



Global wellposedness of the equivariant Chern–Simons–Schrödinger equation

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Abstract. In this article we consider the initial value problem for the m -equivariant Chern–Simons–Schrödinger model in two spatial dimensions with coupling parameter $g \in \mathbb{R}$. This is a covariant NLS type problem that is L^2 -critical. We prove that at the critical regularity, for any equivariance index $m \in \mathbb{Z}$, the initial value problem in the defocusing case ($g < 1$) is globally wellposed and the solution scatters. The problem is focusing when $g \geq 1$, and in this case we prove that for equivariance indices $m \in \mathbb{Z}$, $m \geq 0$, there exist constants $c = c_{m,g}$ such that, at the critical regularity, the initial value problem is globally wellposed and the solution scatters when the initial data $\phi_0 \in L^2$ is m -equivariant and satisfies $\|\phi_0\|_{L^2}^2 < c_{m,g}$. We also show that $\sqrt{c_{m,g}}$ is equal to the minimum L^2 norm of a nontrivial m -equivariant standing wave solution. In the self-dual $g = 1$ case, we have the exact numerical values $c_{m,1} = 8\pi(m + 1)$.

1. Introduction

The two-dimensional Chern–Simons–Schrödinger system is a nonrelativistic quantum model describing the dynamics of a large number of particles in the plane interacting both directly and via a self-generated field. The variables we use to describe the dynamics are the scalar field ϕ , describing the particle system, and the potential A , which can be viewed as a real-valued 1-form on \mathbb{R}^{2+1} . The associated covariant differentiation operators are defined in terms of the potential A as

$$(1.1) \quad D_\alpha := \partial_\alpha + iA_\alpha, \quad \alpha = 0, 1, 2.$$

With this notation, the action integral for the system is

$$(1.2) \quad L(A, \phi) = \frac{1}{2} \int_{\mathbb{R}^{2+1}} \left[\operatorname{Im}(\bar{\phi} D_t \phi) + |D_x \phi|^2 - \frac{g}{2} |\phi|^4 \right] dx dt + \frac{1}{2} \int_{\mathbb{R}^{2+1}} A \wedge dA,$$

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where $g \in \mathbb{R}$ is a coupling constant. The Lagrangian is invariant with respect to the transformations

$$(1.3) \quad \phi \mapsto e^{-i\theta} \phi \quad A \mapsto A + d\theta$$

for compactly supported real-valued functions $\theta(t, x)$.

Computing the Euler–Lagrange equations results in a covariant NLS equation for ϕ , coupled with equations giving the field $F = dA$ in terms of ϕ :

$$(1.4) \quad \begin{cases} D_t \phi = iD_\ell D_\ell \phi + ig|\phi|^2 \phi \\ F_{01} = -\text{Im}(\bar{\phi} D_2 \phi) \\ F_{02} = \text{Im}(\bar{\phi} D_1 \phi) \\ F_{12} = -\frac{1}{2}|\phi|^2 \end{cases}$$

For indices, we use $\alpha = 0$ for the time variable t and $\alpha = 1, 2$ for the spatial variables x_1, x_2 . When we wish to exclude the time variable in a certain expression, we switch from Greek indices to Roman ones. Repeated indices are assumed to be summed, Greek ones over $\{0, 1, 2\}$, and Roman ones over $\{1, 2\}$.

The system (1.4) is a basic model of Chern–Simons dynamics [22], [12], [13], [21]. For further physical motivation for studying (1.4), see [24], [10], [23], [33], and [42].

The Chern–Simons–Schrödinger system (1.4) inherits from (1.2) the gauge invariance (1.3). It is also Galilean-invariant and has conserved *charge*

$$(1.5) \quad \text{chg}(\phi) := \int_{\mathbb{R}^2} |\phi|^2 dx$$

and *energy*

$$(1.6) \quad E(\phi) := \frac{1}{2} \int_{\mathbb{R}^2} \left[|D_x \phi|^2 - \frac{g}{2} |\phi|^4 \right] dx$$

As the scaling symmetry

$$\phi(t, x) \rightarrow \lambda \phi(\lambda^2 t, \lambda x), \quad \phi_0(x) \rightarrow \lambda \phi_0(\lambda x); \quad \lambda > 0,$$

preserves the charge of the initial data ϕ_0 , L_x^2 is the critical space for the main evolution equation of (1.4).

In order for (1.4) to be a well-posed system, the gauge freedom (1.3) has to be eliminated. This is achieved by imposing an additional constraint equation. In the Coulomb gauge, local wellposedness in H^2 is established in [3]. Also given are conditions ensuring finite-time blowup. With a regularization argument, [3] demonstrates global existence (but not uniqueness) in H^1 for small L^2 data. A full local wellposedness theory for H^1 data is proved in [18]. Local wellposedness for data small in H^s , $s > 0$, is established in [32] using the heat gauge. We refer the reader to Section 2 of [32] for a comparison of the Coulomb and heat gauges. At the critical scaling of L^2 , local existence implies global existence for small data;

it is an open problem to determine whether the Chern–Simons–Schrödinger system is wellposed at the critical regularity in any gauge given small but otherwise arbitrary L^2 initial data.

The purpose of this article is to establish the global wellposedness of (1.4) for large L^2 data in a symmetry-reduced setting, and with respect to the Coulomb gauge. We provide a brief introduction to these assumptions here and will formalize them in due course. The Coulomb gauge condition is the requirement that $\nabla \cdot A_x = 0$. Under this gauge choice, we assume that the wavefunction ϕ is equivariant, i.e., in polar coordinates (r, θ) it admits the representation $\phi(t, r, \theta) = e^{im\theta}u(t, r)$ for some $m \in \mathbb{Z}$ and some radial function $u \in L_t^\infty L_x^2$. The integer m we refer to as the degree of equivariance; it is a topological quantity that is invariant under the flow. The case $m = 0$ corresponds to the radial case. The natural defocusing range for this problem is $g < 1$, as H^1 solutions in this range necessarily have positive energy. Positivity of the energy for such g is not immediate from its definition (1.6) but will be shown to be a consequence of the Bogomol’nyi identity (7.1).

It is convenient to rewrite (1.4) as

$$(1.7) \quad \begin{cases} (i\partial_t + \Delta)\phi = -2iA_j\partial_j\phi - i\partial_jA_j\phi + A_0\phi + A_x^2\phi - g|\phi|^2\phi, \\ \partial_tA_1 - \partial_1A_0 = -\text{Im}(\bar{\phi}D_2\phi), \\ \partial_tA_2 - \partial_2A_0 = \text{Im}(\bar{\phi}D_1\phi), \\ \partial_1A_2 - \partial_2A_1 = -\frac{1}{2}|\phi|^2. \end{cases}$$

We study (1.7) in the Coulomb gauge, which is the requirement that

$$(1.8) \quad \partial_1A_1 + \partial_2A_2 = 0.$$

Coupling (1.8) with the curvature constraints leads to

$$\begin{aligned} A_0 &= \Delta^{-1} [\partial_1\text{Im}(\bar{\phi}D_2\phi) - \partial_2\text{Im}(\bar{\phi}D_1\phi)], \\ A_1 &= \frac{1}{2}\Delta^{-1}\partial_2|\phi|^2, \\ A_2 &= -\frac{1}{2}\Delta^{-1}\partial_1|\phi|^2. \end{aligned}$$

We may rewrite A_0 as

$$(1.9) \quad A_0 = \text{Im}(Q_{12}(\bar{\phi}, \phi)) + \partial_1(A_2|\phi|^2) - \partial_2(A_1|\phi|^2),$$

where the null form Q_{12} is defined by

$$Q_{12}(f, g) = \partial_1f \partial_2g - \partial_2f \partial_1g.$$

The equivariance ansatz suggests using polar coordinates. In fact, we will take advantage of both Cartesian coordinates and polar coordinates. Motivated by the transformations

$$\partial_r = \frac{x_1}{|x|} \partial_1 + \frac{x_2}{|x|} \partial_2, \quad \partial_\theta = -x_2\partial_1 + x_1\partial_2,$$

and

$$\partial_1 = (\cos \theta)\partial_r - \frac{1}{r}(\sin \theta)\partial_\theta, \quad \partial_2 = (\sin \theta)\partial_r + \frac{1}{r}(\cos \theta)\partial_\theta,$$

we introduce

$$(1.10) \quad A_r = \frac{x_1}{|x|}A_1 + \frac{x_2}{|x|}A_2, \quad A_\theta = -x_2A_1 + x_1A_2,$$

which are easily seen to satisfy

$$(1.11) \quad A_1 = A_r \cos \theta - \frac{1}{r}A_\theta \sin \theta \quad \text{and} \quad A_2 = A_r \sin \theta + \frac{1}{r}A_\theta \cos \theta.$$

Using these transformations, we may eliminate $A_1, A_2, \partial_1, \partial_2$ in (1.7) in favor of $A_r, A_\theta, \partial_r, \partial_\theta$. In particular,

$$A_j\partial_j = A_r\partial_r + \frac{1}{r^2}A_\theta\partial_\theta, \quad \partial_jA_j = \partial_rA_r + \frac{1}{r}A_r + \frac{1}{r^2}\partial_\theta A_\theta, \quad A_1^2 + A_2^2 = A_r^2 + \frac{1}{r^2}A_\theta^2.$$

The main evolution equation of (1.7) therefore admits the representation

$$(1.12) \quad \begin{aligned} (i\partial_t + \Delta)\phi &= -2i\left(A_r\partial_r + \frac{1}{r^2}A_\theta\partial_\theta\right)\phi - i\left(\partial_rA_r + \frac{1}{r}A_r + \frac{1}{r^2}\partial_\theta A_\theta\right)\phi \\ &\quad + A_0\phi + A_r^2\phi + \frac{1}{r^2}A_\theta^2\phi - g|\phi|^2\phi, \end{aligned}$$

which in more compact form reads

$$(1.13) \quad D_t\phi = i\left(D_r^2 + \frac{1}{r}D_r + \frac{1}{r^2}D_\theta^2\right)\phi + ig|\phi|^2\phi.$$

We also rewrite the $F = dA$ curvature relations in terms of the variables t, r, θ , with

$$(1.14) \quad F_{0r} = \partial_tA_r - \partial_rA_0, \quad F_{0\theta} = \partial_tA_\theta - \partial_\theta A_0, \quad F_{r\theta} = \partial_rA_\theta - \partial_\theta A_r.$$

For instance, we have

$$x_1(\partial_tA_1 - \partial_1A_0) + x_2(\partial_tA_2 - \partial_2A_0) = -x_1 \operatorname{Im}(\bar{\phi}D_2\phi) + x_2 \operatorname{Im}(\bar{\phi}D_1\phi),$$

which reduces to

$$r\partial_tA_r - r\partial_rA_0 = \operatorname{Im}(\bar{\phi}(x_2D_1 - x_1D_2)\phi),$$

so that

$$(1.15) \quad r[\partial_tA_r - \partial_rA_0] = -\operatorname{Im}(\bar{\phi}\partial_\theta\phi) + A_\theta|\phi|^2 = -\operatorname{Im}(\bar{\phi}D_\theta\phi).$$

Similarly, we obtain

$$\partial_tA_\theta - \partial_\theta A_0 = r \operatorname{Im}(\bar{\phi}D_r\phi) \quad \text{and} \quad \partial_1A_2 - \partial_2A_1 = \frac{1}{r}\partial_rA_\theta - \frac{1}{r}\partial_\theta A_r,$$

which implies

$$(1.16) \quad \partial_r A_\theta - \partial_\theta A_r = -\frac{1}{2}|\phi|^2 r.$$

Therefore we may write (1.7) equivalently as

$$(1.17) \quad \begin{cases} (i\partial_t + \Delta)\phi = -2i\left(A_r\partial_r + \frac{1}{r^2}A_\theta\partial_\theta\right)\phi - i\left(\partial_r A_r + \frac{1}{r}A_r + \frac{1}{r^2}\partial_\theta A_\theta\right)\phi, \\ \qquad \qquad \qquad + A_0\phi + A_r^2\phi + \frac{1}{r^2}A_\theta^2\phi - g|\phi|^2\phi, \\ \partial_t A_r - \partial_r A_0 = -\frac{1}{r}\operatorname{Im}(\bar{\phi}D_\theta\phi), \\ \partial_t A_\theta - \partial_\theta A_0 = r\operatorname{Im}(\bar{\phi}D_r\phi), \\ \partial_r A_\theta - \partial_\theta A_r = -\frac{1}{2}|\phi|^2 r. \end{cases}$$

In polar coordinates, the energy (1.6) takes the form

$$(1.18) \quad E(\phi) = \frac{1}{2} \int_0^{2\pi} \int_0^\infty \left(|D_r\phi|^2 + \frac{1}{r^2}|D_\theta\phi|^2 - \frac{g}{2}|\phi|^4 \right) r \, dr \, d\theta.$$

Our next simplification is to restrict to equivariant ϕ . Our formulation of the equivariant ansatz implicitly assumes that we have chosen the Coulomb gauge condition (1.8), which in A_θ, A_r variables takes the form

$$(1.19) \quad \left(\frac{1}{r} + \partial_r\right)A_r + \frac{1}{r^2}\partial_\theta A_\theta = 0$$

In particular, we assume that (A, ϕ) is of the form

$$(1.20) \quad \begin{cases} \phi(t, x) = e^{im\theta}u(t, r), \\ A_1(t, x) = -\frac{x_2}{r}v(t, r), \\ A_2(t, x) = \frac{x_1}{r}v(t, r), \\ A_0(t, x) = w(t, r). \end{cases}$$

The only assumption that we make on m is that $m \in \mathbb{Z}$, and so in particular we include the radial case $m = 0$. This ansatz implies that $A_r = 0$ and that A_θ is a radial function, and so (1.19) is satisfied. Equivariant solutions, of the form (1.20), are also known as *vortex* solutions, and appear in related contexts (see, for instance, [37], [8], [9], [25], [6], and [5]). We also make the natural assumption that A_0 decays to zero at spatial infinity (see the proof of Lemma 2.2 and the references therein for further discussion of this point).

Next we rewrite the system (1.17) assuming the equivariant ansatz (1.20). Thanks to the ansatz, $\partial_\theta\phi = im\phi$ holds identically, and so we make this sub-

stitution where convenient. We obtain

$$(1.21) \quad \left\{ \begin{aligned} (i\partial_t + \Delta)\phi &= \frac{2m}{r^2}A_\theta\phi + A_0\phi + \frac{1}{r^2}A_\theta^2\phi - g|\phi|^2\phi, \\ \partial_r A_0 &= \frac{1}{r}(m + A_\theta)|\phi|^2, \\ \partial_t A_\theta &= r \operatorname{Im}(\bar{\phi}\partial_r\phi), \\ \partial_r A_\theta &= -\frac{1}{2}|\phi|^2r, \\ A_r &= 0. \end{aligned} \right.$$

Definition 1.1 (Equivariant Sobolev spaces). Let $m \in \mathbb{Z}$. For each $s \geq 0$, we define the function space H_m^s to be the Sobolev space of all functions $f \in H_x^s$ that admit the decomposition $f(x) = f(r, \theta) = e^{im\theta}u(r)$. We also will use the notation $L_m^2 = H_m^0$.

Our first main theorem is the following.

Theorem 1.2. *Let $g < 1$ and $m \in \mathbb{Z}$. Then (1.21) is globally wellposed in L_m^2 , and, furthermore, solutions scatter both forward and backward in time.*

For our second main theorem, we use the notation \mathbb{Z}_+ to denote $\{0, 1, 2, \dots\}$. In this theorem for the coupling constant we take $g = 1$, the so-called ‘‘critical coupling’’ or ‘‘self-dual’’ coupling value.

Theorem 1.3. *Let $g = 1$ and $m \in \mathbb{Z}_+$. Let $\phi_0 \in L_m^2$ with $\operatorname{chg}(\phi_0) < 8\pi(m + 1)$. Then (1.21) is globally wellposed in L_m^2 and scatters both forward and backward in time.*

We have a similar statement for the case $g > 1$, though in this case we have not identified the numerical values of threshold constants. We do show, however, that the threshold constant is related to soliton solutions.

Theorem 1.4. *Let $g > 1$ and $m \in \mathbb{Z}_+$. Then there exists a constant $c_{m,g} > 0$ such that if $\phi_0 \in L_m^2$ with $\operatorname{chg}(\phi_0) < c_{m,g}$, then (1.21) is globally wellposed in L_m^2 and scatters forward and backward in time. Moreover, the minimum charge of a nontrivial standing wave solution in the class $L_t^\infty L_m^2$ is equal to $c_{m,g}$.*

The $L_{t,x}^4$ norm plays the role of the scattering norm. Our notions of blowup and scattering are made precise in the remarks preceding Theorem 2.8, which establishes the Cauchy theory for (1.21) that is attainable using standard perturbative techniques. For small data, the sign of $g - 1$ plays no role, and indeed Theorem 2.8 applies to this case. In fact, all results of §§2–6 hold for any $g \in \mathbb{R}$. It is only starting in §7 (in particular, Corollary 7.5) where the value of g plays a role. The system (1.21) admits solitons when $g \geq 1$ and $m \in \mathbb{Z}$ is nonnegative, and so in this sense $-\infty < g < 1$ is the natural defocusing parameter range.

The challenge is to prove Theorems 1.2–1.4 for large data. The first step is to reduce to special localized solutions. Bourgain’s induction-on-energy method for the energy-critical NLS revealed the important role played by solutions simultaneously localized in frequency and space, see [4]. Kenig and Merle ([26], [27]) sub-

sequently streamlined the arguments reducing one’s consideration to such solutions by means of a concentration-compactness argument. Minimal-mass blowup solutions of the mass-critical NLS are studied in [41]. We adopt a concentration-compactness argument, modeled closely after that of Killip, Tao, and Visan [28] for the radial 2-d cubic NLS. Inspiration also comes from the work of Gustafson and Koo [17] on radial 2-d Schrödinger maps into the unit sphere, which, among other things, extends the arguments of [28] so as to handle a nonlocal term.

Definition 1.5. A solution ϕ with lifespan I is said to be almost periodic modulo scaling if there exist a frequency scale function $N : I \rightarrow \mathbb{R}^+$ and a compactness modulus function $C : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$\int_{|x| \geq C(\eta)/N(t)} |\phi(t, x)|^2 dx \leq \eta \quad \text{and} \quad \int_{|\xi| \geq C(\eta)N(t)} |\hat{\phi}(t, \xi)|^2 d\xi \leq \eta$$

for all $t \in I$ and $\eta > 0$.

Here we have used $\hat{f}(\xi)$ to denote the Fourier transform of f in the spatial variable $x \in \mathbb{R}^2$ only. We sometimes use the notation $\mathcal{F}(f)$ instead of \hat{f} .

Remark 1.6. Solutions of (1.7) are invariant under the symmetry group G introduced in Definition 1.6 of [28], which includes the scaling, rotation, translation, and Galilean symmetries (the action of G on ϕ is as specified in [28] and can easily be extended to act on A as well). The equivariance ansatz (1.20) breaks the translation and Galilean symmetries, leaving us with scaling and rotational symmetry. This subgroup is denoted by G_{rad} in Definition 1.6 of [28], as it preserves spherical symmetry; in fact, it preserves m -equivariance for any index $m \in \mathbb{Z}$. Because rotational symmetry corresponds to the action of a compact symmetry group, it may be neglected for our purposes. In fact, it plays no role in Definition 1.14 of [28], which defines almost periodicity modulo G and modulo G_{rad} .

Lemma 1.7. *Suppose that the statement of Theorem 1.2 (or 1.3, 1.4) is not true. Then there exists a critical element, i.e., a maximal-lifespan solution ϕ that is almost periodic modulo scaling and that blows up both forward and backward in time. Furthermore, this critical element can be taken to be m -equivariant. We can also ensure that the lifespan I and the frequency-scale function $N : I \rightarrow \mathbb{R}^+$ match one of the following scenarios:*

- 1) (Self-similar solution) We have $I = (0, +\infty)$ and

$$N(t) = t^{-1/2} \quad \text{for all } t \in I.$$

- 2) (Global solutions) We have $I = \mathbb{R}$ and fall into one of the following two scenarios:

- (a) (Rapid cascade)

$$\liminf_{t \rightarrow -\infty} N(t) = \liminf_{t \rightarrow +\infty} N(t) = 0 \quad \text{and} \quad \sup_{t \in I} N(t) < \infty.$$

- (b) (Soliton-like solution)

$$N(t) = 1 \quad \text{for all } t \in I.$$

Our strategy for proving Theorems 1.2–1.4 is to show that the scenarios described in Lemma 1.7 cannot occur, in the spirit of [28], [31], and [17]. The first step of the program is to establish that the solutions described by Lemma 1.7 are special in that they enjoy extra regularity and in particular are in H^s for each $s > 0$. The energy (1.6) is at the level of H^1 , and its conservation can be exploited in both scenarios. To rule out the global profile, we also use a localized virial identity. This identity can also be adapted to handle the self-similar profile, as described in Section 9 of [28], though we opt instead to rule out the self-similar profile using energy conservation.

The rest of this article is laid out as follows. In the next section, §2, we develop the basic Cauchy theory for (1.21). In §3 we go through the concentration compactness argument behind Lemma 1.7. Next, in §4, we introduce the Littlewood–Paley theory that we will require and we establish how frequency localizations of the nonlinearity $\Lambda(\phi)$, defined in (2.13), depend upon frequency localizations of input functions ϕ . §5 establishes extra regularity for critical elements, a key technical step in the large data theory. In §6, we establish virial and Morawetz identities. These play an important role in §7, which concludes the proof of Theorem 1.2 in the $g < 1$ case by ruling out the blowup scenarios of Lemma 1.7. In §8, we consider the focusing problem, proving Theorems 1.3 and 1.4 along with some auxiliary results.

2. The equivariant Cauchy theory

Throughout this section we assume that ϕ is m -equivariant. A trivial consequence of this that we will repeatedly use is that $|\phi|^2$ is radial. We assume that all spatial L^p spaces are based on the 2-dimensional Lebesgue measure.

Define the operators $[\partial_r]^{-1}$, $[r^{-n}\bar{\partial}_r]^{-1}$, and $[r\partial_r]^{-1}$ by

$$\begin{aligned} [\partial_r]^{-1} &= - \int_r^\infty f(s) ds, & [r^{-n}\bar{\partial}_r]^{-1} f(r) &= \int_0^r f(s) s^n ds, \\ [r\partial_r]^{-1} f(r) &= - \int_r^\infty \frac{1}{s} f(s) ds. \end{aligned}$$

Then straightforward arguments imply

$$(2.1) \quad \|[r\partial_r]^{-1} f\|_{L^p} \lesssim_p \|f\|_{L^p}, \quad 1 \leq p < \infty,$$

$$(2.2) \quad \|r^{-n-1}[r^{-n}\bar{\partial}_r]^{-1} f\|_{L^p} \lesssim_p \|f\|_{L^p}, \quad 1 < p \leq \infty,$$

$$(2.3) \quad \|[\partial_r]^{-1} f\|_{L^2} \lesssim \|f\|_{L^1}.$$

Lemma 2.1 (Bounds on A_θ terms). *We have*

$$(2.4) \quad \|A_\theta\|_{L_x^\infty} \lesssim \|\phi\|_{L_x^2}^2,$$

$$(2.5) \quad \left\| \frac{1}{r} A_\theta \right\|_{L_x^\infty} \lesssim \|\phi\|_{L_x^4}^2,$$

$$(2.6) \quad \left\| \frac{A_\theta}{r^2} \right\|_{L_x^p} \lesssim \|\phi\|_{L_x^{2p}}^2, \quad 1 < p \leq \infty.$$

Proof. We start with

$$(2.7) \quad A_\theta = -\frac{1}{2} \int_0^r |\phi|^2 s \, ds,$$

which we obtain by integrating the $F_{r\theta}$ spatial curvature condition in (1.21) ($F_{r\theta}$ is given in (1.14) and simplifies under (1.20)). To justify the boundary condition, note that (1.10) implies that $A_\theta(r = 0) = 0$ so long as $A_1, A_2 \in L^\infty_{\text{loc}}$. Moreover, in the Coulomb gauge, A_1 and A_2 exhibit $1/|x|$ decay at infinity and so from (1.11) we expect an L^∞ bound for A_θ but not decay. The right hand side of (2.7) is bounded in absolute value by a constant times $\|\phi\|_{L^2_x}^2$, which proves (2.4).

For the second inequality, we get using (2.7) and Cauchy–Schwarz that

$$|A_\theta| \lesssim r \left(\int_0^\infty |\phi|^4 s \, ds \right)^{1/2}.$$

Therefore,

$$\left| \frac{1}{r} A_\theta \right| \lesssim \left(\int_0^\infty |\phi|^4 s \, ds \right)^{1/2}.$$

Finally, to prove (2.6), we use (2.2) with $n = 1$, first writing

$$\frac{|A_\theta|}{r^2} = \frac{1}{2} \cdot \frac{1}{r^2} \int_0^r |\phi|^2 s \, ds = \frac{1}{2} r^{-2} [r^{-1} \bar{\partial}_r]^{-1} |\phi|^2.$$

Therefore,

$$\left\| \frac{A_\theta}{r^2} \right\|_{L^p_x} \lesssim \left\| r^{-2} [r^{-1} \bar{\partial}_r]^{-1} |\phi|^2 \right\|_{L^p_x} \lesssim \| |\phi|^2 \|_{L^p_x} = \|\phi\|_{L^{2p}_x}^2. \quad \square$$

Lemma 2.2 (Bounds on A_0). *Write $A_0 = A_0^{(1)} + A_0^{(2)}$, where*

$$(2.8) \quad A_0^{(1)} := - \int_r^\infty \frac{A_\theta}{s} |\phi|^2 \, ds \quad \text{and} \quad A_0^{(2)} := - \int_r^\infty \frac{m}{s} |\phi|^2 \, ds.$$

Then

$$(2.9) \quad \begin{aligned} \|A_0^{(1)}\|_{L^1_t L^\infty_x} &\lesssim \|\phi\|_{L^4_{t,x}}^4 \\ \|A_0^{(1)}\|_{L^2_{t,x}} &\lesssim \|\phi\|_{L^\infty_t L^2_x}^2 \|\phi\|_{L^4_{t,x}}^2, \quad 1 \leq p < \infty, \end{aligned}$$

and

$$(2.10) \quad \|A_0^{(2)}\|_{L^2_x} \lesssim |m| \|\phi\|_{L^4_x}^2.$$

Proof. The behavior of A_0 is independent of the coordinate system. In particular, it is natural to assume that it decays to zero at infinity as shown in [3], where certain L^p bounds are also established under the assumption of sufficient regularity. This motivates integrating the F_{r0} curvature condition in (1.21) from infinity, which is justified below.

To establish the first inequality of (2.9), rewrite $A_0^{(1)}(r)$ as

$$A_0^{(1)}(r) = - \int_r^\infty \frac{A_\theta}{s^2} |\phi|^2 s ds.$$

Then, bounding $A_\theta(s)/s^2$ in L^2 using (2.6) and putting each ϕ in L^4 , we obtain

$$|A_0^{(1)}(r)| \leq \left\| \frac{A_\theta}{s^2} \right\|_{L_x^2} \|\phi\|_{L^4}^2 \lesssim \|\phi\|_{L_x^4}^4.$$

The bound is independent of r , and integrating in time yields

$$\|A_0^{(1)}\|_{L_t^1 L_x^\infty} \lesssim \|\phi\|_{L_{t,x}^4}^4.$$

The second inequality of (2.9) follows from (2.4) and (2.1) with $p = 2$.

To establish (2.10), we use (2.1) with $f = m|\phi|^2$ and $p = 2$. □

Lemma 2.3 (Quadratic bounds). *We have*

$$(2.11) \quad \left\| \frac{1}{r^2} A_\theta^2 \right\|_{L_t^1 L_x^\infty} \lesssim \|\phi\|_{L_{t,x}^4}^4,$$

$$(2.12) \quad \left\| \frac{1}{r^2} A_\theta^2 \right\|_{L_{t,x}^2} \lesssim \|\phi\|_{L_t^\infty L_x^2}^2 \|\phi\|_{L_{t,x}^4}^2.$$

Proof. The first bound follows from (2.5) and Cauchy-Schwarz. The second is a consequence of (2.4) and (2.6) with $p = 2$. □

Let

$$(2.13) \quad \Lambda(\phi) = 2m \frac{A_\theta}{r^2} \phi + A_0 \phi + \frac{1}{r^2} A_\theta^2 \phi - g|\phi|^2 \phi$$

denote the nonlinearity of the evolution equation of (1.21).

Remark 2.4. The bounds established in the preceding lemmas are very flexible and allow us to control all pieces of the nonlinearity $\Lambda(\phi)$ in $L_{t,x}^{4/3}$ and some pieces of it in $L_t^1 L_x^2$.

Lemma 2.5 (Strichartz estimates). *Let $(i\partial_t + \Delta) = f$ on a time interval I with $t_0 \in I$ and $u(t_0) = u_0$. Call a pair (q, r) of exponents admissible if $2 \leq q, r \leq \infty$, $1/q + 1/r = 1/2$ and $(q, r) \neq (2, \infty)$. Let (q, r) and (\tilde{q}, \tilde{r}) be admissible pairs of exponents. Then*

$$\|u\|_{L_t^\infty L_x^2(I \times \mathbb{R}^2)} + \|u\|_{L_t^q L_x^r(I \times \mathbb{R}^2)} \lesssim \|u_0\|_{L_x^2(\mathbb{R}^2)} + \|f\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(I \times \mathbb{R}^2)},$$

where the prime indicates the dual exponent, i.e., $1/q' := 1 - 1/q$.

These estimates are established in [43] and [14]. The only admissible pair that we use in this section is $(q, r) = (4, 4)$. In the usual way, one may intersect Strichartz spaces. Their dual is then a sum-type space; we use this property in §5. In that section we also use the endpoint estimate, proved in [39] and [40]:

Lemma 2.6 (Endpoint Strichartz estimate). *Let $(i\partial_t + \Delta)u = f$ on a time interval I with $t_0 \in I$ and $u(t_0) = u_0$, and suppose that $m \in \mathbb{Z}$ and $u, f \in L^2_m(\mathbb{R}^2)$. Let (q, r) be an admissible pair of exponents. Then*

$$\|u\|_{L^2_t L^\infty_x(I \times \mathbb{R}^2)} \lesssim \|u_0\|_{L^2_x(\mathbb{R}^2)} + \|f\|_{L^{q'}_t L^{r'}_x(I \times \mathbb{R}^2)}.$$

Though the endpoint estimate was established for radial functions, the proof may be adapted to equivariant functions in a straightforward way by noting properties of Bessel functions (see, for instance, Remark 5.5 for related comments).

Lemma 2.7 (Control of the nonlinearity). *We have*

$$(2.14) \quad \|\Lambda(\phi)\|_{L^{4/3}} \lesssim \|\phi\|_{L^4}^3$$

and

$$(2.15) \quad \|\Lambda(\phi) - \Lambda(\tilde{\phi})\|_{L^{4/3}} \lesssim \|\phi - \tilde{\phi}\|_{L^4} (\|\phi\|_{L^4}^2 + \|\tilde{\phi}\|_{L^4}^2)$$

Proof. The proof is an easy consequence of Strichartz estimates, charge conservation, and the previous lemmas. In particular, we have

$$\begin{aligned} \left\| \frac{2}{r^2} mA_\theta \phi \right\|_{L^{4/3}_{t,x}} &\lesssim |m| \left\| \frac{1}{r^2} A_\theta \right\|_{L^2_{t,x}} \|\phi\|_{L^4_{t,x}} \lesssim |m| \|\phi\|_{L^4_{t,x}}^3, \\ \|A_0 \phi\|_{L^{4/3}_{t,x}} &\lesssim \|A_0\|_{L^2_{t,x}} \|\phi\|_{L^4_{t,x}} \lesssim (|m| + \|\phi\|_{L^\infty_{t,x}}^2) \|\phi\|_{L^4_{t,x}}^3, \\ \left\| \frac{1}{r^2} A_\theta^2 \phi \right\|_{L^{4/3}_{t,x}} &\lesssim \left\| \frac{1}{r^2} A_\theta^2 \right\|_{L^2_{t,x}} \|\phi\|_{L^4_{t,x}} \lesssim \|\phi\|_{L^\infty_{t,x}}^2 \|\phi\|_{L^4_{t,x}}^3, \\ \|g|\phi|^2\phi\|_{L^{4/3}_{t,x}} &\leq |g| \|\phi\|_{L^4_{t,x}}^3, \end{aligned}$$

which establishes (2.14).

The second inequality is easy to show for the nonlinear term $g|\phi|^2\phi$ by using the observation

$$(2.16) \quad \left| |\phi|^2\phi - |\tilde{\phi}|^2\tilde{\phi} \right| \lesssim (|\phi|^2 + |\tilde{\phi}|^2)|\phi - \tilde{\phi}|.$$

To see that others are similar, note that bounds (2.4)–(2.6) for $A_\theta = A_\theta(\phi)$ are linear in $|\phi|^2$. This is also true of the bound for $A_0^{(2)}$ in Lemma 2.2. Applying further decompositions similar to (2.16) allows one to handle the higher-order terms $\frac{1}{r^2}A_\theta^2$ and $A_0^{(1)}$. □

In our analysis, the $L^4_{t,x}$ norm plays the role of a scattering norm. We define $S(\phi) = S_I(\phi) \in [0, \infty]$ of a function $\phi : I \times \mathbb{R}^2 \rightarrow \mathbb{C}$ by

$$(2.17) \quad S_I(\phi) := \|\phi\|_{L^4_{t,x}(I \times \mathbb{R}^2)}^4.$$

If $t_0 \in I$, we also split $S(\phi) = S_{\leq t_0}(\phi) + S_{\geq t_0}(\phi)$ where

$$S_{\geq t_0}(\phi) := \|\phi\|_{L^4_{t,x}((I \cap (-\infty, t_0]) \times \mathbb{R}^2)}^4 \quad \text{and} \quad S_{\leq t_0}(\phi) := \|\phi\|_{L^4_{t,x}((I \cap [t_0, \infty)) \times \mathbb{R}^2)}^4.$$

If $\phi : I \times \mathbb{R}^2 \rightarrow \mathbb{C}$ is a solution of (1.21) on an open time interval I , then we say that ϕ blows up forward in time if $S_{\geq t}(\phi) = \infty$ for all $t \in I$. Similarly, we say that ϕ blows up backward in time if $S_{\leq t}(\phi) = \infty$ for all $t \in I$.

Let $\phi_+ \in L^2$. We say that a solution $\phi : I \times \mathbb{R}^2 \rightarrow \mathbb{C}$ scatters forward in time to ϕ_+ if and only if $\sup I = +\infty$ and $\lim_{t \rightarrow \infty} \|\phi(t) - e^{it\Delta}\phi_+\|_{L^2} = 0$. Similarly, we say that a solution $\phi : I \times \mathbb{R}^2 \rightarrow \mathbb{C}$ scatters backward in time to $\phi_- \in L^2$ if and only if $\inf I = -\infty$ and $\lim_{t \rightarrow -\infty} \|\phi(t) - e^{it\Delta}\phi_-\|_{L^2} = 0$.

Theorem 2.8 (Cauchy theory). *Let $m \in \mathbb{Z}$, $\phi_0 \in L^2_m(\mathbb{R}^2)$, and $t_0 \in \mathbb{R}$. There exists a unique maximal lifespan solution $\phi : I \times \mathbb{R}^2 \rightarrow \mathbb{C}$, $\phi \in L^\infty_t L^2_m$, with $t_0 \in I$, the maximal time interval, $\phi(t_0) = \phi_0$, and with the following additional properties:*

- (1) (Local existence) I is open.
- (2) (Scattering) If ϕ does not blow up forward in time, then $\sup I = +\infty$ and ϕ scatters forward in time to $e^{it\Delta}\phi_+$ for some $\phi_+ \in L^2_m$. If ϕ does not blow up backward in time, then $\inf I = -\infty$ and ϕ scatters backward in time.
- (3) (Small data scattering) There exists $\varepsilon > 0$ such that if $\|\phi_0\|_{L^2} \leq \varepsilon$, then $\|\phi\|_{L^4_{t,x}} \lesssim \|\phi_0\|_{L^2_x}$. In particular, $I = \mathbb{R}$ and the solution scatters both forward and backward in time.
- (4) (Uniformly continuous dependence) For every $A > 0$ and $\varepsilon > 0$ there is a $\delta > 0$ such that if ϕ is an m -equivariant solution satisfying $\|\phi\|_{L^4_{t,x}(J \times \mathbb{R}^2)} \leq A$ with $t_0 \in J$ and if $\phi_0 = \phi(t_0)$, $\tilde{\phi}_0 = \tilde{\phi}(t_0)$ with $\phi_0, \tilde{\phi}_0 \in L^2_m$ satisfying $\|\phi_0 - \tilde{\phi}_0\|_{L^2_x} \leq \delta$, then there exists an m -equivariant solution $\tilde{\phi}$ such that $\|\phi - \tilde{\phi}\|_{L^4_{t,x}(J \times \mathbb{R}^2)} \leq \varepsilon$ and $\|\phi(t) - \tilde{\phi}(t)\|_{L^2} \leq \varepsilon$ for all $t \in J$.
- (5) (Stability) For every $A > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|\phi\|_{L^4_{t,x}(J \times \mathbb{R}^2)} \leq A$, ϕ is m -equivariant and approximates (1.21) in that $\|(i\partial_t + \Delta)\phi - \Lambda(\phi)\|_{L^{4/3}_{t,x}(J \times \mathbb{R}^2)} \leq \delta$, $t_0 \in J$, and $\tilde{\phi}_0 \in L^2_m$ satisfies

$$\|e^{i(t-t_0)\Delta}(\phi(t_0) - \tilde{\phi}_0)\|_{L^4_{t,x}(J \times \mathbb{R}^2)} \leq \delta,$$

then there exists an m -equivariant solution $\tilde{\phi}$ with $\tilde{\phi}(t_0) = \tilde{\phi}_0$ and $\|\phi - \tilde{\phi}\|_{L^4_{t,x}(J \times \mathbb{R}^2)} \leq \varepsilon$.

Proof. The local existence statement follows from (2.14) and a standard iteration argument. The scattering claim (2) follows from (2.14) and from linearizing near the asymptotic states. The remaining claims follow from (2.15) by standard arguments. □

3. Concentration compactness

The purpose of this section is to outline the proof of Lemma 1.7, which proceeds along the lines of the concentration compactness arguments in [26], [41], [28], and [2]. It is worth mentioning that the first part, the existence of critical element, is robust whenever we have a satisfying Cauchy theory like Theorem 2.8, while the second part, the classification of the critical elements, uses only the properties of almost periodic solutions and the scaling symmetry, and hence the argument is essentially independent of the equation.

3.1. Existence of critical element

We start with the symmetry group G_{\max} generated by the scaling transformation $g_\lambda: L^2_m(\mathbb{R}^2) \rightarrow L^2_m(\mathbb{R}^2)$ (as discussed in Remark 1.6, we will ignore the phase rotation)

$$g_\lambda f(r) = \lambda^{-1} f(\lambda^{-1} r).$$

The effect of g_λ is translated to the action g_λ^0 on A_0 and g_λ^θ on A_θ , where

$$g_\lambda^0 A_0(r) = \lambda^{-2} A_0(\lambda^{-1} r) \quad \text{and} \quad g_\lambda^\theta A_\theta(r) = A_\theta(\lambda^{-1} r),$$

and also extends to space-time functions by

$$T_{g_\lambda} f(r, t) = \lambda^{-1} f(\lambda^{-1} r, \lambda^{-2} t).$$

Let us first state the linear profile decomposition.

Proposition 3.1 (Linear profile decomposition). *Let ψ_n , $n = 1, 2, \dots$, be a bounded sequence in L^2_m . Then, after passing to a subsequence if necessary, there exists a sequence of functions $\phi^j \in L^2_m$, group elements $g_n^j \in G_{\max}$, and times $t_n^j \in \mathbb{R}$ such that we have the decomposition*

$$(3.1) \quad \psi_n = \sum_{j=1}^J g_n^j e^{it_n^j \Delta} \phi^j + w_n^J$$

for all $J = 1, 2, \dots$. Moreover, $w_n^J \in L^2_m$ is such that its linear evolution has asymptotically vanishing scattering size

$$(3.2) \quad \lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \|e^{it \Delta} w_n^J\|_{L^4_{t,x}} = 0.$$

Moreover for any $j \neq j'$,

$$(3.3) \quad \frac{\lambda_n^j}{\lambda_n^{j'}} + \frac{\lambda_n^{j'}}{\lambda_n^j} + \frac{|t_n^j (\lambda_n^j)^2 - t_n^{j'} (\lambda_n^{j'})^2|}{\lambda_n^j \lambda_n^{j'}} \rightarrow \infty.$$

Furthermore, for any $J \geq 1$, we have the mass decoupling property

$$(3.4) \quad \lim_{n \rightarrow \infty} \left[M(\psi_n) - \sum_{j=1}^J M(\psi_j) - M(w_n^J) \right] = 0.$$

We omit the proof here since it is the same as that in Theorem 7.3 of [41], where the statement is proved for radial case. Earlier versions of linear profile decomposition can be found in [34] and [1].

We now identify the critical threshold for global wellposedness and scattering. For any $m \geq 0$, define $A(m)$ by

$$A(m) := \sup\{S_I(\phi) : M(\phi) \leq m, \text{ where } \phi \text{ is a solution of (1.21) with maximal life span } I\}.$$

Remark 3.2. We see that $A: [0, \infty) \rightarrow [0, \infty]$ is a monotone nondecreasing function of m , that it is bounded for small m by part (3) and left continuous by part (4) of Theorem 2.8, and thus that there exists a critical $0 < m_0 \leq \infty$ such that $A(m)$ is finite if $m < m_0$ and $A(m) = \infty$ if $m \geq m_0$. Part (2) of Theorem 2.8 also implies that we have global wellposedness and scattering when $m < m_0$.

Proposition 3.3. *Assume $m_0 < \infty$, let $\psi_n : I_n \times \mathbb{R}^2 \rightarrow \mathbb{C}, n \in \mathbb{N}$ be a sequence of m -equivariant solutions to (1.21), and $t_n \in I_n$ a sequence of times such that $\lim_{n \rightarrow \infty} M(\psi_n) = m_0$ and*

$$(3.5) \quad \lim_{n \rightarrow \infty} S_{\geq t_n}(\psi_n) = \lim_{n \rightarrow \infty} S_{\leq t_n}(\psi_n) = \infty.$$

Then there are group elements $g_n \in G_{\max}$ such that $g_n \psi_n(t_n)$ has a subsequence that converges in $L^2_m(\mathbb{R}^2)$.

Proof. The proof is similar to Proposition 2.1 of [41].

We first translate ψ_n and I_n in time, taking $t_n = 0$. Then we apply Proposition 3.1 to $\psi_n(0)$ (after passing to subsequence), to get a linear profile decomposition

$$\psi_n(0) = \sum_{j=1}^J g_n^j e^{it_n^j \Delta} \phi^j + w_n^J.$$

Next, by passing to a further subsequence and using a diagonalization argument, we get for each j a sequence t_n^j converging to $t^j \in [-\infty, \infty]$. If t^j is finite, then we can redefine $\tilde{\phi}^j = e^{it^j \Delta} \phi^j$ and replace the profile $g_n^j e^{it_n^j \Delta} \phi^j$ by $g_n^j \tilde{\phi}^j$, with the difference being pushed into the error term w_n^J . By abuse of notation, we write $\tilde{\phi}^j$ as ϕ^j and hence we will further assume $t_n^j = 0$ for finite t^j .

From the mass decoupling property (3.4) we get

$$\sum_{j \geq 1} M(\phi^j) \leq \lim_{n \rightarrow \infty} M(\psi_n) = m_0,$$

which implies $\sup_j M(\phi^j) \leq m_0$.

Now let us assume

$$(3.6) \quad \sup_j M(\phi^j) \leq m_0 - \epsilon$$

for some $\epsilon > 0$. We will show that this leads to contradiction.

We define the nonlinear profile $v^j : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{C}$ associated to each ϕ^j in the following way:

- If t_n^j is identically zero, we define v^j to be the maximal-lifespan solution of (1.21) with initial data ϕ^j .
- If $t^j = \lim_{n \rightarrow \infty} t_n^j = \pm\infty$, we define v^j to be the maximal-lifespan solution of (1.21) that scatters forward (backward) in time to $e^{it \Delta} \phi^j$.

From the fact $M(\phi^j) \leq m_0 - \epsilon$, we see that v^j are global solutions and we have

$$S(v^j) \leq BM(v^j),$$

where B is a constant depending only on m_0 and ϵ . (This is because $A(m)$ is increasing and finite on $[0, m_0 - \epsilon]$, and when m is small, $A(m)/m$ is bounded because of part (3) of Theorem 2.8.)

Next we will define the approximate solution

$$(3.7) \quad \psi_n^J(t) := \sum_{j=1}^J T_{g_n^j} v^j(\cdot + t_n^j)(t) + e^{it\Delta} w_n^J.$$

From (3.2) and the orthogonal condition (3.3), we get

$$(3.8) \quad \lim_{n \rightarrow \infty} \lim_{n \rightarrow \infty} S(\psi_n^J(t)) = \lim_{J \rightarrow \infty} \sum_{j=1}^J S(v^j) \leq \lim_{J \rightarrow \infty} \sum_{j=1}^J BM(v^j) \leq Bm_0.$$

Now we prove the following facts:

- 1) Asymptotic agreement with initial data: for any $J = 1, 2, \dots$

$$(3.9) \quad \lim_{n \rightarrow \infty} M(\psi_n^J - \psi_n) = 0.$$

By comparing (3.7) with (3.1) and using the triangle inequality, we see that (3.9) follows from

$$\lim_{n \rightarrow \infty} M(T_{g_n^j} v^j(\cdot + t_n^j)(0) - g_n^j e^{it_n^j \Delta} \phi^j) = \lim_{n \rightarrow \infty} M(g_n^j v^j(t_n^j) - g_n^j e^{it_n^j \Delta} \phi^j) = 0,$$

where we used the construction of v^j .

- 2) Asymptotic solvability of the equation: we have

$$\lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \|(i\partial_t + \Delta)\psi_n^J - \Lambda(\psi_n^J)\|_{L_{t,x}^{4/3}} \rightarrow 0.$$

Here $\Lambda(\phi)$ is defined to be the nonlinearity of equation (1.21) with A_0 and A_θ as in (2.7) and (2.8), respectively.

By direct computation, we get

$$(i\partial_t + \Delta)\psi_n^J = \sum_{j=1}^J \Lambda(T_{g_n^j} v^j(\cdot + t_n^j)(t)).$$

So we need to prove

$$(3.10) \quad \lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \|\Lambda(\psi_n^J - e^{it\Delta} w_n^J) - \Lambda(\psi_n^J)\|_{L_{t,x}^{4/3}} \rightarrow 0,$$

and

$$(3.11) \quad \lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \left\| \sum_{j=1}^J \Lambda(T_{g_n^j} v^j(\cdot + t_n^j)(t)) - \Lambda\left(\sum_{j=1}^J T_{g_n^j} v^j(\cdot + t_n^j)(t)\right) \right\|_{L_{t,x}^{4/3}} \rightarrow 0.$$

Now (3.10) follows from (2.15) together with (3.2), (3.8). It is worth mentioning that even though we have nonlocal nonlinearity, we still get (3.11) from asymptotically orthogonal condition (3.3) and the Riemann–Lebesgue characterization of L^p spaces (Similar cases appear in (4.2)–(4.5) of [2]).

Now we use the stability result, part (5) of Theorem 2.8, on ψ_n^J with n, J large enough to conclude that $\psi_n(t)$ exists globally and $S(\psi_n(t)) \leq 3Bm_0$, which contradicts (3.5).

We know (3.6) is false. Therefore we have that

$$\sup_j M(\phi^j) = m_0.$$

So we have exactly one nonzero profile profile, which we call ϕ^1 after relabeling, so that we get

$$\psi_n(0) = g_n^1 e^{it_n^1 \Delta} \phi^1 + w_n^1,$$

with $\lim_{n \rightarrow \infty} M(w_n^1) \rightarrow 0$ because of (3.4).

Now, if $t_n \rightarrow +\infty$ (similar proof when the limit is $-\infty$), we notice that the Strichartz estimate implies that $S(e^{it\Delta}\phi^1) < \infty$, which further implies

$$\lim_{n \rightarrow \infty} S_{\geq 0}(e^{it\Delta} g_n^1 e^{it_n \Delta} \phi^1) = 0.$$

This together with (3.2) implies

$$\lim_{n \rightarrow \infty} S_{\geq 0}(e^{it\Delta} \psi_n(0)) = 0.$$

Taking n large enough, we can invoke the stability argument (part (5) of Theorem 2.8) with 0 as the approximate solution to conclude that $S_{\geq 0}(\psi)$ is finite, which contradicts (3.5).

So we are left with the case where $\lim_{n \rightarrow \infty} t_n^1$ is finite, which we further assume to be 0, hence we get $(g_n^1)^{-1}\psi_n(0)$ converging to ϕ^1 in $L^2(\mathbb{R}^2)$. □

Proposition 3.4. *Assume the critical mass m_0 is finite. Then there exists a maximal-lifespan m -equivariant solution $\phi : I_{\max} \times \mathbb{R}^2 \rightarrow \mathbb{C}$ to equation (1.21) with mass exactly m_0 . which blows up both forward and backward in time. In addition, the orbit $\{\phi(t), t \in I_{\max}\}$ is pre-compact in $L_m^2(\mathbb{R}^2)$ modulo scaling.*

Proof. If m_0 is finite, we can find a sequence of solutions $\psi_n : I_n \times \mathbb{R}^2 \rightarrow \mathbb{C}$ such that $M(\psi_n) \rightarrow m_0-$, $S_{I_n}(\psi_n) \rightarrow \infty$. By taking $t_n \in I_n$ such that $S_{\geq t_n}(\psi_n) = S_{\leq t_n}(\psi_n) = \frac{1}{2}S_{I_n}(\psi_n)$, we can further assume $t_n = 0$ by time translation invariance. Now we use Proposition 3.3 to conclude that by passing to a subsequence, we have elements of $g_n \in G_{\text{rad}}$ such that $g_n\psi_n(0)$ converges to ϕ_0 in $L_m^2(\mathbb{R}^2)$. Moreover, the proof of Proposition 3.3 implies that $M(\phi_0) = m_0$.

Let $\phi(t) : I_{\max} \times \mathbb{R}^2 \rightarrow \mathbb{C}$ be maximal-lifespan solution to (1.21) with initial data ϕ_0 . By the stability argument we have

$$S_{\geq 0}(\phi) = S_{\leq 0}(\phi) = \infty.$$

Now given any sequence of times $t'_n \in I_{\max}$, we get $S_{\geq t'_n}(\phi) = S_{\leq t'_n}(\phi) = \infty$, hence we can apply Proposition 3.3 to $\phi(t'_n)$ to conclude that there exists a subsequence that converges in $L^2_m(\mathbb{R}^2)$ modulo scaling. \square

Remark 3.5. By a standard argument using the Arzelà–Ascoli theorem, the pre-compactness of the orbit $\{\phi(t), t \in I_{\max}\}$ in L^2 implies that the solution is almost periodic modulo scaling as in Definition 1.5. The proof can be found in Lemma 1.17 of [41].

3.2. Classification of the critical element

Now we want to further identify the different types of behavior for almost periodic solutions. The argument was invented in [28]. It is a standard argument by now and has been adapted to many different problems [31], [29], [30], [36], and [35]. Thus we will collect the main results from [28], omitting the details.

We start with the following definition:

Definition 3.6 (Normalized solution). Let $\phi : I \times \mathbb{R}^2 \rightarrow \mathbb{C}$ be an m -equivariant solution to (1.21), which is almost periodic modulo scaling with frequency scale function $N(t)$. We say ϕ is normalized if the life span contains 0 and $N(0) = 1$.

Now we define the normalization of ϕ at time $t_0 \in I$ by

$$\phi^{[t_0]} := T_{g_{N(t_0)}}(\phi(t_0)).$$

The function $\phi^{[t_0]}$ is m -equivariant, almost period modulo scaling and has lifespan

$$I_{\phi^{[t_0]}} = \{s \in \mathbb{R} : t_0 + s/N(t_0)^2 \in I\},$$

with frequency scale function

$$N_{\phi^{[t_0]}}(s) := N(t_0 + s/N(t_0)^2)/N(t_0)$$

and the same compactness modulus function as ϕ .

Now we list the main properties of $N(t)$; see Lemma 3.5 and Corollary 3.6 of [28] for the proofs.

Proposition 3.7 (Compactness of almost periodic solutions). *Let $\phi^{(n)}$ be a sequence of m -equivariant normalized maximal-lifespan solutions to (1.21) which are almost periodic modulo scaling with frequency functions $N^{(n)}(t)$ and a uniform compact modulus function $C : \mathbb{R}^+ \rightarrow \mathbb{R}^+$. Assume we also have uniform mass bound*

$$0 < M(\phi^{(n)}) < \infty.$$

Then, after passing to a subsequence, there exists a nonzero maximal life-span m -equivariant solution ϕ to (1.21) with $0 \in I$ that is almost periodic modulo scaling, such that $\phi^{(n)}$ converges locally uniformly to ϕ .

Proposition 3.8 (Local constancy of N). *Let ϕ be a nonzero maximal lifespan solution to (1.21) with interval I that is almost periodic modulo scaling with frequency scale function $N: I \rightarrow \mathbb{R}^+$. Then there exists a small number δ , depending on ϕ , such that for every $t_0 \in I$, we have*

$$[t_0 - \delta N(t_0)^{-2}, t_0 + \delta N(t_0)^{-2}] \subset I$$

and

$$N(t) \sim_\phi N(t_0) \quad \text{whenever} \quad |t - t_0| \leq \delta N(t_0)^{-2}.$$

Now we are ready to classify the solutions. Given ϕ , the critical element found in Proposition 3.4 with maximal lifespan I and frequency scale function $N_\phi(t)$, we will try to extract a new solution which falls into the classification in Lemma 1.7.

For any $T \geq 0$, we define the quantities

$$\begin{aligned} \text{osc}(T) &:= \inf_{t_0 \in I} \frac{\sup_{t \in I, |t-t_0| \leq T/N(t_0)^2} N(t)}{\inf_{t \in I, |t-t_0| \leq T/N(t_0)^2} N(t)}, \\ a(t_0) &:= \frac{\inf_{t \in I, t \leq t_0} N(t) + \inf_{t \in I, t \geq t_0} N(t)}{N(t_0)}. \end{aligned}$$

Notice that $\text{osc}(T)$ measures the least possible oscillation one can find in $N(t)$ on intervals of normalized duration T , and it is nondecreasing in T .

Now we discuss the cases:

1) When $\lim_{T \rightarrow \infty} \text{osc}(T) < \infty$, we will construct a soliton-like solution. In fact, by the local constancy of $N(t)$, we can find a sequence of times $t_n \in I$, $T_n \rightarrow \infty$,

$$[t_n - T_n/N(t_n)^2, t_n + T_n/N(t_n)^2] \subset I.$$

So we can define the normalized solution $\phi^{[t_n]}$ with lifespan $I_{\phi^{[t_n]}}$ and frequency scale function $N_{\phi^{[t_n]}}(s)$ as in Definition 3.6. Notice that $[-T_n, T_n] \subset I_{\phi^{[t_n]}}$ and $N_{\phi^{[t_n]}}(s) \sim_\phi 1$ on $[-T_n, T_n]$, and hence we can use Proposition 3.7 to find a subsequence that converges locally uniformly to a maximal lifespan solution with mass m_0 and has frequency function $N(t)$ which is bounded, with $0 < \inf N(t) \leq \sup N(t) < \infty$. Hence by modifying the compact modulus function $C(\eta)$, we can take $N(t) \equiv 1$.

2) When $\lim_{T \rightarrow \infty} \text{osc}(T) = \infty$ and $\inf_{t_0 \in I} a(t_0) = 0$, we construct a rapid cascade solution. By the condition on $a(t)$, we find $t_n^- < t_n < t_n^+, t_n^\pm \in I$, such that

$$(3.12) \quad \frac{N(t_n^-)}{N(t_n)} \rightarrow 0 \quad \text{and} \quad \frac{N(t_n^+)}{N(t_n)} \rightarrow 0.$$

We choose t'_n such that $t_n^- < t_n < t_n^+$ and

$$N(t'_n) \sim \sup_{t_n^- < t < t_n^+} N(t).$$

Then we can construct a normalized solution $\phi^{[t'_n]}$ whose lifespan contains $[s_n^-, s_n^+]$ with

$$s_n^\pm := N(t'_n)^2(t_n^\pm - t'_n),$$

and we can check $N_{\phi^{[t'_n]}}(s_n^\pm) \rightarrow 0$ from (3.12) and the definitions of t'_n, s_n^\pm . Now using Proposition 3.7, we find that $\phi^{[t'_n]}$ has a subsequence that converges to a solution, which can be checked to be rapid cascade.

3) When $\lim_{T \rightarrow \infty} \text{osc}(T) = \infty$ and $\inf_{t_0 \in I} a(t_0) > 0$, we construct a self-similar solution. Since the argument is a bit involved, we refer the interested reader to Case III in Section 4 of [28].

Proof of Lemma 1.7. Suppose that the statement of Theorem 1.2 (or 1.3, 1.4) is not true. From Remark 3.2, we know that m_0 is finite. Proposition 3.4 guarantees that there exists an m -equivariant critical element $\phi(t)$ with mass m_0 that blows up forward and backward in time that has pre-compact orbit modulo scaling in $L^2_m(\mathbb{R}^2)$. From Remark 3.5, we know that $\phi(t)$ is almost periodic modulo scaling in the sense of Definition 1.5. Lastly, from the discussion above, we know we can construct from $\phi(t)$ a new m -equivariant critical element with mass m_0 which falls into one of the three scenarios, i.e., self-similar, soliton-like, or rapid cascade. \square

4. Frequency localization

The purpose of this section is to relate Littlewood–Paley frequency-localizations of terms of $\Lambda(\phi)$, defined in (2.13), to frequency localizations of ϕ . This is done in a way that respects the L^p estimates established in §2.

We introduce Littlewood–Paley multipliers in the usual way. In particular, let $\psi : \mathbb{R}^+ \rightarrow [0, 1]$, $\psi \in C^\infty$, equal one on $[0, 1]$ and zero on $[2, \infty)$. For each $\lambda > 0$, define

$$\begin{aligned} \mathcal{F}(P_{\leq \lambda} f)(\xi) &:= \psi(|\xi| \lambda^{-1}) \hat{f}(\xi), & \mathcal{F}(P_{> \lambda} f)(\xi) &:= (1 - \psi(|\xi| \lambda^{-1})) \hat{f}(\xi), \\ \widehat{P_\lambda f}(\xi) &:= (\psi(|\xi| \lambda^{-1}) - \psi(2|\xi| \lambda^{-1})) \hat{f}(\xi). \end{aligned}$$

We similarly define $P_{\leq \lambda}$ and $P_{\geq \lambda}$. Also, for $\lambda > \mu > 0$, set

$$P_{\mu < \cdot \leq \lambda} := P_{\leq \lambda} - P_{\leq \mu}.$$

The standard L^p Bernstein estimates hold for these multipliers, see e.g. Lemma 2.1 of [28].

We record for reference the useful relation

$$\begin{aligned} (4.1) \quad \mathcal{F}(r \partial_r f) &= \mathcal{F}(x \cdot \nabla f) = \mathcal{F}(x^j \partial_j f) = i \partial_{\xi_j} (i \xi_j \hat{f}) \\ &= -2 \hat{f} - \xi \cdot \nabla_\xi \hat{f} = -2 \hat{f} - \rho \partial_\rho \hat{f} = -\rho^{-1} \partial_\rho (\rho^2 \hat{f}), \end{aligned}$$

which is valid when the dimension of the underlying space is 2. Here and throughout we set $\rho := |\xi|$. We also set

$$(4.2) \quad f(r) := -\frac{1}{2} |\phi|^2$$

for short and note the following equalities, which follow from (2.7):

$$(4.3) \quad \frac{1}{r} \partial_r A_\theta = \left(\frac{1}{r} + \partial_r \right) \left(\frac{1}{r} A_\theta \right) = (2 + r \partial_r) \left(\frac{1}{r^2} A_\theta \right) = f(r).$$

Lemma 4.1 (Fourier transforms of A_θ and $r^{-2}A_\theta$). *Let f be given by (4.2). Then*

$$(4.4) \quad \hat{A}_\theta = \rho^{-1} \partial_\rho \hat{f}$$

and

$$(4.5) \quad \mathcal{F}(r^{-2}A_\theta) = -[\rho \partial_\rho]^{-1} \hat{f}.$$

Proof. We invoke (1.10) to get

$$\hat{A}_\theta(\xi) = -i \partial_{\xi_2} \hat{A}_1 + i \partial_{\xi_1} \hat{A}_2,$$

where we interpret the derivatives in the sense of distributions. Upon expansion we write

$$\hat{A}_\theta(\xi) = -\frac{1}{2} \partial_{\xi_2} \left(\frac{\xi_2}{|\xi|^2} \mathcal{F}(|\phi|^2) \right) - \frac{1}{2} \partial_{\xi_1} \left(\frac{\xi_1}{|\xi|^2} \mathcal{F}(|\phi|^2) \right).$$

This simplifies to

$$\hat{A}_\theta(\xi) = -\frac{1}{2} \frac{\xi_j}{|\xi|^2} \partial_{\xi_j} \mathcal{F}(|\phi|^2),$$

so that

$$\hat{A}_\theta(\rho) = -\frac{1}{2\rho} \partial_\rho \mathcal{F}(|\phi|^2),$$

which establishes (4.4); alternatively, one may multiply (4.3) by r^2 and use (4.1). To show (4.5), let

$$F(r) := \frac{A_\theta}{r^2} = \frac{1}{r^2} \int_0^r f(s) s \, ds,$$

where the equality follows from (2.7). This function is differentiable a.e. and satisfies

$$(2 + r \partial_r) F(r) = f(r),$$

as noted in (4.3). Taking Fourier transforms and using (4.1), we obtain

$$\hat{f} = 2\hat{F} + \nabla_\xi \cdot \xi \hat{F} = -\rho \partial_\rho \hat{F}.$$

Because $\phi \in L^4_{t,x}$, it follows that $\phi \in L^4_x$ for a.e. t and hence $f \in L^2_x$ for a.e. t . Therefore, writing $\hat{F} = -[\rho \partial_\rho]^{-1} \hat{f}$, we may invoke (2.1) for a.e. t with $p = 2$ and so conclude that the Fourier transform of A_θ/r^2 has the desired localization properties. \square

Lemma 4.2 (Fourier transform of $A_0^{(1)}$). *Let $G(r) = r^{-2}A_\theta|\phi|^2$. Then*

$$\hat{A}_0^{(1)} = \rho^{-1} \partial_\rho \hat{G}.$$

Proof. Note that

$$r \partial_r A_0^{(1)} = A_\theta |\phi|^2$$

a.e., so that in particular

$$(4.6) \quad \frac{1}{r} \partial_r A_0^{(1)} = \frac{A_\theta}{r^2} |\phi|^2.$$

From this we also obtain

$$(4.7) \quad \partial_r^2 A_0^{(1)} = -\frac{A_\theta}{r^2} |\phi|^2 + \frac{1}{r} \partial_r (A_\theta |\phi|^2),$$

which is valid in the sense of distributions. Combining (4.6) and (4.7) and using the fact that $A_0^{(1)}$ is radial, we conclude

$$A_0^{(1)} = \Delta^{-1} \left[\frac{1}{r} \partial_r (r^2 G) \right] = \Delta^{-1} (2 + r \partial_r) G(r).$$

Invoking (4.1) (with the roles of r and ρ reversed), we get

$$\hat{A}_0^{(1)} = -\frac{1}{\rho^2} (-\rho \partial_\rho \hat{G}) = \frac{1}{\rho} \partial_\rho \hat{G}. \quad \square$$

Lemma 4.3. *The following holds:*

$$P_N ([r \partial_r]^{-1} |P_{<N} \phi|^2) = 0.$$

Proof. The term $A_0^{(2)}$ is nonzero only in the nonradial equivariant case. In particular, we have

$$r \partial_r A_0^{(2)} = m |\phi|^2$$

a.e. from the representation given in Lemma 2.2, and

$$(4.8) \quad \Delta A_0^{(2)} = m \frac{1}{r} \partial_r |\phi|^2$$

in the sense of distributions. The Cartesian coordinate representation

$$(4.9) \quad \Delta A_0^{(2)} = \text{Im}(Q_{12}(\bar{\phi}, \phi)),$$

however, is more convenient for our purposes here. In particular, we see immediately that

$$(4.10) \quad P_N \text{Im}(Q_{12}(P_{<N} \bar{\phi}, P_{<N} \phi)) = 0,$$

so that any contribution to $P_N A_0^{(2)}$ must come from input ϕ -frequencies of at least frequency N . □

Remark 4.4. Together (4.9) and (4.10) suggest splitting each ϕ input in the right-hand side of (4.9) into a sum of Littlewood–Paley frequency localizations. As $Q_{12}(\cdot, \cdot)$ is linear in each argument separately, there are some cross terms to handle, e.g., terms of the form $\text{Im}(Q_{12}(\bar{P}_J \bar{\phi}, P_K \phi))$ with ranges J and K not equal. Whereas the Cartesian representation is well-suited for revealing the frequency

localization, it is the radial representation (4.8) that is used in Lemma 2.2 in proving the L^2 estimate of that lemma, which does not hold for arbitrary (non-equivariant) L^2 data. Therefore, in order to take advantage of this frequency decomposition, we need to ensure that we can apply the L^2 estimate to terms of the form $\text{Im}(Q_{12}(\overline{P_J\phi}, P_K\phi))$. Note that if ϕ is m -equivariant, then so are $P_J\phi$ and $P_K\phi$, so that both inputs of $\text{Im}(Q_{12}(\cdot, \cdot))$ are m -equivariant. In particular, if both ϕ and ψ are m -equivariant, then

$$\text{Im}(Q_{12}(\bar{\phi}, \psi)) = m \frac{1}{r} \partial_r \text{Re}(\bar{\phi}\psi),$$

and so we may use (2.1) as in the proof of (2.10) of Lemma 2.2.

5. Extra regularity

Let us first state two lemmas that characterize some important properties of critical elements.

Lemma 5.1. *Let ϕ be an m -equivariant almost periodic solution to (1.21) on its maximal-lifespan I . Then for all $t \in I$,*

$$(5.1) \quad \phi(t) = \lim_{T \nearrow \sup I} i \int_t^T e^{i(t-t')\Delta} \Lambda(\phi(t')) dt' = - \lim_{T \searrow \inf I} i \int_T^t e^{i(t-t')\Delta} \Lambda(\phi(t')) dt'$$

as a weak limit in L^2_x .

Lemma 5.2. *Let ϕ be an m -equivariant almost periodic solution to (1.21) on its maximal-lifespan I , with frequency scale function $N : I \rightarrow \mathbb{R}^+$. If J is a subinterval of I , then*

$$(5.2) \quad \int_J N(t)^2 dt \lesssim_\phi \int_J \int_{\mathbb{R}^2} |\phi(t, x)|^4 dx dt \lesssim_\phi 1 + \int_J N(t)^2 dt.$$

The proofs of Lemmas 5.1 and 5.2 are similar to arguments in Section 6 of [41] and Lemma 3.9 of [28]. In particular, we need only adapt these arguments to our particular nonlinearity, and hence we omit the details.

5.1. The self-similar case

Our goal in this section is to show that self-similar minimal blowup solutions enjoy extra regularity.

Theorem 5.3. *Let ϕ be a self-similar critical m -equivariant solution of (1.21), almost periodic modulo scaling, with lifespan $I = (0, +\infty)$ and frequency scale function $N(t) = t^{-1/2}$ for $t \in I$. Then, for each $s \geq 0$, $\phi \in L_t^\infty H_m^s(\mathbb{R} \times \mathbb{R}^2)$.*

We adopt the basic setup used in [28] and introduce the quantities

$$\begin{aligned} \mathcal{M}(A) &= \sup_T \|P_{>AT^{-1/2}}\phi(T)\|_{L_x^2(\mathbb{R}^2)}, \\ \mathcal{S}(A) &= \sup_T \|P_{>AT^{-1/2}}\phi(t, x)\|_{L_{t,x}^4([T, 2T] \times \mathbb{R}^2)}, \\ \mathcal{N}(A) &= \sup_T \|P_{>AT^{-1/2}}\Lambda(\phi)(t, x)\|_{L_{t,x}^{4/3}([T, 2T] \times \mathbb{R}^2) + L_t^1[T, 2T]L_x^2(\mathbb{R}^2)}, \\ \tilde{\mathcal{N}}(A) &= \sup_T \|P_{>AT^{-1/2}}\Lambda(\phi)(t, x)\|_{L_{t,x}^{4/3}([T, 2T] \times \mathbb{R}^2)}. \end{aligned}$$

For our definition of Littlewood–Paley multipliers, see §4. The nonlinearity $\Lambda(\phi)$ is defined in (2.13). Whereas $\tilde{\mathcal{N}}(A)$ is used in [28, 17] to prove extra regularity for self-similar solutions, we use the slightly weakened norm $\mathcal{N}(A)$. This is especially helpful when A_θ has high-frequency inputs, as shown in Lemma 5.6.

To prove Theorem 5.3, we will show that

$$(5.3) \quad \mathcal{M}(A) = \sup_T \|P_{>AT^{-1/2}}\phi(T)\|_{L_x^2} <_{\phi, A} A^{-s}$$

for any $s > 0$.

5.1.1. Bounds. Mass conservation gives

$$\mathcal{M}(A) \lesssim_\phi 1,$$

and Strichartz estimates imply

$$(5.4) \quad \mathcal{S}(A) \lesssim_\phi \mathcal{M}(A) + \mathcal{N}(A).$$

The spacetime bound (5.2) establishes

$$\mathcal{S}(A) \lesssim_\phi 1,$$

and this spacetime bound together with Lemma 2.7 implies

$$\mathcal{N}(A) \lesssim \tilde{\mathcal{N}}(A) \lesssim \|\Lambda(\phi)\|_{L^{4/3}([T, 2T] \times \mathbb{R}^2)} \lesssim_\phi 1.$$

Combining the information above, we have

$$(5.5) \quad \mathcal{M}(A) + \mathcal{S}(A) + \mathcal{N}(A) \lesssim_\phi 1.$$

The Strichartz estimate together with the above inequalities implies

$$\|\phi\|_{L_t^2 L_x^\infty([T, 2T] \times \mathbb{R}^2)} \lesssim_\phi 1.$$

In the following lemma we collect some estimates that we will later employ.

Lemma 5.4. *Suppose $1/p = 1/p_1 + 1/p_2 + 1/p_3$ and $1/p_2 + 1/p_3 > 0$. Then the following nonlocal Hölder estimate holds:*

$$(5.6) \quad \left\| q_1 \int_r^\infty q_2 q_3 \frac{d\rho}{\rho} \right\|_p \lesssim \|q_1\|_{p_1} \|q_2\|_{p_2} \|q_3\|_{p_3}.$$

Additionally,

$$(5.7) \quad \left\| \frac{A_\theta}{r^2} \phi \right\|_p \lesssim \|\phi\|_{p_1} \|\phi\|_{p_2} \|\phi\|_{p_3}$$

for $1/p = 1/p_1 + 1/p_2 + 1/p_3$ with $1 < p_i < \infty$.

The Strichartz estimate for equivariant functions f ,

$$(5.8) \quad \|P_N e^{it\Delta} f\|_{L^q} \lesssim N^{1-4/q} \|f\|_{L_x^2}, \quad q \geq \frac{10}{3},$$

is also true, from which easily follows the inhomogeneous estimate

$$(5.9) \quad \|P_N u\|_{L^q} \lesssim N^{1-4/q} (\|f\|_{L_x^2} + \|(i\partial_t + \Delta)u\|_{L^{4/3} + L_t^1 L_x^2}), \quad q \geq \frac{10}{3}.$$

The nonlocal Hölder estimate follows from elementary inequalities, see Section 3 of [17]. Shao [38] proved (5.8) for the range $q > 10/3$, and the endpoint $q = 10/3$ was established by Guo and Wang in [16].

Remark 5.5. There is enough slack in our argument for nonendpoint estimates to suffice. However, when the endpoint estimate is used, the exponents are particularly simple, and so we use this estimate for convenience. Note that both [38] and [16] prove results for radial functions. There they use the fact that the Fourier transform of a radial function may be expressed in terms of a Hankel transform with kernel J_k , a Bessel function of the first kind. When the underlying space is two-dimensional, $k = 0$. In the m -equivariant 2-d setting, the Bessel function required is J_m , which enjoys the same asymptotics at infinity as does J_0 , but is better behaved near the origin. These properties are sufficient for extending the proofs of [38] and [16] to this setting.

We now come to the first main estimate.

Lemma 5.6. *Given A large enough, we have*

$$\mathcal{N}(A) \lesssim_\phi \mathcal{S}(\frac{A}{10}) \mathcal{M}(\sqrt{A}) + A^{-1/10} [\mathcal{M}(\frac{A}{10}) + \mathcal{N}(\frac{A}{10})].$$

Proof. We proceed as in [28]. It suffices to prove

$$\left\| P_{>AT^{-1/2}} \mathcal{N}(\phi)(t, x) \right\|_{L_{t,x}^{4/3} + L_t^1 L_x^2 [T, 2T]} \lesssim \mathcal{S}(\frac{A}{10}) \mathcal{M}(\sqrt{A}) + A^{-\frac{1}{10}} [\mathcal{M}(\frac{A}{10}) + \mathcal{N}(\frac{A}{10})]$$

uniformly in T . To do this, we decompose ϕ into high, intermediate and low frequency pieces, i.e.,

$$\phi = \phi_{\text{hi}} + \phi_{\text{med}} + \phi_{\text{low}},$$

where

$$\phi_{\text{hi}} = \phi_{> \frac{1}{10}AT^{-1/2}}, \quad \phi_{\text{med}} = \phi_{\sqrt{A}T^{-1/2} \leq \cdot \leq \frac{1}{10}AT^{-1/2}}, \quad \phi_{\text{low}} = \phi_{\leq \sqrt{A}T^{-1/2}}.$$

Because of the frequency localization lemmas of §4, we see that having nontrivial $P_{>AT^{-1/2}} \mathcal{N}(\phi)(t, x)$ implies that the nonlinearity must have at least one high frequency input ϕ_{hi} .

As in [28], we split into cases according to whether we have one intermediate input or all low inputs remaining.

It is convenient at this stage to split up the nonlinearity into “cubic” and “quintic” terms, as follows:

$$(5.10) \quad \Lambda_3 := 2m \frac{A_\theta}{r^2} \phi + A_0^{(2)} \phi - g|\phi|^2 \phi, \quad \Lambda_{5,1} := \frac{A_\theta^2}{r^2} \phi, \quad \Lambda_{5,2} := A_0^{(1)} \phi,$$

so that $\Lambda(\phi) = \Lambda_3 + \Lambda_{5,1} + \Lambda_{5,2}$.

Case 1. If we have at least one intermediate input ϕ_{med} in the nonlinearity, then we use the Hölder estimate (5.6). In particular, we use L^4 on ϕ_{hi} and $L_t^\infty L_x^2$ on ϕ_{med} , and so obtain the bound

$$\|\Lambda_3(\phi_{\text{hi}}, \phi_{\text{med}}, \phi)\|_{L^{4/3}[T,2T]} \lesssim \mathcal{S}\left(\frac{A}{10}\right) \mathcal{M}(\sqrt{A}).$$

For the quintic term $\Lambda_{5,1}$, use L^∞ on an A_θ that does not involve ϕ_{hi} and then apply Hölder to $\frac{A_\theta}{r^2} \phi$ in the same way that we do for the cubic terms:

$$\|\Lambda_{5,1}\|_{L^{4/3}[T,2T]} \lesssim_\phi \mathcal{S}\left(\frac{A}{10}\right) \mathcal{M}(\sqrt{A}).$$

We can control the quintic term $\Lambda_{5,2}$ in $L^{4/3}$ using L^∞ on A_θ and Hölder on the other terms provided that A_θ does not have a high frequency input. If A_θ does have a high frequency input, then we estimate $\Lambda_{5,2}$ in $L_t^1 L_x^2$:

$$\begin{aligned} \|\Lambda_{5,2}\|_{L_t^1 L_x^2} &\lesssim \left\| \int_r^\infty \frac{A_\theta}{s^2} |\phi|^2 s \, ds \right\|_{L_t^1 L_x^\infty} \|\phi\|_{L_t^\infty L_x^2[T,2T]} \\ &\lesssim \left\| \frac{A_\theta}{s^2} |\phi|^2 \right\|_{L_{t,x}^1[T,2T]} \|\phi\|_{L_t^\infty L_x^2} \lesssim \mathcal{S}\left(\frac{A}{10}\right) \mathcal{M}(\sqrt{A}). \end{aligned}$$

Here we used the Hölder estimate (5.7), putting the high frequency terms in L^4 , the medium frequency ones in $L^\infty L^2$, and the rest in L^4 and $L^2 L^\infty$.

Altogether, we conclude $\|\Lambda_{5,2}\|_{L^{4/3}+L^1 L^2} \lesssim_\phi \mathcal{S}\left(\frac{A}{10}\right) \mathcal{M}(\sqrt{A})$.

Case 2. For the case where one input is at high frequency and the rest are at low frequency, we adopt the idea of using the Strichartz estimates (5.8), (5.9), as found in Section 3.3 of [17].

For Λ_3 , we use, as in Case 1, $L^{10/3}$ on ϕ_{hi} and L^5 on one of ϕ_{low} :

$$\|\Lambda_3(\phi)\|_{L^{4/3}[T,2T]} \lesssim \|\phi_{\text{hi}}\|_{L^{10/3}[T,2T]} \|\phi_{\text{low}}\|_{L^5[T,2T]} \|\phi\|_{L^4[T,2T]}.$$

Using Bernstein and the inhomogeneous Strichartz estimate (5.9), we get

$$\begin{aligned} \|P_{<M} \phi\|_{L^5_{[T,2T] \times \mathbb{R}^2}} &\lesssim M^{1/5} \left(\|P_{<M} \phi(T)\|_{L^2} + \|P_{<M} \mathcal{N}(\phi)\|_{L^1 L^2 + L^{4/3}[T,2T]} \right), \\ \|P_{>N} \phi\|_{L^{10/3}_{[T,2T] \times \mathbb{R}^2}} &\lesssim N^{-1/5} \left(\|P_{>N} \phi(T)\|_{L^2} + \|P_{>N} \mathcal{N}(\phi)\|_{L^1 L^2 + L^{4/3}[T,2T]} \right). \end{aligned}$$

Taking $N = \frac{1}{10} A T^{-1/2}$ and $M = \sqrt{A} T^{-1/2}$, we obtain

$$\|\Lambda_3(\phi)\|_{L^{4/3}_{[T,2T] \times \mathbb{R}^2}} \lesssim A^{-1/10} [\mathcal{M}\left(\frac{1}{10} A\right) + \mathcal{N}\left(\frac{A}{10}\right)].$$

The quintic pieces of $\Lambda_{5,1}$ and $\Lambda_{5,2}$ with A_θ not involving ϕ_{hi} we handle as in Case 1. In particular, we use L^∞ on A_θ , then apply Hölder to obtain $\|\phi_{hi}\|_{L^{10/3}}$ and $\|\phi_{low}\|_{L^5}$, and then apply Hölder once more to get the $A^{-1/10}$ decay factor.

The quintic term $\Lambda_{5,2}$ with A_θ involving ϕ_{hi} we bound in L^1L^2 as in Case 1:

$$\|\Lambda_{5,2}\|_{L^1_t L^2_x} \lesssim \|\phi_{hi}\|_{L^{10/3}} \|\phi\|_{L^5} \|\phi\|_{L^4}^2 \|\phi\|_{L^\infty L^2} \lesssim A^{-1/10} [\mathcal{M}(\frac{1}{10}A) + \mathcal{N}(\frac{A}{10})]. \quad \square$$

Lemma 5.7. *We have*

$$\lim_{A \rightarrow \infty} \mathcal{M}(A) = \lim_{A \rightarrow \infty} \mathcal{S}(A) = \lim_{A \rightarrow \infty} \mathcal{N}(A) = 0.$$

Proof. The vanishing of the first limit follows from the definition of almost periodicity. The vanishing of the third limit follows from Lemma 5.6 and (5.5), and the vanishing of the second one follows from (5.4). \square

Given the nonlinear estimate in Lemma 5.6, the following ε -regularity result follows using exactly the same arguments employed in Proposition 5.5 and Corollary 5.6 of [28].

Lemma 5.8. *For all $A > 0$,*

$$S(A) \lesssim \eta S(\frac{A}{20}) + A^{-1/40}, \quad \text{and} \quad \mathcal{M}(A) + \mathcal{S}(A) + \mathcal{N}(A) \lesssim A^{-1/40}.$$

Finally, adapting the induction argument, we conclude higher regularity.

Theorem 5.9. *For all $A > 0$ and $s > 0$,*

$$\mathcal{M}(A) \lesssim A^{-s}.$$

5.2. The global critical case

The Fourier transform of an m -equivariant function $f(r, \theta) = e^{im\theta}u(r)$ is given in terms of a Hankel transform of its radial part u . We use polar coordinates (ρ, α) on the Fourier side, obtaining

$$\mathcal{F}(f)(\rho, \alpha) = 2\pi(-i)^m e^{im\alpha} \int_0^\infty u(r) J_m(r\rho) r \, dr.$$

The Fourier transform is an involution on equivariant functions, and so one may also obtain from this an inversion formula. Next, we split the Bessel function J_m into two Hankel functions, corresponding to projections onto outgoing and incoming waves. In particular, we have

$$J_m(|x||\xi|) = \frac{1}{2} H_m^{(1)}(|x||\xi|) + \frac{1}{2} H_m^{(2)}(|x||\xi|),$$

where $H_m^{(1)}$ is the order m Hankel function of the first kind and $H_m^{(2)}$ is the order m Hankel function of the second kind.

Definition 5.10. Let P^+ denote the projection onto outgoing m -equivariant waves:

$$\begin{aligned}
 [P^+ f](x) &:= \frac{1}{4\pi^2} e^{im\theta} \int_{\mathbb{R}^+ \times \mathbb{R}^+} H_m^{(1)}(|x||\xi|) J_m(|\xi||y|) f(|y|) d\xi dy \\
 &= \frac{1}{2} f(x) + \frac{i}{2\pi^2} \int_{\mathbb{R}^2} \left| \frac{y}{x} \right|^m \frac{f(x)}{|x|^2 - |y|^2} dy.
 \end{aligned}$$

Here the second inequality follows from Section 6.521.2 of [15] and analytic continuation.

In a similar way, we can define the projection $[P^- f](x)$ onto incoming waves by replacing $H_m^{(1)}$ with $H_m^{(2)}$. In particular, $P^- f$ is the complex conjugate of $P^+ f$.

We also use the notation P_λ^\pm for the composition $P^\pm P_\lambda$.

As the equivariance class m is clear from context, we omit it from the notation for P^\pm .

Lemma 5.11 (Kernel estimate: incoming/outgoing wave).

- (1) The operator $P^+ + P^-$ acts as the identity on m -equivariant functions belonging to $L^2(\mathbb{R}^2)$.
- (2) For $|x| > \lambda^{-1}$ and $t \gtrsim \lambda^{-2}$,

$$|[P_\lambda^\pm e^{\mp it\Delta}](x, y)| \lesssim \begin{cases} (|x||y||t|)^{-1/2}, & |y| - |x| \sim \lambda t, \\ \frac{\lambda^2}{\langle \lambda|x| \rangle^{1/2} \langle \lambda|y| \rangle^{1/2}} \langle \lambda^2 t + \lambda|x| - \lambda|y| \rangle^{-n}, & \text{otherwise,} \end{cases}$$

for all $n \geq 0$.

- (3) For $|x| \gtrsim \lambda^{-1}, t \lesssim \lambda^{-2}$.

$$|[P_\lambda^\pm e^{\mp it\Delta}](x, y)| \lesssim \frac{\lambda^2}{\langle \lambda|x| \rangle^{1/2} \langle \lambda|y| \rangle^{1/2}} \langle \lambda|x| - \lambda|y| \rangle^{-n}$$

for all $n \geq 0$.

- (4) For $\lambda > 0$ and any equivariant function $f \in L^2(\mathbb{R}^2)$,

$$\|P^\pm P_{\geq \lambda} f\|_{L^2(|x| \geq \frac{1}{100\lambda})} \lesssim \|f\|_{L^2(\mathbb{R}^2)}.$$

This result is established in [28, 31]; see for instance Proposition 6.2 of [28] and Lemma 4.1 of [31]. The spatial cutoff in (4) of Lemma 5.11 is only necessary when $m \neq 0$. The operators P^\pm that act on radial functions are bounded on L^2 . However, their counterparts for $m \neq 0$ are no longer bounded on L^2 because of the worse singularity of $H_m^{(1)}$ (and $H_m^{(2)}$) at the origin.

It is worth comparing the above kernel estimate with the following kernel bound on the linear propagator. We see that when x is far from the origin, we have better decay after decomposing into incoming and outgoing waves.

Lemma 5.12 (Kernel estimate: linear propagator). *For any $n \geq 0$, the kernel of the linear propagator obeys the following estimates:*

$$|[P_\lambda e^{it\Delta}](x, y)| \lesssim \begin{cases} |t|^{-1/2}, & |y - x| \sim \lambda t, \\ \frac{\lambda^2}{|\lambda^2 t|^n \langle \lambda |x - y| \rangle^n}, & \text{otherwise,} \end{cases} \quad \text{for } t \gtrsim \lambda^{-2},$$

and

$$|[P_\lambda e^{it\Delta}](x, y)| \lesssim \lambda^2 \langle \lambda |x - y| \rangle^{-n} \quad \text{for } t \lesssim \lambda^{-2}.$$

With the help of the decay provided by the incoming/outgoing wave decompositions, we can prove the following lemma.

Theorem 5.13. *Let ϕ be a global critical m -equivariant solution of (1.21), almost periodic modulo scaling, and with $N(t) \lesssim 1$ uniformly in $t \in \mathbb{R}$. Then, for each $s \geq 0$, $\phi \in L_t^\infty H_m^s(\mathbb{R} \times \mathbb{R}^2)$.*

It suffices to prove

$$\mathcal{M}(\lambda) := \|\phi_{\geq \lambda}\|_{L_t^\infty L_x^2(\mathbb{R} \times \mathbb{R}^2)} \lesssim \lambda^{-s}.$$

By mass conservation,

$$\|\phi\|_{L_t^\infty L_x^2(\mathbb{R} \times \mathbb{R}^2)} \lesssim_\phi 1,$$

and so $\mathcal{M}(\lambda) \gtrsim 1$. From almost periodicity and from the boundedness of $N(t)$, we get

$$\lim_{\lambda \rightarrow \infty} \|\phi_{\geq \lambda}\|_{L_t^\infty L_x^2(\mathbb{R} \times \mathbb{R}^2)} = 0,$$

which means that $\mathcal{M}(\lambda) \rightarrow 0$.

Theorem 5.13 follows from the following lemma.

Lemma 5.14 (Regularity). *Let ϕ be as in Theorem 5.13 and let $\eta > 0$ be a small number. Then*

$$\mathcal{M}(\lambda) \lesssim \eta \mathcal{M}\left(\frac{\lambda}{8}\right)$$

whenever λ is sufficiently large, depending upon ϕ and η .

We prove this lemma by showing that

$$(5.11) \quad \|\phi_{\geq \lambda}(t_0)\|_{L_x^2(\mathbb{R}^2)} \lesssim \eta \mathcal{M}\left(\frac{\lambda}{8}\right)$$

for any time t_0 and for all λ sufficiently large.

Let us explain the strategy here: we will follow the argument in Section 5 of [31], which is a slight modification of Section 7 of [28]. We first split the estimate into bounding the L^2 norm of $\phi_{\geq \lambda}(t_0)$ in two different regions. Inside the ball $\{|x| \leq \lambda^{-1}\}$ we can use the kernel bound in Lemma 5.12, while outside the ball we enjoy better decay by decomposing into incoming and outgoing waves and using the kernel bound in Lemma 5.11. In each region, we split into three cases: the short time estimate Proposition 5.15, the long time main term estimate Proposition 5.16, and the long time tail estimate Proposition 5.17.

The proof of Propositions 5.15, 5.16 and 5.17, with the help of kernel bounds, involves only Hölder, Strichartz, and weighted Strichartz estimates. We have the same bound as in Section 7 of [28] and so can carry out the same argument there by taking advantage of the decay when outside the ball and by taking advantage of the spatial support smallness when inside the ball.

Now we carry out the ideas in more detail. We can first assume $t_0 = 0$ by time translation. Let $\chi_\lambda(x)$ denote the characteristic function of $[1/\lambda, \infty)$.

For the portion of the frequency localized solution $\phi_{\geq \lambda}$ in the ball $\{|x| \leq \lambda^{-1}\}$, we get (see formula (5–7) in [31]) that

$$\begin{aligned}
 (1 - \chi_\lambda(x))\phi_{\geq \lambda}(0) &= \lim_{T \rightarrow \infty} i \int_0^T (1 - \chi_\lambda(x))e^{-it\Delta} P_{\geq \lambda} \Lambda(\phi)(t) dt \\
 &= i \int_0^\delta (1 - \chi_\lambda(x))e^{-it\Delta} P_{\geq \lambda} \Lambda(\phi)(t) dt \\
 (5.12) \quad &+ \lim_{T \rightarrow \infty} \sum_{\mu \geq \lambda} i \int_\delta^T \int_{\mathbb{R}^2} (1 - \chi_\lambda(x)) [P_\mu e^{-it\Delta}](x, y) P_\mu \Lambda(\phi)(t)(y) dy dt,
 \end{aligned}$$

where for (5.12) we use the integral kernel of $P_\mu e^{-it\Delta}$.

Next, for the portion of $\phi_{\geq \lambda}$ in the region $\{|x| \geq \lambda^{-1}\}$, we split into incoming and outgoing waves propagating backward and forward in time (respectively):

$$\begin{aligned}
 \chi_\lambda(x)\phi_{\geq \lambda}(0) &= \lim_{T \rightarrow \infty} i \int_0^T \chi_\lambda(x) e^{-it\Delta} P_{\geq \lambda} \Lambda(\phi)(t) dt \\
 &= i \int_0^\delta \chi_\lambda(x) P^+ e^{-it\Delta} P_{\geq \lambda} \Lambda(\phi)(t) dt - i \int_{-\delta}^0 \chi_\lambda(x) P^- e^{-it\Delta} P_{\geq \lambda} \Lambda(\phi)(t) dt \\
 &+ \lim_{T \rightarrow \infty} \sum_{\mu \geq \lambda} i \int_\delta^T \int_{\mathbb{R}^2} \chi_\lambda(x) [P_\mu^+ e^{-it\Delta}](x, y) P_\mu \Lambda(\phi)(t)(y) dy dt \\
 &- \lim_{T \rightarrow \infty} \sum_{\mu \geq \lambda} i \int_{-T}^{-\delta} \int_{\mathbb{R}^2} \chi_\lambda(x) [P_\mu^- e^{-it\Delta}](x, y) P_\mu \Lambda(\phi)(t)(y) dy dt.
 \end{aligned}$$

As explained in [31], this is to be interpreted as a weak L^2 limit, and we have

$$f_T \rightarrow f \text{ weakly} \implies \|f\| \leq \limsup \|f_T\|.$$

The following short time estimate works for any spatial region.

Proposition 5.15 (Short-time estimate). *Given any $\eta > 0$ we can find some $\delta = \delta(\phi, \eta) > 0$ such that*

$$\left\| \int_0^\delta e^{-it\Delta} P_{\geq \lambda} \Lambda(\phi)(t) dt \right\|_{L^2} \leq \eta \mathcal{M}(\frac{\lambda}{8})$$

provided λ is large enough depending on ϕ and η .

Similar estimates hold on the time interval $[-\delta, 0]$ and for incoming/outgoing waves under premultiplication by $\chi_\lambda P^\pm$.

The proof is similar to that of Lemma 7.3 of [28], the main difference being that we must use the nonlocal Hölder estimate (5.6) and the estimate (5.7). As in the proof of extra regularity for the self-similar case, we also use the fact that a high frequency output of $\Lambda(\phi)$ implies that there is a high frequency input term. The details of how to perform the decomposition and apply (5.6) and (5.7) are similar, and so we omit the proofs.

To work with the long-time estimate, we break the region of (t, y) integration into two pieces: $|y| \gtrsim \mu|t|$ and $|y| \ll \mu|t|$. In the former region, $P_\mu e^{-it\Delta}$ and $P_\mu^\pm e^{-it\Delta}$ have a stationary point, hence it will give the main contribution. And the later region will be treated as error.

Take χ_k to be the characteristic function for

$$\{(t, y) | 2^k \delta \leq |t| \leq 2^{k+1} \delta, |y| \gtrsim \mu|t|\}.$$

Proposition 5.16 (Long-time estimate: main contribution). *Let $0 < \eta < 1$ and δ be as in Proposition 5.15. Then*

$$\sum_{\mu \geq \lambda} \sum_{k \geq 0} \left\| \int_{2^k \delta}^{2^{k+1} \delta} \int_{\mathbb{R}^2} [P_\mu e^{-it\Delta}](x, y) \chi_k(t, y) P_\mu \Lambda(\phi)(t)(y) dy dt \right\|_{L_x^2} \lesssim \eta \mathcal{M}(\frac{\lambda}{8})$$

for λ large enough depending on ϕ and η . A similar estimate holds under premultiplication by $\chi_k P^\pm$ or on time interval $[-2^{k+1} \delta, -2^k \delta]$.

Now we just need to estimate the tails coming from the region

$$\{(t, y) : 2^k \delta \leq |t| \leq 2^{k+1} \delta, |y| \ll \mu|t|\}.$$

Since this is the non-stationary region, the kernels have rapid decay, as shown in Lemma 5.11 and Lemma 5.12. In particular when $\mu \geq \lambda$, we have

$$|P_\mu e^{-it\Delta}(x, y)| \lesssim \frac{\mu^2}{(\mu^2|t|)^{50}}$$

for $|x| \leq \lambda^{-1}, |y| \ll \mu|t|$ and $|t| \geq \delta \gg N^{-2}$

$$|P_\mu^\pm e^{-it\Delta}(x, y)| \lesssim \frac{\mu^2}{(\mu^2|t|)^{50} \langle \mu|x| - \mu|y| \rangle^{50}}$$

for $|x| > \lambda^{-1}, |y| \ll \mu|t|$ and $|t| \geq \delta \gg N^{-2}$

Let $\tilde{\chi}_k$ denote the characteristic function of this region. Then we have the following tail estimate which reads the same as the estimate for the main contribution.

Proposition 5.17 (Long-time estimate: tails). *Let $0 < \eta < 1$ and δ be as in Proposition 5.15. Then*

$$\sum_{\mu \geq \lambda} \sum_k \left\| \int_\delta^T \int_{\mathbb{R}^2} [P_\mu e^{-it\Delta}](x, y) \tilde{\chi}_k(t, y) P_\mu \Lambda(\phi)(t)(y) dy dt \right\|_{L_x^2} \lesssim \eta \mathcal{M}(\frac{\lambda}{8})$$

for λ large enough depending on ϕ and η . A similar estimate holds under premultiplication $\tilde{\chi}_k P^\pm$ and on time intervals $[-2^{k+1} \delta, -2^k \delta]$.

As explained before, Propositions 5.15–5.17 together establish (5.11).

6. Virial and Morawetz identities

We recall

$$F_{0r} = -\frac{1}{r} \operatorname{Im}(\bar{\phi} D_\theta \phi), \quad F_{0\theta} = r \operatorname{Im}(\bar{\phi} D_r \phi), \quad \text{and} \quad F_{r\theta} = -\frac{1}{2} r |\phi|^2.$$

Because $dF = d^2A = 0$, we have

$$(6.1) \quad \partial_t F_{r\theta} - \partial_r F_{0\theta} + \partial_\theta F_{0r} = 0.$$

To rewrite this in terms of a natural stress-energy tensor, let

$$T_{00} = \frac{1}{2} r |\phi|^2, \quad T_{0r} = r \operatorname{Im}(\bar{\phi} D_r \phi), \quad \text{and} \quad T_{0\theta} = \frac{1}{r} \operatorname{Im}(\bar{\phi} D_\theta \phi).$$

Then (6.1) may be rewritten as $\partial_\alpha T_{0\alpha} = 0$.

Lemma 6.1. *We have*

$$(6.2) \quad \begin{aligned} \partial_t T_{0r} &= -(2 + 2r\partial_r) |D_r \phi|^2 + \frac{1}{2} r g \partial_r |\phi|^4 \\ &+ \frac{1}{r} \partial_r |D_\theta \phi|^2 - \frac{2}{r} \partial_\theta \operatorname{Re}(\overline{D_\theta \phi} D_r \phi) \\ &+ r \partial_r \left[\frac{1}{r^2} \left(\frac{1}{2} \partial_\theta^2 |\phi|^2 - |D_\theta \phi|^2 \right) \right] + \left(\frac{1}{2} r \partial_r^3 + \frac{1}{2} \partial_r^2 - \frac{1}{2r} \partial_r \right) |\phi|^2. \end{aligned}$$

Proof. We write

$$\begin{aligned} \partial_t T_{0r} &= r \operatorname{Im}(\overline{D_t \phi} D_r \phi) + r \operatorname{Im}(\bar{\phi} D_t D_r \phi) \\ &= r \operatorname{Im}(\overline{D_t \phi} D_r \phi) + r \operatorname{Im}(\bar{\phi} D_r D_t \phi) + r F_{0r} |\phi|^2 \\ &= r \operatorname{Im}(\overline{D_t \phi} D_r \phi) + r \operatorname{Im}(\bar{\phi} D_r D_t \phi) + 2F_{\theta r} F_{0r} \end{aligned}$$

and calculate each piece separately, using (1.13).

For the first term, we get

$$\begin{aligned} r \operatorname{Im}(\overline{D_t \phi} D_r \phi) &= -r \operatorname{Re}(\overline{D_r^2 \phi} D_r \phi) - |D_r \phi|^2 - \frac{1}{r} \operatorname{Re}(\overline{D_\theta^2 \phi} D_r \phi) - r g |\phi|^2 \operatorname{Re}(\bar{\phi} D_r \phi) \\ &= -(1 + \frac{1}{2} r \partial_r) |D_r \phi|^2 - \frac{1}{4} g r \partial_r |\phi|^4 - \frac{1}{r} \operatorname{Re}(\overline{D_\theta^2 \phi} D_r \phi). \end{aligned}$$

Now

$$\begin{aligned} \operatorname{Re}(\overline{D_\theta^2 \phi} D_r \phi) &= \partial_\theta \operatorname{Re}(\overline{D_\theta \phi} D_r \phi) - \operatorname{Re}(\overline{D_\theta \phi} D_r D_\theta \phi) - \operatorname{Re}(\overline{D_\theta \phi} i F_{\theta r} \phi) \\ &= \partial_\theta \operatorname{Re}(\overline{D_\theta \phi} D_r \phi) - F_{\theta r} \operatorname{Im}(\bar{\phi} D_\theta \phi) - \frac{1}{2} \partial_r |D_\theta \phi|^2, \end{aligned}$$

and so we can rewrite the first term as

$$\begin{aligned} r \operatorname{Im}(\overline{D_t \phi} D_r \phi) &= -(1 + \frac{1}{2} r \partial_r) |D_r \phi|^2 - \frac{1}{4} g r \partial_r |\phi|^4 \\ &- \frac{1}{r} \partial_\theta \operatorname{Re}(\overline{D_\theta \phi} D_r \phi) - F_{\theta r} F_{0r} + \frac{1}{2r} \partial_r |D_\theta \phi|^2. \end{aligned}$$

For the second term, we get

$$\begin{aligned} r \operatorname{Im}(\bar{\phi} D_r D_t \phi) &= r \operatorname{Re}(\bar{\phi} D_r^3 \phi) + r \operatorname{Re}(\bar{\phi} D_r (\frac{1}{r} D_r \phi)) \\ &\quad + r \operatorname{Re}(\bar{\phi} D_r (\frac{1}{r^2} D_\theta^2 \phi)) + r g \operatorname{Re}(\bar{\phi} D_r (|\phi|^2 \phi)). \end{aligned}$$

Now

$$\begin{aligned} r \operatorname{Re}(\bar{\phi} D_r^3 \phi) &= \frac{1}{2} r \partial_r^3 |\phi|^2 - \frac{3}{2} r \partial_r |D_r \phi|^2, \\ r \operatorname{Re}(\bar{\phi} D_r (\frac{1}{r} D_r \phi)) &= -|D_r \phi|^2 + \left(\frac{1}{2} \partial_r^2 - \frac{1}{2r} \partial_r\right) |\phi|^2, \\ r \operatorname{Re}(\bar{\phi} D_r (\frac{1}{r^2} D_\theta^2 \phi)) &= r \partial_r \left[\frac{1}{r^2} \left(\frac{1}{2} \partial_\theta^2 |\phi|^2 - |D_\theta \phi|^2\right) \right] - \frac{1}{r} \operatorname{Re}(\overline{D_r \phi} D_\theta^2 \phi), \\ r g \operatorname{Re}(\bar{\phi} D_r (|\phi|^2 \phi)) &= \frac{3}{4} r g \partial_r |\phi|^4. \end{aligned}$$

Hence

$$\begin{aligned} r \operatorname{Im}(\bar{\phi} D_r D_t \phi) &= -\left(1 + \frac{3}{2} r \partial_r\right) |D_r \phi|^2 + \left(\frac{1}{2} r \partial_r^3 + \frac{1}{2} \partial_r^2 - \frac{1}{2r} \partial_r\right) |\phi|^2 \\ &\quad + \frac{1}{2r} \partial_r |D_\theta \phi|^2 + r \partial_r \left[\frac{1}{r^2} \left(\frac{1}{2} \partial_\theta^2 |\phi|^2 - |D_\theta \phi|^2\right) \right] \\ &\quad - \frac{1}{r} \partial_\theta \operatorname{Re}(\overline{D_\theta \phi} D_r \phi) - F_{\theta r} F_{0r} + \frac{3}{4} r g \partial_r |\phi|^4. \end{aligned}$$

Combining the above pieces yields (6.2). □

Lemma 6.2 (Virial and Morawetz identities). *A direct calculation relying upon integration by parts verifies the virial identity*

$$(6.3) \quad \partial_t^2 \iint r^2 T_{00} \, dr \, d\theta = 4 \iint \left(|D_r \phi|^2 + \frac{1}{r^2} |D_\theta \phi|^2 - \frac{g}{2} |\phi|^4 \right) r \, dr \, d\theta,$$

and the Morawetz identity

$$(6.4) \quad \partial_t^2 \iint r T_{00} \, dr \, d\theta = \frac{1}{2} \iint \left(\frac{1}{r^2} |\phi|^2 - g |\phi|^4 \right) \, dr \, d\theta.$$

Proof. To prove (6.3), start with

$$\partial_t^2 \iint r^2 T_{00} \, dr \, d\theta = -\partial_t \iint r^2 \left(\partial_r T_{0r} + \partial_\theta T_{0\theta} \right) \, dr \, d\theta = 2 \iint r \partial_t T_{0r} \, dr \, d\theta.$$

Then invoke (6.2) to conclude

$$\iint r \partial_t T_{0r} \, dr \, d\theta = 2 \iint \left(|D_r \phi|^2 + \frac{1}{r^2} |D_\theta \phi|^2 - \frac{g}{2} |\phi|^4 \right) r \, dr \, d\theta = 4E(\phi).$$

To obtain (6.4), write

$$\partial_t^2 \iint r T_{00} \, dr \, d\theta = \iint \partial_t T_{0r} \, dr \, d\theta$$

and then use (6.2). □

Remark 6.3. Under the equivariant ansatz, the components of the stress-energy tensor are radial, so that, in particular the integrands of (6.3) and (6.4) are independent of θ . Under this ansatz, the identity (6.2) admits the simplification

$$(6.5) \quad \begin{aligned} \partial_t T_{0r} = & - (2 + 2r\partial_r)|D_r\phi|^2 + \frac{1}{2}rg\partial_r|\phi|^4 + \frac{1}{r}\partial_r|D_\theta\phi|^2 \\ & - r\partial_r\left(\frac{1}{r^2}|D_\theta\phi|^2\right) + \left(\frac{1}{2}r\partial_r^3 + \frac{1}{2}\partial_r^2 - \frac{1}{2r}\partial_r\right)|\phi|^2. \end{aligned}$$

7. Absence of critical elements

Proposition 7.1. *Let $m \in \mathbb{Z}$ and let $\phi \in H_m^1$ be a nontrivial solution of (1.21) with $g < 1$. Then $E(\phi) > 0$.*

Proof. The main tool required is the so-called Bogomol’nyi identity, which states

$$(7.1) \quad |D_x\phi|^2 = |D_+\phi|^2 + \nabla \times J - F_{12}|\phi|^2,$$

where $D_\pm := D_1 \pm iD_2$ and $J = (J_1, J_2)$ with $J_k := \text{Im}(\bar{\phi}D_k\phi)$. This identity can be motivated by the factorization

$$D_j D_j \phi = (D_1 - iD_2)(D_1 + iD_2)\phi + F_{12}\phi,$$

and both can be verified by direct calculation. Using (7.1) and Green’s theorem, we obtain

$$(7.2) \quad E(\phi) := \frac{1}{2} \int_{\mathbb{R}^2} \left[|D_x\phi|^2 - \frac{g}{2}|\phi|^4 \right] dx = \frac{1}{2} \int_{\mathbb{R}^2} \left[|D_+\phi|^2 + \frac{1}{2}(1-g)|\phi|^4 \right] dx.$$

From this we conclude that if $g < 1$ and ϕ is not zero a.e., then $E(\phi) > 0$. □

7.1. Ruling out the self-similar scenario

As a corollary of (5.3), used to prove Theorem 5.3, we obtain the following.

Corollary 7.2. *Let $g < 1$. Critical equivariant self-similar solutions do not exist.*

Proof. For any $s \geq 0$,

$$\sup_{t \in (0, \infty)} \int_{|\xi| > At^{-1/2}} |\hat{\phi}(\xi, t)|^2 d\xi \leq C_s A^{-s}, \quad A > A_0(s).$$

Therefore

$$(7.3) \quad \|\phi(t, \cdot)\|_{\dot{H}^s(\mathbb{R}^2)} \lesssim t^{-s/2} = [N(t)]^s.$$

Thanks to the following lemma, taking $t \rightarrow \infty$ in (7.3) implies that the conserved energy $E(\phi)$ must be zero and hence the solution ϕ trivial. □

Lemma 7.3. *Let $m \in \mathbb{Z}$ and let $\phi \in L_t^\infty L_m^2$ be a solution of (1.21) with $E(\phi)$, given by (1.6), finite. Then*

$$(7.4) \quad |E(\phi)| \lesssim \|\phi\|_{\dot{H}^1},$$

where the constant depends upon g and the charge $\text{chg}(\phi)$.

Proof. First we note that

$$|D_x \phi|^2 \lesssim |\nabla \phi|^2 + |A_x \phi|^2.$$

To control $A_x \phi$ in L^2 , use $|A_j| = \frac{1}{r}|A_\theta| \lesssim \|\phi\|_{L^4}^2$, where the last inequality follows from (2.5). Therefore,

$$\|D_x \phi\|_{L^2}^2 \lesssim \|\nabla \phi\|_{L^2}^2 + \|\phi\|_{L^4}^2 \|\phi\|_{L^2}^2.$$

The lemma now follows from the Gagliardo–Nirenberg inequality

$$(7.5) \quad \|f\|_{L^4}^4 \lesssim \|\nabla f\|_{L^2}^2 \|f\|_{L^2}^2. \quad \square$$

7.2. Ruling out global critical elements

Let $\chi : \mathbb{R}_+ \rightarrow [0, 1]$ be a smooth cut-off function equal to one on $[0, 1]$ and zero on $[2, \infty)$. For any given $R > 0$, define $\chi_R(r) := \chi(r/R)$. Set

$$I_R(\phi) := \int_0^\infty T_{0r} \chi_R r \, dr.$$

Lemma 7.4 (Localized virial identity). *Let $m \in \mathbb{Z}$, $\phi \in L^\infty H_m^1$, and let $e(\phi)$ denote the energy density*

$$e(\phi) := \frac{1}{2} \left(|D_r \phi|^2 + \frac{1}{r^2} |D_\theta \phi|^2 - \frac{g}{2} |\phi|^4 \right).$$

Then

$$(7.6) \quad \begin{aligned} \frac{d}{dt} I_R(\phi) &= 4 \int e(\phi) r \, dr + 4 \int e(\phi) (\chi_R - 1) r \, dr \\ &\quad - 2 \int \left(|D_r \phi|^2 - \frac{g}{4} |\phi|^4 \right) (r \chi'_R) r \, dr + I_{\text{rem}}, \end{aligned}$$

where

$$I_{\text{rem}} = -\frac{1}{2} \int \frac{|\phi|^2}{r^2} (r^3 \chi_R''') r \, dr - \frac{5}{2} \int \frac{|\phi|^2}{r^2} (r^2 \chi_R'') r \, dr - \frac{3}{2} \int \frac{|\phi|^2}{r^2} (r \chi_R') r \, dr.$$

Proof. By invoking (6.5) and integrating by parts, we obtain

$$\frac{d}{dt} I_R(\phi) = \sum_{j=1}^5 I_5,$$

where

$$\begin{aligned}
 I_1 &:= 2 \int |D_r \phi|^2 \chi_R r \, dr - 2 \int |D_r \phi|^2 (r \chi'_R) r \, dr, \\
 I_2 &:= - \int g |\phi|^4 \chi_R r \, dr + \frac{1}{2} \int g |\phi|^4 (r \chi'_R) r \, dr, \\
 I_3 &:= - \int \frac{1}{r^2} |D_\theta \phi|^2 (r \chi'_R) r \, dr, \\
 I_4 &:= 2 \int \frac{1}{r^2} |D_\theta \phi|^2 \chi_R r \, dr + \int \frac{1}{r^2} |D_\theta \phi|^2 (r \chi'_R) r \, dr,
 \end{aligned}$$

and $I_5 := I_{\text{rem}}$ defined above. Rearranging the terms in I_1 through I_4 yields the result. □

Corollary 7.5. *Let $g < 1$. Global equivariant critical elements do not exist.*

Proof. We divide the proof into two cases:

Case 1. The solution is soliton-like with $N(t) = 1$ for all $t \in \mathbb{R}$.

Invoking Theorem 5.13, we have that for each $s \geq 0$ the estimate

$$(7.7) \quad \|\phi(t, \cdot)\|_{\dot{H}^s(\mathbb{R}^2)} \leq C_s$$

holds uniformly in time. Next, let $\eta > 0$ and take $R = 2C(\eta)/N_0$ so that

$$(7.8) \quad \int_{|x|>R/2} |\phi(t, x)|^2 \, dx < \eta$$

for all time. By interpolating, we can control the energy far from the origin:

$$(7.9) \quad \int_R^\infty \left(|D_r \phi|^2 + \frac{1}{r^2} |D_\theta \phi|^2 - \frac{g}{2} |\phi|^4 \right) r \, dr \lesssim \eta^{1/2}.$$

This suffices for controlling the second term in the right-hand side of (7.6).

To control the third term, first note that the L^4 norm of ϕ can be controlled far from the origin by using (7.5), (7.8), and (7.7). Using L^4 control in (7.9) then permits us to control $|D_r \phi|^2$ far from the origin as well. Finally,

$$r \chi'_R = r [\chi(r/R)]' = \frac{r}{R} \chi'(r/R),$$

which is nonzero only for $R \leq r \leq 2R$; it is also bounded by a constant in this range uniformly in R . Combining these statements suffices for controlling the third term in the right-hand side of (7.6).

To control the fourth term, i.e. I_{rem} , we note that each integrand is bounded by a constant times $|\phi|^2$ integrated against rdr in the range $R \leq r \leq 2R$. Hence we may use (7.8).

Using these estimates, we conclude from (7.6) that

$$\frac{d}{dt} I_R(\phi) \geq \frac{2}{\pi} E(\phi) - C \eta^{1/2}.$$

Therefore, by conservation of energy, we have for η sufficiently small that

$$(7.10) \quad \frac{d}{dt} I_R(\phi) \gtrsim 1.$$

On the other hand, by (7.6) and (7.4),

$$|I_R(\phi)| \lesssim R \|\phi\|_{L^2} \|\phi\|_{\dot{H}^1} \lesssim RC_1$$

holds uniformly in time. This contradicts (7.10) for t sufficiently large.

Case 2. The solution has a rapid cascade so that there exists a sequence t_n such that $N(t_n) \rightarrow 0$.

Given $\eta > 0$ arbitrary, then, by almost periodicity, we can find $C = C(\eta, \phi)$ such that

$$\int_{|\xi| > CN(t)} |\hat{\phi}(\xi)|^2 d\xi \leq \eta.$$

Interpolating with (7.7) for $s = 2$, we get

$$\int_{|\xi| > CN(t)} |\xi|^2 |\hat{\phi}(\xi)|^2 d\xi \lesssim \eta^{1/2},$$

and from mass conservation and Plancherel’s theorem we have

$$\int_{|\xi| \leq CN(t)} |\xi|^2 |\hat{\phi}(\xi)|^2 d\xi \lesssim C^2 N(t)^2.$$

Together these two facts imply

$$\|\nabla\phi\|_{L^2}^2 \lesssim \eta^{1/2} + C^2 N(t)^2.$$

Hence along the sequence t_n , we have $\|\nabla\phi(t_n)\|_{L^2}^2 \rightarrow 0$. By Lemma 7.3, we also obtain

$$|E(\phi(t_n))| \lesssim \|\nabla\phi(t_n)\|_{L^2}^2 \rightarrow 0.$$

This forces a contradiction because of energy conservation and (7.2), which says that when $g < 1$, $E(\phi) > 0$ if $\phi \neq 0$. □

Remark 7.6. (1) The self-similar case can also be ruled out using the Virial identity argument, due to the behavior of $N(t)$ at ∞ . This was demonstrated in Section 9 of [28].

(2) Notice that in the arguments in section 7.1 and section 7.2, the only place we used $g < 1$ is to get the positivity of energy as in Proposition 7.1. When $g \geq 1$, we can still ensure the positivity of energy by requiring the charge of data to be below certain threshold. To be more precise, we use (7.2) and Lemma 8.1 for $g = 1$ and we use Lemma 8.5, 8.6, and 8.8 for $g > 1$ to find the charge threshold. Hence we can carry over the arguments as above to rule out the critical elements for $g \geq 1$.

8. The focusing problem

In the focusing problem we shall restrict ourselves to $m \geq 0$. This is the physically interesting case for (1.21) as written. In fact, the natural Chern–Simons–Schrödinger system for $m < 0$ is not (1.21), but rather an analogous one with the signs in the field constraints flipped. For further discussion of this point, see Chapter II. C., E. of [11].

8.1. The case $g = 1$

Lemma 8.1. *Let $g = 1$ and $m \in \mathbb{Z}_+$. Suppose that $\phi \in L_t^\infty H_m^1$ is a solution of (1.21) with $E(\phi) = 0$. Then ϕ is a soliton.*

Proof. Straightforward calculations reveal

$$(8.1) \quad D_+ = e^{i\theta} \left(D_r + \frac{i}{r} D_\theta \right)$$

and

$$|D_\theta \phi|^2 = (m + A_\theta)^2 |\phi|^2.$$

By (7.2), $E(\phi) = 0$ implies $D_+ \phi = 0$ a.e. For m -equivariant ϕ , this implies

$$\partial_r \phi = \frac{1}{r} (m + A_\theta) \phi.$$

Consequently,

$$\frac{1}{2} \partial_r |\phi|^2 = \frac{1}{r} (m + A_\theta) |\phi|^2 = \partial_r A_0,$$

so that $A_0 = \frac{1}{2} |\phi|^2$. Therefore ϕ is an equivariant solution of the self-dual Chern–Simons–Schrödinger system

$$(8.2) \quad \begin{cases} (D_1 + iD_2)\phi &= 0, \\ A_0 &= \frac{1}{2} |\phi|^2, \\ \partial_1 A_2 - \partial_2 A_1 &= -\frac{1}{2} |\phi|^2. \end{cases}$$

Such solutions constitute static solutions to (1.21) (with $g = 1$). Conversely, any H^1 static solution of (1.21) with $g = 1$ has $E(\phi) = 0$ (for a short proof, see [19]). □

If $m \in \mathbb{Z}_+$, then explicit equivariant solutions are given by

$$\begin{cases} \phi^{(m)}(t, x) &= \sqrt{8\lambda(m+1)} \frac{|\lambda x|^m}{1 + |\lambda x|^{2(m+1)}} e^{im\theta}, \\ A_j^{(m)}(t, x) &= 2(m+1)\lambda^2 \frac{\epsilon_{jk} x_k |\lambda x|^{2m}}{1 + |\lambda x|^{2(m+1)}}, \\ A_0^{(m)}(t, x) &= 4 \left[\frac{\lambda(m+1)|\lambda x|^m}{1 + |\lambda x|^{2(m+1)}} \right]^2. \end{cases}$$

where $\lambda > 0$ is a free scaling parameter and ϵ_{jk} is the anti symmetric tensor with $\epsilon_{12} = 1$. Such solutions are discussed, for instance, in [20] and [22]. For any value $\lambda > 0$, we find

$$\text{chg}(\phi^{(m)}) = 8\pi(m + 1).$$

Uniqueness of these explicit soliton solutions (up to a phase) is discussed in [20] and a proof can be given by combining the arguments of [7] with the equivariance ansatz.

From (7.2) and Lemma 8.1, we see that $E(\phi) > 0$ if $\text{chg}(\phi) < 8\pi(m + 1)$. Hence we can conclude the proof of Theorem 1.3 using arguments from §7.

8.2. The case $g > 1$

Lemma 8.2. *Let $g > 1$. Then there exists a constant $c_g > 0$ such that any nontrivial H^1 solution ϕ of (1.21) with $E(\phi) \leq 0$ satisfies $\text{chg}(\phi) \geq c_g$.*

Proof. Using (1.6) we see that $E(\phi) \leq 0$ implies

$$\frac{2}{g} \|D_x \phi\|_{L_x^2}^2 \leq \|\phi\|_{L_x^4}^4.$$

We can combine this with the covariant Gagliardo–Nirenberg inequality (e.g., see [3], (2.28), for a proof), which states that

$$(8.3) \quad \|\phi\|_{L_x^4}^4 \lesssim \|D_x \phi\|_{L_x^2}^2 \|\phi\|_{L_x^2}^2,$$

to conclude that $E(\phi) \leq 0$ implies

$$\|\phi\|_{L_x^2}^2 \gtrsim \frac{2}{g}. \quad \square$$

Remark 8.3. The constants in Lemma 8.2 are universal (in that they are not dependent upon the equivariance index m or even upon the satisfaction of the equivariance ansatz) but not sharp. In Lemma 8.5 we show that, given an equivariance index $m \in \mathbb{Z}_+$, the sharp charge threshold for the class H_m^1 may be found by minimizing over nontrivial energy zero solutions in that class.

As an interesting application of the Bogomol’nyi identity (7.1), we prove the following inequality, which is similar to an inequality of Byeon, Huh, and Seok (page 1607 of [5]).

Lemma 8.4. *Let ϕ be an H^1 solution of (1.7), $g \in \mathbb{R}$. Then*

$$(8.4) \quad \|\phi\|_{L_x^4}^4 \leq 4 \|D_r \phi\|_{L_x^2} \| \frac{1}{r} D_\theta \phi \|_{L_x^2}.$$

Proof. Integrating (7.1) over \mathbb{R}^2 and using the observation (8.1), we conclude

$$(8.5) \quad \int \frac{1}{2} |\phi|^4 = 2 \int \text{Im}(\overline{r^{-1} D_\theta \phi} D_r \phi).$$

Then (8.4) follows by Cauchy–Schwarz. □

Applying Young’s inequality to (8.4) yields

$$(8.6) \quad \|\phi\|_{L^4_x}^4 \leq 2 \|D_r\phi\|_{L^2_x}^2 + 2 \left\| \frac{1}{r} D_\theta\phi \right\|_{L^2_x}^2 = 2 \|D_x\phi\|_{L^2_x}^2.$$

In particular, in the $g = 1$ case, zero-energy solutions of (1.21) are precisely those that yield equality in (8.6). More generally, using (1.6) and (8.5), we observe that $E(\phi) \leq 0$ implies that ϕ satisfies the following reverse Cauchy–Schwarz inequality:

$$(8.7) \quad \int \left[|\partial_r\phi|^2 + \frac{1}{r^2} |D_\theta\phi|^2 \right] dx \leq 2g \int \operatorname{Im}(\overline{r^{-1}D_\theta\phi} \partial_r\phi) dx.$$

In the following, we will find the sharp threshold $c_{m,g}$ claimed in Theorem 1.4. First notice that for solutions $\phi \in H_m^1$ of (1.21), $E(\phi)$ has the following expression:

$$(8.8) \quad E(\phi) = \int_{\mathbb{R}^2} \left[|\partial_r\phi|^2 + \frac{1}{r^2} \left(m - \frac{1}{2} \int_0^r |\phi|^2 s ds \right)^2 |\phi|^2 - \frac{g}{2} |\phi|^4 \right] r dr.$$

This can be checked directly from the energy formula in polar coordinates (1.18), the equivariant ansatz for ϕ (1.20), and by solving A_r, A_θ in terms of ϕ in (1.21).

Lemma 8.5. *Let $m \in \mathbb{Z}_+$ and, for $\phi \in H_m^1$, then*

$$\inf_{0 \neq \phi \in H_m^1 : E(\phi) \leq 0} \operatorname{chg}(\phi) = \inf_{0 \neq \phi \in H_m^1 : E(\phi) = 0} \operatorname{chg}(\phi)$$

Proof. Let $\phi_n \in H_m^1$ for $n = 1, 2, 3, \dots$ be a minimizing sequence with $\operatorname{chg}(\phi_n) \rightarrow I$, where

$$I := \inf_{0 \neq \phi \in H_m^1 : E(\phi) \leq 0} \operatorname{chg}(\phi).$$

Then, for $\alpha \in \mathbb{R}$,

$$E(\alpha\phi_n) = \alpha^2 \int_{\mathbb{R}^2} \left[|\partial_r\phi_n|^2 + \frac{1}{r^2} \left(m - \alpha^2 \frac{1}{2} \int_0^r |\phi_n|^2 s ds \right)^2 |\phi_n|^2 - \alpha^2 \frac{g}{2} |\phi_n|^4 \right] dx.$$

Because

$$\lim_{\alpha \rightarrow 0} \alpha^{-2} E(\alpha\phi_n) = \int_{\mathbb{R}^2} \left[|\partial_r\phi_n|^2 + \frac{m}{r^2} |\phi_n|^2 \right] dx > 0,$$

there exists $\alpha_n \in (0, 1]$ such that $E(\alpha_n\phi_n) = 0$. Set $\psi_n := \alpha_n\phi_n$. Then $E(\psi_n) = 0$ and $\operatorname{chg}(\psi_n) \leq \operatorname{chg}(\phi_n)$. Passing to a convergent subsequence, we obtain

$$\lim \operatorname{chg}(\psi_n) \leq \lim \operatorname{chg}(\phi_n) = I. \quad \square$$

Lemma 8.6. *The set of minimizers over $E(\phi) = 0$ is nonempty. We denote $c_{m,g}$ to be the minimal charge.*

The proof for $m = 0$ is found in Section 5 of [5] and generalizes to the case $m > 0$. There is a lack of compactness due to the scaling symmetry which is removed by renormalizing the \dot{H}^1 norm. Once this is done, one may pass to a weak limit in H_m^1 that also converges strongly in L_m^q , $q > 2$.

Remark 8.7. Here we can immediately conclude that $E(\phi) > 0$ if and only if $\text{chg}(\phi) < c_{m,g}$. Combining this with the arguments of Section 7 completes the proof for global wellposedness and scattering, which is the first part of the statement in Theorem 1.4.

In the next lemma we characterize energy-zero minimal charge solutions $\phi \in H_m^1 \setminus \{0\}$.

Lemma 8.8. *Let $\phi \in H_m^1 \setminus \{0\}$ with $E(\phi) = 0$ and $\text{chg}(\phi) = c_{m,g}$. Then there exists $\lambda \in \mathbb{R}$ such that $\gamma(t, x) := e^{i\lambda t} \phi(x)$ is a weak solution of (1.21).*

Proof. We use Lagrange multipliers, which necessitates taking the first variation of $E(\phi)$ (in the form of (8.8)). Varying the $\phi(r, \theta)$ terms leads to

$$2 \int \left[\text{Re}(\overline{\partial_r \phi} \partial_r \psi) + \frac{1}{r^2} (m + A_\theta)^2 \text{Re}(\bar{\phi} \psi) - g|\phi|^2 \text{Re}(\bar{\phi} \psi) \right] r \, dr,$$

which upon integration by parts becomes

$$- \int \text{Re}(\bar{\psi} (\partial_r^2 + r^{-1} \partial_r + r^{-2} D_\theta + g|\phi|^2) \phi) r \, dr.$$

The additional contribution from the variation of $\phi(s, \rho)$ is

$$(8.9) \quad - \int \left(\frac{2m}{r^2} - \frac{1}{r^2} \int_0^r |\phi|^s s \, ds \right) \int_0^r \text{Re}(\bar{\phi} \psi) s \, ds |\phi|^2 r \, dr.$$

Let

$$F(r) = - \int_r^\infty \left(\frac{m}{r} - \frac{1}{2r} \int_0^r |\phi|^2 s \, ds \right) |\phi|^2 \, dr.$$

Then (8.9) may be rewritten as

$$-2 \iint_0^r \text{Re}(\bar{\phi} \psi) s \, ds \partial_r F(r) \, dr,$$

which upon integration by parts is seen to be

$$2 \int_0^\infty F(r) \text{Re}(\bar{\phi} \psi) \, dr.$$

With the observation that in fact we may take $A_0 = F(r)$, the proof is complete. □

The above variation is discussed in Chapter II.B. of [11] and is similar to the approach of [5]. Solutions $\psi \in L^\infty H_m^1$ of (1.21) of the form $\gamma(t, x) = e^{i\lambda t} \phi$, $\phi \in H_m^1$, we call standing wave solutions. We remark that we can follow the approach in Appendix A of [5] to show that the standing wave is actually a classical solution. In fact, it was shown in [5] that any critical point of an energy functional, which is very similar to $E(\phi)$ in (8.8), is a classical solution.

Remark 8.9. Notice that standing waves do not scatter, so in light of Remark 8.7, we know their charges are no smaller than $c_{m,g}$. In particular, we conclude that the minimum charge of a nontrivial standing wave solution in the class $L_t^\infty L_m^2$ is equal to $c_{m,g}$.

Remark 8.10. When $g = 1$, static solutions (standing wave solutions with $\lambda = 0$) exist but $\lambda \neq 0$ standing wave solutions do not. When $g > 1$ and $m = 0$, Byeon, Huh, and Seok conjecture that there are no static solutions (see Remark 5.1 of [5]).

We conclude with two Pohozaev-type identities of independent interest.

Lemma 8.11 (Pohozaev identity). *Let $\phi \in L_t^\infty H_m^1$ be a standing wave solution of (1.21). Then*

$$(8.10) \quad \int (\lambda + A_0) |\phi|^2 dx = \frac{g}{2} \int |\phi|^4 dx.$$

Proof. For a standing wave with frequency λ we can write

$$\int (\lambda + A_0) |\phi|^2 dx = \int \text{Im}(\bar{\phi} D_t \phi) dx.$$

Next we replace $D_t \phi$ using the first equation of (1.4) and integrate by parts. □

Through integrating by parts (differently) in (8.10), we can recover the Pohozaev type identity established in the case $m = 0$ in Proposition 2.3 of [5].

Corollary 8.12. *Let $\phi \in L_t^\infty H_m^1$ be a standing wave solution of (1.21). Then*

$$(\lambda - 2mA_0(0)) \int |\phi|^2 dx + 2 \int_0^\infty \frac{1}{r^2} |D_\theta \phi|^2 dx = \frac{g}{2} \int |\phi|^4 dx.$$

Proof. We have

$$\int_0^\infty A_0 |\phi|^2 r dr = \int_0^\infty \left(- \int_r^\infty \frac{m + A_\theta}{s} |\phi|^2(s) ds \right) |\phi|^2(r) r dr.$$

Now write

$$|\phi|^2 r = -2\partial_r \left(m - \frac{1}{2} \int_0^r |\phi|^2 s ds \right).$$

Integrating by parts yields

$$\int_0^\infty A_0 |\phi|^2 r dr = -2mA_0(0) + 2 \int_0^\infty \frac{(m + A_\theta)^2}{r^2} |\phi|^2 r dr. \quad \square$$

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