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On L^p-improving measures

Anthony H. Dooley, Kathryn E. Hare and Maria Roginskaya

Abstract. We give criteria for establishing that a measure is L^p-improving. Many Riesz product measures and Cantor measures satisfy this criteria, as well as certain Markov measures.

1. Introduction

A measure μ on a compact abelian group G is said to be L^p-improving if there are real numbers $q > p$ such that μ acts by convolution as a bounded operator from $L^p(G)$ to $L^q(G)$. Since every measure acts as a bounded operator from L^1 to L^1 and from L^∞ to L^∞, it follows by an interpolation argument that if μ is L^p-improving, then for every $1 < p < \infty$ there is some $q > p$ such that μ maps L^p to L^q boundedly.

While it is clear that Haar measure and, more generally, any absolutely continuous measure with a Radon–Nikodym derivative in $L^{1+\varepsilon}$, for some $\varepsilon > 0$, is L^p-improving, a number of studies have displayed singular measures that are also L^p-improving. For instance, Stein in [11] noted that any measure on the circle group satisfying $|\widehat{\mu}(n)| = O(|n|^{-\varepsilon})$ for some $\varepsilon > 0$ has this property. More interestingly, there are measures whose Fourier transforms do not tend to zero that also have this property. Bonami [2] and Ritter [9], for example, showed that most Riesz product measures have the L^p-improving property. Using a seemingly different approach, Christ [3], extending work of [1] and [8], proved that Cantor measures on Cantor sets with ratios of dissection bounded away from zero are L^p-improving.

In [7], one of the authors characterized L^p-improving measures in terms of the 'size' of the level sets of the Fourier transform, where size was quantified by the concept of $\Lambda(p)$ sets. With this characterization it is easy to see that Stein's example and Ritter and Bonami's Riesz product measures are L^p-improving. It is more difficult to apply this characterization to study Cantor measures. Our objective here is to understand the commonality between these examples of L^p-improving measures. The approach we derive was heavily influenced by Bonami's work. It compliments that of [7], replacing $\Lambda(p)$ sets by more flexible notions.

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All the above mentioned papers have an important combinatorial component, in addition to analytic estimates. The analytic techniques are based on the structure of certain orthogonal subspaces of $L^2(G)$. The purpose of this paper is to encapsulate this structure in a common framework and demonstrate that the L^p-improving property holds whenever this framework is in place.

Here we show that the L^p-improving properties of many Riesz product measures and Cantor measures are special cases of a single theorem. Indeed, with our methods we can obtain the full range of Christ's results for Cantor measures. In the case of Riesz product measures, our hypotheses are more restrictive than those imposed by Ritter and, unlike Bonami, we do not obtain the sharp choices of p, q.

From the same theorem we also deduce the new result that certain Markov measures on infinite product spaces are L^p-improving. These generalizations of product measures were studied in [4] and [5], and are of interest as all G-measures are equivalent to Markov measures.

The structural combinatorial framework is outlined in section 2. In section 3 we prove the needed analytic estimates, and in section 4 see how these apply to Riesz product measures, Cantor measures and Markov measures with suitable properties.

2. Structural framework

We begin by introducing terminology that will be used throughout.

2.1. Index set

By an *index set*, I, we will mean a collection of indices with the following properties:

1) The indices are partitioned into a disjoint collection of subsets of I called *categories*. We will write $C(\alpha)$ for the category containing the index $\alpha \in I$.
2) There is a map $g : I \to \{0, 1, 2, \dots\}$ called the *generation map*.
3) Associated to each $\alpha \in I$ is a subset $R(\alpha) \subseteq I$, known as its set of *roots*. If $g(\alpha) = 0$, then $R(\alpha)$ is empty. Otherwise, if $\beta \in R(\alpha)$, then $g(\beta) < g(\alpha)$. We require that two members of the same category have no roots in common, i.e., if $C(\alpha) = C(\beta)$, then $R(\alpha) \cap R(\beta)$ is empty. Furthermore, at most one member of each category has no roots. In particular, at most one member of each category can be of generation zero.

Example 2.1. An important example is what we will call the 0-1 index set. This index set I will consist of the infinite sequences $\alpha = (\alpha(j))_{j=1}^{\infty}$, with $\alpha(j) = 0, 1$, and only finitely many $\alpha(j) = 1$. The generation of such a sequence α will be the number coordinates equal to 1, i.e., $g(\alpha) = \sum_j \alpha(j)$. In particular, $g(\alpha) = 0$ if and only if α is the sequence of all 0's. The categories can be labelled by the non-negative integers, with category C_k consisting of the set of sequences whose last 1 is in coordinate k; C_0 consisting of the sequence of all 0's. If $\alpha \in C_k$ with $k \neq 0$, then $R(\alpha)$ will be the (unique) sequence whose coordinates all agree with those of α, except for coordinate k. Obviously, if $\beta \in R(\alpha)$, then $g(\beta) = g(\alpha) - 1$.

Example 2.2. A second example we call the triadic index set. In this case, the index set will consist of the set of triadic intervals in $\mathbb{Z}$, meaning the intervals of the form $\alpha = [m3^k, (m+1)3^k) \cap \mathbb{Z}$ where $m, k \in \mathbb{Z}$, $k \geq 0$. The generation of this α is k. The roots of $\alpha = [m3^k, (m+1)3^k) \cap \mathbb{Z}$ are the three triadic subintervals of generation $k-1$ contained in α. The categories are singletons.

2.2. Orthogonality properties

Suppose that G is a compact abelian group. By $L^p(G)$ we will mean the usual L^p spaces with respect to the normalized Haar measure on G.

Assume we have an index set I, as above, and a family of closed, translation invariant subspaces $B_\alpha \subseteq L^2(G)$, $\alpha \in I$. We will say that $\Lambda \subseteq I$ satisfies the *4-orthogonality condition*[1] if whenever $\alpha_1, \dots, \alpha_4 \in \Lambda$, the pairs $(g(\alpha_j), C(\alpha_j))$ are pairwise disjoint and $f_j \in B_{\alpha_j}$, then

$$\int f_1\, f_2\, \overline{f_3\, f_4} = 0.$$

Suppose there is some integer N such that $\Lambda \subseteq \{\alpha \in I : g(\alpha) \leq N\}$. We will say that Λ satisfies the *strong orthogonality condition of order* J if $\Lambda = \bigcup_{j=1}^{J} \Lambda_j$, where: (i) each set Λ_j satisfies the 4-orthogonality condition; (ii) for each j, the subspaces B_α, $\alpha \in \Lambda_j$, are mutually orthogonal in $L^2(G)$; (iii) for each category C and non-negative integer $n \leq N$, let $\Lambda_j^{C,n}$ be the set of all $\alpha \in \Lambda_j$ belonging to the category C and of generation n. Then the set

$$\bigcup\nolimits_{\alpha \in \Lambda_j^{C,n}} R(\alpha)$$

satisfies the strong orthogonality condition of order J.

When $N = 0$, Λ satisfies the strong orthogonality condition of order J if conditions (i) and (ii) are met. This is the base case for the inductive definition since roots are of strictly lower generation.

We remark that if Λ satisfies the strong orthogonality condition of order J, so does any subset of Λ.

Here is an example. The notation $\mathbb{T}$ denotes the circle group, $[0, 1]$.

Lemma 2.3. *Let $\{n_j\}_{j=1}^{\infty}$ be a lacunary sequence of integers with* $\inf n_{j+1}/n_j \geq 3$. *Let I be the* 0-1 *index set. For $\alpha = (\alpha(j))_{j=1}^{\infty} \in I$, let*

$$B_\alpha = \operatorname{span}\Big\{ \exp 2\pi i x \sum_j \varepsilon_j n_j \alpha(j) : \varepsilon_j = \pm 1 \Big\} \subseteq L^2(\mathbb{T}).$$

For any $N \geq 0$, the set $\Lambda_N = \{\alpha \in I : g(\alpha) = N\}$ satisfies the strong orthogonality condition of order 2.

Proof. The lacunarity condition ensures that $\sum_j \varepsilon_j n_j \alpha(j) = \sum_j \varepsilon'_j n_j \alpha'(j)$ with $\varepsilon_j, \varepsilon'_j = \pm 1$ only if $\alpha(j) = \alpha'(j)$ for all j, that is, $\alpha = \alpha'$. Thus the spaces B_α are orthogonal.

[1] We could speak more generally speak about the $2q$-orthogonality condition, but '4' will suffice for our application.

Decompose Λ_N as $\Lambda_N^e \cup \Lambda_N^o$, where Λ_N^e consists of the sequences which belong to a category labelled by an even integer and Λ_N^o consists of those in categories labelled by an odd integer.

If $\alpha_1, \ldots, \alpha_4 \in \Lambda_N^e$ and the tuples $(g(\alpha_i), C(\alpha_i))$ are pairwise disjoint, then since $g(\alpha_i) = N$ for all α_i, the numbers $h_i = C(\alpha_i)$ must be distinct even integers. Without loss of generality, assume $h_1 = \max(h_1, h_2, h_3, h_4)$. Let $\alpha_i = (\alpha_i(j))_j$. The fact that $n_{j+1}/n_j \geq 3$ again implies that for any choice of $\varepsilon_{i,j} = \pm 1$,

$$\Big|\sum_{i,j} \varepsilon_{i,j} n_j \alpha_i(j)\Big| > n_{h_1} - 3 \sum_{j<h_1-1} n_j > \frac{1}{2} n_{h_1} > 0.$$

This shows that if $f_j \in B_{\alpha_j}$, then the 0'th Fourier coefficient of $f_1 f_2 \overline{f_3 f_4}$ equals 0 and therefore $\int f_1 f_2 \overline{f_3 f_4} = 0$. A similar statement holds for Λ_N^o.

As Λ_0 satisfies the strong orthogonality condition of order 2, and the roots of any $\alpha \in \Lambda_N$ with $N > 0$ are of generation $N-1$, the lemma follows by induction. □

2.3. Factoring map

Our main technical result, Lemma 3.3, will be proven by an induction argument (on the generations). One step in doing this is to have a procedure to replace convolution on one generation, with a sum of convolutions on an earlier generation. This is the purpose of the associated factoring map, that we introduce next.

Given an index set I and subspaces $B_\alpha \subseteq L^2(G)$ for $\alpha \in I$, we choose functions $\phi_\alpha \in B_\alpha$ for each $\alpha \in I$. By an *associated factoring map* we mean a family of pairs of linear operators, $\{T^C = (U_j^C, V_j^C)_{j=1}^N\}_C$, $U_j^C : L^4 \to L^4$, $V_j^C : L^2 \to L^2$, such that for each index α in category C and for any $f \in L^2$,

$$f * \phi_\alpha = \sum_{j=1}^{N} U_j^C \Big(V_j^C(f) * \sum_{\beta \in R(\alpha)} \phi_\beta \Big), \tag{2.1}$$

where the sum over the empty set is 1. (Note the choice of N can vary with C.) By the *norm* of the factoring map $\{T^C\}_C$ we will mean $\sup_C \sum_{j=1}^N \|U_j^C\|_{4,4} \|V_j^C\|_{2,2}$, where $\|\cdot\|_{p,p}$ denotes the operator norm as a map from L^p to L^p. (We use 'norm' only as suggestive terminology and will not be concerned with whether or not this is truly a norm.)

We remark that the choice of functions in a particular application will depend on the measure we are studying.

Example 2.4. Assume $n_{j+1}/n_j \geq 3$. Choose the 0-1 index set and the sets B_α as defined in Lemma 2.3. For $\alpha = (\alpha(j))_j \in I$, take

$$\phi_\alpha = \prod_{j=1}^{\infty} (\cos 2\pi n_j x)^{\alpha(j)} .$$

Define $T^{C_h} = (U_j^{C_h}, V_j^{C_h})_{j=1,2}$, where $U_1^{C_h}, U_2^{C_h}$ are multiplication by $(1/2)e^{2\pi i h x}$ and $(1/2)e^{-2\pi i h x}$ respectively, and $V_1^{C_h}$, $V_2^{C_h}$ are multiplication by $e^{\pm 2\pi i h x}$. This is an associated factoring map and its norm is 1. It will be important in the Riesz product example of section 4.

Example 2.5. Let I denote the triadic index set. For $\alpha \in I$, let $B_\alpha = \{f : \operatorname{supp}\widehat{f} \subseteq \alpha\}$ and let ϕ_α be the L^2 function whose Fourier transform is 1 on α and 0 otherwise. For all categories C, take $N = 1$ and $U_1^C = V_1^C = \mathrm{Id}$. When $g(\alpha) > 0$, $\bigcup_{\beta \in R(\alpha)} \beta = \alpha$, thus $U_1(V_1(f) * \sum_{\beta \in R(\alpha)} \phi_\beta) = f * \phi_\alpha$. The norm of this factoring map is also 1.

3. Analytic estimates

3.1. A variation on the $\Lambda(p)$ set concept

Let $p > 2$. A subset E of $\widehat{G}$, the dual group of G, is called a $\Lambda(p)$ set if there is a constant K such that $\|f\|_p \le K \|f\|_2$ whenever $\operatorname{supp}\widehat{f} \subseteq E$. The infimum of such constants is called the $\Lambda(p)$ constant of E. In [7] it was shown that μ is an L^p-improving measure if and only if the sets $\{\gamma \in \widehat{G} : |\widehat{\mu}(\gamma)| > \varepsilon\}$ are $\Lambda(p)$ sets for all $p > 2$ and the $\Lambda(p)$ constants have a suitable growth rate in terms of ε and p.

It is known ([10], [6]) that if $E = \{n_j\} \subseteq \mathbb{Z}$ and

$$\int_{\mathbb{T}} \exp 2\pi i x(n_a + n_b - n_c - n_d)\, dx = 0$$

whenever n_a, n_b, n_c, n_d are distinct integers, then E is a $\Lambda(4)$ set. Hence

$$\Big\| \sum_j a_j\, e^{2\pi i n_j x} \Big\|_4 \le K_0 \Big(\sum_j |a_j|^2 \Big)^{1/2} = K_0 \Big(\sum_j \big\| a_j\, e^{2\pi i n_j x} \big\|_4^2 \Big)^{1/2}$$

for the $\Lambda(4)$ constant, K_0. The following lemma, whose hypotheses inspired the 4-orthogonality condition, is a variation on this.

Lemma 3.1. *Suppose $\{F_j\}$ is a finite set of integrable functions on G with the property that for any set of distinct indices, a, b, c, d, the integral $\int_G F_a F_b \overline{F_c F_d}$ vanishes. Then*

$$\Big\| \sum_j F_j \Big\|_4 \le 3 \Big(\sum_j \|F_j\|_4^2 \Big)^{1/2}.$$

Proof. Let $f = \sum F_j$. The hypothesis ensures that

$$\|f\|_4^4 = \sum_{a,b,c,d} \int F_a\, F_b\, \overline{F_c\, F_d}$$
$$= \sum_a \int |F_a|^4 + \sum_{a \ne b} \int |F_a|^2\, |F_b|^2 + \sum_{a \ne b} \int F_a^2\, \overline{F_b^2}$$
$$+ \sum_{a; c \ne d} \int F_a^2\, \overline{F_c\, F_d} + \overline{F_a^2}\, F_c\, F_d + |F_a|^2\, F_c\, \overline{F_d}. \tag{3.1}$$

Holder's inequality gives $|\int F_a^2 \overline{F_b^2}| \leq \int |F_a|^2 |F_b|^2 \leq \|F_a\|_4^2 \|F_b\|_4^2$, thus

$$2\Big(\sum_a \|F_a\|_4^2\Big)^2 = 2\sum_{a,b} \|F_a\|_4^2 \|F_b\|_4^2$$
$$\geq \sum_a \int |F_a|^4 + \sum_{a\neq b} \int |F_a|^2 |F_b|^2 + \Big|\sum_{a\neq b} \int F_a^2 \overline{F_b^2}\Big|.$$

Next, consider

$$\Big|\sum_{a;c\neq d} \int F_a^2 \overline{F_c F_d}\Big| = \Big|\sum_{a,c} \int F_a^2 \overline{F_c}\,(\overline{f} - \overline{F_c})\Big| = \Big|\sum_a \int F_a^2 \overline{f^2} - \sum_{a,c} \int F_a^2 \overline{F_c^2}\Big|$$
$$\leq \sum_a \int |F_a|^2 |f|^2 + \sum_{a,c} \int |F_a|^2 |F_c|^2$$
$$\leq \sum_a \|F_a\|_4^2 \|f\|_4^2 + \Big(\sum_a \|F_a\|_4^2\Big)^2.$$

Now use the elementary inequality, $xy \leq x^2/5 + 5y^2/4$, which holds whenever $x, y \geq 0$, to get

$$\sum_a \|F_a\|_4^2 \|f\|_4^2 \leq \frac{1}{5}\|f\|_4^4 + \frac{5}{4}\Big(\sum_a \|F_a\|_4^2\Big)^2.$$

Thus

$$\Big|\sum_{a;c\neq d} \int F_a^2 \overline{F_c F_d}\Big| \leq \frac{1}{5}\|f\|_4^4 + \frac{9}{4}\Big(\sum_a \|F_a\|_4^2\Big)^2.$$

A similar argument with the other terms in (3.1) gives that

$$\|f\|_4^4 \leq \frac{3}{5}\|f\|_4^4 + \frac{35}{4}\Big(\sum_a \|F_a\|_4^2\Big)^2.$$

Now simplify. □

3.2. Main results

Recall that a linear map M on $L^2(G)$ is called a *multiplier* if $M(f * g) = M(f) * g = f * M(g)$. Equivalently, M is a multiplier if and only if there are bounded, complex numbers M_γ such that $\widehat{M(f)}(\gamma) = M_\gamma \widehat{f}(\gamma)$ for all $\gamma \in \widehat{G}$ and $f \in L^2(G)$. An example of a multiplier is the action of convolution by a measure μ; in this case $M_\gamma = \widehat{\mu}(\gamma)$ for all $\gamma \in \widehat{G}$.

Let $\rho < 1$. A map M is called *ρ-contractive on $B \subseteq L^2$* if $\|M(f)\|_2 \leq \rho \|f\|_2$ for all $f \in B$. An example of a ρ-contractive multiplier is multiplication by the constant ρ.

Definition 3.2. A multiplier M is said to be L^p-improving if there are real numbers $1 < p < q < \infty$ such that M extends to a bounded operator from $L^p(G)$ to $L^q(G)$. If the multiplier is convolution by the measure μ, we say μ is L^p-improving.

Assumptions and terminology. For the remainder of this section we will assume that I is an index set, B_α for $\alpha \in I$ are a collection of closed, translation invariant subspaces, functions $\phi_\alpha \in B_\alpha$ have been chosen for each $\alpha \in I$, and $\{T^C = (U_j^C, V_j^C)_{j=1}^N\}_C$ is an associated factoring map.

Given $\Lambda \subseteq I$, we will write $\mathcal{P}_\Lambda$ for the orthogonal projection onto $\oplus_{\alpha\in\Lambda} B_\alpha$. We will call multiplier M *compatible* if it commutes with $\mathcal{P}_\alpha$ for each $\alpha \in I$ and commutes with each U_j^C.

Here is the key technical theorem.

Theorem 3.3. *Suppose m_j, for $j = 1, \ldots, n$, are compatible multipliers that are contractions on $L^2(G)$ and ρ-contractive on $\oplus_{g(\alpha)=j} B_\alpha$. Assume $\{T^C\}$ is a factoring map of norm A and $\Lambda_n \subseteq \{\alpha \in I : g(\alpha) \le n\}$ is a finite set that satisfies the strong orthogonality condition of order J. If $\rho \le 1/(3AJ)$ and $M_n = m_1 \circ \cdots \circ m_n$, $M_0 = \mathrm{Id}$, then for all $f \in L^2(G)$,*

$$\Big\| f * M_n\Big(\sum_{\alpha\in\Lambda_n} \phi_\alpha \Big)\Big\|_4 \le 3AJ \left\| \mathcal{P}_{\Lambda_n}(f) \right\|_2 . \tag{3.2}$$

Before proving this, we state the main corollaries we need for later application. We continue to use the notation of the theorem and assume its hypotheses hold.

Corollary 3.4. *The measures $Q_n = M_n\left(\sum_{\alpha\in\Lambda_n} \phi_\alpha\right)$ are uniformly bounded operators from L^2 to L^4. Any measure that is a weak* cluster point of the set of measures $\{Q_n\}$ is L^p-improving.*

Remark 3.5. By measure Q_n we mean the absolutely continuous measure whose Radon–Nikodym derivative is Q_n.

Proof. It is immediate from the theorem that $\|Q_n * f\|_4 \le 3AJ \|f\|_2$ for all $f \in L^2$.

Let μ be the weak* limit of the subnet $\{Q_\beta\}$. As $\|Q_n * f\|_4$ is uniformly bounded for each (fixed) $f \in L^2$, a further subnet of $\{Q_\beta * f\}$ converges weakly in L^4 to some g_f satisfying $\|g_f\|_4 \le \limsup_\beta \|Q_\beta * f\|_4 \le 3AJ\|f\|_2$. A comparison of Fourier transforms shows that $g_f = \mu * f$. Thus $\mu\colon L^2 \to L^4$ is a bounded operator. □

Corollary 3.6. *Suppose that for each n, $\Omega_n \subseteq \{\alpha \in I\colon g(\alpha) \le n\}$ satisfies the strong orthogonality condition of order J. Fix $0 < r < 1$. For each finite subset $F \subseteq I$, let*

$$\mu_F = \sum_{n=0}^{|F|} r^n\, Q_n^{(F)}(x), \quad \textit{where} \quad Q_n^{(F)}(x) = M_n\Big(\sum_{\alpha\in\Omega_n\cap F} \phi_\alpha \Big).$$

The multiplier M defined by $M_\gamma = \lim_F \widehat{\mu_F}(\gamma)$ for all $\gamma \in \widehat{G}$ (with the partial ordering on the net being inclusion) is L^p-improving.

Proof. The proof is similar. Applying the theorem with $\Lambda_n = \Omega_n \cap F$ gives $\|Q_n^{(F)} * f\|_4 \le 3AJ \|f\|_2$ for each $f \in L^2$. Thus

$$\|\mu_F * f\|_4 \le \sum_{n=0}^{|F|} r^n \left\|Q_n^{(F)} * f\right\|_4 \le \frac{3AJ}{1-r} \|f\|_2 .$$

Comparing Fourier transforms, one sees that a weak limit of $\{\mu_F * f\}$ coincides with $M(f)$ and therefore $\|M(f)\|_4 \le 3AJ\,\|f\|_2/(1-r)$. □

Proof of Theorem 3.3. We proceed to prove (3.2) by induction on n.

Base case: $n=0$. As $R(\alpha)$ is empty for any element of generation 0, the factoring map defining property (2.1) implies that whenever α belongs to category C, then for all $f \in L^2$,

$$f * \phi_\alpha = \sum_{j=1}^{N} U_j^C\left(V_j^C(f) * 1\right).$$

Further, $\sum_{j=1}^{N} \|U_j^C\|_{4,4}\|V_j^C\|_{2,2} \le A$.

Since each B_α is translation invariant, so is $B_\alpha^\perp$, hence if $g \in B_\alpha$ and $h \in B_\alpha^\perp$, then $h * g = 0$. In particular, $f * \phi_\alpha = \mathcal{P}_\alpha f * \phi_\alpha \in B_\alpha$ for all $f \in L^2$. Thus

$$\begin{aligned}\|f * \phi_\alpha\|_4 = \|\mathcal{P}_\alpha f * \phi_\alpha\|_4 &\le \sum_{j=1}^{N} \left\|U_j^C\left(V_j^C(\mathcal{P}_\alpha f) * 1\right)\right\|_4 \\ &\le \sum_{j=1}^{N} \left\|U_j^C\right\|_{4,4}\left\|V_j^C(\mathcal{P}_\alpha f) * 1\right\|_4\end{aligned}$$

Since

$$\left\|V_j^C(\mathcal{P}_\alpha f) * 1\right\|_4 \le \left\|V_j^C(\mathcal{P}_\alpha f)\right\|_1 \|1\|_4 \le \left\|V_j^C(\mathcal{P}_\alpha f)\right\|_2,$$

it follows that for all α,

$$\|f * \phi_\alpha\|_4 \le \sum_{j=1}^{N} \left\|U_j^C\right\|_{4,4}\left\|V_j^C\right\|_{2,2}\|\mathcal{P}_\alpha f\|_2 \le A\,\|\mathcal{P}_\alpha f\|_2. \tag{3.3}$$

As Λ_n satisfies the strong orthogonality condition of order J, we can write $\Lambda_n = \bigcup_{j=1}^{J} \Lambda_n^j$, where each Λ_n^j satisfies the 4-orthogonality condition and the subspaces B_α for $\alpha \in \Lambda_n^j$ are orthogonal. If $\alpha_k \in \Lambda_n^j$ are distinct indices of generation zero, then they belong to different categories, so the tuples $(g(\alpha_k), C(\alpha_k))$ are distinct. The definition of 4-orthogonality ensures that if $f_k \in B_{\alpha_k}$, then $\int f_1 f_2 \overline{f_3 f_4} = 0$ and thus Lemma 3.1 implies that

$$\Big\|\sum_{\alpha \in \Lambda_n^j} f_\alpha\Big\|_4 \le 3\left(\sum \|f_\alpha\|_4^2\right)^{1/2} \tag{3.4}$$

whenever $f_\alpha \in B_\alpha$.

We apply this with $f_\alpha = f * \phi_\alpha$, so that combining (3.3) and (3.4) yields

$$\Big\|f * \sum_{\alpha \in \Lambda_n^j} \phi_\alpha\Big\|_4 \le 3\left(\sum \|f * \phi_\alpha\|_4^2\right)^{1/2} \le 3A\left(\sum_{\alpha \in \Lambda_n^j} \|\mathcal{P}_\alpha f\|_2^2\right)^{1/2}.$$

As the spaces B_α, $\alpha \in \Lambda_n^j$, are orthogonal,

$$\sum_{\alpha \in \Lambda_n^j} \|\mathcal{P}_\alpha f\|_2^2 = \Big\|\sum_{\alpha \in \Lambda_n^j} \mathcal{P}_\alpha f\Big\|_2^2 = \|\mathcal{P}_{\Lambda_n^j} f\|_2^2.$$

Thus $\left\|f * \sum_{\alpha\in\Lambda_n^j}\phi_\alpha\right\|_4 \le 3A\|\mathcal{P}_{\Lambda_n^j} f\|_2$, and summing over $j = 1, \dots, J$ gives the desired bound

$$\left\|f * \sum_{\alpha\in\Lambda_n}\phi_\alpha\right\|_4 \le 3AJ\,\|\mathcal{P}_{\Lambda_n} f\|_2\,.$$

Induction step: assume (3.2) holds for $n-1$, $n \ge 1$. The strong orthogonality condition ensures we can write $\Lambda_n = \bigcup_{j=1}^J \Lambda_n^j$, where each Λ_n^j satisfies the three defining conditions. Temporarily fix j and $f \in L^2$. For each category C and non-negative integer $k \le n$, let

$$X_{C,k} = \{\alpha \in \Lambda_n^j : C(\alpha) = C \text{ and } g(\alpha) = k\}.$$

Provided this set is non-empty, let

$$F_{C,k} = f * M_n\Big(\sum_{\alpha\in X_{C,k}}\phi_\alpha\Big).$$

Our first step will be to prove that

$$\|F_{C,k}\|_4 \le A\left\|\mathcal{P}_{X_{C,k}}(f)\right\|_2. \tag{3.5}$$

We begin with $k \ge 1$. Essentially as before,

$$f * \sum_{\alpha\in X_{C,k}}\phi_\alpha = \mathcal{P}_{X_{C,k}}(f) * \sum_{\alpha\in X_{C,k}}\phi_\alpha.$$

Since each m_j is a multiplier,

$$\begin{aligned} F_{C,k} &= M_n\Big(f * \sum_{\alpha\in X_{C,k}}\phi_\alpha\Big) = M_n\Big(\mathcal{P}_{X_{C,k}}(f) * \sum_{\alpha\in X_{C,k}}\phi_\alpha\Big) \\ &= \mathcal{P}_{X_{C,k}}(f) * M_n\Big(\sum_{\alpha\in X_{C,k}}\phi_\alpha\Big). \end{aligned}$$

Thus without loss of generality we can assume $f = \mathcal{P}_{X_{C,k}}(f)$.

With the notation $M_{n,k} = m_{k+1} \circ \cdots \circ m_n$, we can write

$$F_{C,k} = M_{k-1}\Big(m_k\Big(M_{n,k}(f) * \sum_{\alpha\in X_{C,k}}\phi_\alpha\Big)\Big).$$

Again, we can replace $M_{n,k}(f)$ by its orthogonal projection onto $\oplus_{\alpha\in X_{C,k}} B_\alpha$, which we denote by $f_{C,k}$ for notational ease, suppressing the dependence on n as this is fixed. Using the factoring maps and the compatibility assumption we have

$$\begin{aligned} F_{C,k} &= M_{k-1}\Big(m_k(f_{C,k}) * \sum_{\alpha\in X_{C,k}}\phi_\alpha\Big) \\ &= M_{k-1}\Big(\sum_{i=1}^N U_i^C\Big(V_i^C(m_k(f_{C,k})) * \sum_{\alpha\in X_{C,k}}\sum_{\gamma\in R(\alpha)}\phi_\gamma\Big)\Big) \\ &= \Big(\sum_{i=1}^N U_i^C\Big(V_i^C(m_k(f_{C,k})) * M_{k-1}\Big(\sum_{\alpha\in X_{C,k}}\sum_{\gamma\in R(\alpha)}\phi_\gamma\Big)\Big)\Big). \end{aligned}$$

Let $Y_{C,k}$ denote the set of all roots of $\alpha \in X_{C,k}$, that is, $Y_{C,k} = \bigcup_{\alpha \in X_{C,k}} R(\alpha)$. If α, β are distinct indices belonging to category C, then they have no roots in common. Hence

$$\sum_{\alpha \in X_{C,k}} \sum_{\gamma \in R(\alpha)} \phi_\gamma = \sum_{\gamma \in Y_{C,k}} \phi_\gamma .$$

The strong orthogonality condition (iii) implies that the set $Y_{C,k}$ also satisfies the strong orthogonality condition of order J. As $Y_{C,k} \subseteq \{\alpha : g(\alpha) \leq k-1\}$ and $k-1 \leq n-1$, the induction assumption ensures (3.2) holds with Λ replaced by $Y_{C,k}$ and $f = V_i^C(m_k(f_{C,k}))$. With $K = 3AJ$, this gives

$$\begin{aligned}
\|F_{C,k}\|_4 &\leq \sum_{i=1}^{N} \|U_i^C\|_{4,4} \Big\| V_i^C(m_k(f_{C,k})) * M_{k-1}\Big(\sum_{\alpha \in Y_{C,k}} \phi_\gamma \Big) \Big\|_4 \\
&\leq K \sum_{i=1}^{N} \left\|U_i^C\right\|_{4,4} \left\| \mathcal{P}_{Y_{C,k}} \left(V_i^C(m_k(f_{C,k}))\right) \right\|_2 \\
&\leq K \sum_{i=1}^{N} \left\|U_i^C\right\|_{4,4} \left\| V_i^C(m_k(f_{C,k})) \right\|_2 \\
&\leq K \sum_{i=1}^{N} \left\|U_i^C\right\|_{4,4} \left\|V_i^C\right\|_{2,2} \|m_k(f_{C,k})\|_2 \leq KA \|m_k(f_{C,k})\|_2 .
\end{aligned}$$

But m_k is a ρ-contraction on $\oplus_{\alpha \in X_{C,k}} B_\alpha$, hence $\|m_k(f_{C,k})\|_2 \leq \rho \|f_{C,k}\|_2$. Since each m_i is also a contraction on L^2 it follows that whenever $\rho \leq 1/K$, then

$$\|F_{C,k}\|_4 \leq KA\rho \|f_{C,k}\|_2 = KA\rho \|M_{n,k}(f)\|_2 \leq KA\rho \|f\|_2 \leq A \|f\|_2 ,$$

so (3.5) holds.

Now consider the case $k = 0$. Recall that by assumption there is at most one member of each category of generation zero. Thus $X_{C,0}$ is a singleton, say α (or empty). As we saw in the base case argument,

$$\|f * \phi_\alpha\|_4 \leq A \|\mathcal{P}_{B_\alpha} f\|_2 = A \left\|\mathcal{P}_{X_{C,0}} f\right\|_2 .$$

Since M_n is a contraction, the same bound holds for $f * M_n(\phi_\alpha)$. Thus (3.5) holds for all $k \geq 0$.

The strong orthogonality assumption implies (in particular) that the set Λ_n^j satisfies the 4-orthogonality condition, thus Lemma 3.1 applies to the collection of functions $F_{C,k}$. Coupled with (3.5) we deduce that

$$\Big\| \sum_{(C,k)} F_{C,k} \Big\|_4 \leq 3 \Big(\sum_{(C,k)} \|F_{C,k}\|_4^2 \Big)^{1/2} \leq 3A \Big(\sum_{(C,k)} \|\mathcal{P}_{X_{C,k}} f\|_2^2 \Big)^{1/2},$$

where the sum is over those pairs (C,k) for which $X_{C,k}$ is non-empty.

Strong orthogonality also ensures the spaces $\oplus_{\alpha\in X_{C,k}} B_\alpha$ are mutually orthogonal, thus $\sum_{(C,k)} \|\mathcal{P}_{X_{C,k}} f\|_2^2 = \|\mathcal{P}_{\Lambda_n^j} f\|_2^2$. Consequently,

$$\Big\| f * M_n\Big(\sum_{\alpha\in\Lambda_n^j} \phi_\alpha\Big)\Big\|_4 = \Big\| \sum_{(C,k)} F_{C,k}\Big\|_4 \leq 3A\big\|\mathcal{P}_{\Lambda_n^j} f\big\|_2.$$

Summing over $j = 1, \dots, J$ completes the induction step and hence the proof. □

4. Applications

In this section we will see that many Riesz product measures and Cantor measures can be shown to be L^p-improving using this approach. Our results are not meant to be exhaustive or best possible, and in general they are not new, but they illustrate the versatility of the technique.

Before turning to these we remind the reader of some well known facts about L^p-improving measures.

First, we recall that as any measure acts by convolution as a bounded map from L^1 to L^1 and from L^∞ to L^∞, an interpolation argument shows that if the measure μ is L^p-improving, then for every $1 < p < \infty$ there is some $q > p$ such that $\mu\colon L^p \to L^q$ is a bounded operator. Furthermore, by a duality argument, $\mu\colon L^p \to L^q$ if and only if $\mu\colon L^{q'} \to L^{p'}$, with the same operator norm, when p', q' are the conjugate indices to p, q respectively. Consequently, μ is L^p-improving if and only if there is some $p < 2$ such that $\mu\colon L^p \to L^2$ is a bounded operator. It also can be deduced immediately from this that if ν is L^p-improving and $|\widehat{\mu}(\gamma)| \leq |\widehat{v}(\gamma)|$ for all $\gamma \in \widehat{G}$, then μ is also L^p-improving.

Lastly, we mention that if μ^m is L^p-improving for some m-fold convolution power of μ, then an application of Stein's complex interpolation theorem proves that μ is also L^p-improving (see [9] for details).

4.1. Riesz products

Terminology. Let k be a positive integer. We will say $\{\gamma_j\} \subseteq \widehat{G}$ is *k-dissociate* if whenever $\prod \gamma_j^{\varepsilon_j} = 1$ with $\varepsilon_j \in \{0, \pm 1, \pm 2, \dots, \pm k\}$, then all $\gamma_j^{\varepsilon_j} = 1$. (Here the group operation on $\widehat{G}$ is the product.) Recall that 2-dissociate is usually simply called *dissociate*. If $\{n_j\} \subseteq \mathbb{Z}^+$ is lacunary, then $\{e^{2\pi i n_j x}\}$ is k-dissociate if the lacunary ratio is sufficiently large. For example, if $\inf n_{j+1}/n_j \geq 3$, then $\{n_j\}$ is dissociate.

To avoid technicalities, we will assume $\widehat{G}$ has no elements of order two and suppose $\{\gamma_j\} \subseteq \widehat{G}$ is dissociate. By a *Riesz product measure* μ we mean the (unique) weak* limit of the polynomials $P_N = \prod_{j=1}^N (1 + a_j(\gamma_j + \gamma_j^{-1}))$, where $|a_j| \leq 1/2$. It is customary to write $\mu = \prod_{j=1}^\infty (1 + a_j(\gamma_j + \gamma_j^{-1}))$.

Proposition 4.1 ([2], [9]). (a) *Assume $n_{j+1}/n_j \geq 3$ for all j, $|a_j| \leq 1$, and let $\mu = \prod_{j=1}^\infty (1 + a_j \cos 2\pi n_j x)$ be a Riesz product measure on $\mathbb{T}$. Then μ is an L^p-improving measure.*

(b) *If* $\{\gamma_j\} \subseteq \widehat{G}$ *is* 4*-dissociate and* $|a_j| \le 1/2$*, then the Riesz product* $\mu = \prod_{j=1}^{\infty}(1 + a_j(\gamma_j + \gamma_j^{-1}))$ *is an* L^p*-improving measure.*

Proof. (a) Take the index set I, subspaces B_α, functions $\phi_\alpha \in B_\alpha$ and associated factoring map of norm $A = 1$, as in Example 2.4. According to Lemma 2.3, the sets $\Lambda_n = \{\alpha \in I : g(\alpha) = n\}$ satisfy the strong orthogonality condition of order 2. Take $\rho < 1/100$. Let $r < 1$ and consider the Riesz product $\nu = \prod_{j=1}^{\infty}(1 + r\rho \cos 2\pi n_j x)$.

Let m_j denote the multiplier that is multiplication by ρ on L^2; these are compatible multipliers that are ρ-contractions on all of L^2. For $F \subseteq I$ a finite subset, put $Q_n^{(F)} = m_1 \circ \cdots \circ m_n \left(\sum_{\alpha\in\Lambda_n\cap F} \phi_\alpha\right)$. For each integer j, $\widehat{\nu}(j)$ is the limit of the net $\{\widehat{\nu}_F(j)\}$ where $\nu_F = \sum_{k=0}^{|F|} r^k Q_k^{(F)}$. Thus Corollary 3.6 implies ν is L^p-improving.

For sufficiently large m, $|a_j/2|^m \le r\rho$ for all j, and this is enough to ensure $\left|\widehat{\mu^m}(n)\right| \le |\widehat{v}(n)|$ for all n. By the remarks at the beginning of this section, the measure μ^m, and therefore also μ, is L^p-improving.

(b) The strategy of the proof is similar. The 4-dissociate condition allows one to see that if $B_\alpha = \operatorname{span}\{\Pi\gamma_j^{\varepsilon_j\alpha(j)} : \varepsilon_j = \pm 1\}$, then $\Lambda_n = \{\alpha : g(\alpha) = n\}$ satisfies the strong orthogonality condition of order 1. □

4.2. Cantor measures

By a Cantor set, we mean a subset of $[0, 1]$ that has an inductive construction similar to the classical middle-third Cantor set.

Suppose we are given real numbers $\{r_n\}$, $0 < r_n < 1/2$. We begin the inductive construction with the interval $[0, 1]$ and at the first step in the construction remove the open middle interval of length $1 - 2r_1$, keeping the two outer closed intervals of length r_1. Call this set C_1. Proceeding inductively, at step k in the construction the set C_k will consist of 2^k closed intervals of length $r_1 \cdots r_k$. From each of these we remove the open middle subinterval of length $r_1 \cdots r_k(1 - 2r_{k+1})$, keeping the two outer intervals of length $r_1 \cdots r_{k+1}$. The set C_{k+1} is the union of these 2^{k+1} closed intervals of length $r_1 \cdots r_{k+1}$. The Cantor set with ratios of dissection $\{r_j\}$ is the intersection of these sets C_k. The classical middle-third Cantor set is the special case where the ratios of dissection all equal $1/3$.

By a Cantor measure we mean the uniform measure supported on the Cantor set C, meaning that the measure of any of the 2^k closed intervals arising at step k in the construction are assigned measure $1/2^k$. It is well known that when the ratios of dissection are given by $\{r_k\}$, then the Cantor measure is the infinite convolution

$$\mu = \prod_{n=1}^{\infty} \frac{1}{2}(\delta_0 + \delta_{r_1\cdots r_{n-1}(1-r_n)}),$$

where δ_t denotes the point mass measure at t. A simple calculation shows that

$$|\widehat{\mu}(k)| = \prod_{n=1}^{\infty} \left|\cos \pi k r_1 \cdots r_{n-1}(1 - r_n)\right|.$$

Proposition 4.2 ([3]). *If μ is the Cantor measure supported on a Cantor set with ratios of dissection bounded away from 0, then μ is L^p-improving.*

Proof. We will first show how the technique can be applied in the case when $r_k = 1/3$ for each k and then briefly explain the key modifications needed for the general case.

A periodicity argument implies that there is some $\delta > 0$, independent of k, such that $|\cos 2\pi x 3^{-k-1}| \leq 1 - \delta$ for all x belonging to any interval of length $3^{k+1}/4$, except for those x belonging to a (particular) subinterval of length at most $3^k/4$. Choose an integer L such that $(1-\delta)^L < \rho = 1/18$. It will be sufficient to prove that μ^L is L^p-improving.

We will now explain the index set, a slight variation on the triadic index set of Example 2.2. As with the triadic index set, we start with the set of 3-adic intervals in $\mathbb{Z}$, specifically, those of the form $[(2m-1)3^k/8, (2m+1)3^k/8) \cap \mathbb{Z}$, where $m, k \in \mathbb{Z}$, $k \geq 0$. The generation of such an index α is defined as k. Now exclude from this collection those intervals of generation $k \geq 1$ that contain an element x having $\left|\cos 2\pi x 3^{-k-1}\right| > 1 - \delta$. Our comments above imply that from any 3 consecutive intervals of generation k, there are either $0, 1$ or 2 (adjacent) intervals of generation k having this undesirable property. The remaining 3-adic intervals form the index set. As with the triadic index set, each index is its own category. The roots of α of generation k will consist of all the 3-adic intervals contained in α, which belong to the index set and are maximal proper subsets (under inclusion) with this property.

We define the subspaces B_α, functions ϕ_α and the factoring map of norm one as in Example 2.5. If we define multipliers m_k by $\widehat{m_k(f)}(n) = \widehat{f}(n)(\cos 2\pi n 3^{-k-1})^L$, then it is clear by construction that m_k is ρ-contractive on the spaces B_α with $g(\alpha) = k \geq 1$ and a contraction on L^2.

Next, we claim that the set of all roots of any index α satisfies the strong orthogonality condition of order 6. First, we remark that the intervals $\beta \in R(\alpha)$ are disjoint and hence the subspaces B_β are mutually orthogonal. We will show that $R(\alpha)$ can be partitioned into 6 subsets, each of which satisfies the 4-orthogonality condition. This will suffice to prove the claim as categories are singletons. We note that this observation was also a key idea in Christ's original proof.

To see this, observe that our construction guarantees that the roots of index α first monotonically decrease in generation as one moves from left to right (in $\mathbb{R}$), and then monotonically increase. Partition the roots into two groups, those on the decreasing side and those on the increasing side. Furthermore, there are at most 3 (adjacent) intervals of each generation among the roots in each of the two groups. Now further partition each group by selecting every third element as one moves left to right.

This procedure ensures that each of the 6 collections of sets we have constructed contains at most one element of each generation, with any two intervals in a given collection separated by an interval whose length is at least 2 times that of the smaller of the two. A standard geometric series argument shows that each of these collections satisfies the 4-orthogonality condition. This establishes the claim.

To complete the proof, take Λ_n to be the set of roots of the index $\alpha = [-3^n/2, 3^n/2) \cap \mathbb{Z}$. As the union of these sets cover $\mathbb{R}$, the L-fold convolution of μ is the weak* limit of $m_1 \circ \cdots \circ m_n \left(\sum_{\alpha \in \Lambda_n} \phi_\alpha\right)$. Finally, call upon Corollary 3.4.

Now suppose the ratios, r_k, are bounded away from 0. For notational ease, put $\varepsilon_n = r_1 \cdots r_{n-1}(1 - r_n)$. The boundedness away from 0 ensures that there is some even number R such that $9/R \leq \varepsilon_n/\varepsilon_{n+1} \leq R$ for all n. The boundedness also ensures we can choose $\delta > 0$ such that for each n, $|\cos \pi \varepsilon_{n+1} x| \leq 1 - \delta$ for all x belonging to an interval of length $1/(2\varepsilon_{n+1})$, except for x belonging to a subinterval of length at most $1/(6\varepsilon_n)$. To create the index set, for each n we begin with disjoint intervals in $\mathbb{Z}$ with lengths in the range $[1/(6\varepsilon_n), 1/(4\varepsilon_n)]$. These will be of generation n. The intervals of generation n should be a union of the intervals of generation $n-1$. We then apply the exclusion rule, similar to above. Finally, we partition the roots of an index into $3R+2$ sets, each of which can be shown to be 4-orthogonal. □

4.3. Markov measures

For an integer q_j, let $\mathbb{Z}_{q_j}$ denote the multiplicative group of q_j'th roots of unity and let $X = \prod_{j=1}^{\infty} \mathbb{Z}_{q_j}$. The dual of the product space X is the direct sum $\oplus \mathbb{Z}_{q_j}$.

By a product measure, we mean a probability measure $\mu = \prod_{j=1}^{\infty} g_j(x)$ on X where g_j depends only on coordinate j. Product measures are often L^p-improving. Indeed, the following can be proven, using our approach, by adapting the techniques of the next proof.

Proposition 4.3. *If $\varepsilon > 0$ is sufficiently small, then the product measure $\mu = \prod_{j=1}^{\infty} g_j(x)$, on $X = \prod_{j=1}^{\infty} \mathbb{Z}_{q_j}$, is L^p-improving if the real-valued functions g_j satisfy $|1 - g_j(x)| < \varepsilon/q_j$ for all j.*

We turn now to proving a result for the more general class of Markov measures, studied in [4] and [5]. Again, these are probability measures on on $X = \prod_{j=1}^{\infty} \mathbb{Z}_{q_j}$ of the form $\mu = \prod_{j=1}^{\infty} g_j(x)$, but where the probability densities g_j satisfy the following two properties: let $X_n = \{y = (y_j) \in X : y_j = 1 \text{ for all } j \neq n\}$. Then, for all n,

(i) $g_n(xy) = g_n(x)$ for any $y \in X_k$ with $k \neq n, n+1$; and

(ii) $\frac{1}{|X_n|} \sum_{y \in X_n} g_n(xy) = 1$.

Like product measures, Markov measures are also L^p-improving if the functions g_j are close enough to 1.

Proposition 4.4. *If $\varepsilon > 0$ is sufficiently small, then the Markov measure $\mu = \prod_{j=1}^{\infty} g_j(x)$, on $X = \prod_{j=1}^{\infty} \mathbb{Z}_{q_j}$, is L^p-improving if g_j is real valued and $\|1 - g_j(x)\|_2 < \varepsilon/\sqrt{q_j q_{j+1}}$ for all j.*

Proof. We use the 0-1 index set I, but this time with the category C_k consisting of the set of sequences whose first 1 is in coordinate k. The sequence of all 0's will be in the category C_0. The root of α will be the sequence with the first 1 changed to a 0.

Given $\alpha \in I$, let B_α be the functions in $L^2(X)$ with the property that whenever $y \in X_n$ and $\alpha_{n-1} = \alpha_n = 0$, then $f(xy) = f(x)$, while if $\alpha_{n-1} = 0, \alpha_n = 1$, then $\sum_{y\in X_n} f(xy) = 0$. (If $\alpha_{n-1} = 1$, no constraints are imposed.)

As with the Riesz product example, let $\Lambda_k = \{\alpha \in I : g(\alpha) = k\}$. Assume $\alpha \neq \beta$ both belong to Λ_k for for $k \geq 1$. Then (without loss of generality) $0 < C(\alpha) = n < C(\beta)$. If $f_\alpha \in B_\alpha$ and $f_\beta \in B_\beta$, then a change of variables argument shows

$$\int f_\alpha(x) f_\beta(x) dx = \frac{1}{|X_n|} \int \sum_{y\in X_n} f_\alpha(xy) f_\beta(xy)\, dx.$$

But $f_\beta(xy) = f_\beta(x)$ and $\frac{1}{|X_n|}\sum_{y\in X_n} f_\alpha(xy) = 0$ whenever $y \in X_n$. Thus

$$\int f_\alpha(x) f_\beta(x)\, dx = 0,$$

so the spaces B_α and B_β are mutually orthogonal. A similar argument proves Λ_k satisfies the strong orthogonality condition of order 1.

Let $r < 1$ with $\varepsilon/r \leq 1$. Define $\phi_\alpha(x) = \prod_{n=1}^{\infty}((1 - g_n(x))/r)^{\alpha_n}$. It can be verified that $\phi_\alpha \in B_\alpha$. Assume $C(\alpha) = n$. Put $h_n = (1 - g_n)/r$ and suppose the root of α is β. Then $\phi_\alpha = h_n \phi_\beta$. As h_n depends only on coordinates $n, n+1$, we can view $\widehat{h_n}$ as a function on the dual of $\mathbb{Z}_{q_n} \times \mathbb{Z}_{q_{n+1}}$. As this group has order $q_n q_{n+1}$, we can write $\widehat{h_n} = \sum_{j=1}^{q_n q_{n+1}} a_j \delta_{\gamma_j}$ where γ_j are the characters of $\mathbb{Z}_{q_n} \times \mathbb{Z}_{q_{n+1}}$. Direct calculation gives

$$\widehat{f * \phi_\alpha}(\chi) = \widehat{f * h_n \phi_\beta}(\chi) = \widehat{f}(\chi) \sum_{\gamma} \widehat{h_n}(\gamma) \widehat{\phi_\beta}(\chi - \gamma) = \sum_{j=1}^{q_n q_{n+1}} a_j \widehat{f}(\chi) \widehat{\phi_\beta}(\chi - \gamma_j).$$

Define U_j^n on L^2 by $\widehat{U_j^n(f)}(\chi) = a_j \widehat{f}(\chi - \gamma_j)$ and define V_j^n by $\widehat{V_j^n(f)}(\chi) = \widehat{f}(\chi + \gamma_j)$. With this notation, continuing from above we have

$$\begin{aligned} \widehat{f * \phi_\alpha}(\chi) &= \sum_{j=1}^{q_n q_{n+1}} a_j \widehat{V_j^n(f)}(\chi - \gamma_j) \widehat{\phi_\beta}(\chi - \gamma_j) \\ &= \sum_{j=1}^{q_n q_{n+1}} a_j (V_j^n(f) * \phi_\beta)\widehat{\ }(\chi - \gamma_j) = \sum_{j=1}^{q_n q_{n+1}} U_j^n \left(V_j^n(f) * \phi_\beta\right)\widehat{\ }(\chi). \end{aligned}$$

As $R(\alpha)$ is the singleton β, this shows

$$f * \phi_\alpha = \sum_{j=1}^{q_n q_{n+1}} U_j^n \Big(V_j^n(f) * \sum_{\beta \in R(\alpha)} \phi_\beta \Big),$$

so $\{(U_j^n, V_j^n) : j = 1, \ldots, q_n q_{n+1}\}_n$ is a factoring map. One easily sees that

$$\sum_{j=1}^{q_n q_{n+1}} \left\|U_j^n\right\|_{4,4} \left\|V_j^n\right\|_{2,2} = \sum_{j=1}^{q_n q_{n+1}} |a_j| \leq \sqrt{q_n q_{n+1}}\, \|h\|_2 \leq \varepsilon/r \leq 1.$$

Now appeal to Corollary 3.6 in a similar manner to the Riesz product argument. □

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Anthony H. Dooley: School of Mathematics and Physical Sciences, University of Technology Sydney, Ultimo, NSW 2007 Australia.
E-mail: Anthony.Dooley@uts.edu.au

Kathryn E. Hare: Department of Pure Mathematics, University of Waterloo, Waterloo, N2L3G1 Ont., Canada.
E-mail: kehare@uwaterloo.ca

Maria Roginskaya: Department of Mathematical Sciences, Chalmers University of Technology, Gothenburg, Sweden; *and* Department of Mathematical Sciences, University of Gothenburg, Gothenburg, Sweden.
E-mail: maria@chalmers.se

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