Rev. Mat. Iberoam. **32** (2016), no. 4, 1163–1188 © European Mathematical Society
DOI 10.4171/RMI/911

Well-posedness and scattering for nonlinear Schrödinger equations on $\mathbb{R}^d \times \mathbb{T}$ in the energy space

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Abstract. We study the Cauchy problem and the large data H^1 scattering for energy subcritical NLS posed on $\mathbb{R}^d \times \mathbb{T}$.

1. Introduction

In our previous work [19], we considered the nonlinear Schrödinger equation on a product space $\mathbb{R}^d \times M^k$, where M^k is a k-dimensional compact Riemannian manifold. We have seen this problem as a kind of vector valued nonlinear Schrödinger equation on $\mathbb{R}^d$ and we were able to get small data scattering results (cf. also [12] for small data modified scattering results).

Our goal here is to extend this view point to a large data problem in the very particular case when M is the one dimensional torus.

Therefore, our aim in this paper is the study of the local (and global) well-posedness and scattering of the following family of Cauchy problems:

$$(1.1)\qquad \begin{cases} i\partial_t u - \Delta_{x,y} u + u|u|^\alpha = 0, & (t,x,y) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{T}, \quad d \geq 1, \\ u(0,x,y) = f(x,y) \in H^1_{x,y}, \end{cases}$$

where

$$\Delta_{x,y} = \sum_{i=1}^d \partial_{x_i}^2 + \partial_y^2.$$

Concerning the Cauchy theory we shall assume $0 < \alpha < 4/(d-1)$, and for scattering we assume $4/d < \alpha < 4/(d-1)$.

Our first result deals with the Cauchy problem.

Mathematics Subject Classification (2010): 35Q55, 58J37.
Keywords: NLS, scattering, Strichartz estimates, Morawetz estimates.

Theorem 1.1. *Let $d \geq 1$ and $0 < \alpha < 4/(d-1)$ be fixed. Then we have:*

1. *for any initial datum $f \in H^1_{x,y}$, the problem (1.1) has a unique local solution*
$$u(t,x,y) \in \mathcal{C}((-T,T); H^1_{x,y}),$$
where $T = T(\|f\|_{H^1_{x,y}}) > 0$;
2. *the solution $u(t,x,y)$ can be extended globally in time.*

Remark 1.2. Property (2) follows by (1) due to the defocusing character of the nonlinearity (a standard approximation argument is needed to justify the energy conservation). Hence, along the paper, we focus mainly on the proof of (1), i.e., the existence of a unique local solution for any given initial datum. We also notice that the proof of (1) in Theorem 1.1 works also for the focusing NLS.

The proof of the local existence given by Theorem 1.1 goes as follows. First we prove the existence of one unique solution in the space

$$L^q_t L^r_x H^{1/2+}_y \cap \mathcal{C}_t(H^1_{x,y}), \tag{1.2}$$

where (q,r) are Strichartz $\dot{H}^{1/2-}$-admissible for the propagator $e^{it\Delta_x}$. It is of importance for our analysis that a $H^{1/2}$ sub-critical nonlinearity in dimension d is a H^1 sub-critical nonlinearity in dimension $d+1$. Therefore at the x level we perform a $H^{1/2-}$ analysis and at the y level, we perform the (trivial) $H^{1/2+}$ analysis which at the end enables us to perform a H^1 theory in the full sub-critical range of the nonlinearity. Incorporating in a non-trivial way the y dispersive effect in this analysis is a challenging problem. Its solution may allow to extend our analysis to higher dimensional y dependence. A key tool in order to perform a fixed point argument in the space (1.2) are the inhomogenous Strichartz estimates associated with $e^{it\Delta_x}$ (see [5], [8], [20]). The second step is the proof of the unconditional uniqueness in the space $\mathcal{C}_t(H^1_{x,y})$. We underline that the proof of Theorem 1.1 in the range of nonlinearity $0 < \alpha < 4/d$, can be obtained following [18], where it is not needed the use of inhomogeneous Strichartz estimates for $e^{it\Delta_x}$.

In the cases $d = 1$ for every $\alpha > 0$ and $d = 2, 3$ for the H^1-critical nonlinearity $\alpha = 4/(d-1)$, the proof of Theorem 1.1 can also be deduced respectively from the analysis in [2] and [14]. The main point in our approach is that it works in $\mathbb{R}^d \times \mathbb{T}$ for every $d \geq 1$ and moreover it gives some crucial controls of space-time global norms which are of importance for the scattering analysis.

The main result of this paper concerns the long-time behavior of the solutions given by Theorem 1.1.

Theorem 1.3. *Assume $d \geq 1$ and $4/d < \alpha < 4/(d-1)$, $f(x,y) \in H^1_{x,y}$ and let $u(t,x,y) \in \mathcal{C}(\mathbb{R}; H^1_{x,y})$ be the unique global solution to (1.1). Then there exist $f_\pm \in H^1_{x,y}$ such that*

$$\lim_{t\to\pm\infty} \|u(t,x,y) - e^{-it\Delta_{x,y}} f_\pm\|_{H^1_{x,y}} = 0.$$

Remark 1.4. Concerning scattering results for NLS in product spaces we quote [11], where it is studied the quintic NLS on $\mathbb{R} \times \mathbb{T}^2$. We also underline that using the arguments of [14] (see also [11]) one may obtain that for $d = 2, 3$, the result of Theorem 1.3 also holds for the H^1 critical nonlinearity $\alpha = 4/(d-1)$. One may also expect that these arguments provide an alternative (and more complicated) proof of Theorem 1.3 for $d = 2, 3$. For $d \geq 4$, the extension of Theorem 1.3 to the H^1 critical nonlinearity $\alpha = 4/(d-1)$ is an open problem (even for the H^1 local theory).

Remark 1.5. Notice that if one considers (1.1) on $\mathbb{R}^d \times \mathbb{R}$, then it is well-known that H^1-scattering is available for $4/(d+1) < \alpha < 4/(d-1)$ (in contrast with Theorem 1.3 where we require the extra restriction $\alpha > 4/d$). On the other hand the restriction $\alpha > 4/d$ in Theorem 1.3 is quite natural. Indeed, if we choose $f(x, y) = f(x)$ and $0 < \alpha < 4/d$ then the Cauchy problem (1.1) reduces to L^2-subcritical NLS in $\mathbb{R}^d$, and at the best of our knowledge no H^1-scattering result is available in this situation.

It is well known, since the very classical work [9], that a key tool to prove scattering for NLS in the euclidean setting $\mathbb{R}^d$, with nonlinearities which are both energy subcritical and L^2-supercritical, is the proof of the time-decay of the potential energy. In Proposition 1.6 below we prove that this property persists for solutions to NLS on $\mathbb{R}^d \times \mathbb{T}$ in the energy subcritical regime (in particular we do not need to require to the nonlinearity to be L^2-supercritical, see also Remark 1.7 on this point).

A basic tool that we will use is a suitable version in the partially periodic setting of the interaction Morawetz estimates, first introduced in [7] to study the energy critical NLS in the euclidean space $\mathbb{R}^3$. Starting from this work the interaction Morawetz estimates have been exploited in several other papers ([6], [10], [16], [17], [21]), in particular they have been used to provide new and simpler proofs of the classical scattering results from [9] and [15].

We emphasize that we make use of the interaction Morawetz estimates from a different point of view compared with the results above. In particular along the proof of Proposition 1.6 below we are able to treat in a unified and simple way NLS on $\mathbb{R}^d \times \mathbb{T}$ for every $d \geq 1$, without any distinction between the cases $d \leq 3$ and $d > 3$. This distinction is typical in previous papers involving interaction Morawetz estimates in the Euclidean setting (see Remark 1.8 for more details on this point). Moreover it is unclear to us how to proceed, following the approach developed in previous papers related with interaction Morawetz estimates, to prove Proposition 1.6 in the case $d \geq 4$ (see Remark 1.9).

Next we state the key proposition needed to prove Theorem 1.3.

Proposition 1.6. *Let $u(t, x, y) \in \mathcal{C}(\mathbb{R}; H^1_{x,y})$ be a global solution to defocusing NLS posed on $\mathbb{R}^d \times \mathbb{T}$ and with pure power nonlinearity $u|u|^\alpha$, with $0 < \alpha < 4/(d-1)$. Then*

$$\lim_{t \to \pm\infty} \|u(t, x, y)\|_{L^q_{x,y}} = 0, \quad 2 < q < \frac{2(d+1)}{d-1}. \tag{1.3}$$

Remark 1.7. Notice that in contrast with Theorem 1.3, in Proposition 1.6 we do not assume any lower bound on α. Notice also that Proposition 1.6 is not true for the focusing NLS on $\mathbb{R}^d \times \mathbb{T}$ for $\alpha < 4/d$, even if the initial data are assumed to be arbitrarily small in $H^1_{x,y}$. To prove this fact one can think about the solitary waves associated with the L^2 subcritical focusing NLS posed on $\mathbb{R}^d$, and notice that the corresponding $H^1_{x,y}$ norm can be arbitrary small.

Remark 1.8. As already mentioned above, the proof of Proposition 1.6 is based on the use of interaction Morawetz estimates in the partially periodic setting. Let us recall that the interaction Morawetz estimates allow to control the following quantity (see for instance [10]):

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} \left||D_x|^{(3-d)/2}(|u|^2)\right|^2 dx\,dt < \infty \tag{1.4}$$

for u solution to NLS posed in the Euclidean space $\mathbb{R}^d$. Notice that via the Sobolev embedding it implies some a priori bounds of the type

$$\|u(t,x,y)\|_{L^p_t L^q_x} < \infty$$

in the case $d = 1, 2, 3$. This estimate is sufficient to deduce scattering on $\mathbb{R}^d$ for $d = 1, 2, 3$ in the case of the nonlinearity $4/d < \alpha < 4/(d-2)$. Notice also that in higher dimensions $d \geq 4$ we get in (1.4) the control of a negative derivative of $|u|^2$. In this case some extra work is needed in order to retrieve the needed space-time summability that allows to get scattering. Typically the main strategy to overcome this difficulty is to retrieve some information on negative derivative of u via the following estimate (see [17]):

$$\||D_x|^{(3-d)/4} f\|^2_{L^4_x} \leq C \|\,|D_x|^{(3-d)/2}|f|^2\|_{L^2_x}. \tag{1.5}$$

Once a negative derivative of u is estimated, then it can be interpolated with the bound $\|u\|_{L^\infty_t H^1_x}$, and hence we get the needed space-time integrality necessary to prove scattering for $d \geq 4$.

Remark 1.9. We underline that arguing as in [1], where it is studied NLS with a partially confining potential, one can prove the following version of interaction Morawetz estimate:

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} \left| |D_x|^{(3-d)/2}\Big(\int_{\mathbb{T}} |u(t,x,y)|^2 dy\Big)\right|^2 dx\,dt < \infty \tag{1.6}$$

provided that $u(t,x,y)$ solves NLS posed on $\mathbb{R}^d \times \mathbb{T}$. Hence via the Sobolev embedding one can deduce some a priori bounds

$$\|u(t,x,y)\|_{L^p_t L^q_x L^2_y} < \infty$$

in the case $d = 1, 2, 3$. This estimate is sufficient to deduce scattering for $4/d < \alpha < 4/(d-1)$ and $d = 1, 2, 3$ (see the computations in [1] in the case of a partially confining potential). However, as far as we can see, it is unclear how to exploit (1.6) in the case $d \geq 4$.

Remark 1.10. Estimate (1.6) is obtained by controlling a suitable family of multiple integrals of the type:

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{T}} \int_{\mathbb{T}} \cdots \, dx_1 \, dx_2 \, dy_1 \, dy_2 \, dt \,,$$

where the integrand function depends on a test function φ and on the solution u to NLS. Once this test function is suitably chosen then it allows to contract the variables x_1, x_2, y_1, y_2 to x, y, hence we get (1.6). The main point in our analysis is that we combine an argument by the absurd in conjunction with the finiteness of the following quantity

$$\int_{\mathbb{R}} \Big(\sup_{x_0 \in \mathbb{R}^d} \iint_{Q^d(x_0,r)\times(0,2\pi)} |u(t,x,y)|^2 \, dx \, dy \Big)^{(\alpha+4)/2} dt < \infty \tag{1.7}$$

that in turn follows by

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{T}} \int_{\mathbb{T}} \Delta_x \varphi(x_1 - x_2) |u(x_1, y_1)|^{\alpha+2} |u(x_2, y_2)|^2 \, dt dx_1 dy_1 dx_2 dy_2 < \infty, \tag{1.8}$$

where φ is any convex function. In this estimate we choose $\varphi = \langle x \rangle$. Notice that this choice does not allow contraction of the variables (x_1, x_2, y_1, y_2) in (x, y); however it implies (1.7), which is sufficient to conclude the time decay of the potential energy for solutions to NLS in a simpler way and in a more general setting compared with (1.6). We believe that this part of our argument is of independent interest.

Acknowledgement. The authors are grateful to the referee for interesting remarks and suggestions.

2. Some useful functional inequalities

In this section we collect some a-priori estimates for the propagator $e^{-it\Delta_{x,y}}$ and the associated Duhamel operator. At the end we also present an anisotropic Gagliardo–Nirenberg inequality that will be useful in the sequel.

We define $H_x^s H_y^\gamma$ as

$$H_x^s H_y^\gamma = (1 - \Delta_x)^{-s/2} (1 - \partial_y^2)^{-\gamma/2} \, L^2_{x,y},$$

endowed with the natural norm.

In the sequel we shall make extensively use of the argument introduced in [19], that we recall shortly. In [19] it is obtained a suitable version of Strichartz estimates for the linear Schrödinger propagator on the product space $\mathbb{R}_x^d \times M_y^k$. The smoothing is measured in the spaces $L_t^p L_x^q L_y^2$. The basic idea is to project the equation along the eigenfunctions of the Laplace–Beltrami operator on M_y^k, hence getting a sequence of Schrödinger equations on $\mathbb{R}^d$. At the end we can sum-up the corresponding classical Strichartz estimates thanks to a combination of the Minkowski inequality and the Plancherel identity.

Proposition 2.1. *Let $\gamma \in \mathbb{R}$, $0 \le s < d/2$ and $d \ge 1$. Then we have the following homogeneous estimates:*

$$\|e^{-it\Delta_{x,y}} f\|_{L_t^q L_x^r H_y^\gamma} \le C\,\|f\|_{H_x^s H_y^\gamma}, \tag{2.1}$$

provided that the following conditions hold:

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2} - s, \quad q \ge 2, \quad (q,d) \ne (2,2). \tag{2.2}$$

Proof. We claim that by combining the Sobolev embedding with the usual Strichartz estimates on $\mathbb{R}^d$ we get

$$\|e^{-it\Delta_x} h\|_{L_t^q L_x^r} \le C\,\|h\|_{H_x^s}, \tag{2.3}$$

where q, r, s are as in the assumptions. By the same argument as in [19], the estimate (2.3) implies

$$\|e^{-it\Delta_{x,y}} f\|_{L_t^q L_x^r L_y^2} \le C\,\|f\|_{H_x^s L_y^2}.$$

We can conclude by using the fact that $\left(\sqrt{1-\partial_y^2}\,\right)^\gamma$ commutes with the linear Schrödinger equation on $\mathbb{R}^d \times \mathbb{T}$.

Next we give a few details about the proof of (2.3). Given any $q \ge 2$ for $d \ge 3$ (or $q > 2$ for $d = 2$, $q \ge 4$ for $d = 1$), we fix the unique $2 \le \tilde{r} < \infty$ such that

$$\frac{2}{q} + \frac{d}{\tilde{r}} = \frac{d}{2}\,. \tag{2.4}$$

Hence by the usual Strichartz estimates (see [13]) we get:

$$\|e^{-it\Delta_x} h\|_{L_t^q L_x^{\tilde{r}}} \le C\,\|h\|_{L_x^2}.$$

In turn this implies

$$\|e^{-it\Delta_x} h\|_{L_t^q W_x^{s,\tilde{r}}} \le C\,\|h\|_{H_x^s}.$$

Notice that if $s \cdot \tilde{r} < d$ then we conclude by the sharp Sobolev embedding $W_x^{s,\tilde{r}} \subset L_x^r$ (here r is precisely the one that appears in (2.2) once q and s are fixed). In the case $s \cdot \tilde{r} \ge d$ we conclude again by the Sobolev embedding $W_x^{s,\tilde{r}} \subset L_x^p$ for every $\tilde{r} \le p < \infty$, and in particular $W_x^{s,\tilde{r}} \subset L_x^r$. □

Proposition 2.2. *Let $\gamma \in \mathbb{R}$ and $d \ge 1$. Indicate by D both ∂_{x_j}, $j = 1, \dots, d$ and ∂_y. Then we have for $k = 0, 1$ the following estimates:*

$$\|D^k e^{-it\Delta_{x,y}} f\|_{L_t^\ell L_x^p H_y^\gamma} + \left\| D^k \int_0^t e^{-i(t-\tau)\Delta_{x,y}} F(\tau) d\tau \right\|_{L_t^\ell L_x^p H_y^\gamma} \le C\big(\|D^k f\|_{L_x^2 H_y^\gamma} + \|D^k F\|_{L_t^{\tilde{\ell}'} L_x^{\tilde{p}'} H_y^\gamma}\big), \tag{2.5}$$

provided that

$$\frac{2}{\ell} + \frac{d}{p} = \frac{2}{\tilde{\ell}} + \frac{d}{\tilde{p}} = \frac{d}{2}, \quad \ell \ge 2, \quad (\ell, 2) \ne (2,2)\,.$$

Proof. The proof follows by the Strichartz estimates associated with the propagator $e^{-it\Delta_x}$, in conjunction with the argument in [19]. □

Proposition 2.3. *Let $\gamma \in \mathbb{R}$ be fixed and $d \geq 3$. Then we have the following extended inhomogeneous estimates:*

$$\Big\| \int_0^t e^{-i(t-\tau)\Delta_{x,y}} F(\tau) d\tau \Big\|_{L_t^q L_x^r H_y^\gamma} \leq C \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} H_y^\gamma} \tag{2.6}$$

provided that

$$0 < \frac{1}{q}, \frac{1}{r}, \frac{1}{\tilde{q}}, \frac{1}{\tilde{r}} < \frac{1}{2} \tag{2.7}$$

$$\frac{1}{q} + \frac{1}{\tilde{q}} < 1, \quad \frac{d-2}{d} < \frac{r}{\tilde{r}} < \frac{d}{d-2} \tag{2.8}$$

$$\frac{1}{q} + \frac{d}{r} < \frac{d}{2}, \quad \frac{1}{\tilde{q}} + \frac{d}{\tilde{r}} < \frac{d}{2}, \quad \frac{2}{q} + \frac{d}{r} + \frac{2}{\tilde{q}} + \frac{d}{\tilde{r}} = d. \tag{2.9}$$

The same conclusion holds for $d = 1, 2$ provided that we drop the conditions (2.8).

Proof. In the case that f and F do not depend on y, the estimates above are special cases of the inhomogeneous extended Strichartz estimates proved in Theorem 1.4 of [8] (see also [20]). Its extension to the case that we have explicit dependence on y (in f and/or F) follows arguing as in [19]. We underline that in order to apply the technique in [19] we need (2.7), which is not required in [8]. □

Remark 2.4. The interest of using the estimates of [8] is that it allows to avoid to differentiate at a fractional order the nonlinearity $|u|^\alpha u$ with respect to the x variable. Therefore the only fractional Leibniz rule we need is Lemma 4.1 below.

The following result will be useful in the sequel.

Lemma 2.5. *Let $d \geq 1$ and let $u_n(x, y)$ be a sequence such that $\|u_n\|_{H^1_{x,y}} = O(1)$ and $\|u_n\|_{L^p_{x,y}} = o(1)$ for some $2 < p < \infty$. Then for every $2 < r < 2d/(d-1)$ there exists $\delta > 0$ such that $\|u_n\|_{L_x^r H_y^{1/2+\delta}} = o(1)$.*

Proof. By combining the assumption with the Sobolev embedding we obtain that

$$\|u_n\|_{L^q_{x,y}} = o(1) \quad \forall\, 2 < q < \infty \text{ for } d = 1, 2, \tag{2.10}$$
$$\forall\, 2 < q < \frac{2d}{d-2} \text{ for } d \geq 3.$$

First we prove the following estimate, that will be useful in the sequel:

$$\forall \gamma > 0, \quad \exists C = C(\gamma) > 0 \text{ such that } \|v\|_{L_x^{2d/(d-1)} H_y^{1/2-\gamma}} \leq C \|v\|_{H^1_{x,y}}. \tag{2.11}$$

To prove this estimate we develop $v(x, y)$ in Fourier series with respecto to the y variable:

$$v(x, y) = \sum_{n \in \mathbb{Z}} v_n(x)\, e^{iny}.$$

Hence, by the Minkowski inequality, we get

$$\|v\|^2_{L_x^{2d/(d-1)}H_y^{1/2-\gamma}} = \|\sum_{n\in\mathbb{Z}}\langle n\rangle^{1-2\gamma}|v_n(x)|^2\|_{L_x^{d/(d-1)}} \le \sum_{n\in\mathbb{Z}}\langle n\rangle^{1-2\gamma}\|v_n(x)\|^2_{L_x^{2d/(d-1)}},$$

and by the Hausdorff–Young inequality,

$$\begin{aligned}\cdots &\le C\sum_{n\in\mathbb{Z}}\langle n\rangle^{1-2\gamma}\,\|\hat{v}_n(\xi)\|^2_{L_x^{2d/(d+1)}}\\ &\le C\sum_{n\in\mathbb{Z}}\langle n\rangle^{1-2\gamma}\Big(\int|\hat{v}_n(\xi)|^2\langle\xi\rangle^{1+2\gamma}d\xi\Big)\cdot\Big(\int\langle\xi\rangle^{-d-2\gamma d}d\xi\Big)^{1/d} \le C\,\|v\|^2_{H^1_{x,y}}.\end{aligned}$$

Next, we shall prove

$$\exists\; 2 < r_0 < \frac{2d}{d-1} \quad \text{such that} \quad \|u_n\|_{L_x^{r_0}H_y^{\frac{4d-1}{4d}}} = o(1)\,. \tag{2.12}$$

Once (2.12) is proved then we conclude by interpolation between (2.12) and (2.11) in the case $r_0 \le r < 2d/(d-1)$. In the case $2 < r < r_0$ then we can interpolate between (2.12) and the estimate $\|u_n\|_{L_x^2H_y^1} = O(1)$ (that follows by the assumptions). Next we focus on (2.12). Notice that we have the following Gagliardo–Nirenberg inequality:

$$\|v(x,.)\|_{H_y^{s_0}} \le C\,\|v(x,.)\|^{1-s_0}_{L_y^2}\,\|v(x,.)\|^{s_0}_{H_y^1},$$

where we have fixed $s_0 = (4d-1)/(4d)$. In turn, by the Hölder inequality, it gives

$$\big\|\|v(x,.)\|_{H_y^{s_0}}\big\|_{L_x^{r_0}} \le C\big\|\|v(x,.)\|^{1-s_0}_{L_y^2}\big\|_{L_x^{p_0}}\big\|\|v(x,.)\|^{s_0}_{H_y^1}\big\|_{L_x^{2/s_0}},$$

where

$$\frac{1}{r_0} = \frac{1}{p_0} + \frac{s_0}{2}$$

and $p_0 = 8(d+1)$. Since $(1-s_0)p_0 > 2$ we can use the trivial estimate $\|v(.,y)\|_{L_y^2} \le \|v(.,y)\|_{L_y^{(1-s_0)p_0}}$, and we get

$$\begin{aligned}\big\|\|v(x,.)\|_{H_y^{s_0}}\big\|_{L_x^{r_0}} &\le C\,\|v(x,.)\|^{1-s_0}_{L^{(1-s_0)p_0}_{x,y}}\big\|\|v(x,.)\|_{H_y^1}\big\|^{s_0}_{L_x^2}\\ &\le C\,\|v(x,.)\|^{1-s_0}_{L^{(1-s_0)p_0}_{x,y}}\|v\|^{s_0}_{H^1_{x,y}}.\end{aligned}$$

Since $2 < (1-s_0)p_0 < 2d/(d-2)$ for $d \ge 3$, and $2 < (1-s_0)p_0 < \infty$ for $d = 1, 2$, we conclude by (2.10). □

3. Fixing the admissible exponents for the well-posedness analysis

We collect in this section some preparations, useful in the sequel to construct suitable functional spaces in which we shall perform a contraction argument to guarantee existence and uniqueness of solutions to (1.1).

The next proposition will be useful to study the Cauchy problem associated with (1.1) in the regime $0 < \alpha < 4/d$.

Proposition 3.1. *Let $d \geq 1$ and $0 < \alpha < 4/d$ be fixed. Then there exist $(q,r) \in [2,\infty] \times [2,\infty]$ such that*

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \quad (q,d) \neq (2,2), \quad \frac{1}{q'} > \frac{\alpha+1}{q}, \quad \frac{1}{r'} = \frac{\alpha+1}{r}.$$

Proof. Choose $(\frac{1}{q}, \frac{1}{r}) = (\frac{d\alpha}{4(\alpha+2)}, \frac{1}{\alpha+2})$. □

To study the Cauchy problem (1.1) in the regime $4/d \leq \alpha < 4/(d-1)$ we shall need the following proposition.

Proposition 3.2. *Let $d \geq 3$ and $4/d \leq \alpha < 4/(d-1)$ be fixed. Then there exists $0 < s < 1/2$ and $(q,r,\tilde{q},\tilde{r})$ such that:*

$$0 < \frac{1}{q}, \frac{1}{r}, \frac{1}{\tilde{q}}, \frac{1}{\tilde{r}} < \frac{1}{2} \tag{3.1}$$

and

$$\frac{1}{q} + \frac{1}{\tilde{q}} < 1, \qquad \frac{d-2}{d} < \frac{r}{\tilde{r}} < \frac{d}{d-2} \tag{3.2}$$

$$\frac{1}{q} + \frac{d}{r} < \frac{d}{2}, \qquad \frac{1}{\tilde{q}} + \frac{d}{\tilde{r}} < \frac{d}{2} \tag{3.3}$$

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2} - s, \quad \frac{2}{q} + \frac{d}{r} + \frac{2}{\tilde{q}} + \frac{d}{\tilde{r}} = d \tag{3.4}$$

$$\frac{1}{\tilde{q}'} > \frac{\alpha+1}{q}, \qquad \frac{1}{\tilde{r}'} = \frac{\alpha+1}{r}. \tag{3.5}$$

For $d = 1,2$ we get the same conclusion, provided that we drop conditions (3.2). Moreover, we can assume

$$\frac{\alpha}{q} + \frac{\alpha d}{2r} < 1, \qquad \frac{\alpha}{r} < 1. \tag{3.6}$$

In order to ease the reading we postpone its proof to the Appendix, since the (numerological) computations involved are not directly related to the analysis of NLS.

Next we shall also need the following result.

Proposition 3.3. *Let $d \geq 1$ and $4/d \leq \alpha < 4/(d-1)$ be fixed.Then there exist $2 < \ell \leq \infty, 2 \leq p \leq \infty$ such that:*

$$\frac{2}{\ell} + \frac{d}{p} = \frac{d}{2}, \tag{3.7}$$

$$\frac{1}{p'} = \frac{1}{p} + \frac{\alpha}{r}, \tag{3.8}$$

$$\frac{1}{\ell'} > \frac{1}{\ell} + \frac{\alpha}{q}, \tag{3.9}$$

where (q,r) is any couple given by Proposition 3.2.

Proof. The conditions (3.7) and (3.8) imply

$$\frac{1}{\ell} = \frac{\alpha d}{4r}, \quad \frac{1}{p} = \frac{1}{2} - \frac{\alpha}{2r},$$

and hence the condition (3.9) becomes

$$\frac{\alpha d}{2r} + \frac{\alpha}{q} < 1, \tag{3.10}$$

which is verified by (3.6). The last condition to be checked is that if ℓ, p are as above then $\ell, p > 0$. Indeed $\ell > 0$ is trivial and $p > 0$ is equivalent to $\alpha/r < 1$, which follows by (3.6). □

4. Proof of Theorem 1.1

Along this section we need the following lemma to treat the nonlinear term.

Lemma 4.1. *For every $0 < s < 1, \alpha > 0$ there exists $C = C(\alpha, s) > 0$ such that*

$$\|u|u|^\alpha\|_{\dot{H}^s_y} \leq C \|u\|_{\dot{H}^s_y} \|u\|^\alpha_{L^\infty}.$$

Proof. First we prove the following identity:

$$\int_0^{2\pi} \int_{\mathbb{R}} \frac{|u(x+h) - u(x)|^2}{|h|^{1+2s}} \, dx \, dh = c \|u\|^2_{\dot{H}^s_y} \tag{4.1}$$

for a suitable $c > 0$. We apply the Plancherel identity and we get

$$\int_0^{2\pi} |u(x+h) - u(x)|^2 \, dx = \sum_n |e^{inh} - 1|^2 \, |\hat{u}(n)|^2,$$

and hence

$$\int_0^{2\pi} \int_{\mathbb{R}} \frac{|u(x+h) - u(x)|^2}{|h|^{1+2s}} \, dx dh = \sum_n |\hat{u}(n)|^2 \int_{\mathbb{R}} |e^{inh} - 1|^2 \frac{dh}{|h|^{1+2s}}.$$

Next notice that

$$\begin{aligned} \int_{\mathbb{R}} |e^{inh} - 1|^2 \frac{dh}{|h|^{1+2s}} &= \int_{\mathbb{R}} |e^{inh} - 1|^2 \frac{|n|^{1+2s} |n| \, dh}{|n| |nh|^{1+2s}} \\ &= |n|^{2s} \int_{\mathbb{R}} |e^{ir} - 1|^2 \frac{dr}{r^{1+2s}} = c \, |n|^{2s}, \end{aligned}$$

and hence by combining the identities above we get (4.1). Based on (4.1) we get:

$$\begin{aligned} \|u\,|u|^\alpha\|^2_{\dot{H}^s_y} &= c \int_0^{2\pi} \int_{\mathbb{R}} \frac{\left| u\,|u|^\alpha(x+h) - u\,|u|^\alpha(x) \right|^2}{|h|^{1+2s}} \, dx \, dh \\ &\leq C \int_0^{2\pi} \int_{\mathbb{R}} \frac{|u(x+h) - u(x)|^2 \|u\|^{2\alpha}_{L^\infty}}{|h|^{1+2s}} \, dx \, dh \leq C \|u\|^2_{\dot{H}^s_y} \|u\|^{2\alpha}_{L^\infty}. \end{aligned}$$

□

Proof of Theorem 1.1.

First case: $4/d \le \alpha < 4/(d-1)$.

In the sequel we shall denote by $X_T^{1/2+\delta}(q,r)$ the space whose norm is defined as

$$\|u\|_{X_T^{1/2+\delta}(q,r)} = \|u(t,x,y)\|_{L_T^q L_x^r H_y^{1/2+\delta}}, \tag{4.2}$$

with $\delta > 0$, $T > 0$. Here we use the notation $L_T^q(X) = L^q((-T,T);X)$.

From now on (q,r) will be any couple given by Proposition 3.2 and $\delta > 0$ will be in such a way that $1/2+\delta+s \le 1$, where $0 < s < 1/2$ is defined by Proposition 3.2.

We shall also need the following localized norms $Y_T^{(1)}(\ell,p)$ and $Y_T^{(2)}(\ell,p)$:

$$\|u\|_{Y_T^{(1)}(\ell,p)} = \sum_{k=0,1}\sum_{j=1}^{d} \|\partial_{x_j}^k u(t,x,y)\|_{L_t^\ell((-T,T),L_x^p L_y^2)},$$
$$\|u\|_{Y_T^{(2)}(\ell,p)} = \sum_{k=0,1} \|\partial_y^k u(t,x,y)\|_{L_t^\ell((-T,T),L_x^p L_y^2)},$$

where (ℓ,p) are associated with (q,r) via Proposition 3.3.

We also set the global norm

$$\|w\|_{Z_T^{1/2+\delta}} = \|w\|_{X_T^{1/2+\delta}(q,r)} + \|w\|_{Y_T^{(1)}(\ell,p)} + \|w\|_{Y_T^{(2)}(\ell,p)},$$

and we introduce the integral operator:

$$\mathcal{A}_f u = e^{-it\Delta_{x,y}} f + i\int_0^t e^{-i(t-\tau)\Delta_{x,y}}\big(u(\tau)|u(\tau)|^\alpha\big)\, d\tau\,. \tag{4.3}$$

We split the proof in four steps.

Step 1. *For all* $f \in H^1_{x,y}$, *there exist* $T = T(\|f\|_{H^1_{x,y}}) > 0$ *and* $R = R(\|f\|_{H^1_{x,y}}) > 0$ *such that* $\mathcal{A}_f(B_{Z_{T'}^{1/2+\delta}}) \subset B_{Z_{T'}^{1/2+\delta}}$, *for any* $T' < T$.

Let $\tilde q, \tilde r$ be the ones given by Proposition 3.2. We start by noticing that

$$\|u|u|^\alpha\|_{L_T^{\tilde q'} L_x^{\tilde r'} H_y^{1/2+\delta}} \le C\,\Big\|\|u(t,x,.)\|_{H_y^{1/2+\delta}}^{\alpha+1}\Big\|_{L_T^{\tilde q'} L_x^{\tilde r'}}, \tag{4.4}$$

where we used Lemma 4.1. By combining this estimate with (3.5) and with the Hölder inequality we get

$$\|u|u|^\alpha\|_{L_T^{\tilde q'} L_x^{\tilde r'} H_y^{1/2+\delta}} \le C\,\Big\|\|u\|_{L_x^r H_y^{1/2+\delta}}^{\alpha+1}\Big\|_{L_T^{\tilde q'}} \le C\,T^{\beta(\alpha)}\,\|u\|_{L_T^q L_x^r H_y^{1/2+\delta}}^{\alpha+1}, \tag{4.5}$$

with $\beta(\alpha) > 0$ and for some constant $C > 0$ independent on T. By combining this nonlinear estimate with Propositions 2.1 and 2.3, we conclude the following:

$$\|\mathcal{A}_f u\|_{X_T^{1/2+\delta}(q,r)} \le C\,\|f\|_{H_x^s H_y^{1/2+\delta}} + C\,T^{\beta(\alpha)}\,\|u\|_{X_T^{1/2+\delta}(q,r)}^{\alpha+1}. \tag{4.6}$$

A combination of Proposition 2.2 with Proposition 3.3, in conjunction with the Hölder inequality, yield the following estimate:

$$\begin{aligned}
\|\mathcal{A}_f u\|_{Y_T^{(i)}(\ell,p)} &\leq C \sum_{k=0,1} \big(\|D^k f\|_{L^2_{x,y}} + \|D^k(u|u|^\alpha)\|_{L_T^{\ell'} L_x^{p'} L_y^2}\big) \\
&\leq C \sum_{k=0,1} \big(\|D^k f\|_{L^2_{x,y}} + \big\|\|D^k u(t,x,y)\|_{L^2_y} \|u(t,x,y)\|^\alpha_{L^\infty_y}\big\|_{L_T^{\ell'} L_x^{p'}}\big) \\
&\leq C \sum_{k=0,1} \big(\|D^k f\|_{L^2_{x,y}} + \big\|\|D^k u(t,x,y)\|_{L^2_y} \|u(t,x,y)\|^\alpha_{H_y^{1/2+\delta}}\big\|_{L_T^{\ell'} L_x^{p'}}\big) \\
&\leq C \sum_{k=0,1} (\|D^k f\|_{L^2_{x,y}} + T^{\beta(\alpha)} \|D^k u(t,x,y)\|_{L_T^\ell L_x^p L_y^2} \|u(t,x,y)\|^\alpha_{L_T^q L_x^r H_y^{1/2+\delta}}), \qquad (4.7)
\end{aligned}$$

where D stands for $\partial_y, \partial_{x_j}$, $j = 1, \ldots, d$, $k = 0, 1$, and in the third inequality we used the embedding $H_y^{1/2+\delta} \subset L_y^\infty$. Hence we get

$$\|\mathcal{A}_f u\|_{Y_T^{(i)}(\ell,p)} \leq C\, \|f\|_{H^1_{x,y}} + C\, T^{\beta(\alpha)}\, \|u\|_{Y_T^{(i)}(\ell,p)}\, \|u\|^\alpha_{X_T^{1/2+\delta}(q,r)}. \qquad (4.8)$$

We can conclude the proof of this step by combining (4.6) and (4.8).

Step 2. *Let $T, R > 0$ be as in the Step* 1. *Then there exist $\overline{T} = \overline{T}(\|f\|_{H^1_{x,y}}) < T$ such that $\mathcal{A}_f$ is a contraction on $B_{Z_{\overline{T}}^{1/2+\delta}}(0, R)$, equipped with the norm $\|.\|_{L^q_{\overline{T}} L^r_x L^2_y}$.*

Given any $v_1, v_2 \in B_{X_T^{1/2+\delta}(q,r)}(0, R)$ we achieve, by an use of estimate (2.6), the chain of bounds

$$\begin{aligned}
\|\mathcal{A}_f v_1 - \mathcal{A}_f v_2\|_{L^q_T L^r_x L^2_y} &\leq C\, \|v_1|v_1|^\alpha - v_2|v_2|^\alpha\|_{L_T^{\tilde{q}'} L_x^{\tilde{r}'} L^2_y} \\
&\leq C\, \big\|\|v_1 - v_2\|_{L^2_y} (\|v_1\|^\alpha_{L^\infty_y} + \|v_2\|^\alpha_{L^\infty_y})\big\|_{L_T^{\tilde{q}'} L_x^{\tilde{r}'}} \\
&\leq C\, \big\|\|v_1 - v_2\|_{L^r_x L^2_y} (\|v_1\|^\alpha_{L^r_x H_y^{1/2+\delta}} + \|v_2\|^\alpha_{L^r_x H_y^{1/2+\delta}})\big\|_{L_T^{\tilde{q}'}},
\end{aligned}$$

where we used the Sobolev embedding $H_y^{1/2+\delta} \subset L_y^\infty$ and (3.5) at the last step. Again by the Hölder inequality in conjunction with (3.5), we can continue the estimate as follows:

$$\cdots \leq C T^{\beta(\alpha)} \Big(\|v_1\|^\alpha_{L^q_T L^r_x H_y^{1/2+\delta}} + \|v_2\|^\alpha_{L^q_T L^r_x H_y^{1/2+\delta}}\Big) \|v_1 - v_2\|_{L^q_T L^r_x L^2_y} \qquad (4.9)$$

and we can conclude.

Step 3. *The solution exists and is unique in $Z_{\overline{T}}^{1/2+\delta}$, where $\overline{T}$ is as in Step* 2.

We are in position to show existence and uniqueness of the solution by applying the contraction principle to the map $\mathcal{A}_f$ defined on the complete metric space $B_{Z_{\overline{T}}^{1/2+\delta}}(0, R)$, equipped with the topology induced by $\|.\|_{L^q_{\overline{T}} L^r_x L^2_y}$.

Step 4. $u(t, x, y) \in \mathcal{C}((-T, T); H^1_{x,y})$.

Arguing as in the proof of (4.7), we get

$$\|\mathcal{A}_f u\|_{L^\infty((-T,T),L^2_{x,y})} + \sum_{j=1}^{d} \|\partial_{x_j}\mathcal{A}_f u\|_{L^\infty((-T,T),L^2_{x,y})}$$

$$+ \|\partial_y \mathcal{A}_f u\|_{L^\infty((-T,T),L^2_{x,y})} \leq C\,\|f\|_{H^1_{x,y}} + CT^{\beta(\alpha)}\|u\|_{Z_T^{1/2+\delta}}\|u\|^\alpha_{X_T^{1/2+\delta}(q,r)}. \tag{4.10}$$

This estimate it is sufficient to guarantee that $u(t,x,y) \in \mathcal{C}((-T,T);H^1_{x,y})$.

The last step is the proof of unconditional uniqueness of solutions to (1.1).

Step 5. *If $u_1, u_2 \in \mathcal{C}((-T,T);H^1_{x,y})$ are fixed points of $\mathcal{A}_f$, then $u_1 = u_2$.*

By a continuity argument it is sufficient to show that $u_1(t) = u_2(t)$ for a short time $(-\tilde{T},\tilde{T})$, where $\tilde{T}$ depends only on the $H^1_{x,y}$ norms of f.

By taking the difference of the integral equations satisfied by u_1 and u_2 we get

$$(u_1 - u_2)(t,x,y) = \int_0^t e^{-i(t-\tau)\Delta_{x,y}}(u_1(\tau)|u_1(\tau)|^\alpha - u_2(\tau)|u_2(\tau)|^\alpha)\,d\tau\,. \tag{4.11}$$

By an application of Proposition 2.2 we get

$$\|u_1 - u_2\|_{L^\ell_T L^p_x L^2_y} \leq C\,\||u_1|^\alpha u_1 - |u_2|^\alpha u_2\|_{L^{\ell'}_T L^{p'}_x L^2_y} \tag{4.12}$$

provided that $2/\ell + d/p = d/2$, $\ell \geq 2$, $(\ell,d) \neq (2,2)$. We can continue (4.12) as follows:

$$\begin{aligned} \cdots &\leq C\|u_1 - u_2\|_{L^\ell_T L^p_x L^2_y}\Big(\sum_{j=1,2}\|u_j\|^\alpha_{L_T^{\frac{\alpha\ell}{\ell-2}} L_x^{\frac{\alpha p}{p-2}} L^\infty_y}\Big) \\ &\leq C\|u_1 - u_2\|_{L^\ell_T L^p_x L^2_y} T^{\frac{\ell-2}{\ell}}\Big(\sum_{j=1,2}\|u_j\|^\alpha_{L^\infty_T L_x^{\frac{\alpha p}{p-2}} H_y^{1/2+\delta}}\Big), \end{aligned} \tag{4.13}$$

where we used $H_y^{1/2+\delta} \subset L^\infty_y$. We conclude the proof of uniqueness by selecting T small enough and δ, p in such a way that

$$\|v\|_{L_x^{\frac{\alpha p}{p-2}} H_y^{1/2+\delta}} \leq C\,\|v\|_{H^1_{x,y}}.$$

Indeed the estimate above follows by combining (2.11) and the trivial estimate

$$\|v\|_{L^2_x H_y^{1/2+\gamma}} \leq \|v\|_{H^1_{x,y}} \quad \forall \gamma > 0$$

provided that we can select p in such a way that

$$2 < \frac{\alpha p}{p-2} < \frac{2d}{d-1}. \tag{4.14}$$

Notice that the values allowed to p are the following:

$$p \in [2,\infty] \text{ for } d = 1, \quad p \in [2,\infty) \text{ for } d = 2, \quad p \in [2, 2d/(d-2)] \text{ for } d \geq 3\,.$$

Hence for $d=1$ we can trivially satisfy (4.14) for a suitable p. For $d=2$ notice that $\lim_{p\to 2}\frac{\alpha p}{p-2}=\infty$ and $\lim_{p\to\infty}\frac{\alpha p}{p-2}=\alpha$, and we can guarantee (4.14) for a suitable p since $0<\alpha<4/(d-1)=2d/(d-1)$ for $d=2$. In the case $d\geq 3$ we get $\lim_{p\to 2}\frac{\alpha p}{p-2}=\infty$ and $\lim_{p\to 2d/(d-2)}\frac{\alpha p}{p-2}=\alpha d/2$. We conclude since $\alpha d/2<2d/(d-1)$ (indeed it is equivalent to the assumption $\alpha<4/(d-1)$).

Second case: $0<\alpha<4/d$.

The proof is similar to the case $4/d\leq\alpha<4/(d-1)$ with minor changes. In this case the space $X_T^{1/2+\delta}(q,r)$ is selected with a couple (q,r) given by Proposition 3.1. Indeed we use Proposition 3.1 instead of Proposition 3.2, and we use on the Duhamel operator the estimates in Proposition 2.2 instead of the ones in Proposition 2.3. On the linear propagator we use Proposition 2.2 instead of Proposition 2.1. The proof of the unconditional uniqueness provided in the previous step works for every $0<\alpha<4/(d-1)$. □

5. Interaction Morawetz estimates and the proof of Proposition 1.6

Along this section we shall denote by $\int$ the integral with respect to $dxdy$, and by $\iint$ the integral with respect to $dx_1dy_1dx_2dy_2$. For $x\in\mathbb{R}^d$ and $r\geq 0$, we define $Q^d(x,r)$ to be a r dilation of the unit cube centered at x, namely

$$Q^d(x,r)=x+[-r,r]^d\,.$$

The next lemma contains the key global information needed for our analysis.

Lemma 5.1. *Let $u(t,x,y)\in\mathcal{C}(\mathbb{R};H^1_{x,y})$ be as in Proposition 1.6. Then for any $\psi\in C_0^\infty(\mathbb{R}^d)$ we get*

$$\frac{d}{dt}\int\psi(x)|u(t,x,y)|^2\,dx\,dy=-2\,\mathrm{Im}\int\bar u\nabla_x\psi\cdot\nabla_x u\,dx\,dy\,. \tag{5.1}$$

Moreover, we have

$$-2\,\frac{d}{dt}\mathrm{Im}\int\bar u\nabla_x(\langle x\rangle)\cdot\nabla_x u\,dx\,dy \tag{5.2}$$
$$=4\int\nabla_x u D_x^2(\langle x\rangle)\nabla_x\bar u\,dxdy-\int\Delta_x^2(\langle x\rangle)|u|^2\,dxdy+\frac{2\alpha}{\alpha+2}\int\Delta_x(\langle x\rangle)|u|^{\alpha+2}\,dxdy\,.$$

Moreover, for every $r>0$, there exists C such that

$$\int_{\mathbb{R}}\Big(\sup_{x_0\in\mathbb{R}^d}\iint_{Q^d(x_0,r)\times(0,2\pi)}|u(t,x,y)|^2\,dx\,dy\Big)^{(\alpha+4)/2}dt\leq C\,\|f\|^4_{H^1_{x,y}}\,. \tag{5.3}$$

Remark 5.2. We underline that Lemma 5.1 can be extended to the case that the transverse factor is any compact manifold M_y^k and the flat measure dy is replaced by the intrinsic measure $dvol_{M_y^k}$.

Remark 5.3. By analyzing the rather classical proof of (5.1), then one can deduce that the identity (5.1) can be generalized as follows, to the more general case of a function $\psi(t,x)$ that depends on the variables (t,x):

$$\int \psi(t,x)\frac{d}{dt}|u(t,x,y)|^2\,dx\,dy = -2\,\mathrm{Im}\int \bar{u}\nabla_x\psi(t,x)\cdot\nabla_x u\,dx\,dy\,. \tag{5.4}$$

Remark 5.4. By (5.1) we see that the left-hand side in (5.2) can be considered, at least formally, as the second derivative of $\int\int\langle x\rangle|u(t,x,y)|^2\,dxdy$ with respect to time, which is not a well-defined quantity for $u\in H^1_{x,y}$. However, the quantity involved on the left-hand side in (5.2) is well-defined since $\nabla_x\langle x\rangle\in L^\infty_x$ and $u\in H^1_{x,y}$. For this reason we have decided to write in terms of first derivative (5.1) and (5.2). In view of the comments above we can also write the following formal identity, that will be exploited in the sequel along an heuristic computation leading to (5.3):

$$\begin{aligned}\frac{d^2}{dt^2}\int\!\!\int\langle x\rangle|u(t,x,y)|^2\,dx\,dy \\ &= 4\int\nabla_x u D^2_x(\langle x\rangle)\nabla_x\bar{u}\,dx\,dy - \int\Delta^2_x(\langle x\rangle)|u|^2\,dx\,dy \\ &\quad+\frac{2\alpha}{\alpha+2}\int\Delta_x(\langle x\rangle)|u|^{\alpha+2}\,dx\,dy\,.\end{aligned} \tag{5.5}$$

Proof. The proof of (5.1) and (5.2) follows by standard considerations, and we skip it. Concerning the proof of (5.3) we follow [7]. From now on we define $\varphi(x)=\langle x\rangle$ and we make some formal computations. At the end of the proof we shall explain how to make rigorous the arguments below. Write

$$\begin{aligned}&\frac{d}{dt}\int\!\!\int|u(t,x_1,y_1)|^2\varphi(x_1-x_2)|u(t,x_2,y_2)|^2\,dx_1\,dx_2\,dy_1\,dy_2\\ &=\int\Big(\int\frac{d}{dt}|u(t,x_1,y_1)|^2\varphi(x_1-x_2)\,dx_1\,dy_1\Big)|u(t,x_2,y_2)|^2\,dx_2\,dy_2\\ &\quad+\int\Big(\int\frac{d}{dt}|u(t,x_2,y_2)|^2\varphi(x_1-x_2)\,dx_2dy_2\Big)|u(t,x_1,y_1)|^2\,dx_1dy_1\,.\end{aligned}$$

From now on we shall drop the variable t for simplicity and hence we shall write $u(t,x_i,y_i)=u(x_i,y_i)$. By combining the identity above with (5.1) we get

$$\begin{aligned}&\frac{d}{dt}\int\!\!\int|u(x_1,y_1)|^2\varphi(x_1-x_2)|u(x_2,y_2)|^2dx_1dx_2dy_1dy_2\\ &=-2\,\mathrm{Im}\int\!\!\int\bar{u}(x_1,y_1)\nabla_{x_1}u(x_1,y_1)\cdot\nabla_x\varphi(x_1-x_2)|u(x_2,y_2)|^2dx_1dy_1dx_2dy_2\\ &\quad+2\,\mathrm{Im}\int\!\!\int\bar{u}(x_2,y_2)\nabla_{x_2}u(x_2,y_2)\cdot\nabla_x\varphi(x_1-x_2)|u(x_1,y_1)|^2dx_1dy_1dx_2dy_2.\end{aligned} \tag{5.6}$$

Next notice that

$$
\begin{aligned}
&\frac{d^2}{dt^2}\iint |u(x_1,y_1)|^2\varphi(x_1-x_2)|u(x_2,y_2)|^2\,dx_1dx_2dy_1dy_2\\
&\quad = \int\Big(\int \frac{d^2}{dt^2}|u(x_1,y_1)|^2\varphi(x_1-x_2)\,dx_1dy_1\Big)|u(x_2,y_2)|^2\,dx_2dy_2\\
&\qquad + \int\Big(\int \frac{d^2}{dt^2}|u(x_2,y_2)|^2\varphi(x_1-x_2)dx_2dy_2\Big)|u(x_1,y_1)|^2\,dx_1dy_1\\
&\qquad + 2\int\Big(\frac{d}{dt}\int |u(x_1,y_1)|^2\varphi(x_1-x_2)dx_1dy_1\Big)\frac{d}{dt}|u(x_2,y_2)|^2\,dx_2dy_2\\
&\quad = \mathrm{I}+\mathrm{II}+\mathrm{III}.
\end{aligned}
\tag{5.7}
$$

By combining (5.1) and (5.4) we get the following identity

$$
\begin{aligned}
\mathrm{III} &= -4\int\frac{d}{dt}|u(x_2,y_2)|^2\Big(\mathrm{Im}\int \bar u(x_1,y_1)\nabla_{x_1}\varphi(x_1-x_2)\cdot\nabla_{x_1}u(x_1,y_1)\,dx_1dy_1\Big)dx_2dy_2\\
&= 8\,\mathrm{Im}\int \bar u(x_2,y_2)\nabla_{x_2}u(x_2,y_2)\cdot\nabla_{x_2}\big(\mathrm{Im}\int F(x_1,x_2,y_1)dx_1dy_1\big)\,dx_2dy_2\\
&= -8\iint V(x_1,y_1)D_x^2\varphi(x_1-x_2)V(x_2,y_2)\,dx_1dy_1dx_2dy_2,
\end{aligned}
\tag{5.8}
$$

where

$$
\begin{aligned}
F(x_1,x_2,y_1) &= \bar u(x_1,y_1)\nabla_{x_1}\varphi(x_1-x_2)\cdot\nabla_{x_1}u(x_1,y_1)\\
V(x,y) &= \mathrm{Im}(\bar u(x,y)\nabla_x u(x,y))\,.
\end{aligned}
$$

Moreover the term I in the right-hand side of (5.7) can be rewritten as follows (this is based on the formal identity (5.5)):

$$
\begin{aligned}
\mathrm{I} = 4&\iint \nabla_{x_1}u(x_1,y_1)D_x^2\varphi(x_1-x_2)\nabla_{x_1}\bar u(x_1,y_1)|u(x_2,y_2)|^2\,dx_1dy_1dx_2dy_2\\
&-\iint \Delta_x^2\varphi(x_1-x_2)|u(x_1,y_1)|^2|u(x_2,y_2)|^2\,dx_1dy_1dx_2dy_2\\
&+\frac{2\alpha}{\alpha+2}\iint \Delta_x\varphi(x_1-x_2)|u(x_1,y_1)|^{\alpha+2}|u(x_2,y_2)|^2\,dx_1dy_1dx_2dy_2\,,
\end{aligned}
$$

that by the following identity (obtained by integration by parts, see [16]),

$$
\begin{aligned}
-&\iint \Delta_x^2\varphi(x_1-x_2)|u(x_1,y_1)|^2|u(x_2,y_2)|^2\,dx_1dy_1dx_2dy_2\\
&= \iint \nabla_{x_1}(|u(x_1,y_1)|^2)D_x^2\varphi(x_1-x_2)\nabla_{x_2}(|u(x_2,y_2)|^2)\,dx_1dy_1dx_2dy_2,
\end{aligned}
$$

becomes

$$\begin{aligned}(5.9)\quad \mathrm{I} = 4 &\iint \nabla_{x_1} u(x_1,y_1) D_x^2\varphi(x_1 - x_2)\nabla_{x_1}\bar{u}(x_1,y_1)|u(x_2,y_2)|^2\, dx_1dy_1dx_2dy_2 \\ &+ \iint \nabla_{x_1}(|u(x_1,y_1)|^2) D_x^2\varphi(x_1 - x_2)\nabla_{x_2}(|u(x_2,y_2)|^2)\, dx_1dy_1dx_2dy_2 \\ &+ \iint \frac{2\alpha}{\alpha+2}\Delta_x\varphi(x_1 - x_2)|u(x_1,y_1)|^{\alpha+2}|u(x_2,y_2)|^2\, dx_1dy_1dx_2dy_2\,.\end{aligned}$$

By exchanging indices, the term II in the right-hand side of (5.7) can be rewritten as follows:

$$\begin{aligned}(5.10)\quad \mathrm{II} = 4 &\iint \nabla_{x_2} u(x_2,y_2) D_x^2\varphi(x_1 - x_2)\nabla_{x_2}\bar{u}(x_2,y_2)|u(x_1,y_1)|^2 \\ &+ \iint \nabla_{x_1}|u(x_1,y_1)|^2 D_x^2\varphi(x_1 - x_2)\nabla_{x_2}|u(x_2,y_2)|^2 \\ &+ \frac{2\alpha}{\alpha+2}\iint \Delta_x\varphi(x_1 - x_2)|u(x_2,y_2)|^{\alpha+2}|u(x_1,y_1)|^2\, dx_1dy_1dx_2dy_2\,.\end{aligned}$$

Next we introduce the vectors $A(t,x_1,y_1,x_2,y_2)$ and $B(t,x_1,y_1,x_2,y_2)$, defined as follows:

$$A(t,x_1,y_1,x_2,y_2) := u(x_1,y_1)\nabla_{x_2}\bar{u}(x_2,y_2) + \bar{u}(x_2,y_2)\nabla_{x_1}u(x_1,y_1)$$

and

$$B(t,x_1,y_1,x_2,y_2) := u(x_1,y_1)\nabla_{x_2}u(x_2,y_2) - u(x_2,y_2)\nabla_{x_1}u(x_1,y_1)\,.$$

By direct computation we get

$$\begin{aligned}(5.11)\quad & 2AD_x^2\varphi(x_1-x_2)\bar{A} + 2BD_x^2\varphi(x_1 - x_2)\bar{B} \\ &= 4\nabla_{x_1}u(x_1,y_1)D_x^2\varphi(x_1-x_2)\nabla_{x_1}\bar{u}(x_1,y_1)|u(x_2,y_2)|^2 \\ &\quad + 4\nabla_{x_2}u(x_2,y_2)D_x^2\varphi(x_1 - x_2)\nabla_{x_2}\bar{u}(x_2,y_2)|u(x_1,y_1)|^2 \\ &\quad - 8\big(\mathrm{Im}\,\bar{u}(x_1,y_1)\nabla_{x_1}u(x_1,y_1)\big)D_x^2\varphi(x_1-x_2)\big(\mathrm{Im}\,\bar{u}(x_2,y_2)\nabla_{x_2}u(x_2,y_2)\big),\end{aligned}$$

and also

$$\begin{aligned}(5.12)\quad & 2AD_x^2\varphi(x_1-x_2)\bar{A} + 2BD_x^2\varphi(x_1-x_2)\bar{B} \\ &+ 2\nabla_{x_1}|u(x_1,y_1)|^2 D_x^2\varphi(x_1-x_2)\nabla_{x_2}|u(x_2,y_2)|^2) = 4AD_x^2\varphi(x_1-x_2)\bar{A} \geq 0\,.\end{aligned}$$

By combining (5.11) and (5.12) we get

$$\begin{aligned}(5.13)\quad & 4\nabla_{x_1}u(x_1,y_1)D_x^2\varphi(x_1-x_2)\nabla_{x_1}\bar{u}(x_1,y_1)|u(x_2,y_2)|^2 \\ &+ 4\nabla_{x_2}u(x_2,y_2)D_x^2\varphi(x_1-x_2)\nabla_{x_2}\bar{u}(x_2,y_2)|u(x_1,y_1)|^2 \\ &- 8(\mathrm{Im}\,\bar{u}(x_1,y_1)\nabla_{x_1}u(x_1,y_1))D_x^2\varphi(x_1-x_2)(\mathrm{Im}\,\bar{u}(x_2,y_2)\nabla_{x_2}u(x_2,y_2) \\ &+ 2\nabla_{x_1}(|u(x_1,y_1)|^2)D_x^2\varphi(x_1-x_2)\nabla_{x_2}(|u(x_2,y_2)|^2) \geq 0\,,\end{aligned}$$

and hence by (5.8), (5.9), (5.10) and (5.13) we obtain

$$\frac{d^2}{dt^2}\iint |u(x_1,y_1)|^2\varphi(x_1-x_2)|u(x_2,y_2)|^2 dx_1dx_2dy_1dy_2 = \mathrm{I}+\mathrm{II}+\mathrm{III}$$
$$\geq \frac{4\alpha}{\alpha+2}\iint \Delta_x\varphi(x_1-x_2)|u(x_1,y_1)|^{\alpha+2}|u(x_2,y_2)|^2\, dx_1dy_1dx_2dy_2\,.$$

Integration in time gives

$$\frac{d}{dt}\Big(\iint |u(x_1,y_1)|^2\varphi(x_1-x_2)|u(x_2,y_2)|^2 dx_1dx_2dy_1dy_2\Big)_{t=\infty} \tag{5.14}$$
$$-\frac{d}{dt}\Big(\iint |u(x_1,y_1)|^2\varphi(x_1-x_2)|u(x_2,y_2)|^2 dx_1dx_2dy_1dy_2\Big)_{t=0}$$
$$=\int (\mathrm{I}+\mathrm{II}+\mathrm{III})\, dt$$
$$\geq \frac{4\alpha}{\alpha+2}\iint\int \Delta_x\varphi(x_1-x_2)|u(x_1,y_1)|^{\alpha+2}|u(x_2,y_2)|^2 dtdx_1dy_1dx_2dy_2\,.$$

Notice that by (5.6) the left-hand side can be controlled by $C\|f\|^4_{H^1_{x,y}}$ provided that we choose $\varphi=\langle x\rangle$. On the other hand we have $\inf_{Q^d(0,2r)}\Delta_x(\langle x\rangle)>0$, hence we get

$$\int_{\mathbb{R}}\sup_{x_0\in\mathbb{R}^d}\Big(\iint_{(Q^d(x_0,r))^2\times(0,2\pi)^2}|u(x_2,y_2)|^{\alpha+2}|u(x_1,y_1)|^2 dx_1dx_2dy_1dy_2\Big)dt$$
$$\leq C\,\|f\|^4_{H^1_{x,y}},$$

where we used the notation $A^2=A\times A$ for any general set A. In turn by the Hölder inequality we get

$$\int_{Q^d(x_0,r)\times(0,2\pi)}|u(x_2,y_2)|^{\alpha+2}\, dx_2dy_2 \geq C_r\Big(\int_{Q^d(x_0,r)\times(0,2\pi)}|u(x_2,y_2)|^2\, dx_2dy_2\Big)^{\frac{\alpha+2}{2}}$$

and we conclude the proof of (5.3).

Indeed, the computation above is formal since the quantity

$$\iint |u(x_1,y_1)|^2\langle x_1-x_2\rangle|u(x_2,y_2)|^2\, dx_1dx_2dy_1dy_2$$

appearing in (5.6) it is not well-defined for $u\in H^1_{x,y}$. Following the Remark 5.4, we can make rigorous the argument above by writing the following identity:

$$\frac{d}{dt}J(t)=\mathrm{I}+\mathrm{II}+\mathrm{III}\,,$$

where the quantity

$$J(t)=-2\,\mathrm{Im}\iint \bar u(x_1,y_1)\nabla_{x_1}u(x_1,y_1)\cdot\nabla_x\varphi(x_1-x_2)|u(x_2,y_2)|^2\, dx_1dy_1dx_2dy_2$$
$$+2\,\mathrm{Im}\iint \bar u(x_2,y_2)\nabla_{x_2}u(x_2,y_2)\cdot\nabla_x\varphi(x_1-x_2)|u(x_1,y_1)|^2\, dx_1dy_1dx_2dy_2$$

is meaningful for $u\in H^1_{x,y}$. □

Proof of Proposition 1.6. We follow the approach in [21]. First, we write the following localized Gagliardo–Nirenberg inequality (see [18], page 93, eq. (A-5)):

$$\|v\|_{L^{2+4/(d+1)}_{x,y}} \leq C \sup_{x\in\mathbb{R}^d} \big(\|v\|_{L^2_{Q^d(x,1)\times(0,2\pi)}}\big)^{2/(d+3)} \|v\|^{(d+1)/(d+3)}_{H^1_{x,y}}. \tag{5.15}$$

Of course it is sufficient to show that

$$\lim_{t\to\pm\infty} \|u(t,x,y)\|_{L^{2+4/(d+1)}_{x,y}} = 0\,. \tag{5.16}$$

In fact the decay of the $L^q_{x,y}$ norm for $2 < q < 2(d+1)/(d-1)$ follows by combining (5.16) with the bound

$$\sup_{t\in\mathbb{R}} \|u(t,x,y)\|_{H^1_{x,y}} < \infty\,. \tag{5.17}$$

Next, assume by the absurd that (5.16) is false, then by (5.15) and by (5.17) we deduce the existence of a sequence $(t_n, x_n) \in \mathbb{R}\times\mathbb{R}^d$ with $|t_n| \to \infty$ and $\epsilon_0 > 0$ such that

$$\inf_n \|u(t_n,x,y)\|_{L^2_{Q^d(x_n,1)\times(0,2\pi)}} = \epsilon_0\,. \tag{5.18}$$

For simplicity we can assume that $t_n \to \infty$ (the case $t_n \to -\infty$ can be treated by a similar argument).

Notice that by (5.1) in conjunction with (5.17) we get

$$\sup_{n,t} \Big|\frac{d}{dt}\int \chi(x-x_n)|u(t,x,y)|^2\,dxdy\Big| < \infty\,,$$

where $\chi(x)$ is a smooth and non-negative cut-off function taking values in $[0,1]$ such that $\chi(x) = 1$ for $x \in Q^d(0,1)$ and $\chi(x) = 0$ for $x \notin Q^d(0,2)$. By combining this fact with (5.18) then we get the existence of $T > 0$ such that

$$\inf_n \big(\inf_{t\in(t_n,t_n+T)} \|u(t,x,y)\|_{L^2_{Q^d(x_n,2)\times(0,2\pi)}}\big) \geq \epsilon_0/2\,. \tag{5.19}$$

Notice that since $t_n \to \infty$ then we can assume (modulo subsequence) that the intervals (t_n, t_n+T) are disjoint. In particular we have

$$\begin{aligned}\sum_n T(\epsilon_0/2)^{\alpha+4} &\leq \sum_n \int_{t_n}^{t_n+T} \Big(\iint_{Q^d(x_n,2)\times(0,2\pi)} |u(t,x,y)|^2 dx\,dy\Big)^{(\alpha+4)/2} dt\\ &\leq \int \Big(\sup_{z\in\mathbb{R}^d} \iint_{Q^d(z,2)\times(0,2\pi)} |u(t,x,y)|^2 dx\,dy\Big)^{(\alpha+4)/2} dt\end{aligned}$$

and hence we get a contradiction since the left hand side is divergent and the right hand side is bounded by (5.3). □

6. Fixing the admissible exponents for the scattering analysis

In this section we prepare some result useful to prove Theorem 1.3.

Proposition 6.1. *Let $d \geq 1$ and $4/d < \alpha < 4/(d-1)$ be fixed, and $s = \frac{\alpha d - 4}{2\alpha}$. Then there exists $\theta \in (0,1)$ and $(q_\theta, r_\theta, \tilde{q}_\theta, \tilde{r}_\theta)$, in such a way that*

$$0 < \frac{1}{q_\theta}, \frac{1}{r_\theta}, \frac{1}{\tilde{q}_\theta}, \frac{1}{\tilde{r}_\theta} < \frac{1}{2} \tag{6.1}$$

$$\frac{1}{q_\theta} + \frac{1}{\tilde{q}_\theta} < 1, \qquad \frac{d-2}{d} < \frac{r_\theta}{\tilde{r}_\theta} < \frac{d}{d-2} \tag{6.2}$$

$$\frac{1}{q_\theta} + \frac{d}{r_\theta} < \frac{d}{2}, \qquad \frac{1}{\tilde{q}_\theta} + \frac{d}{\tilde{r}_\theta} < \frac{d}{2} \tag{6.3}$$

$$\frac{2}{q_\theta} + \frac{d}{r_\theta} = \frac{d}{2} - s, \qquad \frac{2}{q_\theta} + \frac{d}{r_\theta} + \frac{2}{\tilde{q}_\theta} + \frac{d}{\tilde{r}_\theta} = d, \tag{6.4}$$

$$\frac{1}{(\alpha+1)\tilde{q}_\theta'} = \frac{\theta}{q_\theta}, \qquad \frac{1}{(\alpha+1)\tilde{r}_\theta'} = \frac{\theta}{r_\theta} + \frac{2(1-\theta)}{\alpha d}. \tag{6.5}$$

For $d = 1, 2$ we get the same conclusion provided that we drop conditions (6.2).

Moreover we can also assume that

$$\frac{\alpha}{q_\theta} + \frac{\alpha d}{2 r_\theta} = 1, \qquad \frac{\alpha}{r_\theta} < 1. \tag{6.6}$$

Remark 6.2. By combining Propositions 2.1 and 2.3, we get the following estimate: for every $\gamma \in \mathbb{R}$,

$$\|e^{-it\Delta_{x,y}} f\|_{L_t^{q_\theta} L_x^{r_\theta} H_y^\gamma} + \Big\| \int_0^t e^{-i(t-\tau)\Delta_{x,y}} F(\tau) d\tau \Big\|_{L_t^{q_\theta} L_x^{r_\theta} H_y^\gamma} \leq C\big(\|f\|_{H_x^s H_y^\gamma} + \|F\|_{L_t^{\tilde{q}_\theta'} L_x^{\tilde{r}_\theta'} H_y^\gamma}\big). \tag{6.7}$$

Proof of Proposition 6.1. For the moment we let θ to be free, and at the end we shall select it according with a continuity argument. We fix $(q_\theta, r_\theta) = (q, r)$ (where q, r are given in Lemma 8.1) and we choose $\tilde{q}_\theta$ and $\tilde{r}_\theta$ as follows:

$$\frac{1}{(\alpha+1)\tilde{q}_\theta'} = \frac{\theta}{q}, \quad \frac{1}{(\alpha+1)\tilde{r}_\theta'} = \frac{\theta}{r} + \frac{2(1-\theta)}{\alpha d}.$$

By this choice, (6.4) and (6.5) turn out to be satisfied for every θ. On the other hand, by (8.5) we have

$$\lim_{\theta \to 1} \frac{1}{\tilde{q}_\theta} = \frac{1}{\tilde{q}}, \quad \lim_{\theta \to 1} \frac{1}{\tilde{r}_\theta} = \frac{1}{\tilde{r}},$$

where $\tilde{q}, \tilde{r}$ are given by Lemma 8.1. Hence conditions (6.2) and (6.3) follow by (8.2) and (8.3) provided that we choose θ close enough to the value $\theta = 1$. □

The next proposition, which is a version of Proposition 3.3 where we replace inequality by identity in the last condition, will be useful in the sequel.

Proposition 6.3. *Let $d \geq 1$ and $4/d < \alpha < 4/(d-1)$ be fixed. Then there exist $2 < \ell \leq \infty, 2 \leq p \leq \infty$ such that*

$$\frac{2}{\ell} + \frac{1}{p} = \frac{1}{2}, \quad \frac{1}{p'} = \frac{1}{p} + \frac{\alpha}{r_\theta}, \quad \text{and} \quad \frac{1}{\ell'} = \frac{1}{\ell} + \frac{\alpha}{q_\theta}, \tag{6.8}$$

where (q_θ, r_θ) is any couple given by Proposition 6.1.

The same proof as in Proposition 3.3 can be repeated.

7. Proof of Theorem 1.3

Proposition 7.1. *Let (q_θ, r_θ) be as in Proposition 6.1 and $u(t,x,y) \in \mathcal{C}(\mathbb{R}; H^1_{x,y})$ be the unique global solution to (1.1), with $4/d < \alpha < 4/(d-1)$. Then*

$$u(t,x,y) \in L^{q_\theta}_t L^{r_\theta}_x H^{1/2+\delta}_y \tag{7.1}$$

for some $\delta > 0$.

Proof. We will apply a $H^{1/2+\delta}_y$ valued version of the analysis H^s_x critical analysis of [4]. Notice that in Proposition 6.1 we have $0 < s < 1/2$ and hence by choosing $\delta > 0$ small, we can control $\|.\|_{H^s_x H^{1/2+\delta}_y}$ by $\|.\|_{H^1_{x,y}}$. By combining this fact with Remark 6.2 we get

$$\begin{aligned} \|u(t,x,y)\|_{L^{q_\theta}_{t>t_0} L^{r_\theta}_x H^{1/2+\delta}_y} &\leq C\big(\|u(t_0)\|_{H^1_{x,y}} + \|u|u|^\alpha\|_{L^{\tilde{q_\theta}'}_{t>t_0} L^{\tilde{r}'_\theta}_x H^{1/2+\delta}_y}\big) \\ &\leq C\big(\|u(t_0)\|_{H^1_{x,y}} + \|u\|^{1+\alpha}_{L^{(1+\alpha)\tilde{q_\theta}'}_{t>t_0} L^{(1+\alpha)\tilde{r}'_\theta}_x H^{1/2+\delta}_y}\big), \end{aligned} \tag{7.2}$$

where we have used Lemma 4.1 and we have denoted by $\|f(t)\|_{L^p_{t>t_0}}$ the integral $\int_{t_0}^\infty |f(t)|^p dt$ for any given time-dependent function. By combining (6.5) with the Hölder inequality we can continue the estimate (7.2) as follows:

$$\cdots \leq C\big(\|u(t_0)\|_{H^1_{x,y}} + \|u\|^{\theta(1+\alpha)}_{L^{q_\theta}_{t>t_0} L^{r_\theta}_x H^{1/2+\delta}_y} \|u\|^{(1-\theta)(1+\alpha)}_{L^\infty_{t>t_0} L^{\alpha d/2}_x H^{1/2+\delta}_y}\big).$$

By combining Proposition 1.6 with Lemma 2.5, we deduce that

$$\lim_{t_0 \to \infty} \|u\|_{L^\infty_{t>t_0} L^{\alpha d/2}_x H^{1/2+\delta}_y} = 0\,,$$

and hence for every $\epsilon > 0$ there exists $t_0 = t_0(\epsilon) > 0$ such that

$$\|u(t,x,y)\|_{L^{q_\theta}_{t>t_0} L^{r_\theta}_x H^{1/2+\delta}_y} \leq C\,\|u(t_0)\|_{H^1_{x,y}} + \epsilon \|u\|^{\theta(1+\alpha)}_{L^{q_\theta}_{t>t_0} L^{r_\theta}_x H^{1/2+\delta}_y}\,.$$

We conclude by a continuity argument that $\|u(t,x,y)\|_{L^{q_\theta}_{t>0} L^{r_\theta}_x H^{1/2+\delta}_y} < \infty$. By a similar argument we get $\|u(t,x,y)\|_{L^{q_\theta}_{t<0} L^{r_\theta}_x H^{1/2+\delta}_y} < \infty$. □

Proposition 7.2. *Let (ℓ, p) be as in Proposition 6.3 and let $u(t,x,y)$ be the unique solution to (1.1) with $4/d < \alpha < 4/(d-1)$. Then*

$$\|u(t,x,y)\|_{L^\ell_t L^p_x L^2_y} + \|\partial_y u(t,x,y)\|_{L^\ell_t L^p_x L^2_y} + \|\nabla_x u(t,x,y)\|_{L^\ell_t L^p_x L^2_y} < \infty\,. \tag{7.3}$$

Proof. We show $\|\partial_y u(t,x,y)\|_{L^\ell_t L^p_x L^2_y} < \infty$, the other estimates are similar. By (2.5),

$$\|\partial_y u(t,x,y)\|_{L^\ell_{t>t_0} L^p_x L^2_y} \leq C(\|u(t_0)\|_{H^1_{x,y}} + \|(\partial_y u)|u|^\alpha\|_{L^{\ell'}_{t>t_0} L^{p'}_x L^2_y}).$$

By Proposition 6.3 we can apply the Hölder inequality and we get

$$\begin{aligned}\cdots &\leq C(\|u(t_0)\|_{H^1_{x,y}} + \|(\partial_y u)\|_{L^\ell_{t>t_0} L^p_x L^2_y} \|u\|^\alpha_{L^{q_\theta}_{t>t_0} L^{r_\theta}_x L^\infty_y}) \\ &\leq C(\|u(t_0)\|_{H^1_{x,y}} + \|(\partial_y u)\|_{L^\ell_{t>t_0} L^p_x L^2_y} \|u\|^\alpha_{L^{q_\theta}_{t>t_0} L^{r_\theta}_x H^{1/2+\delta}}).\end{aligned}$$

We conclude by choosing t_0 large enough and by recalling Proposition 7.1. □

Proof of Theorem 1.3. It follows by Proposition 7.2 via a standard argument (see [3]). In fact by using the integral equation associated with (1.1) it is sufficient to prove that

$$\lim_{t_1,t_2\to\infty} \Big\| \int_{t_1}^{t_2} e^{-is\Delta_{x,y}}(u|u|^\alpha)ds \Big\|_{H^1_{x,y}} = 0. \tag{7.4}$$

By combining Proposition 2.2 with a duality argument we get

$$\Big\| \int_{t_1}^{t_2} e^{-is\Delta_{x,y}} F(s)ds \Big\|_{L^2_{x,y}} \leq C\,\|F\|_{L^{\ell'}_{(t_1,t_2)} L^{p'}_x L^2_y},$$

where (l,p) are as in Proposition 6.3. Hence (7.4) follows provided that

$$\lim_{t_1,t_2\to\infty}\Big(\|u|u|^\alpha\|_{L^{\ell'}_{(t_1,t_2)} L^{p'}_x L^2_y} + \|\partial_y(u|u|^\alpha)\|_{L^{\ell'}_{(t_1,t_2)} L^{p'}_x L^2_y} + \|\nabla_x(u|u|^\alpha)\|_{L^{\ell'}_{(t_1,t_2)} L^{p'}_x L^2_y}\Big) = 0\,.$$

This estimate can be proved following the same argument used along the proof of Proposition 7.2, in conjunction with (7.1) and (7.3). □

8. Appendix

This Appendix is devoted to the proof of Proposition 3.2. We need the following.

Lemma 8.1. *Let $d \geq 3$, $4/d \leq \alpha < 4/(d-1)$ be fixed and $s = \frac{\alpha d - 4}{2\alpha}$. Then there exist $(q, r, \tilde{q}, \tilde{r})$ such that:*

$$0 < \frac{1}{q}, \frac{1}{r}, \frac{1}{\tilde{q}}, \frac{1}{\tilde{r}} < \frac{1}{2} \tag{8.1}$$

and

$$\frac{1}{q} + \frac{1}{\tilde{q}} < 1, \qquad \frac{d-2}{d} < \frac{r}{\tilde{r}} < \frac{d}{d-2} \tag{8.2}$$

$$\frac{1}{q} + \frac{d}{r} < \frac{d}{2}, \qquad \frac{1}{\tilde{q}} + \frac{d}{\tilde{r}} < \frac{d}{2} \tag{8.3}$$

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2} - s, \qquad \frac{2}{q} + \frac{d}{r} + \frac{2}{\tilde{q}} + \frac{d}{\tilde{r}} = d, \tag{8.4}$$

$$\frac{1}{\tilde{q}'} = \frac{\alpha+1}{q}, \qquad \frac{1}{\tilde{r}'} = \frac{\alpha+1}{r}. \tag{8.5}$$

For $d = 1, 2$ we get the same conclusion, provided that we drop conditions (8.2).

Moreover we can also assume that

$$\frac{\alpha}{q} + \frac{\alpha d}{2r} = 1, \qquad \frac{\alpha}{r} < 1. \tag{8.6}$$

Remark 8.2. Compared with Proposition 3.2, in Lemma 8.1 we have fixed s and moreover we put identity in (8.5) (compare with (3.5) where we have inequality).

Proof. First we show that by our choice of s (8.6) follows. Indeed we get

$$\frac{\alpha}{q} + \frac{\alpha d}{2r} = \frac{\alpha}{2}\Big(\frac{2}{q} + \frac{d}{r}\Big) = \frac{\alpha}{2}\Big(\frac{d}{2} - s\Big) = 1,$$

where we used (8.4). Notice also that by the second identity in (8.5) we get $\alpha/r < 1$; in fact, $\alpha/r < (\alpha+1)/r + 1/\tilde{r} = 1$.

Moreover the first condition in (8.2) follows by (8.1), and the first condition in (8.3) follows by the first identity in (8.4). Hence since now on we can skip those conditions. It is easy to check that thanks to our choice of s, the identities in (8.4) and (8.5) are not independent. Moreover by (8.4) and (8.5), and by recalling $s = (\alpha d - 4)/(2\alpha)$, we get

$$\frac{1}{\tilde{q}} = -\frac{1}{\alpha} + \frac{(\alpha+1)d}{2r}, \qquad \frac{1}{\tilde{r}} = 1 - \frac{\alpha+1}{r}, \qquad \frac{1}{q} = \frac{1}{\alpha} - \frac{d}{2r}. \tag{8.7}$$

Next we consider two cases:

First case: $d \geq 3$.

Thanks to (8.7), the conditions (8.1), (8.2) (where we skip the first one), (8.3) (where we skip the first one) can be written as follows:

$$\frac{\alpha d}{2} < r < \frac{\alpha d}{(2-\alpha)}, r > 2, \frac{\alpha(\alpha+1)d}{\alpha+2} < r < \frac{\alpha(\alpha+1)d}{2},$$
$$\alpha + 1 < r < 2(\alpha+1), r < \frac{\alpha(\alpha+1)d}{\alpha d - 2}, \frac{d-2}{d} + \alpha + 1 < r < \frac{d}{d-2} + \alpha + 1$$

Hence we conclude tat we can select a suitable r if the condition

$$\max\Big\{\frac{\alpha d}{2}, 2, \frac{\alpha(\alpha+1)d}{\alpha+2}, \alpha+1, \frac{d-2}{d} + \alpha + 1\Big\}$$
$$< \min\Big\{\frac{\alpha d}{(2-\alpha)}, \frac{\alpha(\alpha+1)d}{2}, 2(\alpha+1), \frac{\alpha(\alpha+1)d}{\alpha d - 2}, \frac{d}{d-2} + \alpha + 1\Big\}$$

is satisfied. Since we are assuming $4/d \leq \alpha < 4/(d-1)$, this condition is equivalent to

$$\max\Big\{\frac{\alpha(\alpha+1)d}{\alpha+2}, \frac{d-2}{d} + \alpha + 1\Big\} < \min\Big\{\frac{\alpha d}{(2-\alpha)}, \frac{\alpha(\alpha+1)d}{\alpha d - 2}, \frac{d}{d-2} + \alpha + 1\Big\}. \tag{8.8}$$

Next we notice that

$$\frac{\alpha(\alpha+1)d}{\alpha+2} = \max\Big\{\frac{\alpha(\alpha+1)d}{\alpha+2}, \frac{d-2}{d} + \alpha + 1\Big\}. \tag{8.9}$$

In fact it follows by direct computation for $d = 3, 4$ and for $d \geq 5$ it comes by the following argument. Notice that $\alpha(\alpha+1)d/(\alpha+2) \geq (d-2)/d + (\alpha+1)$ is equivalent to $(\alpha+1)\big(\alpha d/(\alpha+2) - 1\big) \geq (d-2)/d$, that under the constrain $4/d \leq \alpha < 4/(d-1)$ can be written as

$$\inf_{\alpha\in[4/d,4/(d-1))} (\alpha+1)\Big(\frac{\alpha d}{\alpha+2} - 1\Big) \geq \frac{d-2}{d} .$$

In turn this inequality follows provided that $(1+4/d)\big(\frac{d\cdot 4/d}{4/(d-1)+2} - 1\big) \geq (d-2)/d$, and by elementary computations it is equivalent to $(d+4)(d-3) \geq (d+1)(d-2)$, which is satisfied for every $d \geq 5$. Hence by (8.8) and (8.9) we conclude provided that we show

$$\frac{\alpha(\alpha+1)d}{\alpha+2} < \frac{\alpha d}{(2-\alpha)}, \frac{\alpha(\alpha+1)d}{\alpha+2} < \frac{\alpha(\alpha+1)d}{\alpha d - 2}, \frac{\alpha(\alpha+1)d}{\alpha+2} < \frac{d}{d-2} + \alpha + 1 .$$

The first and second inequalities are satisfied for any $0 < \alpha < 4/(d-1)$ and the last one follows by

$$\sup_{\alpha\in[4/d,4/(d-1))} (\alpha+1)\Big(\frac{\alpha d}{\alpha+2} - 1\Big) < \frac{d}{d-2} . \tag{8.10}$$

On the other hand we have

$$\begin{aligned}\sup_{\alpha\in[4/d,4/(d-1))} (\alpha+1)\Big(\frac{\alpha d}{\alpha+2} - 1\Big) &\leq (1+4/(d-1))\Big(\frac{4d/(d-1)}{4/d+2} - 1\Big)\\ &= \frac{(d+3)(d^2-d+2)}{(d-1)^2(d+2)} .\end{aligned}$$

Hence (8.10) follows provided that $\frac{(d+3)(d^2-d+2)}{(d-1)^2(d+2)} < \frac{d}{d-2}$, which is always satisfied.

Second case: $d = 1, 2$.

Arguing as above (recall that we drop (8.2)) we conclude provided that we can select r such that

$$\begin{aligned}&r > \frac{\alpha d}{2}, \quad r > 2, \quad \frac{\alpha(\alpha+1)d}{\alpha+2} < r < \frac{\alpha(\alpha+1)d}{2},\\ &\alpha+1 < r < 2(\alpha+1), \quad r < \frac{\alpha(\alpha+1)d}{\alpha d - 2} .\end{aligned}$$

By elementary computations (see the case $d \geq 3$) and by recalling $4/d \leq \alpha < 4/(d-1)$, the conditions above are equivalent to the following inequality:

$$\max\Big\{(\alpha+1), \frac{\alpha(\alpha+1)d}{\alpha+2}\Big\} < r < \frac{\alpha(\alpha+1)d}{\alpha d - 2} . \tag{8.11}$$

On the other hand, by explicit computation we get

$$\begin{aligned}&\max\Big\{(\alpha+1), \frac{\alpha(\alpha+1)d}{\alpha+2}\Big\} = \frac{\alpha(\alpha+1)2}{\alpha+2} \quad \text{for } d = 2,\\ &\max\Big\{(\alpha+1), \frac{\alpha(\alpha+1)d}{\alpha+2}\Big\} = \alpha+1 \quad \text{for } d = 1,\end{aligned}$$

and (8.11) follows by elementary considerations. □

Proof of Proposition 3.2. We focus on the case $d \geq 3$ (the cases $d = 1, 2$ can be treated by a similar argument). We argue by a continuity argument based on Lemma 8.1. In fact we fix $(1/q, 1/r, 1/\tilde{q}, 1/\tilde{r}, s)$ as in Lemma 8.1 and we look for $(1/(q+\epsilon), 1/r, 1/\tilde{q}_\epsilon, 1/\tilde{r}, s_\epsilon)$ that satisfy conditions of Lemma 3.2, for some $\epsilon > 0$ small enough and $\tilde{q}_\epsilon, s_\epsilon$ will be properly chosen in dependence of ϵ. By our choice it will be clear that $\lim_{\epsilon \to 0} s_\epsilon = s$ and $\lim_{\epsilon \to 0} \tilde{q}_\epsilon = \tilde{q}$. Notice that with this choice the identity in (3.5) is satisfied (compare with (8.5)). Also (3.1), (3.2), (3.3) are satisfied by a continuity argument provided that $\epsilon > 0$ is small enough (recall that $r, \tilde{r}, q, \tilde{q}$ satisfy (8.1), (8.2), (8.3)). Notice also that since $q + \epsilon > q$ then the first identity in (3.4) is satisfied provided that we choose $s_\epsilon > s$ (recall that q, r, s satisfy the first identity in (8.4)) and also (3.6) follows by (8.6).

Next we impose that $1/(q+\epsilon), 1/r, 1/\tilde{q}_\epsilon, 1/\tilde{r}$ satisfy the second identity in (3.4), i.e., $1/(q+\epsilon) + 1/\tilde{q}_\epsilon = \frac{d}{2}(1 - 1/r - 1/\tilde{r}) = \beta$. We claim that $(\alpha+1)/(q+\epsilon) + 1/\tilde{q}_\epsilon < 1$ (notice this is equivalent to the inequality in (3.5)) and it will conclude the proof. Indeed we write $\frac{1}{q+\epsilon} = \frac{1}{q} - \frac{\epsilon}{q(q+\epsilon)}$ and hence $\frac{1}{\tilde{q}_\epsilon} = \beta - \frac{1}{q} + \frac{\epsilon}{q(q+\epsilon)} = \frac{1}{\tilde{q}} + \frac{\epsilon}{q(q+\epsilon)}$, where we used $1/q + 1/\tilde{q} = \beta$ (see the second identity in (8.4)). Hence we get, by the first identity in (8.5),

$$\frac{\alpha+1}{q+\epsilon} + \frac{1}{\tilde{q}_\epsilon} = \frac{\alpha+1}{q} - \frac{\epsilon(\alpha+1)}{q(q+\epsilon)} + \frac{1}{\tilde{q}} + \frac{\epsilon}{q(q+\epsilon)} = -\frac{\epsilon\alpha}{q(q+\epsilon)} + 1 < 1\,. \qquad \square$$

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Received September 13, 2014; revised March 13, 2015.

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N. Tzvetkov is supported by the ERC grant Dispeq, and N. Visciglia by the FIRB project Dispersive Dynamics.