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A characterization of the Gaussian Lipschitz space and sharp estimates for the Ornstein–Uhlenbeck Poisson kernel

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Abstract. The Gaussian Lipschitz space was defined by Gatto and Urbina, by means of the Ornstein–Uhlenbeck Poisson kernel. We give a characterization of this space in terms of a combination of ordinary Lipschitz continuity conditions. The main tools used in the proof are sharp estimates of the Ornstein–Uhlenbeck Poisson kernel and some of its derivatives.

1. Introduction and main results

Let γ be the Gauss measure on $\mathbb{R}^n$ with $n \geq 1$, that is, $d\gamma(x) = \pi^{-n/2} e^{-|x|^2}\, dx$. The Gaussian analog of the Euclidean Laplacian is the *Ornstein–Uhlenbeck operator* $\mathcal{L} = -\frac{1}{2}\Delta + x\cdot\nabla$, where $\nabla = (\partial_{x_1}, \dots, \partial_{x_n})$. The operator $\mathcal{L}$ is the infinitesimal generator of the *Ornstein–Uhlenbeck semigroup* $T_t = e^{-t\mathcal{L}}$, $t > 0$, given by

$$T_t f(x) = \pi^{-n/2} \int_{\mathbb{R}^n} M_{e^{-t}}(x,y) f(y)\, dy$$

for all $f \in L^2(\gamma)$ and $x \in \mathbb{R}^n$, where $M_{e^{-t}}$ is the *Mehler kernel* defined by

$$M_r(x,y) = \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{n/2}} \qquad x,\, y \in \mathbb{R}^n, \quad 0 < r < 1\,.$$

The *Ornstein–Uhlenbeck Poisson semigroup* $\{P_t\}_{t>0}$ is defined by subordination from $\{T_t\}_{t>0}$ as

$$P_t f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}}\, T_{t^2/(4u)} f(x)\, du\,.$$

There is a corresponding *Ornstein–Uhlenbeck Poisson kernel* $P_t(x,y)$, for which

$$P_t f(x) = \int_{\mathbb{R}^n} P_t(x,y) f(y)\, dy\,,$$

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and it is obtained from the Mehler kernel by similar subordination. Transforming variables $s = t^2/(4u)$ and inserting the expression for the Mehler kernel $M_{e^{-s}}$, one gets

$$P_t(x,y) = \frac{1}{2\pi^{(n+1)/2}} \int_0^\infty \frac{t}{s^{3/2}}\, e^{-\frac{t^2}{4s}}\, \frac{e^{-\frac{|y-e^{-s}x|^2}{1-e^{-2s}}}}{(1-e^{-2s})^{n/2}}\, ds\,. \tag{1.1}$$

Gatto and Urbina [3] introduced the Gaussian Lipschitz spaces; see also [2] and [5]. Let $\alpha \in (0,1)$, which will be fixed throughout the paper. A measurable function f in $\mathbb{R}^n$ is said to be in the *Gaussian Lipschitz space* GLip_α if it is bounded and satisfies

$$\|\partial_t P_t f\|_{L^\infty} \le A\, t^{\alpha-1}, \quad t>0, \tag{1.2}$$

for some $A>0$. The norm in $f \in \mathrm{GLip}_\alpha$ is

$$\|f\|_{\mathrm{GLip}_\alpha} = \|f\|_{L^\infty} + \inf\{A:\ A \text{ satisfies } (1.2)\}.$$

The standard Euclidean Lipschitz space $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ consists of all bounded functions f such that for some $C>0$,

$$|f(x)-f(y)| \le C|x-y|^\alpha, \quad x,\, y \in \mathbb{R}^n. \tag{1.3}$$

It is known that the space $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ can be characterized by means of the standard Poisson kernel

$$\mathscr{P}_t(x,y) = c_n\, \frac{t}{(t^2+|x-y|^2)^{(n+1)/2}}\,,$$

where c_n is a dimensional constant that makes $\int_{\mathbb{R}^n} P_t(x,y)\,dy = 1$; see Section V. 4. 2 in Stein [6]. To be precise, an L^∞ function f coincides a.e. with a function in $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ if and only if

$$\|\partial_t \mathscr{P}_t f\|_{L^\infty} \le C\, t^{\alpha-1}$$

for all $t>0$. The main aim of this paper is to describe the Gaussian Lipschitz space by means of a condition like (1.3), as follows.

Theorem 1.1. *Let $\alpha \in (0,1)$. The following statements are equivalent:*

(i) $f \in \mathrm{GLip}_\alpha$;

(ii) *there exists a positive constant K such that for all $x,y \in \mathbb{R}^n$,*

$$|f(x)-f(y)| \le K \min\Big\{|x-y|^\alpha,\ \Big(\frac{|x-y|}{1+|x|+|y|}\Big)^{\alpha/2} + ((|x|+|y|)\sin\theta)^\alpha\Big\}, \tag{1.4}$$

after correction of f on a null set. Here θ denotes the angle between the vectors x and y; if $x=0$ or $y=0$, then θ is understood as 0.

Moreover, the norm $\|f\|_{\mathrm{GLip}_\alpha}$ is equivalent to $|f(0)| + \inf\{K>0:\ K \text{ satisfies } (1.4)\}$.

In one dimension, the inequality (1.4) reads

$$|f(x)-f(y)| \le K \min\Big\{|x-y|^\alpha,\ \Big(\frac{|x-y|}{1+|x|+|y|}\Big)^{\alpha/2}\Big\}.$$

This is a combined Lipschitz condition, with exponent α for short distance $|x-y|$ (in fact, shorter than $1/(1+|x|+|y|)$), and exponent $\alpha/2$, with a different coefficient, for long distance. In higher dimension, the expression $(|x|+|y|)\sin\theta$ describes the "orthogonal component" of the vector $x-y$, since it is the distance from y to the line in the direction x plus the vice versa quantity. To make this more clear, we state an unsymmetric inequality equivalent to (1.4). For $x,\ y \in \mathbb{R}^n$ with $x \neq 0$, we decompose y as $y = y_x + y_x'$, where y_x is parallel to x and y_x' orthogonal to x. If $x = 0$, we let $y_x = y$ and $y_x' = 0$, and this holds for all x in case $n = 1$. As proved in Lemma 2.1 below, (1.5) is equivalent to

$$|f(x)-f(y)| \le K' \min\Big\{|x-y|^\alpha,\ \Big(\frac{|x-y_x|}{1+|x|}\Big)^{\alpha/2} + |y_x'|^\alpha\Big\} \tag{1.5}$$

in any dimension, with a constant $K' > 0$ comparable with K. This means that the combined Lipschitz condition applies in the radial direction, but in the orthogonal direction the exponent is always α. In the proof of Theorem 1.1, we shall use (1.5) instead of (1.4).

Gatto and Urbina defined GLip_α for all $\alpha > 0$. In analogy with the Euclidean case, it seems likely that there are versions of Theorem 1.1 that hold for $\alpha \ge 1$.

In a forthcoming paper [4], the authors obtain a result analogous to Theorem 1.1 but where the Lipschitz space is defined without the boundedness assumption.

The proof of Theorem 1.1 relies on pointwise estimates of the Ornstein–Uhlenbeck Poisson kernel $P_t(x,y)$ and its derivatives, which also have independent interest. Before stating these results, we need some notation.

Throughout the paper, we shall write C for various positive constants which depend only on n and α. Given any two nonnegative quantities A and B, the notation $A \lesssim B$ stands for $A \le CB$ (we say that A is controlled by B), and $A \gtrsim B$ means $B \lesssim A$. If $B \lesssim A \lesssim B$, we write $A \simeq B$. For positive quantities X, we shall write

$$\exp^*(-X)$$

meaning $\exp(-cX)$ for some constant $c = c(n,\alpha) > 0$ whose value may change from one occurrence to another. Then we have for instance $te^{-t} \simeq \exp^*(-t)$ for $t > 1$, since we allow different values of c in the two inequalities defining the $\simeq$ relation. We shall often use inequalities like $\exp^*(-X) \lesssim \exp^*(-X)\exp^*(-X)$.

Theorem 1.2. *For all $t > 0$ and $x,\ y \in \mathbb{R}^n$,*

$$P_t(x,y) \le C[K_1(t,x,y) + K_2(t,x,y) + K_3(t,x,y) + K_4(t,x,y)],$$

where

$$
\begin{aligned}
K_1(t,x,y) &= \frac{t}{(t^2+|x-y|^2)^{(n+1)/2}} \exp^*\left(-t(1+|x|)\right);\\
K_2(t,x,y) &= \frac{t}{|x|}\Big(t^2+\frac{|x-y_x|}{|x|}+|y_x'|^2\Big)^{-(n+2)/2} \exp^*\Big(-\frac{(t^2+|y_x'|^2)|x|}{|x-y_x|}\Big)\\
&\quad \times \chi_{\{|x|>1,\, x\cdot y>0,\, |x|/2\le|y_x|<|x|\}};\\
K_3(t,x,y) &= \min\{1,t\}\exp^*(-|y|^2);\\
K_4(t,x,y) &= \frac{t}{|y_x|}\Big(\log\frac{|x|}{|y_x|}\Big)^{-3/2} \exp^*\Big(-\frac{t^2}{\log(|x|/|y_x|)}\Big)\exp^*(-|y_x'|^2)\\
&\quad \times \chi_{\{x\cdot y>0,\, 1<|y_x|<|x|/2\}}.
\end{aligned}
$$

In Section 6, we consider the sharpness of Theorem 1.2. In particular, we exhibit for each of the four kernels $K_i(t,x,y)$ a set $\widetilde{E}_i$ of points (t,x,y) in which $P_t(x,y)\simeq K_i(t,x,y)$ but where the other three terms $K_j(t,x,y)$ are much smaller; see the proof of Theorem 6.1(b). Thus none of the four terms can be suppressed in Theorem 1.2. It can also be verified that for each i there exist (many) points (t,x) such that the integral of $K_i(t,x,y)$ with respect to y, taken over those y for which $(t,x,y)\in\widetilde{E}_i$, is comparable to $1=\int_{\mathbb{R}^n}P_t(x,y)\,dy$. This means that for these (t,x), the kernel $K_i(t,x,\cdot)$ contains a substantial part of $P_t(x,\cdot)$.

We make some comments about the four terms K_i in Theorem 1.2, focusing on large values of $|x|$.

Consider first small values of t. The term $K_1(t,x,y)$ is for $t<1/(1+|x|)$ essentially the standard Poisson kernel. For us, the most significant term is $K_2(t,x,y)$, since it is the key to the term with exponent $\alpha/2$ in (1.4) and (1.5). In one dimension, one has

$$K_2(t,x,y)\lesssim \widetilde{K}_2(t,x,y):=\frac{1}{t^2|x|}\Big(1+\frac{|x-y|}{t^2|x|}\Big)^{-3/2}$$

for all (t,x,y), and

$$P_t(x,y)\simeq K_2(t,x,y)\simeq \widetilde{K}_2(t,x,y) \tag{1.6}$$

in the set where $x>1$ and $3x/4<y<x-t^2x$ (see Section 6). Notice that $\widetilde{K}_2(t,x,y)$ is a Poisson-like kernel but with a dilation parameter $t^2|x|$ which depends on x, and with a slower decay as $y\to\infty$. Further, for $x>1$ and $t>0$ fixed, the integral in y of each of the three kernels in (1.6) over the interval $(3x/4,x-t^2x)$ is of order of magnitude $1=\int_{\mathbb{R}}P_t(x,y)\,dy$. In higher dimension, $K_2(t,x,y)$ has, as a function of y, a different behavior in the x direction and in the directions orthogonal to x.

Our Poisson kernel P_t can be compared with the standard Poisson kernel $\mathscr{P}_t$ in the following way. Roughly speaking, the main part of the standard Poisson integral $\mathscr{P}_t f(x)$ is essentially the mean value of the function f in a ball of radius t, centered at x. The analog for $P_t f(x)$ is the mean value in a cylinder in the x

direction of length $t^2|x|$, radius t and center $x - t^2x$. This displacement from x of the center is not very significant, since the displacement is not larger than the length.

This displacement comes from the Mehler kernel; the subordination formula says that P_t is a weighted mean in the t variable of values of the Mehler kernel. For small t, the Mehler kernel gives essentially the mean value of the function in a ball of radius $\sqrt{t}$ and center $e^{-t}x \approx x - tx$. So for $t << 1/|x|^2$, the displacement is significant here, since it is much larger than the radius. Actually, it is only this displacement that makes the Mehler kernel essentially different from the standard heat kernel, for small t. Observe that the displacement is in the negative x direction in both cases.

For large t, the Mehler kernel has a dilation factor which is essentially 1, and the displacement is to the origin. As a result, we get for P_t the terms $K_3(t,x,y)$ and $K_4(t,x,y)$, which are large for small y only.

After finishing this paper, we learned that Garrigós et al. [1] (see their Lemmas 4.1 and 4.2) also estimated the kernel $P_t(x,y)$. Their estimates are rather different from ours and intended for other purposes.

From the proof of Theorem 1.2, it will be seen that $t\partial_t P_t$ and $t\partial_{x_i} P_t$ with $1 \le i \le n$ satisfy the same estimates as P_t, as follows.

Theorem 1.3. *Let $i \in \{1, 2, \dots, n\}$. Then for all $t > 0$ and $x, y \in \mathbb{R}^n$,*

$$|t\partial_t P_t(x,y)| + |t\partial_{x_i} P_t(x,y)| \le C\left[K_1(t,x,y) + K_2(t,x,y) + K_3(t,x,y) + K_4(t,x,y)\right].$$

For the derivative of $P_t(x,y)$ with respect to x in the radial direction, i. e., along the vector x, we obtain a sharper estimate than that of Theorem 1.3. This result will be of fundamental importance in the proof of Theorem 1.1. To state it in a simple way, we first observe that P_t is invariant under rotation in the sense that $P_t(Ax, Ay) = P_t(x,y)$ for any orthogonal matrix A. The same is true for all the kernels we use. This means that in our estimates, we can assume without restriction that $x = (x_1, 0, \dots, 0)$ with $x_1 \ge 0$. Then we will write the decomposition of y as $y = (y_1, y') \in \mathbb{R} \times \mathbb{R}^{n-1}$.

Theorem 1.4. *For all $t > 0$, $x = (x_1, 0, \dots, 0) \in \mathbb{R}^n$ with $x_1 \ge 0$ and $y = (y_1, y') \in \mathbb{R}^n$,*

$$|\partial_{x_1} P_t(x,y)| \le C[Z_1(t,x,y) + Z_2(t,x,y) + Z_3(t,x,y) + Z_4(t,x,y)],$$

where

$$Z_1(t,x,y) = \frac{t}{(t^2+|x-y|^2)^{(n+2)/2}} \exp^*(-t(1+|x|));$$

$$Z_2(t,x,y) = \frac{t}{x_1^2}\Big(t^2+\frac{x_1-y_1}{x_1}+|y'|^2\Big)^{-(n+4)/2} \exp^*\Big(-\frac{(t^2+|y'|^2)x_1}{x_1-y_1}\Big) \times \chi_{\{x_1>1,\, x_1/2\le y_1<x_1\}};$$

$$Z_3(t,x,y) = \frac{\min\{t,\ t^{-2}\}}{1+|x|} \exp^*(-|y|^2)\,;$$

$$Z_4(t,x,y) = \frac{t}{x_1\, y_1}\Big(\log\frac{x_1}{y_1}\Big)^{-5/2} \exp^*\Big(-\frac{t^2}{\log(x_1/y_1)}\Big) \exp^*(-|y'|^2)\,\chi_{\{1<y_1<x_1/2\}}\,.$$

The paper is organized as follows. In Section 2, we prove the equivalence between the conditions (1.4) and (1.5) and then give some basic estimates needed later. Section 3 contains the proof of Theorem 1.1, assuming Theorems 1.2, 1.3 and 1.4. The proofs of Theorems 1.2 and 1.3 are given in Section 4. Section 5 contains the proof of Theorem 1.4, which is based on that of Theorem 1.2 but now exploiting also some cancellation in the integral estimates. Finally, Section 6 deals with the sharpness of our estimates for P_t.

2. Auxiliary results

Lemma 2.1. *Let $\alpha \in (0,1)$. The conditions (1.4) and (1.5) are equivalent, and each of them implies that the function f is bounded. More precisely,*

$$\sup_{x\in\mathbb{R}^n} |f(x)-f(0)| \lesssim \inf K \simeq \inf K'. \tag{2.1}$$

Proof. To see that each of the two conditions implies boundedness, it is enough to take $y=0$ in either condition. This also gives the inequality in (2.1) and the analogous inequality for K'.

Let A and B denote the minima appearing in (1.4) and (1.5), respectively. If $|x|+|y|\le 2$, one finds that $A\simeq |x-y|^\alpha \simeq B$. Assume next that $|y|/2<|x|<2|y|$. Then $|y'_x|\simeq (|x|+|y|)\sin\theta$ and it is obvious that $B\lesssim A$. The converse $A\lesssim B$ is easy when $|y'_x|\le |x-y_x|$. When $|y'_x|>|x-y_x|$, we have

$$A\le |x-y|^\alpha \simeq |y'_x|^\alpha \le B\,.$$

Thus it only remains to consider the case when $|x|+|y|>2$ and $|x|/|y|\notin (1/2,2)$. But then A, $B\gtrsim 1$, and via the boundedness we just proved, we see that each of the inequalities (1.4) and (1.5) implies the other for these x, y.

Altogether, this proves the equivalence, and (2.1) also follows. □

Lemma 2.2. *Let* $a,\ T,\ A \in (0,\infty)$, $X \in [0,\infty)$ *and* $\beta \in (1,\infty)$. *Then*

$$\begin{aligned}\mathcal{J} &:= \int_0^a \frac{1}{\sigma^\beta} \exp^*\Big(-\frac{T^2}{\sigma}\Big) \exp^*\Big(-\frac{A^2}{\sigma}\Big) \exp^*(-\sigma X^2)\, d\sigma \\ &\leq M\, \frac{\exp^*(-AT/a)\, \exp^*(-T\,X)}{(T^2+A^2)^{\beta-1}},\end{aligned}$$

where $M > 0$ *is independent of* a, T, A *and* X.

Proof. Notice that $\exp^*(-T^2/\sigma)\exp^*\big(-\sigma X^2\big) \lesssim \exp^*\left(-T\,X\right)$. Via a change of variable $u = (T^2+A^2)/\sigma$, we see that

$$\begin{aligned}\mathcal{J} &\lesssim \exp^*(-T\,X) \int_0^a \frac{1}{\sigma^\beta} \exp^*\Big(-\frac{T^2}{\sigma}\Big) \exp^*\Big(-\frac{A^2}{\sigma}\Big)\, d\sigma \\ &\lesssim \exp^*(-T\,X)\, \frac{1}{(T^2+A^2)^{\beta-1}} \int_{\frac{T^2+A^2}{a}}^\infty u^{\beta-2} \exp^*\left(-u\right) du.\end{aligned}$$

Since $(T^2+A^2)/a \geq 2A\,T/a$ and since $\beta > 1$, the last integral here is controlled by $\exp^*\left(-AT/a\right)$. □

Proposition 2.3. *For all* $x \in \mathbb{R}^n$ *and* $t > 0$, *the* K_i *from Theorem 1.2 satisfy*

$$\int_{\mathbb{R}^n} \left[K_1(t,x,y) + K_2(t,x,y)\right] dy \leq C, \tag{2.2}$$

$$\int_{\mathbb{R}^n} \left[K_3(t,x,y) + K_4(t,x,y)\right] dy \leq C \min\{1,\, t\}\,. \tag{2.3}$$

Proof. Since K_1 is dominated by the standard Poisson kernel, it follows that $\int_{\mathbb{R}^n} K_1(t,x,y)\, dy \lesssim 1$. Also, it is obvious that $\int_{\mathbb{R}^n} K_3(t,x,y)\, dy \lesssim \min\{1,t\}$.

For the estimates of K_2 and K_4, we can make a rotation and assume that $x = (x_1, 0, \ldots, 0)$ with $x_1 > 1$ and write $y = (y_1, y')$. Then

$$\begin{aligned}\int_{\mathbb{R}^n} K_2(t,x,y)\, dy &\lesssim \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \frac{t}{x_1} \Big(t^2 + \frac{|x_1-y_1|}{x_1} + |y'|^2\Big)^{-(n+2)/2} dy'\, dy_1 \\ &\lesssim \int_{\mathbb{R}} \frac{t}{x_1} \Big(t^2 + \frac{|x_1-y_1|}{x_1}\Big)^{-3/2} dy_1 \lesssim 1,\end{aligned} \tag{2.4}$$

and (2.2) is proved.

In the case of K_4, we have

$$\begin{aligned}&\int_{\mathbb{R}^n} K_4(t,x,y)\, dy \\ &\quad\lesssim \int_{1<y_1<x_1/2} \int_{\mathbb{R}^{n-1}} \frac{t}{y_1} \Big(\log\frac{x_1}{y_1}\Big)^{-3/2} \exp^*\Big(-\frac{t^2}{\log(x_1/y_1)}\Big) \exp^*(-|y'|^2)\, dy'\, dy_1 \\ &\quad\simeq \int_{1<y_1<x_1/2} \frac{t}{y_1} \Big(\log\frac{x_1}{y_1}\Big)^{-3/2} \exp^*\Big(-\frac{t^2}{\log(x_1/y_1)}\Big)\, dy_1 \\ &\quad\lesssim \int_{\log 2}^\infty \frac{t}{\sqrt{\tau}} \exp^*\Big(-\frac{t^2}{\tau}\Big) \frac{d\tau}{\tau} \lesssim \min\{1,\, t\}.\end{aligned}$$

This proves (2.3). □

3. Proof of Theorem 1.1

In this section, we assume Theorems 1.2, 1.3 and 1.4 and prove Theorem 1.1. Combining Proposition 2.3 with the pointwise estimates for the x derivatives of the Poisson kernel in Theorems 1.3 and 1.4, we first deduce bounds for the L^1 norms of those derivatives.

Proposition 3.1. (i) *For all* $i \in \{1, 2, \dots, n\}$, $t > 0$ *and* $x \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} |\partial_{x_i} P_t(x, y)|\, dy \le C t^{-1}. \tag{3.1}$$

(ii) *For all* $t > 0$ *and* $x = (x_1, 0, \dots, 0) \in \mathbb{R}^n$ *with* $x_1 \ge 0$,

$$\int_{\mathbb{R}^n} |\partial_{x_1} P_t(x, y)|\, dy \le C t^{-2}(1 + x_1)^{-1}. \tag{3.2}$$

Proof. Notice that (i) follows from Theorem 1.3 and Proposition 2.3.

To prove (ii), we have from Theorem 1.4,

$$|\partial_{x_1} P_t(x, y)| \lesssim Z_1(t, x, y) + Z_2(t, x, y) + Z_3(t, x, y) + Z_4(t, x, y).$$

It is easy to see that

$$\begin{aligned}\int_{\mathbb{R}^n} Z_1(t, x, y)\, dy &\lesssim \int_{\mathbb{R}^n} \frac{t \exp^*(-t(1 + |x|))}{(t^2 + |x - y|^2)^{(n+2)/2}}\, dy \\ &\lesssim t^{-1} \exp^*(-t(1 + |x|)) \lesssim t^{-2}(1 + x_1)^{-1}\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n} Z_3(t, x, y)\, dy &\simeq \int_{\mathbb{R}^n} \frac{\min\{t,\ t^{-2}\}}{1 + |x|} \exp^*(-|y|^2)\, dy \\ &\lesssim \frac{\min\{t,\ t^{-2}\}}{1 + |x|} \lesssim t^{-2}(1 + x_1)^{-1}.\end{aligned}$$

Integrating Z_2 first in y' and then in y_1, we get

$$\begin{aligned}\int_{\mathbb{R}^n} Z_2(t, x, y)\, dy &\lesssim \frac{t}{x_1^2} \int_{x_1/2}^{x_1} \int_{\mathbb{R}^{n-1}} \Big(\frac{x_1}{x_1 - y_1}\Big)^{(n+4)/2} \exp^*\Big(-\frac{(t^2 + |y'|^2)x_1}{x_1 - y_1}\Big)\, dy'\, dy_1 \\ &\lesssim \frac{t}{x_1^2} \int_{x_1/2}^{x_1} \Big(\frac{x_1}{x_1 - y_1}\Big)^{5/2} \exp^*\Big(-\frac{t^2 x_1}{x_1 - y_1}\Big)\, dy_1 \lesssim t^{-2} x_1^{-1}.\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{\mathbb{R}^n} Z_4(t, x, y)\, dy &\lesssim \frac{t}{x_1} \int_0^{x_1/2} \Big(\log \frac{x_1}{y_1}\Big)^{-5/2} \exp^*\Big(-\frac{t^2}{\log \frac{x_1}{y_1}}\Big) \frac{dy_1}{y_1} \\ &\lesssim \frac{t}{x_1} \int_{\log 2}^{\infty} u^{-5/2} \exp^*\Big(-\frac{t^2}{u}\Big)\, du \lesssim t^{-2} x_1^{-1},\end{aligned}$$

where $u = \log(x_1/y_1)$. Combining these estimates and noticing that Z_2 and Z_4 are non-zero only if $x_1 > 1$, we obtain (3.2). □

From this proposition, we deduce two pointwise bounds for the x derivatives of $P_t f$, with f a Gaussian Lipschitz function.

Proposition 3.2. *Let* $\alpha \in (0,1)$ *and* $f \in \mathrm{GLip}_\alpha$ *with norm* 1.

(i) *For all* $i \in \{1, 2, \dots, n\}$, $t > 0$ *and* $x \in \mathbb{R}^n$,

$$|\partial_{x_i} P_t f(x)| \le C\, t^{\alpha-1}. \tag{3.3}$$

(ii) *For all* $t > 0$ *and* $x = (x_1, 0, \dots, 0) \in \mathbb{R}^n$ *with* $x_1 \ge 0$,

$$|\partial_{x_1} P_t f(x)| \le C\, t^{\alpha-2}\, (1 + x_1)^{-1}. \tag{3.4}$$

Proof. To prove (i), we use the semigroup property of the Poisson integral and take derivatives, obtaining

$$\partial_{x_i} \partial_t P_{s+t} f(x) = \partial_{x_i} \partial_t \int_{\mathbb{R}^n} P_s(x,y) P_t f(y)\, dy = \int_{\mathbb{R}^n} \partial_{x_i} P_s(x,y)\, \partial_t P_t f(y)\, dy$$

for s, $t > 0$ and $x \in \mathbb{R}^n$. Now let $s = t$, to get

$$\frac{1}{2}\, \partial_{x_i} \partial_t P_{2t} f(x) = \int_{\mathbb{R}^n} \partial_{x_i} P_t(x,y)\, \partial_t P_t f(y)\, dy\,.$$

By (3.1) and the definition of GLip_α, this implies that for all $t > 0$,

$$|\partial_{x_i} \partial_t P_t f(x)| \lesssim t^{\alpha-2}. \tag{3.5}$$

Since f is bounded, it follows from (3.1) that $\partial_{x_i} P_t f(x) \to 0$ as $t \to \infty$. Thus

$$\partial_{x_i} P_t f(x) = -\int_t^\infty \partial_{x_i} \partial_\tau P_\tau f(x)\, d\tau\,,$$

and (i) is a consequence of this and the preceding inequality.

We prove (ii) by a similar argument, using now (3.2). The only difference is that (3.5) is replaced by $|\partial_{x_1} \partial_t P_t f(x)| \lesssim t^{\alpha-3}(1 + x_1)^{-1}$. □

Proof of Theorem 1.1. To prove that (i) implies (ii), we let $f \in \mathrm{GLip}_\alpha$ with norm 1 and verify (1.5), using Lemma 2.1. We start by modifying f on a null set. Since $f \in L^\infty(\mathbb{R}^n)$ and $\{P_t\}_{t>0}$ is a semigroup to which the Littlewood-Paley-Stein theory applies (see Stein [7]), we know that $P_t f(x) \to f(x)$ as $t \to 0$ for almost all $x \in \mathbb{R}^n$. For each $t > 0$ and all $x \in \mathbb{R}^n$, one has

$$P_t f(x) = P_1 f(x) - \int_t^1 \partial_\tau P_\tau f(x)\, d\tau\,,$$

and this integral has a limit as $t \to 0$ for all x. We define $f(x)$ as $P_1 f(x) - \int_0^1 \partial_\tau P_\tau f(x)\, d\tau$ for all $x \in \mathbb{R}^n$.

Let x, $y \in \mathbb{R}^n$. For any $t > 0$, one has

$$|f(x) - f(y)| \le |f(x) - P_t f(x)| + |P_t f(x) - P_t f(y)| + |P_t f(y) - f(y)|\,. \tag{3.6}$$

Writing the first difference to the right here as an integral and applying the definition of GLip_α, we see that

$$|f(x) - P_t f(x)| = \Big| \int_0^t \partial_\tau P_\tau f(x)\, d\tau \Big| \le \int_0^t \tau^{\alpha-1}\, d\tau \simeq t^\alpha.$$

The same applies to the third difference. For the second difference, Proposition 3.2 (i) yields that

$$|P_t f(x) - P_t f(y)| \le |x-y| \sup_{\theta\in(0,1)} |\nabla P_t f(x + \theta(y-x))| \lesssim |x-y|\, t^{\alpha-1}.$$

Thus

$$|f(x) - f(y)| \lesssim t^\alpha + |x-y|\ t^{\alpha-1}.$$

Taking $t = |x-y|$, we get

$$|f(x) - f(y)| \lesssim |x-y|^\alpha. \tag{3.7}$$

To complete the proof of (1.5), we first make a rotation so that $x = (x_1, 0, \dots, 0)$ with $x_1 \ge 0$. Because of (3.7), it is then enough to show that for all $y = (y_1, y') \in \mathbb{R}\times\mathbb{R}^{n-1}$,

$$|f(x) - f(y)| \lesssim \Big(\frac{|x_1 - y_1|}{1 + x_1}\Big)^{\alpha/2} + |y'|^\alpha. \tag{3.8}$$

Notice that if $|x| = x_1 \le 2$, then (3.8) follows directly from (3.7), since $|f|$ is bounded by 1. If $x_1 > 2$ and $|x_1 - y_1| \ge x_1/2$, the right-hand side of (3.8) is greater than a positive constant, so (3.8) follows again. It only remains to consider the case $x_1 > 2$ and $|x_1 - y_1| < x_1/2$. For such x and y, we write

$$|f(x) - f(y)| \le |f(x) - f(y_1, 0)| + |f(y_1, 0) - f(y)|,$$

and (3.7) implies that $|f(y_1, 0) - f(y)| \lesssim |y'|^\alpha$. To estimate $|f(x) - f(y_1, 0)|$, we apply (3.6) again and proceed as before, but now using (3.4) to estimate the x_1 derivative. This gives that for any $t > 0$,

$$|f(x) - f(y_1, 0)| \lesssim t^\alpha + |x_1 - y_1|\, t^{\alpha-2} \sup_{\theta\in(0,1)} |x_1 + \theta(y_1 - x_1)|^{-1}.$$

Since $|x_1 - y_1| < x_1/2$, the supremum here is no larger than $2x_1^{-1}$. Letting $t = \sqrt{|x_1 - y_1|/x_1}$, we obtain

$$|f(x) - f(y_1, 0)| \lesssim \Big(\frac{|x_1 - y_1|}{x_1}\Big)^{\alpha/2} \simeq \Big(\frac{|x_1 - y_1|}{1 + x_1}\Big)^{\alpha/2},$$

so (3.8) follows, and (1.5) is verified.

We now prove that (ii) implies (i) in Theorem 1.1. Because of Lemma 2.1, we can assume that (1.5) holds with $K' \le 1$; we must then verify (1.2). Using the fact that $\int_{\mathbb{R}^n} \partial_t P_t(x,y)\,dy = 0$ and Theorem 1.3, we can write

$$|t\,\partial_t P_t f(x)| = \Big| \int_{\mathbb{R}^n} t\,\partial_t P_t(x,y)\,[f(y)-f(x)]\,dy \Big| \lesssim \sum_{i=1}^{4} \int_{\mathbb{R}^n} K_i(t,x,y)\,|f(y)-f(x)|\,dy\,.$$

Since the condition (1.5) implies that $f \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} K_1(t,x,y)\,|f(y)-f(x)|\,dy \lesssim \int_{\mathbb{R}^n} \frac{t}{(t+|x-y|)^{n+1}}\,|x-y|^\alpha\,dy \lesssim t^\alpha.$$

From (1.5), we deduce that

$$\begin{aligned}\int_{\mathbb{R}^n} & K_2(t,x,y)\,|f(y)-f(x)|\,dy \\ &\lesssim \int_{\mathbb{R}^n} \frac{t}{|x|}\Big(t^2+\frac{|x-y_x|}{|x|}+|y_x'|^2\Big)^{-(n+2)/2}\Big[\Big(\frac{|x-y_x|}{1+|x|}\Big)^{\alpha/2}+|y_x'|^\alpha\Big]\,dy \\ &\lesssim \int_{\mathbb{R}^n} \frac{t}{|x|}\Big(t^2+\frac{|x-y_x|}{|x|}+|y_x'|^2\Big)^{-(n+2-\alpha)/2}\,dy\,.\end{aligned}$$

After a rotation of coordinates, we can treat the last integral like the one in (2.4); only the exponent is different, and the resulting bound will be Ct^α. Finally, Proposition 2.3 implies that

$$\begin{aligned}\int_{\mathbb{R}^n} & [K_3(t,x,y)+K_4(t,x,y)]\,|f(y)-f(x)|\,dy \\ &\lesssim \|f\|_{L^\infty}\int_{\mathbb{R}^n} [K_3(t,x,y)+K_4(t,x,y)]\,dy \lesssim \min\{1,t\} \lesssim t^\alpha.\end{aligned}$$

We have verified (1.2). □

4. Proof of Theorems 1.2 and 1.3

Since $P_t(x,y)$ and the $K_i(t,x,y)$ are invariant under rotation, we only need to consider $x=(x_1,0\dots,0)$ with $x_1\geq 0$ and write $y=(y_1,y')$ as before. Theorem 1.2 is a consequence of the slightly sharper result in Proposition 4.1 below.

A change of variables $\sigma = 1-e^{-s}$ in (1.1) leads to

$$P_t(x,y) = \frac{1}{2\pi^{(n+1)/2}}\int_0^1 \frac{t}{s(\sigma)^{3/2}}\,e^{-\frac{t^2}{4s(\sigma)}}\,\frac{e^{-\frac{|y-x+\sigma x|^2}{1-e^{-2s(\sigma)}}}}{(1-e^{-2s(\sigma)})^{n/2}}\,e^{s(\sigma)}\,d\sigma\,, \tag{4.1}$$

where $s(\sigma)=\log\frac{1}{1-\sigma}$. In the sequel, we will split the interval of integration into various subintervals, and in each subinterval we use either s or σ as variable of integration.

When $0<y_1<x_1$, the quantity

$$|y-e^{-s}x|^2 = |y-x+\sigma x|^2 = |y_1-x_1+\sigma x_1|^2+|y'|^2$$

has a minimum at the point

$$\sigma_0 := \frac{x_1 - y_1}{x_1} \in (0,1)\,, \tag{4.2}$$

and

$$|y - e^{-s}x|^2 = (\sigma - \sigma_0)^2 x_1^2 + |y'|^2, \qquad 0 < s < +\infty\,.$$

This will be used repeatedly in what follows.

Proposition 4.1. *Let $t > 0$, $x = (x_1, 0, \dots, 0) \in \mathbb{R}^n$ with $x_1 \geq 0$ and $y = (y_1, y') \in \mathbb{R} \times \mathbb{R}^{n-1}$.*

(i) *If $y_1 \notin (0, x_1)$, then*

$$P_t(x,y) \leq C\,[K_1(t,x,y) + K_3(t,x,y)]\,. \tag{4.3}$$

(ii) *If $y_1 \in [x_1/2, x_1)$, then*

$$P_t(x,y) \leq C\,[K_1(t,x,y) + K_2(t,x,y) + K_3(t,x,y)]\,. \tag{4.4}$$

(iii) *If $y_1 \in (0, x_1/2)$, then*

$$P_t(x,y) \leq C\,[K_1(t,x,y) + K_3(t,x,y) + K_4(t,x,y)]\,. \tag{4.5}$$

Proof. To prove (i), let $y_1 \notin (0, x_1)$. We split the integral in (4.1) into integrals over $(0, 1/2)$ and $[1/2, 1)$, called J_1 and J_2.

For J_1, noticing that $\sigma \in (0, 1/2)$ is equivalent to $s(\sigma) \in (0, \log 2)$, we have $1 - e^{-2s(\sigma)} \simeq s(\sigma) \simeq \sigma$ and $e^{s(\sigma)} \simeq 1$. As a result,

$$J_1 \simeq \int_0^{1/2} \frac{t}{\sigma^{(n+3)/2}} \exp^*\left(-\frac{t^2}{\sigma}\right) \exp^*\left(-\frac{|y - x + \sigma x|^2}{\sigma}\right) d\sigma.$$

It follows from $y_1 \notin (0, x_1)$ and $\sigma < 1/2$ that $|y_1 - x_1 + \sigma x_1| \gtrsim \max\{\sigma x_1,\, |x_1 - y_1|\}$, and thus

$$|y - x + \sigma x| \gtrsim \max\{\sigma|x|, |x - y|\}. \tag{4.6}$$

Notice that for $\sigma \in (0,1)$, one has

$$\exp^*\left(-\frac{t^2}{\sigma}\right) \lesssim \exp^*(-t^2) \lesssim \exp^*(-t)\,. \tag{4.7}$$

Combined with Lemma 2.2, this yields that

$$\begin{aligned} J_1 &\lesssim \exp^*(-t) \int_0^{1/2} \frac{t}{\sigma^{(n+3)/2}} \exp^*\left(-\frac{t^2 + |y - x|^2}{\sigma}\right) \exp^*(-\sigma|x|^2)\, d\sigma \\ &\lesssim \frac{t}{(t^2 + |y - x|^2)^{(n+1)/2}} \exp^*(-t(1 + |x|)) \simeq K_1(t,x,y). \end{aligned} \tag{4.8}$$

For J_2 we use the variable s, getting

$$J_2 \simeq \int_{\log 2}^{\infty} \frac{t}{s^{3/2}} \exp^*\left(-\frac{t^2}{s}\right) \exp^*(-|y - e^{-s}x|^2)\, ds\,. \tag{4.9}$$

Since $y_1 \notin (0, x_1)$ and $s \geq \log 2$, one has $|y - e^{-s}x| \simeq |y_1 - e^{-s}x_1| + |y'| \gtrsim |y_1| + |y'| \simeq |y|$ and hence

$$\exp^*(-|y - e^{-s}x|^2) \lesssim \exp^*(-|y|^2)\,. \tag{4.10}$$

Thus,

$$\begin{aligned} J_2 &\lesssim \exp^*(-|y|^2) \int_{\log 2}^{\infty} \frac{t}{s^{3/2}} \exp^*\Big(-\frac{t^2}{s}\Big)\, ds \\ &\lesssim \min\{1, t\} \exp^*(-|y|^2) \simeq K_3(t, x, y)\,. \end{aligned} \tag{4.11}$$

We have proved (4.3) and (i).

Next, we assume $y_1 \in [x_1/2, x_1)$ and prove (ii). With σ_0 given by (4.2) and now satisfying $0 < \sigma_0 \leq 1/2$, we split the integral in (4.1) into integrals over the three intervals $(0, \frac{3}{4}\sigma_0)$, $[\frac{3}{4}\sigma_0, \frac{5}{4}\sigma_0]$ and $(\frac{5}{4}\sigma_0, 1)$, denoted $J_{1,1}$, $J_{1,2}$ and $J_{1,3}$, respectively.

In $J_{1,1}$ we have $1 - e^{-2s(\sigma)} \simeq s(\sigma) \simeq \sigma$ and $e^{2s(\sigma)} \simeq 1$, and also

$$|y - x + \sigma x|^2 = (\sigma - \sigma_0)^2 x_1^2 + |y'|^2 \simeq \sigma_0^2 x_1^2 + |y'|^2 = |x - y|^2.$$

We get

$$J_{1,1} \simeq \int_0^{3\sigma_0/4} \frac{t}{\sigma^{(n+3)/2}} \exp^*\Big(-\frac{t^2}{\sigma}\Big) \exp^*\Big(-\frac{|x-y|^2}{\sigma}\Big)\, d\sigma\,. \tag{4.12}$$

Since $|x - y| \gtrsim \sigma_0 x_1 \geq \sigma x_1$, the last $\exp^*$ expression here allows us to introduce also a factor $\exp^*(-\sigma|x|^2)$ in the integrand. Because of (4.7), we can argue as in (4.8) to get $J_{1,1} \lesssim K_1(t, x, y)$.

In the integral $J_{1,2}$, we have $\frac{3}{4}\sigma_0 \leq \sigma \leq \frac{5}{4}\sigma_0 \leq \frac{5}{8}$ and so $1 - e^{-2s(\sigma)} \simeq s(\sigma) \simeq \sigma \simeq \sigma_0$ and $e^{s(\sigma)} \simeq 1$. Thus,

$$\begin{aligned} J_{1,2} &\simeq \int_{\frac{3}{4}\sigma_0}^{\frac{5}{4}\sigma_0} \frac{t}{\sigma_0^{(n+3)/2}} \exp^*\Big(-\frac{t^2}{\sigma_0}\Big) \exp^*\Big(-\frac{(\sigma-\sigma_0)^2 x_1^2 + |y'|^2}{\sigma_0}\Big)\, d\sigma \\ &\simeq t\Big(\frac{x_1}{x_1 - y_1}\Big)^{(n+3)/2} \exp^*\Big(-\frac{(t^2 + |y'|^2)x_1}{x_1 - y_1}\Big) \int_{\frac{3}{4}\sigma_0}^{\frac{5}{4}\sigma_0} \exp^*\Big(-\frac{x_1^3(\sigma-\sigma_0)^2}{x_1 - y_1}\Big)\, d\sigma, \end{aligned} \tag{4.13}$$

where we inserted the expression (4.2) for σ_0. The last integral, even extended to the whole line, is $O(((x_1 - y_1)/x_1^3)^{1/2})$. The $\exp^*$ expression preceding it is now estimated by a product of two factors. This leads to

$$\begin{aligned} J_{1,2} &\lesssim t\Big(\frac{x_1}{x_1 - y_1}\Big)^{(n+3)/2} \min\Big\{1, \Big(\frac{(t^2 + |y'|^2)x_1}{x_1 - y_1}\Big)^{-(n+2)/2}\Big\} \\ &\quad\times \exp^*\Big(-\frac{(t^2 + |y'|^2)x_1}{x_1 - y_1}\Big)\Big(\frac{x_1 - y_1}{x_1^3}\Big)^{1/2} \\ &\simeq K_2(t, x, y) \end{aligned}$$

for $x_1 = |x| \geq 1$.

The last integral in (4.13) is also $O(\sigma_0) = O((x_1-y_1)/x_1)$, and we get similarly

$$J_{1,2} \lesssim t\Big(\frac{x_1}{x_1-y_1}\Big)^{(n+1)/2} \min\Big\{1,\ \Big(\frac{(t^2+|y'|^2)x_1}{x_1-y_1}\Big)^{-(n+1)/2}\Big\} \exp^*\Big(-\frac{t^2x_1}{x_1-y_1}\Big)$$

$$\lesssim t\,\min\Big\{\Big(\frac{x_1}{x_1-y_1}\Big)^{(n+1)/2},\ \frac{1}{(t^2+|y'|^2)^{(n+1)/2}}\Big\} \exp^*(-t^2), \tag{4.14}$$

where we estimated the $\exp^*$ factor by means of the inequality $x_1/(x_1-y_1) > 1$. For $x_1 < 1$, one has $x_1 - y_1 < 1$ and so $(x_1/(x_1-y_1))^{(n+1)/2} \le (x_1-y_1)^{-(n+1)}$, and also $\exp^*(-t^2) \lesssim \exp^*(-t(1+|x|))$. As a result, $J_{1,2} \lesssim K_1(t,x,y)$.

To treat $J_{1,3}$, we split it into integrals over the intersection of $(\frac54\sigma_0, 1)$ with each of the intervals $(0,1/2]$ and $(1/2,1)$, and denote these by $J_{1,3}^{(1)}$ and $J_{1,3}^{(2)}$, respectively.

For $J_{1,3}^{(1)}$, we may assume that $\frac54\sigma_0 < \frac12$; otherwise $J_{1,3}^{(1)} = 0$. Since here $\sigma \in (\frac54\sigma_0, \frac12]$, we again have $1-e^{-2s(\sigma)} \simeq s(\sigma) \simeq \sigma$ and $e^{s(\sigma)} \simeq 1$. Further, $(\sigma-\sigma_0)x_1 \simeq \sigma x_1 \ge \sigma_0 x_1 = x_1 - y_1$. Thus $(\sigma-\sigma_0)x_1 \simeq \max\{\sigma x_1, x_1-y_1\}$, which implies (4.6), and the argument of (4.8) leads to $J_{1,3}^{(1)} \lesssim K_1(t,x,y)$.

Next, we estimate $J_{1,3}^{(2)}$. For $\max\{\frac54\sigma_0, \frac12\} < \sigma < 1$, we have $|\sigma-\sigma_0|x_1 \simeq \sigma x_1 \simeq x_1 \simeq y_1$ and so $|y-x+\sigma x|^2 \gtrsim |y|^2$. This means that (4.10) holds and, arguing as in (4.11), we conclude that $J_{1,3}^{(2)} \lesssim K_3(t,x,y)$. Altogether, we obtain (4.4) and hence (ii).

Finally, we consider (iii), where $y_1 \in (0, x_1/2)$ and $\sigma_0 \in (1/2,\,1)$. We split the integral in (4.1) into integrals over the intervals $(0, \sigma_0 - \frac{y_1}{4x_1})$, $[\sigma_0 - \frac{y_1}{4x_1}, \sigma_0 + \frac{y_1}{4x_1}]$ and $(\sigma_0 + \frac{y_1}{4x_1}, 1)$, and denote them by $J_{2,1}$, $J_{2,2}$ and $J_{2,3}$, respectively. Notice that $0 < \sigma_0 - \frac{y_1}{4x_1} < \sigma_0 + \frac{y_1}{4x_1} < 1$.

For $J_{2,1}$, we observe that $\sigma < \sigma_0 - \frac{y_1}{4x_1}$ corresponds to $s = s(\sigma) < \log\frac{x_1}{y_1} - \log\frac54$. For such s and σ, one has $e^{-s} > \frac{5y_1}{4x_1}$ and $|y-x+\sigma x| = |y-e^{-s}x| \simeq e^{-s}x_1 + |y'|$. With s as variable of integration, we have

$$J_{2,1} \simeq \int_0^{\log\frac{x_1}{y_1}-\log\frac54} \frac{t}{s^{3/2}} \exp^*\Big(-\frac{t^2}{s}\Big) \frac{\exp^*\Big(-\frac{e^{-2s}x_1^2+|y'|^2}{1-e^{-2s}}\Big)}{(1-e^{-2s})^{n/2}}\,ds\,.$$

Splitting the interval of integration here by intersecting it with $(0,\log 2]$ and $(\log 2,\infty)$, we obtain two integrals denoted $J_{2,1}^{(1)}$ and $J_{2,1}^{(2)}$. For $0 < s \le \log 2$, one has $1-e^{-2s} \simeq s \simeq \sigma$ and $e^{-s}x_1 \simeq x_1 \simeq x_1 - y_1$. This implies (4.6) and, arguing as before, we obtain $J_{2,1}^{(1)} \lesssim K_1(t,x,y)$.

If $\log 2 < s < \log\frac{x_1}{y_1} - \log\frac54$, then $1-e^{-s} \simeq 1$ and $e^{-s}x_1 \gtrsim y_1$, which implies (4.10) and then also $J_{2,1}^{(2)} \lesssim K_3$, as before. We have proved that $J_{2,1} \lesssim K_1 + K_3$.

For $J_{2,2}$, we integrate in s, getting

$$J_{2,2} \simeq \int_{\log\frac{x_1}{y_1}-\log\frac54}^{\log\frac{x_1}{y_1}+\log\frac43} \frac{t}{s^{3/2}} \exp^*\Big(-\frac{t^2}{s}\Big) \frac{\exp\Big(-\frac{|y-e^{-s}x|^2}{1-e^{-2s}}\Big)}{(1-e^{-2s})^{n/2}}\,ds\,.$$

Since now $x_1 > 2y_1$, we see that $\log \frac{x_1}{y_1} - \log \frac{5}{4} \gtrsim 1$, which implies that $s \simeq \log \frac{x_1}{y_1}$ and $1 - e^{-2s} \simeq 1$ in this integral. Let $\tau = \log \frac{x_1}{y_1} - s$, so that $-\log \frac{4}{3} \leq \tau \leq \log \frac{5}{4}$ and

$$|y - e^{-s}x| \simeq |y_1 - e^{-s}x_1| + |y'| = |(1 - e^{\tau})y_1| + |y'| \simeq |\tau|y_1 + |y'|.$$

It follows that

$$\begin{aligned} J_{2,2} &\simeq \frac{t}{(\log \frac{x_1}{y_1})^{3/2}} \exp^*\left(-\frac{t^2}{\log \frac{x_1}{y_1}}\right) \exp^*\left(-|y'|^2\right) \int_{-\log \frac{4}{3}}^{\log \frac{5}{4}} \exp^*\left(-\tau^2 y_1^2\right) d\tau \\ &\lesssim \frac{t}{(\log \frac{x_1}{y_1})^{3/2}} \exp^*\left(-\frac{t^2}{\log \frac{x_1}{y_1}}\right) \exp^*\left(-|y'|^2\right) \frac{1}{y_1} \simeq K_4(t,x,y) \end{aligned} \tag{4.15}$$

when $y_1 > 1$. If $y_1 \in (0,1]$, we control the integral in (4.15) by 1 and obtain

$$\begin{aligned} J_{2,2} &\simeq \frac{t}{(\log \frac{x_1}{y_1})^{3/2}} \exp^*\left(-\frac{t^2}{\log \frac{x_1}{y_1}}\right) \exp^*\left(-|y'|^2\right) \\ &\lesssim \min\{1,t\} \exp^*(-|y'|^2) \simeq K_3(t,x,y). \end{aligned} \tag{4.16}$$

In $J_{2,3}$, we have $s > \log \frac{x_1}{y_1} + \log \frac{4}{3} > \log 2$ and thus $y_1 - e^{-s}x_1 \simeq y_1$, which once more leads to (4.10) and $J_{2,3} \lesssim K_3$.

Summing up, we obtain (4.5) and (iii). □

Proof of Theorem 1.3. By differentiating with respect to t in (1.1), we have

$$t\,\partial_t P_t(x,y) = \frac{1}{2\pi^{(n+1)/2}} \int_0^\infty \frac{t}{s^{3/2}}\, e^{-\frac{t^2}{4s}} \left(1 - \frac{t^2}{2s}\right) \frac{e^{-\frac{|y-e^{-s}x|^2}{1-e^{-2s}}}}{(1-e^{-2s})^{n/2}}\, ds.$$

This expression is similar to that in (1.1), only with an extra factor $1 - t^2/2s$. Since

$$\left|1 - \frac{t^2}{2s}\right| e^{-\frac{t^2}{4s}} \lesssim \exp^*\left(-\frac{t^2}{s}\right),$$

we see that all our estimates for P_t in Proposition 4.1 remain valid for $|t\partial_t P_t|$.

For $i \in \{1, 2, \ldots, n\}$, we have

$$t\,\partial_{x_i} P_t(x,y) = \frac{1}{\pi^{(n+1)/2}} \int_0^\infty \frac{t}{s^{3/2}}\, e^{-\frac{t^2}{4s}}\, \frac{te^{-s}(y_i - e^{-s}x_i)}{1 - e^{-2s}}\, \frac{e^{-\frac{|y-e^{-s}x|^2}{1-e^{-2s}}}}{(1-e^{-2s})^{n/2}}\, ds.$$

Compared with (1.1), the integrand here has an extra factor

$$\frac{te^{-s}(y_i - e^{-s}x_i)}{1 - e^{-2s}} = \frac{t}{\sqrt{s}}\, \frac{\sqrt{s}\, e^{-s}}{\sqrt{1 - e^{-2s}}}\, \frac{y_i - e^{-s}x_i}{\sqrt{1 - e^{-2s}}}.$$

Since the middle factor to the right here is bounded, we can suppress the extra factor if we replace

$$e^{-\frac{t^2}{4s}}\, e^{-\frac{|y-e^{-s}x|^2}{1-e^{-2s}}} \quad \text{by} \quad \exp^*\left(-\frac{t^2}{s}\right) \exp^*\left(-\frac{|y - e^{-s}x|^2}{1 - e^{-2s}}\right)$$

in the integral. Thus

$$|t\partial_{x_i} P_t(x,y)| \lesssim \int_0^\infty \frac{t}{s^{3/2}} \exp^*\Big(-\frac{t^2}{s}\Big) \frac{\exp^*(-\frac{|y-e^{-s}x|^2}{1-e^{-2s}})}{(1-e^{-2s})^{n/2}}\, ds,$$

so the estimates for P_t are valid also for $|t\partial_{x_i} P_t|$. □

5. Proof of Theorem 1.4

Notice that

$$\partial_{x_1} P_t(x,y) = \frac{1}{\pi^{(n+1)/2}} \int_0^\infty \frac{t\, e^{-\frac{t^2}{4s}}}{s^{3/2}} \frac{e^{-s}}{\sqrt{1-e^{-2s}}} \frac{y_1 - e^{-s}x_1}{\sqrt{1-e^{-2s}}} \frac{e^{-\frac{|y-e^{-s}x|^2}{1-e^{-2s}}}}{(1-e^{-2s})^{n/2}}\, ds\,. \tag{5.1}$$

We consider the same cases (i), (ii) (iii) as in Proposition 4.1, and exactly as in the proof of that proposition, we split the integral into parts by splitting the interval of integration. The parts will again be denoted by J_1, J_2, $J_{2,1}^{(1)}$, etc. For all these parts except $J_{1,2}$ and $J_{2,2}$, we follow closely the arguments in Section 4; in particular we often use $\sigma = 1 - e^{-s}$ instead of s.

Since

$$\frac{|y-e^{-s}x|}{\sqrt{1-e^{-2s}}}\, e^{-\frac{|y-e^{-s}x|^2}{1-e^{-2s}}} \lesssim \exp^*\Big(-\frac{|y-e^{-s}x|^2}{1-e^{-2s}}\Big),$$

the absolute value of the integrand in (5.1) is controlled by

$$\frac{t\, e^{-\frac{t^2}{4s}}}{s^{3/2}} \frac{e^{-s}}{\sqrt{1-e^{-2s}}} \frac{\exp^*\Big(-\frac{|y-e^{-s}x|^2}{1-e^{-2s}}\Big)}{(1-e^{-2s})^{n/2}}.$$

Switching to integration with respect to σ, we get instead, since $d\sigma = e^{-s}ds$,

$$\frac{t\, e^{-\frac{t^2}{4s}}}{s^{3/2}} \frac{1}{\sqrt{1-e^{-2s}}} \frac{\exp^*\Big(-\frac{|y-x+\sigma x|^2}{1-e^{-2s}}\Big)}{(1-e^{-2s})^{n/2}},$$

where $s = s(\sigma) = \log(1/(1-\sigma))$. Compared with the integral treated in the proof of Proposition 4.1, we now have an extra factor which for $s < \log 2$, i.e., $\sigma < 1/2$, is controlled by $s^{-1/2} \simeq \sigma^{-1/2}$, and for $s > \log 2$ by e^{-s}.

For the integrals J_1, $J_{1,1}$, $J_{1,3}^{(1)}$ and $J_{2,1}^{(1)}$, we integrate in σ and argue as in Section 4. Because of the extra factor $\sigma^{-1/2}$, the exponent $(n+3)/2$ of σ in (5.1) will now be $(n+4)/2$ in the analogous estimates. As a result, the bound obtained will be $Z_1(t,x,y)$ instead of $K_1(t,x,y)$.

For J_2, $J_{1,3}^{(2)}$ and $J_{2,1}^{(2)}$ and $J_{2,3}$, we use s as variable of integration. Arguments similar to those in Section 4 show that the integrand is now dominated by

$$\frac{t}{s^{3/2}}\, e^{-\frac{t^2}{4s}}\, e^{-s} \exp^*(-|y_1 - e^{-s}x_1|^2) \exp^*(-|y|^2).$$

The interval of integration is $(\log 2, +\infty)$ or a subset of it. Since

$$\frac{t}{s^{3/2}} \exp^*\Big(-\frac{t^2}{s}\Big) \lesssim \min\{t, t^{-2}\},$$

the integrals considered are controlled by

$$\min\{t, t^{-2}\} \exp^*(-|y|^2) \int_{\log 2}^{\infty} e^{-s} \exp^*(-|y_1 - e^{-s}x_1|^2)\, ds \\ \lesssim \min\{t, t^{-2}\} \exp^*(-|y|^2) \min\Big\{1, \frac{1}{x_1}\Big\} \lesssim Z_3(t, x, y)\,.$$

It remains to estimate $J_{1,2}$ and $J_{2,2}$, in which y_1 is as in (ii) and (iii) of Proposition 4.1, respectively.

For $J_{1,2}$, we thus assume $y_1 \in [x_1/2, x_1)$. When $0 \le x_1 \le 1$, we can estimate $|J_{1,2}|$ as in (4.14). But now the four exponents $(n+1)/2$ will be replaced by $(n+2)/2$, and in the next step, we estimate $(x_1/(x_1 - y_1))^{(n+2)/2}$ by $(x_1 - y_1)^{-(n+2)}$. The result will be $|J_{1,2}| \lesssim Z_1(t, x, y)$.

When $x_1 > 1$, we shall estimate

$$J_{1,2} = C \int_{|\sigma-\sigma_0| \le \frac{1}{4}\sigma_0} \frac{t}{s(\sigma)^{3/2}} \exp\Big(-\frac{t^2}{4s(\sigma)}\Big) \frac{(\sigma-\sigma_0)x_1}{1 - e^{-2s(\sigma)}} \frac{e^{-\frac{|\sigma-\sigma_0|^2|x|^2 + |y'|^2}{1 - e^{-2s(\sigma)}}}}{(1 - e^{-2s(\sigma)})^{n/2}}\, d\sigma.$$

Here $\sigma_0 \le 1/2$, and $s(\sigma) \simeq \sigma \le 5/8$ in the integral. Let us make a change of variable $u = (\sigma - \sigma_0)x_1$. Then $\sigma = \sigma(u) = \sigma_0 + u/x_1$, and we write $s(u)$ for $s(\sigma(u))$ so that

$$s(u) = \log \frac{1}{1 - \sigma(u)} = \log \frac{1}{1 - \sigma_0 - u/x_1} = \log \frac{x_1}{y_1 - u}. \tag{5.2}$$

Thus

$$J_{1,2} = \frac{C}{x_1} \int_{|u| \le \frac{x_1 - y_1}{4}} \frac{t}{s(u)^{3/2}} \exp\Big(-\frac{t^2}{4s(u)}\Big) \frac{u}{1 - e^{-2s(u)}} \frac{e^{-\frac{u^2 + |y'|^2}{1 - e^{-2s(u)}}}}{(1 - e^{-2s(u)})^{n/2}}\, du \\ = \frac{C}{x_1} \int_{|u| \le \frac{x_1 - y_1}{4}} u F(s(u), u)\, du,$$

where for $\tau \in (0, \infty)$ and $w \in \mathbb{R}$,

$$F(\tau, w) = \frac{t}{\tau^{3/2}[1 - e^{-2\tau}]^{(n+2)/2}} \exp\Big(-\frac{t^2}{4\tau}\Big) \exp\Big(-\frac{w^2 + |y'|^2}{1 - e^{-2\tau}}\Big).$$

Notice that $F(\cdot, w) = F(\cdot, -w)$ for $w \in \mathbb{R}$. We can write

$$J_{1,2} = \frac{C}{x_1} \int_0^{\frac{x_1 - y_1}{4}} u\, [F(s(u), u) - F(s(-u), u)]\, du, \tag{5.3}$$

and here

$$|F(s(u),u) - F(s(-u),u)| \le |s(u)-s(-u)| \sup_{s(-u)<\tau<s(u)} |\partial_\tau F(\tau,u)| \,. \tag{5.4}$$

From (5.2) and the mean value theorem, we deduce that for $0 < u \le (x_1 - y_1)/4$,

$$|s(u)-s(-u)| \le 2u \sup_{-u<v<u} \frac{1}{y_1 - v} \,. \tag{5.5}$$

With $s(-u) < \tau < s(u)$, we have

$$\partial_\tau F(\tau,u) = F(\tau,u)\Big[-\frac{3}{2\tau} - \frac{(n+2)e^{-2\tau}}{1-e^{-2\tau}} + \frac{t^2}{4\tau^2} + \frac{2(u^2+|y'|^2)e^{-2\tau}}{(1-e^{-2\tau})^2}\Big].$$

Here $(n+2)e^{-2\tau}/(1-e^{-2\tau}) \lesssim \tau^{-1}$, and $\frac{t^2}{4\tau^2}\exp\big(-\frac{t^2}{4\tau}\big) \lesssim \tau^{-1}\exp^*\big(-\frac{t^2}{\tau}\big)$. Further,

$$\begin{aligned}\frac{(u^2+|y'|^2)e^{-2\tau}}{(1-e^{-2\tau})^2} \exp\Big(-\frac{u^2+|y'|^2}{1-e^{-2\tau}}\Big) &\lesssim \frac{e^{-2\tau}}{1-e^{-2\tau}} \exp^*\Big(-\frac{u^2+|y'|^2}{1-e^{-2\tau}}\Big)\\ &\lesssim \frac{1}{\tau}\exp^*\Big(-\frac{u^2+|y'|^2}{1-e^{-2\tau}}\Big),\end{aligned}$$

and so

$$|\partial_\tau F(\tau,u)| \lesssim \frac{t}{\tau^{5/2}[1-e^{-2\tau}]^{(n+2)/2}} \exp^*\Big(-\frac{t^2}{\tau}\Big)\exp^*\Big(-\frac{u^2+|y'|^2}{1-e^{-2\tau}}\Big). \tag{5.6}$$

Recall that $x_1/2 \le y_1 < x_1$. In (5.5) we have $|v| < u \le (x_1-y_1)/4 < y_1/2$ so that $y_1 - v \simeq y_1 \simeq x_1$, and we conclude that

$$|s(u)-s(-u)| \lesssim \frac{u}{x_1}\,.$$

Since all occurring values of $s(\pm u)$ and τ satisfy $s(\pm u) \simeq \tau \simeq \sigma_0$, (5.6) implies

$$\sup_{s(-u)<\tau<s(u)} |\partial_\tau F(\tau,u)| \lesssim \frac{t}{\sigma_0^{(n+7)/2}} \exp^*\Big(-\frac{t^2}{\sigma_0}\Big)\exp^*\Big(-\frac{u^2+|y'|^2}{\sigma_0}\Big).$$

Inserting the last two estimates in (5.4), we obtain

$$|F(s(u),u) - F(s(-u),u)| \lesssim \frac{u}{x_1}\,\frac{t}{\sigma_0^{(n+7)/2}} \exp^*\Big(-\frac{t^2}{\sigma_0}\Big)\exp^*\Big(-\frac{u^2+|y'|^2}{\sigma_0}\Big),$$

which combined with (5.3) implies that

$$\begin{aligned}|J_{1,2}| &\lesssim \frac{t}{x_1^2\,\sigma_0^{(n+7)/2}} \exp^*\Big(-\frac{t^2}{\sigma_0}\Big)\int_0^{\frac{x_1-y_1}{4}} u^2 \exp^*\Big(-\frac{u^2+|y'|^2}{\sigma_0}\Big)\,du\\ &\lesssim \frac{t}{x_1^2}\,\frac{1}{\sigma_0^{(n+4)/2}}\exp^*\Big(-\frac{t^2+|y'|^2}{\sigma_0}\Big)\\ &\lesssim \frac{t}{x_1^2}\,\frac{1}{\sigma_0^{(n+4)/2}}\min\Big\{1,\Big(\frac{\sigma_0}{t^2+|y'|^2}\Big)^{(n+4)/2}\Big\}\exp^*\Big(-\frac{t^2+|y'|^2}{\sigma_0}\Big).\end{aligned}$$

Since $\sigma_0 = (x_1-y_1)/x_1$, we see that the last expression amounts to $Z_2(t,x,y)$.

We shall finally estimate $J_{2,2}$, in which $y_1 \in (0, x_1/2)$. When $0 < y_1 \le 1$, we have an upper estimate for $|J_{2,2}|$ like (4.16), but now with an extra factor $\exp(-\log \frac{x_1}{y_1}) \lesssim (1+x_1)^{-1}$ coming from e^{-s}; recall that $|s - \frac{x_1}{y_1}| \lesssim 1$. Thus

$$|J_{2,2}| \lesssim \frac{t}{(\log \frac{x_1}{y_1})^{3/2}} \exp^*\Big(- \frac{t^2}{\log \frac{x_1}{y_1}}\Big) \frac{1}{1+x_1} \exp^*(-|y'|^2).$$

The first $\exp^*$ factor here is controlled by

$$\min\Big\{1, \Big(\frac{t^2}{\log(x_1/y_1)}\Big)^{-3/2}\Big\}.$$

Since $\log \frac{x_1}{y_1} \gtrsim 1$, this is seen to lead to $|J_{2,2}| \lesssim Z_3(t,x,y)$.

When $y_1 > 1$, we estimate $J_{2,2}$ by modifying the preceding argument for $J_{1,2}$. Instead of (5.3), we get now

$$J_{2,2} = \frac{C}{x_1} \int_0^{y_1/4} u\,[F(s(u),u) - F(s(-u),u)]\, du\,, \tag{5.7}$$

and we still have (5.2), (5.4), (5.5) and (5.6). Since now $0 \le u \le y_1/4$ in (5.5), it follows that $y_1 - v \simeq y_1$ for any $|v| < u$, and thus

$$|s(u) - s(-u)| \lesssim \frac{u}{y_1}\,.$$

In the estimate for $J_{2,2}$ in Section 4, we saw that $s \simeq \log \frac{x_1}{y_1}$, which now means that $s(u) \simeq s(-u) \simeq \log \frac{x_1}{y_1}$, and $\log \frac{x_1}{y_1} > \log 2$. In (5.6), we thus have $\tau \simeq \log \frac{x_1}{y_1}$ so that $1 - e^{-2\tau} \simeq 1$, which implies that

$$\sup_{s(-u)<\tau<s(u)} |\partial_\tau F(\tau,u)| \lesssim t\Big(\log \frac{x_1}{y_1}\Big)^{-5/2} \exp^*\Big(-\frac{t^2}{\log \frac{x_1}{y_1}}\Big) \exp^*(-u^2 - |y'|^2).$$

Inserting the last two estimate in (5.4), we see that

$$\begin{aligned}|F(s(u),u) - &F(s(-u),u)| \\ &\lesssim \frac{t\,u}{y_1}\Big(\log \frac{x_1}{y_1}\Big)^{-5/2} \exp^*\Big(-\frac{t^2}{\log \frac{x_1}{y_1}}\Big) \exp^*\big(-u^2 - |y'|^2\big)\,,\end{aligned}$$

which combined with (5.7) implies that

$$\begin{aligned}|J_{2,2}| &\lesssim \frac{t}{x_1\,y_1}\Big(\log \frac{x_1}{y_1}\Big)^{-5/2} \exp^*\Big(-\frac{t^2}{\log \frac{x_1}{y_1}}\Big) \exp^*\big(-|y'|^2\big) \int_0^{\frac{y_1}{4}} u^2 \exp^*\big(-u^2\big)\, du \\ &\lesssim \frac{t}{x_1\,y_1}\Big(\log \frac{x_1}{y_1}\Big)^{-5/2} \exp^*\Big(-\frac{t^2}{\log \frac{x_1}{y_1}}\Big) \exp^*\big(-|y'|^2\big) \simeq Z_4(t,x,y)\,.\end{aligned}$$

Theorem 1.4 is proved. □

6. Sharpness arguments

We let $K_j(t,x,y)$, $j=1,2,3,4$, be as in Theorem 1.2. Let $\mathbb{R}_+=(0,\infty)$.

Theorem 6.1. (a) *The estimate* $P_t(x,y)\simeq K_1(t,x,y)$ *holds uniformly in the set*

$$E_1=\Big\{(t,x,y)\in\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^n:\\ |x|>1,\ \ x\cdot y>0,\ \ t^2|x|<|x|-|y_x|<\frac{1}{4|x|},\ \ |y_x'|<|x|-|y_x|\Big\}.$$

Similarly, $P_t(x,y)\simeq K_2(t,x,y)$ *uniformly in*

$$E_2=\Big\{(t,x,y)\in\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^n:\ |x|>1,\ x\cdot y>0,\\ t|x|>1,\ t^2|x|<|x|-|y_x|<|x|/4,\ |y_x'|<\sqrt{(|x|-|y_x|)/|x|}\Big\},$$

and $P_t(x,y)\simeq K_3(t,x,y)$ *uniformly in*

$$E_3=\{(t,x,y)\in\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^n:\ \ t>1,\ \ |x|<1,\ \ |y|<1\}.$$

Finally, $P_t(x,y)\simeq K_4(t,x,y)$ *uniformly in*

$$E_4=\Big\{(t,x,y)\in\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^n:|x|>e^{16},\\ t=\sqrt{\log|x|}/2,\ \ |x|^{2/3}\le|y_x|\le|x|^{3/4},\ \ |y_x'|<1\Big\}.$$

(b) *In the estimate in Theorem* 1.2*, none of the terms* $K_i(t,x,y)$, $i=1,2,3,4$, *can be suppressed.*

Proof. To prove (a), we only need to consider $x=(x_1,0,\dots,0)$ with $x_1\ge 0$ and write $y=(y_1,y')$. We shall use several estimates from the proof of Proposition 4.1. Observe that points of E_1 and E_2 belong to (ii) of Proposition 4.1 and satisfy $t<1/2$.

Assume $(t,x,y)\in E_1$. Then

$$x_1>1,\quad t^2x_1<x_1-y_1<x_1^{-1}/4\quad\text{and}\quad |y'|<x_1-y_1.$$

Transforming variables in the integral in (4.12), we get

$$J_{1,1}\simeq\frac{t}{(t^2+|x-y|^2)^{(n+1)/2}}\int_0^B\frac{1}{u^{(n+3)/2}}\exp^*\Big(-\frac{1}{u}\Big)\,du\,,$$

with $B=3(x_1-y_1)/(4x_1(t^2+|x-y|^2))$. One easily verifies that $B^{-1}\lesssim 1$, so that the value of the integral here stays away from 0. Since also $t(1+|x|)\lesssim 1$, it follows that $J_{1,1}\simeq t/(t^2+|x-y|^2)^{(n+1)/2}\simeq K_1(t,x,y)$. Consequently, $P_t(x,y)\gtrsim K_1(t,x,y)$ in E_1.

To obtain the converse inequality, we notice that Proposition 4.1(ii) applies, and its proof shows that $P_t(x,y)\lesssim J_{1,1}+J_{1,2}+J_{1,3}\lesssim K_1(t,x,y)+J_{1,2}+K_3(t,x,y)$.

The inequalities (4.14) now imply that $J_{1,2} \lesssim K_1(t,x,y)$, since $x_1/(x_1-y_1) < (x_1-y_1)^{-2}$ in E_1. Further,

$$K_3(t,x,y) \simeq t\exp^*(-|y|^2) \lesssim t\exp^*(-|x|^2) \lesssim K_1(t,x,y).$$

We conclude that $P_t(x,y) \simeq K_1(t,x,y)$ in E_1.

Now assume $(t,x,y) \in E_2$ so that

$$x_1 > 1, \quad tx_1 > 1, \quad t^2 x_1 < x_1 - y_1 < x_1/4 \quad \text{and} \quad |y'| < \sqrt{(x_1-y_1)/x_1}\,.$$

Then $K_2(t,x,y) \simeq t x_1^{n/2}(x_1-y_1)^{-(n+2)/2}$. Since $x_1(x_1-y_1) > t^2x_1^2 > 1$, a simple scaling shows that the second integral in (4.13) has order of magnitude $((x_1-y_1)/x_1^3)^{1/2}$. The $\exp^*$ factor preceding it is essentially 1, and we conclude that

$$J_{1,2} \simeq t\Big(\frac{x_1}{x_1-y_1}\Big)^{(n+3)/2}\Big(\frac{x_1-y_1}{x_1^3}\Big)^{1/2} \simeq K_2(t,x,y).$$

Thus $P_t(x,y) \gtrsim K_2(t,x,y)$. In E_2 one also has $K_1(t,x,y) \lesssim t/(x_1-y_1)^{n+1}$ and $K_3(t,x,y) \lesssim t\exp^*(-x_1^2)$, and these quantities are controlled by $K_2(t,x,y)$. Proposition 4.1(ii) then shows that $P_t(x,y) \lesssim K_2(t,x,y)$. Thus $P_t(x,y) \simeq K_2(t,x,y)$ in E_2.

Assume next that $(t,x,y) \in E_3$ so that $K_3(t,x,y) \simeq 1$. Now (4.11) is sharp and leads to $J_2 \simeq 1 \simeq K_3$. Also, $K_2(t,x,y) = K_4(t,x,y) = 0$, and $K_1(t,x,y) \lesssim t^{-n} \lesssim 1$. It follows that $P_t(x,y) \simeq K_3(t,x,y)$ in E_3.

Finally let $(t,x,y) \in E_4$. Then the estimate (4.15) is sharp since $y_1 > 1$, and so $J_{2,2} \simeq K_4(t,x,y)$. Further, one verifies that $K_4(t,x,y) \gtrsim x_1^{-3/4}(\log x_1)^{-1}$ and also that $K_1(t,x,y)$ and $K_3(t,x,y)$ are controlled by $\exp^*(-x_1) \lesssim K_4(t,x,y)$. It now follows from Proposition 4.1(iii) that $P_t(x,y) \simeq K_4(t,x,y)$ in E_4.

This completes the arguments for (a).

We prove (b) by finding for each $\varepsilon > 0$ and $i = 1,2,3,4$ a nonempty subset $\tilde{E}_i$ of E_i in which $K_j < \varepsilon P_t$ for $j \neq i$. In the proof below, we fix ε and denote by C_ε various large positive constants which may depend on ε.

Let

$$\widetilde{E}_1 = \Big\{(t,x,y) \in E_1 : \ |x| > C_\varepsilon, \ \ t = \frac{1}{|x|^2}, \ \ \frac{1}{|x|^2} < |x| - |y_x| < \frac{2}{|x|^2}\Big\}.$$

In this set, $P_t(t,x,y) \simeq K_1(t,x,y) \simeq |x|^{2n}$ but also $K_2(t,x,y) \simeq |x|^{3n/2}$ and $K_3(t,x,y) \lesssim 1$, whereas $K_4(t,x,y)$ vanishes. A suitable choice of C_ε yields the desired inequalities.

In a similar way, we define

$$\widetilde{E}_2 = \big\{(t,x,y) \in E_2 : \ |x| > C_\varepsilon, \ \ t = |x|^{-1/2}, \ \ 1 < |x| - |y_x| < 2\big\},$$

and it is enough to observe that in this set $P_t(t,x,y) \simeq K_2(t,x,y) \simeq |x|^{(n-1)/2}$, but $K_1(t,x,y) \lesssim \exp^*(-|x|^{1/2})$ and $K_3(t,x,y) \lesssim \exp^*(-|x|^2)$ and $K_4(t,x,y) = 0$.

The next set is

$$\widetilde{E}_3 = \{(t,x,y) \in E_3 : \; t > C_\varepsilon\},$$

in which $P_t(t,x,y) \simeq K_3(t,x,y) \simeq 1$ but $K_1(t,x,y) \lesssim t^{-n}$ and $K_2(t,x,y) = K_4(t,x,y) = 0$.

Finally,

$$\widetilde{E}_4 = \{(t,x,y) \in E_4 : \; |x| > C_\varepsilon\}.$$

To compare the kernels $K_i(t,x,y)$ on this set, it is enough to consider the last part of the proof of (a).

This ends the proof of (b) and that of the theorem. □

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