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Parabolic BMO and global integrability of supersolutions to doubly nonlinear parabolic equations

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Abstract. We prove that local and global parabolic BMO spaces are equal thus extending the classical result of Reimann and Rychener. Moreover, we show that functions in parabolic BMO are exponentially integrable in a general class of space-time cylinders. As a corollary, we establish global integrability for positive supersolutions to a wide class of doubly nonlinear parabolic equations.

1. Introduction

In 1993, Lindqvist [13] proved that positive $\mathbf{A}_p$-superharmonic functions bounded away from zero are globally integrable to some small power $\epsilon > 0$ on Hölder domains, that is, on domains satisfying a quasihyperbolic boundary condition (defined in [8], see also Definition 2.6). This was qualitatively the most general result in a series of papers investigating global integrability of harmonic functions, starting from Armitage's result on balls [4], and leading via Lipschitz domains [15] and generalizations [16] to Hölder domains of Stegenga and Ullrich [23]. Hölder domains were characterized as the ones with exponentially integrable quasihyperbolic metric in [20], and this was proved to imply exponential integrability of functions BMO in [20] and [10]. Noting the well-known fact visible in Moser's proof [17] for Harnack inequality of elliptic partial differential equations, i.e. that $-\log u \in \mathrm{BMO}$ for positive supersolutions, Lindqvist composed the general result. For more about research related to global integrability, see also [24], [9] and [1].

Contrary to the elliptic theory, in the parabolic case very little or nothing has been done, even in the case of heat equation, at least to the author's knowledge. In this paper, we prove many parabolic analogues of the above mentioned resuls. We will use a version of parabolic BMO, originally introduced by Moser [18]. Then,

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using the John–Nirenberg type inequalities known from [3], we will establish a global version of John–Nirenberg type lemma in space-time cylinders that are Hölder domains in spatial dimensions (corresponding to the non-parabolic results from [10] and [20]). Once we have global John–Nirenberg inequality, we can prove the equivalence of local and global norms generalizing the classical result of Reimann and Rychener [19].

As an application, we will consider equations of the form

$$\frac{\partial(u^{p-1})}{\partial t} = \operatorname{div} A(x,t,u,Du), \tag{1.1}$$

where the function A satisfies certain p-Laplace type growth conditions to be specified in Section 6. This class of equations has been studied for instance in [25] and [11]. The key fact we will use is of course the parabolic BMO-condition, and our results will apply to any equation whose solutions are exponentials of functions in parabolic BMO according to Definition 2.2 or even Definition 2.3.

We conclude the introduction by briefly describing how the parabolic case differs from the elliptic one. The mean oscillation of a function $u \in L_{\mathrm{loc}}(\mathbb{R}^n)$ is defined as

$$\frac{1}{|B|}\int_B |u - u_B|\,dx,$$

with B a ball, and a function is in BMO if its mean oscillation is uniformly bounded. In the definition of parabolic BMO, there is a *time lag* between the domains where we measure the upper and lower deviation from some constant. Thus the picture to have in mind about parabolic oscillation of $u \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ is

$$\frac{1}{|B\times(\theta+I)|}\int_{\theta+I}\int_B (u(x,t)-a_{B\times I})^+\,dx\,dt + \frac{1}{|B\times(I)|}\int_I\int_B (u(x,t)-a_{B\times I})^-\,dx\,dt,$$

where $a_{B\times I}$ is a constant, I an interval so that $B\times I$ respects the appropriate geometry, $\theta+I$ is the interval translated forward in time by the lag parameter $\theta > \operatorname{length}(I)$, and $|\cdot|$ denotes the $n+1$ dimensional Lebesgue measure. The appearance of time lag is a deep fact originating from the time lag phenomenon of parabolic partial differential equations, and it results in that parabolic BMO differs quite a lot from the classical BMO.

The paper is organized as follows: Section 2 introduces notation and known results, Section 3 develops a chaining technique, Section 4 contains the proof of global John–Nirenberg inequality, Section 5 consists of the formulations of its most important consequences, and in Section 6 we apply the results to parabolic differential equations.

2. Definitions and preliminaries

We will start with general conventions. We will work in $\mathbb{R}^{n+1}$. The first n-coordinates will be called *spatial* (usually denoted x) and the last one *temporal* (usually denoted t). We will use the standard notation $|E|$ for the Lebesgue measure of E. In most cases we do not specify its dimension, but it must be clear from the context.

When it comes to integrating, we denote $d\mu = dx\,dt$. The letter C without subscript will be a constant depending only on the quantities we are not keeping track of, and we denote $f \lesssim 1$ if $f \leq C$. Occasional subscripts in this notation, such as $\lesssim_n$, will emphasize the dependencies of the constant. The positive part of a function u is denoted $(u)^+ = (u)_+ = \chi_{\{u>0\}}u$; the negative part is defined by $(u)^- = (u)_- = -\chi_{\{u<0\}}u$.

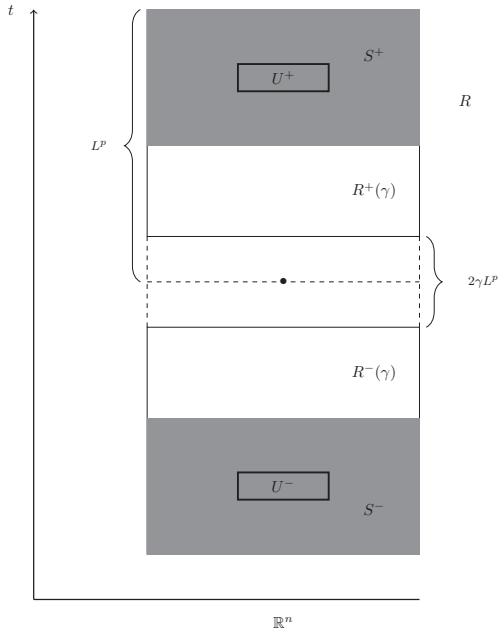

FIGURE 1. A parabolic rectangle R and the arrangement of its subsets $U^+ \subset S^+ \subset R^+(\gamma)$. The lengths are not in correct scales.

The definition of classical BMO is stated in terms of Euclidean cubes. In the parabolic context, the class of cubes must be replaced by that of *parabolic rectangles*. The notation introduced in the next definition is illustrated in Figure 1.

Definition 2.1 (Parabolic rectangle). Let $Q(x, L) \subset \mathbb{R}^n$ be a cube with sidelength L and center x, and let $p > 1$ be fixed. We define a parabolic rectangle centered at (x, t) with sidelength L, its upper half and its upper quarter as

$$R = Q \times (t - L^p, t + L^p), \quad R^+ = Q \times (t, t + L^p), \quad S^+ = \left\{(y, \tau) \in R : \tau > t + \tfrac{1}{2}L^p\right\}.$$

The corresponding lower parts R^- and S^- are defined analogously. The *parabolic scaling* of a rectangle and its quarter are defined as

$$\begin{aligned} \lambda R &= (\lambda Q) \times (t - (\lambda L)^p, t + (\lambda L)^p), \\ \lambda S^+ &= (\lambda Q) \times (t + \tfrac{3}{4}L^p - \tfrac{1}{4}(\lambda L)^p, t + \tfrac{3}{4}L^p + \tfrac{1}{4}(\lambda L)^p). \end{aligned}$$

A special but technical role is played by the sets

$$U^+ = \frac{1}{8}\, S^+,$$

called *upper fragments*. From now on, the symbols Q, $R^\pm$, $S^\pm$ and $U^\pm$ will be reserved for sets introduced in this definition.

It may be useful to notice that S^+ is a metric ball with respect to

$$d((x,t),(y,\tau)) := \max\{\|x-y\|_\infty\,,\, C_p|t-\tau|^{1/p}\}. \tag{2.1}$$

This motivates the way we define its scaling. Moreover, the centers of various $n+1$ dimensional sets mean centers of the corresponding metric balls with respect to (2.1).

The parabolic BMO spaces arise from partial differential equations, and the scaling properties of the equation determine the number p specifying the shape of parabolic rectangles. Since parabolic BMO condition is stated in terms of parabolic rectangles, different values of p lead to different function spaces. In what follows, $p > 1$ is considered to be fixed. The following definition is different from the one given in [7], but it seems to be more suitable when investigating local-to-global phenomena.

Definition 2.2 (Parabolic BMO). Let $\Omega \subset \mathbb{R}^n$ be a domain and $T \in (0,\infty]$. Given $\sigma \geq 1$, a function $u \in L^1_{\mathrm{loc}}(\Omega \times (0,T))$ is in parabolic BMO, denoted $\mathrm{PBMO}^\sigma(\Omega \times (0,T))$ if for each parabolic rectangle R there is a constant a_R such that

$$\sup_{\sigma R \subset \Omega\times(0,T)} \Big(\fint_{S^+} (u-a_R)^+\, d\mu + \fint_{S^-} (a_R-u)^+\, d\mu \Big) =: \|u\|_{\mathrm{PBMO}^\sigma} < \infty. \tag{2.2}$$

In case $\sigma = 1$, it will be omitted in the notation.

The starting point for our considerations is the John–Nirenberg inequality satisfied by $u \in \mathrm{PBMO}$. In sufficient generality it was proved by Aimar [3], who studied BMO-spaces with lag maps. His aim was to develop a unified approach to both classical and parabolic BMO on spaces of homogeneous type. The next technical definition is a special case of Definition 1.4 in [3], and it is included in order to demonstrate that Aimar's results apply to all BMO-type spaces discussed in this paper.

Definition 2.3 (BMO spaces with certain lag mappings). Let $r \in \mathbb{R}$ and $b \in (0,1]$. For an L-sided parabolic rectangle R centered at (y,τ), define

$$T_R(x,t;L) = (x, t-L^p; L), \quad T_S(x,t;L) = \big(x, t - \tfrac{3}{2}L^p; L\big), \quad h(r) = (r)_+^b.$$

A function $u \in L^1_{\mathrm{loc}}(R^*)$ is said to belong to BMO space with lag mapping T_B ($B = S$ or $B = R$) with respect to h if for each parabolic subrectangle $R \subset R^*$, there is a constant a_R such that

$$\sup_{R\subset R^*} \Big(\fint_{B^+} h(u-a_R)\, d\mu + \fint_{B^-} h(a_R-u)\, d\mu \Big) < \infty.$$

Note that the lag maps T_B take a center and a radius of a ball with respect to a metric (2.1), and map them to a new center and a new radius. It is easy to see that with these choices of T_R and T_S, B^- is the metric ball corresponding to the data from T_B and B^+.

The choice $b = 1$ and $B = S$ gives the condition (2.2). Any space of Definition 2.3 satisfies the following John–Nirenberg inequality (Lemma 3.4 in [3]) with certain modifications, but we will state it for the case of PBMO^σ. Aimar's approach allows us to use general $p > 1$ instead of $p = 2$ of [7].

Lemma 2.4 (Aimar [3]). *Let $u \in \mathrm{PBMO}^\sigma(\Omega \times (0,T))$. Then there are constants A and B depending only on $\|u\|_{\mathrm{PBMO}^\sigma}$, p and n such that for each parabolic rectangle R with $\sigma R \subset \Omega \times (0,T)$ the following holds:*

$$|U^+ \cap \{(u - a_R)^+ > \lambda\}| \le A\, e^{-B\lambda}\, |U^+| \quad \textit{and} \tag{2.3}$$

$$|U^- \cap \{(a_R - u)^+ > \lambda\}| \le A\, e^{-B\lambda}\, |U^-|. \tag{2.4}$$

Remark 2.5. The general BMO space with lag mapping (in sense of Definition 2.3) satisfies the same inequalities but $U^\pm$ from Definition 2.1 are replaced by $\frac{1}{8}B^\pm$ from Definition 2.3. This difference is not essential, and all the following arguments will work also in that case. The factor 1/8 is small enough to make the inequalities (2.3) and (2.4) hold, but its role is not important in this paper. In fact, our final result in $\mathbb{R}^{n+1}$ will make the technical notion of fragments unnecessary.

Finally, we will need a geometric condition that ensures that the domains we consider are reasonable enough. The following class of (bounded) domains was first defined by Gehring and Martio in [8].

Definition 2.6 (Quasihyperbolic boundary condition). Let $\Omega \subset \mathbb{R}^n$ be a domain. We define its quasihyperbolic metric as

$$k(x,y) := \inf_{\gamma_{xy}} \int_{\gamma_{xy}} \frac{1}{d(z,\Omega^c)}\, ds(z),$$

where the infimum is over curves connecting x and y. A domain is said to satisfy a *quasihyperbolic boundary condition* if there is a fixed $x_0 \in \Omega$ and a constant K such that, for all $y \in \Omega$,

$$k(x_0,y) \le K \log \frac{K}{d(y,\Omega^c)}.$$

3. Chain lemma

When proving a local-to-global result for a function u on some domain, the most crucial part is to get information about the behavior of u close to the boundary of the domain. We will use a chaining technique composing various ideas from the known results in [14], [22] and [5]. The original chaining techinques do not work as such since the parabolic BMO condition compells us to take into account the special role played by the time variable.

The new problem is that the rectangles in the chain cannot be located as freely as they could in the classical case. The chain has to have a direction in time, each step in spatial dimensions forcing us to take certain step in time. Thus the first challenge is to ensure that for each point in the space-time cylinder there is enough time to move to the spatial point we consider the center. This problem can be solved by imposing an artificial upper bound on the size of spatial steps, which is reflected as the pth power to the time variable due to the parabolic scaling.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain and $\Delta = \Omega \times (0,T)$, $0 < \beta < 1$; $\alpha, \alpha', \delta > 0$ and let $U^+_{(x,t)}$ be an upper fragment of a parabolic rectangle centered at $(x,t) \in \Omega \times (\delta, T)$ with spatial sidelength $l'_{x,t} = \min\{l_{x,t}, \alpha' q\}$. Here,*

$$l_{x,t} := \min\{\beta d(x, \Omega^c), \beta(T-t)^{1/p}, \alpha q\}, \tag{3.1}$$

where $q = \sup_{x\in\Omega} \operatorname{length}(\gamma_{xz})$ is the maximal length of quasihyperbolic geodesics connecting points x to a fixed $z \in \Omega$.

Under these assumptions, the parameter α can be chosen so that there is a chain of parabolic rectangles $\mathcal{P}(U^+_x) = \{R_i\}_{i=1}^{k_{(x,t)}}$ with the following properties:

(i) *$R_1 = R_{(x,t)}$, $R_{k_{(x,t)}}$ is centered at $(z, \tau(x,t))$ and it has spatial sidelength $l_{z,\tau(x,t)}$. For all j we have that $\beta^{-1} R_j \subset \Delta$.*

(ii) *$|U^-_i \cap U^+_{i+1}| \gtrsim_{p,n,\beta} \max\{|R_i|, |R_{i+1}|\}$ as $1 \le i < k_x$.*

(iii) *$0 \le t - \tau(t,x) \le q^p \eta$, where $\eta \eqsim_{n,\beta} \alpha$.*

(iv) *$k_{(x,t)} \lesssim_{p,n,\beta} k(x,z) + \log(T/(T-t)) + \log(\alpha/\alpha' + 1) + 1/\alpha + 1$.*

Proof. We start by assuming that $\alpha' q \ge l_{x,t}$. The first rectangle R_1 is obviously given. We denote its center p_1, and we denote the center of its upper fragment p'_1. Suppose that R_j (centered at $p_j = (y_j, t_j)$) has been chosen. We connect its spatial center y_j to z with the quasihyperbolic geodesic γ and find the point y_{j+1} where γ exits $Q_j = Q(y_j, l_{y_j,t_j})$. Set

$$\begin{aligned}
t'_{j+1} &= t_j - \tfrac{3}{4} l_j^p, \\
p'_{j+1} &= (y_{j+1}, t'_{j+1}), \\
l_{j+1} &= l_{(y_{j+1}, t'_{j+1})}, \\
t_{j+1} &= t'_{j+1} - \tfrac{3}{4} l_{j+1}^p = t_j - \tfrac{3}{4}(l_j^p + l_{j+1}^p), \\
p_{j+1} &= (y_{j+1}, t_{j+1}).
\end{aligned}$$

We define R_{j+1} by extending the spatial cube Q_{j+1} to a parabolic rectangle R_{j+1} centered at p_{j+1} with sidelength l_{j+1} so that its upper fragment is centered at p'_{j+1}, which is also the temporal center of U^-_j (see Figure 2). One of the two consecutive fragments (U^-_j and U^+_{j+1}) has its temporal projection contained in the other, and $y_{j+1} \in \partial Q(y_j, l_j)$, so in order to establish (ii), it suffices to prove that $l_j \eqsim_{\beta,n} l_{j+1}$.

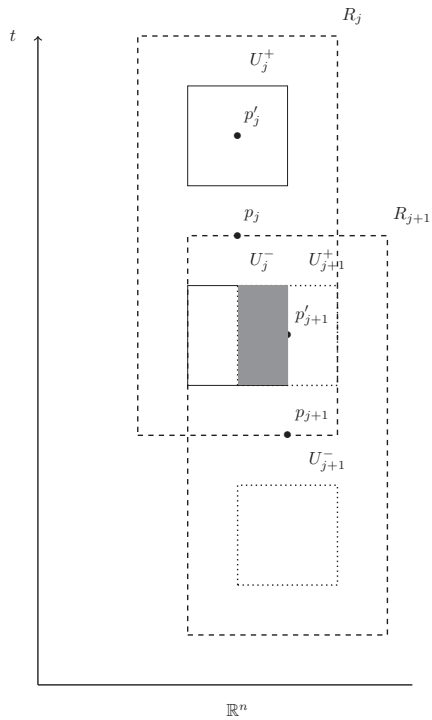

FIGURE 2. Schematic picture on position of two subsequent rectangles in the chain. For clarity, the lengths and distances are not in scale.

We define an auxiliary metric

$$d'((x_1,t_1),(x_2,t_2)) = \max\{|x_1-x_2|, |t_1-t_2|^{1/p}\}.$$

Denoting $\Xi = \Omega^c \times \mathbb{R} \cup \Omega \times (T,\infty)$, the choice condition (3.1) can be reformulated as

$$l_{x,t} = \min\{\beta d'((x,t),\Xi), \alpha q\}.$$

If $l_i = \alpha q = l_{i+1}$, there is nothing to prove, so assume that both $l_i = \beta d'(p'_i,\Xi)$ and $l_{i+1} = \beta d'(p'_{i+1},\Xi)$. Then

$$l_i \le \beta(d'(p'_i,p'_{i+1}) + d'(p'_{i+1},\Xi)) \le \beta\, l_i + l_{i+1} \tag{3.2}$$

and

$$l_{i+1} \le \beta(d'(p'_i,p'_{i+1}) + d'(p'_i,\Xi)) \le (\beta+1)\, l_i. \tag{3.3}$$

If, in turn,

$$l_{i+1} = \beta\, d'(p'_{i+1},\Xi) \le \alpha\, q = l_i,$$

then one direction is clear, and for the other, (3.2) still holds. The last alternative $l_i = \beta\, d'(p'_i,\Xi) \le \alpha q = l_{i+1}$ is done similarly by (3.3). Thus (ii) holds.

For (iii), note that since

$$\sum_i l_i \le Nq,$$

where N depends only on n and β, the choice $\alpha^{p-1} \le \eta/(2N)$ yields the same bound for all $(l_i/q)^{p-1}$, and consequently

$$t - \tau(x,t) \le \sum_i 2\, l_i^p \le \frac{\eta\, q^{p-1}}{N} \sum_j l_j \le \eta\, q^p.$$

To prove (iv), assume first that α does not affect the chain length. It is straightforward to check that for all pairs of consecutive indices $(i, i+1)$ where both sidelengths l_i are not determined by the temporal dimension (that is, one or both of them are determined by the spatial dimensions); l_i, l_{i+1}, $d(x,\Omega^c)$ and $|\gamma_{y_i y_{i+1}}|$ are all comparable for $x \in \gamma_{y_i y_{i+1}}$. We call subchains with this kind of center points p_i *mixed* and label the corresponding segments of γ by γ_{i_j, i_j+1}. Then

$$\begin{aligned} k(x,z) &= \int_\gamma \frac{1}{d(y,\Omega^c)}\, ds(y) = \sum_i \int_{\gamma_{y_i y_{i+1}}} \frac{1}{d(y,\Omega^c)}\, ds(y) \\ &\ge \sum_{j=1}^{k_x} \int_{\gamma_{i_j, i_j+1}} \frac{1}{d(y,\Omega^c)}\, ds(y) \gtrsim k_x. \end{aligned}$$

Multiplying k_x by 2, we may assume that it controls the actual number of points p_i in mixed subchains.

It remains to estimate the gaps between mixed subchains. These are filled by purely temporal subchains, where the sidelengths of consecutive rectangles are determined by the distances from $\Omega \times \{T\}$. Here the fact $t'_{i+1} = t_i - \frac{3}{4} l_i^p = t'_i - \frac{3}{2} l_i^p$ implies

$$l_{i+1}^p = \beta^p (T - t'_{i+1}) = \beta^p (T - t'_i) + \frac{3}{2}\beta^p l_i^p = \Big(1 + \frac{3}{2}\beta^p\Big) l_i^p =: M l_i^p.$$

Given two temporal subchains with no temporal subchain in between, the successor will start with a rectangle greater than the ending rectangle of the predecessor. Moreover, the last rectangle in a purely temporal subchain is also a starting rectangle for the following mixed subchain. By the convention on k_x, these will be counted to both mixed and temporal chains. Without decreasing the actual number of counted rectangles, we may join the temporal subchains by replacing the starting rectangles of the purely temporal subchains by the ending rectangles of the predecessors.

This new chain of rectangles with centers $\{(y_{i_\iota}, t_{i_\iota})\}_{\iota=0}^k$ satisfies

$$T \gtrsim \sum_{\iota=0}^{k} 2\, l_{i_\iota}^p \ge 2\, l_{i_0}^p\, \frac{M^{k+1} - 1}{M - 1} \gtrsim C\, l_{i_0}^p\, M^{k-C}\,,$$

where all dependencies are on β. Especially,

$$k \lesssim \log \frac{T}{l_{i_0}^p} + 1.$$

Adding now the worst contribution of αq-sided rectangles, we have established

$$k_{x,t} \lesssim k(x,z) + \log \frac{T}{T-t} + \frac{1}{\alpha} + 1.$$

Up to the additional assumption on $\alpha' \geq l_{x,t}$, this is (iv).

To get rid of the assumption $\alpha' q \geq l_{x,y}$, we just start the construction by doubling the sidelength at each step until the choice condition of $l_{y,t}$ becomes active. It can be checked that this does not affect other bounds than the number of rectangles, and here the effect is at worst the claimed $\log(\alpha/\alpha' + 1) + 1$. □

Remark 3.2. Given two parabolic rectangles $R = Q \times (t - L^p, t + L^p)$ and $R' = Q \times (t' - L^p, t' + L^p)$ such that $T \geq t' - t \geq ML$ for some big M (say $M \geq 100$), then R' can be connected to R with a chain $\{R_i\}_{i=1}^k$ satisfying (ii) of Lemma 3.1 (with dimensional constant) and $k \leq CM$ where C is a numerical constant. This is practically done by looking at chains constructed as in Lemma 3.1, but using a constant spatial cube Q, and choosing the midpoint of R_{i+1} asking only that $|\mathrm{pr}_t\, R_i \cap R_{i+1}| \geq L^p/M$ is satisfied (pr_t means the projection on the temporal variable). This gives us flexibility to squeeze or stretch the chain in order to synchronize the endpoint rectangles provided by the previous lemma.

4. A global John–Nirenberg inequality

In this section we will prove one of the main results of this paper, the global John–Nirenberg inequality.

Theorem 4.1. *Let Ω satisfy a quasihyperbolic boundary condition. If u is a function in $\mathrm{PBMO}^\sigma(\Omega \times (0,T))$, then for every $\delta > 0$ there is $c \in \mathbb{R}$ and constants A and B depending on δ, σ, p, n, $\|u\|_{\mathrm{PBMO}^\sigma}$ and the data of Ω, such that*

$$|\Omega \times (\delta q^p, T) \cap \{(u-c)^+ > \lambda\}| \leq A\, e^{-B\lambda}\, |\Omega \times (\delta q^p, T)|.$$

Here q is again the maximal length of quasihyperbolic geodesics.

Proof. Choose $\sigma < \beta^{-1}$ and let $\alpha > 0$ be a constant to be determined later. For each $y \in \Omega$, let $5l_y = \min\{\beta d(y, \Omega^c), \alpha q\}$ and denote $Q_y = Q(y, l_y)$. Using 5-covering lemma, we may extract a countable collection $\mathcal{W}_\alpha = \{Q_i\}_i := \{5Q_{y_i}\}_i$ so that the cubes $\{Q_i\}_i$ cover Ω and $\frac{1}{5}Q_i$ are pairwise disjoint. Moreover, we may ask Q_1 to be centered at z, the distinguished point of Ω. The symbols $\mathcal{W}_\zeta$ will refer to similar constructions with additional size bounds $5l_y \leq \zeta q$. We denote $\delta_0 = \delta q^p$.

First look at a fixed time level $\Omega \times \{\delta_0\}$. We extend every cube Q_i to be an upper fragment of a parabolic rectangle R_i having its lower face on $\Omega \times \{\delta\}$. Using Lemma 3.1, we may construct chains $\mathcal{P}(U_i^+)$ connecting these U_i^+, upper fragments of R_i, to rectangles with spatial projections coinciding with Q_1. These rectangles may, however, be centered somewhere in $\Omega \times (-\infty, 0)$, since in the construction of the parabolic chain, connecting the lower fragments of rectangles with the upper fragments of their successors makes the chain flow down to the past. To deal with this, we choose η in (iii) of Lemma 3.1 to be $(10\sigma)^{-10}\delta$ so that the final rectangle will definitely be in $\Omega \times (\frac{1}{2}\delta_0, T)$ and admissible in the definition of PBMO^σ. This imposes an upper bound b on α (depending only on δ and σ).

Next we slice the cylinder $\Omega \times (\delta_0, T)$ both spatially and temporally. We begin with the time. Denote

$$\tau_j = T - 2^{-j}(T - \delta_0), \text{ as } j \geq 0, \quad \text{and} \quad Z_j = \Omega \times (\tau_j, \tau_{j+2}).$$

The union of these Z_j is included in $\Omega \times (\delta_0, T)$ and their overlap is bounded by 2. When it comes to space, we define

$$\Omega_k = \Omega \cap \{d(x, \Omega^c) < 2^{-k}\}.$$

This lets us partition $\mathcal{W}_\alpha$ so that

$$\mathcal{W}_\alpha^k = \{Q_x \in \mathcal{W}_\alpha : x \in \Omega_k \setminus \Omega_{k+1}\}.$$

Since for each $y \in Q(x, l) \in \mathcal{W}_\alpha^k$ we have that

$$d(y, \Omega^c) \leq |x - y| + d(x, \Omega^c) \leq (\beta + 1)\, d(y, \Omega^c) \leq 2^{-k+1},$$

the inclusion

$$\bigcup_{Q \in \mathcal{W}_\alpha^k} Q \subset \Omega_{k-1}$$

will follow.

Then we cover $\Omega \times (\tau_j, \tau_{j+1})$. These subsets of Z_j will in turn cover the whole space-time cylinder $\Omega \times (\delta_0, T)$. Choose

$$\alpha_j = \min\Big\{b, \frac{\beta}{q}\Big(\frac{T - \delta_0}{2^{j+2}}\Big)^{1/p}\Big\}.$$

Take the cover $\mathcal{W}_{\alpha_j}$, extend its cubes to upper fragments of parabolic rectangles U_i^{j+}. At each spatial Q_i, stack these parabolic fragments pairwise disjointly minimal amount to cover the temporal interval (τ_j, τ_{j+1}). At the future end, the stack will not exceed τ_{j+2}. Label the fragments as $\mathcal{Z}_j = \{U_i^{jk+}\}_{ijk}$.

For each U_i^{jk+} we form the chain of Lemma 3.1 with $\alpha = b$ and $\alpha' = \alpha_j$. It has m_i^j rectangles. We want, however, to make all final rectangles coincide not only spatially but also temporally at, say, $\mathfrak{R}$. In order to do that, we must continue the chain of U_i^{jk} with m_j rectangles. According to Remark 3.2, recalling the choice of η,

and carefully checking the interdependence of j and temporal slicing of $\Omega \times (\delta_0, T)$, we see that this can be done with bound

$$m_j \lesssim \sum_{\iota=0}^{j} \frac{2^{-j}\,(T-\delta_0)}{(\alpha_j\, q)^p} \lesssim \frac{j}{\delta}.$$

Call these continued chains $\mathcal{C}(U_i^{jk+})$.

Now we are in position to prove the claim. Recall that a_R is always a constant from the PBMO$^\sigma$ condition (2.2). Let C_0 be the constant in part (ii) of Lemma 3.1. For a while, we denote $\mathcal{C}(U_i^{jk+}) = \{P_\iota\}_{\iota=1}^{m_i^j+m_j}$. We will use an argument from [22]. By a suitable choice of λ_0 (depending only on $\|u\|_{\mathrm{PBMO}^\sigma}$, β, and the dimension) in John–Nirenberg inequalities (2.3) and (2.4), we get

$$|E^-_{\lambda_0,\iota}| := |U^-_\iota \cap \{(a_{P_\iota} - u)^+ > \lambda_0\}| \le \frac{C_0}{2}\,|U^-_\iota|,$$
$$|E^+_{\lambda_0,\iota+1}| := |U^+_{\iota+1} \cap \{(u - a_{P_{\iota+1}})^+ > \lambda_0\}| \le \frac{C_0}{2}\,|U^+_{\iota+1}|,$$

and

$$|(U^-_\iota \cap U^+_{\iota+1}) \setminus (E^-_{\lambda_0,\iota} \cup E^+_{\lambda_0,\iota+1})| > 0.$$

This indicates that there is $p_\iota \in (U^-_\iota \cap U^+_{\iota+1}) \setminus (E^-_{\lambda_0,\iota} \cup E^+_{\lambda_0,\iota+1})$, and consequently

$$\begin{aligned}(a_{P_1} - a_{\mathfrak{R}})^+ &\le \sum_{\iota=1}^{m_i^j+m_j-1} (a_{P_\iota} - a_{P_{\iota+1}})^+ \\ &\le \sum_{\iota=1}^{m_i^j+m_j-1} (a_{P_\iota} - u(p_\iota))^+ + (u(p_\iota) - a_{P_{\iota+1}})^+ \lesssim_{\lambda_0} m_i^j + m_j.\end{aligned}$$

Now for every $\lambda > 0$, we get

$$\begin{aligned}|U_i^{jk+} \cap \{(u - a_{\mathfrak{R}})^+ > \lambda\}| &\le |U_i^{jk+} \cap \{(u - a_{R_i^{jk}})^+ + (a_{P_\iota} - a_{\mathfrak{R}})^+ > \lambda\}| \\ &\lesssim |U_i^{jk+} \cap \{(u - a_{R_j^{jk}})^+ > \lambda/2\}| \\ &\quad + |U_i^{jk+} \cap \{C(m_i^j + m_j) > \lambda/2\}|. \end{aligned} \tag{4.1}$$

By John–Nirenberg, the first term can be estimated by

$$|U_i^{jk} \cap \{(u - a_{R_i^{jk}})^+ > \lambda/2\}| \lesssim |U_i^{jk}|\, e^{-B\lambda}.$$

Moreover,

$$\sum_{i,j,k} |U_i^{jk}| = \sum_j \sum_{U \in \mathcal{Z}_j} |U_i^{jk}| \lesssim \sum_j |Z_j| \lesssim |\Omega \times (\delta_0, T)|,$$

so

$$\sum_{i,j,k} |U_i^{jk} \cap \{(u - a_{R_i^{jk}})^+ > \lambda/2\}| \lesssim e^{-B\lambda}\,|\Omega \times (\delta_0, T)|. \tag{4.2}$$

We then turn to the second one. Since $U_i^{jk} \subset Z_j$, by Lemma 3.1 we have that

$$m_i^j + m_j \lesssim_\alpha k(y_i, z) + \frac{j}{\delta} + j \log \frac{q}{(T-\delta_0)^{1/p}} + 1. \tag{4.3}$$

Denote

$$|U_i^{jk+} \cap \{k(y_i, z) + 1 > C\lambda\}| + |U_i^{jk+} \cap \{j > C\lambda\}| = I_{ijk} + II_{ijk}.$$

Since Ω satisfies a quasihyperbolic boundary condition, it holds that $|\Omega_k| \lesssim 2^{-\nu k}|\Omega|$ for some $\nu > 0$ (see [5] and [12]). On the other hand, the quasihyperbolic boundary condition itself,

$$k(y_i, z) \lesssim \log \frac{K}{d(y_i, z)},$$

gives that if $Q_i \in \mathcal{W}^k_{\alpha_j}$, then $k(y_i, z) \lesssim 1 + k$. Denoting $t_i^{jk} = |\mathrm{pr}_t\, U_i^{jk+}|$, we may compute (since Q_i have bounded overlap)

$$\begin{aligned} \sum_{i,j,k} I_{ijk} &\le \sum_{i,j,k} t_i^{jk} \cdot |Q_i \cap \{k(y_i, z) + 1 > C\lambda\}| \\ &\lesssim (T-\delta_0) \sum_{l > C\lambda - C'} |\Omega_l| \lesssim (T-\delta_0) \sum_{l > C\lambda - C'} 2^{-\nu l}|\Omega| \lesssim |\Omega \times (\delta_0, T)| e^{-B\lambda}. \end{aligned} \tag{4.4}$$

For the term II_{ijk}, we note that since $\frac{1}{5}Q_i$ are pairwise disjoint

$$\begin{aligned} \sum_{i,j,k} II_{ijk} &= \sum_{j > C\lambda} \sum_{i,k} |U_i^{jk+}| \lesssim \sum_{j > C\lambda} |Z_j| \\ &\lesssim \sum_{j > C\lambda} 2^{-j}\, |\Omega \times (\delta_0, T)| \lesssim |\Omega \times (\delta_0, T)|\, e^{-B\lambda}. \end{aligned} \tag{4.5}$$

Summing over i, j and k in (4.1), and plugging in the estimates (4.2), (4.4) and (4.5), we get the claimed

$$|\Omega \times (\delta q^p, T) \cap \{(u - a_{\mathfrak{R}})^+ > \lambda\}| \le A\, e^{-B\lambda}\, |\Omega \times (\delta q^p, T)|. \qquad \square$$

Note that if $\Omega = Q$ is a Euclidean cube and $T = 2l(Q)^p$, we have a parabolic rectangle. In this simple case, one may compute the bounds coming from the quasihyperbolic boundary condition explicitly (compare to [14]). Replacing the quasihyperbolic geodesics by straight lines, and repeating the previous proof, it will be clear that the dependence on the data of Ω will become a dimensional constant. Thus we get a slightly stronger statement for these domains.

Corollary 4.2. *Let R be a parabolic rectangle centered at (x, τ). If u is in $\mathrm{PBMO}^\sigma(R)$, then for every $\delta > 0$ there is $c \in \mathbb{R}$ and constants A and B depending on δ, PBMO^σ-norm of u, σ, p and n such that*

$$|R_\delta^+ \cap \{(u - c)^+ > \lambda\}| \le A\, e^{-B\lambda}\, |R|.$$

Here, $R_\delta^+ = Q \times (\tau - (1-\delta)L^p, \tau + L^p)$.

5. Consequences of the global inequality

In the classical context, the global John–Nirenberg inequality can be regarded as the strongest local-to-global result of BMO. By this we mean that most other results can be deduced directly from it. In this section we state and prove parabolic analogues of the exponential integrability of functions in BMO (see [20] and [10]) and the equivalence of local and global norms (see [19]).

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^n$ satisfy the quasihyperbolic boundary condition of Definition 2.6. If $u \in \mathrm{PBMO}_\sigma(\Omega \times (0,T))$ with $\sigma \geq 1$, then for every $\delta > 0$ there is $\gamma > 0$ and $c \in \mathbb{R}$ such that*

$$\int_{\Omega\times(\delta,T)} e^{\gamma(u-c)^+}\,d\mu < \infty.$$

Proof. We may write

$$\begin{aligned}\int_{\Omega\times(\delta,T)} e^{\gamma(u-c)^+}\,d\mu &= \int_0^\infty |\{\Omega\times(\delta,T) : e^{\gamma(u-c)^+} > \nu\}|\,d\nu \\ &= |\Omega\times(\delta,T)| + \int_1^\infty |\{\Omega\times(\delta,T) : e^{\gamma(u-c)^+} > \nu\}|\,d\nu,\end{aligned}$$

so it suffices to estimate the second term. By Theorem 4.1 we have

$$|\{\Omega\times(\delta,T) : (u-c)^+ > \lambda\}| \leq A\,e^{-B\lambda}\,|\Omega\times(\delta,T)|.$$

Using this, we get

$$\begin{aligned}&\int_1^\infty |\{\Omega\times(\delta,T) : e^{\gamma(u-c)^+} > \nu\}|\,d\nu \\ &\quad= \int_0^\infty e^\lambda |\{\Omega\times(\delta,T) : (u-c)^+ > \lambda/\gamma\}|\,d\lambda \leq A|\Omega\times(\delta,T)| \int_0^\infty e^{\lambda(1-B/\gamma)}\,d\lambda.\end{aligned}$$

Taking γ small enough, we see that this integral is finite. □

Of course, we have a corresponding result for the negative part of the function. Here the gap between the domain of integration and the temporal boundary will be at the positive end.

Corollary 5.2. *Let $\Omega \subset \mathbb{R}^n$ satisfy the quasihyperbolic boundary condition of Definition 2.6. If $u \in \mathrm{PBMO}_\sigma(\Omega \times (0,T))$ with $\sigma \geq 1$, then for every $\delta > 0$ there is $\gamma > 0$ and $c \in \mathbb{R}$ such that*

$$\int_{\Omega\times(0,T-\delta)} e^{\gamma(u-c)^-}\,d\mu < \infty.$$

Proof. From the proof of Theorem 5.1 it is clear that once we have

$$|\{\Omega\times(0,T-\delta) : (u-c)^- > \lambda\}| \leq A\,e^{-B\lambda}\,|\Omega\times(\delta,T)|, \tag{5.1}$$

the claim will follow. In the proof of Theorem 4.1 we used Lemma 3.1. It gave chains $\{R_i\}_i$ where $U_i^- \cap U_{i+1}^+$ had large measure. The same construction could have been done to the reversed direction, that is, so that $U_i^+ \cap U_{i+1}^-$ would have been a large set. Repeating the proof of Theorem 4.1 with this orientation, we get (5.1), and we are done. □

The next result, originally due to Reimann and Rychener [19] (see also Staples [21]), tells that even if we originally assume the $\mathrm{PBMO}^\sigma(\Omega)$ condition with $\sigma > 1$ we actually have the condition with $\sigma = 1$. In other words, even if our original assumption is absolutely local, we still have complete information about the behaviour of a function up to the boundary.

Theorem 5.3. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an arbitrary domain and let $\sigma > 1$. Let $u \in \mathrm{PBMO}_\sigma(\Omega)$. Then $u \in \mathrm{PBMO}(\Omega)$.*

Proof. Take a parabolic rectangle $R \subset \Omega$. By Corollary 4.2, we have

$$|R_\delta^+ \cap \{(u-c)^+ > \lambda\}| \le A\, e^{-B\lambda}\, |R|.$$

Integrating this, we get

$$\int_{R_\delta} (u-c)^+\, dx = \int_0^\infty |R_\delta^+ \cap \{(u-c)^+ > \lambda\}|\, d\lambda \le |R| \int_0^\infty e^{-B\lambda}\, d\lambda \lesssim |R|.$$

Reasoning as in the previous proof, we see that the corresponding inequality holds for R_δ^-. Moreover, choosing $\delta = 5/4$ and making the final rectangles $\mathfrak{R}$ in the proof of Theorem 4.1 coincide, we ensure that the constants c associated to plus and minus parts coincide. □

6. Integrability of supersolutions

In this section we apply the results about PBMO to partial differential equations. More precisely we study equations of the form

$$\frac{\partial(u^{p-1})}{\partial t} = \operatorname{div} A(x,t,u,Du), \quad 1 < p < \infty, \tag{6.1}$$

where $A(x,t,u,Du)$ is a Caratheodory function (see [6]) satisfying the growth conditions

$$A(x,t,u,Du) \cdot Du \ge C_0\, |Du|^p, \tag{6.2}$$

$$|A(x,t,u,Du)| \le C_1\, |Du|^{p-1}. \tag{6.3}$$

We denote by $L^p(0,T;W^{1,p}(\Omega))$ the space of p-integrable functions on $(0,T)$ having their values in the Sobolev space $W^{1,p}(\Omega)$. More concretely, u is in the parabolic space $L^p(0,T;W^{1,p}(\Omega))$ if

$$u(t,\cdot) \in W^{1,p}(\Omega) \text{ for a.e. } t \in (0,T) \quad \text{and} \quad \int_0^T \|u(t,\cdot)\|^p_{W^{1,p}(\Omega)}\, dt < \infty.$$

Definition 6.1. A function $u \in L^p_{\mathrm{loc}}(0,T;W^{1,p}_{\mathrm{loc}}(\Omega))$ is a *supersolution* to (6.1) if

$$\int_0^T \int_\Omega \Big(A(x,t,u,Du)\cdot D\phi - u^{p-1}\frac{\partial \phi}{\partial t}\Big)\, dx\, dt \geq 0$$

for all non-negative $\phi \in C_c^\infty(\Omega \times (0,T))$.

If the integral in the above definition vanishes for all $\phi \in C_c^\infty(\Omega\times(0,T))$, then u is a weak solution. For our purposes, however, it suffices to consider supersolutions.

Following [11] (see also [25] and [18]) one can show that a positive supersolution f of (6.1) that is bounded away from zero has the negative of its logarithm in PBMO. This will imply that f is globally integrable to some small power $\epsilon > 0$, which is a delicate fact since both the equation (6.1) and the definition of supersolution are very local assumptions, that is, they do not say anything about the behavior of f near $\partial\Omega \times (0,T)$. But still, in terms of integrability, f behaves at worst as a power function.

We will use Lemma 6.1 of [11], which is stated for positive supersolutions of doubly nonlinear equation, that is (6.1) with $A(x,t,Du,u) = |Du|^{p-2}Du$, but as the authors of [11] mention, the assumptions that the proof actually requires are the conditions (6.2) and (6.3).

Lemma 6.2 (Kinnunen–Kuusi [11]). *Let $f > \gamma > 0$ be a supersolution to (6.1) on σR where $\sigma > 1$ and R is a parabolic rectangle. Then there are constants C and C' depending only on C_0, C_1, σ, p and n such that*

$$|\{(x,t) \in R^- : \log f > \lambda + \beta + C'\}| \leq \frac{C}{\lambda^{p-1}}\,|R^-|$$

and

$$|\{(x,t) \in R^+ : \log f < -\lambda + \beta - C'\}| \leq \frac{C}{\lambda^{p-1}}\,|R^+|,$$

where β depends on R and f, and $\lambda > 0$ is arbitrary.

A short calculation shows that this estimate leads to the parabolic BMO-space similar to the one first defined in Moser [18]. The difference here is again the p-scaling of time variable. However, having the John–Nirenberg type inequalities of [3] and Theorem 5.3, we will be able to prove that $-\log f \in \mathrm{PBMO}$.

Lemma 6.3. *Let $f > \gamma > 0$ be a supersolution to (6.1) on $\Omega \times (0,T)$ and let $u = -\log f$. Then $u \in \mathrm{PBMO}$ with norm depending only on C_0, C_1, p and n.*

Proof. Let R be a parabolic rectangle such that $\sigma R \subset \Omega \times (0,T)$. Set

$$b = \min\{(p-1)/2, 1\}.$$

Then a straightforward integration gives

$$\begin{aligned}
&\int_{R^+} (u+\beta)_+^b \, d\mu = b\int_0^\infty \lambda^{b-1} |\{(x,t)\in R^+ : u+\beta > \lambda\}| \, d\lambda \\
&= b\int_0^\infty \lambda^{b-1} |\{(x,t)\in R^+ : -\log f + \beta > (\lambda - C') + C'\}| \, d\lambda \\
&\le |R^+|(1+C')^b + b\int_1^\infty (\lambda + C')^{b-1} |\{(x,t)\in R^+ : -\log f + \beta > \lambda + C'\}| \, d\lambda \\
&\le |R^+|(1+C')^b + Cb|R^+| \int_1^\infty (\lambda + C')^{b-p} \, d\lambda,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{R^-} (u+\beta)_-^b \, d\mu = b\int_0^\infty \lambda^{b-1} |\{(x,t)\in R^- : -u-\beta > \lambda\}| \, d\lambda \\
&= b\int_0^\infty \lambda^{b-1} |\{(x,t)\in R^- : \log f - \beta > (\lambda - C') + C'\}| \, d\lambda \\
&\le |R^-|\,(1+C')^b + b\int_1^\infty (\lambda + C')^{b-1} \, |\{(x,t)\in R^- : \log f - \beta > \lambda + C'\}| \, d\lambda \\
&\le |R^-|\,(1+C')^b + C\,b\,|R^-| \int_1^\infty (\lambda + C')^{b-p} \, d\lambda,
\end{aligned}$$

so u satisfies Definition 2.3:

$$\sup_{\sigma R \subset \Omega\times(0,T)} \Big(\fint_{R^+} (u - a_R)_+^b \, d\mu + \fint_{R^-} (a_R - u)_+^b \, d\mu \Big) < \infty.$$

According to Remark 2.5, we get a John–Nirenberg lemma. Even if this differs from (2.3) and (2.4), all the arguments of the previous sections are still valid up to change of some dimensional constants. Thus we may apply Theorem 5.3 to conclude that $u \in \mathrm{PBMO}(\Omega \times (0,T))$. □

Having established the fact $u \in \mathrm{PBMO}(\Omega \times (0,T))$, global integrability of positive supersolutions follows easily.

Theorem 6.4. *Let $f > \gamma > 0$ be a supersolution to (6.1) on $\Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n$ is a domain satisfying a quasihyperbolic boundary condition. Then for each $\delta > 0$ there is $\epsilon > 0$ depending only on p, n, δ, Ω, C_0 and C_1 such that*

$$\int_{\Omega\times(0,T-\delta)} f^\epsilon \, d\mu < \infty. \tag{6.4}$$

Proof. By Lemma 6.3 $-\log f \in \mathrm{PBMO}(\Omega \times (0,T))$, so by Corollary 5.2 there are $c \in \mathbb{R}$ and $\epsilon > 0$ such that

$$\begin{aligned}
\infty > \int_{\Omega\times(0,T-\delta)} e^{\epsilon(-\log f - c)^-} \, d\mu &\ge \int_{\Omega\times(0,T-\delta)} e^{\epsilon((-\log f)^- - (c)^-)} \, d\mu \\
&= C\int_{\Omega\times(0,T-\delta)\cap\{f>1\}} f^\epsilon \, d\mu + C,
\end{aligned}$$

so the finiteness of the integral in (6.4) follows. □

The assumption $f > \gamma > 0$ coming from Lemma 6.2 can be replaced by the assumption that $f > 0$ is lower semicontinuous. Indeed, for a lower semicontinuous $f > 0$ it actually holds that $f > \gamma_R > 0$ in all parabolic rectangles R. Since Lemma 6.2 provides an estimate uniform in γ, we can actually apply it, and get that $-\log f \in \mathrm{PBMO}^\sigma$. Thus the previous theorem can also be stated in the following form.

Theorem 6.5. *Let $f > 0$ be a lower semicontinuous supersolution to (6.1) on $\Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n$ is a domain satisfying a quasihyperbolic boundary condition. Then for each $\delta > 0$ there is $\epsilon > 0$ depending only on p, n, δ, Ω, C_0 and C_1 such that*

$$\int_{\Omega\times(0,T-\delta)} f^\epsilon \, d\mu < \infty.$$

Another extension is to consider increasing limits of positive supersolutions. In this case it suffices to note that, in addition to finiteness, the integral in (6.4) has an upper bound uniform in the supersolutions f except for the quantity e^c. Indeed, in addition to e^c the only dependence is on $\|-\log f\|_{\mathrm{PBMO}}$, which is determined by the structural constants C_0, C_1, p and n. Thus the global integrability of increasing limits of positive supersolutions follows from the monotone convergence theorem provided that the functions in the sequence are uniformly bounded in $\Omega\times(T-\delta,T)$.

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