z^{-\kappa'}, \delta_{S(\mathcal{A}^-)Q})$-basis for some $\mathcal{A}'' \le \mathcal{A}'$, due to Lemma 27 in [1]. (Here we use that $y_Q \in Q$ and $x_Q \in Q$, so $|x_Q - y_Q| \le 2\delta_Q$.) Here, κ and κ' are universal constants.

If Z is chosen to be a large enough universal constant, we conclude that

$$\sigma(S(\mathcal{A}^-)Q) \text{ has an } (\mathcal{A}'', x_Q, \epsilon_1(\mathcal{A}^-), \delta_{S(\mathcal{A}^-)Q})\text{-basis.}$$

Hence, $S(\mathcal{A}^-)Q$ is tagged with $(\mathcal{A}^-, \epsilon_1(\mathcal{A}^-))$.

Recall that $Q \in \overline{CZ}(\mathcal{A}^-)$. In fact, since $\delta_Q \le (1/16)\delta_{\widehat{Q}} \le (1/16)$, condition (e) in Proposition 1 shows that $Q \in CZ(\mathcal{A}^-)$.

Since $\delta_Q \le c_*(\mathcal{A}^-)$, the previous two paragraphs contradict property **(CZ2)** of $CZ(\mathcal{A}^-)$ in Section 3 of [1]. This completes the proof of (1.129) by contradiction. This concludes our analysis of the basis functions $(\varphi_\alpha^Q)_{\alpha\in\mathcal{A}}$.

Part II: Modifying the extension.

Since $\overline{CZ}(\mathcal{A}^-)$ forms a partition of $\mathbb{R}^n$, we have

$$\frac{65}{64}\widehat{Q} \subset \bigcup_{Q\in\mathcal{J}(\widehat{Q})} Q. \tag{1.130}$$

We will need to use a few results from Sections 4.6.4 and 4.6.5 in [1]. Consider:

- Let $CZ = \overline{CZ}(\mathcal{A}^-)$ and $\mathcal{Q} = \mathcal{J}(\widehat{Q})$.
- Take the cube called $\widehat{Q}$ in Sections 4.6.4 and 4.6.5 to be $\frac{65}{64}\widehat{Q}$, with $\widehat{Q}$ as in the present setting.
- Let $\overline{r} = a$, and $A = C$ for a large enough universal constant C.

We have established the good geometry of $\overline{CZ}(\mathcal{A}^-)$, which is a dyadic decomposition of $\mathbb{R}^n$; see Proposition 1. This puts us in the setting of Section 4.6.4.

We point out the **covering condition** stated in (1.130), and the **size condition** stated in (1.110): $\delta_Q \le C\delta_{\widehat{Q}}$ for all $Q \in \mathcal{J}(\widehat{Q})$. These conditions are exactly conditions (4.53) and (4.54) in Section 4.6.5, with the choices of CZ, $\mathcal{Q}$, $\widehat{Q}$, $\overline{r}$, and A as in the above bullet points. This puts us in the setting of Section 4.6.5 of [1].

By Lemma 48 of [1], there exists $\theta_Q \in C^m(\mathbb{R}^n)$ for $Q \in \mathcal{J}(\widehat{Q})$ with

(a) $\sum_{Q\in\mathcal{J}(\widehat{Q})} \theta_Q = 1$ on $\frac{65}{64}\widehat{Q}$,

(b) $\theta_Q = 1$ near x_Q, $\theta_Q = 0$ near $x_{Q'}$ for $Q' \in \mathcal{J}(\widehat{Q}) \setminus \{Q\}$,

(c) $\|\partial^\alpha \theta_Q\|_{L^\infty((1+a)Q)} \le C \cdot \delta_Q^{-|\alpha|}$ for $|\alpha| \le m$, and

(d) $\operatorname{supp} \theta_Q \subset (1+a)Q$.

We set $H := F + \widetilde{F}$ on $\mathbb{R}^n$, where

$$\widetilde{F}(x) := \sum_{Q\in\mathcal{J}(\widehat{Q})} \sum_{\alpha,\beta\in\mathcal{A}} \theta_Q(x)\cdot\varphi_\beta(x)\cdot A^Q_{\alpha\beta}\cdot[\partial^\alpha(P-F)(y_Q)]$$

$$= \sum_{Q\in\mathcal{J}(\widehat{Q})} \sum_{\alpha\in\mathcal{A}} \theta_Q(x)\cdot\varphi^Q_\alpha(x)\cdot[\partial^\alpha(P-F)(y_Q)]. \tag{1.131}$$

Note that H belongs to $\mathbb{X}$.

Since $\varphi_\alpha = 0$ on $E \cap 3\widehat{Q}$, we see that $\widetilde{F} = 0$ on $E \cap 3\widehat{Q}$, hence $H = f$ on $E \cap \frac{65}{64}\widehat{Q}$; see (1.113). This proves the first bullet point in Proposition 5.

Suppose that $Q \in CZ(\mathcal{A}^-)$ and $Q \subset \frac{65}{64}\widehat{Q}$. Then $y_Q = x_Q$, thanks to (1.111). Thus, property (b) of $\{\theta_Q\}$ states that $\theta_Q \equiv 1$ near y_Q, and $\theta_{Q'} \equiv 0$ near y_Q for any $Q' \in \mathcal{J}(\widehat{Q}) \setminus \{Q\}$. Therefore, (1.126) and (1.131) give

$$\partial^\gamma \widetilde{F}(y_Q) = \sum_{\alpha\in\mathcal{A}} \partial^\gamma\varphi^Q_\alpha(y_Q)\cdot\partial^\alpha(P-F)(y_Q)$$

$$= \sum_{\alpha\in\mathcal{A}} \delta_{\alpha\gamma}\cdot\partial^\alpha(P-F)(y_Q) = \partial^\gamma(P-F)(y_Q) \quad \text{for each } \gamma\in\mathcal{A}.$$

Hence,

$$\partial^\gamma H(y_Q) = \partial^\gamma F(y_Q) + \partial^\gamma\widetilde{F}(y_Q) = \partial^\gamma P(y_Q) \quad (\text{with } y_Q = x_Q).$$

This proves the second bullet point in Proposition 5.

Part III: Estimating the norm.

From property (a) of the partition of unity (θ_Q), we may write

$$H = \sum_{Q\in\mathcal{J}(\widehat{Q})} F_Q\cdot\theta_Q \ \text{ on } \ \frac{65}{64}\widehat{Q}, \tag{1.132}$$

$$\text{where } F_Q = F + \sum_{\alpha\in\mathcal{A}} \varphi^Q_\alpha\cdot\partial^\alpha(P-F)(y_Q) \ \text{ on } \frac{65}{64}\widehat{Q}.$$

Before we estimate the semi-norm $\|H\|_{\mathbb{X}(\frac{65}{64}\widehat{Q})}$, we present several estimates.

We apply a version of the Sobolev inequality (see the second inequality in (2.4) from Section 2.3 of [1]), and then we use (1.112) to give

$$\begin{aligned}\delta_{\widehat{Q}}^{|\alpha|+n/p-m}|\partial^\alpha(F-P)(y_Q)| &\le C\cdot\left(\delta_{\widehat{Q}}^{-m}\|F-P\|_{L^p(\frac{65}{64}\widehat{Q})}+\|F-P\|_{\mathbb{X}(\frac{65}{64}\widehat{Q})}\right)\\ &\le C'\cdot\|(f,P)\|_{\frac{65}{64}\widehat{Q}} \quad \text{for } \alpha\in\mathcal{M}.\end{aligned} \tag{1.133}$$

Given $Q, Q'\in\mathcal{J}(\widehat{Q})$ such that $Q\leftrightarrow Q'$, define the rectangular boxes

$$B_1=(1+a)Q\cap(65/64)\widehat{Q} \quad\text{and}\quad B_2=(1+a)Q'\cap(65/64)\widehat{Q}.$$

Since $Q\in\mathcal{J}(\widehat{Q})$, we know that $Q\cap\frac{65}{64}\widehat{Q}\neq\emptyset$ and $\delta_Q\le C\delta_{\widehat{Q}}$ (see (1.110)). Hence, B_1 is a product of n intervals whose lengths are between $c\delta_Q$ and $C\delta_Q$, for universal constants c and C. Thus, the sidelengths of B_1 are between $c\delta_Q$ and $C\delta_Q$.

Similarly, the sidelengths of B_2 are between $c\delta_{Q'}$ and $C\delta_{Q'}$.

Note that δ_Q and $\delta_{Q'}$ differ by at most a factor of 64 thanks to good geometry.

We know that $(1+a)Q\cap(1+a)Q'\neq\emptyset$ because $Q\leftrightarrow Q'$. Since B_1 and B_2 are nonempty, the collection of cubes $\left\{(1+a)Q,(1+a)Q',\frac{65}{64}\widehat{Q}\right\}$ have nonempty pairwise intersections, hence we conclude that there is a common point in these three cubes.[1] Thus, $B_1\cap B_2\neq\emptyset$.

We have proven the following claim.

Claim. For any $Q, Q'\in\mathcal{J}(\widehat{Q})$ with $Q\leftrightarrow Q'$, all the sides of the boxes

$$B_1=(1+a)Q\cap(65/64)\widehat{Q} \quad\text{and}\quad B_2=(1+a)Q'\cap(65/64)\widehat{Q}$$

are between $c\delta_Q$ and $C\delta_Q$ for universal constants c and C. Hence, in particular, B_1 and B_2 have aspect ratio at most a universal constant. Moreover, $B_1\cap B_2\neq\emptyset$. Thus, we may apply Lemma 12 from [1] with K equal to a universal constant.

For each $\beta\in\mathcal{A}$, we have

$$\partial^\beta(J_{y_Q,F_{Q'}}-P)(y_{Q'})=\partial^\beta(F_{Q'}-P)(y_{Q'})=0 \quad \text{(see (1.126), (1.132))}.$$

We conclude that $\partial^\beta(J_{y_Q,F_{Q'}}-P)(y_Q)=0$. Here, we use the monotonicity of $\mathcal{A}$; see (1.1). Hence, $|\partial^\beta(F_{Q'}-P)(y_Q)|=|\partial^\beta(F_{Q'}-J_{y_Q,F_{Q'}})(y_Q)|$. Thus, Lemma 12 from [1] implies that, for $\beta\in\mathcal{A}$, we have

$$\begin{aligned}&|\partial^\beta(F_{Q'}-P)(y_Q)|\\ &\lesssim\delta_Q^{m-|\beta|-n/p}\left(\|F_{Q'}\|_{\mathbb{X}((1+a)Q\cap\frac{65}{64}\widehat{Q})}+\|F_{Q'}\|_{\mathbb{X}((1+a)Q'\cap\frac{65}{64}\widehat{Q})}\right).\end{aligned} \tag{1.134}$$

(Here, we use that $|y_Q-y_{Q'}|\le C\delta_Q$, which is a consequence of $Q\leftrightarrow Q'$ and the good geometry of $\mathcal{J}(\widehat{Q})$.)

[1] This follows from the fact that if three intervals have nonempty pairwise intersections then the three intervals share a point in common.

From (1.123) and (1.132), for $\overline{\beta} \in \mathcal{M}$ we have

$$
\begin{aligned}
&|\partial^{\overline{\beta}}(F_Q - F_{Q'})(y_Q)| \\
&= \Big| \sum_{\beta \in \mathcal{A}} \partial^{\beta}(F-P)(y_Q)\partial^{\overline{\beta}}\varphi_{\beta}^{Q}(y_Q) - \sum_{\alpha,\beta \in \mathcal{A}} \partial^{\alpha}(F-P)(y_{Q'})\omega_{\alpha\beta}^{QQ'}\partial^{\overline{\beta}}\varphi_{\beta}^{Q}(y_Q) \Big| \\
&\le \sum_{\beta \in \mathcal{A}} \big|\partial^{\overline{\beta}}\varphi_{\beta}^{Q}(y_Q)\big| \Big[\Big| \partial^{\beta}(F-P)(y_Q) - \sum_{\alpha \in \mathcal{A}} \partial^{\alpha}(F-P)(y_{Q'})\omega_{\alpha\beta}^{QQ'} \Big| \Big] \\
&= \sum_{\beta \in \mathcal{A}} \big|\partial^{\overline{\beta}}\varphi_{\beta}^{Q}(y_Q)\big| \Big[\Big| \partial^{\beta}(F-P)(y_Q) - \sum_{\alpha \in \mathcal{A}} \partial^{\alpha}(F-P)(y_{Q'})\partial^{\beta}\varphi_{\alpha}^{Q'}(y_Q) \Big| \Big] \\
&\qquad \text{(note that } \omega_{\alpha\beta}^{QQ'} = \partial^{\beta}\varphi_{\alpha}^{Q'}(y_Q); \text{see (1.123) and (1.126))} \\
&= \sum_{\beta \in \mathcal{A}} \big|\partial^{\overline{\beta}}\varphi_{\beta}^{Q}(y_Q)\big| \big[\big| \partial^{\beta}(F_{Q'} - P)(y_Q) \big| \big] \\
&\qquad \text{(see (1.132))} \\
&\le C\Lambda^{D+1} \sum_{\beta \in \mathcal{A}} \delta_Q^{|\beta|-|\overline{\beta}|}\delta_Q^{m-n/p-|\beta|} \big[\|F_{Q'}\|_{\mathbb{X}((1+a)Q \cap \frac{65}{64}\widehat{Q})} + \|F_{Q'}\|_{\mathbb{X}((1+a)Q' \cap \frac{65}{64}\widehat{Q})} \big] \\
&\qquad \text{(see (1.129) and (1.134))}
\end{aligned}
$$

$$
\le C\Lambda^{D+1} \cdot \delta_Q^{m-|\overline{\beta}|-n/p} \big[\|F_{Q'}\|_{\mathbb{X}((1+a)Q \cap \frac{65}{64}\widehat{Q})} + \|F_{Q'}\|_{\mathbb{X}((1+a)Q' \cap \frac{65}{64}\widehat{Q})} \big]. \tag{1.135}
$$

We are now prepared to estimate $\|H\|_{\mathbb{X}(\frac{65}{64}\widehat{Q})}$.

We apply Lemma 49 of [1], which shows that

$$
\begin{aligned}
\|H\|_{\mathbb{X}(\frac{65}{64}\widehat{Q})}^{p} &\lesssim \sum_{Q \in \mathcal{J}(\widehat{Q})} \|F_Q\|_{\mathbb{X}((1+a)Q \cap \frac{65}{64}\widehat{Q})}^{p} \\
&+ \sum_{Q \in \mathcal{J}(\widehat{Q})} \delta_Q^{-mp} \|F_Q - J_{y_Q}F_Q\|_{L^p((1+a)Q \cap \frac{65}{64}\widehat{Q})}^{p} \qquad (1.136) \\
&+ \sum_{\substack{Q,Q' \in \mathcal{J}(\widehat{Q}) \\ Q \leftrightarrow Q'}} \sum_{|\beta| \le m-1} \delta_Q^{(|\beta|-m)p+n} |\partial^{\beta}(J_{y_Q}F_Q - J_{y_{Q'}}F_{Q'})(y_Q)|^{p}. \qquad (1.137)
\end{aligned}
$$

(Here, we take $P_Q = J_{y_Q}F_Q$ in our application of Lemma 49.)

First we estimate the terms in (1.136). A version of the Sobolev inequality (see (2.7) in Section 2.3 of [1]) implies that

$$
\delta_Q^{-m}\|F_Q - J_{y_Q}F_Q\|_{L^p((1+a)Q \cap \frac{65}{64}\widehat{Q})} \le C\|F_Q\|_{\mathbb{X}((1+a)Q \cap \frac{65}{64}\widehat{Q})}.
$$

(Here, we use the fact that the lengths of all sides of the rectangular box $(1+a)Q \cap \frac{65}{64}\widehat{Q}$ are comparable to δ_Q; see the previous **Claim**.)

Next we estimate the terms in (1.137). For $Q, Q' \in \mathcal{J}(\widehat{Q})$ with $Q \leftrightarrow Q'$,

$$\begin{aligned}|\partial^\beta \left[J_{y_Q} F_Q - J_{y_{Q'}} F_{Q'}\right](y_Q)| &= |\partial^\beta \left[F_Q - J_{y_{Q'}} F_{Q'}\right](y_Q)| \\ &\le |\partial^\beta \left[F_Q - F_{Q'}\right](y_Q)| + |\partial^\beta \left[F_{Q'} - J_{y_{Q'}} F_{Q'}\right](y_Q)| \\ &\lesssim |\partial^\beta \left[F_Q - F_{Q'}\right](y_Q)| + \delta_Q^{m-|\beta|-n/p} \left[\|F_{Q'}\|_{\mathbb{X}((1+a)Q \cap \frac{65}{64}\widehat{Q})}\right. \\ &\qquad \left. + \|F_{Q'}\|_{\mathbb{X}((1+a)Q' \cap \frac{65}{64}\widehat{Q})}\right].\end{aligned}$$

(Here, in the last inequality we use Lemma 12 from [1].)

Using our previous estimates on (1.136) and (1.137), we obtain

$$\begin{aligned}&\|H\|^p_{\mathbb{X}(\frac{65}{64}\widehat{Q})} \\ &\lesssim \sum_{\substack{Q,Q' \in \mathcal{J}(\widehat{Q}) \\ Q \leftrightarrow Q'}} \left[\|F_Q\|^p_{\mathbb{X}((1+a)Q \cap \frac{65}{64}\widehat{Q})} + \|F_{Q'}\|^p_{\mathbb{X}((1+a)Q \cap \frac{65}{64}\widehat{Q})}\right. \\ &\qquad \left. + \sum_{\beta \in \mathcal{M}} |\partial^\beta (F_Q - F_{Q'})(y_Q)|^p \delta_Q^{(|\beta|-m)p+n}\right] \\ &\overset{(1.135)}{\le} C\Lambda^{(D+1)p} \sum_{\substack{Q,Q' \in \mathcal{J}(\widehat{Q}) \\ Q \leftrightarrow Q'}} \left[\|F_Q\|^p_{\mathbb{X}((1+a)Q \cap \frac{65}{64}\widehat{Q})} + \|F_Q\|^p_{\mathbb{X}((1+a)Q' \cap \frac{65}{64}\widehat{Q})}\right]. \\ &\overset{(1.122),(1.132)}{\le} C\Lambda^{(D+1)p} \sum_{\substack{Q,Q' \in \mathcal{J}(\widehat{Q}) \\ Q \leftrightarrow Q'}} \left[\|F\|^p_{\mathbb{X}((1+a)Q \cap \frac{65}{64}\widehat{Q})} + \|F\|^p_{\mathbb{X}((1+a)Q' \cap \frac{65}{64}\widehat{Q})}\right. \\ &+ \left. \sum_{\alpha,\beta \in \mathcal{A}} |A^Q_{\alpha\beta}|^p |\partial^\alpha (F - P)(y_Q)|^p \left(\|\varphi_\beta\|^p_{\mathbb{X}((1+a)Q \cap \frac{65}{64}\widehat{Q})} + \|\varphi_\beta\|^p_{\mathbb{X}((1+a)Q' \cap \frac{65}{64}\widehat{Q})}\right)\right] \\ &\overset{(1.121),(1.133)}{\le} C\Lambda^{(2D+1)p} \sum_{\substack{Q,Q' \in \mathcal{J}(\widehat{Q}) \\ Q \leftrightarrow Q'}} \left[\|F\|^p_{\mathbb{X}((1+a)Q \cap \frac{65}{64}\widehat{Q})}\right. \\ &\qquad \left. + \sum_{\beta \in \mathcal{A}} \delta_{\widehat{Q}}^{(m-|\beta|)p-n} \|(f,P)\|^p_{\frac{65}{64}\widehat{Q}} \|\varphi_\beta\|^p_{\mathbb{X}((1+a)Q \cap \frac{65}{64}\widehat{Q})}\right] \\ &\le C\Lambda^{(2D+1)p} \cdot \|(f,P)\|^p_{\frac{65}{64}\widehat{Q}} \left[1 + \sum_{\beta \in \mathcal{A}} \|\varphi_\beta\|^p_{\mathbb{X}(3\widehat{Q})} \delta_{\widehat{Q}}^{(m-|\beta|)p-n}\right] \\ &\le C\Lambda^{(2D+1)p} \cdot \|(f,P)\|^p_{\frac{65}{64}\widehat{Q}}\end{aligned}$$

(by bounded overlap of $\{(1+a)Q : Q \in \mathcal{J}(\widehat{Q})\}$ and by (1.112), (1.114)).

This concludes our estimation of $\|H\|_{\mathbb{X}(\frac{65}{64}\widehat{Q})}$.

Writing $H - P = (F - P) + \widetilde{F}$, we also obtain

$$\begin{aligned}
&\|H - P\|^p_{L^p(\frac{65}{64}\widehat{Q})} \\
&\overset{(1.131)}{\lesssim} \|F - P\|^p_{L^p(\frac{65}{64}\widehat{Q})} \\
&\qquad + \sum_{Q \in \mathcal{J}(\widehat{Q})} \sum_{\alpha,\beta \in \mathcal{A}} \|\varphi_\beta\|^p_{L^p((1+a)Q \cap \frac{65}{64}\widehat{Q})} \cdot |A^Q_{\alpha\beta}|^p \cdot |\partial^\alpha (P - F)(y_Q)|^p \\
&\overset{(1.121),(1.133)}{\lesssim} \|F - P\|^p_{L^p(\frac{65}{64}\widehat{Q})} \\
&\qquad + \Lambda^{Dp} \cdot \sum_{Q \in \mathcal{J}(\widehat{Q})} \sum_{\beta \in \mathcal{A}} \|\varphi_\beta\|^p_{L^p((1+a)Q \cap \frac{65}{64}\widehat{Q})} \delta_{\widehat{Q}}^{(m-|\beta|)p-n} \|(f,P)\|^p_{\frac{65}{64}\widehat{Q}} \\
&\lesssim \|F - P\|^p_{L^p(\frac{65}{64}\widehat{Q})} + \Lambda^{Dp} \cdot \sum_{\beta \in \mathcal{A}} \|\varphi_\beta\|^p_{L^p(3\widehat{Q})} \delta_{\widehat{Q}}^{(m-|\beta|)p-n} \|(f,P)\|^p_{\frac{65}{64}\widehat{Q}} \\
&\qquad\qquad \text{(by bounded overlap of the collection } \{(1+a)Q : Q \in \mathcal{J}(\widehat{Q})\}) \\
&\overset{(1.112),(1.118)}{\lesssim} \Lambda^{(D+1)p} \cdot [\delta_{\widehat{Q}}]^{mp} \|(f,P)\|^p_{\frac{65}{64}\widehat{Q}}.
\end{aligned}$$

Adding together the previous two estimates, we have

$$\|H\|_{\mathbb{X}(\frac{65}{64}\widehat{Q})} + \delta_{\widehat{Q}}^{-m} \|H - P\|_{L^p(\frac{65}{64}\widehat{Q})} \leq C\Lambda^{2D+1} \|(f,P)\|_{\frac{65}{64}\widehat{Q}}.$$

This completes the proof of Proposition 5 □

We recall several facts, and set some notation for the rest of this section.

- Suppose $Q \in CZ(\mathcal{A}^-)$ and $Q \subset (1 + 100t_G)\widehat{Q}$. Then $\frac{65}{64}Q \subset \frac{65}{64}\widehat{Q}$ (see Lemma 11).
- Suppose that $Q', Q'' \in CZ(\mathcal{A}^-)$, $Q' \subset (1 + t_G)\widehat{Q}$, $Q' \leftrightarrow Q''$, and $\delta_{Q'} < t_G \cdot \delta_{\widehat{Q}}$. Then $Q'' \subset (1 + 100t_G)\widehat{Q}$ (by good geometry).
- The sums below are indexed over cubes $Q \in CZ(\mathcal{A}^-)$, and over pairs of cubes $(Q', Q'') \in CZ(\mathcal{A}^-) \times CZ(\mathcal{A}^-)$; and often for ease of notation we choose to not make this indexing explicit.
- For ease of notation, we write $R^{\widehat{Q}}_Q = R^{\widehat{Q}}_Q(f,P)$ (the dependence on (f,P) should be understood).
- We are assuming that t_G is less than a small constant determined by m, n, and p, to be picked later, whereas the universal constant S_1 has already been picked. (See (1.17).)

Proposition 6. *Given* $H \in \mathbb{X}$, *and given* $\{R_{Q^\#} : Q^\# \text{ keystone}\} \subset \mathcal{P}$, *the following inequality holds:*

$$\sum_{\substack{Q \subset (1+100t_G)\widehat{Q} \\ \delta_Q < t_G \cdot \delta_{\widehat{Q}}}} \delta_Q^{-mp} \|H - R_{\mathcal{K}(Q)}\|^p_{L^p(\frac{65}{64}Q)}$$

$$\lesssim \sum_{\substack{Q^\# \text{ keystone} \\ S_1 Q^\# \subset \frac{65}{64}\widehat{Q}}} [\delta_{Q^\#}]^{-mp} \|H - R_{Q^\#}\|^p_{L^p(\frac{65}{64}Q^\#)} + \|H\|^p_{\mathbb{X}(\frac{65}{64}\widehat{Q})}. \tag{1.138}$$

Proof. Let $Q \in CZ(\mathcal{A}^-)$ satisfy

$$Q \subset (1 + 100t_G)\widehat{Q} \text{ and } \delta_Q < t_G \cdot \delta_{\widehat{Q}}. \tag{1.139}$$

Then there exists an exponentially decreasing path connecting Q and $\mathcal{K}(Q)$, as promised by the KEYSTONE-ORACLE. We denote this path by

$$Q = Q(1) \leftrightarrow Q(2) \leftrightarrow \cdots \leftrightarrow Q(L_Q) = \mathcal{K}(Q).$$

Recall that

$$\delta_{Q(\ell')} \le C \cdot (1-c)^{\ell'-\ell} \cdot \delta_{Q(\ell)} \text{ for } \ell' \ge \ell; \tag{1.140}$$

also $Q(\ell) \subset CQ$, and $S_1\mathcal{K}(Q) \subset CQ$, for a universal constant C. From (1.139) we conclude that $\frac{65}{64}CQ \subset \frac{65}{64}\widehat{Q}$, as long as t_G is sufficiently small. Therefore,

$$\frac{65}{64}Q(\ell) \subset \frac{65}{64}\widehat{Q} \quad \text{for all } \ell = 1, \dots, L_Q, \text{ and } \quad S_1\mathcal{K}(Q) \subset \frac{65}{64}\widehat{Q}. \tag{1.141}$$

In particular, note that $\frac{65}{64}Q \subset \frac{65}{64}\widehat{Q}$.

Fix an arbitrary number $\eta \in (0, 1 - n/p)$ depending only on n and p. By the triangle inequality,

$$\begin{aligned}
\delta_Q^{-mp}\|H - R_{\mathcal{K}(Q)}\|^p_{L^p(\frac{65}{64}Q)} &\lesssim \delta_Q^{-mp}\|H - J_{x_Q}H\|^p_{L^p(\frac{65}{64}Q)} \\
&+ \delta_Q^{-mp}\|J_{x_{\mathcal{K}(Q)}}H - R_{\mathcal{K}(Q)}\|^p_{L^p(\frac{65}{64}Q)} \\
&+ \delta_Q^{-mp}\Big\| \sum_{\ell=1}^{L_Q-1} \left(J_{x_{Q(\ell)}}H - J_{x_{Q(\ell+1)}}H\right) \delta_{Q(\ell)}^{-\eta}\delta_{Q(\ell)}^{+\eta} \Big\|^p_{L^p(\frac{65}{64}Q)};
\end{aligned}$$

here, Hölder's inequality shows that

$$\begin{aligned}
&\delta_Q^{-mp}\Big\| \sum_{\ell=1}^{L_Q-1} \left(J_{x_{Q(\ell)}}H - J_{x_{Q(\ell+1)}}H\right) \delta_{Q(\ell)}^{-\eta}\,\delta_{Q(\ell)}^{+\eta} \Big\|^p_{L^p(\frac{65}{64}Q)} \\
&\le \delta_Q^{-mp}\Big(\sum_{\ell=1}^{L_Q-1} \delta_{Q(\ell)}^{\eta p'} \Big)^{p/p'} \sum_{\ell=1}^{L_Q-1} \delta_{Q(\ell)}^{-\eta p}\|J_{x_{Q(\ell)}}H - J_{x_{Q(\ell+1)}}H\|^p_{L^p(\frac{65}{64}Q)} \\
&\lesssim \delta_Q^{\eta p - mp} \sum_{\ell=1}^{L_Q-1} \delta_{Q(\ell)}^{-\eta p}\,\|J_{x_{Q(\ell)}}H - J_{x_{Q(\ell+1)}}H\|^p_{L^p(\frac{65}{64}Q)} \quad \text{(by (1.140))};
\end{aligned}$$

also, the Sobolev inequality shows that

$$\delta_Q^{-mp}\|H - J_{x_Q}H\|^p_{L^p(\frac{65}{64}Q)} \le C\,\|H\|^p_{\mathbb{X}(\frac{65}{64}Q)}.$$

Combining these estimates, we have:

$$\begin{aligned}\delta_Q^{-mp}\|H - R_{\mathcal{K}(Q)}\|^p_{L^p(\frac{65}{64}Q)} \lesssim\ & \|H\|^p_{\mathbb{X}(\frac{65}{64}Q)} + \delta_Q^{-mp}\|J_{x_{\mathcal{K}(Q)}}H - R_{\mathcal{K}(Q)}\|^p_{L^p(\frac{65}{64}Q)}\\ &+ \delta_Q^{\eta p - mp}\sum_{\ell=1}^{L_Q-1}\delta_{Q(\ell)}^{-\eta p}\|J_{x_{Q(\ell)}}H - J_{x_{Q(\ell+1)}}H\|^p_{L^p(\frac{65}{64}Q)}.\end{aligned}$$

We apply Lemma 7 of [1] to all but the first term on the right-hand side of the previous estimate. Thus, we obtain

$$\begin{aligned}\delta_Q^{-mp}\|H - R_{\mathcal{K}(Q)}\|^p_{L^p(\frac{65}{64}Q)} \lesssim\ & \|H\|^p_{\mathbb{X}(\frac{65}{64}Q)}\\ &+ \delta_Q^{-mp}\sum_{|\beta|\le m-1}|\partial^\beta(J_{x_{\mathcal{K}(Q)}}H - R_{\mathcal{K}(Q)})(x_{\mathcal{K}(Q)})|^p\,\delta_Q^{|\beta|p+n}\\ &+ \delta_Q^{\eta p - mp}\sum_{\ell=1}^{L_Q-1}\delta_{Q(\ell)}^{-\eta p}\sum_{|\beta|\le m-1}|\partial^\beta(J_{x_{Q(\ell)}}H - J_{x_{Q(\ell+1)}}H)(x_{Q(\ell)})|^p\,\delta_Q^{|\beta|p+n}.\end{aligned}$$

Let X denote the sum of $\delta_Q^{-mp}\|H - R_{\mathcal{K}(Q)}\|^p_{L^p(\frac{65}{64}Q)}$ over all $Q \in CZ(\mathcal{A}^-)$ with $Q \subset (1 + 100t_G)\widehat{Q}$ and $\delta_Q < t_G\delta_{\widehat{Q}}$.

We now sum the previous estimate over Q. We denote $Q^\# = \mathcal{K}(Q)$, $Q' = Q(\ell)$, and $Q'' = Q(\ell+1)$, and we switch the order of summation in our sum. Using (1.141), we see that

$$\begin{aligned}X \lesssim\ & \sum_{\frac{65}{64}Q\subset\frac{65}{64}\widehat{Q}}\|H\|^p_{\mathbb{X}(\frac{65}{64}Q)}\\ &+ \sum_{\substack{Q^\#\text{ keystone}\\ S_1Q^\#\subset\frac{65}{64}\widehat{Q}}}\ \sum_{|\beta|\le m-1}|\partial^\beta(J_{x_{Q^\#}}H - R_{Q^\#})(x_{Q^\#})|^p\sum_{\substack{\frac{65}{64}Q\subset\frac{65}{64}\widehat{Q}\\ \mathcal{K}(Q)=Q^\#}}\delta_Q^{(|\beta|-m)p+n}\\ &+ \sum_{\substack{Q'\leftrightarrow Q''\\ \frac{65}{64}Q'\subset\frac{65}{64}\widehat{Q}\\ \frac{65}{64}Q''\subset\frac{65}{64}\widehat{Q}}}\delta_{Q'}^{-\eta p}\sum_{|\beta|\le m-1}|\partial^\beta(J_{x_{Q'}}H - J_{x_{Q''}}H)(x_{Q'})|^p\sum_{\substack{\frac{65}{64}Q\subset\frac{65}{64}\widehat{Q}\\ \ell:Q(\ell)=Q'}}\delta_Q^{(\eta+|\beta|-m)p+n}.\end{aligned}$$

Now, for fixed $Q^\#$ we have

$$\sum_{\substack{\frac{65}{64}Q\subset\frac{65}{64}\widehat{Q}\\ \mathcal{K}(Q)=Q^\#}}\delta_Q^{(|\beta|-m)p+n} \le \sum_{\substack{Q\text{ dyadic}\\ Q^\#\subset CQ}}\delta_Q^{(|\beta|-m)p+n} \le C\cdot\left[\delta_{Q^\#}\right]^{(|\beta|-m)p+n}.$$

Also, from Remark 1, which can be found after the KEYSTONE-ORACLE, for fixed Q' we have

$$\sum_{\substack{\frac{65}{64}Q\subset\frac{65}{64}\widehat{Q}\\ \ell:Q(\ell)=Q'}}\delta_Q^{(\eta+|\beta|-m)p+n}\le C\sum_{\substack{\frac{65}{64}Q\subset\frac{65}{64}\widehat{Q}\\ \exists\ell,\ Q(\ell)=Q'}}\delta_Q^{(\eta+|\beta|-m)p+n}$$
$$\le C\sum_{\substack{Q\text{ dyadic}\\ Q'\subset CQ}}\delta_Q^{(\eta+|\beta|-m)p+n}\le C\cdot[\delta_{Q'}]^{(\eta+|\beta|-m)p+n}.$$

Therefore,

$$\begin{aligned}X\lesssim&\sum_{\frac{65}{64}Q\subset\frac{65}{64}\widehat{Q}}\|H\|^p_{\mathbb{X}(\frac{65}{64}Q)}\\&+\sum_{\substack{Q^\#\text{ keystone}\\ S_1Q^\#\subset\frac{65}{64}\widehat{Q}}}\sum_{|\beta|\le m-1}|\partial^\beta(J_{x_{Q^\#}}H-R_{Q^\#})(x_{Q^\#})|^p\cdot\left[\delta_{Q^\#}\right]^{(|\beta|-m)p+n}\\&+\sum_{\substack{Q'\leftrightarrow Q''\\ \frac{65}{64}Q',\frac{65}{64}Q''\subset\frac{65}{64}\widehat{Q}}}\sum_{|\beta|\le m-1}|\partial^\beta(J_{x_{Q'}}H-J_{x_{Q''}}H)(x_{Q'})|^p\cdot[\delta_{Q'}]^{(|\beta|-m)p+n}.\end{aligned}$$

We first estimate the terms in the second sum on the right-hand side above. For fixed $Q^\#$, the Sobolev-type estimate which is stated in Lemma 10 of [1] implies that

$$\sum_{|\beta|\le m-1}|\partial^\beta(J_{x_{Q^\#}}H-R_{Q^\#})(x_{Q^\#})|^p\cdot\left[\delta_{Q^\#}\right]^{(|\beta|-m)p+n}$$
$$\lesssim\|H\|^p_{\mathbb{X}(\frac{65}{64}Q^\#)}+\delta_{Q^\#}^{-mp}\|H-R_{Q^\#}\|^p_{L^p(\frac{65}{64}Q^\#)}.$$

To estimate the terms from the third sum, we apply Lemma 12 of [1], using as the rectangular boxes $B_1=\frac{65}{64}Q'$ and $B_2=\frac{65}{64}Q''$, where $Q',Q''\in CZ(\mathcal{A}^-)$ satisfy $Q'\leftrightarrow Q''$. Thus, we obtain the estimate

$$\sum_{|\beta|\le m-1}|\partial^\beta(J_{x_{Q'}}H-J_{x_{Q''}}H)(x_{Q'})|^p\cdot[\delta_{Q'}]^{(|\beta|-m)p+n}\lesssim\|H\|^p_{\mathbb{X}(\frac{65}{64}Q')}+\|H\|^p_{\mathbb{X}(\frac{65}{64}Q'')}.$$

(Here, we use the fact that $|x_{Q'}-x_{Q''}|\le C\delta_{Q'}$ and that $\frac{65}{64}Q'\cap\frac{65}{64}Q''\ne\emptyset$.)

Thus, we have

$$\begin{aligned}X\lesssim&\sum_{\frac{65}{64}Q\subset\frac{65}{64}\widehat{Q}}\|H\|^p_{\mathbb{X}(\frac{65}{64}Q)}+\sum_{\substack{Q'\leftrightarrow Q''\\ \frac{65}{64}Q',\frac{65}{64}Q''\subset\frac{65}{64}\widehat{Q}}}\left[\|H\|^p_{\mathbb{X}(\frac{65}{64}Q')}+\|H\|^p_{\mathbb{X}(\frac{65}{64}Q'')}\right]\\&+\sum_{\substack{Q^\#\text{ keystone}\\ S_1Q^\#\subset\frac{65}{64}\widehat{Q}}}\|H\|^p_{\mathbb{X}(\frac{65}{64}Q^\#)}+\sum_{\substack{Q^\#\text{ keystone}\\ S_1Q^\#\subset\frac{65}{64}\widehat{Q}}}\delta_{Q^\#}^{-mp}\|H-R_{Q^\#}\|^p_{L^p(\frac{65}{64}Q^\#)}.\end{aligned}$$

From the bounded overlap of the cubes $\frac{65}{64}Q$, $Q \in CZ(\mathcal{A}^-)$, the previous estimate implies that

$$X \lesssim \|H\|^p_{\mathbb{X}(\frac{65}{64}\widehat{Q})} + \sum_{\substack{Q^\# \text{ keystone} \\ S_1 Q^\# \subset \frac{65}{64}\widehat{Q}}} \delta_{Q^\#}^{-mp} \|H - R_{Q^\#}\|^p_{L^p(\frac{65}{64}Q^\#)}.$$

This completes the proof of Proposition 6. □

We are now prepared to prove the conditional inequality.

We seek an estimate on $\big[M_{\widehat{Q}}(f,P)\big]^p$, which is a sum of terms **(I)**–**(IV)** (see (1.65)–(1.68)). We first apply Lemma 7 of [1] to estimate the summands appearing in (1.66), (1.67), (1.68) in terms of L^p norms of polynomials. We then replace (1.66) by a sum over a larger collection of pairs (Q', Q''), as below. Thus, we obtain

$$\begin{aligned}
\big[M_{\widehat{Q}}(f,P)\big]^p \le C(t_G)\cdot \Bigg[& \sum_{\substack{Q\in CZ_{main}(\mathcal{A}^-) \\ Q\subset(1+100t_G)\widehat{Q}}} \big[M_{(Q,\mathcal{A}^-)}(f, R^{\widehat{Q}}_Q)\big]^p \\
& + \sum_{\substack{Q',Q''\in CZ(\mathcal{A}^-) \\ Q',Q''\subset(1+100t_G)\widehat{Q} \\ Q'\leftrightarrow Q''}} \delta_{Q'}^{-mp}\|R^{\widehat{Q}}_{Q'} - R^{\widehat{Q}}_{Q''}\|^p_{L^p(Q')} \\
& + \sum_{\substack{Q\in CZ(\mathcal{A}^-) \\ Q\subset(1+100t_G)\widehat{Q} \\ \delta_Q \ge t_G^2\cdot\delta_{\widehat{Q}}}} \delta_{\widehat{Q}}^{-mp}\|R^{\widehat{Q}}_Q - P\|^p_{L^p(\widehat{Q})} + \delta_{\widehat{Q}}^{-mp}\|R^{\widehat{Q}}_{Q_{sp}} - P\|^p_{L^p(\widehat{Q})}\Bigg].
\end{aligned} \tag{1.142}$$

We pick a function H as in Proposition 5. Our estimates proceed in three stages below.

Stage I. We bound the relevant summands in (1.142).

We consider $Q, Q', Q'' \in CZ(\mathcal{A}^-)$ that satisfy $Q, Q', Q'' \subset (1+100t_G)\widehat{Q}$ and $Q' \leftrightarrow Q''$. We impose either the assumption $Q \in CZ_{main}(\mathcal{A}^-)$ or the assumption $\delta_Q \ge t_G^2 \cdot \delta_{\widehat{Q}}$, depending on which type of summand we seek to bound.

Assume first that $Q \in CZ_{main}(\mathcal{A}^-)$. Then the right-hand estimate in (1.20) implies that

$$M_{(Q,\mathcal{A}^-)}(f, R^{\widehat{Q}}_Q) \le C\cdot\|(f, R^{\widehat{Q}}_Q)\|_{\frac{65}{64}Q} \le C\cdot\big[\|H\|_{\mathbb{X}(\frac{65}{64}Q)} + \delta_Q^{-m}\|H - R^{\widehat{Q}}_Q\|_{L^p(\frac{65}{64}Q)}\big].$$

Here, in the last inequality, we use the definition of the trace seminorm and recall that $H = f$ on $E \cap \frac{65}{64}\widehat{Q}$.

On the other hand, assume that $\delta_Q \ge t_G^2\delta_{\widehat{Q}}$. We apply the triangle inequality, and then we apply estimate (2.5) in Lemma 10 of [1] (a consequence of the Sobolev inequality). Note that $\frac{65}{64}Q \subset \frac{65}{64}\widehat{Q}$, so the application of (2.5) given below is

justified. This gives

$$\begin{aligned}\delta_{\widehat{Q}}^{-m}\|R_Q^{\widehat{Q}}-P\|_{L^p(\widehat{Q})} &\le \delta_{\widehat{Q}}^{-m}\|H-P\|_{L^p(\frac{65}{64}\widehat{Q})}+\delta_{\widehat{Q}}^{-m}\|R_Q^{\widehat{Q}}-H\|_{L^p(\frac{65}{64}\widehat{Q})}\\ &\le \delta_{\widehat{Q}}^{-m}\|H-P\|_{L^p(\frac{65}{64}\widehat{Q})}+C\cdot\left[\delta_Q^{-m}\|R_Q^{\widehat{Q}}-H\|_{L^p(\frac{65}{64}Q)}+\|H\|_{\mathbb{X}(\frac{65}{64}\widehat{Q})}\right].\end{aligned}$$

We now consider the summands in (1.142) which are indexed by pairs (Q', Q''). Using Lemma 11 of [1], we obtain the bound

$$\begin{aligned}\delta_{Q'}^{-m}\|R_{Q'}^{\widehat{Q}}-R_{Q''}^{\widehat{Q}}\|_{L^p(Q')} \le C\cdot\big[&\delta_{Q'}^{-m}\|R_{Q'}^{\widehat{Q}}-H\|_{L^p(\frac{65}{64}Q')}\\ &+\delta_{Q''}^{-m}\|H-R_{Q''}^{\widehat{Q}}\|_{L^p(\frac{65}{64}Q'')}+\|H\|_{\mathbb{X}(\frac{65}{64}Q')}+\|H\|_{\mathbb{X}(\frac{65}{64}Q'')}\big].\end{aligned}$$

Now, at last, we consider the final term on the right-hand side of (1.142). Since $Q_{sp}\subset\widehat{Q}$, we have $\frac{65}{64}Q_{sp}\subset\frac{65}{64}\widehat{Q}$. We now apply a Sobolev-type estimate that is stated in Lemma 10 of [1]. Thus, we obtain

$$\begin{aligned}\delta_{\widehat{Q}}^{-m}\|P-R_{Q_{sp}}^{\widehat{Q}}\|_{L^p(\frac{65}{64}\widehat{Q})} &\le \delta_{\widehat{Q}}^{-m}\|H-P\|_{L^p(\frac{65}{64}\widehat{Q})}+\delta_{\widehat{Q}}^{-m}\|R_{Q_{sp}}^{\widehat{Q}}-H\|_{L^p(\frac{65}{64}\widehat{Q})}\\ &\le \delta_{\widehat{Q}}^{-m}\|H-P\|_{L^p(\frac{65}{64}\widehat{Q})}\\ &\quad+C\cdot\left[\delta_{Q_{sp}}^{-m}\|R_{Q_{sp}}^{\widehat{Q}}-H\|_{L^p(\frac{65}{64}Q_{sp})}+\|H\|_{\mathbb{X}(\frac{65}{64}\widehat{Q})}\right].\end{aligned}$$

We combine (1.142) with the previous four estimates to obtain

$$\begin{aligned}\left[M_{\widehat{Q}}(f,P)\right]^p \le C(t_G)\cdot\Big(&\|H\|_{\mathbb{X}(\frac{65}{64}\widehat{Q})}^p+\delta_{\widehat{Q}}^{-mp}\|H-P\|_{L^p(\frac{65}{64}\widehat{Q})}^p\\ &+\sum_{Q\subset(1+100t_G)\widehat{Q}}\left[\|H\|_{\mathbb{X}(\frac{65}{64}Q)}^p+\delta_Q^{-mp}\|H-R_Q^{\widehat{Q}}\|_{L^p(\frac{65}{64}Q)}^p\right]\Big).\end{aligned}\tag{1.143}$$

Stage II. Observe that

$$\sum_{Q\subset(1+100t_G)\widehat{Q}}\|H\|_{\mathbb{X}(\frac{65}{64}Q)}^p\le C\cdot\|H\|_{\mathbb{X}(\frac{65}{64}\widehat{Q})}^p.$$

Indeed, we have $\frac{65}{64}Q\subset\frac{65}{64}\widehat{Q}$ for any cube $Q\in CZ(\mathcal{A}^-)$ arising above (see Lemma 11); hence, the desired estimate is a consequence of the fact that the cubes $\frac{65}{64}Q$, with $Q\in CZ(\mathcal{A}^-)$, have bounded overlap.

The number of cubes $Q\in CZ(\mathcal{A}^-)$ such that $Q\subset(1+100t_G)\widehat{Q}$ and $\delta_Q\ge t_G\delta_{\widehat{Q}}$ is bounded by a constant $C(t_G)$. Hence,

$$\sum_{\substack{Q\subset(1+100t_G)\widehat{Q}\\ \delta_Q\ge t_G\delta_{\widehat{Q}}}}\delta_Q^{-mp}\|H-R_Q^{\widehat{Q}}\|_{L^p(\frac{65}{64}Q)}^p\le C(t_G)\cdot\delta_{\widehat{Q}}^{-mp}\|H-P\|_{L^p(\frac{65}{64}\widehat{Q})}^p\quad\text{(see (1.63))}.$$

On the other hand, by our definition of $R_Q^{\widehat{Q}}$ (see (1.63)), we obtain

$$\sum_{\substack{Q\subset(1+100t_G)\widehat{Q}\\ \delta_Q<t_G\delta_{\widehat{Q}}}}\delta_Q^{-mp}\|H-R_Q^{\widehat{Q}}\|_{L^p(\frac{65}{64}Q)}^p=\sum_{\substack{Q\subset(1+100t_G)\widehat{Q}\\ \delta_Q<t_G\delta_{\widehat{Q}}}}\delta_Q^{-mp}\|H-R_{\mathcal{K}(Q)}^{\#}\|_{L^p(\frac{65}{64}Q)}^p.$$

We combine (1.143) and the previous three estimates to obtain

$$\begin{aligned}\big[M_{\widehat{Q}}(f,P)\big]^p &\le C(t_G)\cdot\Big(\|H\|^p_{\mathbb{X}(\frac{65}{64}\widehat{Q})}+\delta_{\widehat{Q}}^{-mp}\|H-P\|^p_{L^p(\frac{65}{64}\widehat{Q})}\\ &\qquad+\sum_{\substack{Q\subset(1+100t_G)\widehat{Q}\\ \delta_Q<t_G\delta_{\widehat{Q}}}}\delta_Q^{-mp}\|H-R^{\#}_{\mathcal{K}(Q)}\|^p_{L^p(\frac{65}{64}Q)}\Big)\\ &\le C(t_G)\cdot\Big(\|H\|^p_{\mathbb{X}(\frac{65}{64}\widehat{Q})}+\delta_{\widehat{Q}}^{-mp}\|H-P\|^p_{L^p(\frac{65}{64}\widehat{Q})}\\ &\qquad+\sum_{\substack{Q^{\#}\ \text{keystone}\\ S_1Q^{\#}\subset\frac{65}{64}\widehat{Q}}}(\delta_{Q^{\#}})^{-mp}\|H-R^{\#}_{Q^{\#}}\|^p_{L^p(S_1Q^{\#})}\Big)\end{aligned}\tag{1.144}$$

(see Proposition 6).

Stage III. Let $Q^{\#}\in CZ(\mathcal{A}^-)$ be a keystone cube with $S_1Q^{\#}\subset\frac{65}{64}\widehat{Q}$. Then, as stated in Proposition 5, we have $\partial^{\alpha}H(x_{Q^{\#}})=\partial^{\alpha}P(x_{Q^{\#}})$ for all $\alpha\in\mathcal{A}$. Thus, by Proposition 2, we have

$$(\delta_{Q^{\#}})^{-mp}\|H-R^{\#}_{Q^{\#}}\|^p_{L^p(S_1Q^{\#})}\lesssim\|H\|^p_{\mathbb{X}(S_1Q^{\#})}.\tag{1.145}$$

From Lemma 8, we recall that the cubes $S_1Q^{\#}$ ($Q^{\#}$ keystone) have bounded overlap. Thus, (1.144) and (1.145) imply that

$$\begin{aligned}M_{\widehat{Q}}(f,P)^p &\le C(t_G)\cdot\big(\|H\|^p_{\mathbb{X}(\frac{65}{64}\widehat{Q})}+\|H-P\|^p_{L^p(\frac{65}{64}\widehat{Q})}\delta_{\widehat{Q}}^{-mp}\big)\\ &\le C(t_G)\cdot\Lambda^{(2D+1)p}\cdot\|(f,P)\|^p_{\frac{65}{64}\widehat{Q}}\qquad\text{(see Proposition 5).}\end{aligned}\tag{1.146}$$

Recall that $\epsilon^{\overline{\kappa}}\Lambda^{100D}\le\epsilon^{\overline{\kappa}/2}$ and $\overline{\kappa}\le\kappa_2\le 1$ (see (1.109)). Hence, $\Lambda^{2D+1}\le\epsilon^{-\overline{\kappa}/2}\le\epsilon^{-1}$. This shows that

$$M_{\widehat{Q}}(f,P)\le C(t_G)\cdot(1/\epsilon)\cdot\|(f,P)\|_{\frac{65}{64}\widehat{Q}}$$

This completes the proof of the conditional inequality. This completes the proof of Proposition 4. □

We fix $t_G>0$, depending only on m, n, and p, small enough so that the above results hold. Since we have fixed the constant t_G, all the previous constants of the form $C(t_G)$ or $c(t_G)$ become universal constants C or c. In particular, the constant $a_{new}=a_{new}(t_G)$ from Lemma 11 depends only on m, n, and p. We set

$$a(\mathcal{A})=a_{new}.\tag{1.147}$$

Recall the definition of the convex set $\overline{\overline{\sigma}}(\widehat{Q})$ in (1.69).

Just for the moment, let $\epsilon=\epsilon_0$ be a small enough constant depending only on m, n, and p. From Proposition 4 we obtain the following result.

Proposition 7. *There exist universal constants $\epsilon_0 > 0$ and $C \geq 1$ such that the following holds.*

Let $\widehat{Q}$ be a testing cube. Then the following conclusions hold.

(Unconditional inequality) $\|(f, P)\|_{(1+a(\mathcal{A}))\widehat{Q}} \leq C M_{\widehat{Q}}(f, P)$.

(Conditional inequality) *If $3\widehat{Q}$ is tagged with $(\mathcal{A}, \epsilon_0)$, then*

$$M_{\widehat{Q}}(f, P) \leq C\|(f, P)\|_{\frac{65}{64}\widehat{Q}}.$$

(Unconditional inclusion) $\overline{\overline{\sigma}}(\widehat{Q}) \subset C\sigma((1 + a(\mathcal{A}))\widehat{Q})$.

(Conditional inclusion) *If $3\widehat{Q}$ is tagged with $(\mathcal{A}, \epsilon_0)$, then $\sigma(\frac{65}{64}\widehat{Q}) \subset C\overline{\overline{\sigma}}(\widehat{Q})$.*

Once again, let ϵ be a small parameter. As usual, we assume that ϵ is less than a small enough constant depending only on m, n, and p.

1.5. Tools to fill the gap between geometrically interesting cubes

For the results in this section, the reader may wish to review the definition of testing cubes (see Definition 2).

Proposition 8. *Let $\widehat{Q}$ be a testing cube. If*

$$\Big[\#\Big(\frac{65}{64}\widehat{Q} \cap E\Big) \leq 1 \text{ or } \overline{\overline{\sigma}}(\widehat{Q}) \text{ has an } (\mathcal{A}', x_{\widehat{Q}}, \epsilon, \delta_{\widehat{Q}})\text{-basis for some } \mathcal{A}' \leq \mathcal{A}\Big],$$

then $(1 + a(\mathcal{A}))\widehat{Q}$ is tagged with $(\mathcal{A}, \epsilon^\kappa)$. Otherwise, no cube containing $3\widehat{Q}$ is tagged with $(\mathcal{A}, \epsilon^{1/\kappa})$. Here, κ is a universal constant.

Proof. If $\#(\frac{65}{64}\widehat{Q} \cap E) \leq 1$, then $(1 + a(\mathcal{A}))\widehat{Q}$ is tagged with $(\mathcal{A}, \epsilon)$.

Suppose $\overline{\overline{\sigma}}(\widehat{Q})$ has an $(\mathcal{A}', x_{\widehat{Q}}, \epsilon, \delta_{\widehat{Q}})$-basis for some $\mathcal{A}' \leq \mathcal{A}$. Proposition 7 gives that $\overline{\overline{\sigma}}(\widehat{Q}) \subset C\sigma((1+a(\mathcal{A}))\widehat{Q})$. Thus, $\sigma((1+a(\mathcal{A}))\widehat{Q})$ has an $(\mathcal{A}', x_{\widehat{Q}}, C\epsilon, \delta_{\widehat{Q}})$-basis.

Therefore, $(1 + a(\mathcal{A}))\widehat{Q}$ is tagged with $(\mathcal{A}, \epsilon^\kappa)$. Here, we can arrange that $C\epsilon \leq \epsilon^\kappa$ by taking ϵ sufficiently small.

This proves the first part of Proposition 8.

On the other hand, suppose $Q \supset 3\widehat{Q}$ and suppose Q is tagged with $(\mathcal{A}, \epsilon^{1/\kappa'})$, for some $\kappa' > 0$ to be picked below. Then $3\widehat{Q}$ is tagged with $(\mathcal{A}, \epsilon^{\kappa/\kappa'})$, thanks to Lemma 28 from [1]. Hence, from Proposition 7 we see that

$$\sigma\Big(\frac{65}{64}\widehat{Q}\Big) \subset C \cdot \overline{\overline{\sigma}}(\widehat{Q}). \tag{1.148}$$

Recall that $\frac{65}{64}\widehat{Q} \subset Q$ and that Q is tagged with $(\mathcal{A}, \epsilon^{1/\kappa'})$. Thus, Lemma 28 from [1] shows that $\frac{65}{64}\widehat{Q}$ is tagged with $(\mathcal{A}, \epsilon^{\kappa/\kappa'})$. This means that either $\#(\frac{65}{64}\widehat{Q} \cap E) \leq 1$ or $\sigma(\frac{65}{64}\widehat{Q})$ has an $(\mathcal{A}', x_{\widehat{Q}}, \epsilon^{\kappa/\kappa'}, \delta_{\widehat{Q}})$-basis, with $\mathcal{A}' \leq \mathcal{A}$.

Thus, (1.148) implies that

$$\#\Big(\frac{65}{64}\widehat{Q}\cap E\Big)\le 1 \ \text{ or } \ \overline{\overline{\sigma}}(\widehat{Q}) \text{ has an } (\mathcal{A}', x_{\widehat{Q}}, C\epsilon^{\kappa/\kappa'}, \delta_{\widehat{Q}})\text{-basis.}$$

Hence, either $\#(\frac{65}{64}\widehat{Q}\cap E)\le 1$ or $\overline{\overline{\sigma}}(\widehat{Q})$ has an $(\mathcal{A}', x_{\widehat{Q}}, \epsilon, \delta_{\widehat{Q}})$-basis for some $\mathcal{A}'\le\mathcal{A}$. Here, we have set $\kappa'=\kappa/2$, with κ as in Lemma 28 from [1]; note that $C\epsilon^{\kappa/\kappa'}=C\epsilon^2\le\epsilon$.

This completes the proof of Proposition 8. □

Suppose that $\widehat{Q}_1\subset\widehat{Q}_2$ are testing cubes. We want to understand the tagging of $3\widehat{Q}_2$ in terms of the convex symmetric set $\overline{\overline{\sigma}}(\widehat{Q}_1)$.

Proposition 9. *Suppose that $\widehat{Q}_1\subset\widehat{Q}_2$ are testing cubes. We assume that*

$$\#(3\widehat{Q}_2\cap E)\ge 2 \quad \textit{and} \tag{1.149}$$

$$(1+a(\mathcal{A}))\widehat{Q}_1\cap E = 3\widehat{Q}_2\cap E. \tag{1.150}$$

If $\overline{\overline{\sigma}}(\widehat{Q}_1)$ has an $(\mathcal{A}', x_{\widehat{Q}_1}, \epsilon, \delta_{\widehat{Q}_2})$-basis, then $3\widehat{Q}_2$ is tagged with $(\mathcal{A}', \epsilon^\kappa)$ for a universal constant κ.

Proof. Let $(P_\alpha)_{\alpha\in\mathcal{A}'}$ be an $(\mathcal{A}', x_{\widehat{Q}_1}, \epsilon, \delta_{\widehat{Q}_2})$-basis for $\overline{\overline{\sigma}}(\widehat{Q}_1)$. Thus,

$$P_\alpha\in\epsilon\,\delta_{\widehat{Q}_2}^{-(m-n/p-|\alpha|)}\overline{\overline{\sigma}}(\widehat{Q}_1) \qquad (\alpha\in\mathcal{A}') \tag{1.151}$$

$$\partial^\beta P_\alpha(x_{\widehat{Q}_1}) = \delta_{\beta\alpha} \qquad (\beta,\alpha\in\mathcal{A}') \tag{1.152}$$

$$|\partial^\beta P_\alpha(x_{\widehat{Q}_1})|\le\epsilon\,\delta_{\widehat{Q}_2}^{|\alpha|-|\beta|} \qquad (\alpha\in\mathcal{A}',\ \beta\in\mathcal{M},\ \beta>\alpha). \tag{1.153}$$

The unconditional inclusion and inclusion (2.12) in Section 2.4 of [1] show that

$$\overline{\overline{\sigma}}(\widehat{Q}_1)\subset C\sigma((1+a(\mathcal{A}))\widehat{Q}_1)\subset C\big[\sigma((1+a(\mathcal{A}))\widehat{Q}_1)+\mathcal{B}(x_{\widehat{Q}_1}, 3\delta_{\widehat{Q}_2})\big]\subset C'\sigma(3\widehat{Q}_2).$$

(The application of (2.12) is justified because we assume here that $(1+a(\mathcal{A}))\widehat{Q}_1\cap E=3\widehat{Q}_2\cap E$.)

Thus, (1.151) implies that

$$P_\alpha\in C\,\epsilon\,\delta_{3\widehat{Q}_2}^{-(m-n/p-|\alpha|)}\cdot\sigma(3\widehat{Q}_2) \quad (\alpha\in\mathcal{A}').$$

Thus, due to (1.152) and (1.153), we see that $(P_\alpha)_{\alpha\in\mathcal{A}'}$ is an $(\mathcal{A}', x_{\widehat{Q}_1}, C\epsilon, \delta_{3\widehat{Q}_2})$-basis for $\sigma(3\widehat{Q}_2)$.

It follows from Lemma 27 in [1] that $\sigma(3\widehat{Q}_2)$ has an $(\mathcal{A}'', x_{\widehat{Q}_2}, \epsilon^\kappa, \delta_{3\widehat{Q}_2})$-basis, for some $\mathcal{A}''\le\mathcal{A}'$. By definition, this means that the cube $3\widehat{Q}_2$ is tagged with $(\mathcal{A}', \epsilon^\kappa)$, completing the proof of Proposition 9. □

Corollary 1. *Suppose that* $\widehat{Q}_1 \subset \widehat{Q}_2$ *are testing cubes, and that* (1.149) *and* (1.150) *hold.*

Suppose $\overline{\overline{\sigma}}(\widehat{Q}_1)$ *has an* $(\mathcal{A}', x_{\widehat{Q}_1}, \epsilon, \delta_{\widehat{Q}_2})$*-basis for some* $\mathcal{A}' \leq \mathcal{A}$*. Then* $3\widehat{Q}_2$ *is tagged with* $(\mathcal{A}, \epsilon^\kappa)$ *for a universal constant* κ*.*

Proof. Proposition 9 tells us that $3\widehat{Q}_2$ is tagged with $(\mathcal{A}', \epsilon^\kappa)$. This trivially implies that $3\widehat{Q}_2$ is tagged with $(\mathcal{A}, \epsilon^\kappa)$. □

Proposition 10. *Suppose that* $\widehat{Q}_1 \subset \widehat{Q}_2$ *are testing cubes, and that* (1.149) *and* (1.150) *hold.*

Suppose $3\widehat{Q}_2$ *is tagged with* $(\mathcal{A}, \epsilon)$*. Then* $\overline{\overline{\sigma}}(\widehat{Q}_1)$ *has an* $(\mathcal{A}', x_{\widehat{Q}_1}, \epsilon^{\kappa'}, \delta_{\widehat{Q}_2})$*-basis, for some* $\mathcal{A}' \leq \mathcal{A}$*. Here,* κ' *is a universal constant.*

Proof. We have $3\widehat{Q}_1 \subset 3\widehat{Q}_2$, so Lemma 28 from [1] tells us that $3\widehat{Q}_1$ is tagged with $(\mathcal{A}, \epsilon^\kappa)$. Hence, by the conditional inclusion, we have

$$c \cdot \sigma\Big(\frac{65}{64}\widehat{Q}_1\Big) \subset \overline{\overline{\sigma}}(\widehat{Q}_1). \tag{1.154}$$

Next note that $\frac{65}{64}\widehat{Q}_1 \cap E = 3\widehat{Q}_2 \cap E$, and that $\frac{65}{64}\widehat{Q}_1 \subset 3\widehat{Q}_2$. Therefore, Lemma 15 in [1] gives the inclusion

$$\sigma(3\widehat{Q}_2) \subset C \cdot \Big[\sigma\Big(\frac{65}{64}\widehat{Q}_1\Big) + \mathcal{B}(x_{\widehat{Q}_2}, \delta_{3\widehat{Q}_2})\Big]. \tag{1.155}$$

(Since $|x_{\widehat{Q}_1} - x_{\widehat{Q}_2}| \leq \delta_{\widehat{Q}_2}$, it follows that $\mathcal{B}(x_{\widehat{Q}_1}, \delta_{3\widehat{Q}_2}) \subset C\mathcal{B}(x_{\widehat{Q}_2}, \delta_{3\widehat{Q}_2})$. This shows that (1.155) follows from the conclusion of Lemma 15 in [1].)

Now, $3\widehat{Q}_2$ is assumed to be tagged with $(\mathcal{A}, \epsilon)$, and $\#(3\widehat{Q}_2 \cap E)$ is assumed to be at least 2. Hence, by definition, $\sigma(3\widehat{Q}_2)$ has an $(\mathcal{A}', x_{\widehat{Q}_2}, \epsilon, \delta_{3\widehat{Q}_2})$-basis for some $\mathcal{A}' \leq \mathcal{A}$.

By Lemma 25 of [1], we learn that

$$\sigma(3\widehat{Q}_2) \text{ has an } (\mathcal{A}'', x_{\widehat{Q}_2}, \epsilon^\kappa, \delta_{3\widehat{Q}_2}, \Lambda)\text{-basis, for some } \mathcal{A}'' \leq \mathcal{A}' \leq \mathcal{A}, \tag{1.156}$$

such that $\epsilon^\kappa \Lambda^{100D} \leq \epsilon^{\kappa/2}$ and $\kappa \in [\kappa_1, \kappa_2]$. Here, $\kappa_1, \kappa_2 > 0$ are universal constants.

Inclusions (1.154) and (1.155) show that

$$\sigma(3\widehat{Q}_2) \subset C'' \cdot [\overline{\overline{\sigma}}(\widehat{Q}_1) + \mathcal{B}(x_{\widehat{Q}_2}, \delta_{3\widehat{Q}_2})]. \tag{1.157}$$

From (1.156), (1.157), and Lemma 23 in [1], we see that

$$\overline{\overline{\sigma}}(\widehat{Q}_1) \text{ has an } (\mathcal{A}'', x_{\widehat{Q}_2}, C\epsilon^\kappa\Lambda, \delta_{3\widehat{Q}_2}, C\Lambda)\text{-basis.}$$

We now apply Lemma 26 in [1]. Thus,

$$\overline{\overline{\sigma}}(\widehat{Q}_1) \text{ has an } (\mathcal{A}'', x_{\widehat{Q}_1}, C\epsilon^\kappa\Lambda^{2D+2}, \delta_{3\widehat{Q}_2}, C\Lambda^{2D+1})\text{-basis.}$$

Since $C\epsilon^\kappa\Lambda^{2D+2} \leq \epsilon^{\kappa/3} \leq \epsilon^{\kappa_1/3}$, it follows that

$$\overline{\overline{\sigma}}(\widehat{Q}_1) \text{ has an } (\mathcal{A}'', x_{\widehat{Q}_1}, \epsilon^{\kappa_1/3}, \delta_{\widehat{Q}_2})\text{-basis.} \tag{1.158}$$

(Here, the passage from $\delta_{3\widehat{Q}_2}$ to $\delta_{\widehat{Q}_2}$ is harmless; it just increases the constant "C" in $C\epsilon^{\kappa}\Lambda^{2D+2}$.)

Since $\mathcal{A}'' \le \mathcal{A}$, (1.158) is the conclusion of Proposition 10. □

Combining the results of Propositions 9, 10, we now prove the following.

Proposition 11. *Suppose that $\widehat{Q}_1 \subset \widehat{Q}_2$ are testing cubes and that (1.149), (1.150) hold. Then*

(A) *If $\overline{\overline{\sigma}}(\widehat{Q}_1)$ has an $(\mathcal{A}', \chi_{\widehat{Q}_1}, \epsilon, \delta_{\widehat{Q}_2})$-basis for some $\mathcal{A}' \le \mathcal{A}$, then $(1 + a(\mathcal{A}))\widehat{Q}_2$ is tagged with $(\mathcal{A}, \epsilon^{\kappa})$.*

(B) *If some cube containing $3\widehat{Q}_2$ is tagged with $(\mathcal{A}, \epsilon)$, then $\overline{\overline{\sigma}}(\widehat{Q}_1)$ has an $(\mathcal{A}', \chi_{\widehat{Q}_1}, \epsilon^{\kappa}, \delta_{\widehat{Q}_2})$-basis for some $\mathcal{A}' \le \mathcal{A}$.*

Here, κ is a universal constant.

Proof. First we check (A). If $\overline{\overline{\sigma}}(\widehat{Q}_1)$ has an $(\mathcal{A}', \chi_{\widehat{Q}_1}, \epsilon, \delta_{\widehat{Q}_2})$-basis with $\mathcal{A}' \le \mathcal{A}$, then according to Corollary 1, the cube $3\widehat{Q}_2$ is tagged with $(\mathcal{A}, \epsilon^{\kappa})$. Hence, by Lemma 28 from [1], $(1 + a(\mathcal{A}))\widehat{Q}_2$ is tagged with $(\mathcal{A}, \epsilon^{\kappa'})$, completing the proof of (A).

To check (B), let $Q' \supset 3\widehat{Q}_2$ be tagged with $(\mathcal{A}, \epsilon)$. By Lemma 28 from [1], $3\widehat{Q}_2$ is tagged with $(\mathcal{A}, \epsilon^{\kappa})$. Hence, by Proposition 10, $\overline{\overline{\sigma}}(\widehat{Q}_1)$ has an $(\mathcal{A}', \chi_{\widehat{Q}_1}, \epsilon^{\kappa'}, \delta_{\widehat{Q}_2})$-basis, for some $\mathcal{A}' \le \mathcal{A}$. This completes the proof of (B). □

We apply (A) with ϵ unchanged, and (B) with ϵ replaced by $\epsilon^{1/\kappa}$. Thus we obtain the following result.

Proposition 12. *Let $\widehat{Q} \subset Q$ be testing cubes. Assume that $\#(3Q \cap E) \ge 2$ and that $(1 + a(\mathcal{A}))\widehat{Q} \cap E = 3Q \cap E$.*

Then the following hold, for a universal constant κ.

(A) *If $\overline{\overline{\sigma}}(\widehat{Q})$ has an $(\mathcal{A}', \chi_{\widehat{Q}}, \epsilon, \delta_Q)$-basis for some $\mathcal{A}' \le \mathcal{A}$, then $(1 + a(\mathcal{A}))Q$ is tagged with $(\mathcal{A}, \epsilon^{\kappa})$.*

(B) *If $\overline{\overline{\sigma}}(\widehat{Q})$ does not have an $(\mathcal{A}', \chi_{\widehat{Q}}, \epsilon, \delta_Q)$-basis for any $\mathcal{A}' \le \mathcal{A}$, then no cube containing $3Q$ is tagged with $(\mathcal{A}, \epsilon^{1/\kappa})$.*

The final result in this section is the following algorithm.

ALGORITHM: OPTIMIZE BASIS

We perform one time work at most $CN \log N$ in space CN, after which we can answer queries as follows.

A query consists of a testing cube $\widehat{Q}$ and a set $\mathcal{A} \subset \mathcal{M}$

The response to the query $(\widehat{Q}, \mathcal{A})$ consists of a collection of pairwise disjoint intervals I_ℓ, and numbers a_ℓ and λ_ℓ $(\ell = 1, \dots, \ell_{\max})$, such that the following conditions hold.

- $\bigcup_\ell I_\ell = (0, \infty)$ and $\ell_{\max} \le C$.

- Let $\eta^{(\widehat{Q},\mathcal{A})}(\delta) := a_\ell \delta^{\lambda_\ell}$ for $\delta \in I_\ell$. Then we have:

  (A1) For each $\delta \in (0,\infty)$ there exists $\mathcal{A}' \leq \mathcal{A}$ such that $\overline{\overline{\sigma}}(\widehat{Q})$ has an $(\mathcal{A}', x_{\widehat{Q}}, \eta^{1/2}, \delta)$-basis for all $\eta > C \cdot \eta^{(\widehat{Q},\mathcal{A})}(\delta)$.

  (A2) For each $\delta \in (0,\infty)$ and any $\mathcal{A}' \leq \mathcal{A}$, $\overline{\overline{\sigma}}(\widehat{Q})$ does not have an $(\mathcal{A}', x_{\widehat{Q}}, \eta^{1/2}, \delta)$-basis with $\eta < c \cdot \eta^{(\widehat{Q},\mathcal{A})}(\delta)$.

  (A3) $c \cdot \eta^{(\widehat{Q},\mathcal{A})}(\delta_1) \leq \eta^{(\widehat{Q},\mathcal{A})}(\delta_2) \leq C \cdot \eta^{(\widehat{Q},\mathcal{A})}(\delta_1)$ whenever $\frac{1}{10}\delta_1 \leq \delta_2 \leq 10\delta_1$.

- To answer a query requires work at most $C \log N$.

Explanation. We compute a quadratic form $q_{\widehat{Q}}$ on $\mathcal{P}$ such that there exist universal constants $c > 0$ and $C \geq 1$ so that $\{q_{\widehat{Q}} \leq c\} \subset \overline{\overline{\sigma}}(\widehat{Q}) \subset \{q_{\widehat{Q}} \leq C\}$. (See the algorithm APPROXIMATE NEW TRACE NORM in Section 1.4.6.)

Processing the quadratic form $q^{\widehat{Q}}$ using the algorithm FIT BASIS TO CONVEX BODY (see Section 4.5 of [1]), we compute a piecewise-monomial function $\eta_*^{(\widehat{Q},\mathcal{A}')}(\cdot)$ for each $\mathcal{A}' \leq \mathcal{A}$. We guarantee that $\overline{\overline{\sigma}}(\widehat{Q})$ has an $(\mathcal{A}', x_{\widehat{Q}}, \eta^{1/2}, \delta)$-basis for all $\eta > C \cdot \eta_*^{(\widehat{Q},\mathcal{A}')}(\delta)$, but that $\overline{\overline{\sigma}}(\widehat{Q})$ does not have an $(\mathcal{A}', x_{\widehat{Q}}, \eta^{1/2}, \delta)$-basis for any $\eta < c \cdot \eta_*^{(\widehat{Q},\mathcal{A}')}(\delta)$.

We define

$$\eta^{(\widehat{Q},\mathcal{A})}(\delta) = \eta(\delta) = \min_{\mathcal{A}' \leq \mathcal{A}} \eta_*^{(\widehat{Q},\mathcal{A}')}(\delta) \quad \text{for } \delta \in (0,\infty).$$

It follows that $\overline{\overline{\sigma}}(\widehat{Q})$ has an $(\mathcal{A}', x_{\widehat{Q}}, \eta^{1/2}, \delta)$-basis for some $\mathcal{A}' \leq \mathcal{A}$ whenever $\eta > C \cdot \eta(\delta)$, but that $\overline{\overline{\sigma}}(\widehat{Q})$ does not have an $(\mathcal{A}', x_{\widehat{Q}}, \eta^{1/2}, \delta)$-basis for any $\mathcal{A}' \leq \mathcal{A}$ whenever $\eta < c \cdot \eta(\delta)$. Thus we have proven **(A1)** and **(A2)**.

Recall that $c \cdot \eta^{(\widehat{Q},\mathcal{A}')}(\delta_1) \leq \eta^{(\widehat{Q},\mathcal{A}')}(\delta_2) \leq C \cdot \eta^{(\widehat{Q},\mathcal{A}')}(\delta_1)$ for $\frac{1}{10}\delta_1 \leq \delta_2 \leq 10\delta_1$. Taking the minimum with respect to $\mathcal{A}' \leq \mathcal{A}$ in this inequality, we prove **(A3)**.

Recall that $\eta_*^{(\widehat{Q},\mathcal{A}')}(\delta) = a_{\ell,\mathcal{A}'} \delta^{\lambda_{\ell,\mathcal{A}'}}$ for $\delta \in I_{\ell,\mathcal{A}'}$, where the intervals $I_{\ell,\mathcal{A}'}$ $(\ell = 1, \ldots, \ell_{\max}(\mathcal{A}'))$ form a partition of $(0,\infty)$, for each $\mathcal{A}' \leq \mathcal{A}$. Here, $\ell_{\max}(\mathcal{A}')$ is bounded by a universal constant.

Thus we can partition $(0,\infty)$ into intervals I_ℓ $(\ell = 1, \ldots, \underline{\ell_{\max}})$, for which there exist real numbers a_ℓ, λ_ℓ such that $\eta(\delta) = a_\ell \delta^{\lambda_\ell}$ for $\delta \in I_\ell$. Moreover, $\underline{\ell_{\max}}$ is at most some universal constant. This follows because, for fixed real numbers a, b, λ, γ, the equation $a\delta^\lambda = b\delta^\gamma$ is satisfied either for at most one δ or for all $\delta \in (0,\infty)$. To compute the intervals I_ℓ and the numbers a_ℓ, λ_ℓ we solve at most C equations of the above type, and we make at most C comparisons between the functions $\eta^{(\widehat{Q},\mathcal{A}')}(\delta)$ $(\mathcal{A}' \leq \mathcal{A})$ to compute the minimum value on each of the relevant intervals. This completes the explanation of our algorithm. □

1.6. Computing lengthscales

We say that a dyadic cube $Q \subset \mathbb{R}^n$ is *geometrically interesting* provided that $\operatorname{diam}(3Q \cap E) \geq \lambda \delta_Q$, where we set $\lambda := 1/40$.

ALGORITHM: COMPUTE INTERESTING CUBES

We produce a tree T consisting of all the cubes $Q \in CZ(\mathcal{A}^-)$ that contain points of E, together with all testing cubes $\widehat{Q}$ for which $\operatorname{diam}(3\widehat{Q} \cap E) \geq \lambda \delta_{\widehat{Q}}$; as well as the unit cube Q°.

Here, T is a tree with respect to inclusion. We mark each internal node $Q \in T$ with pointers to its children, and we mark each node $Q \in T$ (except for the root) with a pointer to its parent.

The number of nodes in T is at most CN, and T can be computed with work at most $CN \log N$ in space CN.

We note that all the nodes of T are testing cubes. (This is immediate from the definition of testing cubes - see Definition 2.)

Explanation. We perform the one-time work of the BBD Tree (see Theorem 35 in [1]). Also, we compute representatives arising in the well-separated pairs decomposition using the algorithm MAKE WSPD (see Section 4.2 of [1]). This allows us to compute a sequence of tuples $(x'_\nu, x''_\nu) \in E \times E$ $(\nu = 1, \ldots, \nu_{\max})$ such that, for each $(x', x'') \in E \times E \setminus \{(x, x) : x \in E\}$ there exists ν such that

$$|x'_\nu - x'| + |x''_\nu - x''| \leq 10^{-10}|x' - x''|,$$

and $\nu_{\max} \leq CN$.

We execute the following loop:

- For each $\nu = 1, \ldots, \nu_{\max}$, we compute the sequence of all dyadic cubes $\widetilde{Q}$ such that $x'_\nu, x''_\nu \in 5\widetilde{Q}$ and $|x'_\nu - x''_\nu| \geq \frac{\lambda}{2}\delta_{\widetilde{Q}}$. (There are at most C such cubes for each ν.)

We denote the sequence of all cubes produced above, for all ν, by $Q_1, \ldots, Q_K$. We remove duplicates by sorting, which requires work at most $CN \log N$. Note that we have $K \leq CN$.

Let Q be a geometrically interesting cube. By definition, there exist $x', x'' \in 3Q \cap E$ with $|x' - x''| \geq \lambda \delta_Q$. Hence, there is some ν such that

$$|x'_\nu - x''_\nu| \geq \frac{9}{10}|x' - x''| \geq (\lambda/2)\delta_Q$$

and

$$|x'_\nu - x'| + |x''_\nu - x''| \leq \frac{1}{10}|x' - x''| \leq \frac{\delta_{3Q}}{10}.$$

Therefore, $x'_\nu, x''_\nu \in 5Q$, and hence Q belongs to the list $Q_1, \ldots, Q_K$.

We have proven that all geometrically interesting cubes belong to the list $Q_1, \ldots, Q_K$.

For each $k = 1, \ldots, K$, we compute $\operatorname{diam}(3Q_k \cap E)$ using the BBD Tree. (See Remark 36 in [1].) If $\operatorname{diam}(3Q_k \cap E) < \lambda \delta_{Q_k}$, then we remove Q_k from our list.

We also compute the cube in $CZ(\mathcal{A}^-)$ that contains the center of Q_k. If this cube strictly contains Q_k then we remove Q_k from our list. (This means that Q_k is not a testing cube.)

We denote the sequence of surviving cubes by $\widetilde{Q}_1, \ldots, \widetilde{Q}_{\widetilde{K}}$. As shown above, these are all the testing cubes that are geometrically interesting.

We form a list of all the cubes $Q \in CZ(\mathcal{A}^-)$ that contain points of E, the cubes $\widetilde{Q}_1, \ldots, \widetilde{Q}_{\widetilde{K}}$, and the unit cube Q°. There are at most CN such cubes. By sorting, we can remove duplicates. We organize this list into a tree T using the algorithm MAKE FOREST (see Section 4.1.5 of [1]). We obtain a tree (rather than a forest) because all the cubes have a common ancestor, namely Q°. This algorithm marks Q° as the root of T, and marks each non-root node with a pointer to its parent. In addition, we mark each internal node of T with pointers to its children.

One can easily check that the work and storage of our algorithm are as promised. □

Lemma 13. *Let $Q \subset Q^\circ$ be dyadic, with $\delta_Q \le 1/4$. Suppose that $3Q \cap E \neq \emptyset$ and $\operatorname{diam}(3Q^{++} \cap E) < \lambda\delta_{Q^{++}}$.*

Then $3Q^{++} \cap E = 3Q^{+} \cap E$. Here, Q^{++} denotes the dyadic parent of the dyadic parent of Q.

Proof. For the sake of contradiction, suppose that there exists $x \in E$ with $x \in 3Q^{++}$ and $x \notin 3Q^{+}$. Thus, for each $y \in E \cap 3Q$ we have

$$\operatorname{diam}(3Q^{++} \cap E) \ge |x-y| \ge \operatorname{dist}(\mathbb{R}^n \setminus 3Q^{+}, 3Q) \ge \frac{\delta_Q}{10} = \frac{\delta_{Q^{++}}}{40} = \lambda\delta_{Q^{++}}.$$

This yields a contradiction, completing the proof of the lemma. □

Lemma 14. *Let $Q_1 \subset Q_2$ be dyadic cubes such that Q_2 is the parent of Q_1 in the tree T. Let $a > 0$ be given. Let Q_1^{up} and Q_2^{down} be dyadic cubes.*

Assume that $Q_1 \subsetneq Q_1^{up} \subsetneq Q_2^{down} \subsetneq Q_2$ with $\delta_{Q_2} \ge \Lambda\delta_{Q_2^{down}} \ge \Lambda^2\delta_{Q_1^{up}} \ge \Lambda^3\delta_{Q_1}$ for some $\Lambda \ge 2$.

If Λ exceeds a large enough constant determined by a and n, then $(1+a)Q_1^{up} \cap E = 3Q_2^{down} \cap E$.

Proof. If $\Lambda \ge 4$, then since $Q_1 \subset Q_2^{down}$ and $\delta_{Q_1} \le \frac{1}{\Lambda^2}\delta_{Q_2^{down}}$, we have $Q_1^{+++} \subset Q_2^{down}$.

Fix a sequence of dyadic cubes $Q_{1,1} \subset Q_{1,2} \subset \cdots \subset Q_{1,K}$ with

$$Q_{1,1} = (Q_1)^{+++},\ Q_{1,K} = Q_2^{down},\ \text{ and }\ Q_{1,k} = (Q_{1,k-1})^{+} \text{ for } 2 \le k \le K.$$

Since Q_1 is a testing cube (recall that all the nodes of T are testing cubes), it follows by definition that Q_1 contains a cube in $CZ(\mathcal{A}^-)$. Thus, thanks to (1.6), the set $9Q_1 \cap E$ is nonempty. We have $3Q_{1,k} \supset 3Q_{1,1} = 3Q_1^{+++} \supset 9Q_1$ for any $1 \le k \le K$. Hence, $3Q_{1,k} \cap E \neq \emptyset$ for any $1 \le k \le K$. Moreover, note that $Q_1 \subsetneq Q_{1,k} \subsetneq Q_2$, because Q_1^{+++} and Q_2^{down} are strictly contained between Q_1 and Q_2. Since Q_2 is the parent of Q_1 in the tree T, which contains all the geometrically

interesting testing cubes, we learn that $Q_{1,k}$ is not geometrically interesting, for each $1 \le k \le K$. In particular, we see that $Q_{1,k}^{++} = Q_{1,k+2}$ is not geometrically interesting, hence $\operatorname{diam}(Q_{1,k}^{++} \cap E) < \lambda \delta_{Q_{1,k}^{++}}$, for all $1 \le k \le K-2$. Thus, the hypotheses of Lemma 13 are satisfied by $Q = Q_{1,k}$ for each $1 \le k \le K-2$. We conclude that

$$3Q_{1,2} \cap E = 3Q_{1,3} \cap E = \cdots = 3Q_{1,K} \cap E.$$

That is, $3Q_1^{++++} \cap E = 3Q_2^{\text{down}} \cap E$.

Recall that $Q_1 \subset Q_1^{\text{up}}$ are dyadic cubes with $\delta_{Q_1^{\text{up}}} \ge \Lambda \delta_{Q_1}$. It follows that $3Q_1^{++++} \subset (1+a)Q_1^{\text{up}}$ if Λ is much larger than a^{-1}. Therefore, $3Q_2^{\text{down}} \cap E \subset (1+a)Q_1^{\text{up}} \cap E$. Moreover, the reverse inclusion follows because $Q_1^{\text{up}} \subset Q_2^{\text{down}}$. Therefore, $3Q_2^{\text{down}} \cap E = (1+a)Q_1^{\text{up}} \cap E$. □

1.6.1. Finding enough tagged cubes. We produce the following algorithm.

ALGORITHM: COMPUTE CRITICAL TESTING CUBES

Given $\epsilon > 0$ less than a small enough universal constant, we produce a collection $\widehat{\mathcal{Q}}_\epsilon$ of testing cubes with the following properties.

(a) Each point $x \in E$ belongs to some cube $\widehat{Q}_x \in \widehat{\mathcal{Q}}_\epsilon$.

(b) The number of cubes belonging to $\widehat{\mathcal{Q}}_\epsilon$ is bounded by $C \cdot N$.

(c) If $\widehat{Q} \in \widehat{\mathcal{Q}}_\epsilon$ strictly contains a cube in $CZ(\mathcal{A}^-)$, then $(1+a(\mathcal{A}))\widehat{Q}$ is tagged with $(\mathcal{A}, \epsilon^\kappa)$.

(d) If $\widehat{Q} \in \widehat{\mathcal{Q}}_\epsilon$ and $\delta_{\widehat{Q}} \le c^*$, then no cube containing $S\widehat{Q}$ is tagged with $(\mathcal{A}, \epsilon^{1/\kappa})$.

Here, $c^* > 0$ and $S \ge 1$ are integer powers of 2, depending only on m, n, p; also, $\kappa \in (0,1)$ is a universal constant. The algorithm requires work at most $CN \log N$ in space CN.

Explanation. We introduce a large parameter $\Lambda = 2^{\text{integer}} \ge 1$. We later pick Λ to be a constant determined by m, n, and p, but not yet. We assume that Λ exceeds a large enough constant determined by m, n, and p, and that ϵ is less than a small enough constant determined by Λ, m, n, and p.

We let $\kappa_0, \ldots, \kappa_{20} \in (0,1)$ be constants to be determined later. We assume that κ_0 is less than a small enough constant determined by m, n, and p, and that $\kappa_{j+1} \le \kappa_j^{100}$ for $j = 0, \ldots, 19$.

We first describe the construction of $\widehat{\mathcal{Q}}_\epsilon$.

Let T be the tree constructed in the algorithm COMPUTE INTERESTING CUBES.

We initialize $\widehat{\mathcal{Q}}_\epsilon$ to be the empty collection. Next, for each cube $Q_1 \in T$ other than the root, we perform Steps 0-3 below.

- Step 0: We find the parent Q_2 of Q_1 in the tree T.
- Step 1: If $\delta_{Q_1} \le \Lambda^{-20} \delta_{Q_2}$, then we do the following.

Let Q_1^{up} be the dyadic cube with $Q_1 \subset Q_1^{\text{up}}$ and $\delta_{Q_1^{\text{up}}} = \Lambda \cdot \delta_{Q_1}$. We compute the function $\eta^{(\widehat{Q}_1^{\text{up}}, \mathcal{A})}(\delta)$ using the algorithm OPTIMIZE BASIS (see Section 1.5).

We determine whether or not there exists a number $\delta \in [\Lambda^{10}\delta_{Q_1}, \Lambda^{-10}\delta_{Q_2}]$ with the property that

$$\epsilon^{1/\kappa_5} \le \eta^{(Q_1^{up},\mathcal{A})}(\delta) \le \epsilon^{\kappa_5}.$$

If such a δ exists, we can easily find one. Moreover, we can then find a dyadic cube Q such that

$$Q_1 \subset Q \subset Q_2,\ \delta/2 \le \delta_Q \le 2\delta \quad \text{and} \quad \Lambda^{10}\delta_{Q_1} \le \delta_Q \le \Lambda^{-10}\delta_{Q_2}.$$

We add Q to the collection $\widehat{\mathcal{Q}}_\epsilon$. Note that $c\eta^{(Q_1^{up},\mathcal{A})}(\delta) \le \eta^{(Q_1^{up},\mathcal{A})}(\delta_Q) \le C\eta^{(Q_1^{up},\mathcal{A})}(\delta)$, thanks to condition **(A3)** in the algorithm OPTIMIZE BASIS. Thus, we can guarantee that

$$\left[\epsilon^{1/\kappa_6} \le \eta^{(Q_1^{up},\mathcal{A})}(\delta_Q)\right] \quad \text{and} \quad \left[\eta^{(Q_1^{up},\mathcal{A})}(\delta_Q) \le \epsilon^{\kappa_6}\right]. \tag{1.159}$$

- Step 2: We examine each dyadic cube Q with $Q_1 \subset Q \subset Q_2$, $\delta_Q \le \Lambda^{-10}$, and $\left[\delta_Q \le \Lambda^{10}\delta_{Q_1} \text{ or } \delta_Q \ge \Lambda^{-10}\delta_{Q_2}\right]$.

We compute $\#(E \cap \frac{65}{64}Q)$. This requires work at most $C \log N$ using the BBD Tree. See Remark 36 in [1].

Let Q^{up} be the dyadic cube with $Q \subset Q^{up}$ and $\delta_{Q^{up}} = \Lambda\delta_Q$. We determine whether or not

$$\left[\epsilon^{1/\kappa_5} \le \eta^{(Q^{up},\mathcal{A})}(\delta_{Q^{up}})\right] \text{ and } \left[\#\Big(\frac{65}{64}Q \cap E\Big) \le 1 \text{ or } \eta^{(Q,\mathcal{A})}(\delta_Q) \le \epsilon^{\kappa_5}\right]. \tag{1.160}$$

We add Q to the collection $\widehat{\mathcal{Q}}_\epsilon$ if and only if (1.160) holds.

- Step 3: We examine each dyadic cube Q with $Q_1 \subset Q \subset Q_2$ and $\delta_Q \ge \Lambda^{-10}$.

We compute $\#(E \cap \frac{65}{64}Q)$. This requires work at most $C \log N$ using the BBD Tree.

For each such Q, we determine whether or not

$$\left[\#\Big(\frac{65}{64}Q \cap E\Big) \le 1 \ \text{ or } \ \eta^{(Q,\mathcal{A})}(\delta_Q) \le \epsilon^{\kappa_5}\right]. \tag{1.161}$$

We add Q to the collection $\widehat{\mathcal{Q}}_\epsilon$ if and only if (1.161) holds.

Finally, we perform Steps 4-6 below.

- Step 4: We check whether or not

$$\left[\eta^{(Q^\circ,\mathcal{A})}(\delta_{Q^\circ}) \le \epsilon^{\kappa_5}\right]. \tag{1.162}$$

We add Q° to the collection $\widehat{\mathcal{Q}}_\epsilon$ if and only if (1.162) holds.

- Step 5: We examine all dyadic cubes $Q \subset Q^\circ$ such that $\delta_Q \ge \Lambda^{-10}$.

We can test whether $Q \in CZ(\mathcal{A}^-)$ by querying the $CZ(\mathcal{A}^-)$-ORACLE on the center of Q. We add Q to the collection $\widehat{\mathcal{Q}}_\epsilon$ if and only if $Q \in CZ(\mathcal{A}^-)$.

- Step 6: We examine all cubes $Q \in CZ(\mathcal{A}^-)$ such that $\delta_Q \le \Lambda^{-10}$ and $Q \cap E \ne \emptyset$.

Let Q^{up} be the dyadic cube with $Q \subset Q^{up}$ and $\delta_{Q^{up}} = \Lambda\delta_Q$. We determine whether or not

$$\left[\epsilon^{1/\kappa_5} \le \eta^{(Q^{up},\mathcal{A})}(\delta_{Q^{up}})\right]. \tag{1.163}$$

We add Q to the collection $\widehat{\mathcal{Q}}_\epsilon$ if and only if (1.163) holds.

This completes the construction of $\widehat{\mathcal{Q}}_\epsilon$. We examined at most $C(\Lambda)N$ cubes, and performed work at most $C\log N$ on each cube. Hence, the computation required work at most $C(\Lambda)N\log N$ in space $C(\Lambda)N$. We later choose Λ to be a constant depending only on m, n, and p. We have thus not exceeded the work and storage guarantees of COMPUTE CRITICAL TESTING CUBES. Moreover, we have $\#(\widehat{\mathcal{Q}}_\epsilon) \le C(\Lambda)\cdot N$, which implies condition (b).

If Q belongs to $\mathcal{Q}_\epsilon$, then Q was chosen in one of the six steps above (not including Step 0). We will examine the six cases separately and prove conditions (c) and (d) for the cube Q.

Analysis of Step 1. Suppose that Q was chosen in Step 1. Then Q satisfies (1.159).

We use properties **(A1)** and **(A2)** of the function $\eta^{(Q_1^{up},\mathcal{A})}$ from the algorithm OPTIMIZE BASIS.

From **(A2)** and (1.159), we find that, for any $\mathcal{A}' \le \mathcal{A}$, $\overline{\overline{\sigma}}(Q_1^{up})$ does not have an $(\mathcal{A}', \chi_{Q_1^{up}}, \epsilon^{1/\kappa_7}, \delta_Q)$-basis.

Since $Q_1 \subset Q$ are testing cubes, and $\delta_Q \ge \Lambda^{10}\delta_{Q_1}$, we have $\#(3Q\cap E) \ge \#(9Q_1\cap E) \ge 2$.

Also note that $(1+a(\mathcal{A}))Q_1^{up}\cap E = 3Q\cap E$ if Λ is greater than some constant determined by m, n, p; see Lemma 14.

Hence, Proposition 12 implies that no cube containing $3Q$ is tagged with $(\mathcal{A}, \epsilon^{1/\kappa_8})$. This proves property (d).

To prove property (c), note that **(A1)** and (1.159) imply that

$$\overline{\overline{\sigma}}(Q_1^{up}) \text{ has an } (\mathcal{A}', \chi_{Q_1^{up}}, \epsilon^{\kappa_7}, \delta_Q)\text{-basis for some } \mathcal{A}' \le \mathcal{A}.$$

Thus, Proposition 12 shows that $(1+a(\mathcal{A}))Q$ is tagged with $(\mathcal{A}, \epsilon^{\kappa_8})$.

Analysis of Step 2. Suppose that Q was chosen in Step 2, and let Q^{up} be as in Step 2. Then Q and Q^{up} satisfy (1.160).

We use properties **(A1)** and **(A2)** of the functions $\eta^{(Q,\mathcal{A})}$ and $\eta^{(Q^{up},\mathcal{A})}$ from the algorithm OPTIMIZE BASIS.

Since $Q \subset Q^{up}$ are testing cubes, and $\delta_{Q^{up}} = \Lambda\delta_Q$, we have $\#(E\cap \frac{65}{64}Q^{up}) \ge \#(E\cap 9Q) \ge 2$ for sufficiently large Λ.

From **(A2)** and (1.160), we find that, for any $\mathcal{A}' \le \mathcal{A}$, $\overline{\overline{\sigma}}(Q^{up})$ does not have an $(\mathcal{A}', \chi_{Q^{up}}, \epsilon^{1/\kappa_6}, \delta_{Q^{up}})$-basis. Thus, Proposition 8 implies that no cube containing $3Q^{up}$ is tagged with $(\mathcal{A}, \epsilon^{1/\kappa_7})$. In particular, since $3Q^{up} \subset 100\Lambda Q$, we find that no cube containing $100\Lambda Q$ is tagged with $(\mathcal{A}, \epsilon^{1/\kappa_7})$. This proves property (d).

From **(A1)** and (1.160), we find that either $\#(\frac{65}{64}Q\cap E) \le 1$ or $\overline{\overline{\sigma}}(Q)$ has an $(\mathcal{A}', \chi_Q, \epsilon^{\kappa_6}, \delta_Q)$-basis for some $\mathcal{A}' \le \mathcal{A}$. Thus, Proposition 8 implies that $(1+a(\mathcal{A}))Q$ is tagged with $(\mathcal{A}, \epsilon^{\kappa_7})$. This proves property (c).

Analysis of Step 3. Note that (d) holds vacuously for all the cubes $Q \in \widehat{\mathcal{Q}}_\epsilon$ chosen in Step 3, assuming that $c^* \le \Lambda^{-10}$.

As in the analysis of Step 2, (1.161) implies that $(1+a(\mathcal{A}))Q$ is tagged with $(\mathcal{A}, \epsilon^{\kappa_7})$. This implies property (c) for any Q picked in Step 3.

Analysis of Step 4. Suppose that Q° was chosen in Step 4. Note that (d) holds vacuously for Q°.

As in the analysis of Step 2, (1.162) shows that $(1 + a(\mathcal{A}))Q^\circ$ is tagged with $(\mathcal{A}, \epsilon^{\kappa_7})$. This implies property (c) for Q°.

Analysis of Step 5. We may assume that $c^* \leq \Lambda^{-10}$. Therefore, (c) and (d) are vacuously true for all the cubes $Q \in \widehat{\mathcal{Q}}_\epsilon$ chosen in Step 5.

Analysis of Step 6. Suppose that Q was chosen in Step 6. Note that (c) holds vacuously for Q, since $Q \in CZ(\mathcal{A}^-)$.

Since $\delta_{Q^{up}} = \Lambda\delta_Q$, and since $Q \subset Q^{up}$ are testing cubes, we have $\#(E \cap \frac{65}{64}Q^{up}) \geq \#(E \cap 9Q) \geq 2$.

By (1.163) and property **(A2)** of the function $\eta^{(Q^{up},\mathcal{A})}$ stated in OPTIMIZE BASIS, we find that $\overline{\overline{\sigma}}(Q^{up})$ does not have an $(\mathcal{A}', x_{Q^{up}}, \epsilon^{1/\kappa_6}, \delta_{Q^{up}})$-basis for any $\mathcal{A}' \leq \mathcal{A}$. Then Proposition 8 guarantees that no cube containing $3Q^{up}$ is tagged with $(\mathcal{A}, \epsilon^{1/\kappa_7})$. Therefore, since $3Q^{up} \subset 100\Lambda Q$, we find that no cube containing $100\Lambda Q$ is tagged with $(\mathcal{A}, \epsilon^{1/\kappa_7})$. This implies property (d) for Q, and concludes the analysis of Step 6.

This completes the proof of (c) and (d) in all cases. An inspection of our argument shows that we may take $c^* = \Lambda^{-10}$ and $S = 128\Lambda$.

Next we prove property (a).

Let $x \in E$ be given. Consider the finite sequence of cubes $Q_\nu \in T$ such that

$$x \in Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_{\nu_{max}} = Q^\circ, \tag{1.164}$$

where $Q_0 \in CZ(\mathcal{A}^-)$ and $Q_{\nu+1}$ is the parent of Q_ν in T. (We do not attempt to compute this sequence.)

We will show that there exists $Q' \in \widehat{\mathcal{Q}}_\epsilon$ with $Q_0 \subset Q' \subset Q_{\nu_{max}}$. This will complete the proof of (a).

Note that one of the following cases must occur.

(A) The first extreme case. For all dyadic cubes Q such that $Q_0 \subset Q \subset Q_{\nu_{max}}$, the cube $3Q$ is tagged with $(\mathcal{A}, \epsilon)$.

(B) The second extreme case. For all dyadic cubes Q such that $Q_0 \subset Q \subset Q_{\nu_{max}}$, the cube $3Q$ is not tagged with $(\mathcal{A}, \epsilon)$.

(C) The main case. For some dyadic cube Q such that $Q_0 \subset Q \subsetneq Q_{\nu_{max}}$ we find that exactly one of $3Q$, $3Q^+$ is tagged with $(\mathcal{A}, \epsilon)$.

(A) *In the first extreme case*:

$$3Q^\circ \text{ is tagged with } (\mathcal{A}, \epsilon). \tag{1.165}$$

Notice that $\#(\frac{65}{64}Q^\circ \cap E) = \#(E) \geq 2$. From (1.165) and Proposition 8, we see that $\overline{\overline{\sigma}}(Q^\circ)$ has an $(\mathcal{A}', x_{Q^\circ}, \epsilon^{\kappa_1}, \delta_{Q^\circ})$-basis for some $\mathcal{A}' \leq \mathcal{A}$. Then property **(A2)** from OPTIMIZE BASIS shows that $\eta^{(Q^\circ,\mathcal{A})}(\delta_{Q^\circ}) \leq \epsilon^{\kappa_5}$. Therefore, we decided to include Q° in $\widehat{\mathcal{Q}}_\epsilon$ in Step 4.

This completes the analysis in the first extreme case.

(B) *In the second extreme case*:

$$3Q_0 \text{ is not tagged with } (\mathcal{A}, \epsilon). \tag{1.166}$$

If $\delta_{Q_0} \geq \Lambda^{-10}$, then we decided to include Q_0 in $\widehat{\mathcal{Q}}_\epsilon$ in Step 5.

Otherwise, suppose that $\delta_{Q_0} < \Lambda^{-10}$.

Let Q_0^{up} be a dyadic cube with $Q_0 \subset Q_0^{up} \subset Q^\circ$ and $\delta_{Q_0^{up}} = \Lambda\delta_{Q_0}$.

Note that $3Q_0 \subset (1 + a(\mathcal{A}))Q_0^{up}$, if Λ is sufficiently large. Then (1.166) and Lemma 28 from [1] imply that $(1+a(\mathcal{A}))Q_0^{up}$ is not tagged with $(\mathcal{A}, \epsilon^{1/\kappa_1})$. Hence, Proposition 8 shows that

$$\overline{\overline{\sigma}}(Q_0^{up}) \text{ does not have an } (\mathcal{A}', x_{Q_0^{up}}, \epsilon^{1/\kappa_2}, \delta_{Q_0^{up}})\text{-basis for any } \mathcal{A}' \le \mathcal{A}.$$

Thus, property **(A1)** from OPTIMIZE BASIS shows that $\eta^{(Q_0^{up},\mathcal{A})}(\delta_{Q_0^{up}}) \ge \epsilon^{1/\kappa_5}$. Therefore, we decided to include Q_0 in $\widehat{\mathcal{Q}}_\epsilon$ in Step 6. (Recall that $x \in Q_0$, hence $E \cap Q_0 \neq \emptyset$.)

This completes the analysis in the second extreme case.

(C) *In the main case*: Exactly one of $3Q$, $3Q^+$ is tagged with $(\mathcal{A}, \epsilon)$, thus

$$3Q \text{ is tagged with } (\mathcal{A}, \epsilon^{\kappa_0}) \text{ (see Lemma 28 from [1])}, \tag{1.167}$$

and

$$3Q^+ \text{ is not tagged with } (\mathcal{A}, \epsilon^{1/\kappa_0}) \text{ (again, see Lemma 28 from [1])}. \tag{1.168}$$

We now consider three subcases of the main case.

(C1) The *geometrically interesting* ("GI") subcase: For some ν,

$$Q_\nu \subset Q \subset Q_{\nu+1}, \quad \left[\delta_Q \le \Lambda^{10}\delta_{Q_\nu} \text{ or } \delta_Q \ge \Lambda^{-10}\delta_{Q_{\nu+1}}\right], \text{ and } \delta_Q \le \Lambda^{-10}. \tag{1.169}$$

(C2) The *geometrically uninteresting* ("GUI") subcase: For some ν,

$$Q_\nu \subset Q \subset Q_{\nu+1} \text{ and } \Lambda^{10}\delta_{Q_\nu} \le \delta_Q \le \Lambda^{-10}\delta_{Q_{\nu+1}}. \tag{1.170}$$

(C3) The *near-maximal* ("NM") subcase:

$$\delta_Q \ge \Lambda^{-10}. \tag{1.171}$$

(C1) First consider the *GI subcase*.

From Proposition 8 and (1.167) we see that

$$\#\Big(\frac{65}{64}Q \cap E\Big) \le 1 \text{ or } \overline{\overline{\sigma}}(Q) \text{ has an } (\mathcal{A}', x_Q, \epsilon^{\kappa_1}, \delta_Q)\text{-basis for some } \mathcal{A}' \le \mathcal{A}.$$

Thus, by property **(A2)** from OPTIMIZE BASIS,

$$\#\Big(\frac{65}{64}Q \cap E\Big) \le 1 \text{ or } \eta^{(Q,\mathcal{A})}(\delta_Q) \le \epsilon^{\kappa_5}. \tag{1.172}$$

Pick Q^{up} (dyadic) such that $Q \subset Q^{up} \subset Q^\circ$ and $\delta_{Q^{up}} = \Lambda\delta_Q$. (Recall that $\delta_Q \le \Lambda^{-10}$.) Then $3Q^+ \subset (1 + a(\mathcal{A}))Q^{up}$, assuming that Λ is sufficiently large. Thus, (1.168) shows that

$$(1 + a(\mathcal{A}))Q^{up} \text{ is not tagged with } (\mathcal{A}, \epsilon^{1/\kappa_1}) \quad \text{(see Lemma 28 from [1])}.$$

Therefore, Proposition 8 gives that

$$\overline{\overline{\sigma}}(Q^{up}) \text{ does not have an } (\mathcal{A}', x_{Q^{up}}, \epsilon^{1/\kappa_2}, \delta_{Q^{up}})\text{-basis for any } \mathcal{A}' \leq \mathcal{A}.$$

Hence, using property **(A1)** from OPTIMIZE BASIS,

$$\eta^{(Q^{up},\mathcal{A})}(\delta_{Q^{up}}) \geq \epsilon^{1/\kappa_5} \tag{1.173}$$

From (1.172) and (1.173), we see that Q was included in $\widehat{\mathcal{Q}}_\epsilon$ in Step 2. This completes the analysis in the GI subcase.

(C2) Next consider the *GUI subcase*.

Since $Q_\nu \subset Q$ are testing cubes, and $\delta_Q \geq \Lambda^{10}\delta_{Q_\nu}$, we have $\#(E \cap 3Q) \geq \#(E \cap 9Q_\nu) \geq 2$.

Let Q_ν^{up} denote the dyadic cube with $Q_\nu \subset Q_\nu^{up}$ and $\delta_{Q_\nu^{up}} = \Lambda \cdot \delta_{Q_\nu}$

Note that $(1 + a(\mathcal{A}))Q_\nu^{up} \cap E = 3Q \cap E$, as long as $\Lambda \geq C$ for a large enough universal constant C (see Lemma 14).

From Proposition 12 and assumption (1.167) (from the Main Case), we see that

$$\overline{\overline{\sigma}}(Q_\nu^{up}) \text{ has an } (\mathcal{A}', x_{Q_\nu^{up}}, \epsilon^{\kappa_1}, \delta_Q)\text{-basis, for some } \mathcal{A}' \leq \mathcal{A}.$$

Hence, condition **(A2)** in the algorithm OPTIMIZE BASIS implies that

$$\eta^{(Q_\nu^{up},\mathcal{A})}(\delta_Q) \leq \epsilon^{\kappa_5}. \tag{1.174}$$

Let Q^{up} be a dyadic cube with $Q \subset Q^{up} \subsetneq Q_{\nu+1}$ and $\delta_{Q^{up}} = \Lambda \cdot \delta_Q$. Such a dyadic cube exists because we are assuming that $\delta_Q \leq \Lambda^{-10}\delta_{Q_{\nu+1}}$. Then $(1 + a(\mathcal{A}))Q_\nu^{up} \cap E = 3Q^{up} \cap E$ thanks to Lemma 14.

For large enough Λ, we have $3Q^+ \subset (1 + a(\mathcal{A}))Q^{up}$. Thus, Lemma 28 from [1] and (1.168) imply that $(1 + a(\mathcal{A}))Q^{up}$ is not tagged with $(\mathcal{A}, \epsilon^{1/\kappa_1})$. We apply conclusion (A) in Proposition 12 to the testing cubes $Q_\nu^{up} \subset Q^{up}$ in order to deduce that

$$\overline{\overline{\sigma}}(Q_\nu^{up}) \text{ does not have an } (\mathcal{A}', x_{Q_\nu^{up}}, \epsilon^{1/\kappa_2}, \delta_{Q^{up}})\text{-basis, for any } \mathcal{A}' \leq \mathcal{A}.$$

Thus, property **(A1)** from OPTIMIZE BASIS shows that $\eta^{(Q_\nu^{up},\mathcal{A})}(\delta_{Q^{up}}) \geq \epsilon^{1/\kappa_3}$. Moreover, property **(A3)** from OPTIMIZE BASIS implies that $\eta^{(Q_\nu^{up},\mathcal{A})}(\delta_{Q^{up}}) \leq C(\Lambda)\eta^{(Q_\nu^{up},\mathcal{A})}(\delta_Q)$, hence we have

$$\eta^{(Q_\nu^{up},\mathcal{A})}(\delta_Q) \geq \epsilon^{1/\kappa_5}. \tag{1.175}$$

We are assuming that $\delta_Q \in [\Lambda^{10}\delta_{Q_\nu}, \Lambda^{-10}\delta_{Q_{\nu+1}}]$ (from the GUI subcase). Hence, from (1.174) and (1.175), we see that in Step 1 we included in $\widehat{\mathcal{Q}}_\epsilon$ a dyadic cube Q' such that $Q_\nu \subset Q' \subset Q_{\nu+1}$. This completes the analysis in the GUI subcase.

(C3) Finally, consider the *NM subcase*.

From Proposition 8 and (1.167) we have

$$\#\Big(\frac{65}{64}Q \cap E\Big) \leq 1 \text{ or } \overline{\overline{\sigma}}(Q) \text{ has an } (\mathcal{A}', x_Q, \epsilon^{\kappa_1}, \delta_Q)\text{-basis for some } \mathcal{A}' \leq \mathcal{A}.$$

Then property **(A2)** from OPTIMIZE BASIS implies that

$$\#\Big(\frac{65}{64}Q \cap E\Big) \le 1 \text{ or } \eta^{(Q,\mathcal{A})}(\delta_Q) \le \epsilon^{\kappa_5}. \tag{1.176}$$

Thus, we included Q in $\widehat{\mathcal{Q}}_\epsilon$ in Step 3. This completes the analysis in the NM subcase.

Thus, in all the cases, we see that there exists $Q' \in \widehat{\mathcal{Q}}_\epsilon$ with $Q_0 \subset Q' \subset Q_{\nu_{max}}$, where Q_0 is the unique cube in $CZ(\mathcal{A}^-)$ containing the point $x \in E$. As mentioned before, this completes the proof of (a).

We fix a large enough constant $\Lambda = 2^J \ge 1$, depending only on m, n, and p.

This completes the explanation of the algorithm COMPUTE CRITICAL TESTING CUBES. □

1.6.2. Lengthscales. With the algorithm COMPUTE CRITICAL TESTING CUBES from the previous section, we compute a collection $\widehat{\mathcal{Q}}_\epsilon$ consisting of dyadic subcubes of Q°. We proved that each point of E belongs to a cube in $\widehat{\mathcal{Q}}_\epsilon$. Applying the algorithm PLACING A POINT INSIDE TARGET CUBOIDS (see Section 4.1.5 of [1]), we obtain the following algorithm.

ALGORITHM: COMPUTE LENGTHSCALES

For each $x \in E$ we compute a cube $Q_x \in \widehat{\mathcal{Q}}_\epsilon$ containing x. This requires work at most $CN \log N$ in space CN.

We write $c^* > 0$ and $S \ge 1$ for the universal constants from the algorithm COMPUTE CRITICAL TESTING CUBES. The conclusion of this algorithm implies the next result.

Proposition 13. *For each $x \in E$, the following properties hold.*

(LS1) *Suppose that Q_x strictly contains a cube of $CZ(\mathcal{A}^-)$. Then $(1 + a(\mathcal{A}))Q_x$ is tagged with $(\mathcal{A}, \epsilon^\kappa)$.*

(LS2) *Suppose that $\delta_{Q_x} \le c^*$. Then no cube containing SQ_x is tagged with $(\mathcal{A}, \epsilon^{1/\kappa})$.*

Here, $\kappa > 0$ is a small universal constant.

1.7. Passing from lengthscales to CZ decompositions

For each $x \in E$ we compute the sidelength

$$\Delta_{\mathcal{A}}(x) := \delta_{Q_x}. \tag{1.177}$$

Here, we compute the cube Q_x using the algorithm COMPUTE LENGTHSCALES. Recall that $x \in Q_x$ for each $x \in E$. Since $Q_x \subset Q^\circ$, we know that

$$\Delta_{\mathcal{A}}(x) \in (0, 1] \text{ for all } x \in E. \tag{1.178}$$

Let $Q \subset Q^\circ$ be a testing cube. We say that Q is $OK(\mathcal{A})$ provided that either $Q \in CZ(\mathcal{A}^-)$ or $\Delta_{\mathcal{A}}(x) \ge K\delta_Q$ for all $x \in E \cap 3Q$, where we set $K := 10^9/a(\mathcal{A})$. Recall that we have defined the constant $a(\mathcal{A})$ in equation (1.147). In particular, since $a(\mathcal{A}) \le 1$, we see that $K \ge 1$.

We define a Calderón–Zygmund decomposition $\mathrm{CZ}(\mathcal{A})$ of the unit cube Q° to consist of the maximal dyadic subcubes $Q \subset Q^\circ$ that are $\mathsf{OK}(\mathcal{A})$.

We will prove properties **(CZ1)**–**(CZ5)** in the main technical results for $\mathcal{A}$. We refer the reader to Section 3 of [1] for the statements of **(CZ1)**–**(CZ5)**, which are a component of the main technical results.

First, however, we produce a $\mathrm{CZ}(\mathcal{A})$-ORACLE as described in Section 3 of [1]. The decomposition $\mathrm{CZ}(\mathcal{A})$ coincides with the decomposition $\mathrm{CZ}_{\mathrm{new}}$ from Section 4.6.3 in [1], where we use $\mathrm{CZ}_{\mathrm{old}} = \mathrm{CZ}(\mathcal{A}^-)$ and $\Delta(x) = \Delta_{\mathcal{A}}(x)/K$ in the notation therein. Note that $\Delta(x) \in (0, 1]$ for each $x \in E$, hence we have proven the assumptions in Section 4.6.3 of [1]. The GLORIFIED CZ-ORACLE coincides with the $\mathrm{CZ}(\mathcal{A})$-ORACLE described in Section 3 of [1].

Proposition 14. *The collection* $\mathrm{CZ}(\mathcal{A})$ *is a partition of* Q° *into pairwise disjoint dyadic subcubes.*

Proof. Each point $x \in Q^\circ$ belongs to some cube $Q_0 \in \mathrm{CZ}(\mathcal{A}^-)$. Note that Q_0 is $\mathsf{OK}(\mathcal{A})$, and hence Q_0 is contained in a maximal dyadic subcube $Q \subset Q^\circ$ that is also $\mathsf{OK}(\mathcal{A})$. Thus, each point $x \in Q^\circ$ is contained in some cube $Q \in \mathrm{CZ}(\mathcal{A})$.

Any two distinct cubes $Q, Q' \in \mathrm{CZ}(\mathcal{A})$ are dyadic, hence either Q, Q' are disjoint or one of Q, Q' contains the other. The latter case cannot occur, by definition of $\mathrm{CZ}(\mathcal{A})$. It follows that the cubes in $\mathrm{CZ}(\mathcal{A})$ are pairwise disjoint. □

Our previous decomposition $\mathrm{CZ}(\mathcal{A}^-)$ clearly refines $\mathrm{CZ}(\mathcal{A})$. This establishes property **(CZ5)** for $\mathcal{A}$. We now prove the remaining properties **(CZ1)**–**(CZ4)**.

We prove property **(CZ1)** in the next result.

Proposition 15. *The cubes in* $\mathrm{CZ}(\mathcal{A})$ *have good geometry.*

Proof. For the sake of contradiction suppose that there are cubes $Q, Q' \in \mathrm{CZ}(\mathcal{A})$ such that $Q \leftrightarrow Q'$ and $\delta_Q \le \frac{1}{4}\delta_{Q'}$. It follows that $3Q^+ \subset 3Q'$.

First, suppose $Q' \in \mathrm{CZ}(\mathcal{A}^-)$. Since $\mathrm{CZ}(\mathcal{A}^-)$ refines $\mathrm{CZ}(\mathcal{A})$, there exists a cube $Q'' \in \mathrm{CZ}(\mathcal{A}^-)$ with $Q'' \subset Q$ and $Q'' \leftrightarrow Q'$. Note that $\delta_{Q''} \le \delta_Q \le \frac{1}{4}\delta_{Q'}$. But this contradicts our assumption that the cubes in $\mathrm{CZ}(\mathcal{A}^-)$ satisfy good geometry.

Next, suppose $Q' \notin \mathrm{CZ}(\mathcal{A}^-)$. By definition of $\mathrm{CZ}(\mathcal{A})$ we know that Q^+ is not $\mathsf{OK}(\mathcal{A})$, hence there exists $x \in E \cap 3Q^+$ with $\Delta_{\mathcal{A}}(x) < K\delta_{Q^+}$. Thus, $x \in E \cap 3Q'$ and $\Delta_{\mathcal{A}}(x) < K\delta_{Q'}$. Since also $Q' \notin \mathrm{CZ}(\mathcal{A}^-)$ we see that Q' is not $\mathsf{OK}(\mathcal{A})$. But this contradicts our assumption that $Q' \in \mathrm{CZ}(\mathcal{A})$. □

Proposition 16. *There exists a universal constant* $c_* > 0$ *such that, for any* $Q \in \mathrm{CZ}(\mathcal{A})$*, the following conditions hold.*

(a) *If* Q *is not* c_**-simple then* $3Q$ *is tagged with* $(\mathcal{A}, \epsilon^\kappa)$.

(b) *If* $\delta_Q \le c_*$ *then* WQ *is not tagged with* $(\mathcal{A}, \epsilon^{1/\kappa})$.

Here, $\kappa > 0$ *and* $W \in \mathbb{N}$ *are universal constants.*

Proof. We choose c_* much smaller than the constant c^* from Proposition 13.

We now prove (a). Assume that $Q \in \mathrm{CZ}(\mathcal{A})$ is not c_*-simple. Then there exists $\overline{Q} \in \mathrm{CZ}(\mathcal{A}^-)$ with $\overline{Q} \subset \frac{65}{64}Q$ and $\delta_{\overline{Q}} \le c_*\delta_Q$. For small enough c_* this implies

that $9\overline{Q} \subset 3Q$. Recall (1.6), which implies that $9\overline{Q} \cap E \neq \emptyset$, hence $3Q \cap E \neq \emptyset$. We fix $x \in E \cap 3Q$.

We have $\delta_{Q_x} = \Delta_{\mathcal{A}}(x) \geq K\delta_Q$ with $K = 10^9/a(\mathcal{A})$, because Q is $OK(\mathcal{A})$ and $Q \notin CZ(\mathcal{A}^-)$; see also (1.177).

For any $y \in 3Q$ we have $|y - x| \leq 3\delta_Q \leq \frac{3}{K}\delta_{Q_x}$, because $x \in 3Q$. Moreover, $|x - x_{Q_x}| \leq \frac{1}{2}\delta_{Q_x}$, because $x \in Q_x$. (Recall that x_{Q_x} is the center of Q_x.) Thus,

$$|y - x_{Q_x}| \leq \left[\frac{1}{2} + \frac{3}{K}\right]\delta_{Q_x} \leq \frac{1}{2}\left[1 + a(\mathcal{A})\right]\delta_{Q_x} \quad \text{for any } y \in 3Q.$$

(Here, we use that $K = 10^9/a(\mathcal{A})$.) Hence,

$$3Q \subset (1 + a(\mathcal{A}))Q_x. \tag{1.179}$$

(Recall that we are working with the ℓ^∞ metric.)

We now prove that Q_x strictly contains a cube of $CZ(\mathcal{A}^-)$. Assume for the sake of contradiction that Q_x is contained in a cube in $CZ(\mathcal{A}^-)$. (For a dyadic cube this is the only alternative.) Since Q_x is a testing cube, it follows that Q_x belongs to $CZ(\mathcal{A}^-)$; see Section 1.4.3 where the definition of a testing cube is given. Thanks to **(CZ5)**, we know that $CZ(\mathcal{A}^-)$ refines $CZ(\mathcal{A})$. Hence, there exists $\widetilde{Q} \in CZ(\mathcal{A}^-)$ with $\widetilde{Q} \subset Q$. From (1.179) we have

$$\widetilde{Q} \subset 3Q \subset (1 + a(\mathcal{A}))Q_x \subset \frac{65}{64}Q_x.$$

We know $\widetilde{Q}$ and Q_x are members of $CZ(\mathcal{A}^-)$, which has good geometry, hence we can apply Lemma 46 from [1] to learn that $\frac{1}{2}\delta_{\widetilde{Q}} \leq \delta_{Q_x} \leq 2\delta_{\widetilde{Q}}$. (The good geometry of $CZ(\mathcal{A}^-)$ ensures the hypotheses of Lemma 46 are valid with $\gamma = 1/8$.) Hence, because the cubes in $CZ(\mathcal{A}^-)$ are pairwise disjoint and dyadic, we must have $Q_x = \widetilde{Q}$. Thus, we have

$$(1 + a(\mathcal{A}))Q_x \subsetneq 3\widetilde{Q} \subset 3Q.$$

However, this contradicts (1.179). This completes the proof that Q_x strictly contains a cube of $CZ(\mathcal{A}^-)$.

Hence, from **(LS1)** in Proposition 13 we deduce that $(1 + a(\mathcal{A}))Q_x$ is tagged with $(\mathcal{A}, \epsilon^\kappa)$; hence, $3Q$ is tagged with $(\mathcal{A}, \epsilon^{\kappa'})$ for some universal constant κ', thanks to Lemma 28 from [1] and (1.179). This completes the proof of (a).

We now prove (b). Let S be the universal constant in Proposition 13. Assume that $Q \in CZ(\mathcal{A})$ satisfies $\delta_Q \leq c_*$.

Since Q^+ is not $OK(\mathcal{A})$, there exists $x \in E \cap 3Q^+$ such that

$$\delta_{Q_x} = \Delta_{\mathcal{A}}(x) \leq K\delta_{Q^+}.$$

Hence, because $x \in Q_x$ and $x \in 3Q^+$, we have $SQ_x \subset WQ$ for a large enough integer constant $W \geq 1$ depending only on K and S. Recall that $K = 10^9/a(\mathcal{A})$ is a universal constant. Hence, we can choose W to be a universal constant. Therefore, $\delta_{Q_x} \leq \frac{W}{S}\delta_Q \leq \frac{W}{S}c_* \leq c^*$. Here, we assume that $c_* \leq \frac{S}{W}c^*$.

From **(LS2)** in Proposition 13 it follows that WQ is not tagged with $(\mathcal{A}, \epsilon^{1/\kappa})$.

This completes the proof of the proposition. □

We have proven **(CZ2)** and **(CZ3)** in the main technical results for $\mathcal{A}$, where we set

$$\begin{cases} c_*(\mathcal{A}) = c_*/2, \\ S(\mathcal{A}) = W, \\ \epsilon_1(\mathcal{A}) = \epsilon^{1/\kappa}, \\ \epsilon_2(\mathcal{A}) = \epsilon^{\kappa}. \end{cases} \tag{1.180}$$

Here, κ and W are as in Proposition 16. Note that **(CZ4)** holds vacuously, since we are assuming that $\mathcal{A} \neq \mathcal{M}$. We have thus proven **(CZ1)**–**(CZ5)** for the label $\mathcal{A}$. We will later pick ϵ to be a small enough universal constant, at which point $\epsilon_1(\mathcal{A})$ and $\epsilon_2(\mathcal{A})$ will be determined once and for all.

We let $\mathrm{CZ}_{\mathrm{main}}(\mathcal{A})$ denote the collection of all cubes $Q \in \mathrm{CZ}(\mathcal{A})$ that satisfy $\frac{65}{64}Q \cap E \neq \emptyset$. We note that the collection $\{\frac{65}{64}Q : Q \in \mathrm{CZ}(\mathcal{A})\}$ has bounded overlap, thanks to the good geometry of the cubes in $\mathrm{CZ}(\mathcal{A})$ (see Lemma 15). Hence,

$$\#\left(\mathrm{CZ}_{\mathrm{main}}(\mathcal{A})\right) \leq C \cdot N. \tag{1.181}$$

1.8. Completing the induction

In the previous section, we defined a decomposition $\mathrm{CZ}(\mathcal{A})$ and gave a $\mathrm{CZ}(\mathcal{A})$-ORACLE. Here, we construct the remaining objects in the main results for $\mathcal{A}$.

We first compute a list of all the cubes Q in $\mathrm{CZ}_{\mathrm{main}}(\mathcal{A})$. We just list all the cubes $Q \in \mathrm{CZ}(\mathcal{A})$ for which $E \cap \frac{65}{64}Q \neq \emptyset$. This requires the algorithm FIND MAIN-CUBES from Section 4.6.4 of [1].

We now show that for each $\widehat{Q} \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A})$ we can efficiently collect all the ingredients we need to compute the assists, functionals, and local extension operator relevant to $\widehat{Q}, \mathcal{A}$.

Recall the notion of *supporting data* associated to a testing cube; see Section 1.4.5.

ALGORITHM: PRODUCE ALL SUPPORTING DATA

We produce the supporting data for each cube $\widehat{Q}$ in $\mathrm{CZ}_{\mathrm{main}}(\mathcal{A})$, using work at most $CN \log N$ in space CN.

Explanation. We produce the cubes $Q, Q_{\mathrm{sp}}, Q^{\#}$ and the pairs of cubes (Q', Q'') that arise in (SD1)–(SD5) in Section 1.4.5 for some testing cube $\widehat{Q} \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A})$.

For each $Q \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A}^-)$, we apply the $\mathrm{CZ}(\mathcal{A})$-ORACLE to find the cube $\widehat{Q} \in \mathrm{CZ}(\mathcal{A})$ that contains Q, as well as all the cubes $\widehat{Q}' \in \mathrm{CZ}(\mathcal{A})$ such that $\widehat{Q}' \leftrightarrow \widehat{Q}$. For each such $\widehat{Q}$ (or $\widehat{Q}'$), we check whether $\widehat{Q}$ (or $\widehat{Q}'$) appears in the list $\mathrm{CZ}_{\mathrm{main}}(\mathcal{A})$; if it does, then we check whether $Q \subset (1 + t_G)\widehat{Q}$ (or $(1 + t_G)\widehat{Q}'$). If so, then we add the cube Q to the list of cubes in (SD1) relevant to the testing cube $\widehat{Q}$ (or $\widehat{Q}'$).

Similarly, for each pair $(Q', Q'') \in \mathrm{CZ}(\mathcal{A}^-) \times \mathrm{CZ}(\mathcal{A}^-)$ such that $Q' \leftrightarrow Q''$ but $\mathcal{K}(Q') \neq \mathcal{K}(Q'')$ (the "border disputes"), we look for all possible $\widehat{Q} \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A})$

such that (Q', Q'') arises in (SD2) for the testing cube $\widehat{Q}$. That is, we look for all the $\widehat{Q} \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A})$ such that $Q' \subset (1 + t_G)\widehat{Q}$ and $\delta_{Q'} < t_G \delta_{\widehat{Q}}$.

To find all the $\widehat{Q}$ as above, we need only search among the cubes $\widehat{Q}' =$ the cube of $\mathrm{CZ}(\mathcal{A})$ containing Q', and the cubes of $\mathrm{CZ}(\mathcal{A})$ that touch $\widehat{Q}'$. We obtain all those cubes by making at most C calls to the $\mathrm{CZ}(\mathcal{A})$-ORACLE and doing additional work at most C.

We check each $\widehat{Q}$ obtained as above to see whether $\widehat{Q} \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A})$, and if so whether also $\widehat{Q}$ has the desired relationship with Q'. For each surviving $\widehat{Q}$, we add (Q', Q'') to the list of cubes in (SD2) relevant to that $\widehat{Q}$.

To find all the $Q \in \mathrm{CZ}(\mathcal{A}^-)$ that arise in (SD3), we loop over all the $\widehat{Q} \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A})$. For each fixed $\widehat{Q}$, we examine all the dyadic cubes $Q \subset (1 + t_G)\widehat{Q}$ such that $\delta_Q \geq t_G^2 \delta_{\widehat{Q}}$. (There are only C such Q.) We test Q to see whether it belongs to $\mathrm{CZ}(\mathcal{A}^-)$; if so, then we add Q to the list of cubes in (SD3) relevant to $\widehat{Q}$.

For the supporting data in (SD4), we can loop over all $\widehat{Q} \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A})$. For each such $\widehat{Q}$, we can just take Q_{sp} to be the $\mathrm{CZ}(\mathcal{A}^-)$-cube containing the center of $\widehat{Q}$.

Finally, we loop over all keystone cubes $Q^{\#}$ of $\mathrm{CZ}(\mathcal{A}^-)$. For each such $Q^{\#}$, we look for all the $\widehat{Q} \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A})$ such that $S_1 Q^{\#} \subset (65/64)\widehat{Q}$.

To find all the $\widehat{Q}$ as above, we need only search among the cubes $\widehat{Q}' =$ the cube of $\mathrm{CZ}(\mathcal{A})$ containing $Q^{\#}$, and the cubes of $\mathrm{CZ}(\mathcal{A})$ that touch $\widehat{Q}'$. We obtain all those cubes by making at most C calls to the $\mathrm{CZ}(\mathcal{A})$-ORACLE and doing additional work at most C.

We check each $\widehat{Q}$ obtained as above to see whether $\widehat{Q} \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A})$, and if so whether also $\widehat{Q}$ has the desired relationship with $Q^{\#}$. If those conditions are satisfied, then we add $Q^{\#}$ to the list of cubes in (SD5) relevant to $\widehat{Q}$.

Once we have carried out the above, then for each $\widehat{Q} \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A})$, we have a list of all the cubes $Q, Q_{\mathrm{sp}}, Q^{\#}$ and of all the pairs of cubes (Q', Q'') relevant to the supporting data (SD1)–(SD5) for the given $\widehat{Q}$. Again, see Section 1.4.5.

This uses work $\mathcal{O}(N \log N)$ in space $\mathcal{O}(N)$. This completes our explanation of the algorithm PRODUCE ALL SUPPORTING DATA. □

Next, we define

$$\Omega(\widehat{Q}, \mathcal{A}) \subset \Big[\mathbb{X}(\tfrac{65}{64}\widehat{Q} \cap E)\Big]^* \quad \text{and} \quad \Xi(\widehat{Q}, \mathcal{A}) \subset \Big[\mathbb{X}(\tfrac{65}{64}\widehat{Q} \cap E) \oplus \mathcal{P}\Big]^*,$$

and also a linear extension operator $T_{(\widehat{Q}, \mathcal{A})} : \mathbb{X}(\frac{65}{64}\widehat{Q} \cap E) \oplus \mathcal{P} \to \mathbb{X}$ for each $\widehat{Q} \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A})$. We will prove that these objects satisfy the properties laid out in the third, fourth and fifth bullet points in the main technical results for $\mathcal{A}$ (see Section 3 in [1]).

For each $\widehat{Q} \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A})$, we can define

$$M_{\widehat{Q}}(f, P) = \Big(\sum_{\xi \in \Xi(\widehat{Q}, \mathcal{A})} |\xi(f, P)|^p \Big)^{1/p}.$$

We need to prove the estimates in the fourth bullet point in the main technical results for $\mathcal{A}$. These estimates are

$$c\|(f,P)\|_{(1+a(\mathcal{A}))\widehat{Q}} \le M_{\widehat{Q}}(f,P) \quad \text{and} \tag{1.182}$$

$$M_{\widehat{Q}}(f,P) \le C\|(f,P)\|_{\frac{65}{64}\widehat{Q}}. \tag{1.183}$$

Recall that a testing cube $\widehat{Q}$ is called λ-simple if for every $Q \in CZ(\mathcal{A}^-)$ with $Q \subset \frac{65}{64}\widehat{Q}$ we have $\delta_Q \ge \lambda \cdot \delta_{\widehat{Q}}$. We can determine whether a given cube $\widehat{Q}$ is λ-simple using work at most $C(\lambda)$, and at most $C(\lambda)$ calls to the $CZ(\mathcal{A}^-)$-ORACLE. Here, $C(\lambda)$ is a constant depending only on λ and n.

Let c_* be the universal constant in Proposition 16.

We loop over all the cubes $\widehat{Q} \in CZ_{\text{main}}(\mathcal{A})$. We can determine in time $\mathcal{O}(\log N)$ whether $\widehat{Q}$ is c_*-simple. (Recall that a call to the $CZ(\mathcal{A}^-)$-ORACLE requires work $\mathcal{O}(\log N)$.) The body of our loop separates into two cases depending on the result of the test.

1.8.1. Case I: Non-simple cubes. We suppose $\widehat{Q} \in CZ_{\text{main}}(\mathcal{A})$ is *not* c_*-simple (the non-simple case). We will explain how to construct the objects in the main technical results for $\mathcal{A}$ relevant to $\widehat{Q}$.

We have already computed the supporting data for all the cubes in $CZ_{\text{main}}(\mathcal{A})$. By executing the algorithms COMPUTE NEW ASSISTS and COMPUTE NEW ASSISTED FUNCTIONALS (see Section 1.4.5), we can compute

(a) A list of *assist functionals*: $\Omega(\widehat{Q}) \subset \big[\mathbb{X}\big(E \cap (65/64)\widehat{Q}\big)\big]^*$ (see (1.79)), and

(b) A list of *assisted functionals*: $\Xi(\widehat{Q}) \subset \big[\mathbb{X}\big(E \cap (65/64)\widehat{Q}\big) \oplus \mathcal{P}\big]^*$.

Each functional $\xi \in \Xi(\widehat{Q})$ has $\Omega(\widehat{Q})$-assisted bounded depth, and is written in short form in terms of the assists $\Omega(\widehat{Q})$.

We define $\Omega(\widehat{Q},\mathcal{A}) := \Omega(\widehat{Q})$, $\Xi(\widehat{Q},\mathcal{A}) := \Xi(\widehat{Q})$, and

$$M_{\widehat{Q}}(f,P) = \Big(\sum_{\xi\in\Xi(\widehat{Q})} |\xi(f,P)|^p\Big)^{1/p}.$$

We now prove the estimates (1.182) and (1.183).

The estimate (1.182) is a direct consequence of the unconditional inequality in Proposition 7.

Since $\widehat{Q} \in CZ_{\text{main}}(\mathcal{A})$ and $\widehat{Q}$ is not c_*-simple, we know that $3\widehat{Q}$ is tagged with $(\mathcal{A},\epsilon^\kappa)$ (see Proposition 16). We may assume that $\epsilon^\kappa \le \epsilon_0$, with ϵ_0 as in Proposition 7. Thus, $3\widehat{Q}$ is tagged with $(\mathcal{A},\epsilon_0)$. Hence, the conditional inequality in Proposition 7 implies the estimate (1.183).

Next, we estimate how much work and storage are used to compute the lists $\Omega(\widehat{Q},\mathcal{A})$ and $\Xi(\widehat{Q},\mathcal{A})$ for <u>all</u> the non-simple cubes $\widehat{Q} \in CZ_{\text{main}}(\mathcal{A})$. We will prove that the *total work* is at most $CN\log N$ and that the storage used is at most CN.

We examine the algorithms COMPUTE NEW ASSISTS and COMPUTE NEW ASSISTED FUNCTIONALS (see Section 1.4.5). We see that we can compute all the lists

$\Omega(\widehat{Q},\mathcal{A})$ and $\Xi(\widehat{Q},\mathcal{A})$ for all the non-simple cubes $\widehat{Q} \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A})$, using total work at most

$$\sum_{\widehat{Q}\in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A})} \{\mathfrak{W}_1(\widehat{Q}) + \mathfrak{W}_2(\widehat{Q})\}$$

$$\lesssim \log N \cdot \sum_{\widehat{Q}\in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A})} \Bigg\{ 1 + \sum_{\substack{Q\in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A}^-) \\ Q\subset (1+t_G)\widehat{Q}}} \Big[\sum_{\omega\in\Omega(Q,\mathcal{A}^-)} \mathrm{depth}(\omega) + \#\big(\Xi(Q,\mathcal{A}^-)\big)\Big] + \sum_{\substack{\text{keystone } Q^\# \in \mathrm{CZ}(\mathcal{A}^-) \\ s_1 Q^\# \subset \frac{65}{64}\widehat{Q}}} \sum_{\omega\in\Omega^{\mathrm{new}}(Q^\#)} \mathrm{depth}(\omega) + \#\big\{(Q',Q'') \in \mathrm{BD}(\mathcal{A}^-) :\ Q' \subset (1+t_G)\widehat{Q},\ \delta_{Q'} < t_G \delta_{\widehat{Q}}\big\}\Bigg\}.$$

See (1.80), (1.82), and (1.83), for the definitions of the quantities $\mathfrak{W}_1(\widehat{Q})$ and $\mathfrak{W}_2(\widehat{Q})$. Recall that t_G is now a fixed universal constant, and so $C(t_G)$ in (1.83) is a universal constant C.

Each cube Q in $\mathrm{CZ}_{\mathrm{main}}(\mathcal{A}^-)$, each keystone cube $Q^\# \in \mathrm{CZ}(\mathcal{A}^-)$, and each pair $(Q',Q'') \in \mathrm{BD}(\mathcal{A}^-)$ participates above for at most C distinct $\widehat{Q}$ in $\mathrm{CZ}(\mathcal{A})$. This follows because the collection $\{(65/64)\widehat{Q} : \widehat{Q} \in \mathrm{CZ}(\mathcal{A})\}$ has bounded overlap, which follows from the good geometry of $\mathrm{CZ}(\mathcal{A})$. Thus, by reversing the order of summation in the above expression, we see that the total work is bounded by

$$C\log N \Bigg[\#\big(\mathrm{CZ}_{\mathrm{main}}(\mathcal{A})\big) + \sum_{Q\in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A}^-)} \Big[\sum_{\omega\in\Omega(Q,\mathcal{A}^-)} \mathrm{depth}(\omega) + \#\big(\Xi(Q,\mathcal{A}^-)\big)\Big] + \sum_{\text{keystone } Q^\# \in \mathrm{CZ}(\mathcal{A}^-)} \sum_{\omega\in\Omega^{\mathrm{new}}(Q^\#)} \mathrm{depth}(\omega) + \#\big(\mathrm{BD}(\mathcal{A}^-)\big)\Bigg].$$

According to the main technical results for $\mathcal{A}^-$ and (1.181), the sum of terms inside the curly brackets in the first line above is bounded by CN. According to the algorithm MAKE NEW ASSISTS AND ASSIGN KEYSTONE JETS in Section 1.4.1, the term on the second line above is bounded by CN. According to the KEYSTONE-ORACLE, the term on the last line above is bounded by CN. Hence, with work at most $CN\log N$, we can compute the lists $\Omega(\widehat{Q})$ and $\Xi(\widehat{Q})$ for all the non-simple cubes $\widehat{Q} \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A})$.

Similarly, we see that the computation of the lists $\Omega(\widehat{Q})$ and $\Xi(\widehat{Q})$ for *all* the non-simple cubes $\widehat{Q} \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A})$ requires space at most CN.

Next, we explain how to define a linear extension operator associated to a non-simple $\widehat{Q} \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A})$ as in the main technical results for $\mathcal{A}$.

We define the map $T_{\widehat{Q}} : \mathbb{X}(E \cap \frac{65}{64}\widehat{Q}) \oplus \mathcal{P} \to \mathbb{X}$ as in Proposition 3, and set $T_{(\widehat{Q},\mathcal{A})} := T_{\widehat{Q}}$.

We perform the one-time work of the algorithm COMPUTE NEW EXTENSION OPERATOR (Section 1.4.5). We thus obtain a query algorithm for $T_{\widehat{Q}}$. Given $\underline{x} \in Q^\circ$,

we can compute a short form description of the the $\Omega(\widehat{Q})$-assisted bounded depth linear functional

$$(f, P) \mapsto \partial^{\beta}\left[J_{\underline{x}} T_{\widehat{Q}}(f, P)\right](\underline{x}) \qquad \text{for every } \beta \in \mathcal{M}.$$

This computation requires work at most $C \log N$ per query point.

Proposition 3 states that $T_{\widehat{Q}}(f, P) = f$ on $(1 + a(\mathcal{A}))\widehat{Q} \cap E$, and

$$\|T_{\widehat{Q}}(f, P)\|_{\mathbb{X}((1+a(\mathcal{A}))\widehat{Q})} + \|T_{\widehat{Q}}(f, P) - P\|_{L^p((1+a(\mathcal{A}))\widehat{Q})} \leq C \cdot M_{\widehat{Q}}(f, P)$$

for any $(f, P) \in \mathbb{X}(\frac{65}{64}\widehat{Q} \cap E) \oplus \mathcal{P}$, where

$$M_{\widehat{Q}}(f, P) = \Big(\sum_{\xi \in \Xi(\widehat{Q})} |\xi(f, P)|^p \Big)^{1/p}.$$

This proves **(E1)** and **(E2)** in the main technical results for $\mathcal{A}$.

We have thus treated all the non-simple cubes in $\mathrm{CZ}_{\mathrm{main}}(\mathcal{A})$.

1.8.2. Case II: Simple cubes. We suppose that $\widehat{Q} \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A})$ *is* c_*-simple. We will explain how to construct the objects in the main technical results for $\mathcal{A}$ relevant to $\widehat{Q}$.

We have computed lists $\Omega(Q, \mathcal{A}^-)$ and $\Xi(Q, \mathcal{A}^-)$ of linear functionals on $\mathbb{X}(E \cap (65/64)Q)$ and $\mathbb{X}(E \cap (65/64)Q) \oplus \mathcal{P}$, respectively, for each $Q \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A}^-)$. See the main technical results for $\mathcal{A}^-$. Each functional in $\Xi(Q, \mathcal{A}^-)$ has $\Omega(Q, \mathcal{A}^-)$-assisted bounded depth and is given in short form.

From (1.19) and (1.20), we know that

$$M_{(Q,\mathcal{A}^-)}(f, R) := \Big(\sum_{\xi \in \Xi(Q,\mathcal{A}^-)} |\xi(f, R)|^p \Big)^{1/p} \tag{1.184}$$

satisfies

$$c \cdot \|(f, R)\|_{(1+a)Q} \leq M_{(Q,\mathcal{A}^-)}(f, R) \leq C \cdot \|(f, R)\|_{\frac{65}{64}Q}. \tag{1.185}$$

Here, $a := a(\mathcal{A}^-) \in (0, 1/64]$ is a universal constant in the main technical results for $\mathcal{A}^-$.

Recall that we have fixed a universal constant $t_G \in (0, 1/64]$ satisfying (1.90).

We define

$$\Omega(\widehat{Q}, \mathcal{A}) := \bigcup_{\substack{Q \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A}^-) \\ Q \subset (1+t_G)\widehat{Q}}} \Omega(Q, \mathcal{A}^-). \quad \text{and} \quad \Xi(\widehat{Q}, \mathcal{A}) := \bigcup_{\substack{Q \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A}^-) \\ Q \subset (1+t_G)\widehat{Q}}} \Xi(Q, \mathcal{A}^-).$$

Each $Q \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A}^-)$ participates in the above union for at most C distinct $\widehat{Q} \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A})$. This is a consequence of the bounded overlap of $\{\frac{65}{64}\widehat{Q} : \widehat{Q} \in$

$\mathrm{CZ}_{\mathrm{main}}(\mathcal{A})\}$, since $t_G \le 1/64$. We can thus compute the lists $\Omega(\widehat{Q},\mathcal{A})$ for all c_*-simple cubes $\widehat{Q} \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A})$, using work at most

$$C \cdot \sum_{Q \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A}^-)} \sum_{\omega \in \Omega(Q,\mathcal{A}^-)} \mathrm{depth}(\omega) \le CN,$$

and we can compute the lists $\Xi(\widehat{Q},\mathcal{A})$ for all c_*-simple cubes $\widehat{Q} \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A})$, using work at most

$$C \cdot \sum_{Q \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A}^-)} \big\{1 + \#\big[\Xi(Q,\mathcal{A}^-)\big]\big\} \le CN.$$

(The upper bound by CN on these sums is stated in the main technical results for $\mathcal{A}^-$.) We do not attempt to remove duplicates from the lists $\Omega(\widehat{Q},\mathcal{A})$ and $\Xi(\widehat{Q},\mathcal{A})$, which are computed simply by copying.

When we copy the functionals in the list $\Omega(Q,\mathcal{A}^-)$, for $Q \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A}^-)$, $Q \subset (1+t_G)\widehat{Q}$, into the list $\Omega(\widehat{Q},\mathcal{A})$, we mark each functional in $\Omega(Q,\mathcal{A}^-)$ ($Q \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A}^-)$, $Q \subset (1+t_G)\widehat{Q}$) with a pointer to its position in the list $\Omega(\widehat{Q},\mathcal{A})$. This requires total extra work at most CN.

Each functional $\xi \in \Xi(\widehat{Q},\mathcal{A})$ has $\Omega(Q,\mathcal{A}^-)$-assisted bounded depth for some $Q \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A}^-)$ with $Q \subset (1+t_G)\widehat{Q}$, hence ξ has $\Omega(\widehat{Q},\mathcal{A})$-assisted bounded depth, because $\Omega(Q,\mathcal{A}^-)$ is a sublist of $\Omega(\widehat{Q},\mathcal{A})$. We can compute a short form of ξ in terms of the assists $\Omega(\widehat{Q},\mathcal{A})$ by using the pointers from $\Omega(Q,\mathcal{A}^-)$ into $\Omega(\widehat{Q},\mathcal{A})$ (see Remark 4). This requires a constant amount of work per functional ξ. We assume that this work was carried out when we formed the lists $\Xi(\widehat{Q},\mathcal{A})$.

We fix $\widehat{Q} \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A})$ such that $\widehat{Q}$ is c_*-simple.

As in the main technical results for $\mathcal{A}$, we define

$$\big[M_{(\widehat{Q},\mathcal{A})}(f,P)\big]^p = \sum_{\xi \in \Xi(\widehat{Q},\mathcal{A})} |\xi(f,P)|^p = \sum_{\substack{Q \in \mathrm{CZ}_{\mathrm{main}}(\mathcal{A}^-) \\ Q \subset (1+t_G)\widehat{Q}}} \big[M_{(Q,\mathcal{A}^-)}(f,P)\big]^p. \tag{1.186}$$

We next define an extension operator $T_{(\widehat{Q},\mathcal{A})} : \mathbb{X}(E \cap (65/64)\widehat{Q}) \oplus \mathcal{P} \to \mathbb{X}$. We follow an argument in Section 1.4.5.

We define the *covering cubes*

$$\mathcal{I}_{\mathrm{cov}}(\widehat{Q}) := \big\{Q \in \mathrm{CZ}(\mathcal{A}^-) : Q \subset (1+t_G)\widehat{Q}\big\}.$$

Thanks to our assumption (1.90), we can choose a universal constant $a_{\mathrm{new}} = a_{\mathrm{new}}(t_G)$ satisfying the conclusion of Lemma 11. Hence, since $\widehat{Q}$ is a testing cube, we obtain the following:

Covering property: The cube $(1+a_{\mathrm{new}})\widehat{Q}$ is contained in the union of the cubes $(1+a/2)Q$ over all $Q \in \mathrm{CZ}(\mathcal{A}^-)$ such that $Q \subset (1+t_G)\widehat{Q}$.

Recall that we have defined $a(\mathcal{A}) = a_{\mathrm{new}}$ in (1.147).

We pick cutoff functions $\theta_Q^{\widehat{Q}} \in C^m(\mathbb{R}^n)$, for each $Q \in \mathcal{I}_{\text{cov}}(\widehat{Q})$, with

$$\text{(1.187)} \quad \begin{cases} \sum_{Q \in \mathcal{I}_{\text{cov}}(\widehat{Q})} \theta_Q^{\widehat{Q}} = 1 \text{ on } (1+a_{\text{new}})\widehat{Q}, \\ \text{supp}(\theta_Q^{\widehat{Q}}) \subset (1+a)Q \text{ and } |\partial^\alpha \theta_Q^{\widehat{Q}}| \leq C \cdot \delta_Q^{-|\alpha|} \quad \text{for } |\alpha| \leq m, \text{ and} \\ \theta_Q^{\widehat{Q}} = 1 \text{ near } x_Q, \text{ and } \theta_Q^{\widehat{Q}} = 0 \text{ near } x_{Q'} \text{ for each } Q' \in \mathcal{I}_{\text{cov}}(\widehat{Q}) \setminus \{Q\}. \end{cases}$$

For each $Q \in \mathcal{I}_{\text{cov}}(\widehat{Q})$ we define

$$\text{(1.188)} \quad F_Q^{\widehat{Q}} := \begin{cases} T_{(Q,\mathcal{A}^-)}(f,P), & \text{if } \frac{65}{64}Q \cap E \neq \emptyset \\ P, & \text{if } \frac{65}{64}Q \cap E = \emptyset. \end{cases}$$

We define a linear map $T_{(\widehat{Q},\mathcal{A})} : \mathbb{X}(E \cap \frac{65}{64}\widehat{Q}) \oplus \mathcal{P} \to \mathbb{X}$ by the formula

$$\text{(1.189)} \quad T_{(\widehat{Q},\mathcal{A})}(f,P) := \sum_{Q \in \mathcal{I}_{\text{cov}}(\widehat{Q})} F_Q^{\widehat{Q}} \cdot \theta_Q^{\widehat{Q}}.$$

(Compare to (1.98).)

Here, the maps $T_{(Q,\mathcal{A}^-)}$ are as in the main technical results for $\mathcal{A}^-$; see Section 3 in [1]. Each $T_{(Q,\mathcal{A}^-)}$ has $\Omega(Q,\mathcal{A}^-)$ assisted bounded depth, hence $T_{(Q,\mathcal{A}^-)}$ has $\Omega(\widehat{Q},\mathcal{A})$-assisted bounded depth, since by definition $\Omega(Q,\mathcal{A}^-)$ is a sublist of $\Omega(\widehat{Q},\mathcal{A})$ for each $Q \in \mathcal{I}_{\text{cov}}(\widehat{Q})$.

Therefore, each $T_{(\widehat{Q},\mathcal{A})}$ has $\Omega(\widehat{Q},\mathcal{A})$-assisted bounded depth. We also give a query algorithm for $T_{(\widehat{Q},\mathcal{A})}$: Given $\underline{x} \in Q^\circ$, we compute the map $(f,P) \mapsto J_{\underline{x}} T_{(\widehat{Q},\mathcal{A})}(f,P)$ in short form in terms of the assists $\Omega(\widehat{Q},\mathcal{A})$. We leave details to the reader.

Proposition 17. *Let* $(f,P) \in \mathbb{X}(\frac{65}{64}\widehat{Q} \cap E) \oplus \mathcal{P}$. *Set* $\overline{a} = a_{\text{new}}$. *Then the following properties hold.*

- $T_{(\widehat{Q},\mathcal{A})}(f,P) = f$ *on* $(1+\overline{a})\widehat{Q} \cap E$.
- $\|T_{(\widehat{Q},\mathcal{A})}(f,P)\|_{\mathbb{X}((1+\overline{a})\widehat{Q})} + \delta_{\widehat{Q}}^{-m}\|T_{(\widehat{Q},\mathcal{A})}(f,P) - P\|_{L^p((1+\overline{a})\widehat{Q})} \leq C M_{(\widehat{Q},\mathcal{A})}(f,P)$.

Proof. The proof is analogous to the proof of Proposition 3, except much easier. We spell out the details.

The definition of the linear map in (1.189) is the same as that in (1.98), except that the polynomials $R_Q^{\widehat{Q}}$ used in the functions $F_Q^{\widehat{Q}}$ in (1.95) are replaced by P (compare (1.95) and (1.188)). Thus, to prove our proposition, we may follow parts of the reasoning in the proof of Proposition 3, as long as we substitute $R_Q^{\widehat{Q}}$ everywhere with P.

The functions $F_Q^{\widehat{Q}}$ in (1.188) satisfy

$$\text{(1.190)}\qquad \begin{cases} F_Q^{\widehat{Q}} = f \quad \text{on } (1+a)Q \cap E \\ \|F_Q^{\widehat{Q}}\|_{\mathbb{X}((1+a)Q)} + \delta_Q^{-m}\|F_Q^{\widehat{Q}} - P\|_{L^p((1+a)Q)} \\ \qquad\qquad \le \begin{cases} CM_{(Q,\mathcal{A}^-)}(f,P), & \text{if } \frac{65}{64}Q \cap E \neq \emptyset, \\ 0, & \text{if } \frac{65}{64}Q \cap E = \emptyset. \end{cases} \end{cases}$$

This follows from the main technical results for $\mathcal{A}^-$.

Thus, the function $T_{(\widehat{Q},\mathcal{A})}(f,P)$ defined in (1.189) satisfies the first bullet point of Proposition 17. This is a consequence of the first and second conditions in (1.187), and the first condition in (1.190).

We now prove the second bullet point of Proposition 17.

Let $G = T_{(\widehat{Q},\mathcal{A})}(f,P)$.

The equation (1.101) holds in the present setting if we replace $R_Q^{\widehat{Q}}$ with P, for the same reason as before. (Here, we use the **Covering property**.) Moreover, when we replace $R_Q^{\widehat{Q}}$ with P, the term $A_2(f,P)$ vanishes. Thus, we have

$$\|G\|_{\mathbb{X}((1+\overline{a})\widehat{Q})}^p \lesssim \sum_{\substack{Q\in\mathcal{I}_{cov}(\widehat{Q}) \\ \frac{65}{64}Q\cap E\neq\emptyset}} \left[M_{(Q,\mathcal{A}^-)}(f,P)\right]^p.$$

By definition, the right-hand side is equal to $\left[M_{(\widehat{Q},\mathcal{A})}(f,P)\right]^p$ (see (1.186)). Thus we have proven

$$\|G\|_{\mathbb{X}((1+\overline{a})\widehat{Q})} \le C\cdot M_{(\widehat{Q},\mathcal{A})}(f,P).$$

It remains to show that $\|G-P\|_{L^p((1+\overline{a})\widehat{Q})} \le C\cdot M_{(\widehat{Q},\mathcal{A})}(f,P)$. We proceed directly without referring to the previous arguments. Using (1.189) and the first condition in (1.187), we have

$$G - P = \sum_{Q\in\mathcal{I}_{cov}(\widehat{Q})} \theta_Q^{\widehat{Q}}\cdot(F_Q^{\widehat{Q}} - P) \quad \text{on } (1+\overline{a})\widehat{Q}.$$

Recall that $\theta_Q^{\widehat{Q}}$ is supported on $(1+a)Q$ and $|\theta_Q^{\widehat{Q}}| \le C$ (see (1.187)). Since $\widehat{Q}$ is c_*-simple, at most C cubes Q contribute to the above sum, and $\delta_Q \ge c_*\delta_{\widehat{Q}}$ for each Q. Hence,

$$(\delta_{\widehat{Q}})^{-m}\|G-P\|_{L^p((1+\overline{a})\widehat{Q})}^p \le C\sum_{Q\in\mathcal{I}_{cov}(\widehat{Q})} (\delta_Q)^{-m}\|F_Q^{\widehat{Q}} - P\|_{L^p((1+a)Q)}^p.$$

Hence, using (1.190), we have

$$(\delta_{\widehat{Q}})^{-m}\|G-P\|_{L^p((1+\overline{a})\widehat{Q})}^p \le C\sum_{Q\in\mathcal{I}_{cov}(\widehat{Q})} \left[M_{(Q,\mathcal{A}^-)}(f,P)\right]^p = C\cdot\left[M_{(\widehat{Q},\mathcal{A})}(f,P)\right]^p.$$

This completes the proof of the second bullet point in Proposition 17. □

Lemma 15. *We have*

$$c\|(f,P)\|_{(1+a_{\mathrm{new}})\widehat{Q}} \le M_{(\widehat{Q},\mathcal{A})}(f,P) \le C\|(f,P)\|_{\frac{65}{64}\widehat{Q}}.$$

Proof. The inequality $\|(f,P)\|_{(1+a_{\mathrm{new}})\widehat{Q}} \le CM_{(\widehat{Q},\mathcal{A})}(f,P)$ is an easy consequence of Proposition 17 and the definition of the trace seminorm. Thus, the only task is to prove the second inequality, $M_{(\widehat{Q},\mathcal{A})}(f,P) \le C\|(f,P)\|_{\frac{65}{64}\widehat{Q}}$.

First, the upper bound in (1.185) implies that

$$\left[M_{(\widehat{Q},\mathcal{A})}(f,P)\right]^p \le C \sum_{\substack{Q\in CZ_{\mathrm{main}}(\mathcal{A}^-)\\ Q\subset(1+t_G)\widehat{Q}}} \|(f,P)\|^p_{\frac{65}{64}Q}.$$

Since $\widehat{Q}$ is c_*-simple, each cube Q relevant to the above sum satisfies $\delta_Q \ge c_*\delta_{\widehat{Q}}$. Moreover, Lemma 11 implies that $\frac{65}{64}Q \subset \frac{65}{64}\widehat{Q}$ for each relevant Q; recall (1.90). Hence, each term $\|(f,P)\|_{\frac{65}{64}Q}$ is bounded by $C\|(f,P)\|_{\frac{65}{64}\widehat{Q}}$ thanks to Lemma 14 in [1]. Moreover, the number of terms is at most a universal constant, hence

$$M_{(\widehat{Q},\mathcal{A})}(f,P) \le C\|(f,P)\|_{\frac{65}{64}\widehat{Q}}.$$

This completes the proof of the lemma. □

We have produced lists $\Omega(\widehat{Q},\mathcal{A})$ and $\Xi(\widehat{Q},\mathcal{A})$, and we have defined a linear map $T_{(\widehat{Q},\mathcal{A})}$ that satisfy the conditions in the main technical results for $\mathcal{A}$ (see Section 3 in [1]), for every $\widehat{Q} \in CZ_{\mathrm{main}}(\mathcal{A})$ that is c_*-simple. We have remarked that one can easily produce a query algorithm for $T_{(\widehat{Q},\mathcal{A})}$. We have performed these computations using work at most $CN\log N$ in space CN.

We have thus treated all the simple cubes in $CZ_{\mathrm{main}}(\mathcal{A})$

1.8.3. Closing remarks. All the previously defined objects satisfy the conditions set down in Section 3 of [1] with many of the constants depending on ϵ, and with $\epsilon_2(\mathcal{A}) = \epsilon^\kappa$, $\epsilon_1(\mathcal{A}) = \epsilon^{1/\kappa}$. We have computed a list of assists $\Omega(\widehat{Q},\mathcal{A})$, and a list of assisted functionals $\Xi(\widehat{Q},\mathcal{A})$, and we have given a query algorithm for a linear map $T_{(\widehat{Q},\mathcal{A})}$ for each $\widehat{Q} \in CZ_{\mathrm{main}}(\mathcal{A})$, using one-time work at most $CN\log N$ in space CN. In particular, the bound on the required space implies that

$$\begin{cases} \displaystyle\sum_{\widehat{Q}\in CZ_{\mathrm{main}}(\mathcal{A})} \sum_{\omega\in\Omega(\widehat{Q},\mathcal{A})} \mathrm{depth}(\omega) \le CN, \quad \text{and} \\ \displaystyle\sum_{\widehat{Q}\in CZ_{\mathrm{main}}(\mathcal{A})} \#\left[\Xi(\widehat{Q},\mathcal{A})\right] \le CN. \end{cases}$$

We now fix ϵ to be a universal constant, small enough so that the previous results hold. That completes the induction step, and thus we have achieved the main technical results for $\mathcal{A}$.

2. Proofs of the main theorems

2.1. Extension in homogeneous Sobolev spaces

In this section we prove our main theorem concerning homogeneous Sobolev spaces $\mathbb{X} = L^{m,p}(\mathbb{R}^n)$ $(p > n)$, which reads as follows.

Theorem 1. *Let* $E \subset \mathbb{R}^n$ *satisfy* $N = \#(E) \geq 2$.

We produce lists Ω *and* Ξ*, consisting of functionals on* $\mathbb{X}(E) = \{f : E \to \mathbb{R}\}$*, with the following properties.*

- *The sum of* $\operatorname{depth}(\omega)$ *over all* $\omega \in \Omega$ *is bounded by* CN*. The number of functionals in* Ξ *is at most* CN*.*
- *Each functional* ξ*, in* Ξ *has* Ω*-assisted depth at most* C*. The functionals in* Ω *and* Ξ *are represented in their short form.*
- *For all* $f \in \mathbb{X}(E)$ *we have*

$$c\|f\|_{\mathbb{X}(E)} \leq \Big[\sum_{\xi\in\Xi} |\xi(f)|^p\Big]^{1/p} \leq C\|f\|_{\mathbb{X}(E)}.$$

Moreover, there exists a linear map $T : \mathbb{X}(E) \to \mathbb{X}$ *with the following properties.*

- T *has* Ω*-assisted depth at most* C*.*
- $Tf = f$ *on* E *and* $\|Tf\|_{\mathbb{X}} \leq C\|f\|_{\mathbb{X}(E)}$ *for all* $f \in \mathbb{X}(E)$*.*
- *We produce a query algorithm that operates as follows.*

  Given a point $\underline{x} \in \mathbb{R}^n$*, we compute a short form description of the* Ω*-assisted bounded depth linear map* $\mathbb{X}(E) \ni f \mapsto J_{\underline{x}}(Tf) \in \mathcal{P}$ *using work and storage at most* $C \log N$*.*

The computations above require one-time work at most $CN \log N$ *in space* CN*.*

By translating and rescaling, we may assume without loss of generality that $E \subset \frac{1}{32}Q^\circ$, with $Q^\circ = [0,1)^n$.

We deduce Theorem 1 from the main technical results for $\mathcal{A} = \emptyset$. Recall that we have achieved the following (see Section 3 in [1]).

- There is a decomposition CZ of Q° into dyadic cubes. Every point $x \in Q^\circ$ belongs to a unique cube $Q_x \in CZ$.
- We produce a CZ-ORACLE.

  The CZ-ORACLE accepts a query point $\underline{x} \in Q^\circ$. The response to a query $\underline{x}$ is the list of all $Q \in CZ$ such that $\frac{65}{64}Q$ contains $\underline{x}$. The work and storage required to answer a query are at most $C \log N$.
- If $Q, Q' \in CZ$ and $Q \leftrightarrow Q'$ then $\frac{1}{2}\delta_Q \leq \delta_{Q'} \leq 2\delta_Q$.
- Each point $x \in \mathbb{R}^n$ is contained in at most C of the cubes $\frac{65}{64}Q$, $Q \in CZ$. (This is an easy consequence of the previous bullet point.)

- If $Q \in CZ$ and $\delta_Q \leq c_*$ then SQ is not tagged with $(\emptyset, e_1)$.

  Recall that any cube is tagged with $(\emptyset, e_1)$; see Remark 20 in [1]. Thus, we learn that $\delta_Q > c_*$ for each $Q \in CZ$. In particular, the cardinality of CZ is bounded by some universal constant C.

- Next, we recall the various assists, functionals and local extension operators described in the main technical results.

  For each $Q \in CZ$ with $\frac{65}{64}Q \cap E \neq \emptyset$, we compute a list of assists $\Omega(Q)$ and assisted functionals $\Xi(Q) \subset \left[\mathbb{X}(E \cap \frac{65}{64}Q) \oplus \mathcal{P}\right]^*$. Each ξ in $\Xi(Q)$ has $\Omega(Q)$-assisted bounded depth. We have

  $$\sum_{\xi \in \Xi(Q)} |\xi(f,P)|^p \leq C \cdot \|(f,P)\|^p_{\frac{65}{64}Q}. \tag{2.1}$$

  We compute these lists of functionals using one-time work at most $CN \log N$ in space CN.

  We also define an $\Omega(Q)$-assisted bounded depth linear map $T_Q : \mathbb{X}(\frac{65}{64}Q \cap E) \oplus \mathcal{P} \to \mathbb{X}$ such that

  $$T_Q(f,P) = f \quad \text{on } E \cap (1+a)Q \tag{2.2}$$

  and

  $$\|T_Q(f,P)\|^p_{\mathbb{X}((1+a)Q)} + \delta_Q^{-mp}\|T_Q(f,P) - P\|^p_{L^p((1+a)Q)} \leq C \cdot \sum_{\xi \in \Xi(Q)} |\xi(f,P)|^p. \tag{2.3}$$

  Given a query $\underline{x} \in Q^\circ$, we can compute the linear map $(f,P) \mapsto J_{\underline{x}}T_Q(f,P)$ in short form in terms of the assists $\Omega(Q)$, using work at most $C \log N$.

This completes the description of the objects from Section 3 in [1].

We list the cubes in CZ with the following procedure. For each $x \in (c_*/10)\mathbb{Z}^n \cap Q^\circ$, we use the CZ-ORACLE to list all the cubes Q in CZ such that $x \in \frac{65}{64}Q$. Each $Q \in CZ$ contains at least one point in $(c_*/10)\mathbb{Z}^n \cap Q^\circ$ (because $Q \subset Q^\circ$ and $\delta_Q > c_*$), hence each $Q \in CZ$ arises in an aforementioned list for some x. We concatenate these lists and then sort the resulting list to remove duplicate cubes.

We now construct a suitable partition of unity adapted to the decomposition CZ. Let $a := a(\mathcal{A})$ with $\mathcal{A} = \emptyset$, as in Section 3 of [1]. Recall that

$$0 < a \leq 1/64. \tag{2.4}$$

For each $Q \in CZ$, let $\widetilde{\theta}_Q \in C^m(\mathbb{R}^n)$ be a function such that

1. $0 \leq \widetilde{\theta}_Q \leq 1$ on $\mathbb{R}^n$,
2. $\widetilde{\theta}_Q \geq 1/2$ on Q,
3. $\widetilde{\theta}_Q = 0$ outside $(1+a)Q$,
4. $|\partial^\beta \widetilde{\theta}_Q(x)| \leq C$ for $x \in \mathbb{R}^n$, $|\beta| \leq m$.

Also, let $\eta : [0, \infty) \to \mathbb{R}$ be a C^m function such that

1. $\eta(t) \geq 1/4$ for $t \geq 0$,
2. $\eta(t) = t$ for $t \geq 1/2$,
3. $|(d/dt)^k \eta(t)| \leq C$ for $t \geq 0$, $k \leq m$.

We can satisfy these conditions by choosing $\widetilde{\theta}_Q$ and η to be appropriate spline functions. We assume that the following queries can be answered using work at most C.

ALGORITHM: COMPUTE AUXILIARY FUNCTIONS

Given $Q \in CZ$ and $\underline{x} \in \mathbb{R}^n$, we can compute the jet $J_{\underline{x}}(\widetilde{\theta}_Q)$. Given $t_* \geq 0$ and an integer $0 \leq k \leq m$, we can compute $\frac{d^k \eta}{dt^k}(t_*)$.

For each $Q \in CZ$ we define

$$\theta_Q(x) = \frac{\widetilde{\theta}_Q(x)}{\eta \circ \Psi(x)}, \quad \text{where} \quad \Psi(x) = \sum_{Q \in CZ} \widetilde{\theta}_Q(x).$$

Clearly, we can answer the following query using work at most C.

ALGORITHM: COMPUTE POU2.

Given $Q \in CZ$ and $\underline{x} \in \mathbb{R}^n$, we compute the jet $J_{\underline{x}}(\theta_Q)$.

By examination of the proof of Lemma 48 in [1], we can prove the following properties.

1. $\theta_Q \in C^m(\mathbb{R}^n)$ is well-defined, by property (1) of $\widetilde{\theta}_Q$ and property (1) of η.
2. $\theta_Q = 0$ outside $(1 + a)Q$, by property (3) of $\widetilde{\theta}_Q$.
3. $|\partial^\alpha \theta_Q(x)| \leq C$ for $x \in \mathbb{R}^n$, $|\alpha| \leq m$, by property (4) of $\widetilde{\theta}_Q$ and properties (1),(3) of η.
4. $\sum_{Q \in CZ} \theta_Q = 1$ on Q°.

To prove property (4), recall that the cubes in CZ cover Q°. Hence, properties (1) and (2) above imply that $\Psi(x) = \sum_{Q \in CZ} \widetilde{\theta}_Q(x) \geq 1/2$ for $x \in Q^\circ$. Hence, property (2) of η implies that $\eta \circ \Psi(x) = \Psi(x)$ for $x \in Q^\circ$, which implies that

$$\sum_{Q \in CZ} \theta_Q(x) = \frac{\sum_{Q \in CZ} \widetilde{\theta}_Q(x)}{\Psi(x)} = 1 \quad \text{for } x \in Q^\circ.$$

This completes the proof of property (4) of $\{\theta_Q\}_{Q \in CZ}$.

We let $\Xi^\circ \subset (\mathbb{X}(E) \oplus \mathcal{P})^*$ be the union of the lists $\Xi(Q)$ for all $Q \in CZ$ such that $E \cap \frac{65}{64} Q \neq \emptyset$. Similarly, we let Ω° be the union of the lists $\Omega(Q)$ for all $Q \in CZ$ such that $E \cap \frac{65}{64} Q \neq \emptyset$. Hence,

$$\sum_{\xi^\circ \in \Xi^\circ} |\xi^\circ(f, P)|^p = \sum_{Q \in CZ} \sum_{\xi \in \Xi(Q)} |\xi(f, P)|^p.$$

The functionals in Ξ° have Ω°-assisted bounded depth. We can express each functional ξ in Ξ° in short form in terms of the assists Ω° by sorting. This requires work at most $C \log N$ for each $\xi \in \Xi^\circ$. Because there are at most CN functionals in Ξ°, this requires work at most $CN \log N$ in total.

Given $(f, P) \in \mathbb{X}(E) \oplus \mathcal{P}$ we define

$$T^\circ(f, P) = \sum_{Q \in CZ} \theta_Q \cdot F_Q, \tag{2.5}$$

where $F_Q = T_Q(f, P)$ whenever $\frac{65}{64} Q \cap E \neq \emptyset$, and $F_Q = P$ whenever $\frac{65}{64} Q \cap E = \emptyset$.

Proposition 18. *The following hold.*

- *The sum of* $\mathrm{depth}(\omega)$ *over all* $\omega \in \Omega^\circ$ *is bounded by* CN. *The cardinality of* Ξ° *is bounded by* CN.
- *Given* $\underline{x} \in Q^\circ$, *we can compute a short form description of the* Ω°*-assisted bounded depth linear map*

$$\mathbb{X}(E) \oplus \mathcal{P} \ni (f, P) \mapsto J_{\underline{x}} T^\circ(f, P) \in \mathcal{P}$$

  using work and storage at most $C \log N$.
- *Given* $(f, P) \in \mathbb{X}(E) \oplus \mathcal{P}$ *we have* $T^\circ(f, P) = f$ *on* E, *and*

$$\|T^\circ(f, P)\|^p_{\mathbb{X}(Q^\circ)} + \|T^\circ(f, P) - P\|^p_{L^p(Q^\circ)} \leq C \sum_{\xi \in \Xi^\circ} |\xi(f, P)|^p.$$

- *Given* $(f, P) \in \mathbb{X}(E) \oplus \mathcal{P}$, *we have*

$$\sum_{\xi \in \Xi^\circ} |\xi(f, P)|^p \leq C \cdot \|(f, P)\|^p_{\frac{65}{64} Q^\circ}.$$

Proof. From Section 3 in [1], recall that

$$\sum_{\substack{Q \in CZ \\ \frac{65}{64} Q \cap E \neq \emptyset}} \sum_{\omega \in \Omega(Q)} \mathrm{depth}(\omega) \leq CN \quad \text{and} \quad \sum_{\substack{Q \in CZ \\ \frac{65}{64} Q \cap E \neq \emptyset}} \#\big[\Xi(Q)\big] \leq CN.$$

This implies the conclusion of the first bullet point.

We fix a query point $\underline{x} \in Q^\circ$. Then we have

$$J_{\underline{x}} T^\circ(f, P) = \sum_{\substack{Q \in CZ \\ \frac{65}{64} Q \cap E \neq \emptyset}} J_{\underline{x}} \theta_Q \odot_{\underline{x}} J_{\underline{x}} T_Q(f, P) + \sum_{\substack{Q \in CZ \\ \frac{65}{64} Q \cap E = \emptyset}} J_{\underline{x}} \theta_Q \odot_{\underline{x}} P. \tag{2.6}$$

Recall that we have computed a list of all the $Q \in CZ$, and that there are at most C such cubes. We loop over all $Q \in CZ$, and perform the steps below.

- **Step 1.** For each $\alpha \in \mathcal{M}$ we compute $\partial^\alpha(J_{\underline{x}} \theta_Q)(\underline{x})$ using COMPUTE POU2.

• **Step 2.** If $\frac{65}{64}Q \cap E \neq \emptyset$, then we compute the linear map $(f, P) \mapsto J_{\underline{x}} T_Q(f, P)$ in short form in terms of the assists $\Omega(Q)$ (see the main technical results for $\mathcal{A} = \emptyset$). This means that for each $\alpha \in \mathcal{M}$ we compute linear functionals $\lambda^{Q,\alpha} : \mathcal{P} \to \mathbb{R}$ and $\eta^{Q,\alpha} : \mathbb{X}(E) \to \mathbb{R}$, assists $\omega_1^{Q,\alpha}, \ldots, \omega_d^{Q,\alpha} \in \Omega(Q)$, and numbers $\gamma_1^{Q,\alpha}, \ldots, \gamma_d^{Q,\alpha} \in \mathbb{R}$, such that

$$\partial^\alpha(J_{\underline{x}} T_Q(f, P))(0) = \lambda^{Q,\alpha}(P) + \eta^{Q,\alpha}(f) + \sum_{k=1}^{d} \gamma_k^{Q,\alpha} \cdot \omega_k^{Q,\alpha}(f).$$

We guarantee that $\mathrm{depth}(\eta^{Q,\alpha}) + d$ is at most a universal constant C.

From the first bullet point in Proposition 18, we know that $\Omega^\circ = \cup_{Q \in CZ} \Omega(Q)$ contains at most CN functionals. When we formed the list Ω° by concatenation, we assume that we marked each functional in $\Omega(Q)$ with a pointer to its position in the list Ω°. This requires additional one-time work work at most CN. Thus, the previous formula gives a short form representation of the functional $(f, P) \mapsto \partial^\alpha(J_{\underline{x}} T_Q(f, P))(0)$ in terms of the assists Ω°. Therefore, by Taylor's theorem we can compute a short form representation of the functional $(f, P) \mapsto \partial^\alpha(J_{\underline{x}} T_Q(f, P))(\underline{x})$ in terms of the assists Ω°.

From the definition of the product $\odot_{\underline{x}}$ and the computation in **Step 1**, for each $\alpha \in \mathcal{M}$ we can compute a short form of the functional

$$(f, P) \mapsto \partial^\alpha(J_{\underline{x}} \theta_Q \odot_{\underline{x}} J_{\underline{x}} T_Q(f, P))(\underline{x})$$

in terms of the assists Ω°.

• **Step 3.** If $\frac{65}{64}Q \cap E = \emptyset$, then for each $\alpha \in \mathcal{M}$ we compute a short form of the functional $(f, P) \mapsto \partial^\alpha(J_{\underline{x}} \theta_Q \odot_{\underline{x}} P)(\underline{x})$. Here, we use Taylor's theorem to compute the change-of-coordinate map $(\partial^\alpha P(0))_{\alpha \in \mathcal{M}} \mapsto (\partial^\alpha P(\underline{x}))_{\alpha \in \mathcal{M}}$. Thus, the desired computation is a consequence of the definition of the product $\odot_{\underline{x}}$ and the result of **Step 1**. This concludes the loop over Q.

For $\alpha \in \mathcal{M}$, we compute a short form of the functional $(f, P) \mapsto \partial^\alpha(J_{\underline{x}} T^\circ(f, P))(\underline{x})$ in terms of the assists Ω° by adding together the short form representations of the functionals determined at the end of **Step 2** and **Step 3** (see the formula (2.6)). Therefore, we can compute a short form of the functional $(f, P) \mapsto \partial^\alpha(J_{\underline{x}} T^\circ(f, P))(0)$ in terms of the assists Ω°. This is a consequence of Taylor's theorem and the previous computation. The reader may easily check that the above computation requires work at most $C \log N$ per query $\underline{x} \in Q^\circ$. This completes the proof of the second bullet point in Proposition 18.

Fix $x \in E$. Then

$$T^\circ(f, P)(x) = \sum_{\substack{Q \in CZ \\ \frac{65}{64}Q \cap E \neq \emptyset}} \theta_Q(x) \cdot T_Q(f, P)(x) + \sum_{\substack{Q \in CZ \\ \frac{65}{64}Q \cap E = \emptyset}} \theta_Q(x) \cdot P(x). \tag{2.7}$$

Recall that θ_Q is supported on the cube $(1 + a)Q$, which is contained in $\frac{65}{64}Q$. (See (2.4).)

For the Q arising in the second sum in (2.7) we learn that $\theta_Q(x) = 0$, since the support of θ_Q does not intersect E.

For the Q arising in the first sum in (2.7), if $x \in (1+a)Q$ then $T_Q(f,P)(x) = f(x)$. Otherwise, if $x \notin (1+a)Q$ then $\theta_Q(x) = 0$, due to the support properties of θ_Q.

Hence, $T^\circ(f,P)(x) = \sum_{Q\in CZ} \theta_Q(x) f(x) = f(x)$, because $\sum_{Q\in CZ} \theta_Q = 1$ on E (recall that $E \subset Q^\circ$). Hence, $T^\circ(f,P) = f$ on E, as desired.

We plan to apply Lemma 49 from [1], where $P_Q = P$ and F_Q ($Q \in CZ$) is defined below (2.5). We note that CZ is a dyadic decomposition of Q° which satisfies good geometry. We are also given a CZ-Oracle. These and other properties can be found in the main technical results in Section 3 of [1]. We plan to take $\widehat{Q} = Q^\circ$ in the setting of Section 4.6.5 of [1]. We note that the conditions (4.53) and (4.54) contained therein are obvious consequences of the fact that Q° is decomposed as a union of all $Q \in CZ$. Thus, we may apply Lemma 49 of [1]. This gives that

$$\begin{aligned}
\|T^\circ(f,P)\|^p_{\mathbb{X}(Q^\circ)} &\lesssim \sum_{Q\in CZ} \left[\|F_Q\|^p_{\mathbb{X}((1+a)Q)} + \delta_Q^{-mp} \|F_Q - P\|^p_{L^p((1+a)Q)} \right] \\
&= \sum_{\substack{Q\in CZ \\ \frac{65}{64}Q\cap E\neq\emptyset}} \left[\|T_Q(f,P)\|^p_{\mathbb{X}((1+a)Q)} + \delta_Q^{-mp} \|T_Q(f,P) - P\|^p_{L^p((1+a)Q)} \right] \\
&\lesssim \sum_{\substack{Q\in CZ \\ \frac{65}{64}Q\cap E\neq\emptyset}} \sum_{\xi\in\Xi(Q)} |\xi(f,P)|^p \quad \text{(see (2.3))}. \qquad (2.8)
\end{aligned}$$

(All the terms with $\frac{65}{64}Q \cap E = \emptyset$ vanish, since $F_Q = P$ is an $(m-1)$-st degree polynomial.)

Since $\sum_{Q\in CZ} \theta_Q = 1$ on Q°, we have

$$T^\circ(f,P) - P = T^\circ(f,P) - \sum_{Q\in CZ} \theta_Q \cdot P = \sum_{\substack{Q\in CZ \\ \frac{65}{64}Q\cap E\neq\emptyset}} (T_Q(f,P) - P)\cdot\theta_Q \text{ on } Q^\circ.$$

There are at most C terms in the above sum. Thus, since each θ_Q is supported on $(1+a)Q$ and $\|\theta_Q\|_{L^\infty} \le C$, we have

$$\begin{aligned}
\|T^\circ(f,P) - P\|^p_{L^p(Q^\circ)} &\lesssim \sum_{\substack{Q\in CZ \\ \frac{65}{64}Q\cap E\neq\emptyset}} \|T_Q(f,P) - P\|^p_{L^p((1+a)Q)} \qquad (2.9) \\
&\lesssim \sum_{\substack{Q\in CZ \\ \frac{65}{64}Q\cap E\neq\emptyset}} \sum_{\xi\in\Xi(Q)} |\xi(f,P)|^p.
\end{aligned}$$

Here, in the last inequality we used (2.3). (Recall that $\delta_Q \le 1$ whenever $Q \in CZ$.)

Summing (2.8) and (2.9) shows that

$$\|T^\circ(f,P)\|^p_{\mathbb{X}(Q^\circ)} + \|T^\circ(f,P) - P\|^p_{L^p(Q^\circ)} \lesssim \sum_{\substack{Q\in CZ \\ \frac{65}{64}Q\cap E\neq\emptyset}} \sum_{\xi\in\Xi(Q)} |\xi(f,P)|^p.$$

The right-hand expression is equal to $\sum_{\xi\in\Xi^\circ} |\xi(f,P)|^p$. This completes the proof of the third bullet point in Proposition 18.

From (2.1), recall that

$$\sum_{\substack{Q\in CZ\\ \frac{65}{64}Q\cap E\neq\emptyset}}\sum_{\xi\in\Xi(Q)}|\xi(f,P)|^p\le C\sum_{\substack{Q\in CZ\\ \frac{65}{64}Q\cap E\neq\emptyset}}\|(f,P)\|^p_{\frac{65}{64}Q}. \tag{2.10}$$

We have $\|(f,P)\|_{\frac{65}{64}Q}\lesssim\|(f,P)\|_{\frac{65}{64}Q^\circ}$ for each $Q\in CZ$. Here, we apply Lemma 14 from [1] and use the fact that $\delta_Q\ge c_*$ for all $Q\in CZ$. Since the cardinality of CZ is bounded by a universal constant, we conclude that

$$\sum_{\substack{Q\in CZ\\ \frac{65}{64}Q\cap E\neq\emptyset}}\sum_{\xi\in\Xi(Q)}|\xi(f,P)|^p\le C\cdot\|(f,P)\|^p_{\frac{65}{64}Q^\circ}.$$

This implies the fourth bullet point in Proposition 18. This completes the proof of Proposition 18. □

We will now construct the various assists, functionals, and the extension operator from Theorem 1.

Computing a near-optimal jet. Each functional $\xi_\ell\in\Xi^\circ$ is given in the form

$$\xi_\ell(f,R)=\lambda_\ell(f)+\sum_{a=1}^{I_\ell}\mu_{\ell a}\omega_{\ell a}(f)+\sum_{\alpha\in\mathcal{M}}\check{\mu}_{\ell\alpha}\cdot\partial^\alpha R(0)\qquad\text{for }\ell=1,\dots,L;\ \text{here, } L=\#(\Xi^\circ)\le CN. \tag{2.11}$$

Here, $\omega_{\ell a}\in\Omega^\circ$; λ_ℓ is a linear functional; $\mu_{\ell a}$ and $\check{\mu}_{\ell\alpha}$ are real coefficients; and $\mathrm{depth}(\lambda_\ell)=\mathcal{O}(1)$, $I_\ell=\mathcal{O}(1)$. In this discussion, we write $X=\mathcal{O}(Y)$ to indicate that $X\le CY$ for a universal constant C.

Applying the algorithm OPTIMIZE VIA MATRIX (see Section 2.8 of [1]), we find a matrix $(b_{\alpha\ell})_{\alpha\in\mathcal{M},\,\ell=1,\dots,L}$ such that the sum of the p-th powers of the $|\xi_\ell(f,R)|$ $(\ell=1,\dots,L)$ in (2.11) is essentially minimized for fixed f by setting

$$\partial^\alpha R(0)=\sum_{\ell=1}^{L}b_{\alpha\ell}\Big[\lambda_\ell(f)+\sum_{a=1}^{I_\ell}\mu_{\ell a}\omega_{\ell a}(f)\Big]\equiv\omega_\alpha^{\mathrm{new}}(f). \tag{2.12}$$

We express the functionals $\omega_\alpha^{\mathrm{new}}$ in short form. We first compute real coefficients $(\mu_{\alpha x})_{x\in E}$ $(\alpha\in\mathcal{M})$ so that

$$\omega_\alpha^{\mathrm{new}}(f)=\sum_{x\in E}\mu_{\alpha x}\cdot f(x). \tag{2.13}$$

We achieve this by summing all the coefficients $b_{\alpha\ell}\cdot\mu_{\ell a}$ in (2.12) that correspond to the same functional $\omega=\omega_{\ell a}$. (We accomplish this by sorting over Ω°.) We can convert the resulting expression into the form (2.13), by sorting over E. Hence, we can express each $\omega_\alpha^{\mathrm{new}}$ in short form using work $\mathcal{O}(N\log N)$, since $\sum_{\omega\in\Omega^\circ}\mathrm{depth}(\omega)\le CN$.

Define the map $\mathfrak{R} : \mathbb{X}(E) \to \mathcal{P}$ by the formula $\mathfrak{R}(f)(x) = \sum_{\alpha\in\mathcal{M}} \frac{1}{\alpha!}\omega_\alpha^{\text{new}}(f)\cdot x^\alpha$. Hence, we have the key condition

$$\sum_{\xi\in\Xi^\circ} |\xi(f,\mathfrak{R}(f))|^p \le C\cdot \sum_{\xi\in\Xi^\circ} |\xi(f,R)|^p \quad \text{for any } R\in\mathcal{P}. \tag{2.14}$$

Picking a cutoff function. We choose a function $\theta^\circ \in C^m(\mathbb{R}^n)$ such that

1. $\theta^\circ = 0$ on $\mathbb{R}^n \setminus Q^\circ$,
2. $\theta^\circ = 1$ on E,
3. $|\partial^\alpha\theta^\circ(x)| \le C$ for $x\in\mathbb{R}^n$, $|\alpha|\le m$,
4. Given $\underline{x}\in\mathbb{R}^n$, we can compute $J_{\underline{x}}\theta^\circ$ using work and storage at most C.

We can arrange these conditions by taking θ° to be a spline function that equals 1 on $\frac{1}{32}Q^\circ$ and equals 0 on $\mathbb{R}^n\setminus Q^\circ$; recall that $E\subset\frac{1}{32}Q^\circ$.

Main definitions.

- Let the list $\Omega\subset(\mathbb{X}(E))^*$ consist of all the functionals ω in Ω° and all the functionals of the form $f\mapsto\partial^\beta[\mathfrak{R}(f)](0) = \omega_\beta^{\text{new}}(f)$ for all $\beta\in\mathcal{M}$.
- Let the list $\Xi\subset(\mathbb{X}(E))^*$ consist of all the functionals $f\mapsto\xi^\circ(f,\mathfrak{R}(f))$ where $\xi^\circ\in\Xi^\circ$. Hence,
$$\sum_{\xi\in\Xi}|\xi(f)|^p = \sum_{\xi^\circ\in\Xi^\circ}|\xi^\circ(f,\mathfrak{R}(f))|^p.$$
- Let $T:\mathbb{X}(E)\to\mathbb{X}$ be defined by the formula
$$Tf = \theta^\circ\cdot T^\circ(f,\mathfrak{R}(f)) + (1-\theta^\circ)\cdot\mathfrak{R}(f).$$

We note that the functionals $\xi\in\Xi$ and the map T have Ω-assisted bounded depth.

We can list all the functionals in Ξ and Ω, with each functional expressed in short form, using work and storage at most CN. (We have already computed the functionals in the lists Ω° and Ξ°, and we have computed the map $f\mapsto\mathfrak{R}(f)$, all expressed in short form.)

We give a query algorithm for T. A query consists of a point $\underline{x}\in\mathbb{R}^n$. Then, using property (1) of θ° we can write

$$J_{\underline{x}}(Tf) = \begin{cases} J_{\underline{x}}\theta^\circ \odot_{\underline{x}} J_{\underline{x}}T^\circ + J_{\underline{x}}(1-\theta^\circ)\odot_{\underline{x}}\mathfrak{R}(f) & \text{if } \underline{x}\in Q^\circ \\ \mathfrak{R}(f) & \text{if } \underline{x}\notin Q^\circ. \end{cases}$$

We test whether $\underline{x}\in Q^\circ$ or $\underline{x}\in\mathbb{R}^n\setminus Q^\circ$. If $\underline{x}\in Q^\circ$, then we compute the map $f\mapsto J_{\underline{x}}(Tf)$ in short form in terms of the assists Ω. This uses the query algorithm for T° and property (4) of θ°. Note that we can computate a short form representation of the $\odot_{\underline{x}}$-product or sum of polynomial-valued maps which are given in short form, using work at most C. If $\underline{x}\in\mathbb{R}^n\setminus Q^\circ$, then the map is given by $f\mapsto J_{\underline{x}}(Tf) = \mathfrak{R}(f)$, which is given in short form in terms of the assists Ω. This completes the description of the query algorithm for T. The query work is at most $C\log N$, as promised in Theorem 1.

Main conditions. According to the first bullet point in Proposition 18, we have

$$\#(\Xi) \le CN, \quad \text{and} \quad \sum_{\omega\in\Omega} \operatorname{depth}(\omega) \le CN. \tag{2.15}$$

From property (2) of θ°, and since $T^\circ(f, \mathfrak{R}(f)) = f$ on E, we see that

$$Tf = f \quad \text{on } E. \tag{2.16}$$

We now estimate $\|Tf\|_{\mathbb{X}}$. A standard argument shows that

$$\|Tf\|_{\mathbb{X}} \le C \cdot \left[\|T^\circ(f, \mathfrak{R}(f))\|_{\mathbb{X}(Q^\circ)} + \|T^\circ(f, \mathfrak{R}(f)) - \mathfrak{R}(f)\|_{L^p(Q^\circ)}\right].$$

(See the proof of Lemma 49 in [1].) According to the third bullet point in Proposition 18, we therefore have

$$\|Tf\|_{\mathbb{X}}^p \le C \cdot \sum_{\xi^\circ\in\Xi^\circ} |\xi^\circ(f, \mathfrak{R}(f))|^p = C \cdot \sum_{\xi\in\Xi} |\xi(f)|^p. \tag{2.17}$$

We now observe that

$$\begin{aligned}\sum_{\xi\in\Xi} |\xi(f)|^p = \sum_{\xi^\circ\in\Xi^\circ} |\xi^\circ(f, \mathfrak{R}(f))|^p &\le C \inf_{R\in\mathcal{P}} \sum_{\xi^\circ\in\Xi^\circ} |\xi^\circ(f, R)|^p \quad \text{(see (2.14))} \\ &\le C \inf_{R\in\mathcal{P}} \|(f, R)\|_{\frac{65}{64}Q^\circ}^p \quad \text{(see Proposition 18).}\end{aligned} \tag{2.18}$$

Moreover, by definition of the trace seminorm,

$$\|(f, R)\|_{\frac{65}{64}Q^\circ} = \inf_{F\in\mathbb{X}} \left\{\|F\|_{\mathbb{X}(\frac{65}{64}Q^\circ)} + \|F - R\|_{L^p(\frac{65}{64}Q^\circ)} : F = f \text{ on } E\right\}.$$

Note that $\|F - R\|_{L^p(\frac{65}{64}Q^\circ)} \le C\|F\|_{\mathbb{X}(\frac{65}{64}Q^\circ)}$ if we choose $R = J_{x_{Q^\circ}} F$ (thanks to the Sobolev inequality). Therefore,

$$\inf_{R\in\mathcal{P}} \|(f, R)\|_{\frac{65}{64}Q^\circ} \le C \cdot \inf_{F\in\mathbb{X}} \left\{\|F\|_{\mathbb{X}} : F = f \text{ on } E\right\} = C \cdot \|f\|_{\mathbb{X}(E)}.$$

The previous estimates imply that

$$\sum_{\xi\in\Xi} |\xi(f)|^p \le C \cdot \|f\|_{\mathbb{X}(E)}^p. \tag{2.19}$$

Finally, we have $\|f\|_{\mathbb{X}(E)} = \inf_{F\in\mathbb{X}}\left\{\|F\|_{\mathbb{X}} : F = f \text{ on } E\right\} \le \|Tf\|_{\mathbb{X}}$, thanks to (2.16). This estimate and (2.17) imply that

$$c \cdot \|f\|_{\mathbb{X}(E)}^p \le \sum_{\xi\in\Xi} |\xi(f)|^p. \tag{2.20}$$

In view of (2.15)–(2.20), we have proven Theorem 1. □

2.2. Extension in inhomogeneous Sobolev spaces

Let $E \subset \mathbb{R}^n$ be finite, and let $N = \#(E)$.

The inhomogeneous Sobolev space $W^{m,p}(\mathbb{R}^n) \subset L^{m,p}(\mathbb{R}^n)$ consists of real-valued functions F on $\mathbb{R}^n$ such that $\partial^\alpha F \in L^p(\mathbb{R}^n)$ for all $|\alpha| \leq m$. This space is equipped with the norm

$$\|F\|_{W^{m,p}(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n} \sum_{|\alpha| \leq m} |\partial^\alpha F(x)|^p \, dx \Big)^{1/p}.$$

Let $W^{m,p}(E)$ denote the space of functions $f : E \to \mathbb{R}$, equipped with the trace norm

$$\|f\|_{W^{m,p}(E)} = \inf_{F \in W^{m,p}(\mathbb{R}^n)} \big\{ \|F\|_{W^{m,p}(\mathbb{R}^n)} : F = f \text{ on } E \big\}.$$

We use our extension results for the homogeneous Sobolev space $L^{m,p}(\mathbb{R}^n)$ to obtain analogous results for the inhomogeneous Sobolev space $W^{m,p}(\mathbb{R}^n)$. We will exhibit a query algorithm for a linear extension operator $T : W^{m,p}(E) \to W^{m,p}(\mathbb{R}^n)$ and we will compute a formula that approximates the $W^{m,p}(E)$ trace norm. We will do so using one-time work at most $CN \log N$ in space at most CN. Given $\underline{x} \in \mathbb{R}^n$ and $|\alpha| \leq m - 1$, we will explain how to compute $\partial^\alpha Tf(\underline{x})$ using work at most $C \log N$.

2.2.1. Case I. We assume that $N = \#(E) \geq 2$ and that $E \subset \frac{1}{32} Q^\circ$, where $Q^\circ = [0,1)^n$.

We apply Proposition 18 to define an extension operator $T^\circ : W^{m,p}(E) \oplus \mathcal{P} \to W^{m,p}(Q^\circ)$ and lists $\Xi^\circ \subset (W^{m,p}(E) \oplus \mathcal{P})^*$ and $\Omega^\circ \subset (W^{m,p}(E))^*$.

We define a cutoff function θ° on $\mathbb{R}^n$. As in Section 2.1, we assume that the function $\theta^\circ \in C^m(\mathbb{R}^n)$ satisfies $\theta^\circ = 0$ on $\mathbb{R}^n \setminus Q^\circ$, $\theta^\circ = 1$ on E, and $|\partial^\alpha \theta^\circ(x)| \leq C$ for all $x \in \mathbb{R}^n$ and $|\alpha| \leq m$. Furthermore, we assume that we can compute $J_{\underline{x}} \theta^\circ$ for $\underline{x} \in \mathbb{R}^n$ using work at most C. We accomplish this by taking θ° to be an appropriate spline function.

We define a linear map $T : W^{m,p}(E) \to W^{m,p}(\mathbb{R}^n)$ by

$$Tf := \theta^\circ \cdot T^\circ(f, 0) \quad \text{for any } f \in W^{m,p}(E).$$

Proposition 18 states that $T^\circ(f, 0) = f$ on E. Thus, since $\theta^\circ = 1$ on E we have

$$Tf = f \quad \text{on } E. \tag{2.21}$$

We write $J_{\underline{x}}(Tf) = J_{\underline{x}} \theta^\circ \odot_{\underline{x}} J_{\underline{x}} T^\circ(f, 0)$. Hence, we compute $J_{\underline{x}}(Tf) = 0$ whenever $\underline{x} \in \mathbb{R}^n \setminus Q^\circ$ (since $\theta^\circ \equiv 0$ on $\mathbb{R}^n \setminus Q^\circ$). On the other hand, if $\underline{x} \in Q^\circ$ then we can compute the map $f \mapsto J_{\underline{x}} T^\circ(f, 0)$ in short form in terms of the assists Ω° (see Proposition 18), hence we can compute the map $f \mapsto J_{\underline{x}}(Tf)$ in short form by basic algebra (multiplying polynomials). Thus we have given a query algorithm for T.

Proposition 18 states that

$$\|T^\circ(f, 0)\|^p_{L^{m,p}(Q^\circ)} + \|T^\circ(f, 0)\|^p_{L^p(Q^\circ)} \leq C \sum_{\xi \in \Xi^\circ} |\xi(f, 0)|^p \tag{2.22}$$

and

$$\sum_{\xi\in\Xi^\circ} |\xi(f,0)|^p \le C\cdot \inf_F \left\{ \|F\|^p_{L^{m,p}(\frac{65}{64}Q^\circ)} + \|F\|^p_{L^p(\frac{65}{64}Q^\circ)} : F = f \text{ on } E\right\} \tag{2.23}$$
$$\le C\cdot \inf_F \left\{ \|F\|^p_{W^{m,p}(\frac{65}{64}Q^\circ)} : F = f \text{ on } E\right\}.$$

We write $H = Tf = \theta^\circ H^\circ$ with $H^\circ = T^\circ(f,0)$. Recall that θ° is supported on Q° and that the derivatives of θ° are bounded by a constant C. Hence, applying the Leibniz rule we see that

$$\|H\|^p_{W^{m,p}(\mathbb{R}^n)} \le C\cdot \sum_{|\alpha|+|\beta|\le m} \int_{Q^\circ} |\partial^\alpha H^\circ(x)|^p \cdot |\partial^\beta \theta^\circ(x)|^p dx \le C\Big[\sum_{|\alpha|\le m} \|\partial^\alpha H^\circ\|^p_{L^p(Q^\circ)}\Big].$$

Thus, using a Sobolev-type estimate (see Proposition 9 in [1]) we have

$$\|H\|_{W^{m,p}(\mathbb{R}^n)} \le C\cdot \left[\|H^\circ\|_{L^{m,p}(Q^\circ)} + \|H^\circ\|_{L^p(Q^\circ)}\right].$$

We finally apply (2.22) and insert the definitions of H and H° to see that

$$\|Tf\|^p_{W^{m,p}(\mathbb{R}^n)} \le C\cdot \sum_{\xi\in\Xi^\circ} |\xi(f,0)|^p. \tag{2.24}$$

We set $\Omega = \Omega^\circ$. We define a list $\Xi \subset (\mathbb{X}(E))^*$ consisting of the functionals $f \mapsto \xi(f,0)$ for all $\xi \in \Xi^\circ$. The estimates (2.23) and (2.24) imply that

$$c\|Tf\|^p_{W^{m,p}(\mathbb{R}^n)} \le \sum_{\xi\in\Xi} |\xi(f)|^p \le C \inf_F \left\{ \|F\|^p_{W^{m,p}(\frac{65}{64}Q^\circ)} : F = f \text{ on } E\right\}.$$

Moreover, note that $\|f\|_{W^{m,p}(E)} \le \|Tf\|_{W^{m,p}(\mathbb{R}^n)}$, since $Tf = f$ on E.

All the functionals $\xi \in \Xi$ and the map T have Ω-assisted bounded depth. We have given a query algorithm for T, and we have listed the functionals in Ω. We can list the functionals in Ξ, expressed in short form in terms of the assists Ω, using work and storage at most CN. To see this, note that there are at most CN functionals in Ξ°. We determine a short form representation of the functional $f \mapsto \xi(f,0)$ using the short form representation of $(f,P) \mapsto \xi(f,P)$ by just deleting all the coefficients of the variables $(\partial^\alpha P(0))_{\alpha\in\mathcal{M}}$. This requires work at most C per functional ξ.

According to the first bullet point in Proposition 18 we have

$$\#(\Xi) \le CN, \quad \text{and} \quad \sum_{\omega\in\Omega} \text{depth}(\omega) \le CN.$$

The preceding argument establishes the case $N = \#(E) \ge 2$ in the result below.

Proposition 19. *Assume that we are given a finite subset $E \subset \frac{1}{32}Q^\circ$, with $Q^\circ = [0,1)^n$. Let $N = \#(E)$.*

We compute lists Ω and Ξ, consisting of functionals on $W^{m,p}(E) = \{f\colon E \to \mathbb{R}\}$, with the following properties.

- *The sum of* depth(ω) *over all $\omega \in \Omega$ is bounded by CN. The number of functionals in Ξ is at most CN.*

- *Each functional ξ in Ξ has Ω-assisted bounded depth. The functionals in Ω and Ξ are represented in their short form.*
- *For all $f \in W^{m,p}(E)$ we have*

$$c \cdot \|f\|_{W^{m,p}(E)} \leq \Big[\sum_{\xi \in \Xi} |\xi(f)|^p \Big]^{1/p} \leq C \cdot \inf \big\{ \|F\|_{W^{m,p}(\frac{65}{64} Q^\circ)} : F = f \text{ on } E \big\}.$$

Moreover, there exists a linear map $T : W^{m,p}(E) \to W^{m,p}(\mathbb{R}^n)$ with the following properties.

- *T has Ω-assisted depth at most C.*
- *$Tf = f$ on E and $\|Tf\|^p_{W^{m,p}(\mathbb{R}^n)} \leq C \cdot \sum_{\xi \in \Xi} |\xi(f)|^p$ for all $f \in W^{m,p}(E)$.*
- *We produce a query algorithm that operates as follows.*

  Given a query point $\underline{x} \in \mathbb{R}^n$, we compute a short form description of the Ω-assisted bounded depth map $f \mapsto J_{\underline{x}}(Tf)$ using work and storage at most $C \log(2 + N)$.

The computations above require one-time work at most $CN \log(2 + N) + C$ in space at most $CN + C$.

Proof. We have already established the proposition in case $N = \#(E) \geq 2$.

When E is a singleton $\{x^\circ\}$ (i.e., $N = 1$), we define $\Xi = \{\xi_0\}$ and $\Omega = \emptyset$, where $\xi_0(f) := f(x^\circ)$. We define

$$(Tf)(x) = \theta^\circ(x) \cdot f(x^\circ) \quad \text{for any } f : E \to \mathbb{R},$$

where $\theta^\circ \in C^m(\mathbb{R}^n)$ is supported on Q° and $\theta^\circ(x^\circ) = 1$. Moreover, a simple computation shows that $\|Tf\|_{W^{m,p}(\mathbb{R}^n)} \leq C \cdot |\xi_0(f)|$.

On the other hand,

$$|\xi_0(f)| = |f(x^\circ)| \leq C \cdot \|F\|_{W^{m,p}(\frac{65}{64} Q^\circ)}$$

for any $F \in W^{m,p}(\mathbb{R}^n)$ such that $F(x^\circ) = f(x^\circ)$, thanks to the Sobolev inequality. The above computations require a constant amount of work. This completes the proof of the proposition in the case $N = \#(E) = 1$.

When $E = \emptyset$, we define $\Xi = \Omega = \emptyset$. We define $T : W^{m,p}(E) \to W^{m,p}(\mathbb{R}^n)$ to be the trivial (zero) map defined on a zero-dimensional space. These objects vacuously satisfy the conclusion of the proposition. This completes the proof of the result. □

2.2.2. Case II. Here we strengthen Proposition 19 by removing the hypothesis that E is contained in a unit cube. We employ a standard partition of unity argument.

Assume that E is a finite subset of $\mathbb{R}^n$. Let $N = \#(E)$.

We decompose $\mathbb{R}^n$ into a collection of cubes $\{Q_{\mathbf{k}} : \mathbf{k} \in \mathbb{Z}^n\}$. For each $\mathbf{k} = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$ we define

$$Q_{\mathbf{k}} = \big(k_1 \cdot 2^{-10}, k_1 \cdot 2^{-10} + 1\big] \times \cdots \times \big(k_n \cdot 2^{-10}, k_n \cdot 2^{-10} + 1\big].$$

Note that the union of all the $Q_{\mathbf{k}}$ is equal to $\mathbb{R}^n$. Moreover,

(2.25) each point in $\mathbb{R}^n$ is contained in at most C of the cubes $200Q_{\mathbf{k}}$.

We define $E_{\mathbf{k}} := E \cap Q_{\mathbf{k}}$ for each $\mathbf{k} \in \mathbb{Z}^n$. According to the above,

$$\bigcup_{\mathbf{k}\in\mathbb{Z}^n} E_{\mathbf{k}} = E \quad \text{and} \quad \sum_{\mathbf{k}\in\mathbb{Z}^n} \#(E_{\mathbf{k}}) \le C \cdot N.$$

We define

$$\mathcal{I} := \{\mathbf{k} \in \mathbb{Z}^n : E_{\mathbf{k}} \neq \emptyset\}.$$

For each $x \in E$ we can easily list all the indices $\mathbf{k} \in \mathbb{Z}^n$ such that $x \in Q_{\mathbf{k}}$. We concatenate these lists and remove duplicate indices by sorting. Thus, we can compute the collection $\mathcal{I}$. We know that $\#\mathcal{I} \le C \cdot N$ by (2.25). Moreover, the computation of $\mathcal{I}$ requires work at most $CN\log(N+2)$ in space at most CN. For each $\mathbf{k}$ arising from some $x \in E$ as above, we include x in a list associated to $\mathbf{k}$. In this way we construct the subsets $E_{\mathbf{k}}$ for each $\mathbf{k} \in \mathcal{I}$. This computation requires work at most $CN\log(N+2)$.

For each $\mathbf{k} \in \mathcal{I}$ we do the following. According to Proposition 19, we can compute lists $\Xi_{\mathbf{k}}$ and $\Omega_{\mathbf{k}}$ of linear functionals on $W^{m,p}(E_{\mathbf{k}})$. We also give a query algorithm for a linear extension operator $T_{\mathbf{k}} : W^{m,p}(E_{\mathbf{k}}) \to W^{m,p}(\mathbb{R}^n)$ (see below). The following properties hold.

(a) Each $\xi \in \Xi_{\mathbf{k}}$ has $\Omega_{\mathbf{k}}$-assisted bounded depth.

(b) The sum of depth(ω) over all $\omega \in \Omega_{\mathbf{k}}$ is bounded by $C\#(E_{\mathbf{k}})$. The number of functionals in $\Xi_{\mathbf{k}}$ is at most $C \cdot \#(E_{\mathbf{k}})$.

(c) $T_{\mathbf{k}}$ has $\Omega_{\mathbf{k}}$-assisted bounded depth.

(d) $(T_{\mathbf{k}}f)(x) = f(x)$ for all $x \in E_{\mathbf{k}}$.

(e) For any $f \in W^{m,p}(E_{\mathbf{k}})$ we have

$$\sum_{\xi\in\Xi_{\mathbf{k}}} |\xi(f)|^p \le C \inf\left\{\|F\|^p_{W^{m,p}(200Q_{\mathbf{k}})} : F = f \text{ on } E_{\mathbf{k}}\right\} \tag{2.26}$$

and

$$\|T_{\mathbf{k}}f\|^p_{W^{m,p}(\mathbb{R}^n)} \le C \cdot \sum_{\xi\in\Xi_{\mathbf{k}}} |\xi(f)|^p. \tag{2.27}$$

(f) We can query the extension operator. A query consists of a point $\underline{x} \in \mathbb{R}^n$. We respond to the query $\underline{x}$ with a short form description of the $\Omega_{\mathbf{k}}$-assisted bounded depth map $f \mapsto J_{\underline{x}}(T_{\mathbf{k}}f)$. The query work is at most $C\log(2+\#(E_{\mathbf{k}}))$.

(Here, we use the fact that $E_{\mathbf{k}} \subset \frac{1}{32}(200Q_{\mathbf{k}})$. To achieve the preceding results, we apply Proposition 19 to a rescaled and translated copy of $E_{\mathbf{k}}$. We leave details to the reader.)

The above computations require one-time work at most $C\#(E_{\mathbf{k}})\log(\#(E_{\mathbf{k}})+1)$ and storage at most $C\#(E_{\mathbf{k}})$ for each $\mathbf{k} \in \mathcal{I}$. Thus, the total work and space required are at most $CN\log(N+1)+C$ and $CN+C$, respectively.

We will define an extension operator $T : W^{m,p}(E) \to W^{m,p}(\mathbb{R}^n)$ and lists Ξ and Ω consisting of linear functionals on $W^{m,p}(E)$.

- Let $\Omega \subset (W^{m,p}(E))^*$ be the union of the lists $\Omega_{\mathbf{k}}$ for all $\mathbf{k} \in \mathcal{I}$. (If $E = \emptyset$ then we define $\Omega = \emptyset$.)
- Let $\Xi \subset (W^{m,p}(E))^*$ be the union of the lists $\Xi_{\mathbf{k}}$ for all $\mathbf{k} \in \mathcal{I}$. Hence,

$$\sum_{\xi \in \Xi} |\xi(f)|^p = \sum_{\mathbf{k} \in \mathcal{I}} \sum_{\xi \in \Xi_{\mathbf{k}}} |\xi(f)|^p. \tag{2.28}$$

(If $E = \emptyset$ then we define $\Xi = \emptyset$.)

Remark 5. The sum of depth(ω) over $\omega \in \Omega$ is bounded by

$$C \sum_{\mathbf{k} \in \mathcal{I}} \#(E_{\mathbf{k}}) \leq CN.$$

Also,

$$\#(\Xi) \leq \sum_{\mathbf{k} \in \mathcal{I}} \#(\Xi_{\mathbf{k}}) \leq \sum_{\mathbf{k} \in \mathcal{I}} C \cdot \#(E_{\mathbf{k}}) \leq CN.$$

We choose a partition of unity $\{\theta_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{Z}^n}$ with the following properties.

- $\theta_{\mathbf{k}} \in C^m(\mathbb{R}^n)$, and $\theta_{\mathbf{k}}$ is supported on $Q_{\mathbf{k}}$.
- $|\partial^\alpha \theta_{\mathbf{k}}(x)| \leq C$ for all $|\alpha| \leq m$ and $x \in \mathbb{R}^n$.
- $\sum_{\mathbf{k} \in \mathbb{Z}^n} \theta_{\mathbf{k}} = 1$ on $\mathbb{R}^n$.
- Given $\underline{x} \in \mathbb{R}^n$, $\mathbf{k} \in \mathbb{Z}^n$, we can compute $J_{\underline{x}} \theta_{\mathbf{k}}$ using work at most $C \log(2+N)$.

These conditions are easy to arrange. For instance, we can mimic the construction of $\{\theta_Q\}_{Q \in CZ}$ in Section 2.1. We leave the details to the reader.

We define $T : W^{m,p}(E) \to W^{m,p}(\mathbb{R}^n)$ by the formula

$$(Tf)(x) = \sum_{\mathbf{k} \in \mathcal{I}} (T_{\mathbf{k}} f)(x) \cdot \theta_{\mathbf{k}}(x) \qquad (x \in \mathbb{R}^n).$$

Assume that $\underline{x} \in \mathbb{R}^n$ is given. Note that $J_{\underline{x}} \theta_{\mathbf{k}}$ is nonzero only when $Q_{\mathbf{k}}$ contains $\underline{x}$ (since $\theta_{\mathbf{k}}$ is supported on $Q_{\mathbf{k}}$). We compute a list of all the indices $\mathbf{k} \in \mathcal{I}$ such that $\underline{x} \in Q_{\mathbf{k}}$ using a binary search; this requires work at most $C \log(2+N)$. For each such $\mathbf{k}$, we compute a short form description of the linear map $f \mapsto J_{\underline{x}}(T_{\mathbf{k}} f)$. That requires work at most $C \log(2+N)$. Hence, to compute $J_{\underline{x}}(Tf)$ we can sum the linear maps $J_{\underline{x}} \theta_{\mathbf{k}} \odot_{\underline{x}} J_{\underline{x}}(T_{\mathbf{k}} f)$ over all the relevant indices $\mathbf{k}$. Thus we can compute a short form description of the linear map $f \mapsto J_{\underline{x}}(Tf)$ using work and storage at most $C \log(2 + N)$.

Let $x \in E$. Let $\mathbf{k} \in \mathcal{I}$. Recall that $\theta_{\mathbf{k}}(x) = 0$ if $x \notin Q_{\mathbf{k}}$. Also, $T_{\mathbf{k}} f(x) = f(x)$ if $x \in Q_{\mathbf{k}}$. Thus, we have $\theta_{\mathbf{k}}(x) T_{\mathbf{k}} f(x) = \theta_{\mathbf{k}}(x) f(x)$ unconditionally.

Let $\mathbf{k} \in \mathbb{Z}^n \setminus \mathcal{I}$. By definition of $\mathcal{I}$ we know that $x \notin Q_{\mathbf{k}}$, hence $\theta_{\mathbf{k}}(x) = 0$. Hence,

$$\begin{aligned} Tf(x) &= \sum_{\mathbf{k} \in \mathcal{I}} \theta_{\mathbf{k}}(x) T_{\mathbf{k}} f(x) = \sum_{\mathbf{k} \in \mathcal{I}} \theta_{\mathbf{k}}(x) T_{\mathbf{k}} f(x) + \sum_{\mathbf{k} \in \mathbb{Z}^n \setminus \mathcal{I}} \theta_{\mathbf{k}}(x) f(x) \\ &= \sum_{\mathbf{k} \in \mathbb{Z}^n} \theta_{\mathbf{k}}(x) f(x) = f(x) \quad \text{for any } x \in E. \end{aligned}$$

Proposition 20. *For each* $f \in W^{m,p}(E)$ *we have*

$$\|Tf\|^p_{W^{m,p}(\mathbb{R}^n)} \le C \cdot \sum_{\xi\in\Xi} |\xi(f)|^p.$$

Furthermore,

$$c \cdot \|f\|^p_{W^{m,p}(E)} \le \sum_{\xi\in\Xi} |\xi(f)|^p \le C \cdot \|f\|^p_{W^{m,p}(E)}.$$

Proof. To prove the first estimate, we recall that $Tf = \sum_{\mathbf{k}\in\mathcal{I}} (T_{\mathbf{k}} f) \cdot \theta_{\mathbf{k}}$. Recall that $\theta_{\mathbf{k}}$ is supported on $Q_{\mathbf{k}}$ and that the derivatives of $\theta_{\mathbf{k}}$ are uniformly bounded. Also, note that each point in $\mathbb{R}^n$ is contained in at most C of the cubes $Q_{\mathbf{k}}$ (see (2.25)). Hence, by the Leibniz rule we have

$$\|Tf\|^p_{W^{m,p}(\mathbb{R}^n)} \le C \sum_{\mathbf{k}\in\mathcal{I}} \|T_{\mathbf{k}} f\|^p_{W^{m,p}(Q_{\mathbf{k}})} \overset{(2.27)}{\le} C \sum_{\mathbf{k}\in\mathcal{I}} \sum_{\xi\in\Xi_{\mathbf{k}}} |\xi(f)|^p \overset{(2.28)}{=} C \sum_{\xi\in\Xi} |\xi(f)|^p.$$

Hence,

$$\sum_{\xi\in\Xi} |\xi(f)|^p \ge c\|Tf\|^p_{W^{m,p}(\mathbb{R}^n)} \ge c\|f\|^p_{W^{m,p}(E)}.$$

In the last inequality above, we use the definition of the seminorm $\|\cdot\|_{W^{m,p}(E)}$ and the fact that $Tf = f$ on E.

For the reverse inequality, we use (2.26) and (2.28) and deduce that

$$\begin{aligned}\sum_{\xi\in\Xi} |\xi(f)|^p &= \sum_{\mathbf{k}\in\mathcal{I}} \sum_{\xi\in\Xi_{\mathbf{k}}} |\xi(f)|^p \le C \cdot \sum_{\mathbf{k}\in\mathcal{I}} \inf\left\{\|F\|^p_{W^{m,p}(200Q_{\mathbf{k}})} : F = f \text{ on } E \cap Q_{\mathbf{k}}\right\} \\ &\le C \cdot \inf\left\{ \sum_{\mathbf{k}\in\mathcal{I}} \|F\|^p_{W^{m,p}(200Q_{\mathbf{k}})} : F = f \text{ on } E\right\} \\ &\le C \cdot \inf\left\{\|F\|^p_{W^{m,p}(\mathbb{R}^n)} : F = f \text{ on } E\right\} = C \cdot \|f\|^p_{W^{m,p}(E)}.\end{aligned}$$

Here, we use (2.25) to prove the last inequality.

This completes the proof of Proposition 20. □

The above construction implies our main result for the inhomogeneous Sobolev space. See Proposition 20 and Remark 5.

Theorem 2. *Given a finite subset* $E \subset \mathbb{R}^n$ *with* $N = \#(E)$*, we perform one-time work at most* $CN\log(2+N) + C$ *in space at most* $CN + C$*, after which we have achieved the following.*

We compute lists Ω *and* Ξ*, consisting of functionals on* $W^{m,p}(E) = \{f\colon E \to \mathbb{R}\}$*, with the following properties.*

- *The sum of* $\operatorname{depth}(\omega)$ *over all* $\omega \in \Omega$ *is bounded by* CN*. The number of functionals in* Ξ *is at most* CN*.*
- *Each functional* ξ*, in* Ξ *has* Ω*-assisted bounded depth. The functionals in* Ω *and* Ξ *are represented in their short form.*

- *For all* $f \in W^{m,p}(E)$ *we have*

$$c\|f\|_{W^{m,p}(E)} \leq \Big[\sum_{\xi\in\Xi} |\xi(f)|^p\Big]^{1/p} \leq C\|f\|_{W^{m,p}(E)}.$$

Moreover, there exists a linear map $T : W^{m,p}(E) \to W^{m,p}(\mathbb{R}^n)$ *with the following properties.*

- T *has* Ω*-assisted depth at most* C.
- $Tf = f$ *on* E *and* $\|Tf\|_{W^{m,p}(\mathbb{R}^n)} \leq C \cdot \|f\|_{W^{m,p}(E)}$ *for all* $f \in W^{m,p}(E)$.
- *We produce a query algorithm that operates as follows.*

  Given a query point $\underline{x} \in \mathbb{R}^n$*, we respond with a short form description of the* Ω*-assisted bounded depth map* $f \mapsto J_{\underline{x}}(Tf)$ *using work and storage at most* $C\log(2+N)$.

At last, note that Theorem 1 and Theorem 2 imply the main theorem from the introduction in [1] (Theorem 6). This completes the analysis of our algorithms for the infinite-precision model of computation. In [2] we present an analogue of our algorithms for a finite-precision model of computation.

References

[1] Fefferman, C., Israel, A. and Luli, G. K.: Fitting a Sobolev function to data I. *Rev. Mat. Iberoam.* **32** (2016), no. 1, 273–374.

[2] Fefferman, C., Israel, A. and Luli, G. K.: Fitting a Sobolev function to data III. To appear in *Rev. Mat. Iberoam.*

Received November 19, 2014.

Charles Fefferman: Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton, NJ 08544, USA.
E-mail: cf@math.princeton.edu

Arie Israel: Department of Mathematics, University of Texas at Austin, RLM Hall, 2515 Speedway, Austin, TX 78712, USA.
E-mail: arie@math.utexas.edu

Garving Luli: Department of Mathematics, University of California at Davis, One Shields Avenue, Davis, CA 95616, USA.
E-mail: kluli@math.ucdavis.edu

Charles Fefferman research is supported by NSF grant DMS-1265524 and AFOSR grant FA9550-12-1-0425. Arie Israel research is supported by an NSF postdoctoral fellowship, DMS-1103978. Garving Luli research is supported by NSF grant DMS-1355968.