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Quasicrystals with discrete support and spectrum

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Abstract. We proved recently that a measure on $\mathbb{R}$, whose support and spectrum are both uniformly discrete sets, must have a periodic structure. Here we show that this is not the case if the support and the spectrum are just discrete closed sets.

1. Introduction

By a Fourier quasicrystal one often means an (infinite) pure point measure μ, whose Fourier transform is also a pure point measure (see e.g. [2], [6]).

Consider a (complex) measure μ on $\mathbb{R}^n$ supported on a discrete set Λ:

$$\mu = \sum_{\lambda\in\Lambda} \mu(\lambda)\, \delta_\lambda, \quad \mu(\lambda) \neq 0. \tag{1.1}$$

Assume that μ is a temperate distribution, and that its Fourier transform

$$\widehat{\mu}(t) := \sum_{\lambda\in\Lambda} \mu(\lambda)\, e^{-2\pi i\langle\lambda,t\rangle}$$

(in the sense of distributions) is also a pure point measure, namely

$$\widehat{\mu} = \sum_{s\in S} \widehat{\mu}(s)\, \delta_s, \quad \widehat{\mu}(s) \neq 0. \tag{1.2}$$

The set Λ is called the support of the measure μ, while S is called the spectrum.

The classical example of such a measure comes from Poisson's summation formula. The measure there is the sum of unit masses over a lattice, and the spectrum is the dual lattice. The problem whether other measures of Poisson type may exist, was studied by different authors. See, in particular, [8], [7], [16], [5], [1], [3], [4], and [10]. In the last paper one may find a comprehensive survey and references up to that date.

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Notice that a new peak of interest in the subject appeared after the experimental discovery in the middle of 80's of the physical quasicrystals, demonstrating that an aperiodic atomic structure may have a diffraction pattern consisting of spots.

The "cut-and-project" construction, introduced by Y. Meyer in the beginning of 70's [16], may serve as a good model for this phenomenon, see [17]. It provides many examples of measures with uniformly discrete support and dense countable spectrum.

On the other hand, we proved recently that if both the support and the spectrum of a measure on $\mathbb{R}$ are uniformly discrete sets then the measure has a periodic structure.

Theorem 1 ([11, 12]). *Let μ be a measure on $\mathbb{R}$ satisfying (1.1) and (1.2), and assume that Λ and S are both uniformly discrete sets. Then Λ is contained in a finite union of translates of an arithmetic progression.*

Moreover, it was proved that such a measure can be obtained from Poisson's summation formula by a finite number of shifts, multiplication on exponentials, and taking linear combinations. A similar result was also proved for positive measures in $\mathbb{R}^n$.[1]

The goal of the present note is to establish the sharpness of this result, in the sense that the condition of uniform discreteness cannot be relaxed much. More precisely, we prove the following:

Theorem 2. *There is a (non-zero) real, signed measure μ on $\mathbb{R}$ satisfying (1.1) and (1.2), such that*

(i) *Λ and S are both discrete closed sets;*

(ii) *Λ contains only finitely many elements of any arithmetic progression.*

The condition (ii) indicates that the measure μ is "non-periodic" in a strong sense. In particular, μ cannot be obtained from Poisson's summation formula by the procedures mentioned above. In Section 6 below we discuss some additional properties of the measure in our construction that illustrate its non-periodic nature.

Remarks. 1) It follows from (ii) that Λ may not be covered by any finite union of arithmetic progressions. We will see that in our example, this latter property is true for S as well.

2) The measure μ constructed in the proof, as well as its Fourier transform $\widehat{\mu}$, are translation-bounded measures on $\mathbb{R}$.

2. Notation

A set $\Lambda \subset \mathbb{R}$ is a discrete closed set if it has finitely many points in every bounded interval. The set Λ is called uniformly discrete (u.d.) if $|\lambda - \lambda'| \geqslant \delta(\Lambda) > 0$ for any two distinct points $\lambda, \lambda' \in \Lambda$.

[1] *Note added in proof.* Recently we have strengthened this result so that only the spectrum is assumed to be a uniformly discrete set, while the support is just a discrete closed set. See [13].

By a "measure" on $\mathbb{R}$ we mean a complex, locally finite measure (usually infinite) which is also a temperate distribution. A measure μ is called translation-bounded if

$$\sup_{x\in\mathbb{R}} \int_x^{x+1} |d\mu| < \infty. \tag{2.1}$$

By the "support" of a pure point measure μ we mean the countable set of the non-zero atoms of μ. This should not be confused with the notion of support in the sense of distributions, which is always a closed set. In the construction below this difference will not be important, since both μ and $\widehat{\mu}$ are supported by discrete closed sets.

The Fourier transform on $\mathbb{R}$ will be normalized as follows:

$$\widehat{\varphi}(t) = \int_{\mathbb{R}} \varphi(x)\, e^{-2\pi i t x}\, dx.$$

We denote by $\operatorname{supp}(\varphi)$ the closed support of a Schwartz function φ, and by $\operatorname{spec}(\varphi)$ the closed support of its Fourier transform $\widehat{\varphi}$.

If α is a temperate distribution then $\langle \alpha, \varphi \rangle$ denotes the action of α on a Schwartz function φ. The Fourier transform $\widehat{\alpha}$ is defined by $\langle \widehat{\alpha}, \varphi \rangle = \langle \alpha, \widehat{\varphi} \rangle$.

By a (full-rank) lattice $\Gamma \subset \mathbb{R}^n$ we mean the image of $\mathbb{Z}^n$ under some invertible linear transformation T. The determinant $\det(\Gamma)$ is equal to $|\det(T)|$. The dual lattice Γ^* is the set of all vectors γ^* such that $\langle \gamma, \gamma^* \rangle \in \mathbb{Z}$, $\gamma \in \Gamma$.

3. Interpolation in Paley–Wiener spaces

3.1. Let Ω be a bounded, measurable set in $\mathbb{R}$. We denote by PW_Ω the Paley–Wiener space consisting of all functions $f \in L^2(\mathbb{R})$ whose Fourier transform vanishes a.e. on $\mathbb{R} \setminus \Omega$. Since Ω is bounded, the elements of the space PW_Ω are entire functions of finite exponential type.

A countable set $\Lambda \subset \mathbb{R}$ is called an interpolation set for PW_Ω if for every sequence $\{c(\lambda)\} \in \ell^2(\Lambda)$ there exists at least one $f \in PW_\Omega$ such that $f(\lambda) = c(\lambda)$, $\lambda \in \Lambda$. It is well known that such Λ must be a u.d. set, and there is a constant $K = K(\Lambda, \Omega)$ such that the solution f may be chosen to satisfy $\|f\|_{L^2(\mathbb{R})} \leqslant K \|\{c(\lambda)\}\|_{\ell^2(\Lambda)}$ (the latter follows from standard results in functional analysis).

3.2. We will need to interpolate by Schwartz functions with a given spectrum. Recall that the topology on the Schwartz space on $\mathbb{R}$ is determined by the family of seminorms

$$\|f\|_{m,k} := \sup_{x\in\mathbb{R}} |x^m f^{(k)}(x)| \quad (m, k \geqslant 0).$$

Lemma 3.1. *Let Λ be an interpolation set for PW_Ω where Ω is a compact set in $\mathbb{R}$, and let $\varepsilon > 0$ be given. Then, for any sequence $\{c(\lambda)\}$, $\lambda \in \Lambda$, satisfying*

$$\sup_{\lambda\in\Lambda} |c(\lambda)| \cdot (1 + |\lambda|)^N < \infty \quad (N = 1, 2, 3, \dots), \tag{3.1}$$

one can find a Schwartz function $f \in PW_{\Omega+[-\varepsilon,\varepsilon]}$ which solves the interpolation problem $f(\lambda) = c(\lambda)$, $\lambda \in \Lambda$, and moreover satisfies

$$\|f\|_{m,k} \leqslant C_{m,k} \sup_{\lambda\in\Lambda} |c(\lambda)| \cdot (1+|\lambda|)^m \quad (m,k \geqslant 0) \tag{3.2}$$

for some positive constants $C_{m,k} = C_{m,k}(\Lambda,\Omega,\varepsilon)$ which do not depend on $\{c(\lambda)\}$.

Proof. Choose functions $\varphi_\lambda \in PW_\Omega$ $(\lambda \in \Lambda)$ satisfying

$$\varphi_\lambda(\lambda) = 1, \quad \varphi_\lambda(\lambda') = 0 \quad (\lambda' \in \Lambda,\ \lambda' \neq \lambda)$$

and

$$\sup_{\lambda\in\Lambda} \|\varphi_\lambda\|_{L^2(\mathbb{R})} < \infty.$$

Observe that this implies that

$$M_k := \sup_{\lambda\in\Lambda} \|\varphi_\lambda^{(k)}\|_\infty < \infty \tag{3.3}$$

for every $k \geqslant 0$. Choose a Schwartz function Φ such that $\Phi(0) = 1$ and $\operatorname{spec}(\Phi) \subset (-\varepsilon,\varepsilon)$. Since Λ is a u.d. set we have that

$$L_{m,j} := \sup_{x\in\mathbb{R}} |x|^m \sum_{\lambda\in\Lambda} (1+|\lambda|)^{-m} |\Phi^{(j)}(x-\lambda)| < \infty \tag{3.4}$$

for every $m, j \geqslant 0$. Using (3.1), (3.3) and (3.4), we see that the function

$$f(x) = \sum_{\lambda\in\Lambda} c(\lambda)\, \Phi(x-\lambda)\, \varphi_\lambda(x)$$

is a Schwartz function in $PW_{\Omega+[-\varepsilon,\varepsilon]}$ and satisfies (3.2) with

$$C_{m,k} = \sum_{j=0}^{k} \binom{k}{j} L_{m,j}\, M_{k-j}.$$

Clearly f solves the interpolation problem, so this proves the lemma. □

4. The projection method

4.1. Let Γ be a lattice in $\mathbb{R}^2$. Consider the projections $p_1(x,y) = x$ and $p_2(x,y) = y$, and assume that the restrictions of p_1 and p_2 to Γ are injective, and so their images are dense in $\mathbb{R}$. Let Γ^* be the dual lattice, then the restrictions of p_1 and p_2 to Γ^* are also injective and have dense images.

If I is a bounded interval in $\mathbb{R}$, then the set

$$\Lambda(\Gamma, I) := \{p_1(\gamma) : \gamma \in \Gamma,\ p_2(\gamma) \in I\} \tag{4.1}$$

is called a "model set", or a "cut-and-project" set. Meyer observed ([16], p. 30, see also [17]) that these sets provide examples of non-periodic u.d. sets which support a measure μ, whose Fourier transform is also a pure point measure. Such a measure may be obtained by choosing a Schwartz function φ with $\operatorname{supp}(\widehat{\varphi}) \subset I$, and taking

$$\mu = \sum_{(x,y)\in\Gamma} \widehat{\varphi}(y)\,\delta_x. \tag{4.2}$$

The Fourier transform of μ is then the measure

$$\widehat{\mu} = \frac{1}{\det\Gamma} \sum_{(u,v)\in\Gamma^*} \varphi(v)\,\delta_u. \tag{4.3}$$

However, φ cannot also be supported on a bounded interval, and the support of the pure point measure $\widehat{\mu}$ is generally everywhere dense in $\mathbb{R}$.

Our approach is inspired by Meyer's construction, but an essential difference is that in our example, neither φ nor $\widehat{\varphi}$ will have a bounded support. We will nevertheless see that by a special choice of the function φ, the measure (4.2) and its Fourier transform (4.3) can each be supported by a discrete closed set, obtained by a certain generalization of the cut-and-project construction.

4.2. For completeness of the exposition, we formulate the correspondence between the measures (4.2) and (4.3) for a general Schwartz function φ, including a short proof.

Lemma 4.1. *Let φ be a Schwartz function on $\mathbb{R}$. Then* (4.2) *defines a translation-bounded measure μ on $\mathbb{R}$, whose Fourier transform is the* (*also translation-bounded*) *measure* (4.3).

Proof. Fix $M > 0$ such that every cube with side length 1 contains at most M points of Γ. For $x \in \mathbb{R}$ consider the cubes $B_k(x) := [x, x+1] \times [k, k+1)$, $k \in \mathbb{Z}$. Then

$$\int_x^{x+1} |d\mu| = \sum_{k\in\mathbb{Z}} \sum_{\gamma\in\Gamma\cap B_k(x)} |\widehat{\varphi}(p_2(\gamma))| \leqslant M \sum_{k\in\mathbb{Z}} \sup_{y\in[k,k+1)} |\widehat{\varphi}(y)| =: C(\Gamma,\varphi) < \infty.$$

Hence μ is a translation-bounded measure, and in the same way one can show that the measure in (4.3) is also translation-bounded. It remains to show that the latter measure is indeed the Fourier transform of μ.

Let ψ be a Schwartz function on $\mathbb{R}$. Then

$$\langle \widehat{\mu}, \psi \rangle = \langle \mu, \widehat{\psi} \rangle = \sum_{(x,y)\in\Gamma} \widehat{\psi}(x)\,\widehat{\varphi}(y) = \frac{1}{\det\Gamma} \sum_{(u,v)\in\Gamma^*} \psi(u)\,\varphi(v),$$

where the last equality follows from Poisson's summation formula. As this holds for every Schwartz function ψ, this confirms (4.3). □

4.3. Model sets also play an interesting role in the interpolation theory in Paley–Wiener spaces. It was proved in [18], [19] that there exist "universal" sets Λ of positive density, which serve as a set of interpolation for PW_Ω whenever Ω is a finite union of intervals with sufficiently large measure. An example of such universal interpolation sets can also be obtained by the "cut-and-project" construction:

Theorem M ([14], [15]). *Let I be a bounded interval in $\mathbb{R}$. Then the set $\Lambda(\Gamma, I)$ defined by (4.1) is an interpolation set for PW_Ω whenever Ω is a finite union of intervals such that*

$$\operatorname{mes}(\Omega) > \frac{|I|}{\det \Gamma}.$$

Here $|I|$ denotes the length of the interval I.

5. The construction

5.1. Suppose that we are given a sequence of real numbers

$$0 = a_0 < a_1 < a_2 < a_3 < \cdots, \quad a_n \to \infty \quad (n \to \infty)$$

and also another sequence

$$0 < h_1 < h_2 < h_3 < \cdots, \quad h_n \to \infty \quad (n \to \infty).$$

We partition the plane $\mathbb{R}^2$ into two disjoint sets A, B defined by

$$A = \bigcup_{n=1}^{\infty} \{(x,y) : |x| \geqslant a_{n-1},\ |y| \leqslant h_n\}, \tag{5.1}$$

$$B = \bigcup_{n=1}^{\infty} \{(x,y) : |x| < a_n,\ |y| > h_n\}, \tag{5.2}$$

and consider the two sets

$$\Lambda := \{p_1(\gamma) : \gamma \in \Gamma \cap A\}, \quad Q := \{p_2(\gamma) : \gamma \in \Gamma \cap B\}.$$

Observe that Λ and Q are both discrete closed sets in $\mathbb{R}$ (see Figure 1). Also observe that if φ is a Schwartz function such that $\widehat{\varphi}$ vanishes on Q, then the support of the measure μ in (4.2) is contained in Λ.

Suppose now that we are given two other sequences $\{a_n^*\}$, $\{h_n^*\}$ with properties similar to $\{a_n\}$, $\{h_n\}$, and that these two sequences determine a partition of $\mathbb{R}^2$ into two disjoint sets A^*, B^* defined similarly to A, B. Let

$$S := \{p_1(\gamma^*) : \gamma^* \in \Gamma^* \cap A^*\}, \quad Z := \{p_2(\gamma^*) : \gamma^* \in \Gamma^* \cap B^*\}.$$

As before, S and Z are two discrete closed sets in $\mathbb{R}$ (see Figure 2). If φ vanishes on Z, then according to (4.3) the spectrum of the measure μ is contained in S.

5.2. Our goal will thus be to construct sequences $\{a_n\}$, $\{h_n\}$ and $\{a_n^*\}$, $\{h_n^*\}$ with the properties above, and a Schwartz function φ (not identically zero), such that φ and $\widehat{\varphi}$ are both real-valued, φ vanishes on Z and $\widehat{\varphi}$ vanishes on Q. As we have seen, this would give a non-zero measure μ satisfying property (i) in Theorem 2.

We choose the sequences $\{a_n\}$, $\{h_n\}$ in an arbitrary way, and this choice defines the sets Λ and Q.

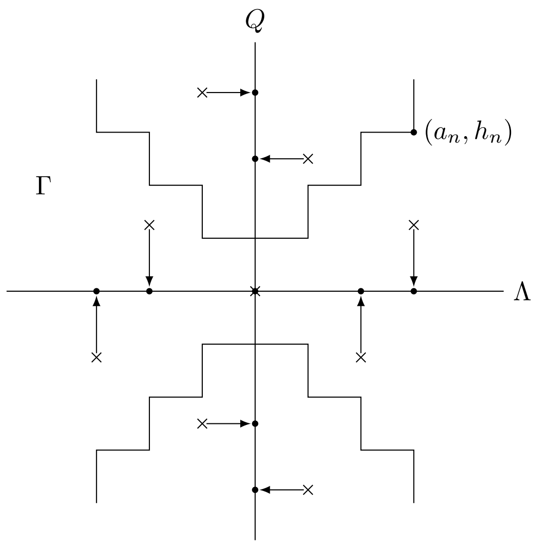

FIGURE 1. Construction of Λ and Q from the lattice Γ.

We also choose the $\{a_n^*\}$ arbitrarily, and for each $n \geqslant 0$ we let Ω_n be a finite union of closed intervals in $\mathbb{R}$, such that

$$\Omega_n = -\Omega_n, \quad \Omega_n \subset \mathbb{R} \setminus Q, \quad \operatorname{mes}(\Omega_n) > \frac{2a_n^*}{\det \Gamma^*}, \quad \Omega_0 \subset \Omega_1 \subset \Omega_2 \subset \cdots$$

(it may be convenient here to notice that $-Q = Q$).

Let φ_0 be a Schwartz function that is real-valued, even,

$$\varphi_0(0) = 1, \quad \operatorname{spec}(\varphi_0) \subset \Omega_0.$$

Now we construct by induction on n the sequence $\{h_n^*\}$ and Schwartz functions φ_n, real-valued and even, such that

(a) $\varphi_n(0) = 1$;

(b) $\operatorname{spec}(\varphi_n) \subset \Omega_n$;

(c) $\|\varphi_n - \varphi_{n-1}\|_{m,k} < 2^{-n} \qquad (0 \leqslant m, k \leqslant n)$;

(d) φ_n vanishes on Z_n, where

$$Z_n := \{p_2(\gamma^*) : \gamma^* \in \Gamma^* \cap B_n^*\}, \quad B_n^* := \bigcup_{k=1}^{n} \{(x, y) : |x| < a_k^*, \, |y| > h_k^*\}.$$

Property (c) ensures that φ_n converges in the Schwartz space to a limit φ, real-valued and even, not identically zero by (a), which vanishes on Z due to (d), and such that $\widehat{\varphi}$ vanishes on Q due to (b). So the measure μ in (4.2) has support in Λ and spectrum in S as required.

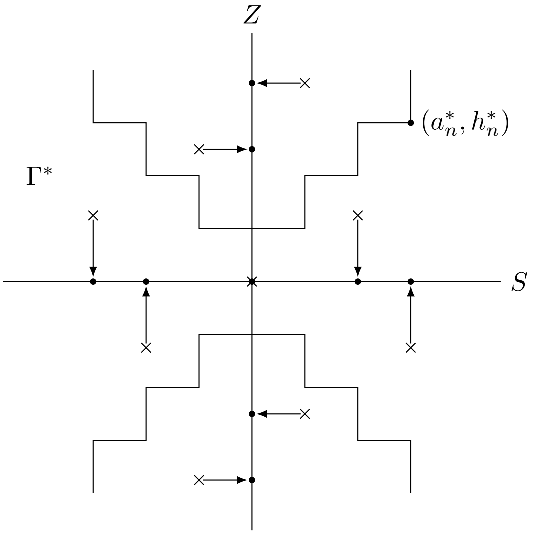

FIGURE 2. A similar construction of S and Z from the dual lattice Γ^*.

To construct the number h_n^* and the function φ_n at the n'th step of the induction, we let J denote a finite union of closed intervals such that

$$\mathrm{mes}(J) > \frac{2a_n^*}{\det \Gamma^*}, \quad J + [-\varepsilon, \varepsilon] \subset \Omega_n$$

for an appropriate $\varepsilon > 0$. Now consider the model set

$$X = X_n := \{p_2(\gamma^*) : \gamma^* \in \Gamma^*, \, |p_1(\gamma^*)| < a_n^*\}.$$

By Theorem M, it is an interpolation set for PW_J. Define

$$C := \sup_{0 \leqslant m,k \leqslant n} C_{m,k}(X, J, \varepsilon)$$

where $C_{m,k}(X, J, \varepsilon)$ is the constant from Lemma 3.1.

Since φ_{n-1} is a Schwartz function, we have

$$\sup_{\lambda \in X} |\varphi_{n-1}(\lambda)| \cdot (1 + |\lambda|)^N < \infty \quad (N = 1, 2, 3, \dots).$$

We choose the number h_n^* sufficiently large such that

$$\sup_{\lambda \in X, |\lambda| > h_n^*} |\varphi_{n-1}(\lambda)| \cdot (1 + |\lambda|)^n < \frac{1}{C \cdot 2^n} \, ,$$

and consider the interpolation problem

$$f(\lambda) = c(\lambda), \quad \lambda \in X, \tag{5.3}$$

where

$$c(\lambda) := \begin{cases} 0, & \lambda \in X, \ |\lambda| \leqslant h_n^*, \\ \varphi_{n-1}(\lambda), & \lambda \in X, \ |\lambda| > h_n^*. \end{cases}$$

By Lemma 3.1, there is a Schwartz function f satisfying (5.3) such that

$$\operatorname{spec}(f) \subset J + [-\varepsilon, \varepsilon] \subset \Omega_n \quad \text{and} \quad \sup_{0 \leqslant m,k \leqslant n} \|f\|_{m,k} < 2^{-n}.$$

Since $\{c(\lambda)\}$ is a real-valued, even sequence, by replacing f with

$$\operatorname{Re}\Big[\frac{f(t) + f(-t)}{2}\Big]$$

we may assume that f is real-valued and even. We then take

$$\varphi_n := \varphi_{n-1} - f.$$

It is clear that φ_n satisfies conditions (b) and (c) above. To check that (a) and (d) are also satisfied, we first use the fact that $f(\lambda) = 0$ for $\lambda \in X$, $|\lambda| \leqslant h_n^*$. It implies that $\varphi_n(0) = \varphi_{n-1}(0) = 1$ and

$$\varphi_n(\lambda) = \varphi_{n-1}(\lambda) = 0, \quad \lambda \in Z_n \cap [-h_n^*, h_n^*],$$

where the latter is true because $Z_n \cap [-h_n^*, h_n^*] \subset Z_{n-1}$ and φ_{n-1} vanishes on Z_{n-1}. On the other hand, since $f(\lambda) = \varphi_{n-1}(\lambda)$ for $\lambda \in X$, $|\lambda| > h_n^*$, we obtain also

$$\varphi_n(\lambda) = 0, \quad \lambda \in Z_n \setminus [-h_n^*, h_n^*].$$

This confirms that conditions (a)–(d) hold, and completes the inductive construction.

6. Arithmetic progressions

6.1. To complete the proof of Theorem 2, it remains to show how to satisfy property (ii) in the construction above. For this we need the following proposition, which provides the relation of the construction to arithmetic progressions.

Lemma 6.1. *Let P be an arithmetic progression in $\mathbb{R}$. Then the set*

$$\{(x, y) \in \Gamma : x \in P\}$$

is contained in a straight line in $\mathbb{R}^2$ which is not parallel to the x-axis.

Proof. Suppose that $\gamma_1, \gamma_2, \gamma_3$ are three distinct points in Γ, whose images under p_1 lie in P. Then there are non-zero integers m_1, m_2, m_3 such that $m_1 + m_2 + m_3 = 0$ and

$$m_1 p_1(\gamma_1) + m_2 p_1(\gamma_2) + m_3 p_1(\gamma_3) = 0.$$

Since p_1 restricted to Γ is injective, this implies that

$$m_1\gamma_1 + m_2\gamma_2 + m_3\gamma_3 = 0,$$

hence the points $\gamma_1, \gamma_2, \gamma_3$ lie on a line in $\mathbb{R}^2$. Since p_2 restricted to Γ is also injective, it follows that the line is not parallel to the x-axis. □

Now recall that the sequences $\{a_n\}$, $\{h_n\}$ have been chosen in an arbitrary way. To satisfy property (ii) we choose them such that the domain A defined by (5.1) contains only a bounded part of any straight line not parallel to the x-axis (for this it is enough that a_n grows much faster than h_n). By Lemma 6.1 this implies that Λ contains only finitely many elements of any arithmetic progression. So Theorem 2 is proved.

6.2. There are several other ways to illustrate the non-periodic nature of the measure in our example, in addition to property (ii) stated above. Observe that (ii) implies:

(iii) Λ may not be covered by any finite union of arithmetic progressions.

This property is true for the spectrum as well, namely:

(iv) Also S may not be covered by a finite union of arithmetic progressions.

In fact, the properties (iii) and (iv) hold due to the following.

Proposition 6.2. *The support of any measure μ of the form (4.2) may not be covered by a finite union of arithmetic progressions, and the same is true for the support of the measure $\widehat{\mu}$, unless the function φ vanishes identically.*

Proof. Indeed, if the support of the measure μ is contained in a finite union of arithmetic progressions, then by Lemma 6.1 the set

$$\Gamma_0 := \{(x, y) \in \Gamma : \widehat{\varphi}(y) \neq 0\}$$

must be contained in a finite union of lines. Thus $p_2(\Gamma_0)$ is a discrete closed set in $\mathbb{R}$. But on the other hand, since $p_2(\Gamma)$ is dense in $\mathbb{R}$, the closure of $p_2(\Gamma_0)$ must be equal to $\operatorname{supp}(\widehat{\varphi})$, a contradiction. Hence the support of μ is not contained in any finite union of arithmetic progressions, and similarly, the same is true for the support of $\widehat{\mu}$. □

In the first version of this paper the property (ii) in Theorem 2 was not mentioned explicitly, being replaced by (iii) and (iv) above. In this weaker form, another proof of our result was given by Kolountzakis [9], who used an infinite sum of appropriately chosen Poisson measures.

One may actually consider stronger versions of properties (iii) and (iv). For instance, given $\varepsilon > 0$ there is a decomposition of μ as the sum of two measures $\mu = \mu_1 + \mu_2$ such that μ_1 is supported on a model set and $\|\mu_2\| < \varepsilon$, where by $\|\cdot\|$ we denote the natural norm on the space of translation-bounded measures defined by (2.1). It follows (again by Lemma 6.1) that μ may not even be approximated, with respect to this norm, to arbitrary degree by measures whose supports are contained in finite unions of arithmetic progressions. The same is true for the measure $\widehat{\mu}$.

7. Remarks

7.1. One may obtain results of similar type also in $\mathbb{R}^n$. For instance, the product $\mu \times \cdots \times \mu$ (n times) of the measure μ in our proof gives an example of a measure in $\mathbb{R}^n$, whose support and spectrum are both discrete closed sets, and the support cannot be covered by any finite union of translated lattices, nor may it contain infinitely many elements of an arithmetic progression.

7.2. By choosing the sequence $\{a_n\}$ increasing sufficiently fast, the measure μ may be constructed with the additional property that the minimal distance between consecutive points of Λ in the interval $(-R, R)$ approaches zero arbitrarily slowly as $R \to \infty$.

7.3. In [11] and [12], we obtained an affirmative answer to Problem 4.1 (a) in [10], which asked whether it is true that a positive measure μ in $\mathbb{R}^n$ may have uniformly discrete support and spectrum only if each of them is contained in a finite union of translates of some lattice.

Problem 4.1 (b) from that paper asks whether the periodic structure is still necessary if the support and the spectrum are just discrete closed sets. Here we basically answer this question in the negative, but without the positivity of the measure μ. At present we do not know whether in our construction one can get a positive measure.

7.4. To each measure satisfying (1.1) and (1.2) corresponds a weighted summation formula

$$\sum_{\lambda \in \Lambda} \mu(\lambda)\, \widehat{f}(\lambda) = \sum_{s \in S} \widehat{\mu}(s)\, f(s), \tag{7.1}$$

which holds for any Schwartz function f. If μ and $\widehat{\mu}$ are translation-bounded measures (or, more generally, measures with polynomial growth) then both series in (7.1) converge absolutely, otherwise an appropriate summation method should be used to sum them.

An interesting summation formula may be found in [7], p. 265, which involves weighted sums of f and $\widehat{f}$ at the nodes $\{\pm(n + 1/9)^{1/2}\}$ $(n = 0, 1, 2, \dots)$. However it is not clear to which class of functions it applies. In particular, whether it corresponds to a temperate distribution. Remark that the nodes in this example contain two arithmetic progressions $3\mathbb{Z} \pm 1/3$.

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