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Bi-Lipschitz parts of quasisymmetric mappings

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Abstract. A natural quantity that measures how well a map $f\colon \mathbb{R}^d \to \mathbb{R}^D$ is approximated by an affine transformation is

$$\omega_f(x,r) = \inf_A \Big(\fint_{B(x,r)} \Big(\frac{|f-A|}{|A'|r} \Big)^2 \Big)^{1/2},$$

where the infimum ranges over all non-zero affine transformations A. This is natural insofar as it is invariant under rescaling f in either its domain or image. We show that if $f\colon \mathbb{R}^d \to \mathbb{R}^D$ is quasisymmetric and its image has a sufficient amount of rectifiable structure (although not necessarily $\mathcal{H}^d$-finite), then $\omega_f(x,r)^2 dxdr/r$ is a Carleson measure on $\mathbb{R}^d \times (0,\infty)$. Moreover, this is an equivalence: if this is a Carleson measure, then, in every ball $B(x,r) \subseteq \mathbb{R}^d$, there is a set E occupying 90% of $B(x,r)$, say, upon which f is bi-Lipschitz (and hence guaranteeing rectifiable pieces in the image).

En route, we make a minor adjustment to a theorem of Semmes to show that quasisymmetric maps of subsets of $\mathbb{R}^d$ into $\mathbb{R}^d$ are bi-Lipschitz on a large subset quantitatively.

1. Introduction

1.1. Background

Recall that a non-constant map $f\colon \mathbb{R}^d \to \mathbb{R}^D$ is *η-quasisymmetric* if there is an increasing homeomorphism $\eta\colon (0,\infty) \to (0,\infty)$ such that, for all $x,y,z \in \mathbb{R}^d$ distinct,

$$\frac{|f(x)-f(y)|}{|f(x)-f(z)|} \leq \eta\Big(\frac{|x-y|}{|x-z|} \Big).$$

The goal of this manuscript is to determine when one can detect or guarantee that a quasisymmetric embedding is bi-Lipschitz on some portion of its support.

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Recall that a subset of $\mathbb{R}^D$ is *d-rectifiable* if it may be covered up to a set of Hausdorff d-dimensional measure zero by Lipschitz images of $\mathbb{R}^d$. In general, the image of a quasisymmetric map can be highly irregular. One example can be obtained as follows: by Assouad's theorem [32], for $\alpha \in (0,1)$ and $d \geq 1$, there are $L = L(d,\alpha)$, $D = D(d)$ and an L-bi-Lipschitz mapping of $\mathbb{R}^d$ equipped with the metric $d(x,y) = |x-y|^\alpha$ into $\mathbb{R}^D$. Such a map can easily be checked to be quasisymmetric, and one can show that the image of such a map is purely k-unrectifiable for any $k = 1, 2, \ldots, d$, in the sense that the image has Hausdorff k-measure zero intersection with any Lipschitz image of $\mathbb{R}^k$. The dimension D depends on d and can be quite larger, but see also [7] or David and Toro [16] for particular "snowflake" embeddings of $\mathbb{R}^d$ into $\mathbb{R}^{d+1}$. In light of these examples, a priori conditions that rule out such examples is a natural question.

Most results in this vein typically deal with a codimension 1 situation. Specifically, they deal with functions that are restrictions of a globally defined quasiconformal map $f\colon \mathbb{R}^d \to \mathbb{R}^d$, $d \geq 2$, and give conditions that guarantee $f(\mathbb{S}^{d-1})$ is $(d-1)$-rectifiable. Before discussing these results, we recall the definition of quasiconformality. For $x \in \mathbb{R}^d$, define

$$K_f(x) = \max\left\{\frac{|Df(x)|^d}{J_f(x)}, \frac{J_f(x)}{\inf_{y\in\mathbb{S}^{d-1}} |Df(x)y|^d}\right\}.$$

For a domain $\Omega \subseteq \mathbb{R}^d$, a map $f\colon \Omega \to \mathbb{R}^d$ that is a homeomorphism onto its image with $f \in W^{1,d}_{\mathrm{loc}}(\Omega)$ and $||K_f(x)||_{L^\infty(\Omega)} \leq K < \infty$ is said to be *K-quasiconformal*. A surjective K-quasiconformal map $f\colon \mathbb{R}^d \to \mathbb{R}^d$ is K-quasiconformal if and only if it is η quasisymmetric, where K and η depend on each other (see [40]). Set $A_t = \{x \in \mathbb{R}^d : 1 - t < |x| < 1 + t\}$, $\tilde{K}_f(t) = \operatorname{esssup}\{K_f(x) : x \in A_t\} - 1$. The smaller this quantity is, the closer f is to being conformal in the t-neighborhood of $\mathbb{S}^{d-1}$.

In [1], it is shown that if $d = 2$, $f|_{\mathbb{B}}$ is conformal, and $\int_0^1 \tilde{K}_f(t)^2 \frac{dt}{t} < \infty$, then $f(\mathbb{S})$ is rectifiable. This was subsequently generalized to higher dimensions (although with a stronger condition on the integral) in [31], where it is shown that for $f\colon \mathbb{R}^d \to \mathbb{R}^d$, $d \geq 2$, if $\int_0^1 \tilde{K}_f(t) \frac{dt}{t} < \infty$, then $f(\mathbb{S}^{d-1})$ is rectifiable. By the recent results in [6], it is only necessary that $\int_0^1 \left(\tilde{K}_f(t) \log(1/\tilde{K}_f(t))\right)^2 \frac{dt}{t} < \infty$. They derive this result from a similar result involving not the quasiconformal dilatation, but the quasisymmetry: if

$$\tilde{H}_f(t) = \sup\left\{\frac{|f(x) - f(y)|}{|f(x) - f(z)|} : x, y, z \in A_t \text{ are distinct and } |x - z| \leq |x - y|\right\},$$

then [6] also shows that $\int_0^1 \tilde{H}_f(t)^2 \frac{dt}{t} < \infty$ implies $f(\mathbb{S}^{d-1})$ is rectifiable.

Reverse implications with these quantities are not possible, as the conditions are too stringent: most quasiconformal mappings with $f(\mathbb{S}^{d-1})$ rectifiable do not have $\lim_{t\to 0} K_f(t) = 0$. Moreover, a result due to Astala, Zinsmeister, and MacManus seems to suggest that loosening these conditions will result in only partial rectifiability of the image. Before stating this result, we review some terminology.

Recall that a *bounded C-chord-arc domain* $\mathscr{U} \subseteq \mathbb{C}$ is a scaled copy of a C-bi-Lipschitz image of the unit ball $\mathbb{B}$, and a *K-quasidisk* is any image of the ball under a K-quasiconformal mapping $f\colon \mathbb{C} \to \mathbb{C}$. A *Bishop–Jones domain* $\Omega \subseteq \mathbb{C}$ is a simply connected domain where, for all $z \in \Omega$ there is a C-chord-arc domain $\mathscr{U} \subseteq \Omega$ containing z such that $\mathscr{H}^1(\partial\mathscr{U} \cap \partial\Omega) \geq a\,\mathrm{dist}(z, \partial\Omega) \geq b\mathscr{H}^1(\partial\mathscr{U})$. Also recall that a measure σ on $\mathbb{R}^d \times (0,\infty)$ is a *Carleson measure* on $\mathbb{R}^d \times (0,\infty)$ if there is an infimal constant $C = C(\sigma)$ (the *Carleson norm* of σ) such that for all $x \in \mathbb{R}^d$ and $r > 0$,

$$\sigma(B(x,r) \times (0,r)) \leq C|B(x,r)|.$$

Theorem 1.1 ([3], [28]). *If $\Omega \subseteq \mathbb{C}$ is a quasidisk, then Ω is a Bishop–Jones domain if and only if there is $f\colon \mathbb{C} \to \mathbb{C}$ quasiconformal, such that $f(\mathbb{H}) = \Omega$ (where $\mathbb{H}$ is the upper half plane in $\mathbb{C}$), f is conformal on the lower half plane, and $\frac{\mu_f(x+iy)^2}{y}dxdy$ is a Carleson measure on $\mathbb{R} \times (0,\infty)$ where $\mu_f = f_{\overline{z}}/f_z$.*

Observe that $|\mu_f(z)| = (K_f(z)-1)/(K_f(z)+1)$, so that if f is K-quasiconformal, then $\tilde{K}_f(z)/(K+1) \leq |\mu_f(z)| \leq \tilde{K}_f(z)$, so one is a Carleson measure exactly when the other is. See Chapters 2 and 3 in [2] for these facts about planar quasiconformal maps and their Beltrami coefficients μ, and [34] for similar results.

The above results do not establish whether when $f\colon \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ is bi-Lipschitz on a subset of $\mathbb{R}^d$, only that their images are rectifiable. For showing a map is bi-Lipschitz on a large piece quantitatively, one typically requires some sort of quantitative differentiability result. To explain this notion, we go by way of a classic example due to Dorronsoro.

Theorem 1.2 ([17]). *Let $f \in L^2(\mathbb{R}^d)$. For $x \in \mathbb{R}^d$, $r > 0$, define*

$$\Omega_f(x,r) = \inf_A \left(\fint_{B(x,r)} \left(\frac{|f-A|}{r}\right)^2 \right)^{1/2}$$

where the infimum is over all affine maps $A : \mathbb{R}^d \to \mathbb{R}$. Then $f \in W^{1,2}(\mathbb{R}^d)$ if and only if

$$\Omega(f) := \int_{\mathbb{R}^d} \int_0^\infty \Omega_f(x,r)^2 \, \frac{dr}{r} \, dx < \infty,$$

in which case, $||\nabla f||_2^2 \sim_d \Omega(f)$.

This is not the exact phrasing of his result, and the original theorem is far more general, but this special case has been more than sufficient for many applications. For the reader's convenience, we provide a well-known proof in Section 7.3 of the appendix.

While Rademacher's theorem, for example, says that at almost every $x \in \mathbb{R}^d$ f is approximately an affine function in small balls around the point x, it does not tell us how soon f is within ε, say, of some affine map. Using Dorronsoro's result and Chebyshev's inequality, however, shows that the largest scale $r > 0$ for which $\Omega_f(x,r) < \varepsilon$ can be estimated from below in terms of $||\nabla f||_2$, d, and $\varepsilon > 0$. Results like this (that quantify how soon a function achieves a certain threshold of regularity, or bounds how often it does not) are examples of *quantitative differentiation* or *coarse differentiation*.

Quantitative differentiation results have been used for embedding problems ([9], [27]), geometric group theory ([18]) and the theory of uniform rectifiability (see [11], [25], Lemma 10.11 in [14], IV.2.2 in [15], and the references therein). While the latter results are more concerned with finding out when a function is approximately affine, there are situations involving, say, a metric space [5], or Carnot groups [27], where "affine" is replaced with some other form of regularity.

In [25], for example, the author shows that if f is 1-Lipschitz, then for every $\delta > 0$ one can partition $[0,1]^d$ into sets $G, K_1, \ldots, K_M$, where $M \leq M(\delta)$, such that $\mathscr{H}_d^\infty(f(G)) < \delta$ and f is $\frac{2}{\delta}$-bi-Lipschitz on each K_j. To prove this, one can use something like Theorem 1.2 and a clever algorithm to sort the domain of f into the desired sets $G, K_1, \ldots, K_M$ (see also [12], p. 62). We will not replicate this method, but the condition in our main result will resemble Dorronsoro's theorem. In particular, instead of Ω_f, we will use a similar quantity: define

$$\omega_f(x,r) = \inf_A \left(\fint_{B(x,r)} \left(\frac{|f-A|}{|A'|r} \right)^2 \right)^{1/2}, \tag{1.1}$$

where the infimum is over affine maps $A : \mathbb{R}^d \to \mathbb{R}^D$ with $|A'| \neq 0$. Here, A' is the derivative of the mapping A, so that $A(x) = A'(x) + A(0)$. The appeal of this quantity, as opposed to Ω_f, is that it is invariant under dilations in the domain *and* scaling the function f in its image: if $s, t > 0$ and $b \in \mathbb{R}^d$, then

$$\omega_f(tx+b, tr) = \omega_g(x,r) \quad \text{if } g(y) = sf(tx+b).$$

Thus, if $\omega_f(x,r)$ is small, then f is well-approximated by a *nontrivial* affine map inside $B(x,r)$, even if the image of $f(B(x,r))$ is very small.

In the main result below, much like Theorem 1.1, we do not give a sufficient condition for when the image of f is rectifiable, but when it contains a uniform amount of rectifiable parts within it in a sense we make precise in the following definition.

Definition 1.3. We will say a set Σ contains *big pieces of d-dimensional bi-Lipschitz images* with constants $\kappa > 0$ and $L \geq 1$ (or BPBI(κ, L, d) for short) if, for all $\xi \in \Sigma$ and $s > 0$, there is $E \subseteq B(\xi, s) \cap \Sigma$ with $\mathscr{H}^d(E) \geq \kappa s^d$ and $g\colon E \to \mathbb{R}^d$ L-bi-Lipschitz. We will simply write BPBI(κ, L) if the dimension d is understood from context.

Note that this "big pieces" terminology is already prevalent in the literature (see [14] and [15]), but usually includes the assumption that Σ is Ahlfors regular, meaning that $\mathscr{H}^d(\Sigma \cap B(x,r))$ is comparable to r^d. We emphasize, however, that the sets we will be dealing with will not necessarily be $\mathscr{H}^d$-finite, let alone regular.

We can now state our main result, which obtains a classification of all quasisymmetric mappings with uniformly rectifiable image in terms of the behavior of ω_f, and can be considered as high dimensional analogue of Theorem 1.1:

Theorem 1.4. *Let $f\colon \mathbb{R}^d \to \mathbb{R}^D$ be quasisymmetric, $d \geq 2$. Then the following are equivalent:*

(1) *The measure $\omega_f(x,r)^2 \frac{dxdr}{r}$ is a C-Carleson measure on $\mathbb{R}^d \times (0,\infty)$.*

(2) *For all $\tau > 0$, there is $L > 0$ such that, for all $x \in \mathbb{R}^d$ and $r > 0$, there is $E \subseteq B(x,r)$ such that*

$$|B(x,r)\backslash E| < \tau |B(x,r)| \quad \text{and} \quad \left(\frac{\operatorname{diam} f(B(x,r))}{\operatorname{diam} B(x,r)}\right)^{-1} f|_E \text{ is } L\text{-bi-Lipschitz.}$$

(3) *There are $c, L > 0$ such that for all $x \in \mathbb{R}^d$ and $r > 0$, there is $E \subseteq B(x,r)$ such that*

$$|E| \geq c|B(x,r)| \quad \text{and} \quad \left(\frac{\operatorname{diam} f(B(x,r))}{\operatorname{diam} B(x,r)}\right)^{-1} f|_E \text{ is } L\text{-bi-Lipschitz.}$$

(4) *The set $f(\mathbb{R}^d)$ has BPBI(κ, L).*

The equivalences are quantitative in the sense that, the constants in each item depend (in addition to D and η) only upon those in the other items.

If $d = 1$, then we just have (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

There is no equivalence in the case of $d = 1$ (that is, (4) $\not\Rightarrow$ (1)), since there are quasisymmetric maps of the real line that are uniformly oscillatory at every scale and location. We will give a counter-example in Proposition 2.3.

We also mention that one can construct a single rectifiable piece in the image (or bi-Lipschitz part of f) without using the full strength of the Carleson measure; indeed, we prove a local version of (1)$\Rightarrow$(2) in Theorem 3.21 below.

A similar result appears in [4], where the authors show that if $f\colon \mathbb{R}^D \to \mathbb{R}^D$ is quasisymmetric, $2 \leq d < D$, $\tilde{H}_f(w,t)^2 \frac{dwdt}{t}$ is a Carleson measure on $\mathbb{R}^d \times (0,\infty)$, where

$$\tilde{H}_f(w,t) = \sup\left\{\frac{|f(x)-f(y)|}{|f(x)-f(z)|} : x,y,z \in B(w,t) \text{ are distinct and } |x-z| \leq |x-y|\right\},$$

then $f(\mathbb{R}^d)$ has big pieces of bi-Lipschitz images, though the implication only holds with $d \geq 2$ and does not have a reverse implication. Also, while $\omega_f(x,r)$ is perhaps not as simple or ideal a quantity to compute than K_f and $\tilde{H}_f$ mentioned above, it does handle a broader class of mappings (maps that are not restrictions of maps $f\colon \mathbb{R}^d \to \mathbb{R}^d$ to the sphere $\mathbb{S}^{d-1}$, for example) and, more importantly, classifies those quasisymmetric mappings that have BPBI in their image. Moreover, the advantage in [6] and [4] is that $\tilde{H}_f$ has the monotonicity property that $\tilde{H}_f(x,r) \leq \tilde{H}_f(y,s)$ whenever $B(x,r) \subseteq B(y,s)$, which does not hold for ω_f. On the other hand, [4] has its own unique challenges: the main tool in our paper is Dorronsoro's theorem, for which $\omega_f(x,t)$ is naturally suited, but it is not clear whether we can apply this using only information about the values $\tilde{H}_f(x,r)$.

Our final result in the vein of finding bi-Lipschitz pieces of quasisymmetric maps is a generalization of the following result of Semmes.

Theorem 1.5 ([37]). *Suppose $E \subseteq \mathbb{R}^d$, $d \geq 2$, and $f\colon E \to \mathbb{R}^d$ is η-quasisymmetric for some η. Then $|E| > 0$ if and only if $|f(E)| > 0$.*

While this is a beautiful result, with just a bit more work one can actually achieve a quantitative version that bounds how small we can make $|f(E)|$ in terms of only η, d, and the density of E.

Proposition 1.6. *Let $E \subseteq Q_0 \subseteq \mathbb{R}^d$, $\rho \in (0, 1/2)$, and set $\delta = |E|/|Q_0| > 0$. Let $f\colon E \to \mathbb{R}^d$ be η-quasisymmetric. Then there is $E' \subseteq E$ compact with $|E'| \geq (1-\rho)|E|$ and $\big(\operatorname{diam} f(E')/\operatorname{diam} E'\big)^{-1} f|_{E'}$ is L-bi-Lipschitz for some L depending on η, d, ρ, and δ.*

We will cite several tools from [37], and with them, the modifications required to obtain Proposition 1.6 are not too difficult, hence the above proposition should really be credited to Semmes; in addition to Dorronsoro's theorem, however, it is a cornerstone to our paper, so we find it worth mentioning.

1.2. Outline of proof

Below we indicate where in the paper to find the proofs of each link in the chain of implications implying Theorem 1.4.

(1) ⇒ (2) We prove this in Theorem 3.1 in Section 3.

(2) ⇒ (3) This case is trivial.

(3) ⇒ (4) Although brief, we prove this implication in Theorem 5.1 in Section 5.

(4) ⇒ (1) This is proven in Theorem 6.1 in Section 6.

Section 4 is devoted to showing Proposition 1.6, a prerequisite for Theorem 5.1. Some basic preliminaries and notation are covered in Section 2, although a few tools will appear throughout whose proofs are delayed to the appendix in Section 7.

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2. Preliminaries

2.1. Notation

Many of the techniques and notation in this paper, if not mentioned or proven here, can be found in [23], [30], and [39].

For nonnegative numbers or functions A and B, we will write $A \lesssim B$ to mean $A \leq CB$ where C is some constant, and $A \lesssim_t B$ if C depends on some parameter t.

Similarly, we will write $A \sim B$ if $A \lesssim B \lesssim A$ and $A \sim_t B$ if $A \lesssim_t B \lesssim_t A$. The Euclidean norm will be denoted by $|\cdot|$ and the ball centered at x of radius r by $B(x,r) = \{y : |x-y| \leq r\}$. Let $\Delta(\mathbb{R}^d)$ denote the collection of dyadic cubes in $\mathbb{R}^d$ of the form

$$\Delta(\mathbb{R}^d) = \bigcup_{n\in\mathbb{Z}} \left\{ \prod_{i=1}^{d} [2^n j_i, 2^n(j_i+1)] \subseteq \mathbb{R}^d : (j_1,\dots,j_d) \in \mathbb{Z}^d \right\}$$

and for $Q_0 \in \Delta(\mathbb{R}^d)$, let $\Delta(Q_0)$ the set of dyadic cubes contained in a dyadic cube Q_0. We will simply write $\Delta = \Delta(\mathbb{R}^d)$ if the dimension is clear from the context. For $Q \in \Delta$, set Q^1 to be the *parent* of Q, that is, the smallest dyadic cube properly containing Q, and inductively, for $N > 1$, define Q^N to be the smallest dyadic cube properly containing Q^{N-1} (so Q^N is the *Nth generation ancestor of Q*). We will also refer to any cube R with $R^1 = Q^1$ as a *sibling* of Q. We will denote the side length of a cube Q by $\ell(Q)$ and its center by x_Q. For $\lambda > 0$, λQ will denote the cube with center x_Q and side length $\lambda\ell(Q)$

For a subset $A \subseteq \mathbb{R}^d$, we will let $|A|$ denote the Lebesgue measure of A, A° its interior, ∂A its boundary, and $\mathbb{1}_A$ the indicator function for A (that is, it is exactly one on A and zero on the complement of A). For a Lebesgue measurable function f and a measurable set A of positive measure, we set $⨍_A f = |A|^{-1}\int_A f$. For $\delta > 0$ and $A \subseteq \mathbb{R}^d$, set

$$\mathscr{H}^d_\delta(A) = w_d \inf \left\{ \sum r_i^d : A \subseteq \bigcup B(x_i, r_i), r_i < \delta \right\},$$

where $w_d = |B(0,1)|$ and define the (*spherical*) *d-dimensional Hausdorff measure*

$$\mathscr{H}^d(A) = \lim_{\delta\to 0} \mathscr{H}^d_\delta(A).$$

If $A, B \subseteq \mathbb{R}^d$, we set

$$\operatorname{diam}(A) = \sup\{|x-y| : x, y \in A\},$$
$$\operatorname{dist}(A,B) = \sup\{|x-y| : x \in A, y \in B\},$$

and for $x \in \mathbb{R}^d$,

$$\operatorname{dist}(x,A) = \operatorname{dist}(\{x\}, A).$$

For an affine transformation $A : \mathbb{R}^d \to \mathbb{R}^D$, we will write $A(x) = A'(x) + A(0)$, where A' is a linear transformation (and the derivative of the map A), and we will let $|A'|$ denote its operator norm.

2.2. Basic facts about Ω_f and ω_f

Let $\Omega \subseteq \mathbb{R}^d$ and $f\colon \Omega \to \mathbb{R}^D$ be a locally bounded continuous function. It will be more convenient throughout the paper to work with dyadic versions of ω_f and Ω_f: for $Q \subseteq \Omega$ a cube, define

$$\omega_f(Q) = \inf_A \left(⨍_Q \frac{|f-A|}{|A'| \operatorname{diam} Q}^2 \right)^{1/2} \quad \text{and} \quad \Omega_f(Q) = \inf_A \left(⨍_Q \frac{|f-A|}{\operatorname{diam} Q}^2 \right)^{1/2},$$

where again the infima are over all nonzero affine maps A. We will use the following monotonicity property often and without mention: if $R \subseteq Q$ and $\ell(R) \geq \delta\ell(Q)$, then $\omega_f(R) \leq \delta^{-d}\omega_f(Q)$. This is easily proven using the definition of ω_f.

Moreover, for any cube Q,

$$\omega_f(Q) \leq \frac{1}{2}. \tag{2.1}$$

To see this, let $A_j = jA + f(x_Q)$ where $A : \mathbb{R}^d \to \mathbb{R}^D$ is a fixed nonzero affine map. Then,

$$\begin{aligned}
\omega_f(Q) &\leq \liminf_{j\to\infty} \Big(\fint_Q \Big(\frac{|f - A_j|}{|A_j'| \operatorname{diam} Q} \Big)^2 \Big)^{1/2} \\
&\leq \liminf_{j\to\infty} \Big(\fint_Q \Big(\frac{|f - f(x_Q)|}{j \operatorname{diam} Q} \Big)^2 \Big)^{1/2} + \liminf_{j\to\infty} \Big(\fint_Q \Big(\frac{|A_j - A_j(x_Q)|}{|A_j'| \operatorname{diam} Q} \Big)^2 \Big)^{1/2} \\
&\leq \liminf_{j\to\infty} \frac{\operatorname{diam} f(Q)}{j \operatorname{diam} Q} + \liminf_{j\to\infty} \Big(\fint_Q \Big(\frac{|A_j'||x - x_Q|}{|A_j'| \operatorname{diam} Q} \Big)^2 \Big)^{1/2} \leq 0 + \frac{1}{2}.
\end{aligned}$$

Lemma 2.1. *Let $\delta > 0$. If f is an η-quasisymmetric embedding of a cube $Q \subseteq \mathbb{R}^d$ into $\mathbb{R}^D$, then there is $\varepsilon_1 = \varepsilon_1(\eta, d, \delta) > 0$ so that if*

$$\fint_Q \frac{|f - A|}{|A'| \operatorname{diam} Q} < \varepsilon_1 \tag{2.2}$$

then

$$|f(x) - A(x)| < \delta\, |A'| \operatorname{diam} Q \quad \textit{for } x \in Q.$$

Moreover,

$$(1 - 2\sqrt{d}\delta)\, |A'|\, \ell(Q) \leq \operatorname{diam} f(Q) \leq (1 + 2\sqrt{d}\delta)\, |A'| \operatorname{diam} Q.$$

We postpone the proof to Section 7.2 in the appendix, and now use it to show that the infimum in the definition of $\omega_f(Q)$ is actually achieved by a nonzero affine map if $\omega_f(Q)$ is small enough.

Lemma 2.2. *Let $\eta\colon (0,\infty) \to (0,\infty)$ be an increasing homeomorphism, and $1 \leq d \leq D$ integers. There is $\varepsilon' = \varepsilon'(\eta, d) > 0$ so that if $Q \in \Delta(\mathbb{R}^d)$ and $f\colon Q \to \mathbb{R}^D$ is η-quasisymmetric with $\omega_f(Q) < \varepsilon'$, then there is an affine transformation $A : \mathbb{R}^d \to \mathbb{R}^D$ so that*

$$\omega_f(Q)^2 = \fint_Q \Big(\frac{|f - A|}{|A'| \operatorname{diam} Q} \Big)^2. \tag{2.3}$$

Proof. Assume $\omega_f(Q) < \varepsilon' := \varepsilon_1(\eta, d, d^{-1/2}/2)/2$. Let A_i is a sequence of affine maps such that

$$\fint_Q \Big(\frac{|f - A_i|}{|A_i'| \operatorname{diam} Q} \Big)^2 \to \omega_f(Q)^2.$$

For i large enough, we know

$$\fint_Q \frac{|f-A_i|}{|A_i'|\operatorname{diam} Q}\left(\fint_Q\left(\frac{|f-A_i|}{|A_i'|\operatorname{diam} Q}\right)^2\right)^{1/2} \leq 2\,\omega_f(Q) \leq \varepsilon_1\Big(\eta, d, \frac{1}{2\sqrt{d}}\Big).$$

Hence, by Lemma 2.1, $|A_i'| \sim \operatorname{diam} f(Q)/\operatorname{diam} Q$, so $|A_i'|$ is uniformly bounded above and below. Moreover, there is $x \in Q$ so that

$$|f(x) - A_i(x)| \leq \varepsilon\, |A_i'|\, \omega_f(Q) \operatorname{diam} Q.$$

Hence, the sequence A_i is uniformly bounded on Q and uniformly bi-Lipschitz, and by Arzela–Ascoli, we may pick a subsequence converging uniformly to a nonconstant affine map A on Q satisfying (2.3). □

2.3. A counter example

Here, we show that if $d = 1$, then (4) $\not\Rightarrow$ (1) in Theorem 1.4.

Proposition 2.3. *There is a quasisymmetric map $f\colon \mathbb{R} \to \mathbb{R}$ such that $\omega_f(Q) \gtrsim 1$ for all $Q \in \Delta(\mathbb{R})$ with $\ell(Q) \leq 1$.*

Proof. To see this, let $\mathscr{I}$ be the set of triadic half-open intervals in $[0, 1)$ obtained inductively by taking an interval I already in $\mathscr{I}$, dividing it into three half-open subintervals I_ℓ, I_m, and I_r (the left, middle, and right intervals) of equal size so that I_m is between the other two, and adding these to $\mathscr{I}$. Now let $\rho \in (0, 1/3)$ and μ be the measure on $\mathbb{R}$ satisfying $\mu([0, 1)) = 1$, $\mu(I_\ell) = \mu(I_r) = \rho\mu(I)$ for all $I \in \mathscr{I}$, and for any $n \in \mathbb{Z}$ and $A \subseteq [n, n+1)$, set $\mu(A) = \mu(A - n)$. This is the so-called *Kahane measure* on $\mathbb{R}$ (although not his exact construction in [26]), and is singular with respect to Lebesgue measure. This is a *doubling measure*, meaning there is $C > 0$ such that $\mu(B(x, 2r)) \leq C\mu(B(x, r))$ and singular with respect to Lebesgue measure (see [21] for a proof of these facts).

Define $f\colon \mathbb{R} \to \mathbb{R}$ by setting $f(x) = \mu([0, t])$ for $x \geq 0$ and $\mu([x, 0])$ for $x \leq 0$. It is not hard to show this is an increasing quasisymmetric mapping since μ is doubling (see Remark 13.20 (b) in [23]). For any $Q \in \Delta(\mathbb{R})$ with $\ell(Q) \leq 1$, we may find a triadic interval $I \subseteq 3Q$ of length at least $\ell(Q)/3$, and if a, b are the endpoints of I and a, c of I_ℓ, then

$$|f(a) - f(c)| = \mu(I_\ell) = \rho\,\mu(I) = \rho\,|f(a) - f(b)|.$$

Let $\delta > 0$ and suppose we may find $x \in \mathbb{R}$ and $Q \in \Delta(\mathbb{R})$ with $\ell(Q) \leq 1$ so that $\omega_f(3Q) < \varepsilon_1(\eta, d, \delta)$. We will show this results in a contradiction if $\delta > 0$ is small enough, proving the proposition. By Lemma 2.1, there is A a nonconstant affine map such that

$$|A'| \sim \frac{\operatorname{diam} f(Q)}{\operatorname{diam} Q} = \frac{\mu(Q)}{\operatorname{diam} Q} \tag{2.4}$$

and

$$||f - A||_{L^\infty(3Q)} < \delta\,|A'| \operatorname{diam} 3Q. \tag{2.5}$$

Hence

$$\rho|f(a)-f(b)| = |f(a)-f(c)| \overset{(2.5)}{>} |A(a)-A(c)| - 2\delta|A'|\operatorname{diam} 3Q$$
$$= \frac{1}{3}|A(a)-A(b)| - 2\delta|A'|\operatorname{diam} 3Q \overset{(2.5)}{>} \frac{1}{3}|f(a)-f(b)| - \frac{8}{3}\delta|A'|\operatorname{diam} 3Q.$$

Thus,

$$\Big(\frac{1}{3}-\rho\Big)\mu(I) = \Big(\frac{1}{3}-\rho\Big)|f(a)-f(b)| < \frac{8}{3}\delta|A'|\operatorname{diam} 3Q \overset{(2.4)}{\sim} \delta\,\mu(Q),$$

and since μ is doubling, we know $\mu(Q) \lesssim_\mu \mu(I)$, hence we have

$$\Big(\frac{1}{3}-\rho\Big)\mu(I) \lesssim_\mu \delta\mu(I),$$

which is a contradiction for δ small enough. □

2.4. Dyadic Carleson conditions

Suppose now $f\colon \mathbb{R}^d \to \mathbb{R}^D$ is a quasisymmetric mapping such that $\omega_f(x,r)^2 \frac{dxdr}{r}$ is a Carleson measure, meaning there is an infimal $C>0$ (the Carleson norm of this measure) such that

$$\int_{B(z,t)}\int_0^t \omega_f(x,r)^2\,\frac{dr}{r}\,dx \le C\,|B(z,t)| \quad \text{for } z\in\mathbb{R}^d \text{ and } t>0. \tag{2.6}$$

If $M>1$, (2.6) is quantitatively equivalent to the condition that there is an infimal C_M such that

$$\sum_{Q\subseteq Q_0} \omega_f(MQ)^2\,|Q| \le C_M\,|Q_0| \tag{2.7}$$

for any dyadic cube Q_0. We show this in the following lemma.

Lemma 2.4. *If $M>1$ and either (2.6) or (2.7) hold, then the other holds, and $C\sim_{d,M} C_M$.*

Proof. We will only show this lemma for $M=3$, as the general case is similar, and we will only show $C_M \lesssim_d C$ as the opposite inequality is proven similarly.

Let A be a nonconstant affine map. Then, for $Q\in\Delta$, $x\in Q$, and $r\in[2\operatorname{diam} Q, 4\operatorname{diam} Q]$,

$$\omega_f(3Q)^2 \le \fint_{3Q}\Big(\frac{|f-A|}{|A'|\operatorname{diam} 3Q}\Big)^2 \le \frac{r^2|B(x,r)|}{|3Q|(\operatorname{diam} 3Q)^2}\fint_{B(x,r)}\Big(\frac{|f-A|}{|A'|r}\Big)^2$$
$$\lesssim_d \fint_{B(x,r)}\Big(\frac{|f-A|}{|A'|r}\Big)^2,$$

and infimizing over non constant affine maps A gives

$$\omega_f(3Q)^2 \lesssim_d \omega_f(x,r) \quad \text{for } x\in Q,\ \ r\in[2\operatorname{diam} Q, 4\operatorname{diam} Q].$$

Thus, for any $Q_0 \in \Delta$,

$$\sum_{Q\subseteq Q_0} \omega_f(3Q)^2\,|Q| \lesssim \sum_{Q\subseteq Q_0} \int_Q \int_{2\operatorname{diam} Q}^{4\operatorname{diam} Q} \omega_f(3Q)^2\,\frac{dr}{r}\,dx$$

$$\lesssim_d \sum_{Q\subseteq Q_0} \int_Q \int_{2\operatorname{diam} Q}^{4\operatorname{diam} Q} \omega_f(x,r)^2\frac{dr}{r}\,dx = \sum_{n\geq 0}\sum_{Q^n=Q_0} \int_Q \int_{2^{-n+1}\operatorname{diam} Q_0}^{2^{-n+2}\operatorname{diam} Q_0} \omega_f(x,r)^2\frac{dr}{r}\,dx$$

$$\leq \int_{Q_0}\int_0^{4\operatorname{diam} Q_0} \omega_f(x,r)^2\,\frac{dr}{r}\,dx \leq C\,|B(x_{Q_0}, 4\operatorname{diam} Q_0)| \lesssim_d C\,|Q_0|.$$

□

We can prove a similar relation for Ω_f.

Lemma 2.5. *For $\Omega\subseteq\mathbb{R}^d$, $f\colon\Omega\to\mathbb{R}^D$, and $Q\subseteq\Omega$, define*

$$\Omega_f(Q)^2 = \inf_A \fint_Q \left(\frac{|f-A|}{\operatorname{diam} Q}\right)^2,$$

where the infimum is over all affine maps $A:\mathbb{R}^d\to\mathbb{R}^D$. If $M>1$ and $f\in W^{1,2}(\mathbb{R}^d,\mathbb{R}^D)$, then

$$\sum_{Q\in\Delta} \Omega_f(MQ)^2\,|Q| \sim_{D,M} ||Df||_2.$$

Proof. Note that Theorem 1.2 holds for functions $f\colon\mathbb{R}^d\to\mathbb{R}^D$ (with D not necessarily equal to one) if we replace ∇f with Df. The proof now is similar to Lemma 2.4, so we omit it. □

3. Carleson condition implies f is bi-Lipschitz on a very large set

In this section, we prove the first part of Theorem 1.4 by establishing that (1) implies (2). We state this implication as a theorem below.

Theorem 3.1. *Suppose $f\colon\mathbb{R}^d\to\mathbb{R}^D$ is η-quasisymmetric and $\omega_f(x,r)^2\,\frac{dr}{r}\,dx$ is a Carleson measure. Then for all $\tau>0$ there is $L>1$ such that for all $x\in\mathbb{R}^d$, $r>0$, there is $E\subseteq B(x,r)$ such that $|B(x,r)\backslash E|<\tau|B(x,r)|$ and $\left(\frac{\operatorname{diam} f(B(x,r))}{\operatorname{diam} B(x,r)}\right)^{-1} f|_E$ is L-bi-Lipschitz.*

3.1. Stopping-time regions

The ideas behind this section are taken from the theory of uniform rectifiability (see [14] and [15], for example). Let

$$M = 30000d.$$

We will keep M fixed throughout the rest of Section 3.

Definition 3.2 ([15], I.3.2). A *stopping-time region* $S \subseteq \Delta$ is a collection of cubes such that

(1) all cubes $Q \in S$ are contained in a maximal cube $Q(S) \in S$;

(2) S is *coherent*, meaning $R \in S$ for all $Q \subseteq R \subseteq Q(S)$ whenever $Q \in S$;

(3) for all $Q \in S$, each of its siblings of Q are also in S.

We let $m(S)$ denote the set of minimal cubes of S, i.e. those cubes $Q \in S$ such that there are no cubes $R \in S$ properly contained in Q. We also set

$$z(S) = Q(S) \backslash \bigcup \{Q : Q \in m(S)\}$$

which a the set of points in $Q(S)$ that are contained in infinitely many cubes in S.

For an η-quasisymmetric map $f : \Omega \to \mathbb{R}^D$ defined on a domain $\Omega \subseteq \mathbb{R}^d$ and $Q \in \Delta$, if $MQ \subseteq \Omega$ and $\omega_f(MQ) < \varepsilon'(\eta, d)$, by Lemma 2.2 we may assign to Q an affine map $A_Q : \mathbb{R}^d \to \mathbb{R}^D$ such that

$$\omega_f(MQ)^2 = \fint_{MQ} \left(\frac{|f - A_Q|}{|A'_Q| \operatorname{diam} MQ} \right)^2.$$

Definition 3.3. For $\Omega \subseteq \mathbb{R}^d$ and $f : \Omega \to \mathbb{R}^D$ η-quasisymmetric, $\varepsilon \in (0, \varepsilon'(\eta, d))$, $\tau \in (0,1)$, we will call a stopping-time region S an (ε, τ)-*region* for f if $MQ(S) \subseteq \Omega$ and if for any $Q \in S$,

(1) $\sum_{Q \subseteq R \subseteq Q(S)} \omega_f(MR)^2 < \varepsilon^2$,

(2) $|A'_{Q(S)} - A'_Q| \leq \tau |A'_{Q(S)}|$, and

(3) all siblings of Q in S satisfy (1) and (2).

Note that, if Q is in a (ε, τ)-region S, then (2) implies

$$(1 - \tau)\, |A'_{Q(S)}| \leq |A'_Q| \leq (1 + \tau)\, |A'_{Q(S)}| \quad \text{for all } Q \in S. \tag{3.1}$$

The first major step toward proving Theorem 3.1 is the following.

Theorem 3.4. *Let $\tau \in (0,1)$, $C_M > 0$, and $\eta : (0,\infty) \to (0,\infty)$ be an increasing homeomorphism. There is $\varepsilon_0 = \varepsilon_0(\eta, D, \tau, C_M) > 0$ so that the following holds. If $\Omega \subseteq \mathbb{R}^d$, $f : \Omega \to \mathbb{R}^D$ is η-quasisymmetric, $0 < \varepsilon < \varepsilon_0$, $Q_0 \in \Delta$, $MQ_0 \subseteq \Omega$, and*

$$\sum_{Q \subseteq Q_0} \omega_f(MQ)^2\, |Q| \leq C_M\, |Q_0|\,, \tag{3.2}$$

then we may partition $\Delta(Q_0)$ into a set of "bad" cubes $\mathscr{B}$ and a collection $\mathscr{F}$ of (ε, τ)-stopping time regions so that

$$\sum_{Q \in \mathscr{B}} |Q| \leq \frac{C_M}{\varepsilon^2}\, |Q_0| \tag{3.3}$$

and

$$\sum_{S\in\mathscr{F}} |Q(S)| \le \Big(4+\frac{2^{d+1}C_M}{\varepsilon^2}\Big)\,|Q_0|. \tag{3.4}$$

Proof. Step 1. We first show that for any $Q_1\in\Delta(Q_0)$, if $\omega_f(MQ_1)<\varepsilon$, we may construct a (ε,τ)-region $S(Q_1)$ with $Q(S(Q_1))=Q_1$. First, enumerate the cubes in $\Delta(Q_1)$ as $\{Q_j\}_{j=1}^\infty$ so that $\ell(Q_i)>\ell(Q_j)$ implies $i<j$. Set $S_1=\{Q_1\}$, and for $j>1$, set $S_j=S_{j-1}\cup\{Q_j\}$ if the following hold:

(a) $Q_j^1\in S$,

(b) $\sum_{Q\subseteq R\subseteq Q_1}\omega_f(MR)^2<\varepsilon^2$,

(c) $|A'_{Q_1}-A'_Q|\le\tau|A'_{Q_1}|$, and

(d) all siblings of Q_j in S satisfy the above properties.

Otherwise, set $S_j=S_{j-1}$. Define $S(Q_1)=\bigcup_{j=1}^\infty S_j$. Clearly, it is a stopping-time region and satisfies (1), (2), and (3) in Definition 3.3. Observe that, when constructed in this way, for $Q\in m(S_1)$, there is a child R of Q such that either (1) or (2) fails.

Step 2. Next, we define the sets $\mathscr{B}$ and $\mathscr{F}$. Set

$$\mathscr{B}=\{Q\subseteq Q_0:\omega_f(MQ)\ge\varepsilon\}$$

and enumerate the cubes $\Delta(Q_0)\backslash\mathscr{B}$ as $\{Q(j)\}_{j=1}^\infty$ so that $\ell(Q(j))<\ell(Q(i))$ implies $i<j$. We let $\mathscr{F}=\bigcup_{j=1}^\infty\mathscr{F}_j$ where the sets $\mathscr{F}_j$ are defined inductively as follows: set $\mathscr{F}_1=\{S(Q_1)\}$ and let $\mathscr{F}_{j+1}=\mathscr{F}_j\cup\{S(Q(j))\}$ if $Q(j)\notin S$ for any $S\in\mathscr{F}_j$; otherwise, set $\mathscr{F}_{j+1}=\mathscr{F}_j$. Note that if $\mathscr{F}_{j+1}\ne\mathscr{F}_j$, then $Q(j+1)\in\mathscr{B}$ or in S for some $S\in\mathscr{F}_j$.

Step 3. We now set out to verify (3.3) and (3.4) for the sets $\mathscr{B}$ and $\mathscr{F}$. The first inequality follows easily, since

$$\sum_{Q\in\mathscr{B}}|Q|\le\varepsilon^{-2}\sum_{Q\in\mathscr{B}}\omega_f(MQ)^2\le C_M\,\varepsilon^{-2}\,|Q_0|\,,$$

so now we focus on (3.4). For $S\in\mathscr{F}$, set

$$m_1(S)=\Big\{Q\in m(S):\sum_{Q'\subseteq R\subseteq Q(S)}\omega_f(MR)^2\ge\varepsilon^2\text{ for some child }Q'\subseteq Q\Big\}$$

and

$$\begin{aligned} m_2(S)&=m(S)\backslash m_1(S)\\ &=\Big\{Q\in m(S):\sum_{Q'\subseteq R\subseteq Q(S)}\omega_f(MR)^2<\varepsilon^2\text{ for all children }Q'\subseteq Q\\ &\qquad\text{but }\frac{|A'_{Q'}-A'_{Q(S)}|}{|A'_{Q(S)}|}>\delta\text{ for some child of }Q'\text{ of }Q\Big\}. \end{aligned} \tag{3.5}$$

Also set

$$M_j(S) = \bigcup_{Q \in m_j(S)} Q, \quad j = 1, 2.$$

Then

$$Q(S) = M_1(S) \cup M_2(S) \cup z(S). \tag{3.6}$$

Lemma 3.5. *There is $\upsilon = \upsilon(D) > 0$ so that if*

$$0 < \varepsilon < \varepsilon_0 := \min\left\{\varepsilon'(d,\eta), \upsilon\, C_M^{-1/2}\tau\right\}, \tag{3.7}$$

and S is an (ε, τ)-region S for an η-quasisymmetric map $f\colon \Omega \to \mathbb{R}^D$ where $MQ(S) \subseteq \Omega \subseteq \mathbb{R}^d$, then

$$|M_2(Q)| < \frac{|Q(S)|}{2}. \tag{3.8}$$

Let us assume this lemma and finish the proof of Theorem 3.4. Let

$$\mathscr{F}_1 = \{S \in \mathscr{F} : |z(S)| \geq |Q(S)|/4\}, \tag{3.9}$$

$$\mathscr{F}_2 = \{S \in \mathscr{F} : |M_1(S)| \geq |Q(S)|/4\}. \tag{3.10}$$

Note that the sets $z(S)$ intersect only at the boundaries of dyadic cubes. To see this, observe that if S and S' were such that they intersected in the interior of a cube, then the interiors of $Q(S)$ and $Q(S')$ intersect, so one must be contained in the other. Suppose $Q(S) \subseteq Q(S')$. Then $Q(S)$ is contained inside a minimal cube of S' (since otherwise $Q(S) \in S \cap S' = \emptyset$), but $z(S)$ is the complement of these minimal cubes and so $z(S') \cap Q(S) = \emptyset$, and thus $z(S) \cap z(S') = \emptyset$, a contradiction. Thus, the $z(S)$ intersect only at the boundaries of dyadic cubes, which have measure zero, hence the $z(S)$ are essentially disjoint. Since they are contained in $Q(S)$,

$$\sum_{S \in \mathscr{F}_1} |Q(S)| \leq 4 \sum_{S \in \mathscr{F}_1} |z(S)| \leq 4\,|Q_0|. \tag{3.11}$$

If $Q \in m_1(S)$, there is a child Q' of Q so that

$$\varepsilon^2 \leq \sum_{Q' \subseteq R \subseteq Q(S)} \omega_f(MR)^2 \leq \omega_f(MQ') + \sum_{Q \subseteq R \subseteq Q(S)} \omega_f(MR)^2.$$

If $\omega_f(MQ')^2 < \varepsilon^2/2$, this implies

$$\frac{\varepsilon^2}{2} < \sum_{Q \subseteq R \subseteq Q(S)} \omega_f(MR)^2 \leq \varepsilon^2,$$

and if $\omega_f(MQ')^2 \geq \varepsilon^2/2$, then

$$\varepsilon^2 \geq \sum_{Q \subseteq R \subseteq Q(S)} \omega_f(MR)^2 \geq \omega_f(MQ)^2 \geq 2^{-d}\omega_f(MQ')^2 \geq \frac{\varepsilon^2}{2^{d+1}},$$

so that in any case,

$$\varepsilon^2 \geq \sum_{Q\subseteq R\subseteq Q(S)} \omega_f(MR)^2 \geq \frac{\varepsilon^2}{2^{d+1}} \quad \text{for all } Q \in m_1(S). \tag{3.12}$$

Hence, since the $Q \in m_2(S)$ have disjoint interiors,

$$\begin{aligned}\sum_{S\in\mathscr{F}_2} |Q(S)| &\leq 4 \sum_{S\in\mathscr{F}_2} |M_1(S)| = 4 \sum_{S\in\mathscr{F}_2} \sum_{Q\in m_1(S)} |Q| \\ &\overset{(3.12)}{\leq} \frac{2^{d+1}}{\varepsilon^2} \sum_{S\in\mathscr{F}_2} \sum_{Q\in m_1(S)} \sum_{Q\subseteq R\subseteq Q(S)} \omega_f(MR)^2\,|Q| \\ &= \frac{2^{d+1}}{\varepsilon^2} \sum_{S\in\mathscr{F}_2} \sum_{R\in S} \omega_f(MR)^2 \sum_{\substack{Q\subseteq R \\ Q\in m_1(S)}} |Q| \leq \frac{2^{d+1}}{\varepsilon^2} \sum_{S\in\mathscr{F}_2} \sum_{R\in S} \omega_f(MR)^2|R| \\ &\leq \frac{2^{d+1}}{\varepsilon^2} \sum_{R\subseteq Q_0} \omega_f(MR)^2|R| \leq \frac{2^{d+1}C_M}{\varepsilon^2}\,|Q_0|. \end{aligned} \tag{3.13}$$

By (3.8), $\mathscr{F} = \mathscr{F}_1 \cup \mathscr{F}_2$, so that

$$\sum_{S\in\mathscr{F}} |Q(S)| = \sum_{i=1,2} \sum_{S\in\mathscr{F}_i} |Q(S)| \overset{(3.11)}{\underset{(3.13)}{\leq}} 4|Q_0| + \frac{2^{d+1}C_M}{\varepsilon^2} \leq \left(4 + \frac{2^{d+1}C_M}{\varepsilon^2}\right)|Q_0|.$$

This finishes the proof of Theorem 3.4, so long as we show Lemma 3.5, which will be the focus of the next few sections. □

3.2. Whitney cubes for stopping-time regions

Before attacking Lemma 3.5, we prove some general properties about stopping-time regions. The reader may just want to familiarize themselves with the notation and lemmas, move on to Section 3.4, and return to the actual proofs on second reading. Many of these estimates can be found in Section 8 of [14].

Let S be a stopping-time region as in Definition 3.2. For $x \in \mathbb{R}^d$, define

$$D_S(x) = \inf\{\operatorname{dist}(x,Q) + \operatorname{diam} Q : Q \in S\}.$$

For $Q \in \Delta$, let

$$D_S(Q) = \inf_{x\in Q} D_S(x).$$

Let R_j be the set of maximal dyadic cubes in $\mathbb{R}^d\backslash\overline{z(S)}$ such that

$$\operatorname{diam} R_j \leq \frac{1}{20} D_S(R_j). \tag{3.14}$$

The R_j are essentially Whitney cubes (see Chapter IV in [38]), though rather than having diameter comparable to their distance from some prescribed set (as is usually how a Whitney decomposition is tailored), they have diameter comparable to their "distances" D_S from S (see (3.17) below).

For each R_j, pick $\tilde{Q}_j \in S$ such that

$$\operatorname{dist}(x_{R_j}, \tilde{Q}_j) + \operatorname{diam} \tilde{Q}_j \leq \frac{3}{2} D_S(x_{R_j}). \tag{3.15}$$

Note that since the R_j has positive diameter, $D_S(x_{R_j}) > 0$, so the above makes sense. Next, pick a maximal parent $Q_j \in S$ of $\tilde{Q}_j$ so that

$$\operatorname{diam} Q_j \leq 3\, D_S(R_j). \tag{3.16}$$

Lemma 3.6. *Let S be a stopping-time region, and define R_j and Q_j as in (3.14), (3.15), and (3.16).*

1) *If $x \in R_j$, then*

$$20 \operatorname{diam} R_j \leq D_S(x) \leq 60 \operatorname{diam} R_j \quad \textit{for all } x \in R_j. \tag{3.17}$$

2) *If $2R_i \cap 2R_j \neq \emptyset$, then*

$$\operatorname{diam} R_i \leq 2 \operatorname{diam} R_j. \tag{3.18}$$

3) *The cubes $2R_j$ have bounded overlap, in the sense that*

$$\mathbb{1}_{\mathbb{R}^d \setminus \overline{z(S)}} \leq \sum_j \mathbb{1}_{2R_j} \lesssim_d \mathbb{1}_{\mathbb{R}^d \setminus \overline{z(S)}}. \tag{3.19}$$

4) *The cubes R_j and Q_j are close, in the sense that*

$$\operatorname{dist}(x_{Q_j}, R_j) \leq 180 \operatorname{diam} R_j. \tag{3.20}$$

5) *For all j,*

$$\operatorname{diam} Q_j \leq 180 \operatorname{diam} R_j. \tag{3.21}$$

6) *If $\operatorname{diam} R_j \leq 2 \operatorname{diam} Q(S)$, then*

$$\operatorname{diam} R_j \leq 2 \operatorname{diam} Q_j. \tag{3.22}$$

7) *If $\operatorname{diam} R_j \geq \operatorname{diam} Q(S)/60$, then $Q_j = Q(S)$.*

Proof. 1) The lower bound in (3.17) follows by definition, so we focus on the upper bound. Observe that, since R_j is maximal, that means there is $y \in R_j^1$ so that

$$\operatorname{diam} R_j^1 > \frac{1}{20} D_S(y).$$

Let $x \in R_j$ be the point closest to y. Since D_S is 1-Lipschitz, we have

$$D_S(y) \geq D_S(x) - |x - y| \geq D_S(x) - \operatorname{diam} R_j$$

and because $\operatorname{diam} R_j^1 = 2 \operatorname{diam} R_j$, we have by the maximality of R_j that

$$2 \operatorname{diam} R_j = \operatorname{diam} R_j^1 > \frac{1}{20} D_S(y) \geq \frac{1}{20}(D_S(x) - \operatorname{diam} R_j),$$

and thus

$$D_S(x) \leq 20(2 + \frac{1}{20}) \operatorname{diam} R_j < 60 \operatorname{diam} R_j.$$

2) If $z \in 2R_i \cap 2R_j$ then

$$|x_{R_i} - x_{R_j}| \leq |x_{R_i} - z| + |z - x_{R_j}| \leq \operatorname{diam} R_i + \operatorname{diam} R_j,$$

so that

$$\begin{aligned} 20 \operatorname{diam} R_i &\overset{(3.17)}{\leq} D_S(x_{R_i}) \leq D_S(x_{R_j}) + |x_{R_i} - x_{R_j}| \\ &\leq 60 \operatorname{diam} R_j + \operatorname{diam} R_i + \operatorname{diam} R_j. \end{aligned}$$

Hence,

$$\operatorname{diam} R_i \leq \frac{61}{19} \operatorname{diam} R_j < 4 \operatorname{diam} R_j.$$

Since R_i and R_j are dyadic cubes, $\operatorname{diam} R_i / \operatorname{diam} R_j$ is a power of two, so in fact, $\operatorname{diam} R_i \leq 2 \operatorname{diam} R_j$, which implies (3.18).

3) Note that for any R_i and $z \in z(S)$, there are infinitely many $Q \in S$ containing z, so $D_S(R_i) \leq |y - z| + \operatorname{diam} Q$ for all such Q, and so $D_S(R_i) \leq |y - z|$ for all $z \in z(S)$, and this implies

$$\operatorname{dist}(R_i, z(S)) \geq D_S(R_j) \overset{(3.17)}{\geq} 20 \operatorname{diam} R_i\,,$$

and so we have $2R_i \subseteq \mathbb{R}^d \backslash \overline{z(S)}$. The rest now follows from this and (3.18).

4) For any j, if $z \in Q_j$ is closest to R_j, then

$$\begin{aligned} \operatorname{dist}(x_{Q_j}, R_j) &\leq \operatorname{dist}(z, R_j) + |x_{Q_j} - z| \leq \operatorname{dist}(x_{R_j}, \tilde{Q}_j) + \frac{\operatorname{diam} Q_j}{2} \\ &\overset{(3.15)(3.16)}{\leq} 3D_S(R_j) \overset{(3.17)}{\leq} 180 \operatorname{diam} R_j\,. \end{aligned}$$

5) This follows from (3.17) and (3.16).

6) This is trivial in the case $Q_j = Q(S)$, so we assume $Q_j \neq Q(S)$, in which case, since $Q_j^1 \in S$ and since Q_j is a maximal cube for which $\operatorname{diam} Q_j \leq 3D_S(R_j)$, we have

$$3D_S(R_j) < \operatorname{diam} Q_j^1 = 2 \operatorname{diam} Q_j\,,$$

so that

$$\operatorname{diam} Q_j > \frac{3}{2} D_S(R_j) \overset{(3.17)}{\geq} 30 \operatorname{diam} R_j.$$

7) Observe that if $Q_j \neq Q(S)$, then any cube $Q \in S$ properly containing Q_j satisfies $\operatorname{diam} Q > 3D_S(x_{R_j})$, so in particular, $\operatorname{diam} Q(S) > 3D_S(x_{R_j})$ Thus,

$$\operatorname{diam} R_j \leq \frac{1}{20} D_S(x_{R_j}) < \frac{1}{60} \operatorname{diam} Q(S). \qquad \square$$

Lemma 3.7. *For all i,*

1) $Q_i \subseteq MR_i$.

2) *If* $\operatorname{diam} R_i \leq 2 \operatorname{diam} Q(S)$, *then*

   a) $R_i \subseteq B(x_{Q_i}, M\ell(Q_i)) \subseteq MQ_i$, *and*

   b) *for all j, if $2R_i \cap 2R_j \neq \emptyset$, we have $R_i \subseteq MQ_j$ and*

$$\operatorname{diam} Q_i \sim \operatorname{diam} Q_j \sim \operatorname{diam} R_i \sim \operatorname{diam} R_j. \tag{3.23}$$

Proof. Before beginning the proof, we recall that we chose $M = 30000d$.

1) If $\tilde{Q}_i$ is as in (3.15), then

$$\begin{aligned} \operatorname{dist}(x_{R_i}, Q_i) + \operatorname{diam} Q_i &\overset{(3.16)}{\leq} \operatorname{dist}(x_{R_i}, \tilde{Q}_i) + 3D_S(x_{R_i}) \overset{(3.15)}{\leq} 5D_S(x_{R_i}) \\ &\overset{(3.14)}{\leq} 300 \operatorname{diam} R_i \end{aligned} \tag{3.24}$$

so that

$$Q_i \subseteq B(x_{R_i}, 300 \operatorname{diam} R_i) \subseteq MR_i. \tag{3.25}$$

2) Assume $\operatorname{diam} R_i \leq 2 \operatorname{diam} Q(S)$.

a) By Lemma 3.6,

$$\begin{aligned} R_i \subseteq B(x_{Q_i}, \operatorname{dist}(x_{Q_i}, R_i) + \operatorname{diam} R_i) &\overset{(3.20)}{\subseteq} B(x_{Q_i}, 181 \operatorname{diam} R_i) \\ &\overset{(3.22)}{\subseteq} B(x_{Q_i}, 362\sqrt{d}\ell(Q_i)) \subseteq MQ_i. \end{aligned} \tag{3.26}$$

b) If $2R_i \cap 2R_j \neq \emptyset$, then $\operatorname{dist}(R_i, R_j) \leq \operatorname{diam} R_i + \operatorname{diam} R_j$ and $\operatorname{diam} R_j \leq 2 \operatorname{diam} R_i$ by (3.18), and so

$$\begin{aligned} \operatorname{dist}(x_{Q_j}, R_i) &\leq \operatorname{dist}(x_{Q_j}, R_j) + \operatorname{diam} R_j + \operatorname{dist}(R_j, R_i) \\ &\overset{(3.20)}{\leq} 180 \operatorname{diam} R_j + \operatorname{diam} R_j + (\operatorname{diam} R_i + \operatorname{diam} R_j) \\ &\overset{(3.18)}{\leq} (180 + 1 + 2 + 1) \operatorname{diam} R_j = 184 \operatorname{diam} R_j. \end{aligned}$$

If $\operatorname{diam} R_j \leq 2 \operatorname{diam} Q(S)$, then

$$184 \operatorname{diam} R_j \overset{(3.22)}{\leq} 368 \operatorname{diam} Q_j \quad \text{and} \quad \operatorname{diam} R_i \leq 2 \operatorname{diam} R_j \leq 4 \operatorname{diam} Q_j;$$

if $\operatorname{diam} R_j > 2 \operatorname{diam} Q(S) > \frac{1}{60} \operatorname{diam} Q(S)$, Lemma 3.6 implies $Q_j = Q(S)$, and since $\operatorname{diam} R_i \leq 2 \operatorname{diam} Q(S) = 2 \operatorname{diam} Q_j$ by assumption,

$$184 \operatorname{diam} R_j \overset{(3.18)}{\leq} 368 \operatorname{diam} R_i \leq 736 \operatorname{diam} Q(S) = 736 \operatorname{diam} Q_j,$$

so that in any case, we have

$$\operatorname{dist}(x_{Q_j}, R_i) \le 184 \operatorname{diam} R_j \le 736 \operatorname{diam} Q_j$$

and

$$\operatorname{diam} R_i \le 4 \operatorname{diam} Q_j. \tag{3.27}$$

Hence,

$$\begin{aligned} R_i &\subseteq B(x_{Q_j}, \operatorname{dist}(x_{Q_j}, R_i) + \operatorname{diam} R_i) \\ &\subseteq B(x_{Q_j}, 736 \operatorname{diam} Q_j + 4 \operatorname{diam} Q_j) \subseteq MQ_j. \end{aligned}$$

Furthermore, (3.23) follows since $\operatorname{diam} R_i \sim_d Q_i$ by 1) and 2a), and

$$\operatorname{diam} Q_j \overset{(3.21)}{\lesssim} \operatorname{diam} R_j \overset{(3.18)}{\lesssim} \operatorname{diam} R_i \overset{(3.27)}{\lesssim} \operatorname{diam} Q_j. \qquad \square$$

3.3. Controlling the distances between affine maps

In this section, we show how if ω_f over two intersecting cubes is small, the approximating affine maps in those cubes are approximately the same.

Lemma 3.8. *If A_1 and A_2 are two affine maps and R is any cube, then*

$$|A_1' - A_2'| \lesssim_d \fint_R \frac{|A_1 - A_2|}{\operatorname{diam} R} \tag{3.28}$$

and

$$|A_1(x) - A_2(x)| \lesssim_d \Big(\fint_R \frac{|A_1 - A_2|}{\operatorname{diam} R} \Big) (\operatorname{dist}(x, R) + \operatorname{diam} R) \quad \textit{for all } x \in \mathbb{R}^d. \tag{3.29}$$

Proof. There is $y \in \frac{1}{2}R$ such that

$$|A_1(y) - A_2(y)| \le \fint_{\frac{1}{2}R} |A_1 - A_2| \le 2^d \fint_R |A_1 - A_2|.$$

Without loss of generality, we may assume $y = 0$. Then, since the norm $|||A||| := \fint_{B(0,\ell(R))} \frac{|A(z)|}{\operatorname{diam} R} dz$ is a norm on the set of linear maps, it is comparable to the usual operator norm, and in a way that is independent of $\ell(R)$. Thus,

$$\begin{aligned} |A_1' - A_2'| &\lesssim_d \fint_R \frac{|A_1'(z) - A_2'(z)|}{\operatorname{diam} R} dz \\ &\le \fint_R \frac{|A_1(z) - A_2(z)|}{\operatorname{diam} R} dz + \fint_R \frac{|A_1(0) - A_2(0)|}{\operatorname{diam} R} dz \\ &\le (1 + 2^d) \fint_R \frac{|A_1(z) - A_2(z)|}{\operatorname{diam} R} dz. \end{aligned}$$

Hence, for $x \in \mathbb{R}^d$,

$$\begin{aligned}|A_1(x) - A_1(x)| &\le |A_1'(x) - A_2'(x)| + |A_1(0) - A_1(0)| \\ &\le |A_1' - A_2'||x| + 2^d \fint_R |A_1 - A_2| \\ &\lesssim_d \fint_R \frac{|A_1(z) - A_2(z)|}{\operatorname{diam} R} |x|\, dz + \fint_R \frac{|A_1(z) - A_2(z)|}{\operatorname{diam} R} \operatorname{diam} R\, dz \\ &= \left(\fint_R \frac{|A_1 - A_2|}{\operatorname{diam} R} \right) (|x-y| + \operatorname{diam} R).\end{aligned}$$

□

Lemma 3.9. *Suppose $Q_1, Q_2 \in \Delta$, $f\colon MQ_1 \to MQ_2 \to \mathbb{R}^D$ is an integrable function, $\max_{i=1,2}\{\omega_f(MQ_i)\} < \varepsilon$ and $R \subseteq MQ_1 \cap MQ_2$. Then*

$$|A_{Q_1}' - A_{Q_2}'| \lesssim \varepsilon \left(\frac{\max_i\{|Q_i|\}}{|R|} \right)^{(d+1)/d} \max_{i=1,2}\{|A_{Q_i}'|\} \tag{3.30}$$

and for all $x \in \mathbb{R}^d$,

$$|A_{Q_1}(x) - A_{Q_2}(x)| \lesssim_d \varepsilon \left(\frac{\max_i\{|Q_i|\}}{|R|} \right)^{(d+1)/d} \max_{i=1,2}\{|A_{Q_i}'|\} (\operatorname{dist}(x,R) + \operatorname{diam} R). \tag{3.31}$$

Proof. We estimate

$$\begin{aligned}\fint_R |A_{Q_1} - A_{Q_2}| &\le \sum_{i=1}^2 \fint_R |A_{Q_i} - f| \le \sum_{i=1}^2 \frac{|MQ_i|}{|R|} \fint_{MQ_i} |f - A_{Q_i}| \\ &= \sum_{i=1}^2 \frac{|MQ_i|}{|R|} \omega_f(MQ_i)\, |A_{Q_i}'| \operatorname{diam} MQ_i \\ &< 2M^{d+1} \left(\frac{\max_i\{|Q_i|\}}{|R|} \right)^{(d+1)/d} \operatorname{diam} R \max_{i=1,2}\{|A_{Q_i}'|\}\, \varepsilon.\end{aligned}$$

Now we invoke Lemma 3.8. □

Lemma 3.10. *Let $f\colon \Omega \to \mathbb{R}^D$ and S be an (ε, τ)-region as in Definition 3.3, and let $\{R_i\}$ be as in Lemma 3.6. If $2R_i \cap 2R_j \neq \emptyset$, then*

$$|A_{Q_i}' - A_{Q_j}'| \lesssim_d \varepsilon\, |A_{Q(S)}'| \tag{3.32}$$

and

$$|A_{Q_i}(x) - A_{Q_j}(x)| \lesssim_d \varepsilon\, |A_{Q(S)}'|\, (\operatorname{dist}(x, R_i) + \operatorname{diam} R_i) \tag{3.33}$$

for all $x \in \mathbb{R}^d$.

Proof. Note that if $\min\{\operatorname{diam} R_i, \operatorname{diam} R_j\} \ge \frac{1}{60} \operatorname{diam} Q(S)$, then $Q_i = Q_j = Q(S)$ by Lemma 3.6, and so (3.32) and (3.33) hold trivially.

Otherwise, if $\operatorname{diam} R_i < \frac{1}{60} \operatorname{diam} Q(S) < 2 \operatorname{diam} Q(S)$, then Lemma 3.7 implies $R_i \subseteq MQ_i \cap MQ_j$ and that $\operatorname{diam} Q_i \sim \operatorname{diam} R_i \sim \operatorname{diam} R_j \sim \operatorname{diam} Q_j$. Hence, the lemma follows from Lemma 3.9 and (3.1). □

3.4. Extensions and the proof of Lemma 3.5

Proof of Lemma 3.5. Let f and S be as in Lemma 3.5 and let R_i, Q_i be as in Lemma 3.6. Note that if R_i and R_j are adjacent in the sense that their boundaries intersect, then by (3.18),

$$\frac{9}{8}R_i \cap \frac{1}{2}R_j = \emptyset. \tag{3.34}$$

This and the fact that

$$\sum_j R_j = \mathbb{R}^d \backslash \overline{z(S)}$$

mean we can pick $\{\phi_j\}$ a partition of unity subordinate to the collection $\{2R_j\}$ so that

$$\phi_j \leq \mathbb{1}_{\frac{9}{8}R_j} \leq \mathbb{1}_{2R_j}, \tag{3.35}$$

$$\sum_j \phi_j \equiv \mathbb{1}_{\mathbb{R}^d \backslash \overline{z(S)}}, \quad \sum_j \nabla\phi_j \equiv 0 \quad \text{on } \mathbb{R}^d \backslash \overline{z(S)}. \tag{3.36}$$

and for all indices α,

$$|\partial^\alpha \phi_j| \lesssim_d \operatorname{diam}(R_j)^{-|\alpha|} \mathbb{1}_{2R_j}. \tag{3.37}$$

Observe that by (3.34), we know that

$$\mathbb{1}_{\frac{1}{2}R_i} \leq \phi_i \leq \mathbb{1}_{(\frac{1}{2}R_j)^c} \quad \text{for all } i \neq j. \tag{3.38}$$

Now, define a map $F_S : \mathbb{R}^d \to \mathbb{R}^D$ by

$$F_S(x) = \sum_j A_{Q_j}(x)\, \phi_j(x)\, \mathbb{1}_{\mathbb{R}^d \backslash \overline{z(S)}} + f(x)\, \mathbb{1}_{\overline{z(S)}}. \tag{3.39}$$

The remainder of the proof depends on two lemmas: one showing that DF_S deviates from A'_S a lot near $M_2(S)$, and the other showing that DF_S does not deviate from A'_S much overall, thus $M_2(S)$ must have small measure.

Lemma 3.11. *For $\varepsilon < \varepsilon'(d, \eta)$, $f : \Omega \to \mathbb{R}^D$ and S an (ε, τ)-region as in Lemma* 3.5,

$$||DF_S - A'_{Q(S)}||_2^2 \gtrsim_d |A'_{Q(S)}|^2\, \tau^2\, |M_2(S)|. \tag{3.40}$$

Lemma 3.12. *For $\varepsilon < \varepsilon'(d, \eta)$, $f : \Omega \to \mathbb{R}^D$ and S an (ε, τ)-region as in Lemma* 3.5,

$$\sum_{Q \in \Delta} \Omega_{F_S}(2Q)^2\, |Q| \lesssim_D \varepsilon^2\, |A'_{Q(S)}|^2. \tag{3.41}$$

We will postpone their proofs to Sections 3.5 and 3.6 for now and complete the proof of Lemma 3.5.

By Lemmas 2.5, 3.11, and 3.12, and since $\Omega_{F_S} = \Omega_{F_S - A_{Q(S)}}$,

$$|M_2(S)| \lesssim_d \frac{||DF_S - A'_{Q(S)}||_2^2}{\tau^2 |A'_{Q(S)}|^2} \lesssim_D \frac{\sum_{Q\in\Delta} \Omega_{F_S - A_S}(2Q)^2\,|Q|}{\tau^2|A'_{Q(S)}|^2} \lesssim_D \left(\frac{\varepsilon}{\tau}\right)^2 |Q(S)|,$$

so that for $\upsilon = \upsilon(D) > 0$ small enough, if $\varepsilon < \upsilon\tau$, we can guarantee that $|M_2(S)| < \frac{1}{2}|Q(S)|$. This proves Lemma 3.5, so long as we prove Lemmas 3.11 and 3.12, which will be the focus of the next two sections. □

3.5. Bounding $M_2(S)$ and the proof of Lemma 3.11

Proof of Lemma 3.11. Let $N = \lceil \log_2(40\sqrt{d})\rceil + 2$. If $Q \in m_2(S)$, let R be the dyadic cube containing x_Q such that $R^N = Q$. Note that if $Q' \in S$, then Q' cannot be properly contained in Q since $Q \in m_1(S) \subseteq m(S)$, so either

1. $Q' \supseteq Q$, in which case
$$\operatorname{diam} Q' + \operatorname{dist}(R, Q') \geq \operatorname{diam} Q,$$
2. or $Q' \not\subseteq Q$, in which case Q' and Q have disjoint interiors, and since $R \subseteq Q$, we have
$$\operatorname{diam} Q' + \operatorname{dist}(R,Q') > \operatorname{dist}(R,Q') \geq \ell(Q) - \ell(R) = (1-2^{-N})\ell(Q) > \frac{\ell(Q)}{2}.$$

Thus, if we infimize over all such $Q' \in S$, we get $D_S(R) \geq \ell(Q)/2$. By our choice of N,

$$D_S(R) \geq \frac{\ell(Q)}{2} = 2^{N-1}\ell(R) = 2^{N-1}d^{-1/2}\operatorname{diam} R > 20 \operatorname{diam} R, \tag{3.42}$$

and hence there must be $R_i \supseteq R$. Since $D_S(Q') \leq \operatorname{diam} Q'$ for all $Q' \supseteq Q$ with $Q' \in S$, we know that $R_i \subseteq Q$ (otherwise (3.42) would not hold). Thus

$$2^{-N} \operatorname{diam} Q = \operatorname{diam} R \leq \operatorname{diam} R_i \leq \operatorname{diam} Q. \tag{3.43}$$

By (3.38),

$$F_S(x) = \sum_j A_{Q_j}(x)\phi_j(x) = A_{Q_i}(x) \quad \text{for } x \in \frac{1}{2}R_i. \tag{3.44}$$

Hence,

$$DF_S(y) = A'_{Q_i} \quad \text{for all } y \in \frac{1}{2}R_i. \tag{3.45}$$

Note that $Q, Q_i \in S$, so that $\omega_f(MQ) < \varepsilon$ and $\omega_f(MQ_i) < \varepsilon$. Since $R \subseteq Q \subseteq Q(S)$, we have $R_i \subseteq Q \cap MQ_i$ by Lemma 3.7, and so

$$\operatorname{diam} Q_i \sim \operatorname{diam} R_i \overset{(3.43)}{\sim}_d \operatorname{diam} Q. \tag{3.46}$$

Hence, Lemma 3.9 implies

$$|A'_{Q_i} - A'_Q| \leq C_1\,\varepsilon\,|A'_{Q(S)}|$$

for some $C_1 = C_1(d) > 0$.

Since $Q \in m_2(S)$, by (3.5) we know that there is a child Q'' of Q for which

$$\omega_f(MQ'') < \varepsilon \quad \text{and} \quad |A'_{Q''} - A'_{Q(S)}| > \tau\,|A'_{Q(S)}|. \tag{3.47}$$

Hence, again by Lemma 3.9, there is $C_2 = C_2(d) > 0$ so that

$$|A'_Q - A'_{Q''}| \le C_2\,\varepsilon\,\max\{|A'_Q|, |A'_{Q''}|\}.$$

This means

$$|A'_{Q''}| \le (1 + C_2\varepsilon)\,|A'_Q| \overset{(3.1)}{\le} (1 + C_2\varepsilon)\,(1+\tau)\,|A'_{Q(S)}|$$

since $Q \in S$, so that

$$|A'_Q - A'_{Q''}| \le \underbrace{C_2(1 + C_2\,\varepsilon)(1+\tau)}_{=:C_3}\,\varepsilon\,|A'_{Q(S)}|.$$

Thus, for $\varepsilon < 2\tau^{-1}(C_1 + C_3)$ and $y \in \frac{1}{2}R_i$,

$$\begin{aligned} |DF_S(y) - A'_{Q(S)}| &\overset{(3.44)}{=} |A'_{Q_i} - A'_{Q(S)}| \ge |A'_{Q(S)} - A'_{Q''}| - |A'_{Q''} - A'_Q| - |A'_Q - A'_{Q_i}| \\ &\overset{(3.47)}{\ge} (\tau - (C_3 + C_1)\varepsilon)|A'_{Q(S)}| \ge \frac{\tau}{2}\,|A'_{Q(S)}|. \end{aligned}$$

Hence,

$$\begin{aligned} \int_Q |DF_S(y) - A'_{Q(S)}|^2 dy &\ge \int_{\frac{1}{2}R_i} \left(\frac{\tau}{2}|A'_{Q(S)}|\right)^2 = 2^{-d}\,|R_i|\,\frac{\tau^2}{4}\,|A'_{Q(S)}|^2 \\ &\overset{(3.46)}{\gtrsim_d} |Q|\,\tau^2\,|A'_{Q(S)}|^2. \end{aligned}$$

Thus,

$$\begin{aligned} ||DF_S - A'_{Q(S)}||_2^2 &\ge \sum_{Q \in m_2(S)} \int_Q |DF_S - A'_{Q(S)}|^2 \\ &\gtrsim_d \tau^2 \sum_{Q\in m_2(S)} |Q|\,|A'_{Q(S)}|^2 = \tau^2\,|m_2(S)|\,|A'_{Q(S)}|^2 \end{aligned}$$

□

3.6. The proof of Lemma 3.12

Throughout this section (and its subsections), we have the standing assumption that $0 < \varepsilon < \varepsilon'(d,\eta)$, $\tau \in (0,1)$, S is an (ε,τ)-region for an η-quasisymmetric map $f\colon \Omega \to \mathbb{R}^D$ as in Lemma 3.5, and F_S is constructed as in (3.39).

To estimate (3.41), we divide the sum into three parts:

$$\sum_{Q\in\Delta} \Omega_{F_S}(2Q)^2\,|Q| = \sum_{i=1}^{3} \sum_{Q\in\Delta_i} \Omega_{F_S}(2Q)^2\,|Q|, \tag{3.48}$$

where

$$\Delta_1 = \left\{Q \in \Delta : \tfrac{1}{20} D_S(Q) < \operatorname{diam} Q \leq \operatorname{diam} Q(S)\right\} \supseteq S, \tag{3.49}$$

$$\Delta_2 = \left\{Q \in \Delta : \operatorname{diam} Q \leq \tfrac{1}{20} D_S(Q)\right\} = \textstyle\bigcup_j \{Q \in \Delta : Q \subseteq R_j\}, \tag{3.50}$$

$$\Delta_3 = \left\{Q \in \Delta : \operatorname{diam} Q > \max\{\tfrac{1}{20} D_S(Q), \operatorname{diam} Q(S)\}\right\}. \tag{3.51}$$

We will estimate each one separately over the next three subsections.

3.6.1. $\boldsymbol{\Delta_1}$. In this section, we focus on proving the following lemma.

Lemma 3.13.

$$\sum_{Q \in \Delta_1} \Omega_{F_S}(2Q)^2 \, |Q| \lesssim_d \varepsilon^2 \, |A'_{Q(S)}|^2. \tag{3.52}$$

Proof. We first need a few technical lemmas.

Lemma 3.14. *If $Q \in \Delta_1$ and $2R_j \cap 2Q \neq \emptyset$, then*

$$\operatorname{diam} R_j \leq \operatorname{diam} Q \leq \operatorname{diam} Q(S). \tag{3.53}$$

Proof. The second inequality follows from the definition of Δ_1, so we focus on the first. Let $y \in Q$ be such that

$$D_S(y) = D_S(Q) < 20 \operatorname{diam} Q, \tag{3.54}$$

and let $x \in R_j$ be closest to y. If $z \in 2R_j \cap 2Q$, then

$$|x - y| \leq |x - z| + |z - y| \leq 2 \operatorname{diam} R_j + 2 \operatorname{diam} Q. \tag{3.55}$$

Thus,

$$\begin{aligned} \operatorname{diam} R_j &\leq \frac{1}{20} D_S(x) \leq \frac{1}{20}(D_S(y) + |x - y|) \\ &\overset{(3.54)}{\underset{(3.55)}{\leq}} \frac{1}{20}(20 \operatorname{diam} Q + 2 \operatorname{diam} R_j + 2 \operatorname{diam} Q) = \frac{11}{10} \operatorname{diam} Q + \frac{1}{10} \operatorname{diam} R_j. \end{aligned}$$

A bit of arithmetic shows that

$$\operatorname{diam} R_j \leq \frac{11}{9} \operatorname{diam} Q < 2 \operatorname{diam} Q.$$

Since $\operatorname{diam} R_j / \operatorname{diam} Q$ is an integer power of two, we in fact know $\operatorname{diam} R_j \leq \operatorname{diam} Q$, which proves the lemma. □

Lemma 3.15. *If $Q \in \Delta_1$, then either $Q \in S$ or $Q \supsetneq R_j$ for some R_j with* $120\sqrt{d} \operatorname{diam} R_j \geq \operatorname{diam} Q$.

Proof. Let $Q \in \Delta_1 \backslash S$ so that $D_S(Q)/20 < \operatorname{diam} Q \leq \operatorname{diam} Q(S)$.

Step 1. We first show that Q is not contained in any R_j. If $Q \subseteq R_j$ for some R_j, then for all $x \in Q \subseteq R_j$,

$$\operatorname{diam} Q \leq \operatorname{diam} R_j \overset{(3.17)}{\leq} \frac{1}{20} D_S(x);$$

infimizing over all $x \in Q$, we get $\operatorname{diam} Q \leq \frac{1}{20} D_S(Q)$, a contradiction since $Q \in \Delta_1$.

Step 2. Next, we show there is R_i so that $x_Q \in R_i \subsetneq Q$. If $Q^\circ \cap z(S) \neq \emptyset$, then $Q \subseteq Q(S)$ since $\operatorname{diam} Q \leq \operatorname{diam} Q(S)$, and there exists $z \in Q^\circ \cap z(S)$. Since $z \in z(S)$, there are arbitrarily small cubes in S containing z (otherwise the smallest one would be a minimal cube, implying $z \notin z(S)$), infinitely many of which intersect Q°, so Q contains a cube in S and by the coherence of S, $Q \in S$, a contradiction since we assumed $Q \in \Delta_1 \backslash S$. Hence, we know $Q^\circ \cap z(S) = \emptyset$. Thus, $Q^\circ \subseteq \mathbb{R}^d \backslash \overline{z(S)} = \bigcup R_j$. Since Q is not contained in any R_j, there is an R_i such that $x_Q \in R_i \subsetneq Q$.

Step 3. Now we estimate the size of R_i. Let $Q' \in S$. If $Q' \subseteq Q$, then $Q \subseteq Q(S)$ since $\operatorname{diam} Q \leq \operatorname{diam} Q(S)$, and by the coherence of S, $Q \in S$, a contradiction since $Q \notin S$. Thus, we know $Q' \nsubseteq Q$, so either Q' and Q have disjoint interiors (in which case $\operatorname{dist}(x_Q, Q') \geq \frac{1}{2}\ell(Q)$) or $Q' \supsetneq Q$ (in which case $\operatorname{diam} Q' \geq 2 \operatorname{diam} Q$). Hence,

$$60 \operatorname{diam} R_j \overset{(3.17)}{\geq} D_S(x_Q) = \inf_{Q' \in S}\{\operatorname{dist}(x_Q, Q') + \operatorname{diam} Q'\} \geq \min\{\ell(Q)/2, 2\operatorname{diam} Q\}$$
$$= \frac{\ell(Q)}{2} = \frac{\operatorname{diam} Q}{2\sqrt{d}},$$

which implies the lemma. □

Lemma 3.16. *For $Q \in \Delta_1$, pick a cube $\tilde{Q} \in S$ as follows. If $Q \in S$, set $\tilde{Q} = Q$. Otherwise, let $\tilde{Q} = Q_j$, where R_j is as in Lemma 3.15. Then $2Q \subseteq M\tilde{Q}$ and $\operatorname{diam} \tilde{Q} \leq 180 \operatorname{diam} Q$.*

Proof. The lemma is clearly true if $Q \in S$, since then $\tilde{Q} = Q$, so suppose $Q \notin S$. Since $\operatorname{diam} R_j \leq \operatorname{diam} Q < 2 \operatorname{diam} Q(S)$, by Lemmas 3.6 and 3.15 we have

$$\operatorname{diam} Q \leq 120\sqrt{d} \operatorname{diam} R_j \overset{(3.22)}{\leq} 240\sqrt{d} \operatorname{diam} Q_j \leq 240 d \ell(Q_j)$$

and

$$\operatorname{dist}(x_{Q_j}, Q) \overset{R_j \subseteq Q}{\leq} \operatorname{dist}(x_{Q_j}, R_j) \overset{(3.20)}{\leq} 180 \operatorname{diam} R_j \overset{(3.22)}{\leq} 360 \operatorname{diam} Q_j = 360\sqrt{d}\, \ell(Q_j).$$

Hence, the above two inequalities give

$$2Q \subseteq B(x_{Q_j}, \operatorname{dist}(x_{Q_j}, Q) + \operatorname{diam} 2Q) \subseteq B(x_{Q_j}, (360\sqrt{d} + 240d)\ell(Q_j)) \subseteq MQ_j.$$

For the last part of the lemma, observe that since $R_j \subseteq Q$,

$$\operatorname{diam} \tilde{Q} = \operatorname{diam} Q_j \overset{(3.21)}{\leq} 180 \operatorname{diam} R_j \leq 180 \operatorname{diam} Q.$$ □

We now proceed with the proof of Lemma 3.13. For $Q \in \Delta_1$, let $\tilde{Q} \in S$ be as in Lemma 3.16. Using the inequality $(a+b)^2 \leq 2(a^2+b^2)$, we get

$$\sum_{Q \in \Delta_1} \frac{\Omega_{F_S}(2Q)^2 |Q|}{|A'_{Q(S)}|^2} \leq \sum_{Q \in \Delta_1} \fint_{2Q} \left(\frac{|F_S - A_{Q(S)}|}{|A'_{Q(S)}| \operatorname{diam} 2Q}\right)^2 |Q|$$
$$\leq \frac{1}{2} \sum_{Q \in \Delta_1} \fint_{2Q} \left(\frac{|f - A_{Q(S)}|}{|A'_{Q(S)}| \operatorname{diam} Q}\right)^2 |Q| + \frac{1}{2} \sum_{Q \in \Delta_1} \fint_{2Q} \left(\frac{|f - F_S|}{|A'_{Q(S)}| \operatorname{diam} Q}\right)^2 |Q|. \tag{3.56}$$

We will estimate the two summands separately, starting with the first.

Recall from (3.1) that for $Q \in \Delta_1$, since $\tilde{Q} \in S$ and $\tau \in (0,1)$, we have $|A'_{\tilde{Q}}| \leq 2|A'_{Q(S)}|$, and by Lemma 3.16, we know $2Q \subseteq M\tilde{Q}$ and $\operatorname{diam} \tilde{Q} \leq 180 \operatorname{diam} Q$. Hence $\operatorname{diam} \tilde{Q} \sim \operatorname{diam} Q$ and so

$$\begin{aligned}
&\sum_{Q\in\Delta_1} \fint_{2Q} \left(\frac{|f - A_{Q(S)}|}{|A'_{Q(S)}| \operatorname{diam} Q}\right)^2 |Q| \\
&\quad \leq \sum_{Q\in\Delta_1} \left(\frac{|A'_{\tilde{Q}}| M \operatorname{diam} \tilde{Q}}{|A'_{Q(S)}| \operatorname{diam} Q}\right)^2 \frac{|M\tilde{Q}|}{|2Q|} \fint_{M\tilde{Q}} \left(\frac{|f - A_{\tilde{Q}}|}{|A'_{\tilde{Q}}| \operatorname{diam} M\tilde{Q}}\right)^2 |Q| \\
&\quad \sim_d \sum_{Q\in\Delta_1} \omega_f(M\tilde{Q})^2\, |Q| \leq \sum_{R\in S} \sum_{Q\in\Delta_1 : \tilde{Q} = R} \omega_f(MR)^2\, |Q| \\
&\quad \lesssim_d \sum_{R\in S} \omega_f(MR)^2\, |R|, \qquad (3.57)
\end{aligned}$$

where in the last line we used the fact that if $R \in S$, then the number of cubes $Q \in \Delta_1$ such that $\tilde{Q} = R$ is uniformly bounded by a number depending only on d (since all those cubes Q have size comparable to $\operatorname{diam} R$ and are contained in MR by Lemma 3.16).

Next, since S is a (ε, τ)-region, we have that

$$\sum_{Q\subseteq R\subseteq Q(S)} \omega_f(MR)^2 < \varepsilon^2$$

for all $Q \in S$, thus

$$\begin{aligned}
\sum_{R\in S} \omega_f(MR)^2 |R| &= \int_{Q(S)} \sum_{R\in S} \omega_f(MR)^2\, \mathbb{1}_R \\
&= \int_{z(S)} \sum_{R\in S} \omega_f(MR)^2\, \mathbb{1}_R + \sum_{Q\in m(S)} \int_Q \sum_{R\in S} \omega_f(MR)^2\, \mathbb{1}_R \\
&= \int_{z(S)} \sum_{x\in R\in S} \omega_f(MR)^2\, dx + \sum_{Q\in m(S)} \sum_{Q\subseteq R\in S} \omega_f(MR)^2\, |Q| \\
&< \varepsilon^2\, |z(S)| + \sum_{Q\in m(S)} \varepsilon^2\, |Q| = \varepsilon^2\, |Q(S)|. \qquad (3.58)
\end{aligned}$$

Thus,

$$\sum_{Q\in\Delta_1} \fint_{2Q} \left(\frac{|f - A_{Q(S)}|}{|A'_{Q(S)}| \operatorname{diam} Q}\right)^2 |Q| \overset{(3.57)}{\lesssim_d} \sum_{R\in S} \omega_f(MR)^2 |R| \overset{(3.58)}{<} \varepsilon^2\, |Q(S)|, \qquad (3.59)$$

which shows that the first sum in (3.56) is at most a constant (depending on d) times $\varepsilon^2\, |Q(S)|$.

For the second sum in (3.56), set

$$I_Q = \{j : 2R_j \cap 2Q \neq \emptyset\}.$$

Recall that $\operatorname{supp}\phi_j \subseteq 2R_j$ and by Lemma 3.6 we have $\sum \mathbb{1}_{2R_j} \lesssim_d \mathbb{1}_{\mathbb{R}^d\setminus\overline{z(S)}}$. Hence, by the definition of F_S, Lemma 3.7, the fact that $f = F_S$ on $\overline{z(S)}$, and because $|A'_{Q_j}| \le 2|A'_{Q(S)}|$ by (3.1), we have

$$\begin{aligned}
&\fint_{2Q}\Big(\frac{|f-F_S|}{|A'_{Q(S)}|\operatorname{diam} Q}\Big)^2 \le \fint_{2Q}\Big(\sum_j \frac{|f-A_{Q_j}|}{|A'_{Q(S)}|\operatorname{diam} Q}\phi_j\Big)^2\\
&\overset{(3.19)}{\lesssim_d} \sum_j \fint_{2Q}\Big(\frac{|f-A_{Q_j}|}{|A'_{Q(S)}|\operatorname{diam} Q}\phi_j\Big)^2 \overset{(3.1)}{\lesssim} \frac{1}{|2Q|}\sum_{j\in I_Q}\int_{2R_j}\Big(\frac{|f-A_{Q_j}|}{|A'_{Q_j}|\operatorname{diam} Q}\Big)^2\\
&= \frac{M^2}{|2Q|}\sum_{j\in I_Q}|MQ_j|\fint_{MQ_j}\Big(\frac{|f-A_{Q_j}|}{|A'_{Q_j}|\operatorname{diam} MQ_j}\frac{\operatorname{diam} Q_j}{\operatorname{diam} Q}\Big)^2\\
&= \frac{M^{d+2}}{2^d}\frac{1}{|Q|}\sum_{j\in I_Q}\omega_f(MQ_j)^2\Big(\frac{\operatorname{diam} Q_j}{\operatorname{diam} Q}\Big)^2|Q_j|. \qquad (3.60)
\end{aligned}$$

Recall by Lemma 3.14 that if $Q\in\Delta_1$ and $2R_j\cap 2Q\neq\emptyset$, then $\operatorname{diam} R_j \le \operatorname{diam} Q$. Hence, if $n\ge 0$ and $I_{j,n}$ is the set of such cubes $Q\in\Delta_1$ with $2R_j\cap 2Q\neq\emptyset$ and $\ell(Q) = 2^n\ell(R_j)$, then $\# I_{j,n}\lesssim_d 1$. Thus, for a fixed j,

$$\sum_{\substack{Q\in\Delta_1\\ 3Q\cap 2R_j\neq\emptyset}}\Big(\frac{\operatorname{diam} Q_j}{\operatorname{diam} Q}\Big)^2 \overset{(3.21)}{\le} \sum_{n\ge 0}\sum_{Q\in I_{j,n}}\Big(\frac{180\operatorname{diam} R_j}{\operatorname{diam} Q}\Big)^2 \lesssim_d \sum_{n\ge 0} 2^{-2n}\lesssim 1. \qquad (3.61)$$

Therefore,

$$\begin{aligned}
&\sum_{Q\in\Delta_1}\fint_{2Q}\Big(\frac{|f-F_S|}{|A'_{Q(S)}|\operatorname{diam} Q}\Big)^2|Q| \overset{(3.60)}{\lesssim_d} \sum_{Q\in\Delta_1}\sum_{j\in I_Q}\omega_f(MQ_j)^2\Big(\frac{\operatorname{diam} Q_j}{\operatorname{diam} Q}\Big)^2|Q_j|\\
&= \sum_{\operatorname{diam} R_j\le\operatorname{diam} Q(S)}\omega_f(MQ_j)^2|Q_j|\sum_{\substack{Q\in\Delta_1\\ 2R_j\cap 2Q\neq\emptyset}}\Big(\frac{\operatorname{diam} Q_j}{\operatorname{diam} Q}\Big)^2\\
&\overset{(3.61)}{\lesssim_d} \sum_{\operatorname{diam} R_j\le\operatorname{diam} Q(S)}\omega_f(MQ_j)^2|Q_j|\\
&\le \sum_{R\in S}\#\{j: Q_j = R, \operatorname{diam} R_j\le\operatorname{diam} Q(S)\}\omega_f(MR)^2|R|\\
&\lesssim_d \sum_{R\in S}\omega_f(MR)^2|R| \le \varepsilon^2|Q(S)|. \qquad (3.62)
\end{aligned}$$

where, to get to the last line, we used the fact that if $\operatorname{diam} R_j\le\operatorname{diam} Q(S)$, then Lemma 3.7 implies $\#\{j: Q_j = R, \operatorname{diam} R_j\le Q(S)\}\lesssim_d 1$.

Combining (3.56), (3.59), and (3.62) together, we obtain

$$\sum_{Q\in\Delta_1}\frac{\Omega_{F_S}(2Q)^2|Q|}{|A'_{Q(S)}|^2}\lesssim_d \varepsilon^2|Q(S)|. \qquad \square$$

3.6.2. Δ_2. In this section, we will focus on proving the following lemma.

Lemma 3.17.

$$\sum_{Q\in\Delta_2} \Omega_{F_S}(2Q)^2\,|Q| \lesssim_d \varepsilon^2\,|A_{Q(S)}|^2\,|Q(S)|. \tag{3.63}$$

The main idea is that near a cube $Q\in\Delta_2$, F_S is smooth and so we can get better control of Ω_{F_S} using Taylor's theorem.

Lemma 3.18. *For all j,*

$$\frac{1}{|R_j|}\sum_{Q\subseteq R_j} \Omega_{F_S}(2Q)^2\,|Q| \lesssim_d \varepsilon^2\,|A'_{Q(S)}|^2. \tag{3.64}$$

Proof. For normed vector spaces spaces U,V, let $\mathscr{L}(U,V)$ denote the set of bounded linear transformations from U into V and write $\mathscr{L}(U)=\mathscr{L}(U,U)$. Then $\mathscr{L}(U,V)$ is also a normed space with the operator norm, which we will also denote $|\cdot|$. For vectors $u,v\in\mathbb{R}^d$, $u\otimes v\in\mathscr{L}(\mathbb{R}^d)$ is the linear transformation defined by $(u\otimes v)(x)=\langle v,x\rangle\,u$; for $A\in\mathbb{R}^d$, $A\otimes v, v\otimes A\in\mathscr{L}(\mathbb{R}^d,\mathscr{L}(\mathbb{R}^d))$ are the linear transformations $(A\otimes v)(x)=\langle v,x\rangle\,A$ and $(v\otimes A)(x)=v\otimes(A(x))$ respectively.

Let $y\in\mathbb{R}^d\backslash\overline{z(S)}$. Since $F_S|_{\mathbb{R}^d\backslash\overline{z(S)}}$ is smooth,

$$|D^2F_S(y)| = \sup_{|u|=|v|=1}\left|\left(\sum_{m,n=1}^{d} u_m v_n \frac{\partial^2 F_{S,l}}{\partial x_m\partial x_n}(y)\right)_{l=1}^{D}\right|, \tag{3.65}$$

where $F_{S,l}$ denotes the lth component of the vector function F_S and $D^2F_S(y)\in\mathscr{L}(\mathbb{R}^d,\mathscr{L}(\mathbb{R}^d))$ is the derivative of the map $y\mapsto DF_S(y)\in\mathscr{L}(\mathbb{R}^d,\mathbb{R}^D)$ at y, (so above, $|\cdot|$ also denotes this operator norm). Let R_i be such that $y\in R_i$. Then, if A denotes the first order Taylor approximation to F_S at $x=x_Q$, then

$$\begin{aligned}
\Omega_{F(S)}(2Q)\,\mathrm{diam}\,2Q &\le \sup_{\substack{|u|=1\\ y\in 2Q}} u\cdot(F_S(y)-A(y)) = \sup_{\substack{|u|=1\\ y\in 2Q}}\sum_{l=1}^{D} u_l(F_{S,l}(y)-A(y))\\
&= \sup_{\substack{|u|=1\\ y\in 2Q}}\sum_{l=1}^{D}\int_0^1\sum_{m=1}^{d} u_l\frac{\partial(F_{S,l}-A)}{\partial x_m}(x+t(y-x))(y_m-x_m)dt\\
&= \sup_{\substack{|u|=1\\ y\in 2Q}}\sum_{l=1}^{D}\int_0^1\int_0^1\sum_{m,n=1}^{d} tu_l\frac{\partial^2 F_{S,l}}{\partial x_m\partial x_n}(x+st(y-x))(y_m-x_m)(y_n-x_n)\,dt\,ds\\
&\le \sup_{y\in 2Q}\int_0^1\int_0^1 t\left|\left(\sum_{m,n=1}^{d}\frac{\partial^2 F_{S,l}}{\partial x_m\partial x_n}(x+st(y-x))(y_m-x_m)(y_n-x_n)\right)_{l=1}^{D}\right|dt\,ds\\
&\overset{(3.65)}{\le} (\mathrm{diam}\,Q)^2\sup_{y\in 2Q}|D^2F_S(y)|. \qquad (3.66)
\end{aligned}$$

For $y \in R_i$, if $\phi_j(y) \neq 0$, then y is also in $2R_j$, and so we may use Lemma 3.10 and the fact that $\partial^\alpha \phi_j \lesssim_{d,\alpha} (\operatorname{diam} R_j)^{-|\alpha|}$ to estimate

$$\begin{aligned}
|D^2 F_S(y)| &= \Big| \sum_j \Big(2A'_{Q_j} \otimes \nabla\phi_j(y) + A_{Q_j}(y) \otimes D^2\phi_j(y)\Big) \Big| \\
&\overset{(3.36)}{=} \Big| \sum_j \Big(2(A'_{Q_j} - A'_{Q_i}) \otimes \nabla\phi_j(y) + (A_{Q_j}(y) - A_{Q_i}(y)) \otimes D^2\phi_j(y)\Big) \Big| \\
&\overset{(3.32),(3.33)}{\lesssim_d} \sum_j \big(\varepsilon|A'_{Q(S)}||\nabla\phi_j(y)| + \varepsilon|A'_{Q(S)}| \operatorname{diam} Q_i |D^2\phi_j(y)|\big) \\
&\overset{(3.37)}{\lesssim_d} \sum_{y\in 2R_j} \Big(\frac{\varepsilon|A'_{Q(S)}|}{\operatorname{diam} R_j} + \frac{\varepsilon|A'_{Q(S)}| \operatorname{diam} Q_i}{(\operatorname{diam} R_i)^2}\Big) \overset{(3.19),(3.21)}{\lesssim}{}_d \frac{\varepsilon\,|A'_{Q(S)}|}{\operatorname{diam} R_i}.
\end{aligned} \tag{3.67}$$

Thus,

$$\begin{aligned}
\sum_{Q\subseteq R_i} \Omega_{F_S}(2Q)^2\,|Q| &\overset{(3.66)}{\leq} \sum_{Q\subseteq R_i} (\operatorname{diam} Q)^2 \Big(\sup_{y\in 2Q} |D^2F_S(y)|\Big)^2 |Q| \\
&\overset{(3.67)}{\lesssim_d} \sum_{Q\subseteq R_i} \Big(\frac{\varepsilon|A'_{Q(S)}| \operatorname{diam} Q}{\operatorname{diam} R_i}\Big)^2 |Q| \\
&= \varepsilon^2 |A'_{Q(S)}|^2 \sum_{n=0}^{\infty} \sum_{\substack{Q\subseteq R_i \\ \ell(Q)=2^{-n}\ell(R_i)}} 2^{-2n}\,|Q| \lesssim \varepsilon^2\,|A'_{Q(S)}|^2\,|R_i|\,.
\end{aligned}$$

□

Define

$$B_S = B(x_{Q(S)}, 3 \operatorname{diam} Q(S)).$$

Lemma 3.19. *If* $\operatorname{dist}(x, Q(S)) \geq 2 \operatorname{diam} Q(S)$*, then* $F_S(x) = A_{Q(S)}(x)$*. In particular, if* $2Q \cap B_S = \emptyset$*, then* $\Omega_{F_S}(2Q) = 0$.

Proof. Let $x \in R_i$. If $\operatorname{dist}(x, Q(S)) \geq 2 \operatorname{diam} Q(S)$, then

$$\operatorname{diam} R_i \overset{(3.17)}{\geq} \frac{1}{60} D_S(x) \geq \frac{1}{60} \operatorname{dist}(x, Q(S)) \geq \frac{1}{30} \operatorname{diam} Q(S),$$

so if $x \in 2R_j$, then $2R_i \cap 2R_j \neq \emptyset$, and

$$\operatorname{diam} R_j \overset{(3.18)}{\geq} \frac{1}{2} \operatorname{diam} R_i \geq \frac{1}{60} \operatorname{diam} Q(S),$$

hence $Q_j = Q(S)$ by Lemma 3.6.

Since this holds for all j with $2R_j \ni x$, we know that

$$F_S(x) = \sum_j A_{Q_j}(x)\phi_j(x) = A_{Q(S)}(x) \text{ if } \operatorname{dist}(x, Q(S)) \geq 2 \operatorname{diam} Q(S).$$

If $2Q \cap B_S = \emptyset$, then $\operatorname{dist}(x, Q(S)) \geq 2 \operatorname{diam} Q(S)$ for all $x \in 2Q$, hence $F_S|_{2Q} \equiv A_{Q(S)}$, so that $\Omega_{F_S}(2Q) = 0$. □

Proof of Lemma 3.17. We estimate

$$\sum_{Q\in\Delta_2}\Omega_{F_S}(2Q)^2|Q| = \sum_j\sum_{Q\subseteq R_j}\Omega_{F_S}(2Q)^2|Q| = \sum_{2R_j\cap B_S\neq\emptyset}\sum_{Q\subseteq R_j}\Omega_{F_S}(2Q)^2|Q|$$
$$\overset{(3.64)}{\lesssim_d} \varepsilon^2|A'_{Q(S)}|^2\sum_{2R_j\cap B_S\neq\emptyset}|R_j|.$$

The lemma will follow from the previous inequality once we verify

$$\sum_{2R_j\cap B_S\neq\emptyset}|R_j|\lesssim_d|Q(S)|. \tag{3.68}$$

If $2R_j\cap B_S\neq\emptyset$, then $\operatorname{dist}(2R_j,Q(S))\leq 3\operatorname{diam}Q(S)$, so that

$$\begin{aligned}\operatorname{diam}2R_j = 2\operatorname{diam}R_j &\overset{(3.17)}{\leq}\frac{1}{10}D_S(R_j)\leq\frac{1}{10}(\operatorname{dist}(R_j,Q(S))+\operatorname{diam}Q(S))\\ &\leq\frac{1}{10}(\operatorname{diam}R_j+\operatorname{dist}(2R_j,Q(S))+\operatorname{diam}Q(S))\\ &\leq\frac{1}{10}(\operatorname{diam}R_j+4\operatorname{diam}Q(S))=\frac{1}{20}\operatorname{diam}2R_j+\frac{2}{5}\operatorname{diam}Q(S),\end{aligned}$$

which implies

$$\operatorname{diam}2R_j\leq\frac{20}{19}\cdot\frac{2}{5}\operatorname{diam}Q(S)<\operatorname{diam}Q(S)\quad\text{if } 2R_j\cap B_S\neq\emptyset. \tag{3.69}$$

Hence,

$$2R_j\subseteq B(x_{Q(S)},3\operatorname{diam}Q(S)+\operatorname{diam}2R_j)\subseteq B(x_{Q(S)},4\operatorname{diam}Q(S))\subseteq 2B_S.$$

This and the disjointness of the R_j imply

$$\sum_{2R_j\cap B_S\neq\emptyset}|R_j|\leq|2B_S|\lesssim_d|Q(S)|,$$

which proves (3.68). □

3.6.3. $\boldsymbol{\Delta_3}$. Finally, we estimate the third sum in (3.48).

Lemma 3.20.

$$\sum_{Q\in\Delta_3}\Omega_{F_S}(2Q)^2|Q|\lesssim_d\varepsilon^2|A'_{Q(S)}|^2|Q(S)|. \tag{3.70}$$

Proof. Again, set $B_S=B(x_{Q(S)},3\operatorname{diam}Q(S))$. For $n\geq 0$, let

$$B_n=\{Q\in\Delta_3: 2Q\cap B_S\neq\emptyset, \ell(Q)=2^n\ell(Q(S))\}.$$

Then by Lemma 3.19,

$$\sum_{Q\in\Delta_3} \Omega_{F_S}(2Q)^2\,|Q| = \sum_{\substack{Q\in\Delta_3\\ 2Q\cap B_S\neq\emptyset}} \Omega_{F_S}(2Q)^2\,|Q| \le \sum_{\substack{Q\in\Delta_3\\ 2Q\cap B_S\neq\emptyset}} \fint_{2Q}\Big(\frac{|F_S - A_{Q(S)}|}{\operatorname{diam} 2Q}\Big)^2 |Q|$$
$$= 2^{-d-2}\sum_{n\ge 0}\sum_{Q\in B_n}\int_{2Q}\Big(\frac{|F_S - A_{Q(S)}|}{\operatorname{diam} Q}\Big)^2 \lesssim_d \sum_{n\ge 0}\int_{\mathbb{R}^d}\Big(\frac{|F_S - A_{Q(S)}|}{2^n \operatorname{diam} Q(S)}\Big)^2.$$

We claim that

$$\int_{\mathbb{R}^d}\Big(\frac{|F_S - A_{Q(S)}|}{\operatorname{diam} Q(S)}\Big)^2 \lesssim_d \varepsilon^2\,|A'_{Q(S)}|^2\,|Q(S)|, \tag{3.71}$$

after which the lemma will follow from

$$\sum_{Q\in\Delta_3} \Omega_{F_S}(2Q)^2\,|Q| \lesssim_d \sum_{n\ge 0}\int_{\mathbb{R}^d}\Big(\frac{|F_S - A_{Q(S)}|}{2^n \operatorname{diam} Q(S)}\Big)^2$$
$$\overset{(3.71)}{\lesssim_d} \sum_{n\ge 0} 2^{-2n}\,\varepsilon^2\,|A'_{Q(S)}|^2\,|Q(S)| \lesssim \varepsilon^2\,|A'_{Q(S)}|^2\,|Q(S)|.$$

Now we prove (3.71). By Lemma 3.19 and the L^2 triangle inequality,

$$\Big(\int\Big(\frac{|F_S - A_{Q(S)}|}{\operatorname{diam} Q(S)}\Big)^2\Big)^{1/2} = \Big(\int_{B_S}\Big(\frac{|F_S - A_{Q(S)}|}{\operatorname{diam} Q(S)}\Big)^2\Big)^{1/2} \tag{3.72}$$
$$\le \Big(\int_{B_S}\Big(\frac{|F_S - f|}{\operatorname{diam} Q(S)}\Big)^2\Big)^{1/2} + \Big(\int_{B_S}\Big(\frac{|f - A_{Q(S)}|}{\operatorname{diam} Q(S)}\Big)^2\Big)^{1/2}.$$

We will estimate the two parts separately. The second part we may bound as follows:

$$\int_{B_S}\Big(\frac{|f - A_{Q(S)}|}{\operatorname{diam} Q(S)}\Big)^2 \le |A'_{Q(S)}|^2 M^2 |MQ(S)| \fint_{MQ(S)}\Big(\frac{|f - A_{Q(S)}|}{|A'_{Q(S)}|\operatorname{diam} MQ(S)}\Big)^2$$
$$= |A'_{Q(S)}|^2\,M^{d+2}\,|Q(S)|\,\omega_f(MQ(S))^2 < |A'_{Q(S)}|^2\,M^{d+2}\,|Q(S)|\,\varepsilon^2,$$

since $Q(S)\in S$ and hence $\omega_f(MQ(S)) < \varepsilon$ by definition. For the first part of (3.72), recall that if $2R_j\cap B_S\neq\emptyset$, then (3.69) implies $\operatorname{diam} R_j < \operatorname{diam} Q(S)$, so Lemma 3.7 implies $R_j\subseteq MQ_j$. This, Lemma 3.19, and the fact that $\operatorname{supp}\phi_j\subseteq 2R_j$ (which have bounded overlap by Lemma 3.6) imply

$$\int_{B_S}\Big(\frac{|F_S - f|}{\operatorname{diam} Q(S)}\Big)^2 \overset{(3.19)}{\lesssim_d} \sum_j \int_{B_S}\Big(\frac{|A_{Q_j} - f|}{\operatorname{diam} Q(S)}\phi_j\Big)^2$$
$$\le \sum_{2R_j\cap B_S\neq\emptyset} |A'_{Q_j}|^2\Big(\frac{\operatorname{diam} Q_j}{\operatorname{diam} Q(S)}\Big)^2 M^2 |MQ_j| \fint_{MQ_j}\Big(\frac{|A_{Q_j} - f|}{|A'_{Q_j}|\operatorname{diam} MQ_j}\Big)^2$$
$$= M^{d+2}\sum_{2R_j\cap B_S\neq\emptyset} |A'_{Q_j}|^2\Big(\frac{\operatorname{diam} Q_j}{\operatorname{diam} Q(S)}\Big)^2 |Q_j|\,\omega_f(MQ_j)^2. \tag{3.73}$$

Next, recall from (3.21), the definition of Δ_3, and (3.69) that if $2R_j \cap B_S \neq \emptyset$, then

$$\operatorname{diam} Q(S) \overset{\Delta_3}{<} \operatorname{diam} Q_j \overset{(3.21)}{\leq} 180 \operatorname{diam} R_j \overset{(3.69)}{<} 180 \operatorname{diam} Q(S). \tag{3.74}$$

Moreover, since $Q_j \in S$, we know $\omega_f(MQ_j) < \varepsilon$ and $|A'_{Q_j}| \leq (1+\tau)|A'_{Q(S)}|$. These facts and (3.73) imply that

$$\int_{B_S} \left(\frac{|F_S - f|}{\operatorname{diam} Q(S)}\right)^2 \lesssim_d \sum_{2R_j \cap B_S \neq \emptyset} |A'_{Q(S)}|^2 \varepsilon^2 |Q_j| \leq 180^d \sum_{2R_j \cap B_S \neq \emptyset} |A'_{Q(S)}|^2 \varepsilon^2 |R_j|$$
$$\overset{(3.68)}{\underset{(3.74)}{\lesssim_d}} |A'_{Q(S)}|^2 \varepsilon^2 |Q(S)|. \qquad \square$$

3.7. Finding a bi-Lipschitz part

In this section, we focus on the following theorem.

Theorem 3.21. *Let $Q_0 \in \Delta(\mathbb{R}^d)$ and let $f\colon MQ_0 \to \mathbb{R}^D$ be η-quasisymmetric such that*

$$\sum_{Q \subseteq Q_0} \omega_f(MQ)^2 |Q| \leq C_M |Q_0|.$$

Then for all $\theta > 0$, there is $L = L(\eta, \theta, D, C_M)$ and $E \subseteq Q_0$ such that $|E| \geq (1-\theta)|Q|$ and $(\operatorname{diam} f(Q_0)/\operatorname{diam} Q_0)^{-1} f|_E$ is L-bi-Lipschitz.

Proof of Theorem 3.21. Recall that ω_f is invariant under dilations and translations in the domain of f and under scaling of f by a constant factor. Moreover, if f is η-quasisymmetric, the map $x \mapsto rf(sx+b)$ is also η-quasisymmetric for any nonzero r, s and any $b \in \mathbb{R}^d$. Thus, it suffices to prove the theorem in the case that $\operatorname{diam} Q_0 = \operatorname{diam} f(Q_0) = 1$ so $\operatorname{diam} f(Q_0)/\operatorname{diam} Q_0 = 1$.

Let $\tau \in (0,1)$, $\delta < d^{-1/2}/4$, and

$$0 < \varepsilon < \min\{\varepsilon_0(\eta, D, \tau, C_M), \varepsilon_1(\eta, d, \delta)\}$$

where ε_1 is as in Lemma 2.1 and ε_0 as in Theorem 3.4. By Theorem 3.4, we may partition $\Delta(Q_0)$ into a set of "bad" cubes $\mathscr{B}$ and a collection of (ε,τ)-regions $\mathscr{F}$ so that

$$\sum_{Q \in \mathscr{B}} |Q| \leq \frac{C_M}{\varepsilon^2}|Q_0| \quad \text{and} \quad \sum_{S \in \mathscr{F}_S} |Q(S)| \leq \left(4 + \frac{2^{d+1} C_M}{\varepsilon^2}\right)|Q_0|. \tag{3.75}$$

Let

$$T = \{Q(S) : S \in \mathscr{F}\} \cup \left(\bigcup_{S \in \mathscr{F}} m(S)\right) \cup \mathscr{B}.$$

Observe that since, for each $S \in \mathscr{F}$, the cubes in $m(S)$ have disjoint interiors, we know $\sum_{Q \in m(S)} |Q| \leq |Q(S)|$, and hence

(3.76)

$$\sum_{Q \in T} |Q| = \sum_{Q \in \mathscr{B}} |Q| + \sum_{S \in \mathscr{F}} \left(|Q(S)| + \sum_{Q \in m(S)} |Q|\right) \overset{(3.75)}{\leq} \left((1 + 2^{d+2})\frac{C_M}{\varepsilon^2} + 8\right)|Q_0|.$$

Let N be an integer. For $Q \in \Delta$, define

$$k(Q) = \#\{R \in T : R \supseteq Q\}, \quad T_N = \{Q \in T : k(Q) \leq N\},$$

and

$$E = Q_0 \backslash \Big(\bigcup_{Q \notin T_N} Q \Big).$$

If $x \in E$, let Q be the smallest cube in T_N containing x. Then $Q = Q(S)$ for some $S \in \mathscr{F}$, for otherwise, if $Q \in \mathscr{B}$ or $Q \in m(S)$ for some $S \in \mathscr{F}$, then the child R of Q containing x is either of the form $Q(S')$ for some $S' \in \mathscr{F}$ or is in $\mathscr{B}$ and hence is also in T, but since Q was minimal in T_N, $R \notin T_N$, which means $k(R) \geq N+1$, implying $z \notin E$, a contradiction. Thus,

$$E = \bigcup_{Q(S) \in T_N} z(S). \tag{3.77}$$

Moreover,

$$\begin{aligned} |Q_0 \backslash E| &= \Big| \bigcup_{Q \in T_N} Q \Big| = \Big| \bigcup_{k(Q)=N+1} Q \Big| = \int \sum_{k(Q)=N+1} \mathbb{1}_Q \leq \int \frac{\sum_{Q \in T_{N+1}} \mathbb{1}_Q}{N+1} \\ &= \frac{\sum_{Q \in T_{N+1}} |Q|}{N+1} \overset{(3.76)}{\leq} \frac{\big((1+2C_d)C_M/\varepsilon^2 + 2C_d\big)}{N+1} |Q_0| < \theta \, |Q_0| \end{aligned}$$

if we set $N = \lceil \theta^{-1} \big((1+2C_d)C_M/\varepsilon^2 + 2C_d\big) \rceil$, so now it suffices to show that f is bi-Lipschitz upon E. Define

$$\mathscr{M} = T_N \cup \bigcup_{Q(S) \in T_N} S.$$

Lemma 3.22. *Let $Q \in \mathscr{M}$. Then*

$$\beta^{-N-1} \leq \frac{\operatorname{diam} f(Q)}{\operatorname{diam} Q} \leq \beta^{N+1}, \tag{3.78}$$

where

$$\beta = \max \Big\{ 2, \eta(2), d^{1/2} \frac{1 + 2\sqrt{d}\delta}{1 - 2\sqrt{d}\delta} (1-\tau)^{-1} \Big\}.$$

Proof. First, we will focus on the case $Q \in T_N$. Let $Q(j) \subsetneq Q(j-1)$ be any sequence of cubes in T_N such that $k(Q(j)) = j$ for $j = 1, 2, \ldots, N$, so that $Q(1) = Q_0 = [0,1]^d$. We claim that for $j = 1, 2, \ldots, N$,

$$\beta^{-j} \leq \frac{\operatorname{diam} f(MQ(j))}{\operatorname{diam} MQ(j)} \leq \beta^j. \tag{3.79}$$

We will prove this inductively using the following lemma:

Lemma 3.23 (Proposition 10.8 in [23]). *If $\Omega \subseteq \mathbb{R}^d$, $f \colon \Omega \to \mathbb{R}^D$ is an η-quasisymmetric map, and $A \subseteq B$ are subsets such that* $0 < \operatorname{diam} A \leq \operatorname{diam} B < \infty$,

$$\frac{1}{2\eta\left(\frac{\operatorname{diam} B}{\operatorname{diam} A}\right)} \leq \frac{\operatorname{diam} f(A)}{\operatorname{diam} f(B)} \leq \eta\Big(\frac{2 \operatorname{diam} A}{\operatorname{diam} B}\Big). \tag{3.80}$$

The lemma is stated more generally in [23] for metric spaces, but this is all we will need.

Let $1 \leq j < N$ and assume we have shown j satisfies (3.79) (also recall that we are assuming $\operatorname{diam} f(Q_0) = \operatorname{diam} Q_0$, and so the $j = 1$ case holds).

1) If $Q(j+1) \in \mathscr{B}$ or $Q(j+1) = Q(S)$ for some $S \in \mathscr{F}$, then $Q(j+1)$ is a child of $Q(j)$. Hence, $MQ(j+1) \subseteq MQ(j)$ and $\operatorname{diam} MQ(j) = 2 \operatorname{diam} MQ(j+1)$, so that by (3.80)

$$\begin{aligned}\frac{\operatorname{diam} f(MQ(j+1))}{\operatorname{diam} MQ(j+1)} &= \frac{2 \operatorname{diam} f(MQ(j+1))}{\operatorname{diam} MQ(j)} \\ &\geq \frac{2\left(2\eta\left(\frac{\operatorname{diam} MQ(j)}{\operatorname{diam} MQ(j+1)}\right)\right)^{-1} \operatorname{diam} f(MQ(j))}{\operatorname{diam} MQ(j)} \\ &= \frac{2 \operatorname{diam} f(MQ(j))}{\eta(2) \operatorname{diam} MQ(j)} \geq \frac{\operatorname{diam} f(MQ(j))}{\beta \operatorname{diam} MQ(j)} \geq \beta^{-j-1}\end{aligned}$$

and since $MQ(j+1) \subseteq MQ(j)$,

$$\frac{\operatorname{diam} f(MQ(j+1))}{\operatorname{diam} MQ(j+1)} = \frac{2 \operatorname{diam} f(MQ(j+1))}{\operatorname{diam} MQ(j)} \leq \beta \frac{\operatorname{diam} f(MQ(j))}{\operatorname{diam} MQ(j)} \leq \beta^{j+1}.$$

2) If $Q(j+1) \in m(S)$ for some S, then $Q(j) = Q(S)$, so in particular, $Q(j), Q(j+1) \in S$. By Lemma 2.1, Lemma 3.23, and since $1/(1-\tau) > 1+\tau$,

$$\begin{aligned}\frac{\operatorname{diam} f(MQ(j+1))}{\operatorname{diam} MQ(j+1)} &\leq (1+2\sqrt{d}\delta)|A'_{Q(j+1)}| \leq (1+2\sqrt{d}\delta)(1+\tau)|A'_{Q(S)}| \\ &\leq \sqrt{d}\frac{1+2\sqrt{d}\delta}{1-2\sqrt{d}\delta}(1+\tau)\frac{\operatorname{diam} f(MQ(j))}{\operatorname{diam} MQ(j)} \leq \beta \frac{\operatorname{diam} f(MQ(j))}{\operatorname{diam} MQ(j)} \leq \beta^{j+1}\end{aligned}$$

and

$$\begin{aligned}\frac{\operatorname{diam} f(MQ(j+1))}{\operatorname{diam} MQ(j+1)} &\geq d^{-\frac{1}{2}}(1-2\sqrt{d}\delta)|A'_{Q(j+1)}| \geq d^{-\frac{1}{2}}(1-2\sqrt{d}\delta)(1-\tau)|A'_{Q(S)}| \\ &\geq d^{-\frac{1}{2}}\frac{1-2\sqrt{d}\delta}{1+2\sqrt{d}\delta}(1-\tau)\frac{\operatorname{diam} f(MQ(j))}{\operatorname{diam} MQ(j)} \geq \beta^{-1}\frac{\operatorname{diam} f(MQ(j))}{\operatorname{diam} MQ(j)} \geq \beta^{-j-1}.\end{aligned}$$

This proves the induction step, and hence proves (3.79).

Now we prove (3.78). If $Q \in T_N$, this follows from (3.79). If $Q \in S$ for some $Q(S) \in T_N$, let $Q(j) \in T_N$ be a nested chain of cubes so that $k(Q(j)) = j$ for all $j < n := k(Q(S))$, so in particular $Q(n) = Q(S)$. Then, since S is a (ε, τ)-region, (3.1) applies, and by Lemma 2.1,

$$\begin{aligned}\frac{\operatorname{diam} f(MQ)}{\operatorname{diam} MQ} &\leq (1+2\sqrt{d}\delta)|A'_Q| \leq (1+2\sqrt{d}\delta)(1+\tau)|A'_{Q(S)}| \\ &\leq \sqrt{d}\frac{1+2\sqrt{d}\delta}{1-2\sqrt{d}\delta}(1+\tau)\frac{\operatorname{diam} f(MQ(n))}{\operatorname{diam} MQ(n)} \leq \beta^{n+1} \leq \beta^{N+1},\end{aligned}$$

and the lower bound follows similarly. □

Let $x, y \in E$ be distinct. We claim there is a chain of cubes Q_j such that $Q_j^j = Q_0$, and

$$x \in Q_j \in \mathscr{M} \quad \text{for all } j \geq 0. \tag{3.81}$$

Since $x \in E$, $x \in z(S)$ for some $S \in \mathscr{F}$ with $Q(S) \in T_N$ by (3.77), and hence x is contained in a chain of cubes $R_j \in S$ such that $R_j^j = Q(S)$. Let n be such that $Q(S)^n = Q_0$ and define $Q_j = R_{j-n}$ for $j \geq n$ (so $Q_j^j = R_{j-n}^j = Q(S)^n = Q_0$) and for $j < n$ let Q_j be the unique ancestor of $Q(S)$ with $Q_j^j = Q_0$. We now just need to show (3.81). For $j \geq n$, $Q_j = R_{j-n} \in S$; for $j < n$, note that since $\mathscr{B}$ and the sets $S' \in \mathscr{F}$ partition $\Delta(Q_0)$, Q_j is always in $\mathscr{B}$ or in some $S' \in \mathscr{F}$. If $Q_j \in \mathscr{B}$ or $Q_j \in m(S')$ for some $S' \in \mathscr{F}$, then $k(Q_j) < k(Q(S)) \leq N$ (note that $S' \neq S$), and so $Q_j \in T_N \subseteq \mathscr{M}$; otherwise, if $Q_j \in S'$ for some $S' \in \mathscr{F}$ and is not a minimal cube, then $k(Q_j) \leq k(Q(S')) < k(Q(S)) \leq N$, and so $Q_j \in \mathscr{M}$. This proves (3.81).

Let j is the largest integer for which $y \in 3Q_j$. since $y \in Q_0 \subseteq 3Q_0$ and $x \neq y$, this integer is well defined. Moreover,

$$|x - y| \geq \frac{\ell(Q_j)}{2} \tag{3.82}$$

for otherwise, $|x-y| < \ell(Q_j)/2 = \ell(Q_{j+1})$ and $x \in Q_{j+1}$ imply $y \in 3Q_{j+1}$, contradicting the maximality of j. Then Lemma 3.22 implies

$$\begin{aligned} |f(x) - f(y)| &\leq \frac{\operatorname{diam} f(MQ_j)}{\operatorname{diam} MQ_j} \operatorname{diam} MQ_j \stackrel{(3.78)}{\leq} \beta^{N+1} M\sqrt{d}\,\ell(Q_j) \\ &\stackrel{(3.82)}{\leq} \beta^{N+1} 2M\sqrt{d}\,|x-y|. \end{aligned} \tag{3.83}$$

Furthermore, by Lemma 3.23, since $x, y \in 3Q_j \subseteq MQ_j$,

$$\begin{aligned} |f(x) - f(y)| &= \frac{\operatorname{diam} f(\{x,y\})}{\operatorname{diam} f(MQ_j)} \frac{\operatorname{diam} f(MQ_j)}{\operatorname{diam} MQ_j} \operatorname{diam} MQ_j \stackrel{\substack{(3.78)\\(3.80)}}{\geq} \frac{|x-y|}{2\eta\left(\frac{\operatorname{diam} MQ_j}{|x-y|}\right)\beta^{N+1}} \\ &\stackrel{\substack{(3.80)\\(3.82)}}{\geq} \frac{|x-y|}{\eta(1)\beta^{N+1}}. \end{aligned} \tag{3.84}$$

Thus (3.83) and (3.84) imply f is $\beta^{N+1}\max\{2M\sqrt{d}, \eta(1)\}$-bi-Lipschitz on E, and this finishes the proof. □

3.8. The proof of Theorem 3.1

Proof of Theorem 3.1. Let $\tau > 0$. Suppose $f\colon \mathbb{R}^d \to \mathbb{R}^D$ is such that $\omega_f(x,r)^2 \frac{dr}{r}\,dx$ is a C-Carleson measure. Let $B(x_0, r_0) \in \mathbb{R}^d$ be any ball and let $Q_0 = [0,1]^d$. Since ω_f is invariant under translations and dilations in the domain, the Carleson norm remains unchanged if we replace $f(x)$ with the function $f\left(\frac{x-x_{Q_0}}{\frac{1}{2}\ell(Q_0)}\right)$, so we may assume without loss of generality that $B(x_0,r_0)=B(x_{Q_0},1/2)$ (that is, the largest ball contained in Q_0). By Lemma 2.4, we know

$$\sum_{Q\subseteq Q_0}\omega_f(MQ)^2\,|Q|\leq C_M\,|Q_0|\,,$$

where $C_M=C_M(C,d)$. Theorem 3.21 implies for all $\theta>0$ there is $E'\subseteq Q_0$ with $|E'|\geq(1-\theta)|Q_0|$ and $(\operatorname{diam} f(Q_0)/\operatorname{diam} Q_0)^{-1}f$ is L-bi-Lipschitz upon E'. By Lemma 3.23, it follows that

$$\frac{\operatorname{diam} f(Q_0)}{\operatorname{diam} Q_0}\sim_{\eta,d}\frac{\operatorname{diam} f(B(x_0,r_0))}{\operatorname{diam} B(x_0,r_0)}.$$

By picking θ small enough, we may guarantee that the set $E=E'\cap B(x_0,r_0)$ satisfies $|E|\geq(1-\tau)|B(x_0,r_0)|$. Since this holds for all $x_0\in\mathbb{R}^d$ and $r_0>0$, Theorem 3.1 is proven. □

4. Finding bi-Lipschitz pieces of a general quasisymmetric map

In this section, we focus on proving Proposition 1.6. For the first few subsections, however, we will recall some basic facts about A_∞ and BMO spaces and review some material from [37], as well as the technical modifications of Semmes' work we will need.

4.1. A_∞-weights

For a locally integrable function w on $\mathbb{R}^d$, we will write, for any measurable subset A, $w(A)=\int_A w$, and $w_A=w(A)/|A|$. We will call w an A_∞-*weight* if it is nonnegative, locally integrable, and there is $q>0$ such that for all cubes $Q\subset\mathbb{R}^d$ and measurable sets $E\subset Q$,

$$w(E)\geq\frac{w(Q)}{1+\exp\left(q|Q|/|E|\right)}\,. \tag{4.1}$$

This is not how A_∞ is described in most texts, but it is equivalent to the usual definition equivalent (see [24]).

An important property we will use is that if $w\in A_\infty$, then $||\log w||_{\mathrm{BMO}}\lesssim_q 1$ (where q is as in (4.1)). Recall that $\log w\in\mathrm{BMO}(\mathbb{R}^d)$ implies there is an infimal number $||\log w||_{\mathrm{BMO}}$ such that for all cubes $Q\subseteq\mathbb{R}^d$,

$$⨍_Q|\log w-(\log w)_Q|\leq||\log w||_{\mathrm{BMO}}. \tag{4.2}$$

Another property is the reverse Jensen's inequality: for w satisfying (4.1),

$$w_Q\leq C_q\,e^{(\log w)_Q},$$

where $C_q>0$ depends on q and d.

For good references on A_∞-weights and BMO with proofs of these facts, see Chapter V in [39] and Chapter VI in [20].

One last technique we will need is following lemma, which is essentially known and is a good exercise with A_∞-weight theory (a similar proof appears in Theorem 3.22 of [19]).

Lemma 4.1. *Let $w \in A_\infty(\mathbb{R}^d)$. For all $\tau \in (0,1)$, and $Q_0 \in \Delta$, there is $E_{Q_0} \subseteq Q_0$ with*

(1) *for all $Q \subseteq Q_0$ with $Q \cap E_{Q_0} \neq \emptyset$, we have $M^{-1} \leq w_Q/w_{Q_0} \leq M$ where $M = \exp(C_q + 2^d ||\log w||_{\mathrm{BMO}}/\tau)$, and*

(2) $|E_{Q_0}| \geq (1-\tau)|Q_0|$.

Proof. Since $w \in A_\infty$, $g{:}\ = \log w \in \mathrm{BMO}(\mathbb{R}^d)$. Let

$$E_{Q_0} = \{x \in Q_0 : M_\Delta(g - g_{Q_0}) \leq 2^d\, ||g||_{\mathrm{BMO}}\, \tau^{-1}\},$$

where M_Δ is the dyadic maximal function

$$M_\Delta h(x) := \sup_{x \in Q \in \Delta} \fint_Q |h|.$$

Since $||M_\Delta||_{L^1 \to L^{1,\infty}} \leq 2^d$, we have

$$\begin{aligned} |Q_0 \backslash E_{Q_0}| &= \{x \in Q_0 : M_\Delta(g - g_{Q_0}) > 2^d\, ||g||_{\mathrm{BMO}}\, \tau^{-1}\} \\ &\leq 2^d\, \frac{\int_{Q_0} |g - g_{Q_0}|}{2^d\, ||g||_{\mathrm{BMO}}\, \tau^{-1}} \leq \tau\, |Q_0|. \end{aligned}$$

Let $Q \subseteq Q_0$ be a dyadic cube such that $Q \cap E_{Q_0} \neq \emptyset$. If $x \in Q \cap E_{Q_0}$, then

$$|g - g_{Q_0}|_Q \leq M_\Delta(g - g_{Q_0})(x) \leq 2^d\, ||g||_{\mathrm{BMO}}\, \tau^{-1}. \tag{4.3}$$

Moreover, since $w \in A_\infty$, we have by (4.2) that

$$\log w_Q \leq C_q + (\log w)_Q. \tag{4.4}$$

Using this and Jensen's inequality, we get

$$\begin{aligned} \log w_Q - \log w_{Q_0} &\overset{(4.4)}{\leq} C_q + (\log w)_Q - (\log w)_{Q_0} = C_q + (g - g_{Q_0})_Q \\ &\overset{(4.3)}{\leq} C_q + 2^d\, ||g||_{\mathrm{BMO}}\, \tau^{-1} \end{aligned}$$

and

$$\begin{aligned} \log w_{Q_0} - \log w_Q &\overset{(4.4)}{\leq} C_q + (\log w)_{Q_0} - (\log w)_Q = C_q - (g - g_{Q_0})_Q \\ &\overset{(4.3)}{\leq} C_q + 2^d\, ||g||_{\mathrm{BMO}}\, \tau^{-1}. \end{aligned}$$

Thus,

$$\left| \log \frac{w_Q}{w_{Q_0}} \right| \leq C_q + 2^d\, \tau^{-1}\, ||g||_{\mathrm{BMO}}. \qquad \square$$

4.2. Metric doubling measures and strong A_∞-weights

We recall the following definition from [35].

Definition 4.2. We say a Borel measure μ on $\mathbb{R}^d$ is C_μ*-doubling* on its support if

$$\mu(B(x,2r)) \leq C_\mu\, \mu(B(x,r))$$

for all $x \in \operatorname{supp}\mu$ and $r>0$. For $E \subseteq \mathbb{R}^d$ closed, we say that a doubling measure μ is a *metric doubling measure* on E if $\operatorname{supp}\mu = E$ and

$$\mu(B(x,|x-y|) \cup B(y,|x-y|))^{1/d} \sim \operatorname{dist}(x,y) \tag{4.5}$$

for some metric $\operatorname{dist}(x,y)$ on E.

In [22], Gehring showed that the pullback of Lebesgue measure under a quasisymmetric map $f\colon \mathbb{R}^d \to \mathbb{R}^d$ is an A_∞-weight; Semmes observed in [35] that this holds more generally for all metric doubling measures on $\mathbb{R}^d$, with a proof essentially the same as Gehring's.

Lemma 4.3 (Proposition 3.4 in [35]). *If ν is a metric doubling measure on $\mathbb{R}^d$, $d \geq 2$, then ν is absolutely continuous and it is given by $w(x)dx$, where $w \in A_\infty$, and q in (4.1) depends upon d, C_ν and the constants in (4.5). We call the weight w a strong A_∞-weight.*

If $f\colon \mathbb{R}^d \to \mathbb{R}^d$, then the pullback of Lebesgue measure under f is an example of a metric doubling measure, where in this case $\operatorname{dist}(x,y) = |f(x)-f(y)|$, and Lemma 4.3 recovers Gehring's original result.

Metric doubling measures and strong A_∞-weights arose in studying the so-called "quasiconformal Jacobian problem" (see [35], [36], and [8] for discussions of this problem). While the aforementioned papers gradually demonstrated the intractability of this problem, its pursuit has developed many useful techniques (and counterexamples) in the theory of quasisymmetric mappings.

4.3. Serious and strong sets

Here we recall some definitions and results from [37] about serious and strong sets.

Definition 4.4. Let $E_0 \subseteq E \subseteq \mathbb{R}^d$. We say E_0 is a *serious subset* of E if there is $C>0$ so that if $x \in E_0$ and $0<t<\operatorname{diam} E_0$, then there is $y \in E$ such that

$$\frac{t}{C} \leq |x-y| \leq t. \tag{4.6}$$

We will call C the *seriousness constant* of the pair (E_0, E). If $E_0 = E$, we say E is a *serious set*.

In Lemma 1.8 of [37], Semmes shows that all compact subsets with positive measure contain a serious subset whose measure is as close to the measure of the original set as you wish, although there is no control given on the seriousness constant of this set. Without too much effort, though, this dependence can be determined, and allows us to make Lemma 4.9 depend quantitatively only on d, η, and the density of E inside a prescribed dyadic cube.

Lemma 4.5. *Let $E \subseteq \mathbb{R}^d$ be a compact set of positive measure contained in a dyadic cube Q_0. Then for each $\delta \in (0,1)$, there is $E_0 \subseteq E$ compact such that*

1) $|E_0| \geq (1-\delta)|E|$, *and*

2) E_0 *is a* $(\delta|E|/|Q_0|)^{1/d}/(8d^{1/2}3^d)$*-serious subset of* E.

Proof. Let Q_j be the collection of maximal cubes contained in Q_0 for which

$$\frac{|E\cap Q_j|}{|Q_j|} < \delta\,\frac{|E|}{|Q_0|}\,,$$

and set

$$E_0 = E \setminus \bigcup Q_j^{\circ}.$$

Observe that this is a countable intersection of bounded closed sets and hence is compact. Since the Q_j have disjoint interiors, we have

$$|E\backslash E_0| = \sum_j |E\cap Q_j^{\circ}| < \delta\,\frac{|E|}{|Q_0|}\sum |Q_j| \leq \delta\,\frac{|E|}{|Q_0|}|Q_0| = \delta\,|E|,$$

which implies the first item of the lemma. Next, set

$$N = \left\lfloor \frac{\log\left(3^d \frac{|Q_0|}{\delta|E|}\right)}{d}\right\rfloor + 1. \tag{4.7}$$

We claim that for any dyadic cube Q intersecting E_0 such that $Q^{N+1} \subseteq Q_0$, we have

$$(Q^{N+1}\backslash 3Q)\cap E \neq \emptyset. \tag{4.8}$$

If not, then since Q^{N+1} is not contained in any Q_j,

$$\frac{|E|}{|Q_0|}\,\delta \leq \frac{|E\cap Q^{N+1}|}{|Q^{N+1}|} \leq \frac{|E\cap 3Q|}{2^{d(N+1)}|Q|} \leq 3^d\, 2^{-d(N+1)} \overset{(4.7)}{\leq} \frac{|E|}{|Q_0|}\,\delta\, 2^{-d},$$

which is a contradiction, hence proving (4.8) and the claim.

Now let $x \in E_0$, $t \leq \operatorname{diam} E$. Let $Q \ni x$ be contained in Q_0 such that

$$\operatorname{diam} Q^{N+1} \leq t < \operatorname{diam} Q^{N+2}.$$

Since $t \leq \operatorname{diam} E \leq \operatorname{diam} Q_0$, $Q^{N+1} \subseteq Q_0$, and by (4.8) there is $y \in (Q^{N+1}\backslash 3Q)\cap E$, so that

$$|x-y| \leq \operatorname{diam} Q^{N+1} \leq t$$

and

$$|x-y| \geq \ell(Q) = 2^{-N-2}d^{-1/2}(\operatorname{diam} Q^{N+2}) \geq 2^{-N-2}\,d^{-1/2}\,t \geq \frac{1}{8\,d^{1/2}\,3^d}\left(\frac{\delta|E|}{|Q_0|}\right)^{1/d} t,$$

and this finishes the second part of the lemma. □

Definition 4.6. A closed set $\tilde{E} \subseteq \mathbb{R}^d$ is a *strong set* if there is a constant $C > 0$ so that for each $x \in \mathbb{R}^d \backslash \tilde{E}$, there is $y \in \tilde{E}$ so that

$$|x - y| \leq C \operatorname{dist}(x, \tilde{E}), \tag{4.9}$$

$$\operatorname{dist}(y, \mathbb{R}^d \backslash \tilde{E}) \geq C^{-1} \operatorname{dist}(x, \tilde{E}). \tag{4.10}$$

Thus, to each point $x \in \tilde{E}^c$, we can assign a ball in $\tilde{E}$ with radius and distance to x comparable to the distance from x to $\tilde{E}$; in Semmes' words, this says $\tilde{E}$ is at least as big as its complement.

Lemma 4.7 (Proposition 1.16 in [37]). *If $\tilde{E} \subseteq \mathbb{R}^d$ is a C-strong set, then for all $x \in \tilde{E}$ and $r > 0$,*

$$|\tilde{E} \cap B(x, r)| \sim_{d,C} r^d. \tag{4.11}$$

Lemma 4.8 (Proposition 1.15 in [37]). *If $\tilde{E} \subseteq \mathbb{R}^d$ is a C-strong set and $g \colon \tilde{E} \to \mathbb{R}^d$ is η-quasisymmetric, then $g(\tilde{E})$ is C'-serious with C' depending on η, C, and d.*

The next lemma is an amalgamation of Propositions 1.10, 1.14, 1.22, and 1.23 from [37].

Lemma 4.9. *Suppose E is a compact subset of $\mathbb{R}^d$, $E_0 \subseteq E$ is a C-serious subset of E, and $f \colon E \to \mathbb{R}^d$ is η-quasisymmetric. Then the following hold:*

1) *There is $\hat{E} \supseteq E_0$ that is $\hat{C}$-serious, with $\hat{C} > 0$ depending only on C and d.*
2) *There is $g \colon \hat{E} \to \mathbb{R}^d$ that agrees with f on E_0 and is $\hat{\eta}$-quasisymmetric, with $\hat{\eta}$ depending on C, d, and η.*
3) *There is a $\tilde{C}$-strong set $\tilde{E} \supseteq \hat{E}$, where $\tilde{C}$ depends only on $\tilde{C}$ and d.*
4) *The map g admits an $\tilde{\eta}$-quasisymmetric extension $G : \tilde{E} \to \mathbb{R}^d$. Here, $\tilde{\eta}$ depends only on $\hat{\eta}$, $\tilde{C}$, and d.*
5) *The measure μ defined by $\mu(A) = |G(A)|$ is a metric doubling measure on $\tilde{E}$, with data depending only on $\tilde{C}$, $\tilde{\eta}$, and d.*
6) *There is a metric doubling measure ν on $\mathbb{R}^d$ such that $\nu(A) = |G(A)|$ for all $A \subseteq \tilde{E}$. The doubling constant C_ν and metric doubling constants of ν depend only on those for ν, d, and $\tilde{C}$.*

Corollary 4.10. *If $E \subseteq Q_0 \subseteq \mathbb{R}^d$ has positive measure, and $f \colon E \to \mathbb{R}^d$ is an η-quasisymmetric map, then Lemma 4.9 still holds and all the implied constants depend on d, η, and $|E|/|Q_0|$.*

Proof. This follows from Lemmas 4.5 and 4.9. □

Lemma 4.11. *With $\nu, \tilde{E}$, f, η, and G as in Lemma 4.9, we have that for all $x \in \tilde{E}$ and $r > 0$,*

$$\nu(B(x, r)) \sim_{d,\tilde{\eta},\tilde{C},C_\nu} (\operatorname{diam} G(\tilde{E} \cap B(x, r)))^d. \tag{4.12}$$

where C_ν is the doubling constant of ν.

Proof. Let $x \in \tilde{E}$, $r > 0$, and Q be a cube containing $B(x,r)$ of side length $2r$.

First, note that since ν is an A_∞-weight and $\tilde{E}$ is strong, (4.1) and (4.11) imply

$$\nu(\tilde{E} \cap B(x,r)) \sim \nu(Q) \sim \nu(B(x,r))$$

with implied constants depending on d, the A_∞-data of ν, and the constants in (4.11). By quasisymmetry, it is not hard to show that there is $\rho < 1$ depending only on $\tilde{\eta}$ so that

$$\begin{aligned} G(\tilde{E}) \cap B(G(x), \rho \operatorname{diam} G(B(x,r) \cap \tilde{E})) &\subseteq G(\tilde{E} \cap B(x,r)) \\ &\subseteq B(G(x), \operatorname{diam} G(\tilde{E} \cap B(x,r)))\,, \end{aligned}$$

and since $G(\tilde{E})$ is also serious by Lemma 4.8, Lemma 4.7 and the above containments imply

$$\nu(B(x,r)) \sim \nu(B(x,r) \cap \tilde{E}) = |G(\tilde{E} \cap B(x,r))| \sim \operatorname{diam}(G(\tilde{E} \cap B(x,r)))^d. \quad \square$$

4.4. A slightly stronger Semmes theorem

We are now in a position to prove Proposition 1.6, which strengthens Semmes' original result, Theorem 1.5. While Semmes shows that if $E \subseteq \mathbb{R}^d$ and $f \colon E \to \mathbb{R}^d$ is quasisymmetric, then $|E| = 0$ if and only if $|f(E)| = 0$, we show here that f is in fact bi-Lipschitz on a large subset of E quantitatively. We restate this below.

Proposition 1.6. Let $E \subseteq Q_0 \subseteq \mathbb{R}^d$, $\rho \in (0, 1/2)$, $d \geq 2$, and set $\delta = |E|/|Q_0| > 0$. Let $f \colon E \to \mathbb{R}^d$ be η-quasisymmetric. Then there is $E'' \subseteq E$ compact with $|E''| \geq (1-\rho)|E|$ and $\big(\operatorname{diam} f(E'')/\operatorname{diam} E''\big)^{-1} f|_{E''}$ is L-bi-Lipschitz for some L depending on η, d, ρ, and δ.

Proof of Proposition 1.6. By Lemma 4.5, there $\hat{E} \subseteq E$ that is $\hat{C}$-serious and

$$|\hat{E}| \geq \left(1 - \frac{\rho}{2}\right)|E|,$$

with $\hat{C}$ depending on d, δ, and ρ. According to Lemma 4.9, $\hat{E} \subseteq \tilde{E}$ for some $\tilde{C}$-serious set $\tilde{E}$, to which f has an $\tilde{\eta}$-quasisymmetric extension $G : \tilde{E} \to \mathbb{R}^d$ and a metric doubling measure ν on $\mathbb{R}^d$ with $\nu(A) = |G(A)|$ for all $A \subseteq \tilde{E}$. We can write $d\nu = w dx$ where w is an A_∞-density by Lemma 4.3. Applying Lemma 4.1 with $\tau = \rho/2$, there is $M > 1$ depending on d, ρ, and the A_∞-data of w and $E' \subseteq Q_0$ with $|E'| \geq (1-\rho/2)|Q_0|$ such that

$$\frac{1}{M} \leq \frac{w_Q}{w_{Q_0}} \leq M \tag{4.13}$$

for all $Q \subseteq Q_0$ such that $Q \cap E \neq \emptyset$. Let $E'' = E' \cap \hat{E}$, so that $|E''| \geq (1-\rho)|E|$. We will now show $(\operatorname{diam} f(E'')/\operatorname{diam} E'')^{-1} f$ is bi-Lipschitz upon E''.

Let $x, y \in E''$ be distinct points and $Q \subseteq Q_0$ be a minimal dyadic cube containing x so that $y \in 3Q$. Since $\tilde{E}$ is serious, and G is $\tilde{\eta}$-quasisymmetric and $\{x, y\}$ and $B(x, |x-y|) \cap \tilde{E}$ have comparable diameters,

$$|f(x)-f(y)| = |G(x)-G(y)| \overset{(3.80)}{\sim_{\tilde{\eta}}} \operatorname{diam} G(B(x,|x-y|)\cap \tilde{E}) \tag{4.14}$$
$$\overset{(4.12)}{\sim}_{\tilde{\eta},d,\tilde{C}} \nu(B(x,|x-y|))^{1/d}.$$

Since $3Q$ is minimal, we know

$$\frac{1}{2}\ell(Q) \leq |x-y| \leq \operatorname{diam} 3Q. \tag{4.15}$$

Hence, since ν is doubling, $\nu(Q) \sim_{d,C_\nu} \nu(B(x,|x-y|))$. Thus, continuing our chain of estimates, (and using the fact that $\ell(Q)^n = |Q|$) we have

$$(4.14) \sim_{d,C_\nu} \nu(Q)^{1/d} = w(Q)^{1/d} = \ell(Q)\,(w_Q)^{1/d} \overset{(4.13)}{\sim}_{M,d} \ell(Q)\,(w_{Q_0})^{1/d}$$
$$\overset{(4.15)}{\sim}_d |x-y|\,(w_{Q_0})^{1/d} = |x-y|\frac{\nu(Q_0)^{1/d}}{\ell(Q_0)}$$
$$\sim_{C_\mu,d} |x-y|\,\frac{\nu(B(x_{Q_0},\operatorname{diam} Q_0))^{1/d}}{\ell(Q_0)}$$
$$\sim_{\tilde{C},\tilde{\eta},d} |x-y|\,\frac{\operatorname{diam} G(B(x_{Q_0},\operatorname{diam} Q_0)\cap\tilde{E})}{\ell(Q_0)}. \tag{4.16}$$

Since $E'' \subseteq Q_0 \cap \tilde{E} \subseteq B(x_{Q_0}, \operatorname{diam} Q_0)$ and $|E''| \geq (1-\rho)|E| \geq \frac{\delta}{2}|Q_0|$, we know $\operatorname{diam} E'' \sim_{d,\delta} \operatorname{diam} Q_0$, and so Lemma 3.23 implies

$$(4.16) \overset{(3.80)}{\sim}_{\tilde{\eta}} |x-y|\,\frac{\operatorname{diam} G(E'')}{\ell(Q_0)} \sim_{d,\delta,\tilde{\eta}} |x-y|\frac{\operatorname{diam} f(E'')}{\operatorname{diam} E''}$$

Combining this with (4.14) and (4.16), we see $|f(x)-f(y)| \sim |x-y|\frac{\operatorname{diam} f(E'')}{\operatorname{diam} E''}$ with implied constants depending on $\tilde{\eta}, \tilde{C}, d, M$, and C_ν. Finally, we recall that these constants depend only on d, η, ρ, and δ. This finishes the proof. □

In the last part of this section, we adapt Proposition 1.6 to the case where f maps a set to a large bi-Lipschitz image of $\mathbb{R}^d$, which is the case we will need later on.

Lemma 4.12. *Suppose $B_0 \subseteq \mathbb{R}^d$, $f\colon B_0 \to \mathbb{R}^D$ is η-quasisymmetric, and there is $E' \subseteq B_0$ such that $\mathscr{H}^d(f(E')) \geq c(\operatorname{diam} f(B_0))^d$, and there is $g\colon f(E') \to \mathbb{R}^d$ that is L-bi-Lipschitz. Then there is $E_0 \subseteq E'$ and $M = M(\eta, d, L, c) \geq 1$ such that $|E_0| \gtrsim_{d,L,\eta,c} |B_0|$ and $\left(\frac{\operatorname{diam} f(E_0)}{\operatorname{diam} E_0}\right)^{-1} f|_{E_0}$ is M-bi-Lipschitz.*

Proof. Let $E_1 = g \circ f(E')$. If B is a ball centered on E_1 with radius $\operatorname{diam} E_1$, then

$$c\,(\operatorname{diam} f(B_0))^d \leq \mathscr{H}^d(f(E')) \leq L^d\,|E_1| \leq L^d\,|B| = L^d\,w_d\,(\operatorname{diam} E_1)^d$$
$$\leq L^{2d}\,w_d\,(\operatorname{diam} f(E'))^d, \tag{4.17}$$

so that

$$\operatorname{diam} f(E') \geq \frac{c^{1/d}}{L^2 w_d^{1/d}} \operatorname{diam} f(B_0). \tag{4.18}$$

Set

$$h := f^{-1} \circ g^{-1} : E_1 \to \mathbb{R}^d.$$

Since f is η-quasisymmetric, $f^{-1} : f(\mathbb{R}^d) \to \mathbb{R}^d$ is η'-quasisymmetric with

$$\eta'(t) = \eta^{-1}(t^{-1})^{-1}$$

(see Proposition 10.6 in [23]), and it is not hard to show using the definition of quasisymmetry that h is $\eta'(L^2\cdot)$-quasisymmetric.

Let Q_1 be a cube containing E_1 with $\ell(Q_1) = \operatorname{diam} E_1$, so that

$$\begin{aligned} |E_1| &\geq L^{-d}\mathscr{H}^d(f(E')) \geq \frac{c}{L^d}(\operatorname{diam} f(B_0))^d \\ &\geq \frac{c}{L^d}(\operatorname{diam} f(E'))^d \geq \frac{c}{L^{2d}}(\operatorname{diam} E_1)^d. \end{aligned} \tag{4.19}$$

By Proposition 1.6, there is $E_1' \subseteq E_1$ with $|E_1'| \geq \frac{1}{2}|E_1|$ upon which $\left(\frac{\operatorname{diam} h(E_1')}{\operatorname{diam} E_1'}\right)^{-1} h$ is L'-bi-Lipschitz, with L' depending on L, c, d, and the function $L^2\eta'$. Let $E_0 = h(E_1') \subseteq Q_0$. Using the facts that $\left(\frac{\operatorname{diam} h(E_1')}{\operatorname{diam} E_1'}\right)^{-1} h$ is L'-bi-Lipschitz, $g^{-1}(E_1') = f(E_0)$, and g is L-bi-Lipschitz, it is not hard to show that $\left(\frac{\operatorname{diam} f(E_0)}{\operatorname{diam} E_0}\right)^{-1} f$ is $L'L^2$-bi-Lipschitz upon E_0.

If B' is a ball centered upon E_1' with radius $\operatorname{diam} E_1'$, then

$$\omega_d(\operatorname{diam} E_1')^d = |B'| \geq |E_1'| \geq \frac{1}{2}|E_1| \overset{(4.19)}{\geq} \frac{c}{2L^{2d}}(\operatorname{diam} E_1)^d$$

and so

$$\operatorname{diam} E_1' \geq \frac{c^{1/d}}{2^{1/d}L^2 w_d^{1/d}} \operatorname{diam} E_1. \tag{4.20}$$

Then

$$\begin{aligned} \operatorname{diam} f(E_0) &\geq L^{-1} \operatorname{diam} g \circ f(E_0) = L^{-1} \operatorname{diam} E_1' \overset{(4.20)}{\geq} \frac{c^{1/d}}{2^{1/d}L^3 w_d^{1/d}} \operatorname{diam} E_1 \\ &\geq \frac{c^{1/d}}{2^{1/d}L^4 w_d^{1/d}} \operatorname{diam} f(E') \overset{(4.18)}{\geq} \frac{c^{\frac{2}{d}}}{2^{1/d}L^6 w_d^{\frac{2}{d}}} \operatorname{diam} f(B_0), \end{aligned} \tag{4.21}$$

where in the first and penultimate inequalities we used the fact that g is L-bi-Lipschitz. By Lemma 3.23,

$$\frac{\operatorname{diam} E_0}{\operatorname{diam} B_0} \geq \left(2\eta'\left(\frac{\operatorname{diam} f(B_0)}{\operatorname{diam} f(E_0)}\right)\right)^{-1} \overset{(4.21)}{\geq} \left(2\,\eta'\left(2^{-1/d}\,L^{-6}\,w_d^{-2/d}\,c^{2/d}\right)\right)^{-1}. \tag{4.22}$$

Furthermore,

$$|E_0| = |h(E_1')| \geq \Big(\frac{\operatorname{diam} h(E_1')}{\operatorname{diam} E_1'}\Big)^d (L')^{-d} |E_1'| \geq \frac{1}{2}\Big(\frac{\operatorname{diam} E_0}{\operatorname{diam} E_1}\Big)^d (L')^{-d} |E_1|$$
$$\overset{(4.19)}{\geq} \frac{c}{2(L')^d L^{2d}} (\operatorname{diam} E_0)^d \overset{(4.22)}{\geq} \frac{c}{2}\left(L^2 L' 2\eta'(2^{-\frac{1}{d}} L^{-6} w_d^{-2/d} c^{2/d})\right)^{-d} (\operatorname{diam} B_0)^d$$
$$\geq \frac{c}{2w_d}\big(L^2 L'\, 2\eta'\,(2^{-1/d} L^{-6} w_d^{-2/d} c^{2/d})\big)^{-d} |B_0|.$$

□

5. Bi-Lipschitz parts imply big-pieces of bi-Lipschitz images

The following theorem proves (3) implies (4) in Theorem 1.4

Theorem 5.1. *Suppose $f\colon \mathbb{R}^d \to \mathbb{R}^D$ is η-quasisymmetric and there are $c, L > 0$ such that for all $x \in \mathbb{R}^d$ and $r > 0$, there is $E \subseteq B(x,r)$ such that $|E| \geq c|B(x,r)|$ and $(\operatorname{diam} f(B(x,r))/\operatorname{diam} B(x,r))^{-1} f|_E$ is L-bi-Lipschitz. Then $f(\mathbb{R}^d)$ has big pieces of bi-Lipschitz images.*

We first need the following lemma.

Lemma 5.2. *Let : $f\colon \mathbb{R}^d \to \mathbb{R}^D$ be η-quasisymmetric. The following are equivalent:*

(1) *The set $f(\mathbb{R}^d)$ has BPBI(κ, L), that is, there is $\kappa > 0$ such that for all $\xi \in f(\mathbb{R}^d)$ and $s > 0$, there is $A \subseteq B(\xi, s) \cap f(\mathbb{R}^d)$ so that $\mathscr{H}^d(A) \geq \kappa s^d$ and an L-bi-Lipschitz map $g\colon A \to \mathbb{R}^d$.*

(2) *There is $c > 0$ such that for all $x \in \mathbb{R}^d$ and $r > 0$, there is $E' \subseteq B(x,r)$ and an L-bi-Lipschitz map $g\colon f(E') \to \mathbb{R}^d$ such that $\mathscr{H}^d(f(E')) \geq c(\operatorname{diam} f(B(x,r)))^d$.*

Proof. Let $f\colon \mathbb{R}^d \to \mathbb{R}^D$ be η-quasisymmetric.

(1) ⇒ (2). Let $x \in \mathbb{R}^d$, $r > 0$, and set

$$s = \sup\{t : B(f(x), t) \cap f(\mathbb{R}^d) \subseteq f(B(x,r))\}.$$

Then by Lemma 3.23 and the fact that $f^{-1}(B(f(x), s)) \subseteq B(x,r)$,

$$\frac{2s}{\operatorname{diam} f(B(x,r))} = \frac{\operatorname{diam} f(f^{-1}(B(f(x),s)))}{\operatorname{diam} f(B(x,r))} \geq \frac{1}{2\eta\left(\frac{\operatorname{diam} f^{-1}(B(x,s))}{\operatorname{diam} B(x,r)}\right)} \geq \frac{1}{2\eta(1)}. \tag{5.1}$$

By assumption, we know there is $E \subseteq B(f(x), s) \cap f(\mathbb{R}^d)$ and $g\colon E \to \mathbb{R}^d$ L-bi-Lipschitz such that

$$\mathscr{H}^d(E) \geq \kappa\, s^d \overset{(5.1)}{\geq} \frac{\kappa}{4^d\, \eta(1)^d} (\operatorname{diam} f(B(x,r)))^d.$$

Letting $E' = f^{-1}(E)$ and $c = \frac{\kappa}{4^d \eta(1)^d}$ proves (2).

(2) $\Rightarrow$ (1). Let $\xi \in f(\mathbb{R}^d)$ and $s > 0$, $x = f^{-1}(\xi)$, and set

$$r = \sup\{t : f(B(x,t)) \subseteq B(\xi, s)\}.$$

Since r is supremal, there is $y \in B(x,r)$ such that $|f(y) - f(x)| = s$. Also, by assumption, there is $E' \subseteq B(x,r)$ so that if $E = f(E') \subseteq B(\xi,s) \cap f(\mathbb{R}^d)$, we have

$$\mathscr{H}^d(E) \geq c\,(\operatorname{diam} f(B(x,r)))^d \geq c\,|f(x) - f(y)|^d = c\,s^d,$$

and so (1) holds with $E = f(E')$ and $\kappa = c$. □

Proof of Theorem 5.1. By Lemma 5.2, it suffices to show that there is $c > 0$ such that for all $x \in \mathbb{R}^d$ and $r > 0$, there is $E' \subseteq B(x,r)$ and an L-bi-Lipschitz map $g\colon f(E') \to \mathbb{R}^d$ such that $\mathscr{H}^d(f(E')) \geq c(\operatorname{diam} f(B(x,r)))^d$. Let $B(x,r) \subseteq \mathbb{R}^d$. By assumption, there is $E' \subseteq B(x,r)$ such that $|E'| \gtrsim |B(x,r)|$ and $\left(\frac{\operatorname{diam} f(B(x,r))}{\operatorname{diam} B(x,r)}\right)^{-1} f$ is L-bi-Lipschitz on E' for some L. Then

$$\begin{aligned}\mathscr{H}^d(f(E')) &\sim_L \left(\frac{\operatorname{diam} f(B(x,r))}{\operatorname{diam} B(x,r)}\right)^d |E'| \gtrsim \left(\frac{\operatorname{diam} f(B(x,r))}{\operatorname{diam} B(x,r)}\right)^d |B(x,r)| \\ &\gtrsim_d (\operatorname{diam} f(B(x,r)))^d.\end{aligned}$$

□

6. Big pieces implies a Carleson estimate

6.1. Preliminaries

In this section, we focus on proving (4) implies (1) in Theorem 1.4 by showing the following.

Theorem 6.1. *Suppose $f\colon \mathbb{R}^d \to \mathbb{R}^D$ is η-quasisymmetric, $d \geq 2$, and $f(\mathbb{R}^d)$ has BPBI(κ, L). Then $\omega_f(x,r)^2 \frac{dr}{r}\,dx$ is a Carleson measure, with Carleson constant depending on D, η, and the constants in the big pieces condition.*

6.2. A reduction using John–Nirenberg and the $\frac{1}{3}$-trick

In this section, we show how to reduce the proof of Theorem 6.1 to the following lemma, which we will prove in the following section.

Lemma 6.2. *Let $d \geq 2$, $f\colon \mathbb{R}^d \to \mathbb{R}^D$ be η-quasisymmetric, and suppose $f(\mathbb{R}^d)$ has BPBI(κ, L). Then for any $v \in \mathbb{R}^d$ and every $Q_0 \in \Delta$, there is $E \subseteq Q_0$ such that $|E| \gtrsim_{\eta,d,\kappa} |Q_0|$ and*

$$\sum_{\substack{R \subseteq Q_0 \\ R \cap E \neq \emptyset}} \omega_{f_v}(R)^2 |R| \lesssim_{d,\eta,\kappa} |Q_0|. \tag{6.1}$$

where f_v is the function $f_v(x) = f(x+v)$.

Proof of Theorem 6.1. Suppose $f\colon \mathbb{R}^d \to \mathbb{R}^D$ is η-quasisymmetric and the image of f has big-pieces of bi-Lipschitz images of $\mathbb{R}^d$.

First, we recall a version of the John–Nirenberg theorem.

Lemma 6.3 (Section IV.1 in [15]). *Let $a : \Delta \to [0,\infty)$ be given, and suppose there are $N, \delta > 0$ such that*

$$\left|\left\{x \in R : \sum_{Q \ni x\,,\, Q \subseteq R} a(Q) \le N\right\}\right| \ge \delta\,|R| \quad \text{for all } R \in \Delta. \tag{6.2}$$

Then

$$\sum_{Q \subseteq R} a(Q)\,|Q| \lesssim_{d,N,\delta} |R| \quad \text{for all } R \in \Delta. \tag{6.3}$$

If we assume Lemma 6.2, then each cube Q contains a set E for which

$$|E| \gtrsim_{d,\eta,\kappa} |Q| \gtrsim_{d,\eta,\kappa} \sum_{\substack{R \cap E \neq \emptyset \\ R \subseteq Q}} \omega_f(R)^2 |R| \gtrsim \sum_{R \subseteq Q} \omega_f(R)^2 |R \cap E| = \int_E \sum_{R \subseteq Q} \omega_f(R)^2 \mathbb{1}_R.$$

Hence, if $E' = \{x \in E : \sum_{x \in R \subseteq Q} \omega_f(R)^2 \le 2C\}$ where C is the product of the implied constants in the above inequalities, we get that $|E'| \ge \frac{1}{2}|E| \gtrsim |Q|$, and so Lemma 6.3 implies

$$\sum_{R \subseteq Q} \omega_f(R)^2 \lesssim_{d,\eta,c,L} |Q| \quad \text{for all } Q \in \Delta. \tag{6.4}$$

Theorem 6.1 does not follow just yet. We would like to employ Lemma 2.4, but this only works if we know

$$\sum_{R \subseteq Q} \omega_f(MR)^2 \lesssim_{d,\eta,c,L} |Q|$$

for some $M > 1$. However, (6.1) implies

$$\sum_{\substack{R \subseteq Q \\ R \in \Delta + v}} \omega_f(R)^2 \lesssim_{d,\eta,c,L} |Q| \quad \text{for all } Q \in \Delta + v, \quad v \in \mathbb{R}^d, \tag{6.5}$$

where $\Delta + v = \{Q + v : Q \in \Delta\}$ is the set of dyadic cubes translated by the vector v.

We now invoke the so-called $\frac{1}{3}$-trick, which says that, for any cube R with $\ell(R) = 2^{-k}/3$, $k \in \{0,1,2,\dots\}$, there is $Q \in \Delta + v$ for some $v \in \{0, 1/3\}^d$ such that $\ell(Q) = 2^{-k}$ and $R \subseteq Q$. For a proof, see [33], pages 339-40. Thus, if $R \in \Delta$ and $\ell(R) = 2^{-k-2}$ for some $k \ge 0$, then $\ell(\frac{4}{3}R) = 2^{-k}/3$, so there is

$$Q_R \in \tilde{\Delta} := \bigcup_{v \in \{0, \frac{1}{3}\}^d} (\Delta + v)$$

with $\ell(Q_R) = 2^{-k}$ containing $\frac{4}{3}R$. Moreover, since $\ell(Q_R) = 4\ell(R)$ and $Q_R \supseteq R$, we know $Q_R \subseteq 12R$ and there there is $C = C(d) > 0$ such that for any $Q \in \tilde{\Delta}$, there are at most C many cubes $R \in \Delta$ such that $Q_R = Q$.

Thus, for any $Q_0 \in \Delta$ with $\ell(Q_0) \leq 1/4$,

$$\sum_{R\subseteq Q_0} \omega_f\left(\frac{4}{3}R\right)^2 |R| \lesssim_d \sum_{R\subseteq Q_0} \omega_f(Q_R)^2 |Q| \lesssim_d \sum_{\substack{Q\in\tilde{\Delta} \\ Q\subseteq 12Q_0}} \omega_f(Q)^2 |Q|$$

$$= \sum_{v\in\{0,\frac{1}{3}\}^d} \sum_{\substack{Q\in\Delta+v \\ Q\subseteq 12Q_0}} \omega_f(Q)^2 |Q| \overset{(6.5)}{\lesssim}_{d,\eta,c,L} \sum_{v\in\{0,\frac{1}{3}\}^d} |Q_0| \lesssim_d |Q_0|. \tag{6.6}$$

Note that this holds for any η-quasisymmetric embedding of $\mathbb{R}^d$ into $\mathbb{R}^D$ whose image has BPBI(κ, L), and since ω_f is dilation and translation invariant, we know that (6.6) holds for any $Q_0 \in \Delta$, not just those with $\ell(Q_0) \leq 1/4$. We can now employ Lemma 2.4 to finish the theorem, at least if we assume Lemma 6.2 holds. □

6.3. Proof of Lemma 6.2

We now devote ourselves to the proof of Lemma 6.2.

Proof of Lemma 6.2. Note that if $f(\mathbb{R}^d)$ has BPBI(κ, L), then so does $f_v(\mathbb{R}^d)$ (where $f_v(x) := f(x+v)$), so without loss of generality, we will assume $v = 0$, since the other cases have the same proof.

Let $Q_0 \in \Delta$. By Lemma 5.2, we know there is

$$E' \subseteq B_0 := B(x_{Q_0}, \ell(Q_0)/2) \subseteq Q_0$$

and $g\colon f(E') \to \mathbb{R}^d$ L-bi-Lipschitz such that $\mathscr{H}^d(f(E')) \geq c(\operatorname{diam} f(B_0))^d$. By Lemma 4.12, there is $E_0 \subseteq B_0$ such that

$$\frac{|E_0|}{|Q_0|} \gtrsim_d \frac{|E_0|}{|B_0|} \gtrsim_{\eta,d,L,c} 1 \tag{6.7}$$

and $\left(\frac{\operatorname{diam} g\circ f(E_0)}{\operatorname{diam} E_0}\right)^{-1} g\circ f|_{E_0}$ is bi-Lipschitz and hence $\left(\frac{\operatorname{diam} f(E_0)}{\operatorname{diam} E_0}\right)^{-1} f|_{E_0}$ is M-bi-Lipschitz for some $M = M(d,\eta,L,c) > 0$. Recall that ω_f (and hence (6.1)) are invariant under a scaling of f in its image, thus we may assume $\operatorname{diam} f(E_0)/\operatorname{diam} E_0 = 1$ without loss of generality, so that f is M-bi-Lipschitz on E_0.

The following theorem of MacManus tells us that we can extend $f|_{E_0}$ to a bi-Lipschitz homeomorphism of $\mathbb{R}^{2D} \to \mathbb{R}^{2D}$.

Theorem 6.4 ([29]). *If K is a compact subset of $\mathbb{R}^D$ and Ψ is an M-bi-Lipschitz map of K into $\mathbb{R}^D$, then Ψ has an extension to a CM^2 bi-Lipschitz map from $\mathbb{R}^{2D}$ onto itself, where C is some universal constant.*

Viewing E_0 as being a subset of $\mathbb{R}^D$, we can extend f from the set E_0 to a CM^2 bi-Lipschitz self-map of $\mathbb{R}^{2D}$. Let $F : \mathbb{R}^d \to \mathbb{R}^{2D}$ be the restriction of this extension to $\mathbb{R}^d$, so that F is CM^2 bi-Lipschitz embedding of $\mathbb{R}^d$ into $\mathbb{R}^{2D}$ that agrees with f on E_0. By Lemma 4.5, we may find $E \subseteq E_0$ compact such that $|E| \geq \frac{1}{2}|E_0| \gtrsim_{\eta,d,L,c} |Q_0|$ and E is $\hat{C}$-serious for some constant $\hat{C}$ depending on the constants in (6.7). We will show E is the desired set such that (6.1) holds.

For $Q\in\Delta$, let A_Q be the orthogonal projection of $F|_Q \in L^2(Q)$ onto the finite dimensional subspace of $L^2(Q)$ consisting of linear $\mathbb{R}^D$ valued functions. Then

$$\Omega_F(Q) = \left(⨍_Q \left(\frac{|F-A_Q|}{\operatorname{diam} Q}\right)^2 \right)^{1/2}.$$

Let $Q_1, Q_2 \subseteq Q$ be such that $Q_j^2 = Q$ and $\operatorname{dist}(Q_1,Q_2) = \frac{1}{2}\operatorname{diam} Q$. Then there are $x_j \in Q_j$ such that

$$|F(x_j)-A_Q(x_j)|^2 \le ⨍_{Q_j} |F-A_Q|^2 \le 2^{2d} ⨍_Q |F-A_Q|^2 = 2^{2d}\left(\Omega_F(Q)\operatorname{diam} Q\right)^2.$$

Then

$$\begin{aligned}\frac{1}{CM^2} &\le \frac{|F(x_1)-F(x_2)|}{|x_1-x_2|}\\ &\le \frac{|F(x_1)-A_Q(x_1)|+|A_Q(x_1)-A_Q(x_2)|+|A_Q(x_2)-F(x_2)|}{\frac{1}{2}\operatorname{diam} Q}\\ &\le 2^{d+2}\,\Omega_F(Q) + 2\,|A_Q'|. \qquad (6.8)\end{aligned}$$

Set

$$\mathscr{B} = \left\{ Q\in\Delta : \Omega_F(Q) \ge \frac{1}{2^{d+3}CM^2}\right\},$$

and set

$$\mathscr{G}_E = \{Q \in \Delta\backslash\mathscr{B} : Q\subseteq Q_0, Q\cap E \neq \emptyset\}$$

so that (6.8) implies

$$|A_Q'| \ge \frac{1}{4CM^2} \quad \text{for all } Q\in\mathscr{G}_E. \qquad (6.9)$$

We now begin the process of showing (6.1) holds for the set E:

$$\begin{aligned}\sum_{\substack{Q\subseteq Q_0\\ Q\cap E\neq\emptyset}} \omega_f(Q)^2\,|Q| &\le \sum_{\substack{Q\subseteq Q_0\\ Q\in\mathscr{B}}} \omega_f(Q)^2\,|Q| + \sum_{Q\in\mathscr{G}_E} \omega_f(Q)^2\,|Q|\\ &\le \sum_{\substack{Q\subseteq Q_0\\ Q\in\mathscr{B}}} \omega_f(Q)^2\,|Q| + \sum_{Q\in\mathscr{G}_E} ⨍_Q \left(\frac{|f-A_Q|}{|A_Q'|\operatorname{diam} Q}\right)^2 |Q|\\ &\le \sum_{\substack{Q\subseteq Q_0\\ Q\in\mathscr{B}}} \omega_f(Q)^2\,|Q| + \sum_{Q\in\mathscr{G}_E} \int_Q \left(\frac{|F-A_Q|+|f-F|}{|A_Q'|\operatorname{diam} Q}\right)^2\\ &\le \sum_{\substack{Q\subseteq Q_0\\ Q\in\mathscr{B}}} \omega_f(Q)^2\,|Q| + 2\sum_{Q\in\mathscr{G}_E} \frac{\Omega_F(Q)^2}{|A_Q'|^2}|Q| + 2\sum_{Q\in\mathscr{G}_E}\int_Q \left(\frac{|f-F|}{|A_Q'|\operatorname{diam} Q}\right)^2. \qquad (6.10)\end{aligned}$$

We will estimate the three summands separately, starting with the first.

Let ϕ_{Q_0} be a smooth bump function such that

$$\mathbb{1}_{3Q_0} \leq \phi_{Q_0} \leq \mathbb{1}_{4\phi_{Q_0}} \quad \text{and} \quad |\partial^\alpha \phi_{Q_0}| \lesssim_{d,\alpha} \ell(Q_0)^{-|\alpha|}.$$

Then by Dorronsoro's theorem, and since $\omega_f(Q) \leq 1$ for all Q,

$$\begin{aligned}
\sum_{\substack{Q\in\mathcal{Q}_0\\ Q\in\mathscr{B}}} \omega_f(Q)^2\,|Q| &\leq \sum_{\substack{Q\in\mathcal{Q}_0\\ Q\in\mathscr{B}}} |Q| \leq \sum_{\substack{Q\in\mathcal{Q}_0\\ Q\in\mathscr{B}}} \left(2^{d+3}CM^2\right)^2 \Omega_F(Q)^2\,|Q| \\
&\lesssim_{d,M} \sum_{Q\subseteq Q_0} \Omega_F(Q)^2\,|Q| = \sum_{Q\subseteq Q_0} \Omega_{\phi_{Q_0}(F-F(x_{Q_0}))}(Q)^2\,|Q| \\
&\lesssim_D ||\nabla(\phi_{Q_0}(F-F(x_{Q_0})))||_2^2 \\
&\leq ||\nabla\phi_{Q_0}(F-F(x_{Q_0})) + \phi_{Q_0}\nabla(F-F(x_{Q_0}))||_2^2 \\
&\lesssim_d \frac{1}{\ell(Q_0)^2}\int_{4Q_0}(F-F(x_{Q_0}))^2 + \int_{4Q_0}|\nabla(F-F(x_{Q_0}))|^2 \\
&\leq \frac{1}{\ell(Q_0)^2}\int_{4Q_0}(CM^2\operatorname{diam}4Q_0)^2 + \int_{4Q_0}(CM^2)^2 \lesssim_{d,M} |Q_0|. \qquad (6.11)
\end{aligned}$$

For the second summand in (6.10), we use (6.9) and Dorronsoro's theorem to estimate

$$\begin{aligned}
\sum_{Q\in\mathscr{G}_E} \frac{\Omega_F(Q)^2}{|A'_Q|^2}\,|Q| &\overset{(6.9)}{\leq} \sum_{Q\in\mathscr{G}_E} (4CM^2)^2\,\Omega_F(Q)^2\,|Q| \\
&\lesssim_{d,M} \sum_{Q\subseteq Q_0} \Omega_F(Q)^2 \overset{(6.11)}{\lesssim_{D,M}} |Q_0|. \qquad (6.12)
\end{aligned}$$

Now we focus on the final sum in (6.10). For any $Q \subseteq Q_0$ such that $Q\cap E \neq \emptyset$, if $x \in Q\cap E$, then there is $y \in E_0$ such that $\hat{C}^{-1}\operatorname{diam} Q \leq |x-y| \leq \operatorname{diam} Q$ (because E is a $\hat{C}$-serious subset of E_0). Hence, if $z \in Q$ is such that $|f(x)-f(z)| \geq \frac{1}{2}\operatorname{diam} f(Q)$,

$$\begin{aligned}
\operatorname{diam} f(Q) &\leq 2\,|f(x)-f(z)| = 2\,\frac{|f(x)-f(z)|}{|f(x)-f(y)|}\,|f(x)-f(y)| \\
&\leq 2\,\eta\Big(\frac{|x-z|}{|x-y|}\Big)\,|f(x)-f(y)| \leq 2\,\eta\Big(\frac{\operatorname{diam} Q}{\hat{C}^{-1}\operatorname{diam} Q}\Big)\,|F(x)-F(y)| \\
&\leq 2\,\eta(\hat{C})\,CM^2\,|x-y| \leq 2\,\eta(\hat{C})\,CM^2\,\operatorname{diam} Q. \qquad (6.13)
\end{aligned}$$

We will require some estimates on the Hölder continuity of f.

Corollary 6.5. *Let $f\colon \mathbb{R}^d \to \mathbb{R}^D$ be η-quasisymmetric and $K \subseteq \mathbb{R}^d$ a bounded set. Then there are constants $C > 0$ and $\alpha \in (0,1)$, depending only on η, such that for all $x, y \in K$ distinct,*

$$\frac{1}{2C}\Big(\frac{|x-y|}{\operatorname{diam} K}\Big)^{1/\alpha} \leq \frac{|f(x)-f(y)|}{\operatorname{diam} f(K)} \leq 2^\alpha\,C\Big(\frac{|x-y|}{\operatorname{diam} K}\Big)^\alpha.$$

We will prove this in Section 7.1 in the appendix.

Let $\{Q_j\}$ be the Whitney cube decomposition for E^c, comprised of those maximal dyadic cubes $Q_j \subseteq E^c$ for which $3Q_j \cap E = \emptyset$. Then it is not too hard to show that, for each j,

$$\ell(Q_j) \leq \text{dist}(x, E) \leq 4 \,\text{diam}\, Q_j \quad \text{for all } x \in Q_j. \tag{6.14}$$

Let $Q \subseteq Q_0$. For $x \in Q$, let x' denote a point in E such that $|x-x'| = \text{dist}(x, E)$, and pick Q_j containing x. By Corollary 6.5, and since $f = F$ on E, there are constants $\alpha \in (0, 1)$ and $C_\eta > 0$ such that

$$\begin{aligned}
|f(x) - F(x)| &\leq |f(x) - f(x')| + |f(x') - F(x')| + |F(x') - F(x)| \qquad (6.15)\\
&\leq 2^\alpha C_\eta \Big(\frac{|x-x'|}{\text{diam}\, Q}\Big)^\alpha \text{diam}\, f(Q) + 0 + CM^2\, |x - x'| \\
&\leq C_\eta \Big(2\,\frac{\text{dist}(x, E)}{\ell(Q)}\Big)^\alpha \text{diam}\, f(Q) + CM^2 \,\text{dist}(x, E) \\
&= C_\eta \Big(2 \frac{\text{dist}(x, E)}{\ell(Q)}\Big)^\alpha \text{diam}\, f(Q) + CM^2\, \frac{\text{dist}(x, E)}{\ell(Q)}\, \ell(Q) \\
&\overset{(6.14)}{\leq} \Big(\frac{4\sqrt{d}\, \ell(Q_j)}{\ell(Q)}\Big)^\alpha C_\eta \,\text{diam}\, f(Q) + CM^2\, \frac{8\sqrt{d}\, \ell(Q_j)}{\ell(Q)}\, \ell(Q) \\
&\overset{(6.13)}{\leq} 8\sqrt{d}\, \Big(\frac{\ell(Q_j)}{\ell(Q)}\Big)^\alpha (2C_\eta\, \eta(\hat{C})\, CM^2 + CM^2)\, \ell(Q) \\
&= (1 + 2C_\eta\, \eta(\hat{C}))\, 8CM^2 \Big(\frac{\ell(Q_j)}{\ell(Q)}\Big)^\alpha \ell(Q). \qquad (6.16)
\end{aligned}$$

Before proceeding, we will need the following geometric lemma.

Lemma 6.6. *Let $\alpha > 0$, $K \subseteq Q_0 \in \Delta(\mathbb{R}^d)$ be any compact subset, and $\{Q_j\}$ be a Whitney decomposition for K^c. For $Q \subseteq Q_0$, define*

$$\lambda_{K,\alpha}(Q) := \sum_{Q_j \subseteq Q} \Big(\frac{\ell(Q_j)}{\ell(Q)}\Big)^{d+\alpha}.$$

Then, for all $Q \subseteq Q_0$,

$$\lambda_{K,\alpha}(Q) \leq \frac{|Q\backslash K|}{|K|} \leq 1, \tag{6.17}$$

$$\sum_{Q \subseteq Q_0} \lambda_{K,\alpha}(Q)|Q| \leq \frac{1}{1-2^{-\alpha}}\, |Q_0\backslash K|. \tag{6.18}$$

Proof. Fix $\alpha > 0$ and set $\lambda = \lambda_{K,\alpha}$. For the first part of the lemma, observe that since the Q_j are disjoint and $\ell(Q_j)/\ell(Q) \leq 1$ if $Q_j \subseteq Q$,

$$\lambda(Q) = \sum_{Q_j \subseteq Q} \Big(\frac{\ell(Q_j)}{\ell(Q)}\Big)^{n+\alpha} \leq \sum_{Q_j \subseteq Q} \Big(\frac{\ell(Q_j)}{\ell(Q)}\Big)^{n} = \frac{1}{|Q|} \sum_{Q_j \subseteq Q} |Q_j| = \frac{1}{|Q|}|Q\backslash K| \leq 1.$$

Now we show (6.18). By Fubini's theorem,

$$\sum_{Q\subseteq Q_0}\lambda(Q)|Q| = \sum_{Q\subseteq Q_0}\sum_{Q_j\subseteq Q}\left(\frac{\ell(Q_j)}{\ell(Q)}\right)^{n+\alpha}|Q| = \sum_{Q_j\subseteq Q_0}\ell(Q_j)^{n+\alpha}\sum_{Q_j\subseteq Q\subseteq Q_0}\frac{|Q|}{\ell(Q)^{n+\alpha}}$$

$$= \sum_{Q_j\subseteq Q_0}\ell(Q_j)^{n+\alpha}\sum_{Q_j\subseteq Q\subseteq Q_0}\ell(Q)^{-\alpha} = \sum_{Q_j\subseteq Q_0}\ell(Q_j)^{n+\alpha}\sum_{j=0}^{\log_2 \ell(Q_0)/\ell(Q_j)}\ell(Q_j)^{-\alpha}2^{-j\alpha}$$

$$\leq \sum_{Q_j\subseteq Q_0}\ell(Q_j)^{n}\frac{1}{1-2^{-\alpha}} = \frac{1}{1-2^{-\alpha}}\,|Q_0\backslash K|.$$

□

We continue with the proof. Since $|f(x)-F(x)|=0$ on E and $E^c=\bigcup Q_j$ since E is closed,

$$\sum_{Q\in\mathscr{G}_E}\int_Q\left(\frac{|f(x)-F(x)|}{\operatorname{diam} Q}\right)^2dx = \sum_{Q\in\mathscr{G}_E}\sum_{Q_j\subseteq Q}\int_{Q_j}\left(\frac{|f(x)-F(x)|}{\operatorname{diam} Q}\right)^2dx$$

$$\overset{(6.16)}{\lesssim}_{\eta,d,M}\sum_{Q\in\mathscr{G}_E}\sum_{Q_j\subseteq Q}\left(\frac{\ell(Q_j)}{\ell(Q)}\right)^{2\alpha}|Q_j| = \sum_{Q\in\mathscr{G}_E}\sum_{Q_j\subseteq Q}\left(\frac{\ell(Q_j)}{\ell(Q)}\right)^{d+2\alpha}|Q|$$

$$= \sum_{Q\in\mathscr{G}_E}\lambda_{E,2\alpha}(Q)|Q| \overset{(6.18)}{\lesssim}_{\alpha}|Q_0|. \tag{6.19}$$

and this bounds the third sum in (6.10).

Combining (6.10), (6.11), (6.12), and (6.19), we obtain

$$\sum_{\substack{Q\subseteq Q_0\\ Q\cap E'\neq\emptyset}}\omega_f(Q)^2\,|Q|\lesssim_{d,M,\eta}|Q_0|.$$

Finally, recall that M and α depend on η, c, D, and L. This finishes the proof. □

7. Appendix

7.1. Hölder estimates: the proof of Corollary 6.5

Lemma 7.1 (Theorem 11.3 in [23]). *An η-quasisymmetric embedding $f\colon\Omega\to\mathbb{R}^D$, where $\Omega\subseteq\mathbb{R}^d$ is connected, is $\tilde\eta$-quasisymmetric with $\tilde\eta$ of the form*

$$\tilde\eta = C\max\{t^\alpha, t^{1/\alpha}\} \tag{7.1}$$

where $C\geq 1$ and $\alpha\in(0,1]$ depend only on η.

The original lemma is stated for A-uniformly perfect spaces (metric spaces X such that $B(x,r)\backslash B(x,r/A)$ is nonempty for all $x\in X$ and $r>0$), and the constants C and α depend also on the constant associated with being uniformly perfect, but connected sets happen to be uniformly perfect with $A=1$ (see the beginning of Chapter 11 of [23] for a discussion and the original statement). As a corollary, we have the following:

Corollary 6.5. *Let $f\colon \mathbb{R}^d \to \mathbb{R}^D$ be η-quasisymmetric and $E \subseteq \mathbb{R}^d$ a bounded set. Then for all $x, y \in E$ distinct,*

$$\frac{1}{2C}\Big(\frac{|x-y|}{\operatorname{diam} E}\Big)^{1/\alpha} \leq \frac{|f(x)-f(y)|}{\operatorname{diam} f(E)} \leq 2^\alpha C\Big(\frac{|x-y|}{\operatorname{diam} E}\Big)^\alpha,$$

where α and C are as in Lemma 7.1.

Proof. Let $x, y \in E$. Pick $y' \in E$ so that $|x-y'| \geq \max\{\frac{1}{2}\operatorname{diam} E, |x-y|\}$. Then

$$\frac{|f(x)-f(y)|}{\operatorname{diam} f(E)} \leq \frac{|f(x)-f(y)|}{|f(x)-f(y')|} \leq \eta\Big(\frac{|x-y|}{|x-y'|}\Big) \leq C\Big(\frac{|x-y|}{|x-y'|}\Big)^\alpha \leq C\,2^\alpha\Big(\frac{|x-y|}{\operatorname{diam} E}\Big)^\alpha.$$

Now, let $y'' \in E$ be such that $|f(x)-f(y'')| \geq \frac{1}{2}\operatorname{diam} f(E)$. Then,

$$\frac{\operatorname{diam} f(E)}{|f(x)-f(y)|} \leq 2\frac{|f(x)-f(y'')|}{|f(x)-f(y)|} \leq 2\eta\Big(\frac{|x-y''|}{|x-y|}\Big). \tag{7.2}$$

If $|x-y''| \leq |x-y|$, then

$$(7.2) \overset{(7.1)}{\leq} 2\,C\Big(\frac{|x-y''|}{|x-y|}\Big)^\alpha \leq 2\,C\Big(\frac{\operatorname{diam} E}{|x-y|}\Big)^\alpha \leq 2\,C\Big(\frac{\operatorname{diam} E}{|x-y|}\Big)^{1/\alpha}.$$

If $|x-y''| > |x-y|$, then

$$(7.2) \overset{(7.1)}{\leq} 2\,C\Big(\frac{|x-y''|}{|x-y|}\Big)^{1/\alpha} \leq 2\,C\Big(\frac{\operatorname{diam} E}{|x-y|}\Big)^{1/\alpha}.$$

Hence, in either case,

$$\frac{\operatorname{diam} f(E)}{|f(x)-f(y)|} \leq 2\,C\Big(\frac{\operatorname{diam} E}{|x-y|}\Big)^{1/\alpha},$$

which proves the lemma. □

7.2. Proof of Lemma 2.1

Lemma 2.1. *Let $\delta > 0$. If f is η-quasisymmetric on a cube $Q \subseteq \mathbb{R}^d$, then there is $\varepsilon_1 = \varepsilon_1(\eta, d, \delta) > 0$ so that if*

$$⨍_Q \frac{|f-A|}{|A'|\operatorname{diam} Q} < \varepsilon_1. \tag{7.3}$$

then

$$|f(x) - A(x)| < \delta|A'|\operatorname{diam} Q.$$

Moreover,

$$(1-2\sqrt{d}\delta)\,|A'|\,\ell(Q) \leq \operatorname{diam} f(Q) \leq (1+2\sqrt{d}\delta)\,|A'|\,\operatorname{diam} Q.$$

Proof. Fix $K > 0$ and let

$$E_K = \{x \in Q : |f(x) - A_Q(x)| \leq K\varepsilon_1|A_Q'|\operatorname{diam} Q\}$$

so that by Chebyshev's inequality,

$$|Q\backslash E_K| \leq \frac{1}{K}|Q|.$$

Note that if $B = B(x,r)$ is any ball contained in E_K^c, not necessarily contained in Q but with center $x \in Q\backslash E_K$, then at least $1/2^d$ percent of it is contained in Q, and so

$$w_d r^d = |B| \leq 2^d\,|Q\backslash E_K| \leq \frac{2^d}{K}\,|Q|,$$

so that

$$r \leq 2(kw_d)^{-d}\,\ell(Q).$$

Pick $K = w_d^{-1} 2^d \varepsilon_1^{-1/d}$ so that $r \leq \varepsilon\ell(Q)$. Then

$$\sup_{x\in Q\backslash E_K} \operatorname{dist}(x, E_K) \leq \varepsilon_1\,\ell(Q).$$

For $x \in Q\backslash E_K$, let $x' \in E_K$ be such that $|x - x'| = \operatorname{dist}(x, E_K)$. Then by Corollary 6.5,

$$\begin{aligned}
|f(x) - A(x)| &\leq |f(x) - f(x')| + |f(x') - A(x')| + |A(x') - A(x)| \\
&\leq 2^\alpha\,C\Big(\frac{|x - x'|}{\operatorname{diam} Q}\Big)^\alpha \operatorname{diam} f(Q) + K\,\varepsilon_1\,|A'|\operatorname{diam} Q + |A'|\,|x - x'| \\
&\leq 2^\alpha\,C\Big(\frac{\varepsilon_1\ell(Q)}{\operatorname{diam} Q}\Big)^\alpha \operatorname{diam} f(Q) + w_d\,2^d\,\varepsilon_1^{1-1/d}\,|A'|\,\operatorname{diam} Q + 2|A'|\,\varepsilon\,\ell(Q) \\
&\leq 2^\alpha\,C\,\varepsilon_1^\alpha \operatorname{diam} f(Q) + (w_d 2^d + 2)\,\varepsilon_!^{1-1/d}\,|A'|\,\operatorname{diam} Q.
\end{aligned}$$

We claim that $|A'|\operatorname{diam} Q \geq \frac{1}{4}\operatorname{diam} f(Q)$ if ε_1 is small enough. If $|A'|\operatorname{diam} Q < \frac{1}{4}\operatorname{diam} f(Q)$, pick $x_0 \in Q$ so that $|f(x_0) - A(x_0)| \leq \varepsilon|A'|\operatorname{diam} Q$ and pick $x \in Q$ so that $|f(x) - f(x_0)| \geq \frac{1}{2}\operatorname{diam} f(Q)$. Then

$$\begin{aligned}
|f(x) - A(x)| &\geq |f(x) - f(x_0)| - |f(x_0) - A(x_0)| - |A(x_0) - A(x)| \\
&\geq \frac{1}{2}\operatorname{diam} f(Q) - \varepsilon_1|A'|\operatorname{diam} Q - |A'|\operatorname{diam} Q \\
&\geq \Big(\frac{1}{2} - \frac{(1+\varepsilon_1)}{4}\Big)\operatorname{diam} f(Q) \geq \frac{1}{8}\operatorname{diam} f(Q)
\end{aligned}$$

if $\varepsilon_1 < 1/2$. However,

$$\begin{aligned}
|f(x) - A(x)| &\leq 2^\alpha\,C\,\varepsilon_1^\alpha \operatorname{diam} f(Q) + (w_d 2^d + 2)\,\varepsilon_1^{1-1/d}\,|A'|\operatorname{diam} Q \\
&< \Big(2^\alpha\,C\,\varepsilon_1^\alpha + \frac{1}{4}\,(w_d\,2^d + 2)\,\varepsilon_1^{1-1/d}\Big)\operatorname{diam} f(Q) < \frac{1}{8}\operatorname{diam} f(Q)
\end{aligned}$$

if $\varepsilon > 0$ is small enough, which is a contradiction. Thus, $|A'|\operatorname{diam} Q \geq \frac{1}{4}\operatorname{diam} f(Q)$, so that

$$|f(x) - A(x)| \leq (2^{\alpha+2}\,C\,\varepsilon^\alpha + (w_d\,2^d + 2)\,\varepsilon^{1-1/d})\,|A'|\operatorname{diam} Q < \delta\,|A'|\operatorname{diam} Q$$

if $\varepsilon > 0$ is picked small enough.

For the last part of the lemma, let $x, y \in Q$ be such that $\operatorname{diam} f(Q) = |f(x) - f(y)|$. Then,

$$\begin{aligned}\operatorname{diam} f(Q) = |f(x) - f(y)| &\leq |A(x) - A(y)| + 2\delta\,|A'|\operatorname{diam} Q\\ &\leq |A'|\,|x - y| + 2\delta \operatorname{diam} Q \leq (1 + 2\delta\sqrt{d}\,)\,|A'|\,\operatorname{diam} Q.\end{aligned}$$

For the opposite inequality, we may assume without loss of generality that $x_Q = 0$. Pick $x \in \partial B(x_Q, \ell(Q)/2)$ so that $|A(x) - A(-x)| = |A'| \operatorname{diam} B_Q$. Then,

$$\begin{aligned}\operatorname{diam} f(B_Q) \geq |f(x) - f(y)| &\geq |A(x) - A(y)| - 2\delta \operatorname{diam} Q\\ &= |A'|\,(1 - 2\delta\sqrt{d}\,)\operatorname{diam} B_Q = |A'|\,(1 - 2\delta\sqrt{d}\,)\,\ell(Q).\end{aligned}$$

□

7.3. Dorronsoro's theorem

Here we prove the following special case of Dorronsoro's theorem. We prove a more general version than what is stated in the introduction by showing we can replace Ω_f with a general L^p-type integral for $p \in [1, 2]$ and obtain the same result. Throughout the paper, however, we only use the $p = 2$ case and write $\Omega_f = \Omega_{2,f}$ for short.

Theorem 1.2 ([17]). *Let $f \in L^2(\mathbb{R}^d)$. For $x \in \mathbb{R}^d$, $r > 0$, and $p \in [1, 2]$, define*

$$\Omega_{p,f}(x, r) = \inf_A \left(\fint_{B(x,r)} \left(\frac{|f - A|}{r} \right)^p \right)^{1/p},$$

where the infimum is over all affine maps $A : \mathbb{R}^d \to \mathbb{R}$. Then $f \in W^{1,2}(\mathbb{R}^d)$ if and only if

$$\Omega_p(f) := \int_{\mathbb{R}^d} \int_0^\infty \Omega_{p,f}(x, r)^2 \,\frac{dr}{r}\, dx < \infty,$$

in which case,

$$||\nabla f||_2^2 \lesssim_d \Omega_1(f) \leq \Omega_q(f) \leq \Omega_2(f) \lesssim_d ||\nabla f||_2 \tag{7.4}$$

for all $q \in [1, 2]$, so in particular, $||\nabla f||_2^2 \sim_d \Omega_q(f)$ for $q \in [1, 2]$.

We should mention, of course, that the original result is far more general; in particular, Dorronsoro gives a characterization of the fractional Sobolev spaces $W^{\alpha,p}$ for all $\alpha > 0$ and $p \in (1, \infty)$. We provide a proof of this special case for the interested reader, since the proof we supply avoids the interpolation theory and reference chasing in [17]; only the basic properties of Sobolev spaces and the Fourier transform are needed. This proof is well known, but not completely written down anywhere to the author's knowledge (although hints at the proof are alluded to in [13]); part of it is also explained in [10].

Proof. Step 1. We first show $||\nabla f||_2^2 \lesssim_d \Omega_1(f)$ supposing that $f \in W^{1,2}(\mathbb{R}^d)$ (we will show later that $\Omega_1(f) < \infty$ implies an L^2 function f is actually in $W^{1,2}$, but we will start with this case). Let ϕ be a radially symmetric C^∞ function supported in $B(0,1)$ such that $\int \phi = 1$. Set $\psi(x) = \phi(x) - 2^d\phi(2x)$, so that it is also supported in $B(0,1)$. Then $\int \psi A = 0$ for any affine function A. For $r > 0$, set $\psi_r(x) = r^{-d}\psi(r^{-1}x)$. Then,

$$\begin{aligned}|\nabla f * \psi_r(x)| &= |\nabla(f-A) * \psi_r(x)| = |(f-A) * \nabla\psi_r(x)|\\ &\leq \Big|\int_{B(x,r)} |f(y)-A(y)| r^{-d-1}\nabla\psi(r^{-1}(x-y))\,dy\Big| \leq w_d\,||\nabla\psi||_\infty \fint_{B(x,r)} \frac{|f-A|}{r},\end{aligned}$$

and infimizing over all affine maps A gives

$$|\nabla f * \psi_r(x)| \leq w_d\,||\nabla\psi||_\infty\,\Omega_{1,f}(x,r). \tag{7.5}$$

Observe that by Fubini's theorem and Plancherel's theorem,

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^d} |\nabla f * \psi_r(x)|^2\,dx\,\frac{dr}{r} &= \int_0^\infty \int_{\mathbb{R}^d} |\widehat{\nabla f}(\xi)|^2\,|\hat\psi(r\xi)|^2\,d\xi\,\frac{dr}{r}\\ &= \int_{\mathbb{R}^d} |\widehat{\nabla f}(\xi)|^2 \Big(\int_0^\infty |\hat\psi(r\xi)|^2\,\frac{dr}{r}\Big)\,d\xi.\end{aligned}$$

Since ψ is radially symmetric, so is $\hat\psi$, thus, if $e_1 \in \mathbb{R}^d$ denotes the first standard basis vector,

$$\int_0^\infty |\hat\psi(r\xi)|^2\,\frac{dr}{r} = \int_0^\infty |\hat\psi(r|\xi|e_1)|^2\,\frac{dr}{r} = \int_0^\infty |\hat\psi(re_1)|^2\,\frac{dr}{r} =: c_\psi < \infty.$$

The reason this is finite is because $\hat\psi$ is a Schwartz function, $\hat\psi(0) = \int\psi = 0$, and $\hat\psi$ is differentiable at zero, so $|\psi(\xi)| \lesssim |\xi|/(|1+|\xi|^3)$. Thus,

$$\begin{aligned}\omega_d||\nabla\psi||_\infty \int_{\mathbb{R}^d}\int_0^\infty \Omega_{1,f}(x,r)^2\,\frac{dr}{r} &\geq \int_0^\infty \int_{\mathbb{R}^d} |\nabla f * \psi_r(x)|^2\,dx\,\frac{dr}{r}\\ &= c_\psi \int_{\mathbb{R}^d} |\widehat{\nabla f}(\xi)|^2\,d\xi = c_\psi \int_{\mathbb{R}^d} |\nabla f(x)|^2\,dx.\end{aligned}$$

This proves the first inequality in (7.4).

Step 2. Now just suppose $f \in L^2(\mathbb{R}^d)$, we will show that $\Omega_1(f) < \infty$ implies f has a weak gradient ∇f that is in L^2. Let ϕ be a nonnegative C^∞ bump function supported in $B(0,1)$ with $\int\phi = 1$. Observe that since $||\hat\phi||_\infty \leq \int\phi = 1$, we have

$$||f * \phi_t||_2 = ||\hat f\hat\phi_t||_2 \leq ||\hat f||_2 = ||f||_2$$

and $||\hat{\nabla}\phi||_\infty \leq ||\nabla\phi||_1 < \infty$, so that

$$\begin{aligned}||\nabla(f*\phi_t)||_2^2 = ||f * \nabla\phi_t||_2^2 &= \int_{\mathbb{R}^d} |\hat f(\xi)|^2\,|\widehat{\nabla\phi_t}(\xi)|^2\,d\xi\\ &= \int_{\mathbb{R}^d} |\hat f(\xi)|^2\,t^{-2}\,|\widehat{\nabla\phi}(t\xi)|^2\,d\xi \leq t^{-2}\,||\nabla\phi||_1 \int_{\mathbb{R}^d} |\hat f(\xi)|^2 = \frac{||f||_2^2}{t^2}.\end{aligned}$$

Thus, $f*\phi_t \in W^{1,2}(\mathbb{R}^d)$, and so we know that

$$||\nabla f * \phi_t||_2^2 \lesssim_d \int_{\mathbb{R}^d}\int_0^\infty \Omega_{1,f*\phi_t}(x,r)^2\,\frac{dr}{r}\,dx.$$

Suppose $r \geq t$. Since $\int \phi = 1$, and since affine functions are harmonic, we have $\phi_t * A = A$ for any affine function, thus

$$\fint_{B(x,r)} \frac{|f*\phi_t - A|}{r} = \fint_{B(x,r)} \frac{|(f-A)*\phi_t|}{r} \leq \fint_{B(x,r)} \int_{\mathbb{R}^d} \frac{|f(z)-A(z)|}{r} \phi_t(y-z)\, dz\, dy$$
$$= \frac{|B(x,2r)|}{|B(x,r)|} \int_{\mathbb{R}^d} \fint_{B(x,2r)} \frac{|f(z)-A(z)|}{r} \phi_t(y-z)\, dy\, dz = 2^d \fint_{B(x,2r)} \frac{|f(z)-A(z)|}{r}\, dz,$$

and infimizing over affine maps A gives

$$\Omega_{1,f*\phi_t}(x,r) \leq 2^d\, \Omega_{1,f}(x,2r) \quad \text{for } r \geq t. \tag{7.6}$$

Now suppose $r < t$. Then by Taylor's theorem, and since $\partial^\alpha \phi_t * A = 0$ for any affine map A and $|\alpha| \geq 1$,

$$\begin{aligned}\frac{\Omega_{1,f*\phi_t}(x,r)}{r} &\leq \frac{r^2 \max_{|\alpha|=2} ||\partial^\alpha f * \phi_t||_{L^\infty(B(x,t))}}{r^2} = \max_{|\alpha|=2} ||f * \partial^\alpha \phi_t||_{L^\infty(B(x,t))} \\
&= \max_{|\alpha|=2} ||(f-A) * \partial^\alpha \phi_t||_{L^\infty(B(x,t))} \\
&\leq \max_{|\alpha|=2} \sup_{z \in B(x,t)} \int |f(y)-A(y)|\, |\partial^\alpha \phi_t(z-y)|\, dy \\
&= t^{-2} \max_{|\alpha|=2} \sup_{z\in B(x,t)} \int_{B(x,r+t)} |f(y)-A(y)|\, |(\partial^\alpha \phi)_t(z-y)|\, dy \\
&\leq t^{-2} \max_{|\alpha|=2} \sup_{z\in B(x,t)} \int_{B(x,2t)} |f(y)-A(y)|\, \frac{||\partial^\alpha \phi||_\infty}{t^d}\, dy \\
&= t^{-1}\, w_d \max_{|\alpha|=2} ||\partial^\alpha \phi||_\infty \fint_{B(x,2t)} \frac{|f(y)-A(y)|}{t}\, dy\,,\end{aligned}$$

and infimizing over affine maps A gives

$$\Omega_{1,f*\phi_t}(x,r) \leq \frac{r}{t} w_d \max_{|\alpha|=2} ||\partial^\alpha \phi||_\infty\, \Omega_{1,f}(x,2t).$$

Thus,

$$\begin{aligned}||\nabla f * \phi_t||_2^2 &\lesssim_d \int_{\mathbb{R}^d} \Big(\int_0^t \Omega_{1,f*\phi_t}(x,r) \frac{dr}{r} + \int_t^\infty \Omega_{1,f*\phi_t}(x,r)^2 \frac{dr}{r} \Big)\, dx \\
&\lesssim_d \int_{\mathbb{R}^d} \Big(\int_0^t \frac{r}{t^2} \Omega_{1,f}(x,2t)^2\, dr + \int_t^\infty \Omega_{1,f}(x,2r)^2 \frac{dr}{r} \Big)\, dx \\
&\leq \frac{1}{2} \int_{\mathbb{R}^d} \Omega_{1,f}(x,2t)^2\, dx + \Omega_1(f)^2.\end{aligned}$$

Since $\Omega_{1,f}(x,t) \leq 2^d\, \Omega_{1,f}(x,t+s)$ for all $s \in [0,t]$, we have

$$\begin{aligned}\int_{\mathbb{R}^d} \Omega_{1,f}(x,2t)^2 dx &\lesssim \int_{\mathbb{R}^d} \int_{2t}^{4t} \Omega_{1,f}(x,2t)^2 \frac{dr}{r}\, dx \lesssim_d \int_{\mathbb{R}^d} \int_{2t}^{4t} \Omega_{1,f}(x,r)^2 \frac{dr}{r}\, dx \\
&\leq \int_{\mathbb{R}^d} \int_0^\infty \Omega_{1,f}(x,r)^2 \frac{dr}{r}\, dx = \Omega_1(f)^2.\end{aligned}$$

Hence, $||\nabla f * \phi_t||_2^2 \lesssim_d \Omega_1(f)^2$, and since $\hat{\phi}(t\xi) \to 1$ uniformly on compact subsets of $\mathbb{R}^d$ as $t \to 0$, we have, for any $R > 0$ and $B_R = B(0,R)$,

$$\begin{aligned}\int_{B_R} |\hat{f}(\xi)|^2\,|\xi|^2\,d\xi &= \lim_{t\to 0}\int_{B_R} |\hat{f}(\xi)|^2\,|\xi|^2|\,\hat{\phi}(t\xi)|^2\,d\xi = \lim_{t\to 0}\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2\,|\xi|^2|\,\hat{\phi}(t\xi)|^2\,d\xi \\ &= \lim_{t\to 0}\int_{\mathbb{R}^d} |\nabla f * \phi_t|^2 \lesssim_d \int_{\mathbb{R}^d}\int_0^\infty \Omega_{1,f}(x,r)^2\,\frac{dr}{r} = \Omega_1(f)^2.\end{aligned}$$

Letting $R \to \infty$, we get $\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2|\xi|^2 d\xi \lesssim_d \Omega_1(f)^2$, which implies $f \in W^{1,2}(\mathbb{R}^d)$.

Step 3. Note that the second and third inequalities in (7.4) follow from Jensen's inequality since $\Omega_{p,f}(x,r) \le \Omega_{q,f}(x,r)$ if $p \le q$, and hence $\Omega_p(f) \le \Omega_q(f)$.

Step 4. It remains to prove the last inequality $\Omega_2(f) \lesssim_d ||\nabla f||_2^2$. To do so, we follow the hint in [13].

Assume $f \in W^{1,2}(\mathbb{R}^d)$ and let ϕ be a radially symmetric nonnegative function supported in $B(0,1)$ such that $\int \phi = 1$. Define an affine map

$$A_{x,r}(y) = \phi_r * \nabla f(x)\cdot(y-x) + f * \phi_r(x).$$

Then by Tonelli's theorem, change of variables, Plancherel's theorem, and the fact that, for $p \in [1,2]$,

$$\Omega_{2,f}(x,r)^2 \le ⨍_{B(x,r)} \frac{|f(y) - A_{x,r}|^2}{r^2}\,dy,$$

we have

$$\begin{aligned}\omega_d \int_{\mathbb{R}^d}\int_0^\infty \Omega_{2,f}(x,r)^2\,\frac{dt}{t}\,dx &\le \int_{\mathbb{R}^d}\int_0^\infty \omega_d ⨍_{B(x,r)} \frac{|f(y)-A_{x,r}|^2}{r^2}\,dy\,\frac{dr}{r}\,dx \\ &= \int_{\mathbb{R}^d}\int_0^\infty\int_{B(x,r)} |f(y) - \phi_r * \nabla f(x)\cdot(y-x) - f*\phi_r(x)|^2\,dy\,\frac{dr}{r^{d+3}}\,dx \\ &= \int_{B(0,r)}\int_0^\infty\int_{\mathbb{R}^d} |f(y+x) - \phi_r * \nabla f(x)\cdot y - f*\phi_r(x)|^2\,dx\,\frac{dr}{r^{d+3}}\,dy \\ &= \int_{B(0,r)}\int_0^\infty\int_{\mathbb{R}^d} |\hat{f}(\xi)e^{-2\pi i y\cdot\xi} - \hat{\phi}(r\xi)\hat{f}(\xi)(-2\pi i y\cdot\xi) \\ &\qquad - \hat{f}(\xi)\,\hat{\phi}(r\xi)|^2\,d\xi\,\frac{dr}{r^{d+3}}\,dy\end{aligned}$$

$$= \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \int_{B(0,r)}\int_0^\infty |e^{-2\pi i y\cdot\xi} - \hat{\phi}(r\xi)(-2\pi i y\cdot\xi) - \hat{\phi}(r\xi)|^2\,\frac{dr}{r^{d+3}}\,dy\,d\xi. \tag{7.7}$$

If we show

$$\int_{B(0,r)}\int_0^\infty |e^{-2\pi i y\cdot\xi} - \hat{\phi}(r\xi)(-2\pi i y\cdot\xi) - \hat{\phi}(r\xi)|^2\,\frac{dr}{r^{d+3}}\,dy \lesssim |\xi|^2\,, \tag{7.8}$$

then the theorem will follow since

$$(7.7) < \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2\,|\xi|^2\,d\xi = \int_{\mathbb{R}^d} |\nabla f(\xi)|^2\,\partial\xi = \int_{\mathbb{R}^d} |\nabla f|^2.$$

We begin proving (7.8). Again, since ϕ is radially symmetric, so is $\hat\phi$, and hence $\hat\phi(\xi)=\hat\Phi(|\xi|)$ for some function $\hat\Phi\colon [0,\infty)\to[0,\infty)$. We will abuse notation and write $\hat\Phi(r)=\hat\phi(r)$. By two changes of variables (once in r, then in y), and again writing $B_r=B(0,r)$,

$$\int_{B_r}\int_0^\infty |e^{-2\pi i y\cdot\xi}-\hat\phi(r|\xi|)(-2\pi i y\cdot\xi)-\hat\phi(r|\xi|)|^2\,\frac{dr}{r^{d+3}}\,dy \tag{7.9}$$

$$\begin{aligned}
&=|\xi|^{d+2}\int_{B_{t/|\xi|}}\int_0^\infty |e^{-2\pi i y\cdot\xi}-\hat\phi(t)(-2\pi i y\cdot\xi)-\hat\phi(t)|^2\,\frac{dt}{t^{d+3}}\,dy\\
&=|\xi|^2\int_{B_t}\int_0^\infty |e^{-2\pi i y\cdot\xi/|\xi|}-\hat\phi(t)\big(-2\pi i y\cdot\tfrac{\xi}{|\xi|}\big)-\hat\phi(t)|^2\,\frac{dt}{t^{d+3}}\,dy\\
&=|\xi|^2\int_0^\infty\int_{B_t}\Big|e^{-2\pi i y\cdot\xi/|\xi|}(1-\hat\phi(t))+\hat\phi(t)\big(e^{-2\pi i y\cdot\xi/|\xi|}-2\pi i y\cdot\tfrac{\xi}{|\xi|}-1\big)\Big|^2\,dy\,\frac{dt}{t^{d+3}}\,.
\end{aligned}$$

Since $\hat\phi$ is a Schwartz function and $\hat\phi(0)=\int\phi=1$, and

$$\frac{d}{dt}\,\hat\phi(0)=(\widehat{-2\pi i t\,\phi}(t))|_{t=0}=-\int 2\pi i t\,\phi(t)\,dt=0$$

since ϕ is radially symmetric, we thus have by Taylor's theorem,

$$|e^{-2\pi i y\cdot\xi/|\xi|}\,(1-\hat\phi(t))|\lesssim\min\Big\{t^2,\frac{1}{1+|t|^2}\Big\}.$$

Again by Taylor's theorem, since $1+a$ is the first two terms of the Taylor series for e^a, and since we always have $|y|\le t$ in the domain of the integral, we get

$$\Big|\hat\phi(y)\Big(e^{-2\pi i y\cdot\xi/|\xi|}-2\pi i y\cdot\frac{\xi}{|\xi|}-1\Big)\Big|\lesssim\frac{1}{1+t^4}\Big|-2\pi i y\cdot\frac{\xi}{|\xi|}\Big|^2\le\frac{t^2}{1+t^4}\,,$$

so that

$$\begin{aligned}
(7.9)&\lesssim|\xi|^2\int_0^\infty\int_{B(0,t)}\Big(\min\Big\{t,\frac{1}{1+|t|^2}\Big\}+\frac{t^2}{1+t^4}\Big)^2\,dy\,\frac{dt}{t^{d+3}}\\
&=|\xi|^2\int_0^\infty\Big(\min\Big\{t^2,\frac{1}{1+|t|^2}\Big\}+\frac{t^2}{1+t^4}\Big)^2\,\frac{dt}{t^3}\lesssim|\xi|^2\,,
\end{aligned}$$

which proves (7.8). □

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