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Sharp L^p estimates for Schrödinger groups

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Abstract. Consider a non-negative self-adjoint operator H in $L^2(\mathbb{R}^d)$. We suppose that its heat operator e^{-tH} satisfies an off-diagonal algebraic decay estimate, for some exponents $p_0 \in [0,2)$. Then we prove sharp $L^p \to L^p$ frequency truncated estimates for the Schrödinger group e^{itH} for $p \in [p_0, p_0']$.

In particular, our results apply to every operator of the form $H = (i\nabla + A)^2 + V$, with a magnetic potential $A \in L^2_{\rm loc}(\mathbb{R}^d, \mathbb{R}^d)$ and an electric potential V whose positive and negative parts are in the local Kato class and in the Kato class, respectively.

1. Introduction

It is well known that the Schrödinger group $e^{it\Delta}$ is bounded on $L^p(\mathbb{R}^d)$ only for $p = 2$ (if $t \neq 0$). However, frequency truncated estimates still hold, which can for instance be phrased in the form

$$\|e^{it\Delta}\varphi(2^{-k}D)f\|_{L^p} \lesssim (1 + 2^{2k}|t|)^s \, \|f\|_{L^p}, \quad k \in \mathbb{Z}, \ t \in \mathbb{R}, \tag{1.1}$$

for $1 \le p \le \infty$, $s = d\,|1/2 - 1/p|$, where $\varphi \in C_c^\infty(\mathbb{R}^d)$ is a cut-off function. This result follows at once from the stationary phase theorem and is sharp, both for the growth in t and for the loss of derivatives, i.e., the factor 2^{2k} in the right-hand side (see [5], [29], and [41]).

In this paper we show a far-reaching generalization of this result, to every self-adjoint non-negative operator H in $L^2(\mathbb{R}^d)$, whose heat operator e^{-tH} satisfies a mild smoothness effect and a mild off-diagonal decay.

Concerning strong (p,p) estimates of spectral multipliers, recently much attention has been devoted to minimal assumptions on H. A condition which is nowadays common in the literature, after [18], [21], and that already covers a lot of interesting operators, is a pointwise Gaussian estimate for the heat kernel $p_t(x,y)$ of e^{-tH}, namely

$$|p_t(x,y)| \lesssim t^{-d/m} \exp\Big(- b\big(t^{-1/m}\,|x-y|\big)^{m/(m-1)}\Big), \qquad t > 0, \ x, y \in \mathbb{R}^d, \tag{1.2}$$

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for some $b > 0$, $m > 1$. For example, every Schrödinger operator with an electromagnetic potential, under very natural assumptions, satisfies such estimate with $m = 2$ ([6], [13]). As another example, the operator $H = (-\Delta)^k$, with $k \geq 1$ integer, satisfies (1.2) with $m = 2k$.

Recently, motivated by Schrödinger operators with bad potentials [37] or higher order operators with measurable coefficients [16], the assumptions on H were further weakened in the form of the so-called generalized Gaussian estimates, namely

$$\begin{aligned} &\|\mathbf{1}_{B(x,t^{1/m})}\, e^{-tH}\mathbf{1}_{B(y,t^{1/m})}\|_{L^{p_0}\to L^{p_0'}} \\ &\qquad \lesssim t^{-\frac{d}{m}(1/p_0-1/p_0')} \exp\Big(-b\big(t^{-1/m}\,|x-y|\big)^{m/(m-1)}\Big), \end{aligned} \tag{1.3}$$

for some $p_0 \in [1, 2)$ and every $t > 0$, $x, y \in \mathbb{R}^d$, where $b > 0$ and $m > 1$; see [2], [3], [27], and [28]. When $p_0 = 1$, (1.3) is in fact equivalent to (1.2) [4].

In the present paper we will consider even weaker estimates, allowing off-diagonal algebraic decay.

For every $j \in \mathbb{Z}$, let $\mathcal{Q}_j$ be the collection of all dyadic cubes in $\mathbb{R}^d$ with side-length 2^{-j}.

Assumption (H) *Assume that H is a self-adjoint non-negative operator in $L^2(\mathbb{R}^d)$, whose heat operator satisfies the following estimates. There exist $p_0 \in [1, 2)$, $m > 0$ such that for every $t > 0$ and $j \in \mathbb{Z}$, with $2^{-j} \leq t^{1/m} < 2^{-j+1}$, we have*

$$\sup_{Q'\in\mathcal{Q}_j} \sum_{Q\in\mathcal{Q}_j} \|\mathbf{1}_Q e^{-tH}\mathbf{1}_{Q'}\|_{L^{p_0}\to L^{p_0'}} \lesssim 2^{jd\,(1/p_0-1/p_0')} \tag{1.4}$$

and

$$\sup_{Q'\in\mathcal{Q}_j} \sum_{Q\in\mathcal{Q}_j} (1+2^j \mathrm{dist}\,(Q,Q'))^N \|\mathbf{1}_Q e^{-tH}\mathbf{1}_{Q'}\|_{L^2\to L^2} \lesssim 1, \quad N = \lfloor d/2\rfloor + 1. \tag{1.5}$$

Of course, if the pointwise bound (1.2) holds, then (1.4) is satisfied with $p_0 = 1$, as well as (1.5). Also, (1.3) implies (1.4) and (1.5). Notice however that there are operators satisfying (1.4) and (1.5) but not (1.3). For example, for the fractional Laplacian $H = (-\Delta)^\alpha$, $\alpha > 0$, (1.4) is satisfied with $p_0 = 1$ for every $\alpha > 0$, whereas (1.5) holds for $2\alpha > \lfloor d/2\rfloor + 1$ (both with $m = 2\alpha$); see Section 5 below.

Now, we can state our main result.

Theorem 1.1. *Assume the hypothesis* **(H)**. *Let $p \in [p_0, p_0']$ and $s = d|1/2 - 1/p|$. Then the operator e^{-itH} satisfies the estimate*

$$\|e^{-itH}\varphi(2^{-k}H)f\|_{L^p} \lesssim (1+2^k|t|)^s\,\|f\|_{L^p}, \quad k\in\mathbb{Z},\ t\in\mathbb{R}, \tag{1.6}$$

uniformly for φ in bounded subsets of $C_c^\infty(\mathbb{R})$.

This result is sharp both for the growth in t and for the loss of derivatives, in the sense that when $H = -\Delta$ (and k is replaced by $2k$) it reduces to that mentioned above, which is sharp. Moreover, it is new even for operators satisfying (1.2) with $m = 2$.

Besides the case of the free Laplacian, estimate (1.6) was first proved for $H = -\Delta + V$ in Theorem 1.4 of [25], and in Theorem 5.2 of [25], under several assumptions on V. Actually, we shall see in Theorem 5.2 below that we can consider any operator for the form

$$H = (i\nabla + A) + V,$$

with a magnetic potential $A \in L^2_{\rm loc}(\mathbb{R}^d; \mathbb{R}^d)$, and an electric potential V with positive and negative parts $V_+ \in \mathcal{K}_{\rm loc}(\mathbb{R}^d)$, $V_- \in \mathcal{K}(\mathbb{R}^d)$, where $\mathcal{K}(\mathbb{R}^d)$ is the Kato class (see Definition 5.1 below). Although this operator is just bounded from below, the conclusions of Theorem 1.1 still hold for H at least for $k \geq 0$.

The proof is inspired by [25]. In short, by duality and interpolation we are reduced to prove the result for $p = p_0$. Then we exploit the smoothing effect (1.4) of the heat operator to reduce matters to a continuity result in certain amalgam spaces of functions with an upgraded L^2 local regularity ($p_0 < 2$) and an ℓ^{p_0} decay at infinity (on average). Technically, we need a version of these spaces adpapted to different scales, as in [42]. Then, the off-diagonal decay (1.5) of the heat operator is used to prove the desired estimates in such spaces.

As an intermediate step, we prove strong (p_0, p_0) estimates for $\varphi(H)$, $\varphi \in C^\infty_c(\mathbb{R})$. It would be interesting to know whether the assumption (**H**) (or a variant of it) is still sufficient for more general spectral multipliers theorems, where φ satisfies Hörmander type conditions ([6], [18], [21], [27]). We plan to investigate this issue in a subsequent work.

As another remark, we observe that we could allow an exponential factor $\exp(ct)$, $c > 0$, in the right-hand sides of (1.4) and (1.5), provided the conclusion of Theorem 1.1 is restricted to $k \geq 0$ (see Section 5.1).

As a consequence of Theorem 1.1 we obtain, by a standard scaling argument (see [24], p. 193), an estimate in Sobolev spaces adapted to the operator H.

Corollary 1.2. *Assume the hypothesis* (**H**). *Let $p \in [p_0, p_0']$ and $s = d|1/2 - 1/p|$. Then for every $\varepsilon > 0$ the operator e^{-itH} satisfies the estimates*

$$\|e^{-itH}(I + H)^{-s-\varepsilon} f\|_{L^p} \lesssim (1 + |t|)^s \, \|f\|_{L^p}, \quad t \in \mathbb{R}. \tag{1.7}$$

This result improves Theorem 1.4 in [24], and (at least in the Euclidean setting) Theorem 5.2 in [7], and Theorem 1.3 (b) in [3], where an additional ε-loss occurred in the exponent of t (and moreover the stronger estimates (1.3) were assumed).

Another interesting issue is the validity of (1.7) with $\varepsilon = 0$. Indeed, for $H = -\Delta$ in $\mathbb{R}^d$ and $1 < p < \infty$, the estimate (1.7) was proved with $\varepsilon = 0$ (and $t = 1$) in [31], but this sharp form seems out of reach in the present generality, even for fixed t. However, using some results of time-frequency analysis we will see that estimates such as (1.7) with $\varepsilon = 0$ indeed hold for a large class of propagators, essentially any operator which is bounded in the so-called modulation spaces (see Section 5 for the definition). This is just a remark, which however seems to be new. Details and examples are given in Section 5.

The paper is organized as follows. In Section 2 we prove some preliminary results and we define the above mentioned amalgam spaces, together with a criterion of boundedenss. In Section 3 we prove strong (p, p) estimates for the operator $\varphi(H)$, for $p \in [p_0, p_0']$, and $\varphi \in C_c^\infty(\mathbb{R})$. This will be used in the proof of Theorem 1.1, which is given in Section 4. Finally in Section 5 we discuss examples of operators which Theorem 1.1 applies to, and the above mentioned connection with time-frequency analysis.

2. Preliminary results

2.1. Some remarks on assumption (H)

For future reference, we collect here some comments on the assumption **(H)**.

Remark 2.1. Let us notice that for a linear operator A and $1 \le p, q \le \infty$ we have

$$\|\mathbf{1}_Q A\|_{L^p \to L^q} \le \|\mathbf{1}_{Q'} A\|_{L^p \to L^q}, \qquad \|A\,\mathbf{1}_Q\|_{L^p \to L^q} \le \|A\,\mathbf{1}_{Q'}\|_{L^p \to L^q}$$

if $Q \subset Q'$ are measurable sets. Moreover, if $\mathbf{1}_Q = \sum_{k=1}^m \mathbf{1}_{Q_k}$ (pointwise almost everywhere) then

$$\|\mathbf{1}_Q A\|_{L^p \to L^q} \le \sum_{k=1}^m \|\mathbf{1}_{Q_k} A\|_{L^p \to L^q}, \quad \|A\,\mathbf{1}_Q\|_{L^p \to L^q} \le \sum_{k=1}^m \|A\,\mathbf{1}_{Q_k}\|_{L^p \to L^q}.$$

This implies that, for any given $M \in \mathbb{N}$ the estimates (1.4) and (1.5) hold for every j with $2^{-j-M} \le t^{1/m} < 2^{-j+M}$, where the constant implicit in the notation $\lesssim$ will depend on M.

Remark 2.2. The estimate (1.4) is equivalent to the couple of estimates

$$\sup_{Q' \in \mathcal{Q}_j} \sum_{Q \in \mathcal{Q}_j} \|\mathbf{1}_Q e^{-tH} \mathbf{1}_{Q'}\|_{L^{p_0} \to L^2} \lesssim 2^{jd\,(1/p_0 - 1/2)} \tag{2.1}$$

and

$$\sup_{Q \in \mathcal{Q}_j} \sum_{Q' \in \mathcal{Q}_j} \|\mathbf{1}_Q e^{-tH} \mathbf{1}_{Q'}\|_{L^{p_0} \to L^2} \lesssim 2^{jd\,(1/p_0 - 1/2)}. \tag{2.2}$$

Indeed, (2.1) follows from (1.4) and Hölder's inequality, because $|Q| = 2^{-jd}$. Moreover, if (1.4) holds as stated then it also holds, by duality, with Q and Q' exchanged in the sum and supremum, so that (2.1) holds with the same exchange, which is (2.2).

Viceversa, assume (2.1) and (2.2). Then we have

$$\begin{aligned}
\|\mathbf{1}_Q e^{-tH} \mathbf{1}_{Q'}\|_{L^{p_0} \to L^{p_0'}} &\le \sum_{Q'' \in \mathcal{Q}_j} \|\mathbf{1}_Q e^{-(t/2)H} \mathbf{1}_{Q''} e^{-(t/2)H} \mathbf{1}_{Q'}\|_{L^{p_0} \to L^{p_0'}} \\
&\le \sum_{Q'' \in \mathcal{Q}_j} \|\mathbf{1}_Q e^{-(t/2)H} \mathbf{1}_{Q''}\|_{L^2 \to L^{p_0'}} \|\mathbf{1}_{Q''} e^{-(t/2)H} \mathbf{1}_{Q'}\|_{L^{p_0} \to L^2},
\end{aligned}$$

and (1.4) follows from Remark 2.1, (2.1) and the dual version of (2.2).

Remark 2.3. The estimate (1.5) implies, by duality,

$$\sup_{Q\in\mathcal{Q}_j}\sum_{Q'\in\mathcal{Q}_j}(1+2^j\operatorname{dist}(Q,Q'))^N\|\mathbf{1}_Q e^{-tH}\mathbf{1}_{Q'}\|_{L^2\to L^2}\lesssim 1. \tag{2.3}$$

Remark 2.4. Let

$$S_\lambda f(x)=f(\lambda x),\quad \lambda>0.$$

Observe that if the assumption (**H**) holds for the operator H then it also holds for the operator

$$H_k:=2^k S_{\lambda_k} H S_{\lambda_k}^{-1},\quad \lambda_k=2^{k/m},\quad k\in\mathbb{Z},$$

uniformly with respect to k.

Indeed, by the spectral calculus we have

$$e^{-tH_k}=S_{\lambda_k}e^{-2^k tH}S_{\lambda_k}^{-1}$$

and for $Q,Q'\in\mathcal{Q}_j$, with $2^{-j}\le t^{1/m}<2^{-j+1}$, we have

$$\mathbf{1}_Q\, e^{-tH_k}\,\mathbf{1}_{Q'}=S_{\lambda_k}\,\mathbf{1}_{\lambda_k Q}\,e^{-2^k tH}\,\mathbf{1}_{\lambda^k Q'}\,S_{\lambda_k}^{-1},$$

with $\lambda Q:=\{\lambda x:\ x\in Q\}$.

Moreover, for a linear operator A and $1\le p,q\le\infty$ we have

$$\|S_\lambda A S_\lambda^{-1}\|_{L^p\to L^q}=\lambda^{d\,(1/p-1/q)},\quad \lambda>0.$$

Hence, it is sufficient to apply the assumption (**H**) with t replaced by $2^k t$. More precisely, the involved cubes should be those dyadic having sidelength 2^{-M}, with $2^{-M}\le(2^k t)^{1/m}<2^{-M+1}$, but one can cover each of the cubes $\lambda_k Q'$ and $\lambda_k Q'$ (whose sidelength is exactly $(2^k t)^{1/m}$) by using 2^d of such dyadic cubes (cf. Remark 2.1).

2.2. Amalgam spaces

Let us recall some useful results concerning amalgam spaces ([16], [22], [24], [42]). For $1\le p,q\le\infty$, $j\in\mathbb{Z}$, consider the space $X_j^{p,q}$ of measurable functions in $\mathbb{R}^d$ equipped with the norm

$$\|f\|_{X_j^{p,q}}:=\Big(\sum_{Q\in\mathcal{Q}_j}\|\mathbf{1}_Q f\|_{L^q}^p\Big)^{1/p}$$

(with obvious changes if $q=\infty$). As above $\mathcal{Q}_j$ denotes the collection of dyadic cubes of sidelength 2^{-j}. We also set $X^{p,q}=X_0^{p,q}$.

Notice that $X_j^{p,p}=L^p$ for every j and p. Moreover we will need the following embeddings.

Proposition 2.5. *For $1\le p\le q\le\infty$, $j\in\mathbb{Z}$, we have*

$$\|f\|_{X^{p,q}}\le\max\{1,2^{-jd\,(1/p-1/q)}\}\|f\|_{X_j^{p,q}}. \tag{2.4}$$

Proof. Consider first the case $j < 0$. Then we prove that, if $\tilde{Q} \in \mathcal{Q}_j$,

$$\Big(\sum_{\mathcal{Q}_0 \ni Q \subset \tilde{Q}} \|\mathbf{1}_Q f\|_{L^q}^p \Big)^{1/p} \leq 2^{-jd\,(1/p-1/q)} \|\mathbf{1}_{\tilde{Q}} f\|_{L^q}.$$

Since

$$\|\mathbf{1}_{\tilde{Q}} f\|_{L^q} = \Big(\sum_{\mathcal{Q}_0 \ni Q \subset \tilde{Q}} \|\mathbf{1}_Q f\|_{L^q}^q \Big)^{1/q},$$

the result follows from Hölder's inequality for finite sequences, with $N = 2^{-jd}$ elements.

Consider now the case $j \geq 0$. We prove that, if $\tilde{Q} \in \mathcal{Q}_0$,

$$\|\mathbf{1}_{\tilde{Q}} f\|_{L^q} \leq \Big(\sum_{\mathcal{Q}_j \ni Q \subset \tilde{Q}} \|\mathbf{1}_Q f\|_{L^q}^p \Big)^{1/p}.$$

Again, the left-hand side is equal to $\big(\sum_{\mathcal{Q}_j \ni Q \subset \tilde{Q}} \|\mathbf{1}_Q f\|_{L^q}^q \big)^{1/q}$, and the result follows, because $p \leq q$. □

Here is an elementary criterion for boundedness on $X^{p,q}$.

Proposition 2.6. *Let A be a linear operators satisfying, for some $1 \leq q_1, q_2 \leq \infty$, and $j \in \mathbb{Z}$,*

$$\sup_{Q' \in \mathcal{Q}_j} \sum_{Q \in \mathcal{Q}_j} \|\mathbf{1}_Q A \mathbf{1}_{Q'}\|_{L^{q_1} \to L^{q_2}} = M_1 < \infty\,,$$

and

$$\sup_{Q \in \mathcal{Q}_j} \sum_{Q' \in \mathcal{Q}_j} \|\mathbf{1}_Q A \mathbf{1}_{Q'}\|_{L^{q_1} \to L^{q_2}} = M_2 < \infty\,.$$

Then, for every $1 \leq p \leq \infty$,

$$\|Af\|_{X_j^{p,q_2}} \leq M_1^{1-\theta}\, M_2^{\theta}\, \|f\|_{X_j^{p,q_1}}, \qquad \frac{1}{p} = 1 - \theta\,.$$

Proof. The desired estimate follows at once from the definition of the spaces $X_j^{p,q}$ and Schur's test for operators acting on sequences. □

2.3. A criterion for boundedness on $X^{p,2}$

We recall here a result from [24] which gives a sufficient condition for a linear operator A to be bounded on the spaces $X^{p,2}$, $1 \leq p \leq 2$.

Theorem 2.7. *Let A be a bounded operator on L^2, and for any $l = 1, \ldots, d$, define the commutator $\mathrm{Ad}_l(A) = [x_l, A]$. Suppose that for some $M \geq 1$ we have*

$$\|\mathrm{Ad}_l^k(A)\|_{L^2 \to L^2} \leq M^k, \quad 0 \leq k \leq \lfloor d/2 \rfloor + 1, \quad 1 \leq l \leq d\,.$$

Then, for $1 \leq p \leq 2$,

$$\|Af\|_{X^{p,2}} \leq C_0\, M^{d\,(1/p-1/2)}\, \|f\|_{X^{p,2}}, \tag{2.5}$$

where the constant C_0 depends only on d and upper bounds for $\|A\|_{L^2 \to L^2}$.

Proof. The proof can be found in [24], but the result is not explicitly stated there in the present form. Hence, for the benefit of the reader we point out detailed references of the main steps.

By the computations at the end of page 261 in [25], the assumption implies the L^2 weighted estimate

$$\|\langle \cdot - n\rangle^k A \langle \cdot - n\rangle^{-k}\|_{L^2\to L^2} \lesssim_d M^k, \quad \forall n \in \mathbb{Z}^d, \quad k = \lfloor d/2 \rfloor + 1,$$

By formula (3.7) in [25], this last formula implies that

$$|||A|||_k := \|A\|_{L^2\to L^2} + \sup_{n\in\mathbb{Z}^d} \|\langle \cdot - n\rangle^k A \mathbf{1}_{Q_n}\|_{L^2\to L^2} \lesssim_d M^k, \quad k = \lfloor d/2 \rfloor + 1,$$

where $Q_n = n + [0,1]^d$. Since $k > d/2$ we can apply the interpolation inequality in Theorem 2.4 of [25], with $\beta = k$, and we obtain

$$\|Af\|_{X^{1,2}} \lesssim_d \|A\|_{L^2\to L^2}^{1-d/(2(\lfloor d/2\rfloor+1))} M^{d/2} \|f\|_{X^{1,2}}\,.$$

By interpolation with the $L^2 \to L^2$ estimate, we deduce (2.5). □

3. Boundedness of $\varphi(H)$

This section is devoted to the proof of the following result, which is an intermediate step for Theorem 1.1.

Theorem 3.1. *Let H satisfy the assumption* (**H**). *Then for every $p \in [p_0, p_0']$ we have*

$$\|\varphi(2^k H) f\|_{L^p} \lesssim \|f\|_{L^p}\,. \tag{3.1}$$

uniformly for φ in bounded subsets of $C_c^\infty(\mathbb{R})$ and $k \in \mathbb{Z}$.

First of all we observe that it suffices to consider the case $k = 0$. Indeed, as observed in Remark 2.4, the same assumption **(H)** holds for the operator

$$H_k := 2^k S_{\lambda_k} H S_{\lambda_k}^{-1}, \quad \lambda_k = 2^{k/m}, \quad k \in \mathbb{Z}\,,$$

uniformly with respect to k. On the other hand, by the spectral calculus

$$\varphi(H_k) = S_{\lambda_k} \varphi(2^k H)\, S_{\lambda_k}^{-1}$$

and intertwining with S_{λ_k} preserves the (p,p) norm.

Let us therefore prove (3.1) for $k = 0$.

We begin with an easy lemma, which is definitively based on the assumption (1.4).

Proposition 3.2. *We have the estimates*

$$\|e^{-tH} f\|_{X^{p_0,2}} \lesssim (1 + t^{-\frac{d}{m}(1/p_0 - 1/2)})\,\|f\|_{L^{p_0}}, \quad t > 0\,.$$

Proof. By (1.4), Remark 2.2 and Proposition 2.6 we have, for $2^{-j} \leq t^{1/m} < 2^{-j+1}$,

$$\|e^{-tH} f\|_{X_j^{p_0,2}} \lesssim 2^{dj\,(1/p_0 - 1/2)} \|f\|_{X_j^{p_0,p_0}}.$$

Since $X_j^{p_0,p_0} = L^{p_0}$, the embedding in Proposition 2.5 gives the desired conclusion. □

Consider now the resolvent operator

$$R = (I + H)^{-1}.$$

Proposition 3.3. *If $\beta > \frac{d}{m}(1/p_0 - 1/2)$, we have*

$$\|R^\beta f\|_{X^{p_0,2}} \lesssim \|f\|_{L^{p_0}}.$$

Proof. It is sufficient to use the formula

$$R^\beta = \frac{1}{\Gamma(\beta)} \int_0^{+\infty} t^{\beta-1}\, e^{-t}\, e^{-tH}\, dt, \tag{3.2}$$

together with Proposition 3.2 and Minkowski's inequality for integrals. □

The criterion in Theorem 2.7 allows one to transfer estimates in amalgam spaces from R to $e^{-i\xi R}$, $\xi \in \mathbb{R}$. Here we will use the assumption (1.5).

Proposition 3.4. *We have*

$$\|e^{-i\xi R} f\|_{X^{p_0,2}} \lesssim (1 + |\xi|)^{d(1/p_0 - 1/2)} \|f\|_{X^{p_0,2}}, \quad \xi \in \mathbb{R}.$$

Proof. By the criterion in Theorem 2.7, it is sufficient to prove that

$$\|\mathrm{Ad}_l^k(e^{-i\xi R}) f\|_{L^2} \lesssim (1 + |\xi|)^k \|f\|_{L^2}$$

for $k = 0, 1, \ldots, \lfloor d/2 \rfloor + 1$, $l = 1, \ldots, d$.

Using the formula

$$\mathrm{Ad}_l^1(e^{-i\xi R}) = -i \int_0^\xi e^{-isR}\, \mathrm{Ad}_l^1(R)\, e^{-i(\xi - s)R}\, ds$$

repeatedly, we are reduced to prove that

$$\|\mathrm{Ad}_l^k(R) u\|_{L^2} \lesssim \|u\|_{L^2} \tag{3.3}$$

for $k = 0, 1, \ldots, \lfloor d/2 \rfloor + 1$.

Now, setting $R(x, y)$ for the distribution kernel of R, that of the operator $\mathrm{Ad}_l^k(R)$ is $(x_l - y_l)^k R(x, y)$. Using the integral representation (3.2) with $\beta = 1$, and Minkowski's inequality for integrals we see that it is sufficient to prove that, for $2^{-j} \leq t^{1/m} < 2^{-j+1}$, the operator

$$Af(x) := 2^{jk} \int_{\mathbb{R}^d} (x_l - y_l)^k\, p_t(x, y)\, f(y)\, dy$$

is bounded on $L^2 = X_j^{2,2}$ uniformly with respect to $j \in \mathbb{Z}$, for $k = 1, 2, \dots, \lfloor d/2 \rfloor + 1$ where $p_t(x, y)$ denotes the distribution kernel of e^{-tH} (in fact, the exponential factor in (3.2) compensates the factor $2^{-jk} \le t^{k/m}$ which has been introduced).

Now, to prove the boundedness of A on $L^2 = X_j^{2,2}$, we use Proposition 2.6. Hence, by self-adjointness we are reduced to prove that

$$\sup_{Q' \in \mathcal{Q}_j} \sum_{Q \in \mathcal{Q}_j} \|\mathbf{1}_Q\, A\, \mathbf{1}_{Q'}\|_{L^2 \to L^2} \le C \tag{3.4}$$

for a constant C independent of j.

To this end, observe that, if $Q = z_Q + 2^{-j}[0, 1]^d$, $Q' = z_{Q'} + 2^{-j}[0, 1]^d$, then

$$\begin{aligned}\mathbf{1}_Q A \mathbf{1}_{Q'} = \sum_{\alpha+\beta+\gamma=k} \frac{k!}{\alpha!\beta!\gamma!} &(2^j(z_{Q,l} - z_{Q',l}))^\alpha \\ &\times (2^j(x_l - z_{Q,l}))^\beta\, \mathbf{1}_Q e^{-tH}\, \mathbf{1}_{Q'} (2^j(z_{Q',l} - y_l))^\gamma.\end{aligned} \tag{3.5}$$

Formula (3.4) then follows from the assumption (1.5) and the elementary estimates

$$\|(2^j(x_l - z_{Q,l}))^\beta \mathbf{1}_Q\|_{L^2 \to L^2} \lesssim 1, \quad \|\mathbf{1}_{Q'}(2^j(z_{Q',l} - y_l))^\gamma\|_{L^2 \to L^2} \lesssim 1. \qquad \square$$

Now we continue with the proof of Theorem 3.1. Writing

$$\varphi(R) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{i\xi R}\, \widehat{\varphi}(\xi)\, d\xi$$

and using Proposition 3.4 we deduce that

$$\|\varphi(R) f\|_{X^{p_0,2}} \lesssim \|(1 + |\xi|)^{d(1/p_0 - 1/2)}\, \widehat{\varphi}\|_{L^1}\, \|f\|_{X^{p_0,2}}\, . \tag{3.6}$$

Now, given $\varphi \in C_0^\infty(\mathbb{R})$ we can find $\psi \in C_0^\infty(\mathbb{R}_+)$ such that $\psi((\lambda + 1)^{-1}) = \varphi(\lambda)$ for $\lambda \ge 0$, so that $\varphi(H) = \psi(R)$. Moreover, for $\beta > \frac{d}{m}\,(1/p_0 - 1/2)$, write

$$R^{-\beta} \varphi(H) = R^{-\beta} \psi(R) =: \tilde{\psi}(R).$$

The operator R^β is then bounded $L^{p_0} \to X^{p_0,2}$ by Proposition 3.3, whereas $\tilde{\psi}(R)$ is bounded on $X^{p_0,2}$ by (3.6). Since $X^{p_0,2} \hookrightarrow L^{p_0}$, $f(H) = \tilde{\psi}(R) R^\beta$ is bounded on L^{p_0}, and therefore on every L^p, $p \in [p_0, p_0']$, by duality and interpolation with the L^2 case.

The uniformity of the estimate when φ varies in bounded subsets of $C_c^\infty(\mathbb{R})$ also follows from (3.6) and this last argument.

This concludes the proof of Theorem 3.1.

4. Proof of the main result (Theorem 1.1)

By using the same scaling argument as in the proof of Theorem 3.1 it is sufficient to prove that

$$\|e^{-itH} \varphi(H)\|_{L^p \to L^p} \le C\,(1 + |t|)^{d|1/2 - 1/p|}, \tag{4.1}$$

uniformly for φ in bounded subsets of $C_0^\infty(\mathbb{R})$.

We will adopt the notation $\mathrm{Ad}(A) = [x_j, A]$, $j = 1, \dots, d$, for the commutator $[x_j, A]$, as in Theorem 2.7, omitting the subscript j for simplicity. Below we will prove that, for $k = \lfloor d/2 \rfloor + 1$,

$$\|\mathrm{Ad}^l(R^{2^{k+1}-2}e^{-itH})\|_{L^2\to L^2} \le C\,(1+|t|)^l, \quad l \le k\,. \tag{4.2}$$

This implies, from Theorem 2.7, that

$$\|R^{2^{k+1}-2}e^{-itH}\|_{X^{p_0,2}\to X^{p_0,2}} \le C\,(1+|t|)^{d(1/p_0-1/2)}.$$

Combining this estimate with Proposition 3.3 and Theorem 3.1 we obtain

$$\begin{aligned}
&\|e^{-itH}\varphi(H)\|_{L^{p_0}\to X^{p_0,2}}\\
&\qquad\lesssim \|R^{2^{k+1}-2}e^{-itH}\|_{X^{p_0,2}\to X^{p_0,2}}\|R^{\beta-2^{k+1}+2}\|_{L^{p_0}\to X^{p_0,2}}\|R^{-\beta}\varphi(H)\|_{L^{p_0}\to L^{p_0}}\\
&\qquad\lesssim (1+|t|)^{d(1/p_0-1/2)},
\end{aligned}$$

where we choose

$$\beta > 2^{k+1} - 2 + \frac{d}{m}\left(\frac{1}{p_0} - \frac{1}{2}\right)$$

in order to apply Proposition 3.3. Using the inclusion $X^{p_0,2} \hookrightarrow L^{p_0}$ we deduce (4.1) for $p = p_0$. By duality and interpolation with the L^2 case we get (4.1).

It remains to prove (4.2). We will use induction on $k = 0, 1, \dots, \lfloor d/2 \rfloor + 1$. First we prove the following result.

Proposition 4.1. *For every $k = 1, \dots, \lfloor d/2 \rfloor + 1$, the operator $\mathrm{Ad}(R^{2^{k+1}-2}e^{-itH})$ is given by a (finite) linear combination of operators of the following type:*

$$R^{\mu_1}\,R^{2^k-2}e^{-itH}\,\mathrm{Ad}(R^{\mu_2})\,R^{\mu_3}, \quad \mu_1,\mu_2,\mu_3 \in \mathbb{N},$$
$$R^{\mu_1}\,\mathrm{Ad}(R^{\mu_2})\,R^{2^k-2}e^{-itH}\,R^{\mu_3}, \quad \mu_1,\mu_2,\mu_3 \in \mathbb{N},$$

and

$$\int_0^t R^{2^k-2}\,e^{-isH}\,\mathrm{Ad}(R)\,R^{2^k-2}e^{-i(t-s)H}\,ds\,.$$

Proof. The result is obtained by induction on k. It is true for $k = 1$. Indeed,

$$\mathrm{Ad}(Re^{-itH}R) = \mathrm{Ad}(R)e^{-itH}R + Re^{-itH}\mathrm{Ad}(R) - i\int_0^t e^{-isH}R[x_j, H]Re^{-i(t-s)H}\,ds\,.$$

Now, $R[x_j, H]R = [R, x_j] = -\mathrm{Ad}(R)$, so that the result for $k = 1$ is verified.

Let us assume it holds for $k-1$ and compute

$$\begin{aligned}
\mathrm{Ad}(R^{2^{k+1}-2}e^{-itH}) &= \mathrm{Ad}(R^{2^{k-1}}(R^{2^k-2}e^{-itH})R^{2^{k-1}})\\
&= \mathrm{Ad}(R^{2^{k-1}})\;R^{2^k-2}e^{-itH}\;R^{2^{k-1}} + R^{2^{k-1}}\;\mathrm{Ad}(R^{2^k-2}e^{-itH})\,R^{2^{k-1}}\\
&\quad + R^{2^{k-1}}\;R^{2^k-2}\,e^{-itH}\;\mathrm{Ad}(R^{2^{k-1}})\,.
\end{aligned}$$

The first and the last term are of the desired form. The second one is also of the desired form by the inductive hypothesis, because

$$R^{2^{k-1}}R^{2^{k-1}-2} = R^{2^k-2}. \qquad \square$$

Let us now prove (4.2) by induction on $k = 0, 1, \ldots, \lfloor d/2 \rfloor + 1$. The result is trivially true for $k = 0$. Assume it holds for $k-1$, and write, for $l \le k$,

$$\mathrm{Ad}^l(R^{2^{k+1}-1}e^{-itH}) = \mathrm{Ad}^{l-1}\big(\mathrm{Ad}(R^{2^{k+1}-1}e^{-itH})\big).$$

Using the above Proposition 4.1, the formula

$$\mathrm{Ad}^{l-1}(A_1 \cdots A_n) = \sum_{m_1+\ldots+m_n=l-1} \frac{(l-1)!}{m_1! \cdots m_n!} \mathrm{Ad}^{m_1}(A_1) \cdots \mathrm{Ad}^{m_n}(A_n),$$

as well as the inductive hypothesis and (3.3), we obtain (4.2).

5. Examples and concluding remarks

5.1. Schrödinger operators

Here is our main example. We recall the definition of the Kato class from page 453 of [39].

Definition 5.1. A real-valued measurable function in $\mathbb{R}^d$ is called to lie in the Kato class $\mathcal{K}(\mathbb{R}^d)$ if and only if

a) if $d = 3$,

$$\lim_{r\downarrow 0} \sup_{x\in\mathbb{R}^d} \int_{|x-y|<r} \frac{|V(y)|}{|x-y|^{d-2}}\, dy = 0\,;$$

b) if $d = 2$,

$$\lim_{r\downarrow 0} \sup_{x\in\mathbb{R}^d} \int_{|x-y|<r} \log|x-y|^{-1}|V(y)|\, dy = 0\,;$$

c) if $d = 1$,

$$\sup_{x\in\mathbb{R}^d} \int_{|x-y|<1} |V(y)|\, dy < \infty\,.$$

We moreover define $\mathcal{K}_{\mathrm{loc}}(\mathbb{R}^d)$ as the space of functions V such that $V\mathbf{1}_B \in \mathcal{K}(\mathbb{R}^d)$ for every ball B.

It follows from Hölder's inequality that $L^p_{unif}(\mathbb{R}^d) \subset \mathcal{K}(\mathbb{R}^d)$ if $p > d/2$ ($d \ge 2$), where the uniform L^p spaces are defined by the norm

$$\|V\|^p_{L^p_{unif}} = \sup_{x\in\mathbb{R}^d} \int_{|x-y|<1} |V(y)|^p\, dy < \infty\,.$$

For example, $1/|x|^\alpha$ belongs to $\mathcal{K}(\mathbb{R}^d)$ if $0 < \alpha < 2$.

Now, we have the following result.

Theorem 5.2. *Consider the operator*

$$H = (i\nabla - A)^2 + V,$$

where $A \in L^2_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d)$, $V = V_+ - V_-$ *(positive and negative parts),*

$$V_+ \in \mathcal{K}_{\text{loc}}(\mathbb{R}^d), \quad V_- \in \mathcal{K}(\mathbb{R}^d)\,.$$

Then H *has a self-adjoint extension in* $L^2(\mathbb{R}^d)$*, bounded from below.*

Moreover for $1 \le p \le \infty$*, and for* $s = d\,|1/p - 1/2|$*, the operator* e^{-itH} *satisfies the estimate*

$$\|e^{-itH}\varphi(2^{-k}H)f\|_{L^p} \lesssim (1 + 2^k|t|)^s\,\|f\|_{L^p}, \quad k \ge 0,\ t \in \mathbb{R}\,, \tag{5.1}$$

uniformly for φ *in bounded subsets of* $C_c^\infty(\mathbb{R})$*.*

Proof. The first part of the statement is proved in Sections B1 and B13 of [39]. Moreover, combining the estimates for the heat kernel in Proposition B.6.7 of [39] (case $A \equiv 0$) with the diamagnetic inequality (Theorem B.13.2 in [39]), one sees that the operator e^{-tH} has a measurable kernel $p_t(x, y)$ satisfying

$$|p_t(x, y)| \lesssim t^{-d/2}\exp(ct)\exp\Big(-\frac{|x-y|^2}{4t}\Big), \quad t > 0,\ x, y \in \mathbb{R}^d,$$

for some $c > 0$. Hence, for some $c' > 0$, the operator $H + c'I$ in non-negative and satisfies the assumption **(H)** with $p_0 = 1$, $m = 2$. We can then apply Theorem 1.1 to the operator $H + c'I$, with the cut-off $\varphi(x - 2^{-k}c')$ and obtain the desired conclusion for $k \ge 0$, since the functions $\varphi(x - 2^{-k}c')$ for $k \ge 0$ vary in a bounded subset of $C_c^\infty(\mathbb{R})$ if φ does. □

5.2. Operators with variable coefficients

Following [13], consider the operator

$$-Hf = \nabla^b \cdot (a(x)\nabla^b f) - c(x)f, \quad \nabla^b = \nabla + ib(x)$$

where $a(x) = [a_{jk}(x)]_{j,k=1}^d$, $b(x) = (b_1(x), \ldots, b_d(x))$ and $c(x)$ satisfy:

$$a, b, c \text{ are real-valued, } a_{jk} = a_{kj} \text{ and } NI \ge a(x) \ge \nu I \text{ for some } N \ge \nu > 0\,.$$

Suppose moreover $d \ge 3$, and the following conditions in terms of Lorentz spaces:

$$a \in L^\infty,\ b \in L^4_{\text{loc}} \cap L^{d,\infty},\ \nabla \cdot b \in L^2_{\text{loc}},\ c \in L^{d/2,1},\ \|c_-\|_{L^{d/2,1}} < \varepsilon\,.$$

Then if $\varepsilon > 0$ is sufficiently small the operator H extends to a non-negative self-adjoint operator in $L^2(\mathbb{R}^d)$ and its heat kernel e^{-tH} satisfies the Gaussian bound

$$|p_t(x, y)| \lesssim t^{-d/2}\exp\Big(-\frac{|x-y|^2}{Ct}\Big), \quad t > 0,\ x, y \in \mathbb{R}^d,$$

for some $C > 0$; see Propositions 6.1 and 6.2 in [13]. Hence Theorem 1.1 applies with $p_0 = 1$.

We also refer to [15], [14], and [34] for early results on Gaussian bounds for variable coefficient second-order operators.

5.3. Higher order operators (constant coefficients)

Consider the operator

$$H = (-\Delta)^k, \quad k \in \mathbb{N},\ k \geq 1.$$

Let us verify that the assumption **(H)** is verified with $p_0 = 1$, $m = 2k$. It is sufficient to show that the heat kernel $p_t(x,y)$ satisfies the estimate

$$|p_t(x,y)| \lesssim t^{-d/(2k)} \exp\big(- b(t^{-1/(2k)}|x-y|)^{\frac{2k}{2k-1}}\big), \qquad t>0,\ x,y \in \mathbb{R}^d,$$

for some $b>0$. This is known, but we provide here a proof for the sake of completeness and also because this method of proof extends to any elliptic, homogeneous and non-negative constant coefficient operator. We leave this generalization to the interested reader.

By homogeneity we are reduced to prove that, if $f(\xi) = \exp\big(-|\xi|^{2k}\big)$, then its inverse Fourier transform verifies

$$|f^\vee(x)| \lesssim \exp\big(- b|x|^{\frac{2k}{2k-1}}\big), \qquad x \in \mathbb{R}^d.$$

It is easy to see that this inequality is equivalent to

$$|x^\alpha f^\vee(x)| \leq C^{|\alpha|+1}(\alpha!)^{1-\frac{1}{2k}}, \quad \alpha \in \mathbb{N}^d,\ x \in \mathbb{R}^d,$$

for some constant $C>0$ independent of α (see e.g. Proposition 6.1.7 in [33]). This in turn follows if we prove that

$$\|\partial^\alpha f\|_{L^1} \leq C^{|\alpha|+1}(\alpha!)^{1-\frac{1}{2k}}, \quad \alpha \in \mathbb{N}^d. \tag{5.2}$$

This can be verified by means of the Cauchy estimates, as follows.

Observe that f has an entire extension $f(\zeta) = \exp(-(\zeta_1^2 + \cdots + \zeta_d^2)^k)$, for $\zeta = (\zeta_1, \dots, \zeta_d) \in \mathbb{C}^d$, satisfying

$$|f(\zeta)| \leq \exp\big(-(1/2)|\mathrm{Re}\,\zeta|^{2k} + C\,|\mathrm{Im}\,\zeta|^{2k}\big), \quad \zeta \in \mathbb{C}^d. \tag{5.3}$$

Therefore, given $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$, we consider the polydisk

$$\tilde{B}(\xi, R) = \prod_{j=1}^d B(\xi_j, R) = \{\zeta \in \mathbb{C}^d : |\zeta_j - \xi_j| \leq R\},$$

with $R = (1+|\alpha|)^{1/(2k)}$, $\alpha \in \mathbb{N}^d$. Observe that (5.3) implies

$$\sup_{\zeta \in \tilde{B}(\xi,R)} |f(\zeta)| \leq e^{-(1/2)(|\xi|-R)_+^{2k} + C(\sqrt{d}R)^{2k}}, \tag{5.4}$$

where $(\cdot)_+$ denotes the positive part.

The Cauchy integral formula

$$\partial_\xi^\alpha f(\xi) = \frac{\alpha!}{(2\pi i)^d} \int \cdots \int_{\partial B(\xi_1,R)\times\cdots\times\partial B(\xi_d,R)} \frac{f(\zeta_1, \dots, \zeta_d)}{(\zeta_1-\xi_1)^{\alpha_1+1}\cdots(\zeta_d-\xi_d)^{\alpha_d+1}}\, d\zeta_1 \cdots d\zeta_d$$

and the estimate (5.4) yield

$$
(5.5)\quad |\partial_\xi^\alpha f(\xi)| \le e^{-(1/2)(|\xi|-R)_+^{2k}} \frac{\alpha!\, e^{C(\sqrt{d}R)^{2k}}}{R^{|\alpha|}} \le e^{-(1/2)(|\xi|-R)_+^{2k}} \frac{C_1^{|\alpha|+1}\alpha!}{(1+|\alpha|)^{|\alpha|/(2k)}}.
$$

Using Stirling's formula, we have

$$
\frac{1}{(1+|\alpha|)^{|\alpha|/(2k)}} \le \frac{C_2^{|\alpha|}}{(\alpha!)^{1/(2k)}},
$$

which combined with (5.5) gives

$$
|\partial_\xi^\alpha f(\xi)| \le C^{|\alpha|+1}\,(\alpha!)^{1-1/(2k)}\; e^{-(1/2)(|\xi|-R)_+^{2k}}, \quad \alpha\in\mathbb{N}^d.
$$

Integrating separately for $|\xi| > R$ and $|\xi| \le R$ gives the desired estimate (5.2) (the factor coming from the volume of the ball of radius R is absorbed by taking a slightly bigger constant C in (5.2)).

5.4. Higher order operators (measurable coefficients)

Following [16], let $k \ge 1$ and consider the operator

$$
H = \sum_{|\alpha|\le k,\, |\beta|\le k} D^\alpha(a_{\alpha,\beta}(x)D^\beta f),
$$

where $a_{\alpha,\beta}(x) = \overline{a_{\beta,\alpha}(x)}$ are complex-valued, bounded measurable functions. The associated quadratic form is

$$
Q(f,f) = \int_{\mathbb{R}^d} \sum_{|\alpha|\le k,\, |\beta|\le k} a_{\alpha,\beta}(x)\, D^\beta f(x)\, \overline{D^\alpha f(x)}\, dx.
$$

We suppose that $Q = Q_0 + Q_1 + Q_2$ where Q_0, Q_1, Q_2 have the same form as Q, but Q_0 is homogeneous and elliptic of degree $2k$ and has constant coefficients, Q_1 is homogeneous of degree $2k$ and is non-negative in the sense that

$$
\sum_{|\alpha|=k,\, |\beta|=k} a_{1,\alpha,\beta}(x)\, v_\beta\, \overline{v_\alpha} \ge 0
$$

for all $v_\alpha \in \mathbb{C}$, $x \in \mathbb{R}^d$, whereas Q_2 contains lower order terms.

We distinguish two cases. If $2k > d$ then it was proved in Lemma 19 of [16] that the heat kernel satisfies the estimate

$$
|p_t(x,y)| \lesssim t^{-d/(2k)} \exp(ct) \exp\big(-b(t^{-1/(2k)}|x-y|)^{\frac{2k}{2k-1}}\big), \quad t>0,\ x,y\in\mathbb{R}^d,
$$

for some $b, c > 0$.

In the case $2k < d$ the above estimate can fail. However, let

$$
p_0 = \frac{2d}{d+2k}.
$$

Then $p_0 < 2$, and

$$\begin{aligned}\|\mathbf{1}_{B(x,t^{1/(2k)})}e^{-tH}\mathbf{1}_{B(y,t^{1/(2k)})}\|_{L^2\to L^{p_0'}}\\ \lesssim t^{-\frac{d}{2k}(1/2-1/p_0')}\exp(ct)\exp\Big(-b\big(t^{-1/(2k)}|x-y|\big)^{\frac{2k}{2k-1}}\Big),\end{aligned}$$

for some $b, c > 0$; see Lemma 24 in [16]. Hence, by arguing as in Remark 2.2 we see that (1.3) holds with $m = 2k$ except for a further factor $\exp(ct)$ in the right-hand side.

By the same arguments as in Section 5.1 we deduce that in both cases the conclusion of Theorem 3.1 holds (if $2k > d$ with $p_0 = 1$, if $2k < d$ with $p_0 = 2d/(d+2k)$) at least for all $k \geq 0$.

5.5. Fractional Laplacian

Consider the fractional Laplacian

$$H = (-\Delta)^\alpha, \quad \alpha > 0\,.$$

By using standard results on homogeneous distributions it is easy to see that its heat kernel satisfies

$$0 < p_t(x,y) \lesssim t^{-d/(2\alpha)}\,(1+t^{-1/(2\alpha)}\,|x-y|)^{-(d+2\alpha)},$$

Hence we see that (1.4) is satisfied with $p_0 = 1$ for every $\alpha > 0$, whereas (1.5) holds for $2\alpha > \lfloor d/2 \rfloor + 1$ (both with $m = 2\alpha$).

5.6. Sharp estimates in Sobolev spaces

In this section we investigate the validity of (1.7) with $\varepsilon = 0$ (and for fixed $t \neq 0$), namely with the optimal loss of derivatives. We rely on results from time-frequency analysis, so that we start by recalling the relevant function spaces.

For $1 \leq p \leq \infty$, $s \in \mathbb{R}$, let L^p_s stand for the usual Sobolev (or Bessel potential) space, i.e.,

$$\|f\|_{L^p_s} := \|(1-\Delta)^s f\|_{L^p}.$$

We also recall the definition of modulation spaces ([19], [20], [36], [43]). They can be defined similarly to the Besov spaces, but for a different geometry: the dyadic annuli in the frequency domain are replaced by isometric boxes which allows a finer analysis in many respects. The construction goes as follows (cf. [43]).

Let $\rho \in \mathcal{S}(\mathbb{R}^d)$ be a smooth function, with values in the interval $[0,1]$, $\rho(\xi) = 1$ on the box $\{|\xi_j| \leq 1/2,\ j = 1,\ldots,d\}$, and $\rho(\xi) = 0$ away from the box $\{|\xi_j| \leq 1,$ $j = 1,\ldots,d\}$. Let Q_k the the unit cube centered in $k \in \mathbb{Z}^d$ and let ρ_k be the translation of ρ:

$$\rho_k(\xi) = \rho(\xi - k), \quad k \in \mathbb{Z}^d.$$

We have $\rho_k(\xi) = 1$ on Q_k, so that $\sum_{k\in\mathbb{Z}^d}\rho_k(\xi) \geq 1$ for every $\xi \in \mathbb{R}^d$. We then define the functions (symbols)

$$\sigma_k(\xi) = \rho_k(\xi)\Big(\sum_{m\in\mathbb{Z}^d}\rho_m(\xi)\Big)^{-1},$$

and the corresponding Fourier multipliers

$$\square_k = \mathcal{F}^{-1}\sigma_k\,\mathcal{F},$$

where $\mathcal{F}$ denotes the Fourier transform in $\mathbb{R}^d$.

Now, for $1 \le p, q \le \infty$, $s \ge 0$, one defines the modulation spaces

$$M^{p,q}_s = \Big\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{M^{p,q}_s} := \Big(\sum_{k\in\mathbb{Z}^d} \langle k\rangle^{sq} \|\square_k f\|^q_{L^p} \Big)^{1/q} < \infty \Big\} \tag{5.6}$$

(with obvious changes if $q = \infty$). We also set $M^{p,q}$ for $M^{p,q}_0$. We recapture in particular the L^2-based Sobolev spaces $M^{2,2}_s = H^s$, whereas the space $M^{\infty,1}$ coincides with the so-called Sjöstrand's class ([30], [40]).

The inclusion relations with the Besov spaces are well understood, see e.g. [43]. The results below instead rely in an essential way on the *sharp* inclusion relations with the Sobolev spaces, which were recently proved in Theorem 1.3 of [26]:

$$L^p_s \hookrightarrow M^p \hookrightarrow L^p \quad \text{for } 1 < p \le 2,\ s = 2d\left(\frac{1}{p} - \frac{1}{2}\right). \tag{5.7}$$

As a direct consequence of these embeddings we have the following result.

Theorem 5.3. *Let A be a linear bounded operator $M^p \to M^p$, for some $1 < p \le 2$. Then A extends to a bounded operator $L^p_s \to L^p$, with $s = 2d\,(1/p - 1/2)$, and*

$$\|A\|_{L^p_s \to L^p} \lesssim \|A\|_{M^p \to M^p}.$$

The interesting fact is that this simple result gives optimal estimates in many cases.

Example 5.4. Consider the operator $e^{it\Delta}$, $t \in \mathbb{R}$. It was proved, e.g. in Proposition 6.6 of [43], that the following estimates in modulation spaces hold true:

$$\|e^{it\Delta}\|_{M^p \to M^p} \lesssim (1 + |t|)^{d\,|1/p-1/2|}, \quad 1 \le p \le \infty.$$

Combining this result with Theorem 5.3 we therefore have

$$\|e^{it\Delta}\|_{L^p_s \to L^p} \lesssim (1 + |t|)^{d\,|1/p-1/2|}$$

for $1 < p < \infty$, $s = 2d\big|1/p - 1/2\big|$ (the case $p > 2$ follows by duality).

The estimates in the previous examples are sharp, and certainly known. However one can treat similarly a class of Fourier integral operators A, generalizing the propagator $e^{i\Delta}$ and defined as follows.

Consider an operator A of the form

$$Af(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\Phi(x,\eta)}\, a(x,\eta)\, \widehat{f}(\eta)\, d\eta, \tag{5.8}$$

with a phase $\Phi \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, real-valued and satisfying

$$|\partial_z^\alpha \Phi(z)| \le C_\alpha \quad |\alpha| \ge 2,\ z \in \mathbb{R}^{2d} \tag{5.9}$$

as well as

$$\left|\det\left(\frac{\partial^2\Phi}{\partial x_j \partial \eta_j}\right)\right| \ge \delta > 0 \quad \text{in } \mathbb{R}^d \times \mathbb{R}^d, \tag{5.10}$$

and a symbol a in the class $S^0_{0,0}$, i.e.,

$$|\partial_z^\alpha a(z)| \le C_\alpha, \quad \alpha \in \mathbb{N}^d,\ z \in \mathbb{R}^{2d}. \tag{5.11}$$

Operators of this type arise as propagators for Schrödinger equations with an Hamiltonian having quadratic growth (see [23]), in contrast with Fourier integral operators with a phase positively homogeneous of degree 1 in η, which are instead applied in the study of hyperbolic problems. For the latter class, the problem of the local and global L^p continuity is well understood ([8], [11], [17], [38]). Instead, we are not aware of similar results for operators A of the above form, except for the L^2 results in [1] and [35]. Now, it was proved in [9] that, under the above assumption, A and its adjoint are bounded on M^p, for every $1 \le p \le \infty$. As a consequence of Theorem 5.3, we have therefore the following result.

Theorem 5.5. *Let A be a Fourier integral operator as in (5.8)–(5.11). Then for every $1 < p < \infty$, $s = 2d\big|1/p - 1/2\big|$, we have*

$$\|Af\|_{L^p} \lesssim \|f\|_{L^p_s}, \quad 1 < p \le 2\,,$$

$$\|Af\|_{L^p_{-s}} \lesssim \|f\|_{L^p}, \quad 2 \le p < \infty\,.$$

In general none of the estimates in the above theorem holds for every $1 < p < \infty$. We refer to [32] for counterexamples and for applications of this circle of ideas to the study of L^p-boundedness of Feynman path integrals.

Similarly, one can consider Schrödinger operators with rough Hamiltonians. We consider here a simple case, to avoid technicalities.

Theorem 5.6. *Consider a potential $V(x)$ satisfying $\partial^\alpha V \in M^{\infty,1}$, $|\alpha| = 2$. Let $H = -\Delta + V(x)$, $1 < p < \infty$, $s = 2d\big|1/p - 1/2\big|$. Then*

$$\|e^{iH} f\|_{L^p} \lesssim \|f\|_{L^p_s}, \quad 1 < p \le 2,$$

$$\|e^{iH} f\|_{L^p_{-s}} \lesssim \|f\|_{L^p}, \quad 2 \le p < \infty.$$

Indeed, it was proved in [10] that the propagator e^{itH} is bounded in M^p, $1 \le p \le \infty$, $t \in \mathbb{R}$. Much more general Hamiltonians can be treated similarly; we refer to [10] and the bibliography therein for more details.

Finally we point out that ideas strictly related to those in this last section, but involving the so-called Wiener amalgam spaces rather than modulation spaces, have been recently employed in [12] to prove the L^p-boundedness, with optimal loss of derivatives, of pseudodifferential operators with non-smooth symbols.

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