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Focal points and sup-norms of eigenfunctions II: the two-dimensional case

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Abstract. We use a purely dynamical argument on circle maps to improve a result in our accompanying article, [5], on real analytic surfaces possessing eigenfunctions that achieve maximal sup norm bounds. The improved result is that there exists a 'pole' p so that all geodesics emanating from p are smoothly closed.

1. Introduction and main results

In the accompanying article [5], the authors gave a dynamical characterization of compact real analytic Riemannian manifolds (M^n, g) of dimension n possessing Δ_g-eigenfunctions

$$(\Delta + \lambda_{j_k}^2) e_{j_k} = 0, \quad ||e_{j_k}||_{L^2} = 1$$

of maximal sup norm growth,

$$\|e_{j_k}\|_{L^\infty(M)} \geq C_g \, \lambda_{j_k}^{(n-1)/2}. \tag{1.1}$$

Here, C_g is a positive constant independent of λ_j. The main result of [5] (recalled more precisely below) is that if (M, g) possesses such a sequence $\{e_{j_k}\}$, then there must exist *self-focal points* p at which all geodesics from p loop back to p at some time. The minimal such time is called the first return time T_p. Moreover, there must exist a self-focal point for which the first return map

$$\eta_p : S^*_p M \to S^*_p M$$

preserves an L^1 measure on the unit co-sphere $S^*_p M$ at p. The purpose of this addendum is to add a purely dynamical argument to the main result of [5] to prove the stronger:

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Theorem 1.1. *Let* (M, g) *be a compact real analytic compact surface without boundary. If there exists a sequence of* L^2*-normalized eigenfunctions,* $(\Delta + \lambda_{j_k}^2)e_{j_k} = 0$*, satisfying* $\|e_{j_k}\|_{L^\infty(M)} \geq C_g \, \lambda_{j_k}^{1/2}$*, then* (M, g) *possesses a pole, i.e., a point* p *so that every geodesic starting at* p *returns to* p *at time* $2T_p$ *as a smoothly closed geodesic.*

Thus, (M, g) is a $C_{2T_p}^p$-manifold in the terminology of [1] (Definition 7.7(e)). Theorem 1.1 proves the conjecture stated on page 152 of [3] in the case of real analytic surfaces. We also remark that the conclusion of the theorem remains valid if there exists a sequence of quasimodes of order $o(\lambda)$ saturating the sup-norms (see [5] for the terminology).

Theorem 1.1 follows by combining the main result of [5] with the following:

Proposition 1.2. *Let* (S^2, g) *be a two-dimensional real analytic Riemannian surface. Suppose that* $p \in S^2$ *is a self-focal point and that the first return map* $\eta_p : S_p^* S^2 \to S_p^* S^2$ *preserves a probability measure which is in* $L^1(S_p^* S^2)$*. Then* η_p^2 *is the identity map, and in particular all geodesics through* p *are smoothly closed with the common period* $2T_p$*.*

The proof only involves dynamics and not eigenfunctions of Δ_g. To explain how Theorem 1.1 follows from Proposition 1.2, we first recall some of the definitions and main result of [5] to establish notation. We then give the proof of Proposition 1.2.

A natural question is whether all real analytic Riemannian surfaces with maximal eigenfunction growth are surfaces of revolution. There are many P_ℓ^m metrics besides surfaces of revolution. A second question is whether the proposition has some kind of generalization to higher dimensions.

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2. Background on maximal eigenfunction growth

Let $\eta_t(x, \xi) = (x(t), \xi(t))$ denote the homogeneous Hamilton flow on $T^*M \backslash 0$ generated by $H(x, \xi) = |\xi|_g$. Since η_t preserves the unit cosphere bundle $S^*M = \{|\xi|_g = 1\}$, it defines a flow on S^*M which preserves Liouville measure. For a given $x \in M$, let $\mathcal{L}_x \subset S_x^*M$ denote the set of *loop directions*, i.e., unit directions ξ for which $\eta_t(x, \xi) \in S_x^*M$ for some time $t \neq 0$. Also, let $d\mu_x$ denote the measure on S_x^*M induced by the Euclidean metric g_x and let $|\mathcal{L}_x| = \mu_x(\mathcal{L}_x)$.

We say that p is a *self-focal point* if there exists a time $\ell > 0$ so that $\eta_\ell(p, \xi) \in S_p^*M$ for all $\xi \in S_p^*M$, i.e., if

$$\mathcal{L}_p = S_p^*M.$$

We let T_p be the minimal such time, and write

$$\eta_{T_p}(p, \xi) = (p, \eta_p(\xi)), \quad \xi \in S_p^*M. \tag{2.1}$$

Under the assumption that g is real analytic, the *first return map*

$$\eta_p : S_p^* M \to S_p^* M \tag{2.2}$$

is also real analytic.

The key property of interest is that η_p is conservative in the following sense:

Definition 2.1. We say ν_p is conservative if it preserves a measure $\rho d\mu_p$ on $S_p^* M$ which is absolutely continuous with respect to μ_p and of finite mass:

$$\exists \rho \in L^1(\mu_p), \quad \eta_p^* \rho \, d\mu_p = \rho \, d\mu_p. \tag{2.3}$$

Theorem 2.2 ([5]). *Suppose that (M, g) is a compact real analytic manifold without boundary which possesses a sequence $\{e_{j_k}\}$ of eigenfunctions satisfying (1.1). Then (M, g) possesses a self-focal point p whose first return map η_p is conservative.*

3. Proof of Proposition 1.2

It is not particularly important to the proof, but we may assume with no loss of generality that M is diffeomorphic to S^2. The proof is a standard one on manifolds with focal points; we refer to [4] for background. We also assume throughout that η_p is real analytic, since that is the case in our setting; most of the statements below are true for smooth circle maps.

We note that η_p is the restriction of the geodesic flow G^T to the invariant set $S_p^* S^2$. This circle is contained in a symplectic transversal S_p to the geodesic flow. On the symplectic transversal G^T is a symplectic map which is invertible. Hence η_p is invertible. Thus, $(D\eta_p)_\omega$ is non-zero for all $\omega \in S_p^* S^2$. It follows that η_p is either orientation preserving or orientation reversing.

Next we use time reversal invariance of the geodesic flow to show that η_p is conjugate to its inverse.

Lemma 3.1. *η_p is reversible (conjugate to its inverse).*

Proof. Let $\tau(x, \xi) = (x, -\xi)$ on $S^* M$. Then on all of $S^* M$, we have $\tau G^t \tau = G^{-t}$. Indeed, let Ξ_H be the Hamilton vector field of $H(x, \xi) = |\xi|_g$. Then $H \circ \tau = H$, i.e., H is time reversal invariant. We claim that

$$\tau_* \Xi_H = -\Xi_H.$$

Written in Darboux coordinates,

$$\Xi_H = \sum_j \frac{\partial H}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial H}{\partial x_j} \frac{\partial}{\partial \xi_j}.$$

If we let $(x, \xi) \to (x, -\xi)$ and use invariance of the Hamiltonian we see that the vector field changes sign. Now, G^{-t} is the Hamilton flow of $-\Xi_H$ and that is $\tau_* \Xi_H$. But the Hamilton flow of the latter is $\tau G^t \tau$.

Since $S_p^* M$ is invariant under τ, we just restrict the identity $\tau G^t \tau = G^{-t}$ to $S_p^* M$ to see that η_p is reversible. □

3.1. Orientation preserving case

First let us assume that η_p is orientation preserving. Then it has a rotation number. We recall that the rotation number of a circle homeomorphism is defined by

$$r(f) = \Big(\lim_{n\to\infty} \frac{F^n(x) - x}{n} \Big) \mod 1.$$

Here, $F\colon \mathbb{R} \to \mathbb{R}$ is a lift of f, i.e., a map satisfying $F(x+1) = F(x)$ and $f = \pi \circ F$, where $\pi\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is the standard projection. The rotation number is independent of the choice of F or of x. It is rational if and only if f has a periodic orbit. For background, see [2].

Lemma 3.2. *The rotation number of η_p is either 0 or π.*

Proof. For a circle homeomorphism, the rotation number $\tau(f^{-1})$ is always $-\tau(f)$. Since η_p is reversible, $\tau(\eta_p) = -\tau(\eta_p)$, i.e., its rotation number can only be $0, \pi$. □

Lemma 3.3. *η_p^2 has fixed points.*

Proof. The rotation number of η_p^2 is 0. But it is known that $\tau(f) = 0$ if and only if f has a fixed point. See [2], Theorem 2.4. □

We now complete the proof that $\eta_p^2 = Id$ if η_p is orientable. Since η_p is real analytic, this is the case if η_p^2 has infinitely many fixed points, so we may assume that $\mathrm{Fix}(\eta_p^2)$ is finite (and non-empty). We write $\#\mathrm{Fix}(\eta_p^2) = N$ and denote the fixed points by p_j.

If $N = 1$, i.e., η_p^2 has one fixed point Q, then $S^1\backslash\{Q\}$ is an interval and η_p^2 is a monotone map of this interval. So every orbit is asymptotic to the fixed point of η_p^2.

Let μ be the L^1 invariant measure for η_p^2 and let $K = \operatorname{supp}\mu$. We can decompose K into N subsets K_j such that $\eta_p^2(K_j) \to p_j$. K_j is the basin of attraction of p_j. Then

$$\mu(K_j) = \mu(\eta^{2p}(K)_j) \to \mu(\{p_j\}).$$

But $p_j \in K_j$ so it must be that $K_j = \{p_j\}$. This shows that μ cannot be L^1, concluding the proof.

3.2. Orientation reversing case

The square η_p^2 of an orientation reversing diffeomorphism of $S_p^*S^2$ is an orientation preserving diffeomorphism. If η_p preserves the measure $d\mu$ then so does η_p^2. Thus we reduce to the orientation preserving case.

Remark. Keith Burns pointed out to us that the orientation reversing case cannot occur. If η_p is orientation reversing, so is $-\eta_p$. An orientation reversing homeomorphism of the circle must have a fixed point $\xi \in S_p^*M$. But then $G^T(p,\xi) = (p,-\xi)$ for some $T > 0$, which is impossible.

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