Focal points and sup-norms of eigenfunctions

Christopher D. Sogge and Steve Zelditch

Abstract. If (M, g) is a compact real analytic Riemannian manifold, we give a necessary and sufficient condition for there to be a sequence of quasimodes of order $o(\lambda)$ saturating sup-norm estimates. In particular, it gives optimal conditions for existence of eigenfunctions satisfying maximal sup norm bounds. The condition is that there exists a self-focal point $x_0 \in M$ for the geodesic flow at which the associated Perron–Frobenius operator $U_{x_0} \colon L^2(S_{x_0}^* M) \to L^2(S_{x_0}^* M)$ has a nontrivial invariant L^2 function. The proof is based on an explicit Duistermaat–Guillemin–Safarov pre-trace formula and von Neumann's ergodic theorem.

1. Introduction and main results

Let (M, g) be a compact boundaryless Riemannian manifold. We then let $\Phi_t(x, \xi) = (x(t), \xi(t))$ denote the homogeneous Hamilton flow on $T^*M \backslash 0$ associated to the principal symbol $|\xi|_g$ of $\sqrt{-\Delta}$, with $\Delta = \Delta_g$ denoting the Laplace–Beltrami operator of (M, g). Since Φ_t preserves the unit cosphere bundle $S^*M = \{|\xi|_g = 1\}$, it defines a flow on S^*M which preserves the Liouville measure. For a given $x \in M$, let $\mathcal{L}_x \subset S_x^* M$ denote those unit directions ξ for which $\Phi_t(x, \xi) \in S_x^* M$ for some time $t \neq 0$, and let $|\mathcal{L}_x|$ denote its surface measure $d\mu_x$ in $S_x^* M$ induced by the Euclidean metric g_x. Thus, $\mathcal{L}_x$ denotes the initial directions of geodesic loops through x.

In [23] it was shown that

$$(1.1) \qquad \|e_\lambda\|_{L^\infty(M)} = o(\lambda^{(n-1)/2}),$$

if

$$|\mathcal{L}_x| = 0 \quad \text{for all } x \in M,$$

whenever e_λ is an L^2-normalized eigenfunction of frequency λ, i.e.,

$$(1.2) \qquad (\Delta + \lambda^2)e_\lambda = 0, \quad \text{and} \quad \int_M |e_\lambda|^2 \, dV = 1.$$

Here, $dV = dV_g$ denotes the volume element associated with the metric. Moreover, there is a corresponding bound for the $L^2 \to L^\infty$ norm of projection operators onto shrinking spectral bounds, i.e.,

$$(1.3) \qquad \big\| \chi_{[\lambda, \lambda + o(1)]} \big\|_{L^2(M) \to L^\infty(M)} = o(\lambda^{(n-1)/2}).$$

By this we mean that, given $\varepsilon > 0$, we can find a $\delta(\varepsilon) > 0$ and $\Lambda_\varepsilon < \infty$ so that

$$(1.4) \qquad \big\| \chi_{[\lambda, \lambda + \delta(\varepsilon)]} f \big\|_{L^\infty(M)} \le \varepsilon \, \lambda^{(n-1)/2} \, \| f \|_{L^2(M)}, \quad \lambda \ge \Lambda_\varepsilon,$$

with $\chi_{[\lambda, \lambda + \delta]} : L^2(M) \to L^2(M)$ denoting projection onto frequencies (i.e., eigenvalues of $\sqrt{-\Delta}$) in the interval $[\lambda, \lambda + \delta]$.

In [22] this result was improved by showing that if $\mathcal{R}_x \subset \mathcal{L}_x \subset S_x^* M$ is the set of recurrent directions over x and if $|\mathcal{R}_x| = 0$ for all $x \in M$, then we have (1.1).

We shall now give a simple and natural further refinement in the real analytic case which involves an ergodicity condition. The real analytic case is simple to analyze due to the fact that if $x \in M$ then there are just two extreme possibilities regarding the nature of $\mathcal{L}_x \subset S_x^* M$, the loop directions. Indeed, as shown in [23], either $|\mathcal{L}_x| = 0$ or $\mathcal{L}_x = S_x^* M$. In the second case there is also a minimal time $\ell > 0$ so that $\Phi_\ell(x, \xi) \in S_x^* M$, meaning that all geodesics starting at x loop back at exactly this minimal time ℓ. We call such a point a *self-focal point*. We note that there may exist more than one self-focal point and the minimal common return time ℓ of the loops may depend on x but for simplicity of notation we do not give it a subscript. If we then write

$$(1.5) \qquad \Phi_\ell(x, \xi) = (x, \eta_x(\xi)), \quad \xi \in S_x^* M,$$

then the *first return map*

$$(1.6) \qquad \eta_x : S_x^* M \to S_x^* M$$

above our self-focal point is real analytic. Following Safarov [17], we can associate to this first return map the *Perron–Frobenius operator* $U_x \colon L^2(S_x^* M, d\mu_x) \to L^2(S_x^* M, d\mu_x)$ by setting

$$(1.7) \qquad U_x f(\xi) = f(\eta_x(\xi)) \sqrt{J_x(\xi)}, \quad f \in L^2(S_x^* M, d\mu_x),$$

where $J_x(\xi)$ denotes the Jacobian of the first return map, i.e., $\eta_x^* d\mu_x = J_x(\xi) d\mu_x$. Clearly U_x is a unitary operator and

$$(1.8) \qquad \eta_x^*(f d\mu_x) = U_x(f) d\mu_x.$$

The key assumption underlying our results is contained in the following:

Definition 1.1. A self-focal point $x \in M$ is said to be *dissipative* if U_x has no invariant function $f \in L^2(S_x^* M)$. Equivalently, Φ_ℓ has no invariant L^1 measure with respect to $d\mu_x$.

The dissipative condition is a spectral condition on U_x. If U_x has any L^2 eigenfunction g then $U_x g = e^{i\theta} g$ form some $e^{i\theta} \in L^2$ and then $U_x|g| = |U_x g| = |g|$.

Hence the dissipative condition is the condition that the spectrum of U_x is purely continuous. For this reason, one might prefer the term 'weak mixing'; but that might create the wrong impression that Φ_ℓ is weak mixing with respect to some given invariant measure. The term 'dissipative' refers to the Hopf decomposition of Φ_ℓ on S_x^*M into conservative and dissipative parts. As discussed in §1.5, lack of an L^2 eigenfunction does not necessarily imply that the conservative part is of measure zero, so the term 'dissipative point' does not precisely mean that Φ_x is a dissipative dynamical system.

   Our main result then is the following.

Theorem 1.2. *Let (M, g) be a real analytic compact boundaryless manifold of dimension $n \geq 2$. Then (1.1) and (1.3) are true if and only if every self-focal point is dissipative.*

Remark. The condition that every self-focal point is dissipative implies that

$$(1.9) \qquad \lambda^{-1} \, \|\nabla e_\lambda\|_{L^\infty(M)} = o(\lambda^{(n-1)/2}).$$

For expository simplicity we will not prove it here, but the proof is almost the same.

   On any surface of revolution, the invariant eigenfunctions have sup norms of order $\simeq \lambda^{(n-1)/2}$ and achieve their suprema at the poles. The norm of the gradients achieve the bound (1.9) at a distance C/λ away from the poles. The dissipative condition rules out such examples.

   Theorem 1.2 is stronger than the result in [22]. Indeed, if $|\mathcal{R}_x| = 0$ then U_x cannot have an invariant L^2 function and so (1.3) would hold. The reason is that existence of $f \in L^2$ with $U_x f = f$ implies (by (1.8)) that $|f|^2 d\mu_x$ is a finite invariant measure in the class of $d\mu_x$. By the Poincaré recurrence theorem, $|f|^2 d\mu_x$ - almost every point would be recurrent, and thus the set of recurrent points would have positive μ_x measure.

   Theorem 1.2 is in a sense the optimal result on the problem of relating sup-norms of eigenfunctions to properties of the geodesic flow. As will be discussed at the end of the introduction, the result can only be sharpened by further understanding of the dynamical question of relating existence of an invariant $f \in L^2(S_x^*M, d\mu_x)$ to the structure of loops of (M, g).

1.1. Geometry of loops and self-focal points

Associated to a self-focal point x is the flowout manifold

$$(1.10) \qquad \Lambda_x = \bigcup_{0 \leq t \leq \ell} G^t S_x^*M.$$

It is an immersed Lagrangian submanifold of S^*M whose projection

$$\pi : \Lambda_x \to M$$

has a "blow-down singularity" at $t = 0, t = \ell$. For this reason self-focal points were called blow-down points in [22].

We may view Λ_{x_0} as the embedding of the mapping cylinder $\mathcal{C}_x$ of η_x, i.e., as

$$(1.11) \qquad \mathcal{C}_x = S_x^* M \times [0,\ell]/ \cong, \quad \text{where } (\xi, \ell) \cong (\Phi_\ell(x,\xi), 0)$$

In [22] it is proved that the map

$$\iota_x(\xi, t) = G^t(x,\xi) : \mathcal{C}_x \to \Lambda_x \subset S_x^* M$$

is a Lagrange immersion whose image is $\Lambda_x \subset S_x^* M$. If ℓ is the minimal period of all loops (i.e., if there are no exceptionally short loops) then $\iota_z|_{S_z^* \times (0,\ell)}$ is an embedding.

Focal points come in two basic kinds, depending on the first return map η_x. We say that x is a *pole* if

$$\eta_x = \mathrm{Id}_x : S_x^* M \to S_x^* M.$$

Equivalently, the set $\mathcal{CL}_x$ of smoothly closed geodesics based at x is all of $S_x^* M$,

$$(1.12) \qquad x \text{ is a pole if } \mathcal{CL}_x = S_x^* M, \quad \mathcal{CL}_x = \{\xi \in S_x^* M : \Phi_t(x,\xi) = (x,\xi)\}.$$

On the other hand, it is possible that $\eta_x = \mathrm{Id}$ only on a set of zero measure in $\mathcal{L}_x$, which in the analytic case means that it is almost nowhere the identity and $\mathcal{L}_x$ must have codimension ≥ 1. We call such a η_x *twisted*. Thus,

$$(1.13) \qquad x \text{ is self-focal with a twisted return map if } \mathrm{codim}\,(\mathcal{CL}_x) \geq 1.$$

Examples of poles are the poles x of a surface of revolution (in which case all geodesic loops at x are smoothly closed). Examples of self-focal points with twisted return map are the four umbilic points of two-dimensional tri-axial ellipsoids, from which all geodesics loop back at time 2π but are almost never smoothly closed [13]. The only smoothly closed directions are the geodesic (and its time reversal) defined by the middle length 'equator'. There are topological restrictions on manifolds possessing a self-focal point. In [1] a manifold with such a point is denoted a $F_\ell^{x_0}$ (or $Y_\ell^{x_0}$-)-manifold; if ℓ is the least common return time for all loops it is denoted by $L_\ell^{x_0}$. If (M, g) has a focal point x_0 from which all geodesics are simple (non-intersecting) loops, then the integral cohomology ring $H^*(M, \mathbb{Z})$ is generated by one element [15]. For an $F_\ell^{x_0}$ manifold, $H^*(M, \mathbb{Q})$ has a single generator (Theorem 4 of [1]). Most results on manifolds with self-focal points consider only the special case of Zoll metrics; see [2] for classic results and [16] for some recent results and references.

In the case of a triaxial ellipsoid $E \subset \mathbb{R}^3$, the first return map η_x is a totally dissipative expanding map of the circle with two fixed points, one a source and one a sink. It has invariant δ-measures at the fixed points and an infinite locally L^1 invariant measure on each component of the complement. According to Theorem 1.2, the eigenfunctions of E cannot achieve maximal sup norm bounds. In fact, the result of [22] already rules out maximal eigenfunction growth on the ellipsoid.

Note also that the analog of (1.3) is automatic in the Euclidean case since, by Plancherel's theorem and duality,

$$\Big\| \int_{\{\xi \in \mathbb{R}^n : |\xi| \in [\lambda, \lambda + \delta]\}} e^{ix \cdot \xi} \hat{f}(\xi) \, d\xi \Big\|_{L^\infty(\mathbb{R}^n)} \leq c_n \sqrt{\delta} \, \lambda^{(n-1)/2} \|f\|_{L^2(\mathbb{R}^n)}, \quad \lambda \geq 1.$$

Thus, by Theorem 1.2, the condition as to whether or not there is a self-focal point $x \in M$ so that $U_x g = g$ for some $0 \neq g \in L^2(S_x^* M)$, with U_x as in (1.7), provides a necessary and sufficient condition determining when the spectral projection operators on M also enjoy improved sup-norm bounds over shrinking intervals. Besides [23] and [22], earlier works on related problems are in [3], [4], [20], [24] and [25]. We should also point out that in (1.4) one needs that $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ since, for any compact n-dimensional Riemannian manifold (M, g), by (5.1.12) in [19], one has

$$\limsup_{\lambda \to \infty} \lambda^{-(n-1)/2} \|\chi_{[\lambda, \lambda+1]}\|_{L^2(M) \to L^\infty(M)} > 0.$$

1.2. Coherent states associated to self-focal points

The proof of Theorem 1.2 extends beyond eigenfunctions to certain kinds of quasi-modes, which play a fundamental role in the proof. To construct them, we fix $\rho \in \mathcal{S}(\mathbb{R})$ satisfying

(1.14) $\rho \geq 0, \ \rho(0) = 1, \quad 0 \leq \hat{\rho} \leq 1, \ \text{and} \ \hat{\rho}(t) = 0 \ \text{if} \ t \notin (-1, 1).$

We set

$$P = \sqrt{-\Delta},$$

and consider the operators

(1.15) $$\rho(T(\lambda - P)) = \frac{1}{T} \int_{\mathbb{R}} \hat{\rho}\Big(\frac{t}{T}\Big) e^{it\lambda} \, e^{-itP} \, dt.$$

If we freeze the second component of the Schwartz kernel of (1.15) at a focal point x_0 we obtain a semi-classical Lagrangian quasi-mode

(1.16) $$\psi_{\lambda, T}^{x_0}(x) := \rho\big(T(\lambda - \sqrt{\Delta})\big)(x, x_0)$$

associated to the Lagrangian submanifold (1.10) in the sense of [5] and [6]. That is, it is a semi-classical oscillatory integral with large parameter λ whose phase generates (1.10). More precisely, $\{\psi_{\lambda, T}^{x_0}\}$ is a one-parameter family of quasi-modes depending on the parameter T (as well as the semi-classical parameter λ). We refer to (1.16) as *coherent states* centered at x_0. In terms of an orthonormal basis of real valued eigenfunctions $\{e_j\}_{j=0}^\infty$,

$$\psi_{\lambda, T}^{x_0}(x) = \lambda^{-(n-1)/2} \sum_{j=0}^\infty \rho\big(T(\lambda - \lambda_j)\big) e_j(x) e_j(x_0).$$

The following lemma explains the precise sense in which (1.16) are quasi-modes:

Lemma 1.3. *For each T, $\psi_{\lambda,T}^{x_0}$ is a semi-classical Lagrangian distribution associated to the Lagrangian* (1.10) *whose principal symbol pulls back under ι_x to*

$$\frac{1}{T}\, e^{i\lambda t}\, \hat{\rho}\Big(\frac{t}{T}\Big)\, |dt|^{1/2} \otimes |d\mu_x|^{1/2}.$$

One has

(1.17) $$\|\psi_{\lambda,T}^{x_0}\|_{L^2} \le C,$$

and, moreover, for any ε there exists T_0 so that for $T \ge T_0$,

(1.18) $$\|(\Delta + \lambda^2)\, \psi_{\lambda,T}^{x_0}\|_{L^2} \le C\, \varepsilon \lambda.$$

For fixed T, (1.16) is therefore a very kind of weak quasi-mode but as a family it behaves like a quasi-mode of order $o(\lambda)$ as $T \to \infty$.

Further, the value of $\psi_{\lambda,T}^{x_0}$ at its peak point x_0 equals

(1.19) $$\lambda^{-(n-1)/2} \sum_{j=0}^{\infty} \rho\big(T(\lambda - \lambda_j)\big)(e_j(x_0))^2,$$

and as we will see, this peak value is of maximal growth if and only if U_x (1.7) has an invariant L^2 function.

The fact that $\psi_{\lambda,T}$ is a semi-classical Lagrangian quasi-mode is an immediate consequence of the well-known parametrix construction for e^{-itP} (see (2.4)), from which it is clear that one may express (1.16) as an oscillatory integral with large parameter λ. The symbol can also be evaluated as in the Lemma directly from this expression and the principal symbol of e^{-itP} computed in [7]. Note that the symbol is invariant under the geodesic flow on the long time interval where $\hat{\rho}(t/T) \equiv 1$. As $T \to \infty$ one gets a kind of time average over orbits and we will see that the family (1.16) behaves as a family of approximate quasi-modes with invariant symbols. It is natural to ask whether the invariant L^2 function can be 'quantized' to construct a quasi-mode of order $o(\lambda)$; we discuss this in §1.6.

It is also not hard to see that (1.18) is valid. Indeed, since $\rho \in \mathcal{S}(\mathbb{R})$, we have for any $N = 1, 2, \dots$,

$$\int_M \big|(\Delta + \lambda^2)\,\psi_{\lambda,T}^{x_0}(x)\big|^2\, dV = \lambda^{-(n-1)} \sum_j (\lambda^2 - \lambda_j^2)^2 \big(\rho(T(\lambda - \lambda_j))\big)^2 (e_j(x_0))^2$$

$$\le C_N\, \lambda^{-(n-1)} \sum_j (\lambda^2 - \lambda_j^2)^2 \,(1 + T|\lambda - \lambda_j|)^{-N} (e_j(x_0))^2$$

$$\le C_N\, T^{-2}\, \lambda^{-(n-1)} \sum_j (\lambda + \lambda_j)^2 \,(1 + T|\lambda - \lambda_j|)^{-N} (e_j(x_0))^2$$

$$\le C_N\, T^{-2}\, \lambda^2,$$

if $N - 2 > n$, using in the last step the fact that

$$\sum_{\lambda_j \in [\mu, \mu+1]} (e_j(x))^2 \le C\,(1 + \mu)^{n-1}, \quad \mu \ge 0.$$

Using this fact one also easily obtains (1.17).

1.3. Outline of the proof of Theorem 1.2

We now outline the strategy we shall employ in proving Theorem 1.2 and explain the role of the quasi-modes (1.16).

First, we shall prove the $o(\lambda^{(n-1)/2})$ bounds (1.1) and (1.3) under the assumption that every self-focal point is dissipative. As we pointed out before, because of our real analyticity assumption, there are two extreme cases: *self-focal points* and *non-focal points* (where $|\mathcal{L}_x| = 0$). Thus, it is natural to split our estimates for eigenfunctions or quasi-modes into two cases. The first, which is new, is to prove favorable bounds in a ball around each dissipative self-focal point. The second, which was dealt with in [23], is to prove such bounds near a non-focal point.

After we prove (1.3), we shall prove the converse direction: the $o(\lambda^{(n-1)/2})$ bounds at self-focal points for which U_x (1.7) has no nonzero invariant L^2 functions can be turned around to show that there are $\Omega(\lambda^{(n-1)/2})$ bounds if nontrivial invariant functions do exist.

The main estimate is as follows: If all focal points are dissipative, then for any $\varepsilon > 0$, we prove that there are finite $T_0 = T_0(\varepsilon)$ and $\Lambda_0 = \Lambda_0(\varepsilon)$ so that (as in (1.19))

$$(1.20) \qquad \sum_{j=0}^{\infty} \rho\big(T_0(\lambda - \lambda_j)\big)(e_j(y))^2 \le \varepsilon\, \lambda^{n-1} \quad \lambda \ge \Lambda_0,\ y \in M,$$

Since $\rho(0) = 1$, it is plain that this estimate implies the first assertion, (1.1), of the theorem. As mentioned above, we split the proof of (1.20) into two very different cases: (i) (1.20) is valid in a neighborhood of a given dissipative self-focal point, and (ii) (1.20) is valid in a neighborhood of any non-focal point.

The role of the quasi-modes (1.16) in the proof of the first part of Theorem 1.2 is explained by the following.

Proposition 1.4. *Let x be a dissipative self-focal point in our real analytic compact boundaryless manifold of dimension $n \ge 2$. Then for any $\varepsilon > 0$, we can find a neighborhood $\mathcal{N}(x, \varepsilon)$ of x and finite numbers $T = T(x, \varepsilon)$ and $\Lambda = \Lambda(x, \varepsilon)$ so that*

$$(1.21) \qquad \sum_{j=0}^{\infty} \rho\big(T(\lambda - \lambda_j)\big)(e_j(y))^2 \le \varepsilon\, \lambda^{n-1}, \quad \text{if } y \in \mathcal{N}(x, \varepsilon),\ \text{and}\ \ \lambda \ge \Lambda.$$

The proof of the estimate (1.21) at a self-focal point y uses the dissipative assumption and the von Neumann mean ergodic theorem. To extend the estimate to points y in a neighborhood of a self-focal point, we use the rather explicit formula for the wave trace formula for large times T of Safarov [17], [18], which shows the leading part of the left side of (1.21) varies smoothly. Consequently, if the inequality in (1.21) holds at the self-focal point, it holds in some neighborhood of the self-focal point when λ is sufficiently large.

Remark. The proof of Theorem 1.2 and Proposition 1.4 shows that the suprema of $|e_\lambda(x)|$, respectively $|\nabla e_\lambda(x)|$, is obtained in a ball of radius $O(1/\lambda)$ around a self-focal point. But it does not show that the suprema are obtained at a self-focal

point, and obviously both suprema cannot be attained at the same point. Since e_λ oscillates on the scale $1/\lambda$, it is a priori possible that it vanishes at the self-focal point and takes its suprema in a ball of radius C/λ around the self-focal point. Thus, Proposition 1.4 shows that not only is $|e_\lambda(y)|$ small at the dissipative self-focal point but also that there is "propagation of smallness" to a neighborhood of such a point.

The other case, which we need to handle in order to prove the first part of Theorem 1.2, is to show that we also have these types of bounds near every non-focal point:

Proposition 1.5. *Let $x \in M$ and assume that $|\mathcal{L}_x| = 0$. Then, given any $\varepsilon > 0$, we can find a neighborhood $\mathcal{N}(x, \varepsilon)$ of x and finite numbers $T = T(x, \varepsilon)$ and $\Lambda = \Lambda(x, \varepsilon)$ so that (1.21) is valid.*

Proposition 1.5 is a special case of Theorem 1.1 in [23], but we shall present its simple proof for the sake of completeness. As in [23], it uses an idea coming from Ivrii's [12] proof of his generalization of the Duistermaat–Guillemin theorem [7].

Before outlining the proofs of the propositions, let us see how they imply (1.3). We first claim that (1.20) is valid if and only if, given $\varepsilon > 0$, there are numbers $\delta_0 = \delta_0(\varepsilon)$ and $\Lambda_0 = \Lambda_0(\varepsilon)$ so that, for every $y \in M$,

$$(1.22) \qquad \sum_{\{j:\, |\lambda - \lambda_j| \le \delta_0\}} \big(e_j(y)\big)^2 \le \varepsilon\, \lambda^{n-1}, \quad \text{if} \ \ \lambda \ge \Lambda_0.$$

This clearly implies (1.3) since

$$\big\|\chi_{[\lambda, \lambda+\delta]}\big\|^2_{L^2(M) \to L^\infty(M)} = \sup_{y \in M} \sum_{\{j:\, \lambda_j \in [\lambda, \lambda+\delta]\}} \big(e_j(y)\big)^2.$$

We shall also make use of this fact when we prove the first part of Corollary 1.6.

To prove our claim that (1.20) is equivalent to (1.22), we first note that, by (1.14), $\rho(s) \ge 1/2$ when $|s| \le \delta$, for some fixed $\delta > 0$. From this it is clear that (1.20) implies (1.22). Since $\rho \in \mathcal{S}(\mathbb{R})$, it follows that for any $N = 1, 2, 3, \ldots$ there is a constant C_N so that

$$\sum_{j=0}^{\infty} \rho(T_0(\lambda - \lambda_j))(e_j(y))^2 \le C_N \sum_{j=0}^{\infty} \big(1 + T_0|\lambda - \lambda_j|\big)^{-N}(e_j(y))^2.$$

As a result if we take δ_0 in (1.22) to be $1/T_0$ and $N = n+1$, we conclude that (1.22) implies (1.20).

We now use a compactness argument to show that the propositions imply (1.22). Proposition 1.4 implies that, if $\varepsilon > 0$ is given and if x is a self-focal point then there must be a neighborhood $\mathcal{N}(x, \varepsilon)$ of x and numbers $\delta_x > 0$ and $\Lambda_x < \infty$ so that

$$(1.23) \qquad \sum_{|\lambda_j - \lambda| \le \delta_x} \big(e_j(y)\big)^2 \le \varepsilon\, \lambda^{n-1}, \quad \text{if} \ y \in \mathcal{N}(x, \varepsilon), \text{ and } \ \lambda \ge \Lambda_x.$$

Proposition 1.5 implies the same conclusion for any non-focal point x. Since $\{\mathcal{N}(x, \varepsilon)\}_{x \in M}$ is an open covering of M, by Heine–Borel, there must be a finite sub-

covering. In other words, there must be points $x_j \in M$, $1 \leq j \leq N$, so that $M \subset \bigcup_{j=1}^{N} \mathcal{N}(x_j, \varepsilon)$. By (1.23), if $\Lambda_0 = \max\{\Lambda_{x_1}, \ldots, \Lambda_{x_N}\}$, and $\delta_0 = \min\{\delta_{x_1}, \ldots, \delta_{x_N}\}$, then we must have (1.22). As discussed above, this implies (1.20), hence (1.3), and therefore (1.1).

1.4. Generalization to fat quasi-modes

With a small additional effort, the proof of Theorem 1.2 establishes a more general result for growth rates of "fat quasi-modes" or quasi-modes of order $o(\lambda)$. They are defined as follows: when $n = 2$ and $n = 3$, we say that a sequence $\{\phi_{\lambda_k}\}$ of *quasi-modes of order $o(\lambda)$* if

$$(1.24) \qquad \int_M |\phi_{\lambda_k}|^2 \, dV = 1, \quad \text{and} \ \ \|(\Delta + \lambda_k^2)\phi_{\lambda_k}\|_{L^2(M)} = o(\lambda_k).$$

In higher dimensions, $n \geq 4$, as explained in [22], pp. 164–165, one needs to modify this definition in order to get natural results by requiring that

$$(1.25) \quad \int |\phi_{\lambda_k}|^2 \, dV = 1, \ \ \text{and} \ \ \|S_{[2\lambda_k, \infty)}\phi_\lambda\|_{L^\infty(M)} + \|(\Delta + \lambda^2)\phi_{\lambda_k}\|_{L^2(M)} = o(\lambda_k)$$

if $S_{[2\lambda_k, \infty)} \colon L^2(M) \to L^2(M)$ denotes the projection onto the $[2\lambda_k, \infty)$ frequencies. As was shown in [22], condition (1.24) automatically implies (1.25) if $n = 2$ or $n = 3$, and, moreover, if $|\mathcal{R}_x| = 0$ for each $x \in M$, then

$$(1.26) \qquad\qquad \|\phi_\lambda\|_{L^\infty(M)} = o(\lambda^{(n-1)/2})$$

whenever ϕ_λ is a sequence of quasi-modes of order $o(\lambda)$.

As we shall see, an equivalent formulation of Theorem 1.2 is the following.

Corollary 1.6. *Let (M, g) be a compact real analytic boundaryless manifold of dimension $n \geq 2$. Then (1.26) holds for quasi-modes of order $o(\lambda)$ if and only if every self-focal point of M is dissipative.*

We shall prove this easy consequence of Theorem 1.2 in the final section of the paper and also state a natural problem about whether the above quasi-mode condition can be weakened if the sup-norms are $\Omega(\lambda^{(n-1)/2})$.

1.5. Hopf decomposition and existence of a finite invariant measure in the class of $d\mu_x$

In this section we explain the term 'dissipative' in Definition 1.1.

Let Φ be an invertible measurable map of a measure space (X, μ). A set W is called wandering if the sets $\Phi^{-k}W$ for $k \geq 0$ are disjoint, i.e., if no point of W returns to W. Φ is called *conservative* if there exists no wandering set of positive measure. Φ is called *recurrent* if for all Borel sets A, almost every point of A belongs to the set A_{rec} of points returning at least once to A,

$$A_{\text{rec}} := A \cap \bigcup_{k=1}^{\infty} \Phi^{-k}(A).$$

It is called infinitely recurrent if almost all points of A belong to

$$A_{inf} := \{x \in A : \Phi^k \in A, \text{ for infinitely many } k \geq 1\} = A \cap \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \Phi^{-k}(A).$$

The recurrence theorem ([14], Theorem 3.1) states that the following are equivalent:

- Φ is conservative;
- Φ is recurrent;
- Φ is infinitely recurrent;

The Hopf decomposition states:

Theorem 1.7 ([14], Theorem 3.2). *If Φ is null-preserving and non-singular, then there exists a decomposition of X into two disjoint measurable sets C and D (the conservative and dissipative parts), so that*

- *C is Φ-invariant;*
- *$\Phi|_C$ is conservative;*
- *There exists a wandering set W so that $D = \bigcup_{-\infty}^{\infty} \Phi^k W$.*

We are of course interested in the Hopf decomposition for $\Phi = \Phi_\ell, X = S_x^* M$ and $\mu = \mu_x$. The term 'dissipative' in Definition 1.1 is intended to suggest that $(\Phi_\ell, S_x^* M, d\mu_x)$ is dissipative if there exists no invariant $f \in L^2(S_x^* M, d\mu_x)$ for the unitary operator U_x. It is obvious that if such an invariant f exists, then $\mu_x(C) > 0$. However, in the class of measurable dynamical systems, there exist examples (due to Halmos) of conservative systems without finite invariant measures. The recent review [8] contains several equivalent criteria for existence of a finite invariant measure in the class of μ. It is not clear to us whether $\mu_x(C) > 0$ in our setting implies existence of an invariant L^2 function, and we do not claim that Φ_ℓ is dissipative if x is a dissipative self-focal point.

1.6. Potential further improvements

There are three potential avenues for improvement of Theorem 1.2.

The first is purely geometric or dynamical. It is possible that existence of an invariant L^2 function for U_x at some self-focal point x implies that x is a pole.[1] Furthermore, it is possible that there do not exist real analytic Riemannian manifolds (M, g) with $\dim M \geq 3$ which have self-focal points with twisted first return maps. For instance, it does not appear that ellipsoids of dimension ≥ 3 have self-focal points.

It would also simplify the proof if we knew that there are only a finite number of self-focal points with twisted first return maps for analytic (M, g). It is not hard to prove that if there does exist an infinite number of such points, then the first return times tend to infinity as they approach a limit point. All known examples have only a finite number of self-focal points with twisted first return maps. For further comments on the geometry we refer to [27].

---

[1] In the two-dimensional case this is proven in an addendum, [26]; however, the case of higher dimensions remains open.

Second, it is a natural question if one can use an invariant L^2 function f in $L^2(S_x^*M)$ to construct a better quasi-mode. The classical construction of a quasi-mode satisfying

$$||(\Delta + \lambda^2)\psi_\lambda||_{L^2} \le C$$

as an oscillatory integral [6] requires a smooth invariant density on (1.10). On the one hand, it is possible that the invariant f is always C^∞. On the other hand, it is possible that one can quantize a 'rough' density to construct a quasi-mode. In the real analytic case, there is a possible construction using the FBI transform which we plan to investigate in future work. See the discussion in §4 for further details.

Finally, it is natural to generalize the argument of this article to general C^∞ metrics. Our assumption that g is real analytic is only to simplify the geometric analysis of loopsets. In general, they can be badly behaved, and that requires further approximations and arguments. We plan to give the general proof in a subsequent article. However, the present one contains the main analytical ideas.

1.7. Organization

The paper is organized as follows. In the next section we shall prove that we have the o-bounds (1.3) provided that the Perron–Frobenius operators (1.7) above every self-focal point have no nonzero invariant functions. Then in §3, we shall see that this proof can be easily modified to give the remaining part of Theorem 1.2, by showing that we cannot have (1.22) if $y = x$ is a self-focal point where the operator U in (1.7) has a nontrivial $L^2(S_x^*M)$ function. Finally, in §4 we shall give the simple argument showing that Theorem 1.2 implies Corollary 1.6 and discuss further problems for quasi-modes.

2. Proof of $o(\lambda^{(n-1)/2})$ bounds

As we noted above, to prove the o-bounds posited in Theorem 1.2, it is natural to split the analysis into two cases: bounds near self-focal points and near non-focal points. Let us start by giving the argument for the former.

2.1. Analysis near self-focal points with no invariant L^2 functions

In this subsection we shall prove Proposition 1.4. So let us assume that x is a self-focal point, and that $\ell > 0$ is the first return time for the geodesic flow through x. Thus, $\Phi_\ell(S_x^*M) = S_x^*M$, with ℓ being the minimal such time. We also are assuming that the associated Perron–Frobenius U has no nontrivial invariant $L^2(S_x^*M)$ functions.

Our assumption about ℓ does not rule out the existence of subfocal times $0 < t_0 < \ell$ and directions $\xi \in S_x^*M$ with $\Phi_{t_0}(x, \xi) \in S_x^*M$ (i.e., loops through x of length shorter than ℓ). On the other hand, the set $\mathcal{E}_x \subset S_x^*M$ of such directions must be closed and of measure zero. As a result, it will be a simple matter to modify the argument for the case of no subfocal times to handle the general case. We shall do so at the end of this subsection.

To simplify the notation, we note that, after possibly rescaling the metric, we may, and shall take ℓ to be equal to one. We then let $\eta = \eta_x : S_x^* M \to S_x^* M$, as in (1.5), be the first return map and $U = U_x$, as in (1.7), the associated unitary operator on $L^2(S_x^* M)$. Assuming that the operator U in (1.7) has no nonzero invariant functions, we must show that there is a neighborhood $\mathcal{N}(x)$ of x in M so that we have (1.21).

Using the Fourier transform, we can rewrite the left hand side of (1.21) as

$$(2.1) \qquad \sum_j \rho(T(\lambda - \lambda_j)) e_j(x)\, e_j(y) = \frac{1}{2\pi T} \int \hat{\rho}(t/T) \left(e^{itP}\right)(x,y)\, e^{-it\lambda}\, dt,$$

and, because of our assumptions, the integrand vanishes if $|t| \notin (-T, T)$. Here $\left(e^{itP}\right)(x,y)$ denotes the kernel of the half-wave operator e^{itP}. It follows from Hörmander's theorem [9] on the propagation of singularities and our temporary assumption that there are no subfocal times, that $(t, y) \to \left(e^{itP}\right)(y,y)$ is smooth when $y = x$ and $t \in \mathbb{R} \backslash \mathbb{Z}$.

On the other hand, $\left(e^{itP}\right)(x,x)$ will be singular when $t = \nu \in \mathbb{Z}$. We recall that for t near a given such ν and y near x, one can write down a parametrix for this half-wave operator which is a finite sum of terms of the form

$$(2.2) \qquad (2\pi)^{-n} \int_{\mathbb{R}^n} e^{iS_{\nu,j}(t,y,\xi) - iz\cdot\xi}\, a_{\nu,j}(t,y,z,\xi)\, d\xi, \quad j = 1, \dots, N(\nu),$$

where, up to Maslov factors, i^{s_ν} (see [7], p. 68), the principal symbols of the $a_{\nu,j}$ are nonnegative zero order symbols and the phase functions $S_{\nu,j}(t,y,\xi)$ are real-valued, homogeneous of degree one in ξ and generating functions for portions of the canonical relation associated with P. That means that, if $p(x,\xi)$ is the principal symbol of P, then

$$(2.3) \qquad \partial_t S_{\nu,j}(t,y,\xi) = p(y, \nabla_x S_{\nu,j}(t,y,\xi)).$$

We therefore can write, for $|t| \leq T$ and y in a small neighborhood of x,

$$(2.4) \;\; \left(e^{itP}\right)(y,y) = (2\pi)^{-n} \sum_{|\nu| \leq T} \sum_{j=1}^{N(\nu)} \int_{\mathbb{R}^n} e^{iS_{\nu,j}(t,y,\xi) - iy\cdot\xi}\, a_{\nu,j}(t,y,y,\xi)\, d\xi + O_T(1),$$

where, because of our assumption that there are no subfocal points, we may assume that

$$(2.5) \qquad a_{\nu,j}(t,x,x,\xi) = 0, \quad \text{if } t \notin (\nu - \delta, \nu + \delta),$$

with $\delta > 0$ small but fixed.

By the Hamilton–Jacobi equations associated with $p(x,\xi)$, this means that if $t = \nu\ell = \nu$ and we have that $a_{\nu,j}(\nu, x, x, \xi) \neq 0$ for some $\nu = 0, \pm 1, \dots, \pm T$, then

$$(2.6) \qquad \nabla_x S_{\nu,j}(\nu, x, \xi) = \eta^\nu(\xi), \quad \xi \in S_x^* M,$$

where $\eta(\xi)$ is as in (1.5), and $\eta^\nu = \eta \circ \eta \circ \cdots \circ \eta$ is the ν-fold composition of η if $\nu > 0$, while $\eta^\nu = \eta^{-\nu}$ if $\eta < 0$, and $\eta^0(\xi) = \xi$. Additionally, since $e^{i\nu P} \circ (e^{i\nu P})^* = I$, one

sees, by a theorem of Hörmander (see e.g., [19], Theorem 6.1.4) that we must also have that for ν as above

$$(2.7) \qquad \sum_{j=1}^{N(\nu)} |a_{\nu,j}(\nu, x, x, \eta^{\nu}(\xi))| = \sqrt{J^{\nu}(\xi)} \mod S_{1,0}^{-1},$$

$$J^{\nu}(\xi) = |\det(d\eta^{\nu}(\xi)/d\xi)|, \ \ \xi \in S_x^* M.$$

Based on this, we conclude that in order to prove (1.21), it suffices to show that, given $\varepsilon > 0$, there is a neighborhood $\mathcal{N}(x, \varepsilon)$ of x so that if T is large and fixed, we have

$$(2.8) \quad T^{-1} \bigg| \sum_{|\nu| \le T} \sum_{j=1}^{N(\nu)} \lambda^n \iint \hat{\rho}(t/T) \, e^{i\lambda(S_{\nu,j}(t,y,\xi) - y\cdot\xi - t)} \, a_{\nu,j}(t,y,y,\lambda\xi) \, d\xi \, dt \bigg|$$

$$\le \varepsilon \, \lambda^{n-1} + O_{\varepsilon,T}(\lambda^{n-2}), \quad y \in \mathcal{N}(x,\varepsilon).$$

To prove this, as in [7] and [17]–[18], we shall use stationary phase in the r and t variables, if we write $\xi = r\omega$, where $\omega \in S_y^* M$. Given our assumption that $\eta^{\nu}(S_x^* M) = S_x^* M$, $\nu = 0, \pm 1, \pm 2, \dots$, this is all we can do. In view of this, it is natural to rewrite (2.8) in the equivalent form

$$(2.9) \qquad T^{-1} \bigg| \sum_{|\nu| \le T} \sum_{j=1}^{N(\nu)} A_{\nu,j}(y) \bigg| \le \varepsilon \, \lambda^{-1} + O_{\varepsilon,T}(\lambda^{-2}), \quad y \in \mathcal{N}(x,\varepsilon),$$

where

$$(2.10) \qquad A_{\nu,j}(y) = (2\pi)^{-1} \iint \hat{\rho}(t/T) \, e^{i\lambda(S_{\nu,j}(t,y,\xi) - x\cdot\xi - t)} \, a_k(t,y,y,\lambda\xi) \, d\xi \, dt.$$

We shall get the gain λ^{-1} in the right side of (2.9) from stationary phase in these two variables, while an additional gain of $O(\varepsilon)$ at $y = x$ will come from von Neumann's ergodic theorem if $T = T(\varepsilon)$ is large and fixed. Uniformity over a small neighborhood of x (depending on ε and T) will come from stationary phase with parameters, due to the fact that the Hessian of the phase function in the t, r variables is nondegenerate for every fixed direction $\omega \in S_x^* M$ at x. We shall use the fact that if the leading coefficient in the stationary phase expansion is small at a self-focal point, then it is small in a neighborhood of this point.

Let

$$(2.11) \qquad \psi_{\nu,j}(t,y,\xi) = S_{\nu,j}(t,y,\xi) - y \cdot \xi - t$$

denote the phase function in (2.10). Let us first argue that (2.9) is valid when y is our self-focal point x. We follow an argument in [17], [18].

If we fix a direction $\omega \in S_x^* M$ (so that $p(x,\omega) = 1$) and write $\xi = r\omega$, then by the Euler homogeneity relations we have

$$(2.12) \qquad \partial_r \psi_{\nu,j}(t,x,r\omega) = 0 \iff S_{\nu,j}(t,x,\xi) - x \cdot \xi = 0.$$

In other words,

$$(2.13) \qquad \partial_r \psi_{\nu,j}(t,x,\xi) = 0 \implies \psi_{\nu,j}(t,x,\xi) = -t.$$

Additionally, by (2.3),

$$(2.14) \qquad \partial_t \psi_{\nu,j}(t,y,\xi) = 0 \implies p(y, \nabla_x S_{\nu,j}(t,y,\xi)) = 1,$$

meaning that if $a_{\nu,j}(\nu,x,x,\xi) \neq 0$ and if $\eta^\nu(\xi) = \nabla_x S_{\nu,j}(t,x_0,\xi))$ is as in (2.6), then $\eta^\nu(\xi) \in S_x^* M$. The Hessian of $\psi_{\nu,j}$ in the t,r variables therefore must be of the form

$$(2.15) \qquad H(x) = \begin{pmatrix} 0 & p(x,\eta^\nu(\xi)) \\ p(x,\eta^\nu(\xi)) & 0 \end{pmatrix},$$

in view of (2.3) and the fact that $p(x,\eta^\nu(\xi))$ is homogeneous of degree one. Thus,

$$(2.16) \qquad \det H(x) = -1.$$

Therefore, by stationary phase (e.g., Theorem 7.7.5 in [11]), we have

$$(2.17) \quad A_{\nu,j}(x) = (2\pi)^{-1} \iint \int_0^\infty \int_{-T}^T e^{i\lambda \psi_{\nu,j}(t,x,r\omega)} \hat\rho\left(\frac{t}{T}\right) a_{\nu,j}(t,x,x,\lambda r\omega) r^{n-1} dr\, dt\, d\omega$$

$$= \lambda^{-1} \int e^{-i\lambda\nu} \hat\rho\left(\frac{t}{T}\right) a_{\nu,j}(\nu,x,x,\lambda\omega)\, d\omega + O_T(\lambda^{-2}).$$

By (2.7),

$$(2.18) \qquad \sum_{j=1}^{N(\nu)} |a_{\nu,j}(\nu,x,x,\lambda\omega)| = \sqrt{J^\nu(\omega)} + O_T(\lambda^{-1}).$$

By (2.17), (2.18) and the fact that $0 \leq \hat\rho \leq 1$, we have

$$T^{-1} \sum_{|\nu| \leq T} \sum_{j=1}^{N(\nu)} |A_{\nu,j}(x)| \leq \lambda^{-1} T^{-1} \sum_{|\nu| \leq T} \sum_{j=1}^{N(\nu)} \int_{S_x^* M} |a_{\nu,j}(\nu,x,x,\omega)|\, d\omega + O_T(\lambda^{-2})$$

$$(2.19) \qquad\qquad = \lambda^{-1} T^{-1} \sum_{|\nu| \leq T} \int_{S_x^* M} \sqrt{J^\nu(\omega)}\, d\omega + O_T(\lambda^{-2}).$$

Therefore, we would have (2.9) if we could show that, given $\varepsilon > 0$, we can chose $T \gg 1$ so that

$$(2.20) \qquad T^{-1} \sum_{|\nu| \leq T} \int_{S_{x_0}^* M} \sqrt{J^\nu(\omega)}\, d\omega < \frac{\varepsilon}{2}.$$

If $\mathbf{1}$ denotes the function on $S_x^* M$ which is identically one, and if $U \colon L^2(S_x^* M) \to L^2(S_x^* M)$ is as in (1.7), then we can rewrite the left hand side of (2.20) as

$$\left\langle T^{-1} \sum_{|\nu| \leq T} U^\nu \mathbf{1}, \mathbf{1} \right\rangle.$$

Since $U \colon L^2(S_x^*M) \to L^2(S_x^*M)$ is unitary, by von Neumann's ergodic theorem,

$$\frac{1}{2T} \sum_{|\nu| \leq T} U^\nu \mathbf{1} \to \Pi(\mathbf{1}), \quad \text{in } L^2(S_x^*M), \quad \text{as } T \to +\infty,$$

where $\Pi(\mathbf{1})$ denotes the projection of $\mathbf{1}$ onto the U-invariant subspace of $L^2(S_x^*M)$. Our assumption that this operator has no nonzero invariant functions means that $\langle \Pi(\mathbf{1}), \mathbf{1} \rangle = 0$, and therefore we have (2.20) if $T = T(\varepsilon)$ is sufficiently large.

We have shown that

$$(2.21) \qquad T^{-1} \sum_{|\nu| \leq T} \sum_{j=1}^{N(\nu)} |A_{\nu,j}(x)| \leq \varepsilon \lambda^{-1}/2 + O_T(\lambda^{-2}).$$

Let us now argue that because of (2.16), this implies that we can find a neighborhood $\mathcal{N}(x, \varepsilon)$ of x so that (2.9) is valid which would finish the proof of (2.8).

The situation at x was simple since the phase functions $\psi_{\nu,j}$, as a function of r and t, only had critical points on the support of the amplitudes exactly at $r = 1$ and $t = \nu$. If we assume that y is sufficiently close to x, and, as we may, that the $a_{\nu,j}$ satisfy (2.5) with $\delta > 0$ sufficiently small, it follows from (2.16) and the implicit function theorem that there are unique pairs $(t_{\nu,j}(y, \omega), r_{\nu,j}(y, \omega))$, which depend smoothly on y and $\omega \in S_y^*M$ so that if we replace x by y in the oscillatory integral in (2.17), the unique stationary point of the phase $(t, r) \to \psi_{\nu,j}(t, y, r\omega)$ occurs at this point. Also, assuming that y is close to x we will have that $r_{\nu,j}(y, \omega)$ is very close to 1 and $t_{\nu,j}(y, \omega)$ is very close to ν if $a_{\nu,j}(\nu, x, r\omega) \neq 0$. Let $H_{\nu,j}(y, \omega) = H_{\nu,j}(t_{\nu,j}(y, \omega), r_{\nu,j}(y, \omega))$ denote the Hessian of the phase function at the stationary point. It then follows that $\det H_{\nu,j}$ is close to -1 if y is close to x. Therefore by stationary phase (e.g. [11], Theorem 7.7.6), for y close to x, we have the following analog of (2.17):

$$A_{\nu,j}(y) = \int \int_0^\infty \int_{-T}^T e^{i\lambda \psi_{\nu,j}(t, y, r\omega)} \hat{\rho}(t/T) a_{\nu,j}(t, y, y, \lambda r\omega) r^{n-1} dr\, dt\, d\omega$$

$$(2.22) \quad = \lambda^{-1} \int |\det H_{\nu,j}|^{-1/2} e^{-i\lambda t_{\nu,j}(y,\omega)} \hat{\rho}(t_{\nu,j}(y, \omega)/T)$$

$$\times a_{\nu,j}(t_{\nu,j}(y, \omega), y, y, \lambda r_{\nu,j}(y, \omega)\omega)\, r_{\nu,j}(y, \omega)^{n-1} d\omega + O_T(\lambda^{-2}),$$

where the constant in the $O_T(\lambda^{-2})$ error term can be chosen to be uniform for y sufficiently close to x. We conclude that

$$\left| \sum_{|\nu| \leq T} \sum_{j=1}^{N(\nu)} \left(|A_{\nu,j}(y)| - |A_{\nu,j}(x)| \right) \right|$$

$$\leq \lambda^{-1} \sum_{|\nu| \leq T} \sum_{j=1}^{N(\nu)} \int \Big| |\det H_{\nu,j}(y, \omega)|^{-1/2} \hat{\rho}(t_{\nu,j}(y, \omega))$$

$$\times |a_{\nu,j}(t_{\nu,j}(y, \omega), y, y, \lambda r_{\nu,j}(y, \omega)\omega)|\, r_{\nu,j}(y, \omega)^{n-1}$$

$$- \hat{\rho}(\nu)|a_{\nu,j}(\nu, x, x, \lambda\omega)| \Big|\, d\omega + O_T(\lambda^{-2}).$$

Since $a_{\nu,j} \in S^0_{1,0}$ and $t_{\nu,j}(y,\omega)$ and $r_{\nu,j}(y,\omega)$ are smooth functions for y near x which satisfy $t_{\nu,j}(x,\omega) = \nu$ and $r_{\nu,j}(x,\omega) = 1$, it follows that each of the preceding integrals is a smooth function of y near x, which vanishes when $y = x$. Therefore, by (2.21), we deduce that (2.9) must be valid if $\mathcal{N}(x,\varepsilon)$ is a sufficiently small neighborhood of x, since at this point, T has been fixed.

Handling the contribution of sub-focal times

To complete the proof of Proposition 1.4, we must remove the assumption that at our self-focal point we have $\Phi_t(x,\xi) \notin S^*_x M$ for any ξ if t is not an integer multiple of the return time ℓ. As before, we may assume that $\ell = 1$.

We note that if $\delta > 0$ is smaller than the injectivity radius of (M,g), it follows that we can never have $\Phi_t(x,\xi) \in S^*_x M$ for some $t \in [\nu - \delta, \nu + \delta]$ with $\nu \in \mathbb{Z}$ since $\Phi_\nu(x,\xi) \in S^*_x M$. Thus, in any such time interval, $(t,y) \to (e^{itP})(y,y)$ is smooth at $y = x$ if $t \in [\nu - \delta, \nu + \delta] \backslash \{\nu\}$.

To use this, fix $\beta \in C^\infty_0(\mathbb{R})$ satisfying

$$(2.23) \qquad \beta(s) = 1, \;\; |s| \leq \frac{3}{4}\delta, \;\; \text{and} \;\; \text{supp } \beta \subset (-\delta, \delta).$$

We then can write

$$\rho\big(T(\lambda - P)\big)(y,y) = K(T,\lambda;y) + R(T,\lambda;y),$$

where

$$(2.24) \qquad K(T,\lambda;y) = \frac{1}{2\pi T} \sum_{|\nu| \leq T} \int \beta(t-\nu)\,\hat{\rho}(t/T)\,\big(e^{itP}\big)(y,y)\,e^{-it\lambda}\,dt,$$

and

$$(2.25) \qquad R(T,\lambda;y) = \frac{1}{2\pi T} \sum_{|\nu| \leq T} \int \big(1 - \beta(t-\nu)\big)\,\hat{\rho}(t/T)\,\big(e^{itP}\big)(y,y)\,e^{-it\lambda}\,dt.$$

Since for every $\nu \in \mathbb{Z}$, $|\nu| \leq T$, $\beta(t-\nu)\big(e^{itP}\big)(y,y)$ can be written as in the right side of (2.4) where the amplitude satisfies (2.5) and (2.7), it is clear that the proof of (1.21) under the assumption that there are no subfocal times shows that, given $T = T(\varepsilon)$ large enough and fixed, there is a neighborhood $\mathcal{N}_0$ of x so that

$$(2.26) \qquad |K(T,\lambda;y)| \leq \varepsilon\lambda^{n-1} + O_T(\lambda^{n-2}), \quad y \in \mathcal{N}_0.$$

As a result, the proof of Proposition 1.4 would be complete if we could show that for a given fixed T there is a a neighborhood $\mathcal{N}_1 = \mathcal{N}_1(\varepsilon, T)$ of x so that

$$(2.27) \qquad |R(T,\lambda;y)| \leq \varepsilon\lambda^{n-1} + O_T(\lambda^{n-2}), \quad y \in \mathcal{N}_1.$$

For if we take $\mathcal{N} = \mathcal{N}_0 \cap \mathcal{N}_1$, we then have

$$\rho\big(T(\lambda - P)\big)(y,y) \leq 2\varepsilon\lambda^{n-1} + O_T(\lambda^{n-2}), \quad y \in \mathcal{N},$$

which implies (1.21), as $\varepsilon > 0$ is arbitrary.

To prove (2.27), we note that the set

$$\mathcal{E}_x = \Big\{ (x,\xi) \in S_x^* M \colon \Phi_t(x,\xi) = (x,\eta), \text{ some } \eta, \ t \in [-T,T] \setminus \bigcup_{|\nu| \le T} (\nu - \delta/2, \nu + \delta/2) \Big\}$$

is closed. Moreover, since we are assuming that (M,g) is real analytic and we are avoiding the focal times $\nu \in \mathbb{Z}$, this subset of $S_x^* M$ must have measure zero (see [23], p. 416). Therefore, working in local coordinates in the cotangent bundle, given $\varepsilon_0 > 0$, we can choose a $C^\infty(\mathbb{R}^n \setminus 0)$ function $b(\xi)$ which is homogeneous of degree zero so that if $B(\xi) = 1 - b(\xi)$, then

$$(2.28) \qquad 0 \le b \le 1, \quad \int_{S^{n-1}} b \, d\omega < \varepsilon_0, \quad \mathcal{E}_x \cap \operatorname{supp} B = \emptyset.$$

The last condition means that if we write $\Phi_t(x,\xi) = (x(t), \xi(t))$ then if $\xi \in \operatorname{supp} B$

$$\gamma_\xi = \Big\{ x(t) \colon \ t \in [-T,T] \setminus \bigcup_{|\nu| \le T} (\nu - \delta/2, \nu + \delta/2) \Big\}$$

is a union of geodesic segments all of which are disjoint from x. Since the geodesic distance from γ_ξ to x is a continuous function of $\xi \in S_x^* M$, it follows that

$$\min_{\xi \in \operatorname{supp} B} d_g(x, \gamma_\xi) > 0.$$

Similarly, there must be a neighborhood $\mathcal{N}$ of x such that for $y \in \mathcal{N}$ if $(y(t), \xi(t)) = \Phi_t(y,\xi)$ then

$$(2.29) \qquad \Big\{ y(t) \colon \ t \in [-T,T] \setminus \bigcup_{|\nu| \le T} (\nu - \delta/2, \nu + \delta/2) \Big\} \notin \mathcal{N}, \quad \text{if } \ \xi \in \operatorname{supp} B.$$

Therefore, by Hörmander's propagation of singularities theorem [9], if $\Psi \in C_0^\infty$ equals one near x but is supported in $\mathcal{N}$, it follows that if we let $B(y,D)$ be the operator with symbol $B(y,\xi) = \Psi(y)B(\xi)$, then

$$\sum_{|\nu| \le T} \big(1 - \beta(t - \nu) \big) \big(B(y,D) \circ e^{itP} \big)(y,y) \in C^\infty([-T,T] \times M),$$

due to the fact that $\beta(t - \nu)$ equals one on $[\nu - \tfrac{3}{4}\delta, \nu + \tfrac{3}{4}\delta]$. Consequently,

$$\frac{1}{2\pi T} \sum_{|\nu| \le T} \int \big(1 - \beta(t - \nu) \big) \, \hat\rho(t/T) \big(B(y,D) \circ e^{itP} \big)(y,y) \, e^{-it\lambda} \, dt = O_{T,B}(1).$$

Since, if $b(y,D)$ is the operator with symbol $\Psi(y)b(\xi)$, we have

$$\Psi(y) R(T,\lambda; y)$$
$$= \frac{1}{2\pi T} \sum_{|\nu| \le T} \int \big(1 - \beta(t-\nu) \big) \, \hat\rho(t/T) \big((B(y,D) + b(y,D)) \circ e^{itP} \big)(y,y) \, e^{-it\lambda} \, dt,$$

we would therefore obtain (2.27) if we could show that

$$\frac{1}{2\pi T}\Big| \int \sum_{|\nu|\leq T} \big(1 - \beta(t-\nu)\big)\, \hat\rho(t/T)\, \big(b \circ e^{itP}\big)(y,y)\, e^{-it\lambda}\, dt\Big|$$

$$\text{(2.30)} \qquad\qquad\qquad\qquad\qquad\qquad\qquad \leq \varepsilon\lambda^{n-1} + O_{T,b}(\lambda^{n-2}).$$

If

$$m_{T,\beta}(\tau) = \sum_{|\nu|\leq T} \int \big(1 - \beta(t-\nu)\big)\hat\rho(t/T)e^{-it\tau}\, d\tau,$$

then the quantity in the left side of (2.30) is

$$\Big| \sum_{j=0}^{\infty} m_{T,\beta}(\lambda - \lambda_j)\, (be_j)(y)\, e_j(y)\Big|.$$

Since $m_{T,\beta} \in \mathcal{S}(\mathbb{R})$, it follows that

$$|m_{T,\beta}(\tau)| \leq C_{T,\beta,N}(1 + |\tau|)^{-N}, \quad N = 1, 2, 3, \ldots.$$

As a result, the left side of (2.30) is bounded by

$$C_{T,\beta,N} \sum_{j=0}^{\infty}(1 + |\lambda - \lambda_j|)^{-N}|be_j(y)|\, |e_j(y)|$$

$$\leq C_{T,\beta,N}\Big(\sum_{j=0}^{\infty}(1 + |\lambda - \lambda_j|)^{-N}|be_j(y)|^2\Big)^{1/2} \Big(\sum_{j=0}^{\infty}(1 + |\lambda - \lambda_j|)^{-N}|e_j(y)|^2\Big)^{1/2},$$

using the Cauchy–Schwarz inequality in the last step. Recall that, by the generalized local Weyl formula (see, e.g., Theorem 5.2.3 in [21]), there is a constant C depending only on (M,g) so that if A is a classical zero order pseudodifferential operator with principal symbol $a(x,\xi)$, we have

$$\text{(2.31)} \qquad \sum_{\lambda_j\in[\lambda,\lambda+1]} |Ae_j(y)|^2 \leq C\lambda^{n-1} \int_{p(y,\xi)\leq 1} |a(y,\xi)|^2\, d\xi + O_A(\lambda^{n-2}).$$

Combining this with the preceding inequality implies that the left side of (2.30) is

$$\leq C_{M,T,\beta}\, \lambda^{n-1}\Big(\int_{S^{n-1}} |b|^2\, d\omega\Big)^{1/2} + O_{M,T,\beta,b}(\lambda^{n-2}),$$

and, therefore, if ε_0 in (2.28) is small enough, we obtain (2.30), which completes the proof.

2.2. Analysis near non-focal points

We shall now give the proof of Proposition 1.5. It is very similar to the argument that we just gave to prove bounds for the contribution of subfocal times for the estimate near self-focal points.

We are assuming that at a given $x \in M$ we now have $|\mathcal{L}_x| = 0$, and we need to show that (1.21) is valid. Recall that $\mathcal{L}_x \subset S_x^* M$ is the set of unit directions such that $\Phi_t(x, \xi) \in S_x^* M$ for some $t \neq 0$. In other words, the set of initial unit directions for geodesic loops through x.

Note that we then have that

$$\mathcal{L}_x^T = \left\{ \xi \in S_x^* M : \ \Phi_t(x, \xi) \in S_x^* M \text{ for some } t \in [-T, T] \backslash \{0\} \right\}$$

is closed and of measure zero. It follows that, given $\varepsilon_0 > 0$, we can find a $b \in C^\infty(\mathbb{R}^n \backslash 0)$ which is homogeneous of degree zero so that we have the following analog of (2.28):

$$(2.32) \qquad 0 \leq b \leq 1, \quad \int_{S^{n-1}} b \, d\omega < \varepsilon_0, \quad \mathcal{L}_x^T \cap \operatorname{supp} \beta = \emptyset.$$

The last condition means that if $\xi \in \operatorname{supp} B$ and $\delta > 0$ the geodesic two segments given by

$$\gamma_\xi = \{ x(t) : \ \delta \leq |t| \leq T \}$$

are disjoint from x if $\Phi_t(x, \xi) = (x(t), \xi(t))$. As before, this yields

$$\min_{\xi \in \operatorname{supp} B} d_g(x, \gamma_\xi) > 0,$$

and so there must be a neighborhood $\mathcal{N}$ of x so that if $y \in \mathcal{N}$ and $(y(t), \xi(t)) = \Phi_t(y, \xi)$ then

$$\{ y(t) : \ \delta \leq |t| \leq T \} \notin \mathcal{N}, \quad \text{if } \ \xi \in \operatorname{supp} B.$$

Therefore, if $\Psi \in C_0^\infty$ equals one near x but is supported in $\mathcal{N}$ and if we let $B(y, D)$ be the operator with symbol $B(y, \xi) = \Psi(y) B(\xi)$, by propagation of singularities,

$$(2.33) \qquad \left(B \circ e^{itP} \right)(y, y) \in C^\infty(\{\delta \leq |t| \leq T\} \times M).$$

To use this, fix $\beta \in C_0^\infty(\mathbb{R})$ which equals one on $[-2\delta, 2\delta]$. Since for $T \geq 1$ we have the uniform bounds

$$\left| \frac{1}{2\pi T} \int \beta(t) \rho(t/T) \, e^{it\tau} \, dt \right| \leq C_N \, T^{-1}(1 + |\tau|)^{-N}, \quad N = 1, 2, 3, \ldots,$$

it follows from the sharp local Weyl law that

$$\left| \frac{1}{2\pi T} \int \beta(t) \rho(t/T) \left(e^{itP} \right)(y, y) \, e^{-it\lambda} \, dt \right|$$

$$\leq C \, T^{-1} \sum_{j=0}^{\infty} (1 + |\lambda - \lambda_j|)^{-n-1} (e_j(y))^2 \leq C \, T^{-1} \lambda^{n-1}.$$

Therefore, by (2.1), and the fact that if $b(y, \xi) = \Psi(y) b(\xi)$, then $B(y, D) + b(y, D) = \Psi(y)$, with Ψ equal to one near x, we conclude that we would have (1.21) if we could show that for fixed $T \geq 1$ we have

$$(2.34) \qquad \frac{1}{2\pi T} \int (1 - \beta(t)) \rho(t/T) \left(B \circ e^{itP} \right)(y, y) \, e^{-it\lambda} \, dt = O_{B,T}(1),$$

and

$$(2.35) \quad \Big| \frac{1}{2\pi T} \int \big(1 - \beta(t)\big) \rho(t/T) \, \big(b \circ e^{itP}\big)(y, y) \, e^{-it\lambda} \, dt \Big| \leq C_T \sqrt{\varepsilon_0} \, \lambda^{n-1} + O_{b,T}(\lambda^{n-2}).$$

The first bound (2.34) just follows from (2.33) and the fact that β equals one on $[-2\delta, 2\delta]$. If

$$m_{T,\beta}(\tau) = \frac{1}{2\pi T} \int \big(1 - \beta(t)\big) \rho(t/T) \, e^{it\tau} \, dt,$$

then the left side of (2.35) equals

$$\Big| \sum_{j=0}^{\infty} m_{T,\beta}(\lambda - \lambda_j) \, (b e_j)(y) \, e_j(y) \Big|.$$

Since $m_{T,\beta} \in \mathcal{S}(\mathbb{R})$, we can use the Cauchy–Schwarz inequality to see that for every $N = 1, 2, 3, \ldots$ this is dominated by a constant depending on T, β and N times

$$\Big(\sum_{j=0}^{\infty} (1 + |\lambda - \lambda_j|)^{-N} |b e_j(y)|^2 \Big)^{1/2} \Big(\sum_{j=0}^{\infty} (1 + |\lambda - \lambda_j|)^{-N} |e_j(y)|^2 \Big)^{1/2}.$$

Thus, (2.35) follows from (2.31), which completes the proof of Proposition 1.5.

3. $\Omega(\lambda^{(n-1)/2})$ bounds at self-focal points with U-invariant functions

In this section we shall finish the proof of Theorem 1.2. Since we have just shown that we have (1.3) when at every self-focal point $x \in M$ the associated Perron–Frobenius operator $U = U_x$ in (1.7) has no nonzero $L^2(S_x^* M)$-invariant functions, we would be done if we could establish the following.

Proposition 3.1. *Let (M, g) be a compact boundaryless real analytic Riemannian manifold of dimension $n \geq 2$. Assume that x is a self-focal point and that $\ell > 0$ is the first return time for the geodesic flow. Suppose further that, if $U \colon L^2(S_x^* M) \to L^2(S_x^* M)$ is the associated Perron–Frobenius map over x, we have*

$$(3.1) \qquad\qquad U g = g, \quad some \ \ 0 \neq g \in L^2(S_x^* M).$$

Let β denote the number of conjugate points counted with multiplicity along $\gamma(t)$, $0 < t \leq \ell$, with $\gamma(t)$ being a unit speed geodesic starting at x. Then if

$$(3.2) \qquad\qquad \mu_k = \frac{2\pi}{\ell}\big(k + \beta/4\big),$$

there is a $c = c(M) > 0$ so that

$$(3.3) \qquad \liminf_{k \to \infty} \mu_k^{-(n-1)} \Big(\sum_{\lambda_j \in [\mu_k, \mu_k + \delta]} \big((e_j(x))^2 \big) \Big) \geq c, \quad if \ \delta > 0.$$

The number β here is independent of the geodesic starting at x, and it is also commonly referred to as the Maslov index of the geodesic.

Since
$$\big\|\chi_{[\mu_k,\mu_k+\delta]}\big\|^2_{L^2(M)\to L^\infty(M)} = \sup_{y\in M}\sum_{\lambda_j\in[\mu_k,\mu_k+\delta]}\big(e_j(y)\big)^2,$$

it is clear that if (3.3) is valid then we cannot have (1.4), which is the remaining part of Theorem 1.2.

To prove the Proposition, as above, let $\rho\in\mathcal{S}(\mathbb{R})$ be as in (1.14). Then clearly, we would have (3.3) if we could show that there is a *uniform* constant $c>0$ so that whenever $T\gg 1$ is fixed we have

$$(3.4)\qquad \sum_{j=0}^\infty \rho\big(T(\mu_k-\lambda_j)\big)\big((e_j(x))^2 \geq c\,\mu_k^{n-1},\quad k\geq N_T,$$

for some $N_T<\infty$. One obtains (3.3) from this by taking $T=\delta^{-1}$ after recalling that $\rho\geq 0$ and $\rho(0)=1$.

To prove (3.4), as before, we may assume that $\ell=1$. For the sake of simplicity, we shall also assume that there are no sub-focal times, since by the argument at the end of §2.1, their contributions to (3.4) will be $o(\nu_k^{n-1})$. We therefore, are assuming that
$$\Phi_t(\xi,\xi)\neq(x,\xi),\ \forall\xi\in S_x^*M,\quad\text{if }\ t\notin\mathbb{Z}.$$

Then, by (2.17), we have that

$$(2\pi)^n\sum_{j=0}^\infty \rho\big(T(\mu_k-\lambda_j)\big)\big((e_j(x))^2$$

$$= \mu_k^{n-1}\,T^{-1}\sum_{\nu=-\infty}^\infty \int e^{-i\nu\mu_k}\hat\rho(\nu/T)\sum_{j=1}^{N(\nu)} a_{\nu,j}(\nu,x,x,\mu_k\omega)\,d\omega + O_T(\mu_k^{n-2}).$$

Since (see §3 of [7], (3.2.15) in [10] and [2]) we have the more precise version of (2.18)

$$(3.5)\qquad \sum_{j=1}^{N(\nu)} a_{\nu,j}(\nu,x,x,\mu_k\omega)=i^{\nu\beta}\sqrt{J^\nu(\omega)}+O_T(\mu_k^{-1}),$$

if $\hat\rho(\nu/T)\neq 0$, and since μ_k is given by (3.2) with $\ell=1$, we conclude that, modulo $O_T(\mu_k^{n-2})$ terms, the left side of (3.4) equals $(2\pi)^{-n}$ times

$$\mu_k^{n-1}\Big(T^{-1}\sum_{\nu=-\infty}^\infty \int \hat\rho(\nu/T)\sqrt{J^\nu(\omega)}\,d\omega\Big) = \mu_k^{n-1}\Big(T^{-1}\sum_{\nu=-\infty}^\infty \hat\rho(\nu/T)\,\langle U^\nu\mathbb{1},\mathbb{1}\rangle\Big).$$

Since $\hat\rho$ is nonnegative and $\hat\rho(0)>0$, it follows that there is a constant $c_0>0$, which is independent of T, so that

$$T^{-1}\sum_{\nu=-\infty}^\infty \hat\rho(\nu/T)\,\langle U^\nu\mathbb{1},\mathbb{1}\rangle \geq c_0\,M^{-1}\sum_{\nu=-M}^M \langle U^\nu\mathbb{1},\mathbb{1}\rangle,\quad M=c_0 T.$$

By von Neumann's ergodic theorem we have that

$$\lim_{M \to \infty} \frac{1}{2M} \sum_{\nu=-M}^{M} \langle U^\nu \mathbb{1}, \mathbb{1} \rangle = \langle \Pi(\mathbb{1}), \mathbb{1} \rangle,$$

where $\Pi(1)$ denotes the projection of $\mathbb{1}$ onto the U-invariant subspace of $L^2(S_x^* M)$. Since our assumption (3.1) gives that

$$\langle \Pi(\mathbb{1}), \mathbb{1} \rangle > 0,$$

we conclude that (3.4) must be valid, which completes the proof of Proposition 3.1.

4. Bounds for quasi-modes

In this section, we shall prove Corollary 1.6. Recall that we are assuming that ϕ_λ are quasi-modes satisfying (1.25), i.e.,

$$(4.1) \quad \int |\phi_\lambda|^2 \, dV = 1, \quad \text{and} \quad \|S_{[2\lambda,\infty)}\phi_\lambda\|_{L^\infty(M)} + \|(\Delta + \lambda^2)\phi_\lambda\|_{L^2(M)} = o(\lambda),$$

with $S_{[2\lambda,\infty)}$ denoting the projection on to frequencies in $[2\lambda, \infty)$. By Theorem 1.2, if there is a a self-focal point x for which the operators in (1.7) have a nontrivial invariant function satisfying (3.1), then we know that there is a uniform constant $c > 0$ so that, given any $\delta > 0$ we have

$$\limsup_{\lambda \to \infty} \lambda^{-(n-1)/2} \left\| \chi_{[\lambda, \lambda+\delta]} \right\|_{L^2(M) \to L^\infty(M)} \geq c.$$

Thus, in this case, if $\delta_j \to 0$, we can find ϕ_{λ_j} with spectrum in $[\lambda_j, \lambda_j + \delta]$ satisfying $\|\phi_{\lambda_j}\|_2 = 1$ and $\|\phi_{\lambda_j}\|_\infty \geq c\, \lambda_j^{(n-1)/2}$. Consequently since the $\{\phi_{\lambda_j}\}$ are clearly quasi-modes of order $o(\lambda)$, to finish the proof we just need to prove that (1.26) is valid when all the self-focal points are dissipative.

To do this, we just use the fact that, by Lemma 2.5 in [22], we must have

$$\|\phi_\lambda\|_{L^\infty(M)} = o(\lambda^{(n-1)/2})$$

in this case, which completes the proof.

Remark. As we mentioned in §1.6, it would be interesting to see whether we can obtain $\Omega(\mu_k^{(n-1)/2})$ bounds for quasi-modes of *order zero* whenever there is a self-focal point $x \in M$ satisfying (3.1). We recall these are ones satisfying the stronger variant of (1.25), which says that

$$\|(\Delta + \mu_k^2)\phi_{\mu_k}\|_{L^2(M)} = O(1), \quad \text{and} \quad \int |\phi_{\mu_k}|^2 \, dV = 1.$$

Such quasi-modes were constructed in [22] when the map $\xi \to \eta(\xi)$ equals the identity map on an open subset of $S_x^* M$, which, of course, is a stronger condition than (3.1). It is plausible that in the real analytic setting, one may construct quasi-modes of order zero corresponding to a self-focal point such that U_x has an

invariant L^2 function. The idea is to use $|f|^2 d\mu_x$ as the 'symbol' or invariant measure in the quasi-mode construction. In the real analytic setting, one can try to 'quantize' $(\Lambda_x, |f|^2 d\mu_x)$ using Toeplitz quantization in a Grauert tube around M; we plan to investigate this in future work.

References

[1] Béard-Bergery, L.: Quelques exemples de variétés riemanniennes o toutes les géodésiques issues d'un point sont fermées et de même longueur, suivis de quelques résultats sur leur topologie. *Ann. Inst. Fourier (Grenoble)* **27** (1977), no. 1, 231–249.

[2] Besse, A. L.: *Manifolds all of whose geodesics are closed.* Results in Mathematics and Related Areas 93, Springer-Verlag, Berlin-New York, 1978.

[3] Blair, M. and Sogge, C. D.: On Kakeya–Nikodym averages, L^p-norms and lower bounds for nodal sets of eigenfunctions in higher dimensions. *J. European Math. Soc.(JEMS)* **17** (2015), no. 10, 2513–2543.

[4] Bourgain, J., Shao, P., Sogge, C. D. and Yao, X.: On L^p-resolvent estimates and the density of eigenvalues for compact Riemannian manifolds. *Comm. Math. Phys.* **333** (2015), no. 3, 1483–1527.

[5] Colin de Verdière, Y.: Quasi-modes sur les variétés Riemanniennes. *Invent. Math.* **43** (1977), no. 1, 15–52.

[6] Duistermaat, J. J.: Oscillatory integrals, Lagrange immersions and unfolding of singularities. *Comm. Pure Appl. Math.* **27** (1974), 207–281.

[7] Duistermaat, J. J. and Guillemin, V. W.: The spectrum of positive elliptic operators and periodic bicharacteristics. *Invent. Math.* **29** (1975), 39–79.

[8] Eigen, S., Hajian, A., Ito, Y. and Prasad, V.: Existence and non-existence of a finite invariant measure. *Tokyo J. Math.* **35** (2012), no. 2, 339–358.

[9] Hörmander, L.: On the existence and the regularity of solutions of linear pseudo-differential equations. *Enseignement Math. (2)* **17** (1971), 99–163.

[10] Hörmander, L.: Fourier integral operators. I. *Acta Math.* **127** (1971), 79–183.

[11] Hörmander, L.: *The analysis of linear partial differential operators I. Distribution theory and Fourier analysis.* Reprint of the second (1990) edition. Classics in Mathematics, Springer-Verlag, Berlin, 2003.

[12] Ivrii, V.: The second term of the spectral asymptotics for a Laplace–Beltrami operator on manifolds with boundary. (Russian) *Funktsional. Anal. i Prilozhen.* **14** (1980), 25–34.

[13] Klingenberg, W.: *Riemannian geometry.*. De Gruyter Studies in Mathematics 1, Walter de Gruyter, Berlin–New York, 1982.

[14] Krengel, U.: *Ergodic theorems.* De Gruyter Studies in Mathematics 6, Walter de Gruyter, Berlin, 1985.

[15] Nakagawa, H.: A note on theorems of Bott and Samelson. *J. Math. Kyoto Univ.* **7** (1967), 205–220.

[16] Olsen, J.: Three dimensional manifolds all of whose geodesics are closed. *Ann. Global Anal. Geom.* **37** (2010), no. 2, 173–184.

[17] SAFAROV, Y. G.: Asymptotics of a spectral function of a positive elliptic operator without a nontrapping condition. (Russian) *Funktsional. Anal. i Prilozhen.* **22** (1988), 53–65, 96. Translation in *Funct. Anal. Appl.* **22** (1988), no. 3, 213–223 (1989).

[18] SAFAROV, Y. G. AND VASSILIEV, D.: The asymptotic distribution of eigenvalues of partial differential operators. Translations of Mathematical Monographs 155, American Mathematical Society, Providence, RI, 1997.

[19] SOGGE, C. D.: *Fourier integrals in classical analysis.* Cambridge Tracts in Mathematics 105, Cambridge University Press, Cambridge, 1993.

[20] SOGGE, C. D.: Kakeya–Nikodym averages and L^p-norms of eigenfunctions. *Tohoku Math J. (2)* **63** (2011), 519–538.

[21] SOGGE, C. D.: *Hangzhou lectures on eigenfunctions of the Laplacian.* Annals of Math Studies 188, Princeton University Press, Princeton, NJ, 2014.

[22] SOGGE, C. D., TOTH, J. A. AND ZELDITCH, S.: About the blowup of quasimodes on Riemannian manifolds. *J. Geom. Anal.* **21** (2011), 150–173.

[23] SOGGE, C. D. AND ZELDITCH, S.: Riemannian manifolds with maximal eigenfunction growth. *Duke Math. J.* **114** (2002), 387–437.

[24] SOGGE, C. D. AND ZELDITCH, S.: Concerning the L^4 norms of typical eigenfunctions on compact surfaces. In *Recent development in geometry and analysis,* 407–423. Adv. Lect. Math. (ALM) 23, Int. Press, Somerville, MA, 2012.

[25] SOGGE, C. D. AND ZELDITCH, S.: *On eigenfunction restriction estimates and L^4-bounds for compact surfaces with nonpositive curvature.* In *Advances in analysis: the legacy of Elias M. Stein,* 447–461. Princeton Math. Ser. 50, Princeton University Press, Princeton, NJ, 2014.

[26] SOGGE, C. D. AND ZELDITCH, S.: Focal points and sup-norms of eigenfunctions II: the two-dimensional case. *Rev. Mat. Iberoam.* **32** (2016), no. 3, 995–999.

[27] ZELDITCH, S.: CBMS Lectures (to appear).

CHRISTOPHER D. SOGGE: Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218, USA.
E-mail: sogge@jhu.edu

STEVE ZELDITCH: Department of Mathematics, Northwestern University, Evanston, IL 60208, USA.
E-mail: s-zelditch@northwestern.edu