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Rigidity of fiber-preserving quasisymmetric maps

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Abstract. We show that fiber-preserving quasisymmetric maps are biLipschitz. As an application, we show that quasisymmetric maps on Carnot groups with reducible first stratum are biLipschitz.

1. Introduction

In this paper we study the rigidity property of quasisymmetric maps that preserve a foliation. We show that, under quite general conditions, such quasisymmetric maps are biLipschitz. We then give an application to quasisymmetric maps between Carnot groups.

Quasisymmetric maps that preserve a foliation arise when one studies the rigidity property of quasiisometries between negatively curved solvable Lie groups, see [10], [12], [15], and [18]. It is well known that a negatively curved space has an ideal boundary, and quasiisometries between negatively curved spaces correspond to quasisymmetric maps between the ideal boundaries. Rigidity properties of the quasiisometries correspond to the rigidity properties of the quasisymmetric maps. The ideal boundary of a negatively curved solvable Lie group is a nilpotent Lie group equipped with a homogeneous distance. Recent results suggest that very often the quasisymmetric maps on such nilpotent Lie groups must be biLipschitz, [9], [3], [4], [12], [15], [16], [18], [19].

There are usually two steps in the proof of the above mentioned rigidity property of the quasisymmetric maps. The first step is to show that the quasisymmetric map preserves a certain foliation. The second step is to show that a quasisymmetric map must be biLipschitz if it preserves a foliation. The main purpose of this paper is to take a closer look at the second step in the context of quasisymmetric maps between general metric spaces. We hope that this will be useful in the eventual (hopefully) complete solution of the rigidity of quasiisometries between negatively curved solvable Lie groups.

One often has to deal with quasimetrics instead of metrics while studying the ideal boundary of negatively curved spaces. For this reason we state our main

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result for quasimetric spaces. Recall that a function $d\colon X \times X \to [0,\infty)$ is a *quasimetric* on a set X if

(1) d is symmetric, that is, $d(x_1,x_2) = d(x_2,x_1)$ for all $x_1,x_2 \in X$;

(2) $d(x_1,x_2) = 0$ if and only if $x_1 = x_2$;

(3) there exists some constant $M \geq 1$ such that $d(x_1,x_3) \leq M \cdot (d(x_1,x_2) + d(x_2,x_3))$ for all $x_1,x_2,x_3 \in X$.

We call M a quasimetric constant of d. A subset A of a quasimetric space X is called *closed* if for any $x \in X$ and for any sequence $\{a_j\} \subset A$, the condition $d(a_j,x) \to 0$ implies $x \in A$.

To state our main result, we introduce the notion of "fibered quasimetric space". To do that we first specify the concept of parallelism for sets. We say that two closed subsets U and V of a quasimetric space X are *parallel* if there is a constant $a > 0$ such that $d(u,V) = d(v,U) = a$ for every $u \in U$ and every $v \in V$. Recall that $d(u,V) := \inf\{d(u,v) : v \in V\}$. It is easy to check that in this case we have $d(U,V) = HD(U,V) = a$, where

$$d(U,V) := \inf\{d(u,v) \,|\, u \in U,\ v \in V\},$$

and $HD(U,V)$ denotes the Hausdorff distance between U and V:

$$HD(U,V) := \sup(\{d(u,V) : u \in U\} \cup \{d(v,U) : v \in V\}).$$

The following definition provides the setting for our main theorem.

Definition 1.1 (Fibered quasimetric space). Let $\alpha > 0$, $L \geq 1$. We say that a quasimetric space X is an (α, L)-*fibered quasimetric space* if X admits a cover $\mathcal{U}$ by closed pairwise disjoint subsets, called *fibers*, with the following properties:

(1.2) Fibers are snow-flake equivalent to unbounded geodesic spaces: for each $U \in \mathcal{U}$, there exists an unbounded geodesic space $(\widetilde{U}, d)$ such that U is L-biLipschitz to $(\widetilde{U}, d^\alpha)$.

(1.3) Parallel fibers are not isolated: for any $U \in \mathcal{U}$, there is a sequence $U_i \in \mathcal{U}$ such that U_i and U are distinct, parallel, and $HD(U_i,U) \to 0$ as $i \to \infty$.

(1.4) Fibers have positive distance: for any two distinct $U,V \in \mathcal{U}$, $d(U,V) > 0$.

(1.5) Non-parallel fibers diverge: if $U,V \in \mathcal{U}$ are not parallel, $HD(U,V) = \infty$.

Given $K \geq 1$ and $C > 0$, we say that a bijection $F\colon X \to Y$ between two quasimetric spaces is a (K,C)-*quasi-similarity* if

$$\frac{C}{K} \cdot d(x_1,x_2) \leq d(F(x_1),F(x_2)) \leq CK \cdot d(x_1,x_2), \quad \forall x_1,x_2 \in X.$$

Clearly a map is a quasi-similarity if and only if it is biLipschitz. The point here is that often there is control on K but not on C. In this case, the notion of quasi-similarity provides more information about the distortion. The same notion also appears in other works with a different name: K-conformally biLipschitz with scale factor C.

The following is our main result and states that any fiber-preserving quasisymmetric map between fibered quasimetric spaces is a quasi-similarity (and hence biLipschitz). See Section 2.1 for the definition of quasisymmetric map.

Theorem 1.1. *Let X, Y be (α, L)-fibered quasimetric spaces for some $\alpha > 0$ and $L \geq 1$. Suppose $F\colon X \to Y$ is an η-quasisymmetric map that sends fibers of X homeomorphically onto fibers of Y. Then F is a (K, C)-quasi-similarity, where K depends only on η, α, L and the quasimetric constants of X, Y.*

We remark that for the validity of Theorem 1.1 the condition $d(u, V) = d(v, U) = a$ for parallel fibers can not be replaced by the weaker condition $HD(U, V) < \infty$. See the end of Section 3 for an example.

Theorem 1.1 in particular implies that all quasiconformal maps of the sub-Riemannian Heisenberg groups that send vertical lines to vertical lines must be biLipschitz, see Proposition 4.5. See [13] and [1] for the construction of such maps. We remark that there exist biLipschitz maps of Heisenberg groups that send vertical lines to curves that are not vertical lines, see [17].

Our main application of Theorem 1.1 is to quasisymmetric maps on Carnot groups with reducible first stratum. Let G be a Carnot group. See Section 2.2 for a brief introduction to Carnot groups. Let $V_1 \oplus \cdots \oplus V_r$ be the stratification of the Lie algebra $\mathfrak{g}$ of G. For each $t > 0$, the standard dilation $\lambda_t\colon \mathfrak{g} \to \mathfrak{g}$ is defined by $\lambda_t(v) = t^i v$ for $v \in V_i$. A Lie algebra isomorphism $A\colon \mathfrak{g} \to \mathfrak{g}$ is called a *strata-preserving* automorphism if $A(V_i) = V_i$, for $i = 1, \ldots, r$. Let $\mathrm{Aut}_*(\mathfrak{g})$ be the group of strata-preserving automorphisms of $\mathfrak{g}$. We say V_1 is *reducible* (or the first stratum of $\mathfrak{g}$ is reducible) if there is a non-trivial proper linear subspace $W_1 \subset V_1$ such that $A(W_1) = W_1$ for every $A \in \mathrm{Aut}_*(\mathfrak{g})$. See the last part of Section 4.1 for more information on Carnot groups with reducible first stratum.

The main consequence of Theorem 1.1 is the following theorem. Recall that a Carnot group is a metric space equipped with a particular Carnot–Carathéodory distance, see Section 2.2.

Theorem 1.2. *Let G be a Carnot group with reducible first stratum. Then every quasisymmetric map $F\colon G \to G$ is a quasi-similarity, quantitatively.*

Theorem 1.2 is quantitatively in the sense that for any Carnot group G and for any $\eta \in \mathrm{Homeo}([0, \infty))$ there exists a constant K depending only on G and η such that every η-quasisymmetric map $F\colon G \to G$ is a (K, C)-*quasi-similarity* for some C.

We remark that in [3], Cowling and Ottazzi proved that all smooth quasiconformal maps on rigid Carnot groups are affine maps (an affine map is the composition of a left translation and a strata-preserving automorphism) and so in particular are biLipschitz. On the other hand, it is shown in [19] that if N is a non-rigid Carnot group other than the Euclidean groups and certain quotients of the product of the same Heisenberg group or the same complexified Heisenberg group, then every quasisymmetric map of N is biLipschitz. We note that Carnot groups with reducible first stratum include both rigid and non-rigid groups (see end of Section 4.1 for more details). In Theorem 1.2 there is no assumption on the regularity of the quasisymmetric map.

In Section 2 we recall some basic definitions and facts that are used later. In Section 3 we prove Theorem 1.1. In Section 4 we prove Theorem 1.2 and also give two other applications of Theorem 1.1: quasiconformal maps of Heisenberg groups that send vertical lines to vertical lines, and a new proof of a theorem of Dymarz [5] on quasisymmetric maps of ideal boundary of certain amenable hyperbolic locally compact groups.

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2. Preliminaries

In this section we collect definitions and results needed later.

2.1. Quasiconformal and quasisymmetric maps

In this subsection we recall the definitions of quasiconformal and quasisymmetric maps.

A map $F\colon X \to Y$ between two quasimetric spaces is continuous if for any $x \in X$ and any sequence $\{x_j\} \subset X$, the condition $\lim_{j\to\infty} d(x, x_j) = 0$ implies $\lim_{j\to\infty} d(F(x), F(x_j)) = 0$.

Let $F\colon X \to Y$ be a bijection between two quasimetric spaces. For $x \in X$ and $t > 0$, define

$$H_F(x,t) = \frac{\sup\{d(F(x'),F(x)) | d(x',x) \le t\}}{\inf\{d(F(x'),F(x)) | d(x',x) \ge t\}}.$$

The map F is called λ-*quasiconformal* if both F and F^{-1} are continuous and

$$\limsup_{t\to 0} H_F(x,t) \le \lambda$$

for all $x \in X$. We say F is quasiconformal if it is λ-quasiconformal for some $\lambda \ge 1$.

Let $\eta\colon [0,\infty) \to [0,\infty)$ be a homeomorphism. A bijection $F\colon X \to Y$ between two quasimetric spaces is η-*quasisymmetric* if both F and F^{-1} are continuous and for all distinct triples $x, y, z \in X$, we have

$$\frac{d(F(x),F(y))}{d(F(x),F(z))} \le \eta\Big(\frac{d(x,y)}{d(x,z)}\Big).$$

If $F\colon X \to Y$ is an η-quasisymmetry, then $F^{-1}\colon Y \to X$ is an η_1-quasisymmetry, where $\eta_1(t) = (\eta^{-1}(t^{-1}))^{-1}$, see Theorem 6.3 in [14]. A bijection between quasimetric spaces is said to be quasisymmetric if it is η-quasisymmetric for some η.

We remark that quasisymmetric maps between general quasimetric spaces are quasiconformal. In the case of Carnot groups, and more generally Ahlfors regular Loewner spaces, a quasiconformal homeomorphism is locally quasisymmetric, see Theorem 4.7 in [6].

2.2. Stratified Lie algebras and Carnot groups

In this subsection we review the basic definitions related to Carnot groups.

A *stratified Lie algebra* is a finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ together with a direct sum decomposition $\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_s$ of nontrivial vector subspaces such that $[V_1, V_i] = V_{i+1}$ for all $1 \leq i \leq s$, where we set $V_{s+1} = \{0\}$. The integer s is called step or degree of nilpotency of $\mathfrak{g}$. Every stratified Lie algebra $\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_s$ admits a one-parameter family of automorphisms $\lambda_t \colon \mathfrak{g} \to \mathfrak{g}$, $t \in (0, \infty)$, where $\lambda_t(x) = t^i x$ for $x \in V_i$.

Let $\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_s$ and $\mathfrak{g}' = V_1' \oplus V_2' \oplus \cdots \oplus V_s'$ be two stratified Lie algebras. A Lie algebra isomorphism $\phi\colon \mathfrak{g} \to \mathfrak{g}'$ is such that $\phi(V_i) = V_i'$ for all $1 \leq i \leq s$ if and only if it commutes with λ_t for all $t > 0$; that is, if $\phi \circ \lambda_t = \lambda_t \circ \phi$. We call such isomorphisms *strata preserving*.

Let G be a connected, simply connected Lie group whose Lie algebra is stratified as $\mathrm{Lie}(G) = V_1 \oplus \cdots \oplus V_s$. The subspace V_1 defines a left-invariant distribution $HG \subset TG$ on G. We fix a left-invariant inner product on HG. An absolutely continuous curve γ in G whose velocity vector $\gamma'(t)$ is contained in $H_{\gamma(t)}G$ for almost every t is called a *horizontal curve*. Any horizontal curve has an associated length defined using the left-invariant inner product, i.e., by integrating the norm of its tangent vector. Since V_1 generates the whole Lie algebra, any two points of G can be connected by horizontal curves. Let $p, q \in G$; the *Carnot–Carathéodory metric* between p and q is denoted by $d_c(p, q)$ and is defined as the infimum of length of horizontal curves joining p and q. We call *Carnot group* the data of G, its stratification, the inner product on its first stratum, and consequently the distance d_c.

Since the inner product on HG is left invariant, the Carnot metric on G is left invariant as well. Different choices of inner product on HG result in Carnot metrics that are biLipschitz equivalent. The Hausdorff dimension of G with respect to a Carnot metric is given by $\sum_{i=1}^{s} i \cdot \dim(V_i)$.

Recall that, for a simply connected nilpotent Lie group G with Lie algebra $\mathfrak{g}$, the exponential map $\exp\colon \mathfrak{g} \to G$ is a diffeomorphism. Under this identification the Lesbegue measure on $\mathfrak{g}$ is a Haar measure on G. Furthermore, the exponential map induces a one-to-one correspondence between Lie subalgebras of $\mathfrak{g}$ and connected Lie subgroups of G.

Let G be a Carnot group with Lie algebra $\mathfrak{g} = V_1 \oplus \cdots \oplus V_s$. Since $\lambda_t \colon \mathfrak{g} \to \mathfrak{g}$ $(t > 0)$ is a Lie algebra automorphism and G is simply connected, there is a unique Lie group automorphism $\Lambda_t \colon G \to G$ whose differential at the identity is λ_t. For each $t > 0$, Λ_t is a similarity with respect to the Carnot metric: $d(\Lambda_t(p), \Lambda_t(q)) = t\, d(p, q)$ for any two points $p, q \in G$. For a Lie group isomorphism $f\colon G \to G'$ between two Carnot groups, the corresponding Lie algebra isomorphism $f_*\colon \mathrm{Lie}(G) \to \mathrm{Lie}(G')$ is strata preserving if and only if f commutes with Λ_t for all $t > 0$; that is, if $f \circ \Lambda_t = \Lambda_t \circ f$.

2.3. The Baker–Campbell–Hausdorff formula

In this subsection we review the BCH formula. Let G be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. The exponential map $\exp\colon \mathfrak{g} \to G$ is a diffeomorphism. One can then pull back the group operation from G to get a group structure on $\mathfrak{g}$. This group structure can be described by the Baker–Campbell–Hausdorff formula (BCH formula in short), which expresses the product $X * Y$ ($X, Y \in \mathfrak{g}$) in terms of the iterated Lie brackets of X and Y. The group operation in G will be denoted by $\cdot$. The pull-back group operation $*$ on $\mathfrak{g}$ is defined as follows. For $X, Y \in \mathfrak{g}$, define

$$X * Y = \exp^{-1}(\exp X \cdot \exp Y).$$

Then the first a few terms of the BCH formula ([2], page 11) is given by

$$X * Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \cdots .$$

2.4. Pansu differentiability theorem

In this subsection we recall the definition of Pansu differential and the Pansu differentiability theorem.

Definition 2.1. Let G and G' be two Carnot groups endowed with Carnot metrics, and $U \subset G$, $U' \subset G'$ open subsets. A map $F\colon U \to U'$ is *Pansu differentiable* at $x \in U$ if there exists a strata preserving homomorphism $L\colon G \to G'$ such that

$$\lim_{y \to x} \frac{d(F(x)^{-1} * F(y),\, L(x^{-1} * y))}{d(x, y)} = 0.$$

In this case, the strata preserving homomorphism $L\colon G \to G'$ is called the *Pansu differential* of F at x, and is denoted by $dF(x)$.

The following result (except the terminology) is due to Pansu [9].

Theorem 2.1. *Let G and G' be Carnot groups, and let $U \subset G$ and $U' \subset G'$ be open subsets. Let $F\colon U \to U'$ be a quasiconformal map. Then F is almost everywhere Pansu differentiable. Furthermore, at a.e. $x \in U$, the Pansu differential $dF(x)\colon G \to G'$ is a strata preserving isomorphism.*

3. Fiber-preserving quasisymmetric maps

In this section we prove Theorem 1.1. At the end of this section we give an example that shows for the validity of Theorem 1.1 the condition $d(u, V) = d(v, U) = a$ for parallel fibers can not be replaced by the weaker condition $HD(U, V) < \infty$.

Let X, Y be (α, L)-fibered quasimetric spaces and $F\colon X \to Y$ an η-quasisymmetric map that sends fibers of X onto fibers of Y. We shall prove that F is a (K, C)-quasi-similarity, where K depends only on η, α, L and the quasimetric constants of X, Y. We first show that F maps parallel fibers to parallel fibers (Lemma 3.1), then show that the restriction of F to each fiber is a quasi-similarity (Lemma 3.4), and finally show that F is itself a quasi-similarity (Lemma 3.5).

Lemma 3.1. *If two fibers U and V in X are parallel, then $F(U)$ and $F(V)$ are parallel fibers in Y.*

Proof. Suppose $F(U)$ and $F(V)$ are not parallel. By Condition (1.5), after possibly switching U and V, we may assume that there exist a sequence $x_i \in U$ such that $d(F(x_i), F(V)) \to \infty$. Since U and V are parallel, for each i, there exists $v_i \in V$ such that $d(x_i, v_i) = d(U, V)$. Since by Condition (1.2) the set U is path connected and unbounded, there are points on U at arbitrary distance from any point. Hence, for each i, there exists $u_i \in U$ such that $d(x_i, u_i) = d(x_i, v_i)$.

Since $d(F(x_i), F(V)) \to \infty$, we have $d(F(x_i), F(v_i)) \to \infty$. The quasisymmetry condition and the fact $d(x_i, u_i) = d(x_i, v_i)$ imply that $d(F(x_i), F(u_i)) \to \infty$.

Because of Condition (1.3), we can take a fiber $U' \neq F(U)$ parallel to $F(U)$. Again by Condition (1.2) choose $y_i \in U'$ so that $d(F(x_i), y_i) = d(F(U), U')$. Then $d(F(x_i), y_i)/d(F(x_i), F(u_i)) \to 0$ as $i \to \infty$. Now the quasisymmetry condition for F^{-1} implies

$$\frac{d(x_i, F^{-1}(y_i))}{d(U, V)} = \frac{d(x_i, F^{-1}(y_i))}{d(x_i, u_i)} \to 0 \quad \text{as } i \to \infty.$$

It follows that $d(x_i, F^{-1}(y_i)) \to 0$. Since $x_i \in U$ and $F^{-1}(y_i) \in F^{-1}(U')$, we have $d(U, F^{-1}(U')) = 0$, contradicting Condition (1.4) and the fact $U' \neq F(U)$. □

The next two results are similar to Lemma 15.3 and Corollary 15.4 in [11]. The proofs are modifications of their arguments.

Lemma 3.2. *There exists $K_1 \geq 1$ depending only on η, α and L so that for any two parallel fibers U_1, U_2 in X and any $p, q \in U_1$ satisfying $d(p, q) \geq L \cdot d(U_1, U_2)$ and $d(F(p), F(q)) \geq L \cdot d(F(U_1), F(U_2))$ we have*

$$\frac{1}{K_1} \cdot \frac{d(F(U_1), F(U_2))}{d(U_1, U_2)} \leq \frac{d(F(p), F(q))}{d(p, q)} \leq K_1 \cdot \frac{d(F(U_1), F(U_2))}{d(U_1, U_2)}. \tag{3.1}$$

Proof. By Condition (1.2), there exists a geodesic space $(\widetilde{X}_1, d_1)$ and an L-biLipschitz map $f_1 \colon (U_1, d) \to (\widetilde{X}_1, d_1^{\alpha})$. The assumption $d(p, q) \geq L \cdot d(U_1, U_2)$ implies

$$d_1(f_1(p), f_1(q)) \geq (d(U_1, U_2))^{1/\alpha}.$$

There is some integer $k \geq 2$ such that

$$(k-1) \cdot (d(U_1, U_2))^{1/\alpha} \leq d_1(f_1(p), f_1(q)) < k \cdot (d(U_1, U_2))^{1/\alpha}.$$

The biLipschitz property of f_1 implies

$$\frac{1}{L}(k-1)^{\alpha} \cdot d(U_1, U_2) \leq d(p, q) < L\, k^{\alpha} \cdot d(U_1, U_2). \tag{3.2}$$

Since $(\widetilde{X}_1, d_1)$ is a geodesic space, there are points $p = p_0, p_1, \ldots, p_k = q$ in U_1 such that $\frac{1}{2} d(U_1, U_2)^{1/\alpha} \leq d_1(f_1(p_i), f_1(p_{i+1})) \leq d(U_1, U_2)^{1/\alpha}$. We have

$$\frac{1}{L\, 2^{\alpha}} \cdot d(U_1, U_2) \leq d(p_i, p_{i+1}) \leq L \cdot d(U_1, U_2).$$

Since U_1 and U_2 are parallel, Lemma 3.1 implies that $F(U_1)$ and $F(U_2)$ are also parallel. Let $q_i \in U_2$ be a point such that $d(F(p_i), F(q_i)) = d(F(U_1), F(U_2))$. We have

$$d(p_i, p_{i+1}) \le L \cdot d(U_1, U_2) \le L \cdot d(p_i, q_i).$$

Since F is η-quasisymmetric, we have

$$d(F(p_i), F(p_{i+1})) \le \eta(L) \cdot d(F(p_i), F(q_i)) = \eta(L) \cdot d(F(U_1), F(U_2)).$$

By Condition (1.2) again, there exists some geodesic space $(\widetilde{Y}_1, \rho_1)$ and an L-bi-Lipschitz map $g_1 \colon (F(U_1), d) \to (\widetilde{Y}_1, \rho_1^\alpha)$. It follows that

$$\rho_1(g_1 \circ F(p_i),\ g_1 \circ F(p_{i+1})) \le (L\eta(L))^{1/\alpha} \cdot d(F(U_1), F(U_2))^{1/\alpha}.$$

The triangle inequality for the metric space $(\widetilde{Y}_1, \rho_1)$ implies

$$\rho_1(g_1 \circ F(p),\ g_1 \circ F(q)) \le k(L\eta(L))^{1/\alpha} \cdot d(F(U_1), F(U_2))^{1/\alpha}.$$

Now the biLipschitz property of g_1 implies

$$d(F(p), F(q)) \le L \cdot \rho_1^\alpha(g_1 \circ F(p), g_1 \circ F(q)) \le L^2\eta(L)k^\alpha \cdot d(F(U_1), F(U_2)). \tag{3.3}$$

Now the second inequality in (3.1) follows from (3.2) and (3.3). Finally, we notice that the second inequality for F^{-1} is equivalent to the first inequality for F. □

Lemma 3.3. *The following holds for any two distinct parallel fibers U_1, U_2 and any two distinct $p, q \in U_1$:*

$$\frac{1}{K_1^3} \cdot \frac{d(F(U_1), F(U_2))}{d(U_1, U_2)} \le \frac{d(F(p), F(q))}{d(p, q)} \le K_1^3 \cdot \frac{d(F(U_1), F(U_2))}{d(U_1, U_2)}, \tag{3.4}$$

where K_1 is the constant in Lemma 3.2.

Proof. Let U_1 and U_2 be two distinct parallel fibers, and $p, q \in U_1$ be distinct. Pick two points $p_0, q_0 \in U_1$ that satisfies $d(p_0, q_0) > L \cdot d(U_1, U_2)$ and $d(F(p_0), F(q_0)) > L \cdot d(F(U_1), F(U_2))$. By Lemma 3.2 we have

$$\frac{1}{K_1} \cdot \frac{d(F(U_1), F(U_2))}{d(U_1, U_2)} \le \frac{d(F(p_0), F(q_0))}{d(p_0, q_0)} \le K_1 \cdot \frac{d(F(U_1), F(U_2))}{d(U_1, U_2)}. \tag{3.5}$$

By Condition (1.3) there exist a sequence of fibers $U_{\lambda_i} \neq U_1$ parallel to U_1 such that U_{λ_i} converges to U_1. For sufficiently large i, we have

$$\min\{d(p, q), d(p_0, q_0)\} > L \cdot d(U_1, U_{\lambda_i})$$

and

$$\min\{d(F(p), F(q)), d(F(p_0), F(q_0))\} > L \cdot d(F(U_1), F(U_{\lambda_i})).$$

Now Lemma 3.2 applied to U_1, U_{λ_i} and p, q yields

$$\frac{1}{K_1} \cdot \frac{d(F(U_1), F(U_{\lambda_i}))}{d(U_1, U_{\lambda_i})} \le \frac{d(F(p), F(q))}{d(p, q)} \le K_1 \cdot \frac{d(F(U_1), F(U_{\lambda_i}))}{d(U_1, U_{\lambda_i}))}.$$

Similarly, we have

$$\frac{1}{K_1} \cdot \frac{d(F(U_1), F(U_{\lambda_i}))}{d(U_1, U_{\lambda_i})} \le \frac{d(F(p_0), F(q_0))}{d(p_0, q_0)} \le K_1 \cdot \frac{d(F(U_1), F(U_{\lambda_i}))}{d(U_1, U_{\lambda_i}))}.$$

It follows that

$$\frac{1}{K_1^2} \cdot \frac{d(F(p_0), F(q_0))}{d(p_0, q_0)} \leq \frac{d(F(p), F(q))}{d(p, q)} \leq K_1^2 \cdot \frac{d(F(p_0), F(q_0))}{d(p_0, q_0)}. \tag{3.6}$$

Now (3.4) follows from (3.5) and (3.6). □

Lemma 3.3 says that for any fiber U of X, the restriction $F|_U$ is a (K_1^3, C)-quasi-similarity for some constant $C > 0$ that may depend on the fiber U. The next lemma states that this constant C can be chosen to be independent of the fiber.

Lemma 3.4. *There is some constant $C > 0$ such that $F|_U$ is a (K_2, C)-quasi-similarity for any fiber U of X, where K_2 is a constant depending only on η, α, L and the quasimetric constants of X, Y.*

Proof. Fix a fiber U_0 (in X) and let U be an arbitrary fiber in X. By Lemma 3.3, there are constants $C_0, C > 0$ such that $F|_{U_0}$ is a (K_1^3, C_0)-quasi-similarity and $F|_U$ is a (K_1^3, C)-quasi-similarity. It suffices to show that there is a constant D depending only on η, α, L, and the quasimetric constants of X and Y, such that $C_0/D \leq C \leq DC_0$.

Let $M \geq 1$ be a quasimetric constant for both X and Y. Fix $x \in U$ and $x_0 \in U_0$, and pick $y \in U$ and $y_0 \in U_0$ such that

$$d(x_0, y_0) = d(x, y) = 10M \cdot d(x, x_0).$$

The generalized triangle inequality in X applied to x, x_0, y_0 implies

$$\frac{1}{2M} \cdot d(x, y) = \frac{1}{2M} \cdot d(x_0, y_0) \leq d(x, y_0) \leq 2M \cdot d(x_0, y_0) = 2M \cdot d(x, y).$$

The quasisymmetry condition now implies

$$d(F(x_0), F(y_0))/\eta(2M) \leq d(F(x), F(y_0)) \leq \eta(2M) \cdot d(F(x_0), F(y_0))$$

and

$$d(F(x), F(y))/\eta(2M) \leq d(F(x), F(y_0)) \leq \eta(2M) \cdot d(F(x), F(y)).$$

It follows that

$$\frac{1}{(\eta(2M))^2} \cdot d(F(x_0), F(y_0)) \leq d(F(x), F(y)) \leq (\eta(2M))^2 \cdot d(F(x_0), F(y_0)).$$

This together with the quasi-similarity property of $F|_{U_0}$ and $F|_U$ implies $C_0/D \leq C \leq DC_0$, where $D = (\eta(2M))^2 K_1^6$. □

Lemma 3.5. *F is a (K, C)-quasi-similarity, where K depends only on η, α, L and the quasimetric constants of X, Y.*

Proof. Let $p, q \in X$ be arbitrary. Let U, V be fibers in X such that $p \in U$, $q \in V$ (we may have $U = V$). Pick $x \in U$ such that $d(p, x) = d(p, q)$. Then

$$d(F(p), F(q)) \leq \eta(1) \cdot d(F(p), F(x)) \leq \eta(1) K_2 C \cdot d(p, x) = \eta(1) K_2 C \cdot d(p, q).$$

So we have the upper bound for $d(F(p), F(q))$. The same argument applied to F^{-1} yields the lower bound for $d(F(p), F(q))$. □

The proof of Theorem 1.1 is now complete.

An example. Recall that two closed subsets U and V of a quasimetric space are defined to be parallel if $d(u,V) = d(v,U)$ for any $u \in U, v \in V$. The following example shows that the conclusion of Theorem 1.1 fails if we replace "parallel" with the weaker condition $HD(U,V) < \infty$.

Let $X = Y = \mathbb{C}$ be the complex plane with the usual metric. It is well known that for any $\alpha > -1$, the map $f_\alpha \colon \mathbb{C} \to \mathbb{C}$ given by $f_\alpha(z) = |z|^\alpha z$ is a quasisymmetric map. Let $\alpha > -1$, $\alpha \neq 0$, define a map $F_\alpha \colon \mathbb{C} \to \mathbb{C}$ as follows:

$$F_\alpha(z) = \begin{cases} f_\alpha(z) & \text{if } |z| \leq 1, \\ z & \text{if } |z| \geq 1. \end{cases}$$

Then it is clear that F_α is a quasiconformal map, and hence is quasisymmetric. The fibers in X are horizontal lines, and the fibers in Y are images of horizontal lines under F_α. A direct calculation shows that all fibers of Y are 10-biLipschitz to the real line. So Condition (1.2) is satisfied by both X and Y. All the conditions in Definition 1.1 are satisfied, provided we replace "parallel" with the weaker condition $HD(U,V) < \infty$. However, F_α is not biLipschitz due to the distortion around the origin.

4. Applications

4.1. Quasisymmetric maps on Carnot groups with reducible first stratum

In this section we use Theorem 1.1 to prove Theorem 1.2. Specifically, we show that every quasisymmetric map on Carnot groups with reducible first stratum is biLipschitz.

Let G be a Carnot group with Lie algebra $\mathfrak{g} = V_1 \oplus \cdots \oplus V_s$. Assume there is a non-trivial proper linear subspace $W_1 \subset V_1$ that is invariant under the action of the group of strata-preserving automorphisms of $\mathfrak{g}$. Let $\mathfrak{h}$ be the subalgebra of $\mathfrak{g}$ generated by W_1. We denote the connected Lie subgroup of G with Lie algebra $\mathfrak{h}$ by H, and refer to it as the *subgroup generated* by W_1. Notice that H is also a Carnot group and its lie algebra can be written as $\mathfrak{h} = W_1 \oplus \cdots \oplus W_s$. In general, there is some integer $1 \leq \bar{s} \leq s$ such that $W_{\bar{s}} \neq 0$ and $W_j = 0$ for $j > \bar{s}$. Let $\langle \cdot, \cdot \rangle$ be an inner product on V_1 and d the left-invariant sub-Riemannian Carnot metric on G determined by $\langle \cdot, \cdot \rangle$.

We recall the following:

Proposition 4.1 ([16], Proposition 3.4). *Let G and G' be two Carnot groups. Let W_1 and W_1' be two subspaces of the first strata of the stratifications of* Lie(G) *and* Lie(G')*, respectively. Let H and H' be the groups generated by W_1 and W_1', respectively. Let $F \colon G \to G'$ be a quasisymmetric map. If $dF(x)(W_1) \subset W_1'$ for a.e. $x \in G$, then F sends each left coset U of H into a left coset of H'.*

Now let $F \colon G \to G$ be an η-quasisymmetric map. By Pansu's differentiability theorem, at almost everywhere $x \in G$ the map F is Pansu differentiable and the Pansu differential $dF(x) \colon \mathfrak{g} \to \mathfrak{g}$ is a strata-preserving automorphism. The assumption on W_1 in Theorem 1.2 implies that $dF(x)(W_1) = W_1$. Now Proposition 4.1 implies that F sends left cosets of H to left cosets of H, where H is the subgroup generated by W_1.

Theorem 1.2 shall follow from Theorem 1.1 once we verify the conditions in Definition 1.1. Here the fibers are left cosets of H. Since H is also a Carnot group, (H, d) is a geodesic space. So Condition (1.2) is satisfied for $\alpha = L = 1$. Conditions (1.3), (1.4) and (1.5) will be verified in Lemmas 4.2, 4.3 and 4.4, respectively.

Given a Lie algebra $\mathfrak{n}$ and a subalgebra $\mathfrak{s} \subset \mathfrak{n}$, the normalizer of $\mathfrak{s}$ in $\mathfrak{n}$ is defined by

$$\mathcal{N}_{\mathfrak{n}}(\mathfrak{s}) = \{X \in \mathfrak{n} : [X, \mathfrak{s}] \subset \mathfrak{s}\}.$$

It is easy to see that $\mathcal{N}_{\mathfrak{n}}(\mathfrak{s})$ is a Lie subalgebra of $\mathfrak{n}$ and $\mathfrak{s}$ is an ideal in $\mathcal{N}_{\mathfrak{n}}(\mathfrak{s})$. Let N be a connected Lie group with Lie algebra $\mathfrak{n}$. Let S be the Lie subgroup of N with Lie algebra $\mathfrak{s}$. Let d be a left invariant distance on N. Lemmas 4.2 and 4.3 are valid for all connected and simply connected nilpotent Lie groups N with a left invariant distance. In particular, they hold true for the Carnot group G and left cosets of H.

Lemma 4.2. *For any proper Lie subalgebra $\mathfrak{s}$ of a nilpotent Lie algebra $\mathfrak{n}$, we have*

$$\mathcal{N}_{\mathfrak{n}}(\mathfrak{s}) \neq \mathfrak{s}. \tag{4.1}$$

Consequently, for any left coset U of S, there exist a sequence of left cosets U_i of S that are parallel to U and converge to U.

Proof. For the first claim we induct on the degree of nilpotency. The claim holds when $\mathfrak{n}$ is abelian since in this case $\mathcal{N}_{\mathfrak{n}}(\mathfrak{s}) = \mathfrak{n}$. Suppose the claim holds for all k-step nilpotent Lie algebras. Let $\mathfrak{n}$ be $(k+1)$-step. We assume $\mathcal{N}_{\mathfrak{n}}(\mathfrak{s}) = \mathfrak{s}$ and will derive a contradiction from this. Notice the center $C(\mathfrak{n})$ of $\mathfrak{n}$ lies in $\mathcal{N}_{\mathfrak{n}}(\mathfrak{s}) = \mathfrak{s}$. So we have a proper Lie subalgebra $\mathfrak{s}/C(\mathfrak{n})$ of the k-step nilpotent Lie algebra $\mathfrak{n}/C(\mathfrak{n})$. By the induction hypothesis, we have $\mathcal{N}_{\mathfrak{n}/C(\mathfrak{n})}(\mathfrak{s}/C(\mathfrak{n})) \neq \mathfrak{s}/C(\mathfrak{n})$. Let $\pi\colon \mathfrak{n} \to \mathfrak{n}/C(\mathfrak{n})$ be the natural projection. It is easy to check that $\pi^{-1}(\mathcal{N}_{\mathfrak{n}/C(\mathfrak{n})}(\mathfrak{s}/C(\mathfrak{n}))) = \mathcal{N}_{\mathfrak{n}}(\mathfrak{s})$. Since $\pi^{-1}(\mathfrak{s}/C(\mathfrak{n})) = \mathfrak{s}$, we obtain $\mathcal{N}_{\mathfrak{n}}(\mathfrak{s}) \neq \mathfrak{s}$, contradicting the assumption $\mathcal{N}_{\mathfrak{n}}(\mathfrak{s}) = \mathfrak{s}$.

Regarding the second part of the lemma, let K be the connected subgroup of N with Lie algebra $\mathcal{N}_{\mathfrak{n}}(\mathfrak{s})$. Since $\mathfrak{s}$ is an ideal in $\mathcal{N}_{\mathfrak{n}}(\mathfrak{s})$, we see that S is a normal subgroup of K. From (4.1) we deduce that K contains S properly. There exist $k_i \in K \backslash S$ such that $k_i \to e$. So for any left coset $U = gS$, the left cosets $U_i = gk_iS$ converge to U. Finally we claim that U_i is parallel to U. Indeed, for any $p = gs_0 \in U$,

$$d(p, U_i) = d(gs_0, gk_iS) = d(e, s_0^{-1}k_iS) = d(e, k_i \cdot k_i^{-1}s_0^{-1}k_iS) = d(e, k_iS);$$

and for any $q = gk_is_0 \in U_i$,

$$\begin{aligned} d(q, U) &= d(gk_is_0, gS) \\ &= d(Sk_is_0, e) = d(Sk_is_0k_i^{-1} \cdot k_i, e) = d(Sk_i, e) = d(e, Sk_i) = d(e, k_iS), \end{aligned}$$

where in the last equality we used the fact that $Sk_i = k_iS$. □

Notice that the last part of the preceding proof shows that for any $g \in N$ and any $k \in K$, the two left cosets gS and gkS are parallel.

Next lemma says that two different left costs of S can never get arbitrarily close.

Lemma 4.3. *If U and V are two distinct left cosets of S, then $d(U, V) > 0$.*

Proof. Set $\mathfrak{s}_0 = \mathfrak{s}$ and define inductively $\mathfrak{s}_j = \mathcal{N}_{\mathfrak{n}}(\mathfrak{s}_{j-1})$. From (4.1) we have that $\mathfrak{s}_j$ properly contains $\mathfrak{s}_{j-1}$ unless $\mathfrak{s}_{j-1} = \mathfrak{n}$. Since $\mathfrak{n}$ is finite dimensional, there is some k such that $\mathfrak{s}_{k-1} \neq \mathfrak{n}$ and $\mathfrak{s}_k = \mathfrak{n}$. Let S_j be the connected Lie subgroup of N with Lie algebra $\mathfrak{s}_j$. Then S_{j-1} is a proper normal subgroup of S_j for $j \leq k$. It follows that for any $g \in S_j \setminus S_{j-1}$, the two left cosets S_{j-1} and gS_{j-1} are parallel. Hence $d(S_{j-1}, gS_{j-1}) > 0$.

Now let U and V be two distinct left cosets of S. After applying a left translation we may assume $U = S$ and $V = gS$ for some $g \notin S$. There is some $j \leq k$ such that $g \in S_j \setminus S_{j-1}$. It follows that

$$d(U, V) = d(S, gS) \geq d(S_{j-1}, gS_{j-1}) > 0,$$

since $S \subset S_{j-1}$. □

For the proof of the next lemma we will work under the assumption of Theorem 1.2.

Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathfrak{g} = V_1 \oplus \cdots \oplus V_s$ such that V_i and V_j are perpendicular to each other for $i \neq j$. Define a "quasi norm" on $\mathfrak{g}$ as follows:

$$\left\| \sum_i x_i \right\| = \sum_i \langle x_i, x_i \rangle^{1/(2i)}, \quad \text{where } x_i \in V_i.$$

Next define $\bar{d}(x, y) = \|(-x) * y\|$ for $x, y \in \mathfrak{g}$. It is well known that every Carnot metric d on $\mathfrak{g}$ is biLipschitz equivalent to $\bar{d}$: there exists some constant $C_0 \geq 1$ such that

$$\frac{1}{C_0} \cdot \bar{d}(x, y) \leq d(x, y) \leq C_0 \cdot \bar{d}(x, y)$$

for any $x, y \in \mathfrak{g}$. Here we identified G and $\mathfrak{g}$ via the exponential map.

Lemma 4.4. *If U and V are two left cosets of H that are not parallel, then $HD(U, V) = \infty$.*

Proof. Suppose U and V are not parallel and $HD(U, V) < \infty$. We shall obtain a contradiction. After applying a left translation we may assume $U = H$ and $V = gH$ for some $g \in G \setminus H$. The last part of the proof of Lemma 4.2 shows that for any k in the normalizer K of H the two fibers H and $kH = Hk$ are parallel. So $g \notin K$. Below we will identify $\mathfrak{g}$ with G via the exponential map and do calculations in the Lie algebra using BCH formula. So we let $Y \notin \mathcal{N}_{\mathfrak{g}}(\mathfrak{h})$ be such that $HD(\mathfrak{h}, Y * \mathfrak{h}) < \infty$. Since

$$HD(Y * \mathfrak{h},\ Y * \mathfrak{h} * (-Y)) \leq d(e, -Y),$$

we see that $HD(\mathfrak{h},\ Y * \mathfrak{h} * (-Y)) \leq C$ for some constant $C > 0$.

We write $Y = \sum_{i=1}^{s} Y_i$ with $Y_i \in V_i$ and let $j \geq 1$ be the index such that $Y_j \notin \mathcal{N}_{\mathfrak{g}}(\mathfrak{h})$ and $Y_i \in \mathcal{N}_{\mathfrak{g}}(\mathfrak{h})$ for all $i < j$. By replacing Y with $Y*(-Y_1-\cdots-Y_{j-1})$, we may assume $Y_i = 0$ for $i < j$. Since $[Y_j, \mathfrak{h}] \not\subset \mathfrak{h}$ and $\mathfrak{h}$ is generated by the first stratum W_1, there is some $X \in W_1$ such that $[Y_j, X] \notin \mathfrak{h}$. Notice that $[Y_j, X] \in V_{j+1}$. Since $HD(\mathfrak{h}, Y*\mathfrak{h}*(-Y)) \leq C$, for any $t \in \mathbb{R}$, there is some $X' \in \mathfrak{h}$ (X' may depend on t) such that

$$d((-X')*Y*(tX)*(-Y), 0) = d(Y*(tX)*(-Y), X') \leq C. \tag{4.2}$$

We next calculate $Z := Y*(tX)*(-Y)$ and $A := (-X')*Z$. We have

$$Z = e^{\operatorname{ad} Y}(tX) = t\{X + [Y, X] + \cdots + \frac{1}{k!}(\operatorname{ad} Y)^k(X) + \cdots + \frac{1}{(s-1)!}(\operatorname{ad} Y)^{s-1}(X)\},$$

where $\operatorname{ad} Y\colon \mathfrak{g} \to \mathfrak{g}$ is the linear map given by $\operatorname{ad} Y(B) = [Y, B]$ for $B \in \mathfrak{g}$. Write $Z = Z_1 + \cdots + Z_s$ with $Z_i \in V_i$. Since $Y_i = 0$ for $i < j$, we have

$$\frac{1}{k!}(\operatorname{ad} Y)^k(X) \in V_{j+2} \oplus \cdots \oplus V_s \quad \text{for all } \ k \geq 2.$$

So the terms Z_i with $i \leq j+1$ is determined by $t(X + [Y, X])$. Since $X \in V_1$ we have $Z_1 = tX$, $Z_i = 0$ for $2 \leq i \leq j$ and $Z_{j+1} = t[Y_j, X]$.

Write $A = A_1 + \cdots + A_s$ with $A_i \in V_i$. Notice that $[-X', Z]$ and all the iterated brackets in the BCH formula for $(-X')*Z$ are the sum of an element of $\mathfrak{h}$ and an element in $V_{j+2} \oplus \cdots \oplus V_s$. So A_{j+1} is completely determined by $-X' + Z$. It follows that A_{j+1} is the sum of an element of W_{j+1} and $t[Y_j, X]$ (recall $\mathfrak{h} = W_1 \oplus \cdots \oplus W_s$). Write $[Y_j, X] = B + B^{\perp}$, where $B \in W_{j+1}$ and $B^{\perp} \in V_{j+1}$ is perpendicular to W_{j+1} with respect to the inner product $<,>$ on $\mathfrak{g}$. Since $[Y_j, X] \notin \mathfrak{h}$, we have $B^{\perp} \neq 0$. Hence A_{j+1} equals the sum of $tB^{\perp}$ and an element of W_{j+1}. It follows that

$$d((-X')*Y*(tX)*(-Y), 0) \geq \frac{1}{C_0}\cdot ||(-X')*Y*(tX)*(-Y)|| \geq \frac{1}{C_0}\cdot |tB^{\perp}|^{\frac{1}{j+1}} \to \infty,$$

as $t \to \infty$ since $B^{\perp} \neq 0$. This contradicts (4.2). □

We have verified all the conditions in Definition 1.1 and the proof of Theorem 1.2 is now complete.

Remark 4.3. Lemma 4.4 is equivalent to the following statement: if the Hausdorff distance between H and gHg^{-1} is finite, then g lies in the normalizer of H. We believe this is true for any left invariant distance on any connected, simply connected nilpotent Lie group N and any proper Lie subgroup H.

We next make some remarks on Carnot groups with reducible first stratum. For some constructions and more examples, see Section 4.2 and Section 6 of [20].

One way to construct Carnot groups with reducible first stratum is to use the notion of rank. For an element x in a Lie algebra $\mathfrak{n}$, let $\operatorname{rank}(x)$ be the rank of the linear transformation $\operatorname{ad}(x)\colon \mathfrak{n} \to \mathfrak{n}$, $\operatorname{ad}(x)(y) = [x, y]$. That is, $\operatorname{rank}(x)$ is the dimension of the image of $\operatorname{ad}(x)$. For a stratified Lie algebra $\mathfrak{n} = V_1 \oplus \cdots \oplus V_s$, define

$$r(\mathfrak{n}) = \min\{\operatorname{rank}(x) : x \in V_1\}.$$

Let $W_1 \subset V_1$ be the linear subspace spanned by elements $x \in V_1$ with $\mathrm{rank}(x) = r(\mathfrak{n})$. It is clear that $A(W_1) = W_1$ for every $A \in \mathrm{Aut}_*(\mathfrak{n})$. It follows that if $W_1 \neq V_1$, then $\mathfrak{n}$ has reducible first stratum. For example, if $\mathfrak{h}_n$ is the n-th Heisenberg algebra and $m \geq 1$, then $\mathfrak{n} = \mathbb{R}^m \oplus \mathfrak{h}_n$ satisfies $W_1 = \mathbb{R}^m \neq V_1$ and so has reducible first stratum. Another example is the model filiform algebra $\mathfrak{f}_n$ with $n \geq 3$, see [21].

Carnot groups with reducible first stratum include both rigid and non-rigid Carnot groups. Recall that a C^2 map between open subsets of Carnot groups is contact if the differentials send horizontal subspaces into horizontal subspaces. A Carnot group (and its Lie algebra) is called *rigid* if the space of contact maps is finite dimensional, and is called non-rigid otherwise. In [7] Ottazzi showed that if $\mathfrak{n}$ has rank at most one, then $\mathfrak{n}$ is non-rigid. Furthermore, Ottazzi and Warhurst in [8] proved that a Carnot group is non-rigid if and only if the complexification $\mathfrak{n} \otimes \mathbb{C}$ of its Lie algebra has rank at most one (as a complex Lie algebra). Hence the above two examples $\mathbb{R}^m \oplus \mathfrak{h}_n$ and $\mathfrak{f}_n$ $(n \geq 3)$ are non-rigid Carnot groups with reducible first stratum. It is shown in [19] that most non-rigid Carnot group have reducible first stratum. For an example of rigid stratified Lie algebra, consider the direct sum of two stratified Lie algebras $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ with $r(\mathfrak{n}_1) < r(\mathfrak{n}_2)$. The elements of minimal rank of $\mathfrak{n}$ are contained in $\mathfrak{n}_1$ and so $\mathfrak{n}$ has reducible first stratum. By the result of Ottazzi and Warhurst, $\mathfrak{n}$ is rigid when $r(\mathfrak{n}_1) \geq 3$. This construction also shows that every stratified Lie algebra embeds in a stratified Lie algebra with reducible first stratum.

Of course there are many Carnot groups that do not have reducible first stratum. Examples include Heisenberg groups and free nilpotent Lie groups.

Theorem 1.2 generalizes all previous results by the second-named author on the rigidity of quasiconformal maps between Carnot groups, [16], [19], [20]. In particular, the non-rigid Carnot groups covered in [19] are exactly those non-rigid Carnot groups with reducible first stratum. There are many rigid Carnot groups with reducible first stratum, and there are also many rigid Carnot groups that do not have reducible first stratum.

4.2. Other applications

In this subsection we give a couple of other applications of Theorem 1.1. The first is to quasiconformal maps of Heisenberg groups that send vertical lines to vertical lines. The second is to quasisymmetric maps of ideal boundary of certain amenable hyperbolic locally compact groups.

We first show that quasiconformal maps of the Heisenberg groups $H^n = \mathbb{R}^{2n} \times \mathbb{R}$ that permute vertical lines are biLipschitz. These maps are lifts of those biLipschitz maps of $\mathbb{R}^{2n}$ that preserve the standard symplectic form on $\mathbb{R}^{2n}$. The reader is referred to [13] and [1] for more details.

Proposition 4.5. *Let $F\colon H^n \to H^n$ be a quasiconformal map of the Heisenberg group. If F maps vertical lines to vertical lines, then F is biLipschitz.*

Proof. Recall that by Theorem 4.7 in [6], F is quasisymmetric. The fibers are the vertical lines. There is some $L \geq 1$ such that the vertical lines are L-biLipschitz to

$(\mathbb{R}, |\cdot|^{1/2})$, where $|\cdot|$ is the usual metric on $\mathbb{R}$. So Condition (1.2) in Definition 1.1 is satisfied for $\alpha = 1/2$ and L. It is easy to see that the other three conditions in Definition 1.1 are also satisfied. By Theorem 1.1, F is biLipschitz. □

We next give a simple proof of Dymarz's theorem.

Theorem 4.6 ([5], Theorem 1). *Let N be a Carnot group equipped with a left invariant Carnot metric d and $\mathbb{Q}_m$ ($m \geq 2$) the m-adics with standard metric d_m. Then every quasisymmetric map $N \times \mathbb{Q}_m \to N \times \mathbb{Q}_m$ is biLipschitz. Here the metric on $N \times \mathbb{Q}_m$ is given by $d'((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), d_m(y_1, y_2)\}$.*

Recall that the standard metric d_m on $\mathbb{Q}_m$ is given by:

$$d_m\Big(\sum a_i m^i, \sum b_i\, m^i\Big) = m^{-(k+1)},$$

where k is the smallest index for which $a_i \neq b_i$.

Proof. The fibers of $N \times \mathbb{Q}_m$ are the subsets $N \times \{y\}$ ($y \in \mathbb{Q}_m$). Notice that $(\mathbb{Q}_m, d_m)$ is perfect and totally disconnected. So the fibers of $N \times \mathbb{Q}_m$ are exactly the connected components of $N \times \mathbb{Q}_m$. Hence every quasisymmetric map F of $N \times \mathbb{Q}_m$ permutes the fibers. Each fiber is isometric to a Carnot group and so is geodesic. Hence Condition (1.2) is satisfied for $\alpha = L = 1$. The perfectness of $\mathbb{Q}_m$ implies that parallel fibers are not isolated (Condition (1.3)). Distance between distinct fibers is clearly positive (Condition (1.4)). All fibers are parallel, so Condition (1.5) is vacuous. By Theorem 1.1 every quasisymmetric map of $N \times \mathbb{Q}_m$ is biLipschitz. □

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