Rev. Mat. Iberoam. **32** (2016), no. 4, 1137–1162
DOI 10.4171/RMI/910

Interpolatory estimates, Riesz transforms and wavelet projections

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Abstract. We prove that directional wavelet projections and Riesz transforms are related by interpolatory estimates. The exponents of interpolation depend on the Hölder estimates of the wavelet system. This paper complements and continues previous work on Haar projections.

1. Introduction

This paper is concerned with wavelet systems, directional wavelet projections and their estimates in terms of Riesz transforms. We continue and extend the methods introduced in [18] and [13].

Let $\mathcal{F}$ denote the $L^2(\mathbb{R}^n)$ normalized Fourier transform. The Riesz transform R_i is the Fourier multiplier defined by

$$\mathcal{F}(R_i(u))(\xi) = -\sqrt{-1}\,\frac{\xi_i}{|\xi|}\,\mathcal{F}(u)(\xi), \quad \text{where } 1 \le i \le n, \quad \xi = (\xi_1, \dots, \xi_n). \tag{1.1}$$

Let $\mathcal{A} = \{0,1\}^n \setminus \{(0,0,\dots,0\}$, and let $\mathcal{S}$ be the collection of dyadic cubes in $\mathbb{R}^n$. We let

$$\{\varphi_Q^{(\varepsilon)} : Q \in \mathcal{S}, \varepsilon \in \mathcal{A}\}$$

denote an admissible wavelet system of Hölder exponent $0 < \alpha \le 1$ and decay estimates of order $\delta > 0$. (The definition is given in (1.2) below.) For a fixed direction $\varepsilon \in \mathcal{A}$ the associated orthogonal wavelet projection is defined as

$$W^{(\varepsilon)}(u) = \sum_{Q \in \mathcal{S}} \langle u, \varphi_Q^{(\varepsilon)} \rangle\, \varphi_Q^{(\varepsilon)}\, |Q|^{-1}, \quad u \in L^2(\mathbb{R}^n).$$

The results of this paper give pointwise estimates for the directional wavelet projection $W^{(\varepsilon)}$ in terms of the Riesz transforms.

Mathematics Subject Classification (2010): Primary 42B25, 42B37, 42C40; Secondary 49J40, 49J45.
Keywords: Riesz transform, directional wavelet projection, interpolatory estimates, rearrangement operators.

Admissible wavelet systems. We specify now the wavelet systems we use in this paper. Recall that $I \subset \mathbb{R}$ is a dyadic interval if there exist natural numbers $k, m \in \mathbb{Z}$ so that $I = [(k-1)2^m, k2^m[$. Let $I_1, \dots, I_n$ be dyadic intervals in $\mathbb{R}$ so that $|I_i| = |I_j|$. Define the dyadic cube $Q \subset \mathbb{R}^n$, as

$$Q = I_1 \times \cdots \times I_n.$$

We let $s(Q)$ denote the side length of Q, thus $s(Q) = |I_1|$. Let $\mathcal{S}$ denote the collection of all dyadic cubes in $\mathbb{R}^n$ and put $\mathcal{A} = \{\varepsilon \in \{0,1\}^n : \varepsilon \neq (0, \dots .0)\}$. We say that

$$\{\varphi_Q^{(\varepsilon)} : Q \in \mathcal{S}, \varepsilon \in \mathcal{A}\}$$

is an admissible wavelet system if $\{\varphi_Q^{(\varepsilon)}/\sqrt{|Q|} : Q \in \mathcal{S}, \varepsilon \in \mathcal{A}\}$ is an orthonormal basis in $L^2(\mathbb{R}^n)$ and there exists $C > 0, \delta > 0$ and $0 < \alpha \leq 1$ so that the following conditions hold:

1) Localization with decay estimates:

$$|\varphi_Q^{(\varepsilon)}(x)| \leq C\Big(1 + \frac{\operatorname{dist}(x,Q)}{s(Q)}\Big)^{-n(1+\delta)}, \quad x \in \mathbb{R}^n. \tag{1.2a}$$

2) Hölder estimates of order α: if $x, t \in \mathbb{R}^n$, and $|x - t| \leq s(Q)$, then

$$|\varphi_Q^{(\varepsilon)}(x) - \varphi_Q^{(\varepsilon)}(t)| \leq Cs(Q)^{-\alpha}|x-t|^{\alpha}\Big(1 + \frac{\operatorname{dist}(x,Q)}{s(Q)}\Big)^{-n(1+\delta)}. \tag{1.2b}$$

3) Sectional oscillation for $i \in \{j \leq n : \varepsilon_j = 1\}$:

$$|\mathbb{E}_i(\varphi_Q^{(\varepsilon)})(x)| \leq Cs(Q)\Big(1 + \frac{\operatorname{dist}(x,Q)}{s(Q)}\Big)^{-n(1+\delta)}, \tag{1.2c}$$

and for $t \in \mathbb{R}^n$, with $|x - t| \leq s(Q)$,

$$|\mathbb{E}_i(\varphi_Q^{(\varepsilon)})(x) - \mathbb{E}_i(\varphi_Q^{(\varepsilon)})(t)| \leq C|x-t|^{\alpha}s(Q)^{1-\alpha}\Big(1 + \frac{\operatorname{dist}(x,Q)}{s(Q)}\Big)^{-n(1+\delta)},$$

where $\mathbb{E}_i$ denotes integration with respect to the variable x_i,

$$\mathbb{E}_i(f)(x) = \int_{-\infty}^{x_i} f(x_1, \dots, s, \dots, x_n)\, ds, \quad x = (x_1, \dots, x_i, \dots x_n). \tag{1.2d}$$

We refer to $\delta > 0$ and $0 < \alpha \leq 1$ as the decay and Hölder exponents of a wavelet system satisfying (1.2).

Directional wavelet projections. We fix an admissible wavelet system $\{\varphi_Q^{(\varepsilon)} : Q \in \mathcal{S}, \varepsilon \in \mathcal{A}\}$. For a given direction $\varepsilon \in \mathcal{A}$, let $W^{(\varepsilon)}$ denote the associated projection on $L^2(\mathbb{R}^n)$,

$$W^{(\varepsilon)}(u) = \sum_{Q \in \mathcal{S}} \langle u, \varphi_Q^{(\varepsilon)} \rangle\, \varphi_Q^{(\varepsilon)}\, |Q|^{-1}, \quad u \in L^2(\mathbb{R}^n).$$

We summarize next the main estimates in [18] and [13], and relate them to the results of the present paper.

Review of [18]. If the Hölder exponent of the wavelet system satisfies $0 < \alpha < 1$, the following Hilbertian estimates for $W^{(\varepsilon)}$ are obtained with the method introduced in [18]:

$$\|W^{(\varepsilon)}(u)\|_2 \le A\,\|u\|_2^{1-\alpha}\,\|R_{i_0}(u)\|_2^{\alpha} + A\,\frac{\|u\|_2^{1-\alpha}\|R_{i_0}(u)\|_2^{\alpha} - \|R_{i_0}(u)\|_2}{2^{1-\alpha}-1}, \tag{1.3}$$

whenever $\varepsilon = (\varepsilon_1, \dots \varepsilon_n)$ and $\varepsilon_{i_0} = 1$. We have $A = A(\alpha, \delta) \to \infty$ as $\alpha \to 0$ or $\delta \to 0$. The Lipschitz case when $\alpha = 1$ is of particular interest. It appears as the limit as $\alpha \to 1$ of the estimates (1.3). By L'Hôpital's rule, (1.3) implies

$$\|W^{(\varepsilon)}(u)\|_2 \le A(1,\delta)\Big(1 + \log\frac{\|u\|_2}{\|R_{i_0}u\|_2}\Big)\|R_{i_0}u\|_2. \tag{1.4}$$

If $0 < \alpha < 1$, is fixed and if one is not interested in the limiting behavior as $\alpha \to 1$, then a simplified form of (1.3) is as follows:

$$\|W^{(\varepsilon)}(u)\|_2 \le \frac{A(\alpha,\delta)}{1-\alpha}\,\|u\|_2^{1-\alpha}\,\|R_{i_0}u\|_2^{\alpha}, \tag{1.5}$$

The estimates (1.3), (1.4) and (1.5) were proven in [18] by cotlarization of the operator $W^{(\varepsilon)}$.

The present paper extends the $L^2(\mathbb{R}^n)$ estimates (1.3) to the scale of $L^p(\mathbb{R}^n)$ spaces. We use below the abbreviation $\|R_{i_0}\|_p = \|R_{i_0} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)\|$. Our main result asserts that, for Hölder exponents $0 < \alpha < 1$ and $1 < p < \infty$,

$$\begin{aligned}&\|W^{(\varepsilon)}(u)\|_p\\ &\le C\|R_{i_0}\|_p^{-\alpha}\|u\|_p^{1-\alpha}\|R_{i_0}(u)\|_p^{\alpha} + C\,\frac{\|R_{i_0}\|_p^{1-\alpha}\|u\|_p^{1-\alpha}\|R_{i_0}(u)\|_p^{\alpha} - \|R_{i_0}(u)\|_p}{2^{1-\alpha}-1}\end{aligned} \tag{1.6}$$

whenever $\varepsilon = (\varepsilon_1, \dots \varepsilon_n)$ and $\varepsilon_{i_0} = 1$. The asymptotic behavior of the constants $C = C(p, \alpha, \delta)$ is as follows:

$$C(p,\alpha,\delta) = \frac{p^2 C(\alpha,\delta)}{p-1} \quad \text{and} \quad \mathrm{C}(\alpha,\delta) \to \infty \quad \text{as} \quad \alpha \to 0 \quad \text{or} \quad \delta \to 0\,.$$

Again the estimates for the Lipschitz case $\alpha = 1$ appear as limit of (1.6) by using L'Hôpital's rule:

$$\|W^{(\varepsilon)}(u)\|_p \le C\Big(1 + \log\frac{\|u\|_p\|R_{i_0}\|_p}{\|R_{i_0}u\|_p}\Big)\|R_{i_0}u\|_p\,, \tag{1.7}$$

where $C = C(p, 1, \delta)$. For fixed $0 < \alpha < 1$, a simplified version of (1.6) is the following:

$$\|W^{(\varepsilon)}(u)\|_p \le \frac{C}{1-\alpha}\,\|u\|_p^{1-\alpha}\,\|R_{i_0}u\|_p^{\alpha}\,. \tag{1.8}$$

Specializing (1.6), (1.7) and (1.8) to the case $p = 2$ gives back (1.3), (1.4) and (1.5).

Review of [18] and [13]. We next compare the inequalities (1.7) and (1.8) to the interpolatory estimates for directional Haar projections [13]. Let

$$\{h_Q^{(\varepsilon)} : Q \in \mathcal{S}, \varepsilon \in \mathcal{A}\}$$

be the isotropic Haar system supported on dyadic cubes. (See Section 4 for the definition.) The directional Haar projection is defined by

$$P^{(\varepsilon)}(u) = \sum_{Q \in \mathcal{S}} \langle u, h_Q^{(\varepsilon)} \rangle \, h_Q^{(\varepsilon)} \, |Q|^{-1}, \quad u \in L^2(\mathbb{R}^n).$$

In [13] we proved that for $1 < p < \infty$ and $\tau_p = \max\{1/2, 1/p\}$,

$$\|P^{(\varepsilon)}(u)\|_p \leq C(p) \, \|u\|_p^{\tau_p} \, \|R_{i_0} u\|_p^{1-\tau_p}, \tag{1.9}$$

when $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ and $\varepsilon_{i_0} = 1$. Comparing (1.9) to (1.8) we observe that:

1. In (1.8) the interpolation exponents for $W^{(\varepsilon)}$ depend just on the order α of the Hölder estimates, and not on the value of p.
2. By contrast in (1.9), the $P^{(\varepsilon)}$ estimates show a critical transition at $p = 2$. The exponents in (1.9) are best possible, as shown in Section 8 of [13]. Hence (1.9) does not arise as the limit of $\alpha \to 0$ from the estimates (1.8).
3. As Hölder estimates are not available for the Haar system we exploit in [13] that the discontinuities of Haar functions are concentrated at an $(n-1)$ dimensional set, and that
$$\int_{\mathbb{R}^n} |h_Q^{(\varepsilon)}(x - s) - h_Q^{(\varepsilon)}(x)|^p \, dx \leq C \, |s| \cdot |Q|^{(n-1)/n}. \tag{1.10}$$
4. It remains an open problem to prove interpolatory estimates for directional projections $W^{(\varepsilon)}$ when the underlying wavelets satisfy decay estimates *only.* The particular interest in this question comes from theorems of G. Gripenberg [10] and P. Wojtaszczyk [22] who proved that wavelets with decay (1.2a) form an unconditional basis in $L^p, (1 < p < \infty)$ – *without* using assumptions on smoothness.

Outlook. In [18] and [13], proving weak semi-continuity of separately convex functionals – as conjectured by J. Ball and F. Murat [1] and L. Tartar [21] – provided the initial motivation for estimating Haar projections in terms of Riesz transforms. For further motivation we refer to the analysis of Sverak's counterexamples to quasi-convexity in [19].

In the course of development [18] and [13], the inequalities (1.9) gave rise to general questions of *ordering* singular integral operators on a given space by means of interpolatory estimates. This includes the following problems:

1. Determination of the best possible exponents in interpolatory estimates. See Section 8 in [13] for the sharp exponents between Haar projections and Riesz transforms.
2. Extensions to vector valued singular integral operators. R. Lechner [12] obtained the UMD version of [18], [13].

3. Presently interpolatory estimates between singular integral operators are known only for the setting of $\mathbb{R}^n$. For singular integrals over non commutative groups (e.g., Heisenberg group, homogeneous Lie groups) such estimates are open. See M. Christ [2], [3], M. Christ and D. Geller [4], P. G. Lemarie [14], and Folland and Stein [8].

The results of the present paper and [18], [13] are the first steps in this direction.

2. The main results

Theorem 2.1 is the main result of this paper. The partial coercivity of Riesz transforms (2.3) follows immediately from Theorem 2.1. We use below the abbreviation $\|R_{i_0}\|_p = \|R_{i_0} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)\|$, to denote the norm of the Riesz transform on $L^p(\mathbb{R}^n)$.

Theorem 2.1. *Let $1 < p < \infty$, $1 \le i_0 \le n$, and $\varepsilon = (\varepsilon_1, \dots \varepsilon_n) \in \mathcal{A}$ with $\varepsilon_{i_0} = 1$. If $0 < \alpha < 1$, then for any $u \in L^p(\mathbb{R}^n)$,*

$$\begin{aligned} &\|W^{(\varepsilon)}(u)\|_p \\ &\le C\|R_{i_0}\|_p^{-\alpha}\|u\|_p^{1-\alpha}\|R_{i_0}(u)\|_p^{\alpha} + C\frac{\|R_{i_0}\|_p^{1-\alpha}\|u\|_p^{1-\alpha}\|R_{i_0}(u)\|_p^{\alpha} - \|R_{i_0}(u)\|_p}{2^{1-\alpha}-1}, \end{aligned} \tag{2.1}$$

where $C = C(p, \alpha, \delta)$ and

$$C(p,\alpha,\delta) = \frac{p^2 C(\alpha,\delta)}{p-1} \quad \text{with} \quad C(\alpha,\delta) \to \infty \quad \text{as} \quad \alpha \to 0 \quad \text{or} \quad \delta \to 0.$$

If $\alpha = 1$, then

$$\|W^{(\varepsilon)}(u)\|_p \le C\,\|R_{i_0}\|_p^{-1}\|R_{i_0}(u)\|_p + C\,\log\Big(\frac{\|u\|_p\|R_{i_0}\|_p}{\|R_{i_0}(u)\|_p}\Big)\|R_{i_0}(u)\|_p\,, \tag{2.2}$$

where $C = C(p, 1, \delta)$. The estimate (2.2) appears as the limit of (2.1) as $\alpha \to 1$.

Remark 2.2. Clearly (2.1) implies that

$$\|W^{(\varepsilon)}(u)\|_p \le \frac{C}{1-\alpha}\,\|u\|_p^{1-\alpha}\,\|R_{i_0}(u)\|_p^{\alpha},$$

and (2.2) yields

$$\|W^{(\varepsilon)}(u)\|_p \le C\log\Big(1 + \frac{\|u\|_p}{\|R_{i_0}(u)\|_p}\Big)\|R_{i_0}(u)\|_p\,.$$

Partial coercivity of Riesz transforms. The estimates of Theorem 2.1 imply partial coercivity for the Riesz transforms. On the closure of $W^{(\varepsilon)}(L^p(\mathbb{R}^n))$, the Riesz transform R_{i_0} is invertible provided that $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ and $\varepsilon_{i_0} = 1$. Indeed, since $W^{(\varepsilon)}$ is a projection, Theorem 2.1 gives

$$\|v\|_p \le C(p,\alpha,\delta)\|R_{i_0}v\|_p, \quad v \in W^{(\varepsilon)}(L^p(\mathbb{R}^n))\,. \tag{2.3}$$

By (1.9) the same holds when $W^{(\varepsilon)}$ is replaced by $P^{(\varepsilon)}$. For the concept and background we refer to T. Kato [11], F. Murat [20], B. Dacorogna [6]. The interpretation of Theorem 2.1 as a partial coercivity estimate for Riesz transforms (2.3) emphasizes the connection to [3].

The outline of the proof. We use the pattern of reduction applied previously in [18] and [13]. In the present paper we exploit properties of the discrete Calderón reproducing formula going back to Frazier and Jawerth [9]. We start the proof of Theorem 2.1 with a multi-scale analysis of $W^{(\varepsilon)}$ using a discrete Calderón reproducing formula. See [9]. We fix $v, w \in C^\infty(\mathbb{R}^n)$ so that $\operatorname{supp}\mathcal{F}v$, $\operatorname{supp}\mathcal{F}w \subseteq [1/2, 2]$ and

$$1 = \sum_{\ell\in\mathbb{Z}} (\mathcal{F}v)(2^\ell\zeta)\,(\mathcal{F}w)(2^\ell\zeta)\,.$$

For any multi-index $\gamma \in \mathbb{N}^n$ and $N \in \mathbb{N}$, there exists $A = A(\gamma, N)$ so that

$$|\partial_\gamma v(x)| + |\partial_\gamma w(x)| \le A\,(1+|x|)^{-N}.$$

Put $v_\ell(x) = 2^{\ell n}v(2^\ell x)$, $w_\ell(x) = 2^{\ell n}w(2^\ell x)$, and form the convolution product

$$d_\ell(x) = v_\ell * w_\ell(x), \quad \ell \in \mathbb{Z}\,.$$

For any multi-index $\gamma \in \mathbb{N}^n$ and $N \in \mathbb{N}$, there exists $A = A(\gamma, N)$ so that

$$|\partial_\gamma d_\ell(x)| \le A\,2^{(n+|\gamma|)(\ell)}\,(1+2^\ell|x|)^{-N}. \tag{2.4}$$

Finally we put

$$\Delta_\ell(u) = u * d_\ell\,. \tag{2.5}$$

Then, as obtained by Frazier and Jawerth [9],

$$u = \sum_{\ell=-\infty}^{\infty} \Delta_\ell(u), \quad u \in L^p(\mathbb{R}^n)\,,$$

where convergence holds in $L^p(\mathbb{R}^n)$.

We denoted by $\mathcal{S}$ the collection of all dyadic cubes in $\mathbb{R}^n$. Let $j \in \mathbb{Z}$ and consider the following subcollection of $\mathcal{S}$:

$$\mathcal{S}_j = \{Q \in \mathcal{S} : |Q| = 2^{-nj}\}\,. \tag{2.6}$$

The cubes in $\mathcal{S}_j$ are pairwise disjoint. For $\ell \in \mathbb{Z}$, $\varepsilon \in \mathcal{A}$, and $Q \in \mathcal{S}_j$, define

$$f^{(\varepsilon)}_{Q,\ell} = \Delta_{j+\ell}(\varphi^{(\varepsilon)}_Q)\,, \tag{2.7}$$

and

$$T^{(\varepsilon)}_\ell(u) = \sum_{Q\in\mathcal{S}} \langle u, f^{(\varepsilon)}_{Q,\ell}\rangle\,\varphi^{(\varepsilon)}_Q\,|Q|^{-1}. \tag{2.8}$$

Thus we arrive at our basic Littlewood–Paley decomposition for the directional wavelet projection,

$$W^{(\varepsilon)}(u) = \sum_{\ell=-\infty}^{\infty} T_\ell^{(\varepsilon)}(u)\,.$$

Let $1 \le i_0 \le n$ and $\mathcal{A}_{i_0} = \{\varepsilon \in \mathcal{A} : \varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \quad \text{and} \quad \varepsilon_{i_0} = 1\}$. As observed in [18] and [13], for $\varepsilon \in \mathcal{A}_{i_0}$ we get

$$T_\ell^{(\varepsilon)} R_{i_0}^{-1} = T_\ell^{(\varepsilon)} R_{i_0} + \sum_{\substack{i=1\\ i\neq i_0}}^{n} T_\ell^{(\varepsilon)}\, \mathbb{E}_{i_0}\partial_i R_i\,,$$

where R_i denotes the i-th Riesz transform, ∂_i denotes the differentiation with respect to the x_i variable and $\mathbb{E}_{i_0}$ the integration with respect to the x_{i_0}-th coordinate. See (1.2d). Hence, putting

$$k_Q^{(\ell,i)} = \Delta_{j+\ell}\big(\mathbb{E}_{i_0}\partial_i \varphi_Q^{(\varepsilon)}\big), \quad Q \in \mathcal{S}_j, \tag{2.9}$$

we obtain the representation

$$T_\ell^{(\varepsilon)} R_{i_0}^{-1}(u) = T_\ell^{(\varepsilon)} R_{i_0}(u) + \sum_{Q\in\mathcal{S}} \sum_{\substack{i=1\\ i\neq i_0}}^{n} \langle R_i(u), k_Q^{(\ell,i)}\rangle\, \varphi_Q^{(\varepsilon)}\, |Q|^{-1}. \tag{2.10}$$

The following two theorems record the norm estimates for the operators $T_\ell^{(\varepsilon)}$ and $T_\ell^{(\varepsilon)} R_{i_0}^{-1}$ by which we obtain Theorem 2.1. First we treat the case $\ell > 0$. It displays the crucial dependence on the Hölder exponent of the admissible wavelet system. Below and throughout the paper the constants $C(p,\alpha,\delta) > 0$ satisfy the conditions

$$C(p,\alpha,\delta) = \frac{p^2 C(\alpha,\delta)}{p-1}, \quad \text{where} \quad C(\alpha,\delta) \to \infty, \quad \text{as} \quad \alpha \to 0, \quad \text{or} \quad \delta \to 0.$$

Theorem 2.3. *Let $\delta > 0$ and $0 < \alpha \le 1$ be the decay and Hölder exponents of the admissible wavelet system specified in (1.2). Let $1 < p < \infty$, $\ell \ge 0$ and $\varepsilon \in \mathcal{A}$. Then $T_\ell^{(\varepsilon)}$ satisfies the norm estimate*

$$\|T_\ell^{(\varepsilon)}\|_p \le C(p,\alpha,\delta)\, 2^{-\ell\alpha}. \tag{2.11}$$

Let $1 \le i_0 \le n$, and $\varepsilon \in \mathcal{A}_{i_0}$. Then,

$$\|T_\ell^{(\varepsilon)} R_{i_0}^{-1}\|_p \le C(p,\alpha,\delta)\, 2^{\ell-\ell\alpha}. \tag{2.12}$$

When $\ell < 0$ we get exponents independent of the Hölder condition.

Theorem 2.4. *Let $\delta > 0$ and $0 < \alpha \le 1$ be the decay and Hölder exponents of the admissible wavelet system specified in (1.2). Let $1 < p < \infty$. Let $\ell \le 0$. Then for $\varepsilon \in \mathcal{A}$, the operator $T_\ell^{(\varepsilon)}$ satisfies the norm estimate*

$$\|T_\ell^{(\varepsilon)}\|_p \le C(p,\alpha,\delta)\, 2^{-|\ell|}\, |\ell|. \tag{2.13}$$

If moreover $1 \le i_0 \le n$, *and* $\varepsilon \in \mathcal{A}_{i_0}$, *then*

$$\|T_\ell^{(\varepsilon)} R_{i_0}^{-1}\|_p \le C(p,\alpha,\delta)\, 2^{-|\ell|}\, |\ell|. \tag{2.14}$$

Proof of Theorem 2.1. Theorems 2.3 and 2.4 yield the proof of Theorem 2.1 as follows. Fix $u \in L^p(\mathbb{R}^n)$. Define $M \in \mathbb{N}$ by the relation

$$2^{M-1} \le \frac{\|u\|_p \|R_{i_0}\|_p}{\|R_{i_0}(u)\|_p} \le 2^M. \tag{2.15}$$

First we fix the Hölder exponent wavelet system as $0 < \alpha < 1$. Thereafter we consider the limit as $\alpha \to 1$. By Theorems 2.3 and 2.4, there exists $C = C(p,\alpha,\delta)$ so that

$$\sum_{\ell=M}^{\infty} \|T_\ell^{(\varepsilon)}\|_p \le C\, 2^{-M\alpha},$$

and

$$\sum_{\ell=-\infty}^{M-1} \|T_\ell^{(\varepsilon)} R_{i_0}^{-1}\|_p \le C \left(\sum_{\ell=0}^{M-1} 2^{\ell-\alpha\ell} \right) = C\, \frac{2^{M-\alpha M}-1}{2^{1-\alpha}-1}.$$

The constants $C(p,\alpha,\delta)$ stay bounded as $\alpha \to 1$. Since $W^{(\varepsilon)}(u) = \sum_{\ell=-\infty}^{\infty} T_\ell^{(\varepsilon)}(u)$, triangle inequality gives

$$\begin{aligned} \|W^{(\varepsilon)}(u)\|_p &\le \sum_{\ell=M}^{\infty} \|T_\ell^{(\varepsilon)}\|_p \|u\|_p + \sum_{\ell=-\infty}^{M-1} \|T_\ell^{(\varepsilon)} R_{i_0}^{-1}\|_p\, \|R_{i_0}(u)\|_p \\ &\le C\Big(2^{-M\alpha}\|u\|_p + \frac{2^{M-\alpha M}-1}{2^{1-\alpha}-1} \|R_{i_0}(u)\|_p \Big). \end{aligned} \tag{2.16}$$

Inserting the value of M specified in (2.15) gives the following upper bound for (2.16):

$$C\,\|R_{i_0}\|_p^{-\alpha} \|u\|_p^{1-\alpha} \|R_{i_0}(u)\|_p^{\alpha} + C\, \frac{\|R_{i_0}\|_p^{1-\alpha}\|u\|_p^{1-\alpha}\|R_{i_0}(u)\|_p^{\alpha} - \|R_{i_0}(u)\|_p}{2^{1-\alpha}-1}. \tag{2.17}$$

The term arising in (2.17) has a well defined limit as $\alpha \to 1$. Indeed, by L'Hôpital's rule,

$$\lim_{\alpha\to 1} \frac{2^{M-\alpha M}-1}{2^{1-\alpha}-1} = M,$$

and hence

$$\lim_{\alpha\to 1} (2.17) = C\,\|R_{i_0}\|_p^{-1}\|R_{i_0}(u)\|_p + C\, \log\Big(\frac{\|u\|_p\|R_{i_0}\|_p}{\|R_{i_0}(u)\|_p}\Big)\|R_{i_0}(u)\|_p, \tag{2.18}$$

where $C = C(p,1,\delta)$. ∎

Remark 2.5. The somewhat complicated form of (2.17) was used to obtain the limit estimate (2.18). For fixed $\alpha < 1$ we may simplify the upper bound (2.17) as follows

$$\|W^{(\varepsilon)}(u)\|_p \leq \frac{C}{1-\alpha} \|u\|_p^{1-\alpha} \|R_{i_0}(u)\|_p^{\alpha}, \quad 1 < p < \infty. \tag{2.19}$$

Also (2.18) may be simplified further,

$$\|W^{(\varepsilon)}(u)\|_p \leq C \log\left(1 + \frac{\|u\|_p}{\|R_{i_0}(u)\|_p}\right) \|R_{i_0}(u)\|_p, \quad 1 < p < \infty. \tag{2.20}$$

Remark 2.6. An alterative proof of (2.18) may be deduced directly from Theorem 2.3 and Theorem 2.4. Define $M \in \mathbb{N}$ by (2.15). Then

$$\sum_{\ell=M}^{\infty} \|T_\ell^{(\varepsilon)}\|_p \leq C(p,\delta)\, 2^{-M} \quad \text{and} \quad \sum_{\ell=-\infty}^{M-1} \|T_\ell^{(\varepsilon)} R_{i_0}^{-1}\|_p \leq C(p,\delta) M.$$

Hence, by the triangle inequality,

$$\|W^{(\varepsilon)}(u)\|_p \leq C(p,\delta)\, 2^{-M} \,\|u\|_p + C(p,\delta)\, M \,\|R_{i_0}(u)\|_p. \tag{2.21}$$

As M is given by (2.15), the right hand side of (2.21) is dominated by

$$C(p,\delta)\Big(\|R_{i_0}\|_p^{-1} \|R_{i_0}(u)\|_p + \log\Big(\frac{\|u\|_p \|R_{i_0}\|_p}{\|R_{i_0}(u)\|_p}\Big) \|R_{i_0}(u)\|_p\Big), \quad 1 < p < \infty.$$

The paper is organized as follows. In Section 3 we prove point-wise estimates for the decay and smoothness of the systems $\{f_{Q,\ell}^{(\varepsilon)}\}$ and $\{k_Q^{(\ell,i)}\}$ defined in (2.7) and (2.9). The cases $\ell > 0$ and $\ell \leq 0$ are given different treatment.

In Section 4 we present two general tools used to reduce estimates for integral operators to those of rearrangements.

In Section 5 we combine the preparatory theorems of Section 3 and Section 4 to prove Theorem 2.3. Section 6 contains the proof of Theorem 2.4.

3. Wavelets and convolution

Our basic concern are the norm estimates for the operators $T_\ell^{(\varepsilon)}$ and $T_\ell^{(\varepsilon)} R_{i_0}^{-1}$ as formulated in Theorem 2.3 and Theorem 2.4. We showed in the introduction that this amounts to proving estimates for operators of the following form: first,

$$X(u) = \sum_{Q \in \mathcal{S}} \langle u, k_Q^{(\ell,i)} \rangle \, \varphi_Q^{(\varepsilon)} \, |Q|^{-1},$$

where

$$k_Q^{(\ell,i)} = \Delta_{j+\ell}(\mathbb{E}_{i_0} \partial_i \varphi_Q^{(\varepsilon)}),$$

with $Q \in \mathcal{S}_j$, and $\varepsilon \in \mathcal{A}_{i_0}$, with $i \neq i_0$.

And second,

$$Y(u) = \sum_{Q\in\mathcal{S}} \langle u, f^{(\varepsilon)}_{Q,\ell}\rangle\, \varphi^{(\varepsilon)}_Q\, |Q|^{-1},$$

where $f^{(\varepsilon)}_{Q,\ell} = \Delta_{j+\ell}(\varphi^{(\varepsilon)}_Q)$, with $Q \in \mathcal{S}_j$. We recall also that in (2.5) the operator $\Delta_{j+\ell}$ is defined as convolution with $d_{j+\ell}$, where for any multi-index $\gamma \in \mathbb{N}^n$ and $N \in \mathbb{N}$ there exists $A = A(\gamma, N)$ so that

$$|\partial_\gamma d_{j+\ell}(x)| \le A\, 2^{(n+|\gamma|)(j+\ell)}\, (1 + 2^{j+\ell}|x|)^{-N}. \tag{3.1}$$

In Lemma 3.1 through Lemma 3.5 we record the point-wise estimates for the systems $\{f^{(\varepsilon)}_{Q,\ell}\}$ and $\{k^{(\ell,i)}_Q\}$ as needed for the purpose of this paper. Those are the basis for the norm inequalities of the operators X and Y defined above.

3.1. Point-wise estimates for $\Delta_{j+\ell}(\varphi^{(\varepsilon)}_Q)$

The following lemma records basic point-wise estimates for $f^{(\varepsilon)}_{Q,\ell}$, $\ell \ge 0$ and its gradient.

Lemma 3.1. *Assume that* $\{\varphi^{(\varepsilon)}_Q : Q\in\mathcal{S}\}$ *satisfies* (1.2)*. The system* $\{f^{(\varepsilon)}_{Q,\ell} : Q\in\mathcal{S}, \ell \ge 0\}$ *defined by* (2.7) *satisfies these basic estimates:*

$$|f^{(\varepsilon)}_{Q,\ell}(x)| \le C\, 2^{-\alpha\ell}\Big(1 + \frac{\operatorname{dist}(x,Q)}{s(Q)}\Big)^{-n(1+\delta)}, \quad x \in \mathbb{R}^n. \tag{3.2a}$$

$$|\nabla f^{(\varepsilon)}_{Q,\ell}(x)| \le C\, 2^{-\alpha\ell}\, 2^{j+\ell}\Big(1 + \frac{\operatorname{dist}(x,Q)}{s(Q)}\Big)^{-n(1+\delta)}, \quad x \in \mathbb{R}^n. \tag{3.2b}$$

$$\int_{\mathbb{R}^n} f^{(\varepsilon)}_{Q,\ell}(x)dx = 0. \tag{3.2c}$$

Proof. Let $x \in \mathbb{R}^n$, $Q \in \mathcal{S}_j$ and $\ell > 0$. Let $A_x = \{t : |x-t| \le C\, 2^{-j-\ell}\}$. Since $\int d_{j+\ell}(x-t)dt = 0$ and $t \to d_{j+\ell}(x-t)$ is centered at A_x we get

$$\begin{aligned}
|d_{j+\ell} * \varphi^{(\varepsilon)}_Q(x)| &= \Big|\int_{\mathbb{R}^n} d_{j+\ell}(x-t)(\varphi^{(\varepsilon)}_Q(t) - \varphi^{(\varepsilon)}_Q(x))\, dt\Big| \\
&\le \int_{\mathbb{R}^n} |d_{j+\ell}(x-t)| \cdot |\varphi^{(\varepsilon)}_Q(t) - \varphi^{(\varepsilon)}_Q(x)|\, dt \\
&\le C \int_{\mathbb{R}^n} |d_{j+\ell}(x-t)|\, dt\; \frac{\operatorname{diam}(A_x)^\alpha}{s(Q)^\alpha}\Big(1 + \frac{\operatorname{dist}(x,Q)}{s(Q)}\Big)^{-n(1+\delta)}.
\end{aligned}$$

Invoking that $\operatorname{diam}(A_x) \le C\, 2^{-j-\ell}$, $s(Q) = 2^{-j}$ and $\int_{\mathbb{R}^n} |d_{j+\ell}(x-t)|dt \le C$ yields (3.2a).

In a similar fashion we obtain the remaining estimates (3.2b). Put $\widetilde{d}_{j+\ell} = 2^{-(j+\ell)}\nabla d_{j+\ell}$. Repeating the above argument with $d_{j+\ell}$ replaced by $\widetilde{d}_{j+\ell}$ we get

$$\begin{aligned}|\nabla d_{j+\ell} * \varphi_Q^{(\varepsilon)}(x)| &= 2^{j+\ell}\left|\int_{\mathbb{R}^n} \widetilde{d}_{j+\ell}(x-t)(\varphi_Q^{(\varepsilon)}(t) - \varphi_Q^{(\varepsilon)}(x))\,dt\right| \\ &\le 2^{j+\ell}\int_{\mathbb{R}^n} |\widetilde{d}_{j+\ell}(x-t)|\cdot|\varphi_Q^{(\varepsilon)}(t) - \varphi_Q^{(\varepsilon)}(x)|\,dt \\ &\le C\,2^{j+\ell}\int_{\mathbb{R}^n} |\widetilde{d}_{j+\ell}(x-t)|\,dt\ \frac{\operatorname{diam}(A_x)^\alpha}{s(Q)^\alpha}\Big(1+\frac{\operatorname{dist}(x,Q)}{s(Q)}\Big)^{-n(1+\delta)} \\ &\le C\,2^{-\alpha\ell}\,2^{j+\ell}\Big(1+\frac{\operatorname{dist}(x,Q)}{s(Q)}\Big)^{-n(1+\delta)}.\end{aligned}$$

□

Compactly supported wavelets. Let $\{\psi_K^{(\beta)}/\sqrt{|K|}\colon K\in\mathcal{S},\beta\in\mathcal{A}\}$ be an orthonormal basis in $L^2(\mathbb{R}^n)$, satisfying $\int\psi_K^{(\beta)}=0$ and the following structure conditions:

$$\operatorname{supp}\psi_K^{(\beta)} \subseteq C\cdot K,\quad |\psi_K^{(\beta)}|\le C,\quad \operatorname{Lip}(\psi_K^{(\beta)})\le C\,s(K)^{-1}. \tag{3.3}$$

We often write in place of $\{\psi_K^{(\beta)}\}$ just $\{\psi_K\}$. The existence of compactly supported wavelets was proven by I. Daubechies, see [15].

Low frequency slices of $\boldsymbol{\Delta_{j+\ell}(\varphi_Q^{(\varepsilon)})}$. Here we prove point-wise estimates for decay and regularity of the low frequency slices of $f_{Q,\ell}^{(\varepsilon)}$ when $\ell\ge 0$. We define those slices using a compactly supported wavelet basis $\{\psi_K\}$ satisfying (3.3). Fix $k\in\mathbb{Z}\setminus\mathbb{N}$ and define

$$p_Q = \sum_{K\in\mathcal{S}_{j+k}} \langle f_{Q,\ell}^{(\varepsilon)},\psi_K\rangle\,\psi_K\,|K|^{-1},\quad Q\in\mathcal{S}_j\,,\quad j\in\mathbb{Z}\,. \tag{3.4}$$

Note that for $k\in\mathbb{Z}\setminus\mathbb{N}$ and $Q\in\mathcal{S}_j$, $j\in\mathbb{Z}$, there exists a unique cube $K_0=K_0(Q)$ so that

$$K_0\supseteq Q\,,\quad K_0\in\mathcal{S}_{j+k}\,. \tag{3.5}$$

Pointwise estimates for p_Q and ∇p_Q are as follows.

Lemma 3.2. *Let $\ell\in\mathbb{N}$ and $k\in\mathbb{Z}\setminus\mathbb{N}$. Let $\gamma_0=\min\{\delta n/2,1\}$. Then the system of slices defined by (3.4) satisfies the following estimates:*

$$|p_Q(t)|\le C\,2^{-\alpha\ell}\,2^{k(n+\gamma_o)}\Big(1+\frac{\operatorname{dist}(t,K_0)}{s(K_0)}\Big)^{-n(1+\delta)}, \tag{3.6a}$$

$$|\nabla p_Q(t)|\le C\,s(K_0)^{-1}\,2^{-\alpha\ell}\,2^{k(n+\gamma_o)}\Big(1+\frac{\operatorname{dist}(t,K_0)}{s(K_0)}\Big)^{-n(1+\delta)}, \tag{3.6b}$$

where $Q\in\mathcal{S}$ and $K_0=K_0(Q)$ is defined by (3.5).

Proof. Fix a dyadic cube $Q \in \mathcal{S}_j$, $j \in \mathbb{Z}$. Determine $K_0 \in \mathcal{S}_{j+k}$ so that $Q \subseteq K_0$. For any $\mu \in \mathbb{Z}^n$ and $K = K_0 + \mu \cdot s(K_0)$ we prove the coefficient estimate

$$|\langle f^{(\varepsilon)}_{Q,\ell}, \psi_K\rangle||K|^{-1} \le 2^{-\alpha\ell}\, 2^{-|k|(n+\gamma_0)}\, (1+|\mu|)^{-n(1+\delta)}. \tag{3.7}$$

Consider first the case $|\mu| \ge 4$. Then Lemma 3.1 gives

$$|\langle f^{(\varepsilon)}_{Q,\ell}, \psi_K\rangle|\, |K|^{-1} \le 2^{-\alpha\ell}\, |\mu|^{-n(1+\delta)}\, 2^{kn(1+\delta)}. \tag{3.8}$$

Note that (3.8) implies (3.7) by arithmetic.

Next consider $|\mu| \le 4$. We use that $f^{(\varepsilon)}_{Q,\ell}$ is of vanishing mean and rewrite

$$\langle f^{(\varepsilon)}_{Q,\ell}, \psi_K\rangle = \int_{\mathbb{R}^n} f^{(\varepsilon)}_{Q,\ell}(t)(\psi_K(t) - \psi_K(t_Q))\, dt, \tag{3.9}$$

where $t_Q \in Q$. We decompose the domain of integration as follows. Let $A_0(Q) = Q$ and $A_i(Q) = 2^i \cdot Q \setminus 2^{i-1} \cdot Q$ where $2^i \cdot Q$ is the cube with side-length $2^i s(Q)$ and the same center as Q. Thus defined the sets $A_i(Q) = 2^i \cdot Q \setminus 2^{i-1} \cdot Q$ form a decomposition of $\mathbb{R}^n$. Hence the right hand side of (3.9) is bounded by

$$\sum_{i=0}^{|k|} \int_{A_i(Q)} |f^{(\varepsilon)}_{Q,\ell}(t)(\psi_K(t) - \psi_K(t_Q))|\, dt + \sum_{i=|k|+1}^{\infty} \int_{A_i(Q)} |f^{(\varepsilon)}_{Q,\ell}(t)|\, dt\,. \tag{3.10}$$

For $i \le |k|$ we exploit the Lipschitz estimates for ψ_K and use that $\operatorname{diam}(A_i(Q)) \le C\, 2^i s(Q) \le C s(K)$. Thus we get

$$|\psi_K(t) - \psi_K(t_Q)| \le C(2^i s(Q))^{\gamma_0} (\mathrm{Lip}\psi_K)^{\gamma_0}, \quad t \in A_i(Q)\,.$$

Invoking also the basic estimates of Lemma 3.1 gives

$$\begin{aligned} &\int_{A_i(Q)} |f^{(\varepsilon)}_{Q,\ell}(t)\, (\psi_K(t) - \psi_K(t_Q))|\, dt \\ &\qquad \le 2^{-\alpha\ell}\, 2^{-in(1+\delta)}\, (2^i s(Q))^{\gamma_0}\, (\mathrm{Lip}\psi_K)^{\gamma_0}\, |2^i \cdot Q|\,. \end{aligned} \tag{3.11}$$

We take the sum over $i \le |k|$ in (3.11) and get the following upper bound for the first sum in (3.10):

$$2^{-\alpha\ell}\, \frac{(s(Q))^{\gamma_0}|Q|}{(s(K))^{\gamma_0}} \sum_{i=0}^{k} 2^{-in\delta + i\gamma_0} \le C\, 2^{-\alpha\ell}\, \frac{(s(Q))^{n+\gamma_0}}{(s(K))^{\gamma_0}}.$$

On the other hand, if $i \ge |k|$ we have again by Lemma 3.1 that

$$\int_{A_i(Q)} |f^{(\varepsilon)}_{Q,\ell}(t) \le C\, |Q|\, 2^{-\alpha\ell}\, 2^{-in\delta}.$$

Since $|K|2^{-n|k|} = |Q|$, summing over $i \ge |k|$ gives

$$|Q| \sum_{i=|k|+1}^{\infty} 2^{-\alpha\ell}\, 2^{-in\delta} \le C\, |K|\, 2^{-\alpha\ell}\, 2^{-|k|n(\delta+1)}.$$

Recall now that we are treating the case $|\mu| \leq 4$. Hence we may rephrase the above as

$$|\langle f^{(\varepsilon)}_{Q,\ell}, \psi_K \rangle| \, |K|^{-1} \leq C \, 2^{-\alpha\ell} \, 2^{-|k|(n+\gamma_0)} \, (1 + |\mu|)^{-n(1+\delta)}.$$

By the definition (3.4) of the slices, we obtain the pointwise estimates (3.6) from the estimate (3.7). □

The scalar products $\langle \Delta_{j+\ell}(\varphi^{(\varepsilon)}_Q), \Psi_K \rangle$, $K \in \mathcal{S}_{j+k}$, $0 \leq k \leq \ell$. Here we record a short but crucial consequence of Lemma 3.1. It is here where we explicitly exploit that our multi-scale analysis $\{d_\ell\}$ is based on Calderón's reproducing formula and admits a factorization as

$$d_\ell = v_\ell * w_\ell.$$

Lemma 3.3. *Let $j \in \mathbb{Z}, \ell \in \mathbb{N}$, and $0 \leq k \leq \ell$. For $Q \in \mathcal{S}_j$, $K \in \mathcal{S}_{j+k}$,*

$$|\langle \Delta_{j+\ell}(\varphi^{(\varepsilon)}_Q), \Psi_K \rangle| \leq C \, 2^{-\alpha\ell} \, 2^{k-\ell} \, |K| \left(1 + \frac{\operatorname{dist}(K,Q)}{s(Q)}\right)^{-n(1+\delta)}. \tag{3.12}$$

Proof. Recall that $\Delta_{j+\ell}(\varphi^{(\varepsilon)}_Q) = d_{j+\ell} * \varphi^{(\varepsilon)}_Q$ where $d_{j+\ell} = v_{j+\ell} * w_{j+\ell}$. Hence

$$\langle \Delta_{j+\ell}(\varphi^{(\varepsilon)}_Q), \Psi_K \rangle = \langle w_{j+\ell} * \varphi^{(\varepsilon)}_Q, v_{j+\ell} * \Psi_K \rangle \, .$$

By Lemma 3.1 we get

$$|w_{j+\ell} * \varphi^{(\varepsilon)}_Q(x)| \leq 2^{-\alpha\ell} \left(1 + \frac{\operatorname{dist}(x,Q)}{s(Q)}\right)^{-n(1+\delta)}. \tag{3.13}$$

Similarly, using that $j + \ell \geq j + k \geq j$, the proof of Lemma 3.1 gives routinely the estimate

$$|v_{j+\ell} * \psi_K(x)| \leq 2^{k-\ell} \left(1 + \frac{\operatorname{dist}(x,K)}{s(K)}\right)^{-4n}. \tag{3.14}$$

Taking into account that $s(K) \leq s(Q)$ we get

$$\int_{\mathbb{R}^n} \left(1 + \frac{\operatorname{dist}(x,Q)}{s(Q)}\right)^{-n(1+\delta)} \left(1 + \frac{\operatorname{dist}(x,K)}{s(K)}\right)^{-4n} dx$$
$$\leq C \, |K| \left(1 + \frac{\operatorname{dist}(K,Q)}{s(Q)}\right)^{-n(1+\delta)}.$$

Combining this with the pointwise estimates (3.13) and (3.14) gives (3.12). □

3.2. Point-wise estimates for $\Delta_{j+\ell}(\mathbb{E}_{i_0} \partial_i \varphi^{(\varepsilon)}_Q)$

We turn to the analysis of the system $k^{(\ell,i)}_Q = \Delta_{j+\ell}(\mathbb{E}_{i_0} \partial_i \varphi^{(\varepsilon)}_Q)$ as defined by (2.9). The cases $\ell \geq 0$ and $\ell \leq 0$ will be treated separately. We begin with the case $\ell \geq 0$.

Lemma 3.4. *Let $\varepsilon \in \mathcal{A}_{i_0}$. The system $\{k_Q^{(\ell,i)} : Q \in \mathcal{S}, i \neq i_0, \ell \geq 0\}$ defined by (2.9) satisfies the structural conditions*

$$|k_Q^{(\ell,i)}(x)| \leq C\, 2^{\ell-\alpha\ell}\Big(1+\frac{\operatorname{dist}(x,Q)}{s(Q)}\Big)^{-n(1+\delta)}, \tag{3.15a}$$

$$|\nabla k_Q^{(\ell,i)}(x)| \leq C\, 2^{j+\ell}2^{\ell-\alpha\ell}\Big(1+\frac{\operatorname{dist}(x,Q)}{s(Q)}\Big)^{-n(1+\delta)}, \tag{3.15b}$$

$$\int_{\mathbb{R}^n} k_Q^{(\ell,i)}(x)dx = 0\,, \tag{3.15c}$$

with $C > 0$ independent of $Q \in \mathcal{S}$, $i \neq i_0$, or $\ell \geq 0$.

Proof. Fix $Q \in \mathcal{S}$, and $x \in \mathbb{R}^n$. Put $e_Q = \mathbb{E}_{i_0}\varphi_Q^{(\varepsilon)}$, then

$$k_Q^{(\ell,i)}(x) = \int_{\mathbb{R}^n} \partial_i d_{j+\ell}(x-t)\, e_Q(t)\, dt\,.$$

By (1.2) for admissible wavelets we get for $x \in \mathbb{R}^n$,

$$|e_Q(x)| \leq C\, s(Q)\Big(1+\frac{\operatorname{dist}(x,Q)}{s(Q)}\Big)^{-n(1+\delta)}, \tag{3.16}$$

and for $t \in \mathbb{R}^n$, with $|x-t| \leq s(Q)$,

$$|e_Q(x)-e_Q(t)| \leq C\, |x-t|^{\alpha} s(Q)^{1-\alpha}\Big(1+\frac{\operatorname{dist}(x,Q)}{s(Q)}\Big)^{-n(1+\delta)}. \tag{3.17}$$

Hence, with (3.1) and (3.17), we obtain (3.15a) and (3.15b) by repeating the proof of Lemma 3.1. It remains to check (3.15c). Since $\Delta_{j+\ell}$ commutes with differentiation, $k_Q^{(\ell,i)} = \partial_i\Delta_{j+\ell}(e_Q)$. Hence the decay of e_Q and $\Delta_{j+\ell}e_Q$ imply

$$\int_{\mathbb{R}^n} k_Q^{(\ell,i)}(x)\, dx = 0\,,$$

that is (3.15c). □

Next we treat the case $\ell \leq 0$.

Lemma 3.5. *The family $\{k_Q^{(\ell,i)} : Q \in \mathcal{S}, i \neq i_0, \ell \leq 0\}$ satisfies the conditions*

$$|k_Q^{(\ell,i)}(x)| \leq C\, 2^{-(n+1)|\ell|}\Big(1+2^{-|\ell|}\frac{\operatorname{dist}(x,Q)}{s(Q)}\Big)^{-n(1+\delta)}, \tag{3.18a}$$

$$|\nabla k_Q^{(\ell,i)}(x)| \leq C\, 2^{j-|\ell|}\, 2^{-(n+1)|\ell|}\Big(1+2^{-|\ell|}\frac{\operatorname{dist}(x,Q)}{s(Q)}\Big)^{-n(1+\delta)}, \tag{3.18b}$$

$$\int_{\mathbb{R}^n} k_Q^{(\ell,i)}(x)\, dx = 0\,, \tag{3.18c}$$

where $C > 0$ is independent of $Q \in \mathcal{S}$, $i \neq i_0$, or $\ell \leq 0$.

Proof. Fix $Q \in \mathcal{S}$, and $x \in \mathbb{R}^n$. Put again $e_Q = \mathbb{E}_{i_0}\varphi_Q^{(\varepsilon)}$ so that

$$k_Q^{(\ell,i)}(x) = \int_{\mathbb{R}^n} e_Q(x-t)\partial_i d_{j+\ell}(t)\,dt\,.$$

By (1.2) we have

$$|e_Q(x)| \le C\,(1+\ \mathrm{dist}(x,Q)/s(Q))^{-n(1+\delta)}\,s(Q) \tag{3.19}$$

By (3.1) we get

$$\|\partial_i d_{j+\ell}\|_1 \le C\,2^{j-|\ell|}, \quad \text{and} \quad \|\partial_i d_{j+\ell}\|_\infty \le C\,2^{n(j-|\ell|)}\,2^{j-|\ell|}. \tag{3.20}$$

We distinguish between the following cases:

1. $\mathrm{dist}(x,Q) \ge 2^{|\ell|}s(Q)$.
2. $\mathrm{dist}(x,Q) \le 2^{|\ell|}s(Q)$.

In the first case select $\mu \in \mathbb{Z}^n$ so that $x \in 2^{|\ell|}\cdot Q + \mu 2^{|\ell|}s(Q)$. Then we have with (3.19) and (3.20)

$$\begin{aligned}\int_{\mathbb{R}^n} e_Q(x-t)\partial_i d_{j+\ell}(t)dt &\le 2^{-|\ell|}|\mu|^{-n(1+\delta)}2^{-n|\ell|(1+\delta)}\\ &\le \left(1+2^{-|\ell|}\ \mathrm{dist}(x,Q)/s(Q)\right)^{-n(1+\delta)}\,2^{-n|\ell|(2+\delta)}.\end{aligned} \tag{3.21}$$

In the second case we have $\mathrm{dist}(x,Q) \le 2^{|\ell|}s(Q)$. Select k_0 so that

$$2^{k_0}s(Q) \le\ \mathrm{dist}(x,Q) \le 2^{k_0+1}s(Q)\,.$$

We may assume that $k_0 \ge 1$. Define the disk

$$A_0 = \{y \in \mathbb{R}^n : |y-x| \le\ \mathrm{dist}(x,Q)\}$$

and the annuli

$$A_k = \{y \in \mathbb{R}^n : 2^{k-1}(\ \mathrm{dist}(x,Q)) \le |y-x| \le 2^k(\ \mathrm{dist}(x,Q))\}\,.$$

Use (3.19) and (3.20) to obtain

$$\begin{aligned}\int_{\mathbb{R}^n} |e_Q\,(x-t)\partial_i d_{j+\ell}(t)|\,dt &\le \sum_{k=0}^{\infty}\int_{A_k\setminus A_{k-1}} |e_Q(x-t)\partial_i d_{j+\ell}(t)|\,dt\\ &\le C\sum_{k=0}^{\infty}|A_k|\left(1+2^k\ \mathrm{dist}(x,Q)/s(Q)\right)^{-n(1+\delta)}s(Q)\|\partial_i d_{j+\ell}\|_\infty\\ &\le C\sum_{k=0}^{\infty}(2^k\,2^{k_0})^{-n\delta}\,2^{-(n+1)|\ell|}.\end{aligned} \tag{3.22}$$

Clearly (3.22) gives (3.18a). The gradient estimates (3.18b) follow from (3.1) and (3.18a). □

4. Review of basic dyadic operations

In this section we prove two auxiliary results on rearrangement operators. The norm estimates for the operators $T_\ell^{(\varepsilon)}$, $T_\ell^{(\varepsilon)} R_{i_0}^{-1}$ will be obtained as applications of Proposition 4.2 and Theorem 4.3.

4.1. The Haar system

We recall the definition of the isotropic Haar system and its equivalence to admissible wavelet systems. We use [7], [15], [17] as sources. Let I be a dyadic interval and h_I be the L^∞ normalized Haar function supported on I. Thus $h_I = 1$ on the left half of I and $h_I = -1$ on the right half of I. Given a dyadic cube $Q = I_1 \times \cdots \times I_n$ and a direction $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathcal{A}$ we define the Haar function

$$h_Q^{(\varepsilon)}(x) = \prod_{i=1}^{n} h_{I_i}^{\varepsilon_i}(x_i), \quad x = (x_1, \dots, x_n).$$

The Haar system $\{h_Q^{(\varepsilon)} : Q \in \mathcal{S}, \varepsilon \in \mathcal{A}\}$ is a complete orthogonal system in $L^2(\mathbb{R}^n)$. Given $f \in L^p(\mathbb{R}^n)$,

$$C_p^{-p} \|f\|_{L^p(\mathbb{R}^n)}^p \le \int \Big(\sum_{\varepsilon\in\mathcal{A},\, Q\in\mathcal{S}} \langle f, h_Q^{(\varepsilon)} \rangle^2 1_Q |Q|^{-2} \Big)^{p/2} \le C_p^p \|f\|_{L^p(\mathbb{R}^n)}^p, \tag{4.1}$$

where $C_p \le Cp^2/(p-1)$. As is well known, wavelets, Calderón–Zygmund operators and Haar functions are related by L^p equivalence. Any admissible wavelet system $\{\varphi_Q^{(\varepsilon)} : Q \in \mathcal{S}, \varepsilon \in \mathcal{A}\}$ is equivalent to the Haar system $\{h_Q^{(\varepsilon)} : Q \in \mathcal{S}, \varepsilon \in \mathcal{A}\}$ in $L^p(\mathbb{R}^n)$. For any choice of finite sums,

$$f = \sum_{\varepsilon\in\mathcal{A},\, Q\in\mathcal{S}} a_Q^{(\varepsilon)} h_Q^{(\varepsilon)} \quad \text{and} \quad g = \sum_{\varepsilon\in\mathcal{A},\, Q\in\mathcal{S}} a_Q^{(\varepsilon)} \varphi_Q^{(\varepsilon)},$$

we have equivalent norms

$$C(p,\alpha,\delta)^{-1} \|f\|_{L^p(\mathbb{R}^n)} \le \|g\|_{L^p(\mathbb{R}^n)} \le C(p,\alpha,\delta) \|f\|_{L^p(\mathbb{R}^n)}. \tag{4.2}$$

where $C(p,\alpha,\delta) \le C(\alpha,\delta)p^2/(p-1)$. See [15], [7], and [17].

4.2. Rearrangements I

We present two quick applications of Semenov's theorem. The aim is to estimate series formed by block bases of compactly supported wavelets, Proposition 4.2.

Semenov's theorem. Let $1 < p < \infty$, and let $\mu \in \mathbb{Z}^n$. Semenov's theorem (see [17] and [16]) asserts that the rearrangement operator defined as the linear extension of

$$T_\mu : h_Q^{(\varepsilon)} \to h_{Q+\mu s(Q)}^{(\varepsilon)}$$

gives rise to a bounded operator on $L^p(\mathbb{R}^n)$ with

$$\|T_\mu\|_p \le C(p)\log(2+|\mu|),$$

where $C(p) \le Cp^2/(p-1)$. For our purposes, the logarithmic dependence on $|\mu|$ is crucial.

Proposition 4.1. *Let $\{\varphi_Q^{(\varepsilon)} : Q \in \mathcal{S}, \varepsilon \in \mathcal{A}\}$ denote the wavelet system defined by (1.2). Let $1 < p < \infty$, let $\mu \in \mathbb{Z}^n$, and $f \in L^p(\mathbb{R}^n)$. Then*

$$\Big\|\Big(\sum_{Q\in\mathcal{S}}\langle f,\varphi_Q^{(\varepsilon)}\rangle^2 1_{Q+\mu s(Q)}|Q|^{-2}\Big)^{1/2}\Big\|_p \le C(p,\alpha,\delta)\log(2+|\mu|)\|f\|_p\,.$$

Proof. By Semenov's theorem and (4.1), we have square function estimates as follows:

$$\Big\|\Big(\sum_{Q\in\mathcal{S}}\langle f,\varphi_Q^{(\varepsilon)}\rangle^2 1_{Q+\mu s(Q)}|Q|^{-2}\Big)^{1/2}\Big\|_p \le C\Big\|\Big(\sum_{Q\in\mathcal{S}}\langle f,\varphi_Q^{(\varepsilon)}\rangle^2 1_{Q}|Q|^{-2}\Big)^{1/2}\Big\|_p, \tag{4.3}$$

where $C = C(p)\log(2+|\mu|)$. By (4.2) the Haar and wavelet systems are equivalent, so that with (4.1), the right hand side of (4.3) is bounded by

$$C(p,\alpha,\delta)\log(2+|\mu|)\|f\|_p\,. \qquad \square$$

Block bases of compactly supported wavelets. We consider again a compactly supported wavelet system $\{\psi_K^{(\beta)} : K \in \mathcal{S}, \beta \in \mathcal{A}\}$ in $L^2(\mathbb{R}^n)$ satisfying the structure conditions (3.3).

Let $\{c_K(Q) : K, Q \in \mathcal{S}\}$ be a sequence of coefficients. Define the following block basis:

$$\widetilde{\psi}_Q = \sum_{K\in\mathcal{S}_{j+k}} c_K(Q)\psi_K^{(\beta)}, \quad Q \in \mathcal{S}_j\,, \quad j \in \mathbb{Z}\,, \quad k \in \mathbb{N}\,, \tag{4.4}$$

and form the operator

$$S_0(f) = \sum_{Q\in\mathcal{S}}\langle f,\varphi_Q\rangle\,\widetilde{\psi}_Q\,|Q|^{-1}.$$

Our aim is to prove that S_0 is bounded on $L^p(\mathbb{R}^n)$ whenever $\{c_K(Q) : K, Q \in \mathcal{S}\}$ satisfies (4.5). To this end we split the block basis along integer translates of Q and estimate with Semenov's theorem. To be precise, let $\mu \in \mathbb{Z}^n$. Then put

$$\mathcal{A}(Q,\mu) = \{K \in \mathcal{S}_{j+k} : K \subseteq Q + \mu s(Q)\} \quad \text{and} \quad \widetilde{\psi}_{Q,\mu} = \sum_{K\in\mathcal{A}(Q,\mu)} c_K(Q)\psi_K^{(\beta)}.$$

Define next

$$S_\mu(f) = \sum_{Q\in\mathcal{S}}\langle f,\varphi_Q\rangle\,\widetilde{\psi}_{Q,\mu}\,|Q|^{-1}.$$

Proposition 4.2. *Let $\delta > 0, j \in \mathbb{Z}, k \in \mathbb{N}$. Assume that the sequence of coefficients $\{c_K(Q) : K, Q \in \mathcal{S}\}$ defining (4.4) satisfies*

$$|c_K(Q)| \le \Big(1 + \frac{\operatorname{dist}(K,Q)}{s(Q)}\Big)^{-n(1+\delta)}, \quad Q \in \mathcal{S}_j\,,\ K \in \mathcal{S}_{j+k}\,. \tag{4.5}$$

Then for any $\mu \in \mathbb{Z}^n$,

$$\|S_\mu\|_p \le C(p,\alpha,\delta)(1+|\mu|)^{-n(1+\delta)}\log(2+|\mu|)\,,$$

and consequently

$$\|S_0\|_p \le C(p,\alpha,\delta)\,. \tag{4.6}$$

Proof. Put

$$\sigma^2(\widetilde{\psi}_{Q,\mu}) := \sum_{K \in \mathcal{A}_{(Q,\mu)}} |c_K(Q)|^2\,|\psi_K^{(\beta)}|^2.$$

By (4.5) we have

$$\sigma^2(\widetilde{\psi}_{Q,\mu}) \le C(1+|\mu|)^{-2n(1+\delta)} \sum_{|\mu-\nu|\le C} 1_{Q+\nu s(Q)}\,. \tag{4.7}$$

By the unconditionality of wavelet-bases,

$$\|S_\mu(f)\|_p \le C(p,\alpha,\delta)\Big\|\Big(\sum_{Q\in\mathcal{S}} \langle f,\varphi_Q\rangle^2 \sigma^2(\widetilde{\psi}_{Q,\mu})|Q|^{-2}\Big)^{1/2}\Big\|_p\,. \tag{4.8}$$

By (4.7) and Proposition 4.1, the right-hand side of (4.8) is bounded by

$$C(p,\alpha,\delta)(1+|\mu|)^{-n(1+\delta)}\log(2+|\mu|)\|f\|_p\,.$$

Since

$$S_0(f) = \sum_{\mu\in\mathbb{Z}^n} S_\mu(f),$$

this gives

$$\|S_0\|_p \le C(p,\alpha,\delta)\sum_{\mu\in\mathbb{Z}^n}\log(2+|\mu|)(1+|\mu|)^{-n(1+\delta)}$$

and (4.6) holds true. □

4.3. Rearrangements II

We review here an auxiliary result on a rearrangement operator S that is induced by mapping a dyadic cube to one of its dyadic predecessors. The operator was introduced and studied in [13]. We define S in (4.10) and record its norm estimates. Let $\lambda \in \mathbb{N}$ and let $Q \in \mathcal{S}$ be a dyadic cube. The λ-th dyadic predecessor of Q, denoted $Q^{(\lambda)}$, is given by the relation

$$Q^{(\lambda)} \in \mathcal{S}, \quad |Q^{(\lambda)}| = 2^{n\lambda}|Q|, \quad Q \subset Q^{(\lambda)}.$$

Let $\tau : \mathcal{S} \to \mathcal{S}$ be the map that associates to each $Q \in \mathcal{S}$ its λ-th dyadic predecessor. Thus

$$\tau(Q) = Q^{(\lambda)}, \quad Q \in \mathcal{S}.$$

Clearly $\tau : \mathcal{S} \to \mathcal{S}$ is not injective. We canonically split $\mathcal{S} = \mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_{2^{n\lambda}}$ such that the restriction of τ to each of the collections $\mathcal{Q}_k$, is injective: given $Q \in \mathcal{S}$, form

$$\mathcal{U}(Q) = \big\{W \in \mathcal{S} : W^{(\lambda)} = Q\big\}.$$

Thus $\mathcal{U}(Q)$ is a covering of Q and contains exactly $2^{n\lambda}$ pairwise disjoint dyadic cubes. We enumerate them, rather arbitrarily, as $W_1(Q), \ldots, W_{2^{n\lambda}}(Q)$. For $1 \le k \le 2^{n\lambda}$, define

$$\mathcal{Q}_k = \{W_k(Q) : Q \in \mathcal{S}\}.$$

Note that $\tau : \mathcal{Q}_k \to \mathcal{S}$ is a bijection, and

$$\tau(W_k(Q)) = Q, \quad W_k(Q) \in \mathcal{Q}_k, \quad Q \in \mathcal{S}.$$

Let $1 \le k \le 2^{n\lambda}$ and let $\{F_Q^{(k)} : Q \in \mathcal{S}\}$ be any family of functions satisfying $\int F_Q^{(k)}(x)\,dx = 0$ and the following structural conditions: there exists $C > 0$, $\delta > 0$ and $0 < \alpha \le 1$ so that, for each $Q \in \mathcal{S}$,

$$|F_Q^{(k)}(x)| \le C\Big(1 + \frac{\operatorname{dist}(x,Q)}{s(Q)}\Big)^{-n(1+\delta)}, \tag{4.9a}$$

and for $|x - t| \le s(Q)$,

$$|F_Q^{(k)}(x) - F_Q^{(k)}(t)| \le C s(Q)^{-\alpha} |x-t|^{\alpha} \Big(1 + \frac{\operatorname{dist}(x,Q)}{s(Q)}\Big)^{-n(1+\delta)}. \tag{4.9b}$$

We emphasize that $F_Q^{(k)}$ may depend on k, by contrast the structural conditions (4.9) are independent of the value of k. Define the operator S by the equation

$$S(g) = \sum_{k=1}^{2^{n\lambda}} \sum_{Q \in \mathcal{Q}_k} \big\langle g, F_{\tau(Q)}^{(k)} \big\rangle \varphi_Q\, |Q|^{-1}, \tag{4.10}$$

where $\{\varphi_Q\}$ is an admissible wavelet system satisfying (1.2). The operator S is the transposition of the rearrangement operator defined by τ followed by a Calderón–Zygmund integral. The next theorem records the operator norm of S, particularly its joint (n,λ)-dependence, on $L^p(\mathbb{R}^n)$.

Theorem 4.3. *Let $1 < p < \infty$. The operator S defined by (4.10) is bounded on $L^p(\mathbb{R}^n)$. The norm estimates depend on the value of $\lambda \in \mathbb{N}$ and the dimension of the ambient space $\mathbb{R}^n$ as follows:*

$$\|S\|_p \le C(p,\alpha,\delta)\, \lambda^{1/2}\, 2^{n\lambda}. \tag{4.11}$$

Proof. Just transfer Theorem 5.2 in [13] from compactly supported wavelets to those satisfying (4.9). □

5. Proof of Theorem 2.3

In this section we prove Theorem 2.3. Section 5.1 is devoted to the estimates (2.11) for the operator $T_\ell^{(\varepsilon)}$, $\ell \geq 0$. Thereafter we discuss the reduction of the estimates (2.12) for $T_\ell^{(\varepsilon)} R_{i_0}^{-1}$, $\varepsilon \in \mathcal{A}_{i_0}$, to those of $T_\ell^{(\varepsilon)}$.

5.1. Estimates for $T_\ell^{(\varepsilon)}$

We prove here (2.11) asserting that $T_\ell^{(\varepsilon)}$, $\ell \geq 0$ satisfies the norm estimates

$$\|T_\ell^{(\varepsilon)}\|_p \leq C(p,\alpha,\delta)\, 2^{-\ell\alpha}. \tag{5.1}$$

We do this by performing a further decomposition of the operator $T_\ell^{(\varepsilon)}$.

The decomposition of $T_\ell^{(\varepsilon)}$, $\ell \geq 0$. We decompose the operator $T_\ell^{(\varepsilon)}$, $\ell \geq 0$ into a series of operators $T_{\ell,m}, m \in \mathbb{Z}$ using a compactly supported wavelet system $\{\psi_K^{(\beta)} : K \in \mathcal{S}, \beta \in \mathcal{A}\}$. We assume that $\{\psi_K^{(\beta)}/\sqrt{|K|}\}$ is an orthonormal basis in $L^2(\mathbb{R}^n)$, satisfying $\int \psi_K^{(\beta)} = 0$ and the structure conditions

$$\operatorname{supp} \psi_K^{(\beta)} \subseteq C \cdot K\,, \quad |\psi_K^{(\beta)}| \leq C\,, \quad \operatorname{Lip}(\psi_K^{(\beta)}) \leq C \operatorname{diam}(K)^{-1}.$$

We suppress the superindices (β) and, in place of $\{\psi_K^{(\beta)}\}$, we write just $\{\psi_K\}$.

Fix $m \in \mathbb{Z}$, $j \in \mathbb{Z}$, and $Q \in \mathcal{S}_j$. Put

$$\widetilde{\psi}_Q = \sum_{K \in \mathcal{S}_{j+\ell+m}} \big\langle \Delta_{j+\ell}(\varphi_Q^{(\varepsilon)}), \psi_K \big\rangle\, \psi_K\, |K|^{-1} \tag{5.2}$$

and

$$T_{\ell,m}(f) = \sum_{Q \in \mathcal{S}} \langle f, \widetilde{\psi}_Q\rangle\, \varphi_Q^{(\varepsilon)}\, |Q|^{-1}. \tag{5.3}$$

By construction,

$$T_\ell = \sum_{m=-\infty}^{\infty} T_{\ell,m}\,.$$

For fixed $\ell \geq 1$ we consider below three cases:

$$-\infty < m \leq -\ell - 1\,, \quad -\ell \leq m \leq 0\,, \quad \text{and} \quad m \geq 0\,.$$

We prove accordingly that

$$\sum_{m=-\infty}^{-\ell-1} ||T_{\ell,m}||_p \leq C(p,\alpha,\delta)\, 2^{-\ell\alpha}\,, \quad \sum_{m=-\ell}^{0} \|T_{\ell,m}\|_p \leq C(p,\alpha,\delta)\, 2^{-\ell\alpha} \tag{5.4}$$

and

$$\sum_{m=0}^{\infty} \|T_{\ell,m}\|_p \le C(p,\alpha,\delta)\, 2^{-\ell\alpha}. \tag{5.5}$$

The estimates (5.4) and (5.5) yield $\|T_\ell\|_p \le C(p,\alpha,\delta)2^{-\ell\alpha}$ as claimed.

Proposition 5.1. *Let $1 < p < \infty$. Let $\delta > 0$ and $\alpha > 0$ be fixed in the definition of the admissible wavelet system. Put $\gamma_0 = \min\{n\delta/2, 1\}$. For $\ell \ge 0$, and $m < -\ell$ the operator $T_{\ell,m}$ satisfies the norm estimate*

$$\|T_{\ell,m}\|_p \le C(p,\alpha,\delta)\, 2^{-\ell\alpha}\, 2^{-|m+\ell|\gamma_0} \sqrt{|m+\ell|}\,. \tag{5.6}$$

Proof. Let $j \in \mathbb{Z}$ and fix a dyadic cube $Q \in \mathcal{S}_j$. Since $\ell + m < 0$ there exists a unique cube $K_0 \in \mathcal{S}_{j+\ell+m}$, so that $Q \subseteq K_0$. Lemma 3.2 gives the pointwise estimates

$$|\widetilde{\psi}_Q(t)| \le C\, 2^{-\alpha\ell}\, 2^{-|m+\ell|(n+\gamma_o)} \left(1 + \frac{\operatorname{dist}(t,K_0)}{s(K_0)}\right)^{-n(1+\delta)}, \tag{5.7a}$$

and

$$|\nabla\widetilde{\psi}_Q(t)| \le C\ \operatorname{diam}(K_0)^{-1}\, 2^{-\alpha\ell}\, 2^{-|m+\ell|(n+\gamma_o)} \left(1 + \frac{\operatorname{dist}(t,K_0)}{s(K_0)}\right)^{-n(1+\delta)}. \tag{5.7b}$$

We next invoke rearrangement operators. Let $\tau\colon \mathcal{S} \to \mathcal{S}$ be the map that associates to $Q \in \mathcal{S}$ its $|m+\ell|$-th dyadic predecessor, denoted $Q^{|m+\ell|}$. Thus

$$\tau(Q) = Q^{|m+\ell|}.$$

In section 4.3 we defined the canonical splitting of $\mathcal{S}$ as

$$\mathcal{S} = \mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_{2^{n|m+\ell|}},$$

so that for each fixed $k \le 2^{n|m+\ell|}$, the map $\tau\colon \mathcal{Q}_k \to \mathcal{S}$ is a bijection. Fix now $k \le 2^{n|m+\ell|}$ and define the family of functions $\{F_W^{(k)} : W \in \mathcal{S}\}$ by the equations

$$F^{(k)}_{\tau(Q)} = 2^{\alpha\ell}\, 2^{|m+\ell|(n+\gamma_0)}\, \widetilde{\psi}_Q, \quad Q \in \mathcal{Q}_k. \tag{5.8}$$

Let $A = 2^{n|m+\ell|}$ and define the rearrangement operator S by

$$S(f) = \sum_{k=1}^{A} \sum_{Q\in\mathcal{Q}_k} \langle f, F^{(k)}_{\tau(Q)}\rangle\, \varphi_Q\, |Q|^{-1}.$$

By (5.7), $\{F_W^{(k)} : W \in \mathcal{S}\}$ satisfies the structure estimates (4.9). Apply Theorem 4.3 with $\lambda = |m+\ell|$, to obtain

$$\|S\|_p \le C\, 2^{n|m+\ell|} \sqrt{|m+\ell|}\,,$$

where $C = C(p, \alpha, \delta)$. Hence with (5.8) we get

$$\begin{aligned} \|T_{\ell,m}(f)\|_p &\leq C\, 2^{-\alpha\ell - |m+\ell|(n+\gamma_0)}\, \|S(f)\|_p \\ &\leq C\, 2^{-\alpha\ell - |m+\ell|\gamma_0} \sqrt{|m+\ell|}\|f\|_p\,, \end{aligned} \tag{5.9}$$

where again $C = C(p, \alpha, \delta)$. □

We treat next the case $m \geq 0$ and $\ell \geq 0$. Here we estimate the transposed operator of $T_{\ell,m}$ which is given by

$$T^*_{\ell,m}(f) = \sum_{Q \in \mathcal{S}} \langle f, \varphi_Q^{(\varepsilon)} \rangle\, \widetilde{\psi}_Q\, |Q|^{-1}. \tag{5.10}$$

Proposition 5.2. *Let $1 < p < \infty$. For $m \geq 0$ and $\ell \geq 0$, we have*

$$\|T_{\ell,m}\|_p \leq C(p, \alpha, \delta)\, 2^{-m}\, 2^{-\ell\alpha}. \tag{5.11}$$

Proof. Fix $\ell \geq 0$ and $m \geq 0$. Let $j \in \mathbb{Z}$ and choose a dyadic cube $Q \in \mathcal{S}_j$. The structure estimates for $\Delta_{j+\ell}(\varphi_Q^{(\varepsilon)})$ in Lemma 3.1 translate into coefficient estimates as follows. If $K \in \mathcal{S}_{j+\ell+m}$, then

$$|\langle \Delta_{j+\ell}(\varphi_Q^{(\varepsilon)}), \psi_K \rangle| \cdot |K|^{-1} \leq C\, 2^{-m}\, 2^{-\alpha\ell} \left(1 + \frac{\operatorname{dist}(K, Q)}{s(Q)}\right)^{-n(1+\delta)}. \tag{5.12}$$

Using (5.12) and applying Proposition 4.2 to $T^*_{\ell,m}$ gives the norm estimate

$$\|T_{\ell,m}\|_p \leq C(p, \alpha, \delta)\, 2^{-m}\, 2^{-\ell\alpha}.$$

□

Next consider $\ell \geq 0$, $-\ell \leq m \leq 0$. The ingredients of the previous proof are applied again.

Proposition 5.3. *Let $1 < p < \infty$. Let $\ell \geq 0$ and $-\ell \leq m \leq 0$. Then,*

$$\|T_{\ell,m}\|_p \leq C(p, \alpha, \delta)\, 2^{m}\, 2^{-\ell\alpha}. \tag{5.13}$$

Proof. We estimate the transposed operator $T^*_{\ell,m}$ given by (5.10). Fix $\ell \geq 0$ and $-\ell \leq m \leq 0$. Let $j \in \mathbb{Z}$ and choose dyadic cubes $Q \in \mathcal{S}_j$ and $K \in \mathcal{S}_{j+\ell+m}$. Next we apply Lemma 3.3 and get

$$|\langle \Delta_{j+\ell}(\varphi_Q^{(\varepsilon)}), \psi_K \rangle| \cdot |K|^{-1} \leq C\, 2^{m}\, 2^{-\alpha\ell} \left(1 + \frac{\operatorname{dist}(K, Q)}{s(Q)}\right)^{-n(1+\delta)}.$$

Applying Proposition 4.2 to $T^*_{\ell,m}$ gives $\|T^*_{\ell,m}\|_p \leq C(p, \alpha, \delta)\, 2^{m}\, 2^{-\ell\alpha}$. □

The proof of Theorem 2.3. Part 1. The estimate (2.11) is now obtained as follows. The assertions of Proposition 5.1, Proposition 5.2 and Proposition 5.3 imply

$$\sum_{m=-\infty}^{\infty} ||T_{\ell,m}||_p \le C(p,\alpha,\delta)\, 2^{-\ell\alpha}, \tag{5.14}$$

Since

$$T_\ell^{(\varepsilon)}(f) = \sum_{m=-\infty}^{\infty} T_{\ell,m}(f)\,, \tag{5.15}$$

we get (2.11).

5.2. Estimates for $T_\ell^{(\varepsilon)} R_{i_0}^{-1}$

Here we prove (2.12). We fix $\ell \ge 0$, $1 \le i_0 \le n$, and $\varepsilon \in \mathcal{A}_{i_0}$. We now prove the norm estimates for $T_\ell^{(\varepsilon)} R_{i_0}^{-1}$ *by reduction* to the estimates for the operator $T_\ell^{(\varepsilon)}$

Let $j \in \mathbb{Z}$ and $Q \in \mathcal{S}_j$. Recall that we put

$$k_Q^{(\ell,i)} = \Delta_{j+\ell}\left(\mathbb{E}_{i_0}\partial_i \varphi_Q^{(\varepsilon)}\right).$$

Proposition 5.4. *Let* $1 < p < \infty$. *Let* $1 \le i \ne i_0 \le n$ *and* $\varepsilon \in \mathcal{A}_{i_0}$. *For* $\ell \ge 0$ *the operator* X *defined by*

$$X(f) = \sum_{Q\in\mathcal{S}} \langle f, k_Q^{(\ell,i)}\rangle\, \varphi_Q^{(\varepsilon)}\, |Q|^{-1},$$

satisfies the norm estimates

$$||X||_p \le C(p,\alpha,\delta)\, 2^{+\ell-\alpha\ell}. \tag{5.16}$$

Proof. It remains to compare the structure conditions (3.15) for the system $k_Q^{(\ell,i)}$ defining X with those for $f_Q^{(\ell,i)}$ defining the operator $T_\ell^{(\varepsilon)}$. This gives

$$||X||_p \le 2^\ell\, \|T_\ell^{(\varepsilon)}\|_p\,,$$

which implies (5.16). □

The proof of Theorem 2.3. Part 2. The estimate (2.12) is obtained as follows. Proposition 5.4, in combination with the norm estimate (2.11) and the representation (2.10), imply that for $\ell > 0$,

$$||T_\ell^{(\varepsilon)} R_{i_0}^{-1}||_p \le C(p,\alpha,\delta)\, 2^{+\ell-\alpha\ell}.$$

6. Proof of Theorem 2.4

In this section we prove Theorem 2.4. For $\ell \le 0$ we obtain the norm estimates for $T_\ell^{(\varepsilon)} R_{i_0}^{-1}$ and $T_\ell^{(\varepsilon)}$ by the same method. Let $i \ne i_0$ and $\varepsilon \in \mathcal{A}_{i_0}$ and let $\ell \le 0$. We show that then

$$||T_\ell^{(\varepsilon)}||_p + ||T_\ell^{(\varepsilon)} R_{i_0}^{-1}||_p \le C(p, \alpha, \delta)\, 2^{-|\ell|}\, |\ell|.$$

We use the representations (2.8) and (2.10), and recall that we put

$$f_{Q,\ell}^{(\varepsilon)} = \Delta_{j+\ell}(\varphi_Q^{(\varepsilon)})\,, \quad \text{and} \quad k_Q^{(\ell,i)} = \Delta_{j+\ell}(\mathbb{E}_{i_0}\partial_i \varphi_Q^{(\varepsilon)})\,, \quad Q \in \mathcal{S}_j\,.$$

We showed in Lemma 3.5 that $\{k_Q^{(\ell,i)} : Q \in \mathcal{S}, \ell \le 0\}$ satisfies conditions (3.18). It is easy to see that also the family $\{f_{Q,\ell}^{(\varepsilon)} : Q \in \mathcal{S}, \ell \le 0\}$ satisfies the structural conditions (3.18).

Now we choose $\{g_{Q,\ell} : Q \in \mathcal{S}\}$ satisfying the structure conditions (3.18). Define the operator

$$X(f) = \sum_{Q\in\mathcal{S}} \langle u, g_{Q,\ell}\rangle\, \varphi_Q^{(\varepsilon)}\, |Q|^{-1}.$$

In view of the preceding discussion, the L^p estimates for X will apply to both $T_\ell^{(\varepsilon)}$ and $T_\ell^{(\varepsilon)} R_{i_0}^{-1}$.

To estimate X, we consider again the rearrangement $\tau\colon \mathcal{S} \to \mathcal{S}$ that maps $Q \in \mathcal{S}$ to its $|\ell|$-th dyadic predecessor. Let $\mathcal{Q}_1, \dots, \mathcal{Q}_{2^{n|\ell|}}$ be the canonical splitting of $\mathcal{S}$ so that for fixed $k \le 2^{n|\ell|}$ the map $\tau\colon \mathcal{Q}_k \to \mathcal{S}$ is bijective. Fix $k \le 2^{n|\ell|}$. Determine the family $\{F_W^{(k)} : W \in \mathcal{S}\}$ by the equations

$$F_{\tau(Q)}^{(k)} = 2^{(n+1)|\ell|} g_{Q,\ell}\,, \quad Q \in \mathcal{Q}_k\,. \tag{6.1}$$

Define the operator

$$S(u) = \sum_{k=1}^{2^{n|\ell|}} \sum_{Q\in\mathcal{Q}_k} \langle u, F_{\tau(Q)}^{(k)}\rangle\, \varphi_Q^{(\varepsilon)}\, |Q|^{-1}\,.$$

Apply Theorem 4.3 to S with $\lambda = |\ell|$. This yields

$$\|S\|_p \le C(p, \alpha, \delta)\, 2^{n|\ell|}\, |\ell|\,. \tag{6.2}$$

Comparing the structure conditions gives

$$\|X\|_p \le C(p, \alpha, \delta)\, 2^{-(n+1)|\ell|}\, \|S\|_p\,. \tag{6.3}$$

Consequently, our upper bounds for $\|X\|_p$ follow from (6.2). Indeed,

$$\|X\|_p \le C(p, \alpha, \delta)\, 2^{-|\ell|}\, |\ell|.$$

The above estimate for X gives finally

$$||T_\ell^{(\varepsilon)}||_p + ||T_\ell^{(\varepsilon)} R_{i_0}^{-1}||_p \le C(p, \alpha, \delta)\, 2^{-|\ell|}\, |\ell|\,. \qquad \square$$

References

[1] Ball, J. M. and Murat, F.: Remarks on rank-one convexity and quasiconvexity. In *Ordinary and partial differential equations, Vol. III (Dundee, 1990),* 25–37. Pitman Res. Notes Math. Ser. 254, Longman Sci. Tech., Harlow, 1991.

[2] Christ, M.: Characterization of H^1 by singular integrals: necessary conditions. *Duke Math. J.* **51** (1984), no. 3, 599–609.

[3] Christ, M.: On the regularity of inverses of singular integral operators. *Duke Math. J.* **57** (1988), no. 2, 459–484.

[4] Christ, M. and Geller, D.: Singular integral characterizations of Hardy spaces on homogeneous groups. *Duke Math. J.* **51** (1984), no. 3, 547–598.

[5] Ciesielski, Z.: Haar orthogonal functions in analysis and probability. In *A. Haar memorial conference, Vol. I, II (Budapest, 1985),* 25–56. Colloquia Mathematica Societatis János Bolyai 49, North-Holland, Amsterdam, 1987.

[6] Dacorogna, B.: *Weak continuity and weak lower semicontinuity of nonlinear functionals.* Lecture Notes in Mathematics 922, Springer-Verlag, Berlin, 1982.

[7] Figiel, T.: Singular integral operators: a martingale approach. In *Geometry of Banach spaces (Strobl, 1989),* 95–110. London Math. Soc. Lecture Note Ser. 158, Cambridge Univ. Press, Cambridge, 1990.

[8] Folland, G. B. and Stein, E. M.: *Hardy spaces on homogeneous groups.* Mathematical Notes 28, Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1982.

[9] Frazier, M. and Jawerth, B.: Decomposition of Besov spaces. *Indiana Univ. Math. J.* **34** (1985), no. 4, 777–799.

[10] Gripenberg, G.: Wavelet bases in $L^p(\mathbb{R})$. *Studia Math.* **106** (1993), no. 2, 175–187.

[11] Kato, T.: On a coerciveness theorem by Schulenberger and Wilcox. *Indiana Univ. Math. J.* **24** (1974/75), 979–985.

[12] Lechner, R.: An interpolatory estimate for the UMD-valued directional Haar projection. *Dissertationes Math. (Rozprawy Mat.)* **503** (2014), 1–60.

[13] Lee, J., Müller, P. F. X. and Müller, S.: Compensated compactness, separately convex functions and interpolatory estimates between Riesz transforms and Haar projections. *Comm. Partial Differential Equations* **36** (2011), no. 4, 547–601.

[14] Lemarié, P. G.: Base d'ondelettes sur les groupes de Lie stratifiés. *Bull. Soc. Math. France* **117** (1989), no. 2, 211–232.

[15] Meyer, Y. and Coifman, R.: *Wavelets.* Cambridge Studies in Advanced Mathematics 48, Cambridge University Press, Cambridge, 1997.

[16] Müller, P. F. X.: Rearrangements of the Haar system that preserve BMO. *Proc. London Math. Soc. (3)* **75** (1997), no. 3, 600–618.

[17] Müller, P. F. X.: *Isomorphisms between H^1 spaces.* Mathematics Institute of the Polish Academy of Sciences, Mathematical Monographs (New Series) 66, Birkhäuser Verlag, Basel, 2005.

[18] Müller, S.: Rank-one convexity implies quasiconvexity on diagonal matrices. *Internat. Math. Res. Notices* **1999**, no. 20, 1087–1095.

[19] Müller, S.: Quasiconvexity is not invariant under transposition. *Proc. Roy. Soc. Edinburgh Sect. A* **130** (2000), no. 2, 389–395.

[20] Murat, F.: Compacité par compensation: condition nécessaire et suffisante de continuité faible sous une hypothèse de rang constant. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **8** (1981), no. 1, 69–102.

[21] Tartar L.: Some remarks on separately convex functions. In *Microstructure and phase transition*, 191–204. IMA Vol. Math. Appl. 54, Springer, New York, 1993.

[22] Wojtaszczyk, P.: Wavelets as unconditional bases in $L_p(\mathbb{R})$. *J. Fourier Anal. Appl.* **5** (1999), no. 1, 73–85.

Received September 8, 2014.

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P. F .X. Müller is supported – in part – by the Austrian Science foundation (FWF), Projects P23987 and P22549, and by the Hausdorff Research Institute for Mathematics, Bonn. S. Müller is supported by the Hausdorff Center for Mathematics and by the SFB 1060 "The mathematics of emergent effects".