



Qualitative properties and classification of nonnegative solutions to $-\Delta u = f(u)$ in unbounded domains when $f(0) < 0$

Alberto Farina and Berardino Sciunzi

Abstract. We consider nonnegative solutions to $-\Delta u = f(u)$ in unbounded Euclidean domains under zero Dirichlet boundary conditions, where f is merely locally Lipschitz continuous and satisfies $f(0) < 0$. In the half-plane, and without any other assumption on u , we prove that u is either one-dimensional and periodic or positive and strictly monotone increasing in the direction orthogonal to the boundary. Analogous results are obtained if the domain is a strip. As a consequence of our main results, we answer affirmatively to a conjecture and to an open question posed by Berestycki, Caffarelli and Nirenberg. We also obtain some symmetry and monotonicity results in the higher-dimensional case.

1. Introduction and main results

We study qualitative properties of *nonnegative* solutions to $-\Delta u = f(u)$ in unbounded Euclidean domains, where f is *merely locally Lipschitz continuous* and satisfies $f(0) < 0$. In particular, we are interested in proving the one-dimensional symmetry, the monotonicity and/or the periodicity of the considered solutions.

Let us start by considering the case of $\mathbb{R}_+^2 := \{(x, y) \in \mathbb{R}^2 : y > 0\}$, the open half-plane:

$$(1.1) \quad \begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}_+^2, \\ u \geq 0 & \text{in } \mathbb{R}_+^2, \\ u = 0 & \text{on } \partial\mathbb{R}_+^2. \end{cases}$$

As a consequence of more general results that we shall state later, we prove:

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Theorem 1.1. *Assume that f is locally Lipschitz continuous on $[0, +\infty)$ with $f(0) < 0$. Let $u \in C^2(\overline{\mathbb{R}_+^2})$ be a solution to (1.1). Then, either u is one-dimensional and periodic, or u is positive and strictly monotone increasing in the direction orthogonal to the boundary with $\partial_y u > 0$ in \mathbb{R}_+^2 .*

The theorem above provides a complete picture of the situation in our general framework.

Theorem 1.2. *Let $u \in C^2(\overline{\mathbb{R}_+^2})$ be a solution to*

$$(1.2) \quad \begin{cases} -\Delta u = u - 1 & \text{in } \mathbb{R}_+^2, \\ u \geq 0 & \text{in } \mathbb{R}_+^2, \\ u = 0 & \text{on } \partial\mathbb{R}_+^2. \end{cases}$$

Then

$$u(x, y) = 1 - \cos y.$$

Theorem 1.2 provides an affirmative answer to an extended version of a *conjecture* posed by Berestycki, Caffarelli and Nirenberg (see p. 73 of [2]). It recovers and improves upon a result of [15] (cf. also [2]) since here no a-priori bound on the solutions is required.

The techniques developed here allow us to consider also problems defined in strips $\Sigma_{2b} := \{(x, y) \in \mathbb{R}^2 : y \in (0, 2b)\}$, $b > 0$. The following result answers affirmatively to an *open question* raised by Berestycki, Caffarelli and Nirenberg (see p. 486 in [1]).

Theorem 1.3. *Let $u \in C^2(\overline{\Sigma_{2b}})$ be a solution to*

$$(1.3) \quad \begin{cases} -\Delta u = f(u), & \text{in } \Sigma_{2b}, \\ u \geq 0, & \text{in } \Sigma_{2b}, \\ u = 0, & \text{on } \partial\Sigma_{2b}, \end{cases}$$

with f locally Lipschitz continuous on $[0, +\infty)$ (no restriction on the sign of $f(0)$). Then,

- (i) *if $f(0) < 0$, either u is positive on Σ_{2b} , symmetric about $\{y = b\}$ with $\partial_y u > 0$ in Σ_b , or u is one-dimensional and periodic; in this case, $2b$ is necessarily a multiple of the period of u ;*
- (ii) *if $f(0) \geq 0$, either u vanishes identically or it is positive on Σ_{2b} , symmetric about $\{y = b\}$ with $\partial_y u > 0$ in Σ_b .*

Remark 1.4. Note that the theorem above also applies when $f(0) \geq 0$ and it is *new* even in this case.

Before proceeding further, let us briefly recall the main difficulties that one has to face when working in our general framework.

When $f(0) < 0$, nonnegative solutions are natural and must be taken into account. Indeed, in this case, nontrivial, nonnegative solutions (vanishing somewhere) can exist and sometimes, they are the only nonnegative solutions of the considered problem (this is the case when $f(u) = u - 1$, for instance). These phe-

nomena are strongly related to the absence of both the *strong maximum principle* and the *Hopf lemma*. Hence a deeper and different analysis with respect to the case $f(0) \geq 0$ is needed. Further difficulties in the analysis are added by the fact that the solution u is *not* assumed to be bounded and f is *merely locally Lipschitz continuous*. Indeed, in the study of the qualitative properties of the solutions to semilinear problems in unbounded domains, it is always assumed some *a priori* bound on u and/or the *global Lipschitz character* of f . These properties ensure, for instance, the possibility to use elliptic estimates to study the asymptotic behaviour u , by means of the translation invariance of the considered problem and/or to use some comparison principles on unbounded cylindrical domains having small cross section. The lack of those properties in our general framework will require a different approach to the problem.

To obtain our results we shall use a *rotating plane method* (inspired by [2], and especially by [8]) combined with the *unique continuation principle* (see [17] and the references therein). These tools are described and developed in sections 2 and 3.

To continue the description of our results, we denote by $p := (x, y)$ a general point in the plane and, for a nonnegative solution u of (1.1), we say that u satisfies the property (\mathcal{P}_μ) if *there exists a real number $\mu > 0$ and a point $p \in \{y = \mu\}$ such that $u(p) \neq 0$* .

Equivalently:

$$(\mathcal{P}_\mu) \quad \text{holds if} \quad \{y = \mu\} \cap \{u \neq 0\} \neq \emptyset.$$

Since u cannot be identically zero by the assumption $f(0) < 0$, we see that the above property (\mathcal{P}_μ) is satisfied for some $\mu > 0$.

We shall prove in Theorem 6.1 that the set

$$(1.4) \quad \Lambda^* = \Lambda^*(u) := \{\lambda > 0 : (\mathcal{P}_\mu) \text{ holds for every } 0 < \mu \leq \lambda\}$$

is not empty (in any dimension $N \geq 2$). Therefore we have

$$(1.5) \quad \lambda^* = \lambda^*(u) := \sup \Lambda^* \in (0, +\infty].$$

Note that, by a continuity argument, if λ^* is finite, we easily get that $\{y = \lambda^*\} \subseteq \{u = 0\}$.

Recalling the notation $\Sigma_\lambda := \{(x, y) \in \mathbb{R}^2 \mid 0 < y < \lambda\}$, we have the following.

Theorem 1.5. *Assume that f is locally Lipschitz continuous on $[0, +\infty)$ with $f(0) < 0$. Let $u \in C^2(\overline{\mathbb{R}_+^2})$ be a solution to (1.1). Then*

(i) *if $\lambda^* = +\infty$, then u is positive and strictly monotone increasing in the y direction, with $u > 0$ and $\partial_y u > 0$ in \mathbb{R}_+^2 .*

(ii) *If $\lambda^* < +\infty$, u is one-dimensional and periodic, i.e.,*

$$u(x, y) = u_0(y) \quad \forall (x, y) \in \mathbb{R}_+^2,$$

where $u_0 \in C^2(\mathbb{R}, [0, \infty))$ is periodic of period λ^* . Moreover, u_0 is the unique solution of

$$(1.6) \quad -u_0'' = f(u_0) \quad \text{in } [0, \infty), \quad u_0'(0) = u_0(0) = 0 = u_0'(\lambda^*) = u_0(\lambda^*).$$

Also, u is symmetric with respect to $\{y = \lambda^/2\}$ with $\partial_y u > 0$ in Σ_{λ^*} .*

We shall provide two different proofs of the above theorem. Note that Theorem 1.1 is an immediate consequence of Theorem 1.5. Both of them provide a complete classification of the solutions to problem (1.1) and they significantly extend the results of [2], where it is always assumed that f is *globally Lipschitz continuous* and/or the *solution u is positive and bounded*, and the partial results obtained in [10], which hold for *bounded solutions and $f \in C^1$* .

Next we have two results concerning the one-dimensional symmetry for solutions to (1.1).

Theorem 1.6. *Assume that f is locally Lipschitz continuous on $[0, +\infty)$. Let $u \in C^2(\overline{\mathbb{R}_+^2})$ be a solution to (1.1) with $|\nabla u| \in L^\infty(\mathbb{R}_+^2)$. Then u is one-dimensional, i.e.,*

$$u(x, y) = u(y) \quad \forall (x, y) \in \mathbb{R}_+^2.$$

The above theorem recovers and improves upon a result of [16], where only positive solutions were considered (cf. also [2]).

Theorem 1.7. *Assume that $f \in C^1([0, +\infty))$ with $f(0) < 0$ such that*

$$f'(t) \geq c > 0 \quad \text{for any } t \in (0, \infty),$$

and let $u \in C^2(\overline{\mathbb{R}_+^2})$ be a solution to (1.1). Then u is one-dimensional and periodic, i.e.,

$$u(x, y) = u_0(y) \quad \forall (x, y) \in \mathbb{R}_+^2$$

with u_0 as in Theorem 1.5. In particular, there are no positive solutions to (1.1).

Note that Theorem 1.2 is obtained by setting $f(u) = u - 1$ in the previous theorem.

We can now turn to the case of the strips Σ_{2b} , $b > 0$. Precisely, we consider the following problem:

$$(1.7) \quad \begin{cases} -\Delta u = f(u), & \text{in } \Sigma_{2b}, \\ u \geq 0, & \text{in } \Sigma_{2b}, \\ u = 0, & \text{on } \partial\Sigma_{2b}. \end{cases}$$

We let Λ^* be defined by (1.4), considering there values of λ such that $0 < \lambda < 2b$, thus $\lambda^* \in (0, 2b]$.

We shall prove:

Theorem 1.8. *Assume that f is locally Lipschitz continuous on $[0, +\infty)$ with $f(0) < 0$. Let $u \in C^2(\overline{\Sigma_{2b}})$ be a solution to (1.7). Then,*

- (i) *if $\lambda^* = 2b$, it follows that u is positive in Σ_{2b} with $\partial_y u > 0$ in Σ_b . Furthermore u is symmetric with respect to $\{y = b\}$, i.e., $u(x, y) = u(x, 2b - y)$ for any $0 \leq y \leq 2b$.*

- (ii) *If $\lambda^* < 2b$, u is one-dimensional and periodic, i.e.,*

$$u(x, y) = u_0(y) \quad \forall (x, y) \in \Sigma_{2b}$$

where $u_0 \in C^2(\mathbb{R}, [0, \infty))$ is periodic of period λ^* . Moreover, u_0 is the unique solution of

$$(1.8) \quad -u_0'' = f(u_0) \quad \text{in } [0, 2b], \quad u_0(0) = u_0'(0) = 0.$$

Also, u is symmetric with respect to $\{y = \lambda^*/2\}$ with $\partial_y u > 0$ in $\Sigma_{\lambda^*/2}$. Finally, $2b$ is necessarily a multiple of the period λ^* .

The techniques used to prove Theorem 1.8 also cover the case $f(0) \geq 0$. Since the result appears to be new even in this case, we explicitly state it in the next Theorem 1.9.

Theorem 1.9. *Assume that f is locally Lipschitz continuous on $[0, +\infty)$ with $f(0) \geq 0$. Let $u \in C^2(\overline{\Sigma_{2b}})$ be a solution to (1.7).*

Then, either u vanishes identically or it is positive on Σ_{2b} , symmetric about $\{y = b\}$ with $\partial_y u > 0$ in Σ_b .

Note that Theorem 1.3 is a consequence of the combination of Theorem 1.8 and Theorem 1.9.

The next result concerns the qualitative properties of nonnegative solutions on coercive epigraphs. It holds true in every dimension $N \geq 2$. Let us recall that a domain $\Omega \subset \mathbb{R}^N$ is a *smooth coercive epigraph* if, up to a rotation of the space, there exists $g \in C^2(\mathbb{R}^{N-1}, \mathbb{R})$ such that $\Omega := \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N > g(x')\}$ and $\lim_{|x'| \rightarrow +\infty} g(x') = +\infty$.

Theorem 1.10. *Let $N \geq 2$ and let $\Omega \subset \mathbb{R}^N$ denote a smooth coercive epigraph. Assume that f is locally Lipschitz continuous on $[0, +\infty)$ with $f(0) < 0$. Let $u \in C^2(\overline{\Omega})$ be a solution to*

$$(1.9) \quad \begin{cases} -\Delta u = f(u), & \text{in } \Omega, \\ u \geq 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Then, u is positive and strictly monotone increasing in the x_N direction, with $\partial_{x_N} u > 0$ in Ω .

Previous results in this case have been obtained in [3], [11] under the condition that u is a positive solution (cf. also [13]).

Further results concerning the higher dimensional case are provided in the last section.

In this work we focused on the case $f(0) < 0$, where very few results were available. For the more classical case $f(0) \geq 0$, f globally Lipschitz continuous and/or u bounded, we refer to [1], [2], [3], [4], [6], [7], [9], [11], [12], [13], [14], [16], and [18], and the references therein.

The paper is organized as follows: in Section 2 we state and prove some preliminary results needed for the application of the rotating plane technique. In Section 3 we give the first proof of Theorem 1.5. Here we exploit the unique continuation

principle only to prove the last assertion of the statement. We provide the second proof of Theorem 1.5 in Section 4 exploiting there the unique continuation principle to start the rotating plane procedure. Section 5 is devoted to the proof of Theorems 1.1–1.3 and Theorems 1.6–1.9. The results in higher dimensions are treated in Section 6.

2. Preliminary results

In this section we assume that f is merely locally Lipschitz continuous. No restrictions are imposed on the sign of $f(0)$.

In the proof of our main result we will exploit a rotating plane technique. This will be strongly based on the use of weak and strong maximum principles, see e.g. [20] and [21]. Since we are not assuming that the solution is globally bounded and since we are not assuming that the nonlinearity f is globally Lipschitz continuous, we need the following version of the weak comparison principle in domains of small measure.

Proposition 2.1 (Weak comparison principle in small domains). *Assume $N \geq 2$. Let us consider a bounded domain $D \subset \mathbb{R}^N$ and $u, v \in C^2(\overline{D})$ such that*

$$(2.1) \quad -\Delta u - f(u) \leq -\Delta v - f(v) \quad \text{in } D.$$

Then there exists $\vartheta = \vartheta(D, u, v, f) > 0$ such that, for any domain $D' \subset D$ with $u \leq v$ on $\partial D'$ and $\mathcal{L}(D') \leq \vartheta$ (where $\mathcal{L}(\cdot)$ denotes the Lebesgue measure of a set), it follows

$$u \leq v \quad \text{in } D'.$$

Proof. We use $(u - v)^+ \in H_0^1(D')$ as test function in the weak formulation of (2.1) and get

$$\begin{aligned} \int_{D'} |\nabla(u - v)^+|^2 dx &\leq \int_{D'} \frac{f(u) - f(v)}{(u - v)} ((u - v)^+)^2 dx \\ &\leq C(D, u, v, f) \int_{D'} ((u - v)^+)^2 dx, \end{aligned}$$

where the positive constant $C(D, u, v, f)$ can be determined exploiting the fact that u, v are bounded on \overline{D} and f is locally Lipschitz continuous on $[0, +\infty)$.

An application of Poincaré inequality gives

$$\int_{D'} |\nabla(u - v)^+|^2 dx \leq C(D, u, v, f)(C_N(\mathcal{L}(D'))^{2/N}) \int_{D'} |\nabla(u - v)^+|^2 dx,$$

where $C_N > 0$ is a constant depending only on the Euclidean dimension N .

For $\mathcal{L}(D')$ small such that $C(D, u, v, f)(C_N(\mathcal{L}(D'))^{2/N}) < 1$, we get that $(u - v)^+ \equiv 0$ and the thesis. \square

Now we focus on the two-dimensional case and fix some notations. Given $x_0 \in \mathbb{R}$, $s > 0$ and $\theta \in (0, \pi/2)$, let $L_{x_0,s,\theta}$ be the line, with slope $\tan(\theta)$, passing through (x_0, s) . Also, let V_θ be the vector orthogonal to $L_{x_0,s,\theta}$ such that $(V_\theta, e_2) > 0$ and $\|V_\theta\| = 1$. We denote by

$$(2.2) \quad \mathcal{T}_{x_0,s,\theta}$$

the (open) triangle delimited by $L_{x_0,s,\theta}$, $\{y = 0\}$ and $\{x = x_0\}$. We also define

$$u_{x_0,s,\theta}(x) = u(T_{x_0,s,\theta}(x)), \quad x \in \mathcal{T}_{x_0,s,\theta}$$

where $T_{x_0,s,\theta}(x)$ is the point symmetric to x with respect to $L_{x_0,s,\theta}$, and

$$(2.3) \quad w_{x_0,s,\theta} = u - u_{x_0,s,\theta}.$$

It is immediate to see that $u_{x_0,s,\theta}$ still fulfills $-\Delta u_{x_0,s,\theta} = f(u_{x_0,s,\theta})$ and

$$(2.4) \quad -\Delta w_{x_0,s,\theta} = c_{x_0,s,\theta} w_{x_0,s,\theta}$$

on the triangle $\mathcal{T}_{x_0,s,\theta}$, where we have set

$$(2.5) \quad c_{x_0,s,\theta}(x) := \begin{cases} \frac{f(u(x)) - f(u_{x_0,s,\theta}(x))}{u(x) - u_{x_0,s,\theta}(x)} & \text{if } w_{x_0,s,\theta} \neq 0, \\ 0 & \text{if } w_{x_0,s,\theta} = 0. \end{cases}$$

Note that $|c_{x_0,s,\theta}| \leq C(\mathcal{T}_{x_0,s,\theta}, u, f)$ on the triangle $\mathcal{T}_{x_0,s,\theta}$, where $C(\mathcal{T}_{x_0,s,\theta}, u, f)$ is a positive constant which can be determined by exploiting the fact that u and $u_{x_0,s,\theta}$ are bounded on $\overline{\mathcal{T}_{x_0,s,\theta}}$ and f is locally Lipschitz continuous on $[0, +\infty)$.

Let us remark that, exploiting also the Dirichlet boundary condition and the fact that u is nonnegative, it follows that $w_{x_0,s,\theta} = 0$ on $\overline{\mathcal{T}_{x_0,s,\theta}} \cap L_{x_0,s,\theta}$, while $w_{x_0,s,\theta} \leq 0$ on $\overline{\mathcal{T}_{x_0,s,\theta}} \cap \{y = 0\}$.

In what follows we shall make repeated use of a refined version of the *moving plane technique* [22] (see also [5] and [19]). Actually we will exploit a *rotating plane technique* and a *sliding plane technique* developed in [8].

Let us give the following definition:

Definition 2.2. Given x_0, s and θ as above, we say that the condition $(\mathcal{HT}_{x_0,s,\theta})$ holds in the triangle $\mathcal{T}_{x_0,s,\theta}$ if

$$w_{x_0,s,\theta} < 0 \text{ in } \mathcal{T}_{x_0,s,\theta}, \quad w_{x_0,s,\theta} \leq 0 \text{ on } \partial(\mathcal{T}_{x_0,s,\theta}), \text{ and} \\ w_{x_0,s,\theta} \text{ is not identically zero on } \partial(\mathcal{T}_{x_0,s,\theta}),$$

with $w_{x_0,s,\theta}$ defined in (2.3).

We have the following.

Lemma 2.3 (Small perturbations). *Let (x_0, s, θ) and $\mathcal{T}_{x_0,s,\theta}$ be as above and assume that $(\mathcal{HT}_{x_0,s,\theta})$ holds. Then there exists $\bar{\mu} = \bar{\mu}(x_0, s, \theta) > 0$ such that*

$$(2.6) \quad \begin{cases} |\theta - \theta'| + |s - s'| < \bar{\mu}, \\ w_{x_0,s',\theta'} \leq 0 \text{ on } \partial(\mathcal{T}_{x_0,s',\theta'}), \\ w_{x_0,s',\theta'} \text{ is not identically zero on } \partial(\mathcal{T}_{x_0,s',\theta'}) \end{cases} \implies (\mathcal{HT}_{x_0,s',\theta'}) \text{ holds.}$$

Proof. In order to exploit Proposition 2.1, let us fix a bounded domain $D \subset \mathbb{R}_+^2$ such that $\mathcal{T}_{x_0, s', \theta} \subset D$ for all $s' \in (0, s + 1)$ and $\theta' \in (\theta/2, \pi/2)$. Now pick a small $\epsilon = \epsilon(\theta, s) > 0$ such that $\mathcal{L}(\mathcal{T}_{x_0, s+\epsilon, \theta-\epsilon} \setminus \mathcal{T}_{x_0, s-\epsilon, \theta+\epsilon}) < \vartheta/10$, and then a compact set $K \subset \mathcal{T}_{x_0, s-\epsilon, \theta+\epsilon}$ such that $\mathcal{L}(\mathcal{T}_{x_0, s-\epsilon, \theta+\epsilon} \setminus K) < \vartheta/10$, where ϑ is given by Proposition 2.1. Therefore, for all (s', θ') satisfying $|\theta - \theta'| + |s - s'| < \epsilon$, we have $\mathcal{L}(\mathcal{T}_{x_0, s', \theta'} \setminus K) < \vartheta/5$. Also, since by assumption $w_{x_0, s, \theta} < 0$ in $\mathcal{T}_{x_0, s, \theta}$ we get $w_{x_0, s, \theta} \leq \rho < 0$ on the compact set K . Therefore, we can find $\bar{\mu} \in (0, \epsilon)$ such that, for all (s', θ') satisfying $|\theta - \theta'| + |s - s'| < \bar{\mu}$, we have $w_{x_0, s', \theta'} \leq \rho/2 < 0$ on the compact set K , and $\mathcal{L}(\mathcal{T}_{x_0, s', \theta'} \setminus K) < \vartheta/5$. Since $w_{x_0, s', \theta'} \leq 0$ on $\partial(\mathcal{T}_{x_0, s', \theta'} \setminus K)$, we can apply Proposition 2.1 to get that

$$w_{x_0, s', \theta'} \leq 0 \quad \text{in } \mathcal{T}_{x_0, s', \theta'} \setminus K,$$

and therefore in the triangle $\mathcal{T}_{x_0, s', \theta'}$. Also by the strong comparison principle, we get

$$w_{x_0, s', \theta'} < 0 \quad \text{in } \mathcal{T}_{x_0, s', \theta'},$$

and the proof is completed. □

Let us now show that, from the fact that we can make small translations and rotations of $\mathcal{T}_{x_0, s, \theta}$ towards $\mathcal{T}_{x_0, s', \theta'}$, if $(s', \theta') \approx (s, \theta)$, then we can also make larger translations and rotations. We have the following.

Lemma 2.4 (The sliding-rotating technique). *Let (x_0, s, θ) be as above and assume that $(\mathcal{H}\mathcal{T}_{x_0, s, \theta})$ holds. Let $(\hat{s}, \hat{\theta})$ be fixed and assume that there exists a continuous function $g(t) = (s(t), \theta(t)) : [0, 1] \rightarrow (0, +\infty) \times (0, \pi/2)$, such that $g(0) = (s, \theta)$ and $g(1) = (\hat{s}, \hat{\theta})$. Assume that*

$$(2.7) \quad w_{x_0, s(t), \theta(t)} \leq 0 \quad \text{on } \partial(\mathcal{T}_{x_0, s(t), \theta(t)}) \text{ for every } t \in [0, 1),$$

and that $w_{x_0, s(t), \theta(t)}$ is not identically zero on $\partial(\mathcal{T}_{x_0, s(t), \theta(t)})$ for every $t \in [0, 1)$.

Then

$$(\mathcal{H}\mathcal{T}_{x_0, \hat{s}, \hat{\theta}}) \text{ holds.}$$

Proof. By the assumptions and exploiting Lemma 2.3, we obtain the existence of $\bar{t} > 0$ small such that, for $0 \leq t \leq \bar{t}$, $(\mathcal{H}\mathcal{T}_{x_0, s(t), \theta(t)})$ holds. We now set

$$\bar{T} \equiv \{\tilde{t} \in [0, 1] \text{ such that } (\mathcal{H}\mathcal{T}_{x_0, s(t), \theta(t)}) \text{ holds for any } 0 \leq t \leq \tilde{t}\},$$

and

$$\bar{t} = \sup \bar{T}.$$

We claim that actually $\bar{t} = 1$. To prove this, assume $\bar{t} < 1$ and note that in this case we have

$$\begin{aligned} w_{x_0, s(\bar{t}), \theta(\bar{t})} &\leq 0 \quad \text{in } \mathcal{T}_{x_0, s(\bar{t}), \theta(\bar{t})}, \\ w_{x_0, s(\bar{t}), \theta(\bar{t})} &\leq 0 \quad \text{on } \partial(\mathcal{T}_{x_0, s(\bar{t}), \theta(\bar{t})}), \end{aligned}$$

by continuity, and that $w_{x_0,s(\bar{t}),\theta(\bar{t})}$ is not identically zero on $\partial(\mathcal{T}_{x_0,s(\bar{t}),\theta(\bar{t})})$ by assumption. Hence, by the strong maximum principle, we see that

$$w_{x_0,s(\bar{t}),\theta(\bar{t})} < 0 \quad \text{in} \quad \mathcal{T}_{x_0,s(\bar{t}),\theta(\bar{t})}.$$

Therefore $(\mathcal{HT}_{x_0,s(\bar{t}),\theta(\bar{t})})$ holds and using once again Lemma 2.3, we can find a sufficiently small $\varepsilon > 0$ so that $(\mathcal{HT}_{x_0,s(t),\theta(t)})$ holds for any $0 \leq t \leq \bar{t} + \varepsilon$, which contradicts the definition of \bar{t} . □

3. First proof of Theorem 1.5

Given any $x_0 \in \mathbb{R}$, let us set

$$(3.1) \quad Q_h(x_0) = \{(x, y) : |x - x_0| \leq h, 0 \leq y \leq 2h\}.$$

Since $\partial_{xx}u(x_0, 0) = 0$, we have that

$$-\partial_{yy}u(x_0, 0) = -\Delta u(x_0, 0) = f(u(x_0, 0)) = f(0) < 0,$$

since $f(0) < 0$ by assumption. Recalling that $u \in C^2(\overline{\mathbb{R}_+^2})$, we conclude that we can take $\bar{h} > 0$ small such that

$$\partial_{yy}u > 0 \quad \text{in} \quad Q_{\bar{h}}(x_0).$$

Exploiting again the fact that $u \in C^2(\overline{\mathbb{R}_+^2})$, we can consequently find $\bar{\theta} = \bar{\theta}(\bar{h}) \in (0, \pi/2)$ such that

$$(3.2) \quad \frac{\partial}{\partial V_\theta} \left(\frac{\partial u}{\partial V_\theta} \right) > 0 \quad \text{in} \quad Q_{\bar{h}}(x_0) \quad \text{for} \quad -\bar{\theta} \leq \theta \leq \bar{\theta}.$$

Also, since we assumed that u is nonnegative in \mathbb{R}_+^2 , it follows that

$$(3.3) \quad \frac{\partial u}{\partial V_\theta}(x, 0) \geq 0 \quad \text{for any} \quad -\bar{\theta} \leq \theta \leq \bar{\theta} \quad \text{and for any} \quad x \in \mathbb{R}.$$

By combining (3.2) and (3.3), we deduce the strict monotonicity of u in the V_θ -direction, for every $x \in Q_{\bar{h}}(x_0) \cap \mathbb{R}_+^2$ and every $\theta \in [-\bar{\theta}, \bar{\theta}]$. From the analysis above, we find the existence of (possible very small)

$$(3.4) \quad \bar{s} = \bar{s}(\bar{\theta}) > 0$$

such that, for any $0 < s \leq \bar{s}$:

- i) both the triangle $\mathcal{T}_{x_0,s,\bar{\theta}}$ and its reflection with respect to $L_{x_0,s,\bar{\theta}}$ are contained in $Q_{\bar{h}}(x_0)$ (as well as their reflections with respect to the axis $\{x = x_0\}$),
- ii) both the segment $\{(x_0, y) : 0 \leq y \leq s\}$ and its reflection with respect to $L_{x_0,s,\theta}$ are contained in $Q_{\bar{h}}(x_0)$ for every $\theta \in (0, \bar{\theta}]$,
- iii) $u < u_{x_0,s,\bar{\theta}}$ in $\mathcal{T}_{x_0,s,\bar{\theta}}$,
- iv) $u \leq u_{x_0,s,\theta}$ on $\partial(\mathcal{T}_{x_0,s,\theta})$ for every $\theta \in (0, \bar{\theta}]$,
- v) $u < u_{x_0,s,\theta}$ on the set $\{(x_0, y) : 0 < y < s\}$, for every $\theta \in (0, \bar{\theta}]$.

Note that, from iii)–iv), we have that

$$(3.5) \quad \forall s \in (0, \bar{s}), \quad (\mathcal{HT}_{x_0, s, \bar{\theta}}) \text{ holds.}$$

Next we prove a result that allows to start the moving plane procedure. We recall the notation

$$u_\lambda(x, y) := u(x, 2\lambda - y).$$

Lemma 3.1 (Monotonicity near the boundary). *There exists $\hat{\lambda} > 0$ such that, for any $0 < \lambda \leq \hat{\lambda}$, we have*

$$(3.6) \quad u < u_\lambda \quad \text{in } \Sigma_\lambda.$$

Furthermore,

$$(3.7) \quad \partial_y u > 0 \quad \text{in } \Sigma_{\hat{\lambda}}.$$

Proof. Let $\bar{\theta}$ given by (3.2) and $\bar{s} = \bar{s}(\bar{\theta})$ as in (3.4). We showed that, for any $0 < s < \bar{s}$, $(\mathcal{HT}_{x_0, s, \bar{\theta}})$ holds. We use now Lemma 2.4 as follows: for any fixed $s \in (0, \bar{s})$ and $\theta' \in (0, \bar{\theta})$, we consider the rotation

$$g(t) = (s(t), \theta(t)) := (s, t\theta' + (1 - t)\bar{\theta}), \quad t \in [0, 1].$$

Recalling that $(\mathcal{HT}_{x_0, s, \bar{\theta}})$ holds by (3.5), we deduce that also $(\mathcal{HT}_{x_0, s, \theta'})$ holds. Therefore, by the fact that $0 < \theta' < \bar{\theta}$ is arbitrary and by a continuity argument, we pass to the limit for $\theta' \rightarrow 0$ and get

$$u(x, y) \leq u_s(x, y) \quad \text{in } \Sigma_s \cap \{x \leq x_0\} \text{ for } 0 < s < \bar{s}.$$

The invariance of the considered problem with respect to the axis $\{x = x_0\}$ enables us to use the same argument to treat the case of negative θ , yielding

$$u(x, y) \leq u_s(x, y) \quad \text{in } \Sigma_s \cap \{x \geq x_0\} \text{ for } 0 < s < \bar{s},$$

possibly reducing \bar{s} . Thus $u(x, y) \leq u_s(x, y)$ in Σ_s for every $s \in (0, \bar{s})$. The desired conclusion (3.6) then follows by taking $\hat{\lambda}$ such that $0 < \hat{\lambda} < \min\{\bar{s}, \lambda^*/2\}$. Here we have used in a crucial way that the property $(\mathcal{P})_\lambda$ holds for every $\lambda \in (0, \hat{\lambda}]$, so that the case $u \equiv u_\lambda$ in Σ_λ is not possible. Moreover, by Hopf’s lemma, for every $\lambda \in (0, \hat{\lambda}]$ and every $x \in \mathbb{R}$, we get

$$(3.8) \quad 2\partial_y u(x, \lambda) = \frac{\partial(u - u_\lambda)}{\partial y}(x, \lambda) > 0.$$

This proves (3.7). □

To proceed further we need *some notations*. Let λ^* be defined as in (1.5) (a self-contained independent proof of the fact that λ^* is well defined will be provided in Theorem 6.1). In the case $\lambda^* = \infty$, we set

$$\Lambda = \{\lambda > 0 : u < u_\lambda, \quad \text{in } \Sigma_{\lambda'}, \quad \forall \lambda' < \lambda\}.$$

If λ^* is finite we use the same notation but considering values of λ such that $0 < \lambda < \lambda^*/2$, namely,

$$\Lambda = \{ \lambda < \lambda^*/2 : u < u_{\lambda'} \text{ in } \Sigma_{\lambda'} \quad \forall \lambda' < \lambda \}.$$

By Lemma 3.1, we know that Λ is not empty and we can define

$$(3.9) \quad \bar{\lambda} = \sup \Lambda.$$

The proof of the theorem will be done if we show that $\bar{\lambda} = +\infty$, when $\lambda^* = \infty$ (respectively, $\bar{\lambda} = \lambda^*/2$, when λ^* is finite). Therefore we argue by contradiction and assume that $\bar{\lambda} < +\infty$, when $\lambda^* = \infty$ (respectively, $\bar{\lambda} < \lambda^*/2$, when λ^* is finite).

First, as above, we deduce that u is strictly monotone increasing in the e_2 -direction in $\Sigma_{\bar{\lambda}}$, with

$$(3.10) \quad \partial_y u > 0 \text{ in } \Sigma_{\bar{\lambda}}.$$

To proceed further we need to prove the following.

Lemma 3.2. *Let λ^* and $\bar{\lambda}$ be as above. Assume that there is a point $x_0 \in \mathbb{R}$ satisfying $u(x_0, 2\bar{\lambda}) > 0$. Then there exists $\bar{\delta} > 0$ such that: for any $-\bar{\delta} \leq \theta \leq \bar{\delta}$ and for any $0 < \lambda \leq \bar{\lambda} + \bar{\delta}$, we have*

$$u(x_0, y) < u_{x_0, \lambda, \theta}(x_0, y), \quad \text{for } 0 < y < \lambda.$$

Proof. First we note that $\partial_y u(x_0, \bar{\lambda}) > 0$. In fact, by construction $u < u_{\bar{\lambda}}$ in $\Sigma_{\bar{\lambda}}$. Therefore, by Hopf's lemma, we have

$$(3.11) \quad 2\partial_y u(x_0, \bar{\lambda}) = \frac{\partial(u - u_{\bar{\lambda}})}{\partial y}(x_0, \bar{\lambda}) > 0.$$

We argue now by contradiction. If the lemma were false, we would find a sequence of small $\delta_n \rightarrow 0$ and $-\delta_n \leq \theta_n \leq \delta_n$, $0 < \lambda_n \leq \bar{\lambda} + \delta_n$, $0 < y_n < \lambda_n$ with

$$u(x_0, y_n) \geq u_{x_0, \lambda_n, \theta_n}(x_0, y_n).$$

Possibly considering subsequences, we may and do assume that $\lambda_n \rightarrow \tilde{\lambda} \leq \bar{\lambda}$. Also $y_n \rightarrow \tilde{y}$ for some $\tilde{y} \leq \tilde{\lambda}$. Considering the construction of $Q_{\tilde{h}}(x_0)$ as above and in particular taking into account (3.2) and (3.3), we deduce that $\tilde{\lambda} \geq 0$ and, by continuity, it follows that $u(x_0, \tilde{y}) \geq u_{\tilde{\lambda}}(x_0, \tilde{y})$. Consequently $y_n \rightarrow \tilde{\lambda} = \tilde{y}$, since we know that $u < u_{\lambda'}$ in $\Sigma_{\lambda'}$ for any $\lambda' \leq \bar{\lambda}$ and we assumed that $u(x_0, 2\bar{\lambda}) > 0$ so that in particular $u(x_0, 0) = 0 < u(x_0, 2\bar{\lambda})$. By the mean value theorem, since $u(x_0, y_n) \geq u_{x_0, \lambda_n, \theta_n}(x_0, y_n)$, it follows

$$\frac{\partial u}{\partial V_{\theta_n}}(x_n, y_n) \leq 0$$

at some point $\xi_n \equiv (x_n, y_n)$ lying on the line from (x_0, y_n) to $T_{x_0, \lambda_n, \theta_n}(x_0, y_n)$, recalling that the vector V_{θ_n} is orthogonal to the line $L_{x_0, \lambda_n, \theta_n}$. Since $V_{\theta_n} \rightarrow e_2$ as $\theta_n \rightarrow 0$. Taking the limit it follows that

$$\partial_y u(x_0, \tilde{\lambda}) \leq 0$$

which is impossible by (3.11) and (3.10). □

End of the first proof of Theorem 1.5. Since we are assuming that $\bar{\lambda} < +\infty$, when $\lambda^* = \infty$ (respectively, $\bar{\lambda} < \lambda^*/2$, when λ^* is finite), we can find $x_0 \in \mathbb{R}$ such that $u(x_0, 2\bar{\lambda}) > 0$. Let $Q_{\bar{h}}(x_0)$ be constructed as above and pick $\bar{\theta}$ given by (3.2). Let also $\bar{\delta}$ as in Lemma 3.2. Then fix $\theta_0 > 0$ with $\theta_0 \leq \bar{\delta}$ and $\theta_0 \leq \bar{\theta}$. Let us set

$$s_0 := s_0(\theta_0),$$

such that the triangle $\mathcal{T}_{x_0, s_0, \theta_0}$ and its reflection with respect to L_{x_0, s_0, θ_0} is contained in $Q_{\bar{h}}(x_0)$ and consequently $(\mathcal{HT}_{x_0, s_0, \theta_0})$ holds. It is convenient to assume that $s_0 \leq \hat{\lambda}$ with $\hat{\lambda}$ as in Lemma 3.1. For any

$$s_0 < s \leq \bar{\lambda} + \bar{\delta}, \quad 0 < \theta < \theta_0,$$

we carry out the *sliding-rotating technique* exploiting Lemma 2.4 with

$$g(t) = (s(t), \theta(t)) := (ts + (1 - t)s_0, t\theta + (1 - t)\theta_0) \quad t \in [0, 1].$$

By Lemma 3.2, we deduce that the boundary conditions required to apply Lemma 2.4 are fulfilled and therefore, by Lemma 2.4, we get that $(\mathcal{HT}_{x_0, s, \theta})$ holds. We can now argue as in the proof of Lemma 3.1 and deduce that $u(x, y) < u_\lambda(x, y)$ in Σ_λ for any $0 < \lambda \leq \bar{\lambda} + \bar{\delta}$. This provides a contradiction unless $\bar{\lambda} = +\infty$ (respectively, $\bar{\lambda} = \lambda^*/2$, if λ^* is finite). Arguing e.g. as in the proof of Lemma 3.1, we deduce

$$\partial_y u > 0 \quad \text{in } \mathbb{R}_+^2 \quad \text{if } \lambda^* = +\infty,$$

while

$$\partial_y u > 0 \quad \text{in } \Sigma_{\lambda^*/2} \quad \text{if } \lambda^* < +\infty.$$

As a consequence of the monotonicity result, we deduce that u is positive in \mathbb{R}_+^2 if $\lambda^* = +\infty$.

In we assume that λ^* is finite, we deduce by continuity that

$$u \leq u_{\lambda^*/2} \quad \text{in } \Sigma_{\lambda^*/2}.$$

By the strong comparison principle, we deduce that: either $u < u_{\lambda^*/2}$ or $u \equiv u_{\lambda^*/2}$, in $\Sigma_{\lambda^*/2}$. Note that, by the definition of λ^* , we have that $\{y = \lambda^*\} \subseteq \{u = 0\}$, that also implies $\{y = \lambda^*\} \subseteq \{\nabla u = 0\}$ since u is nonnegative. If $u < u_{\lambda^*/2}$ in $\Sigma_{\lambda^*/2}$, we get by the Hopf boundary lemma (see [20]) that $\partial_y(u_{\lambda^*/2} - u) > 0$ on $\{y = 0\}$. Since $\partial_y(u_{\lambda^*/2}) = 0$ on $\{y = 0\}$ (by the fact that $\{y = \lambda^*\} \subseteq \{\nabla u = 0\}$) this provides a contradiction with the fact that u is nonnegative. Therefore it occurs $u \equiv u_{\lambda^*/2}$, in $\Sigma_{\lambda^*/2}$.

Note now that, since $\{y = \lambda^*\} \subseteq \{u = 0\} \cap \{\nabla u = 0\}$, by symmetry we deduce

$$\{y = 0\} \subseteq \{u = 0\} \cap \{\nabla u = 0\}.$$

Therefore we deduce that u is one-dimensional by the *unique continuation principle* (see for instance Theorem 1 of [17] and the references therein). Indeed, for every $t \in \mathbb{R}$, the function $u^t(x, y) := u(x + t, y)$ is a nonnegative solution of (1.1) with $u^t = \nabla u^t = 0$ on $\partial\mathbb{R}_+^2$ and the unique continuation principle implies that $u \equiv u^t$ on \mathbb{R}_+^2 . This immediately gives that u depends only on the variable y , i.e.,

$$u(x, y) = u_0(y) \quad \forall (x, y) \in \mathbb{R}_+^2,$$

where $u_0 \in C^2([0, +\infty))$ is the unique solution of $u_0'' + f(u_0) = 0$ with $u_0'(0) = u_0(0) = 0$. The remaining part of the statement, namely the properties of u_0 , follows by a simple ODE analysis. \square

4. Second proof of Theorem 1.5

This proof makes use of the *unique continuation principle* to start the moving plane procedure.

Second proof of Theorem 1.5. If we assume that

$$\nabla u(x, 0) = 0 \quad \text{for any } x \in \mathbb{R},$$

then it follows that u coincides with u_0 by the unique continuation principle, with u_0 as in the statement. A simple analysis of the associated ordinary differential equation shows in this case that u_0 is monotone increasing if $\lambda^* = \infty$, while u_0 is periodic when λ^* is finite. Therefore the proof is done in this case and we reduce to consider the case:

$$\text{there exists } x_0 \in \mathbb{R} \quad \text{such that} \quad \nabla u(x_0, 0) \neq 0.$$

Necessarily in this case we have that $\partial_y u(x_0, 0) > 0$ since the case $\partial_y u(x_0, 0) < 0$ is not possible because u is nonnegative. Setting as above (see (3.1))

$$(4.1) \quad Q_h(x_0) = \{(x, y) : |x - x_0| \leq h, 0 \leq y \leq 2h\},$$

recalling that $u \in C^2(\overline{\mathbb{R}_+^2})$, we can therefore fix $\bar{h} > 0$ small such that

$$\partial_y u > 0 \quad \text{in } Q_{\bar{h}}(x_0).$$

Exploiting again the fact that $u \in C^2(\overline{\mathbb{R}_+^2})$, we can consequently consider $\bar{\theta} = \bar{\theta}(\bar{h})$ small such that

$$(4.2) \quad \frac{\partial u}{\partial V_{\bar{\theta}}} > 0 \quad \text{in } Q_{\bar{h}}(x_0) \quad \text{for } -\bar{\theta} \leq \theta \leq \bar{\theta}.$$

From the above (4.2), we immediately deduce the existence of \bar{s} as in (3.4) and satisfying properties i)–v) at the beginning of the first proof of Theorem 1.5, so that the moving plane procedure can be started.

To proceed further note that, for $\bar{\lambda}$ defined as above, it occurs that

$$u(x_0, 2\bar{\lambda}) > 0.$$

In fact, if this is not the case, then we have $\nabla u(x_0, 2\bar{\lambda}) = 0$. Consequently,

$$\partial_y u(x_0, 0) = \partial_y (u - u_{\bar{\lambda}})(x_0, 0) \leq 0,$$

while $\partial_y u(x_0, 0) > 0$ by construction. Therefore Lemma 3.2 can be exploited and, since the remaining part of the proof can be repeated verbatim, we omit it. \square

5. Proof of Theorems 1.1–1.3 and Theorems 1.6–1.9

Proof of Theorem 1.1. By Theorem 1.5 we have that, either $\lambda^* < +\infty$ and thus u is one-dimensional and periodic or $\lambda^* = +\infty$ and u satisfies $u > 0$ and $\partial_y u > 0$ in \mathbb{R}_+^2 . \square

Proof of Theorem 1.6. Let us first consider the case $f(0) < 0$. By Theorem 1.1, either u is one-dimensional and periodic, and we are done, or $u > 0$ and $\partial_y u > 0$ in \mathbb{R}_+^2 . Therefore, since we are assuming that $|\nabla u|$ is bounded, we are in position to apply Theorem 1.2 in [16] to conclude that u is one-dimensional. When $f(0) \geq 0$, either u is identically zero, or $u > 0$ and $\partial_y u > 0$ in \mathbb{R}_+^2 by Theorem 1.1 in [8]. The desired conclusion then follows by applying once again Theorem 1.2 in [16]. \square

Proof of Theorem 1.7. Let us assume by contradiction that $\lambda^* = +\infty$ in Theorem 1.5, hence $\partial_y u > 0$ in \mathbb{R}_+^2 . This implies that

$$\int_{\mathbb{R}_+^2} |\nabla \varphi|^2 - f'(u)\varphi^2 dx \geq 0 \quad \text{for any } \varphi \in C_c^\infty(\mathbb{R}_+^2).$$

Since we are assuming that $f'(t) \geq c > 0$ for any $t \in (0, \infty)$, it follows that

$$(5.1) \quad \int_{\mathbb{R}_+^2} |\nabla \varphi|^2 dx \geq c \int_{\mathbb{R}_+^2} \varphi^2 dx \quad \text{for any } \varphi \in C_c^\infty(\mathbb{R}_+^2).$$

Let us now consider an arbitrary open ball Ω such that $\bar{\Omega} \subset \mathbb{R}_+^2$ and let $\lambda_1(\Omega)$ be the first eigenvalue of the Laplace operator in Ω under zero Dirichlet boundary conditions. By the variational characterization of $\lambda_1(\Omega)$ and by (5.1), it would follow that

$$\lambda_1(\Omega) \geq c > 0,$$

and this is clearly impossible since $\lambda_1(\Omega)$ approaches zero when Ω is chosen arbitrary large. Therefore, necessarily, it occurs that λ^* is finite, u is one-dimensional and periodic and u cannot be positive. \square

Proof of Theorem 1.2. By Theorem 1.7, u is one-dimensional and periodic with profile u_0 satisfying (1.6). A simple ODE analysis shows that

$$u_0(y) = 1 - \cos y,$$

and the result is proved. □

Proof of Theorem 1.8. The proof is analogous to the proof of Theorem 1.5. Let us only provide a few details.

Consider first the case $\lambda^* = 2b$. The moving plane procedure can be started as in Theorem 1.5 exploiting Lemma 3.1. Then we define $\bar{\lambda}$ as in (3.9) and we deduce that necessarily $\bar{\lambda} = b$ arguing by contradiction exactly as in the proof of Theorem 1.5 and exploiting the assumption $\lambda^* = 2b$. Note that there is no need of changes in the proof of Lemma 3.2. Therefore we deduce that $u \leq u_\lambda$ in Σ_λ for any $0 < \lambda < b$ and, by continuity, we have

$$u \leq u_b \quad \text{in } \Sigma_b.$$

As a consequence of the moving plane procedure we also deduce that u is monotone non-decreasing in Σ_b . Actually, arguing as in Lemma 3.1 we have $\partial_y u > 0$ in Σ_b . In particular u is positive in the entire strip Σ_{2b} .

Performing the moving plane method in the opposite direction $(0, -1)$ and observing that the corresponding λ^* is still equal to $2b$ (by the positivity of u), we derive in the same way that

$$u \geq u_b \quad \text{in } \Sigma_b,$$

and this implies that u is symmetric with respect to $\{y = b\}$.

Let us now consider the case $\lambda^* < 2b$. In this case arguing as above we deduce that u is positive in Σ_{λ^*} , $u \leq u_\lambda$ in Σ_λ for any $0 < \lambda < \lambda^*/2$ and

$$u \equiv u_{\lambda^*/2} \quad \text{in } \Sigma_{\lambda^*/2}.$$

As above $\{y = \lambda^*\} \subseteq \{u = 0\} \cap \{\nabla u = 0\}$, that gives, by symmetry,

$$\{y = 0\} \subseteq \{u = 0\} \cap \{\nabla u = 0\}.$$

Now we deduce that u is one-dimensional by the *unique continuation principle* and consequently $u(x, y) \equiv u_0(y)$ for u_0 defined as in the statement. □

Proof of Theorem 1.9. The proof is completely analogous to the proof of Theorem 1.8. Actually, it is easier in this case since we are assuming that $f(0) \geq 0$ so that the *strong maximum principle* and the *Hopf lemma* can be exploited. In particular we can start the moving plane procedure recovering (4.2) via the Hopf lemma. Then we complete the proof repeating verbatim the proof of Theorem 1.8 and exploiting the fact that in this case, by the strong maximum principle, the solution is either trivial or positive. □

Proof of Theorem 1.3. The proof of Theorem 1.3 follows combining Theorem 1.8 and Theorem 1.9. □

6. Some results in any dimension $N \geq 2$

In this section we state and prove some results for any dimension $N \geq 2$. We continue to assume that f is locally Lipschitz continuous and satisfies $f(0) < 0$. Let us consider the problem

$$(6.1) \quad \begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}_+^N, \\ u \geq 0 & \text{in } \mathbb{R}_+^N, \\ u = 0 & \text{on } \partial\mathbb{R}_+^N, \end{cases}$$

and let us denote by (x', y) a point in \mathbb{R}^N , with $x' = (x_1, \dots, x_{N-1})$ and $y = x_N$. For a fixed solution u , we consider the property (\mathcal{P}_μ) as in the introduction, and we have the following

Theorem 6.1. *Let $u \in C^2(\overline{\mathbb{R}_+^N})$ be a solution to (6.1) and let us set*

$$(6.2) \quad \Lambda^* = \Lambda^*(u) := \{\lambda > 0 : (\mathcal{P}_\mu) \text{ holds for every } 0 < \mu \leq \lambda\}.$$

Then

$$\Lambda^* \neq \emptyset.$$

Proof. We prove the theorem by contradiction, and therefore we assume that there exists a sequence of positive numbers μ_n such that μ_n tends to zero as $n \rightarrow \infty$ and (\mathcal{P}_{μ_n}) fails, namely

$$\{y = \mu_n\} \subset \{u = 0\}.$$

For $x'_0 \in \mathbb{R}^{N-1}$ fixed, by the fact that u is nonnegative, it follows that

$$\nabla u(x'_0, \mu_n) = 0.$$

Exploiting the Dirichlet condition, it also follows that the real valued function $u(x'_0, t)$ with $t \in [0, \mu_n]$, has an interior local maximum $t_n \in (0, \mu_n)$. Therefore $\partial_y u(x'_0, t_n) = 0$.

By the mean value theorem we deduce that

$$\partial_{yy} u(x'_0, \xi_n) = 0 \quad \text{for some } \xi_n \in [t_n, \mu_n].$$

Since $u \in C^2(\overline{\mathbb{R}_+^N})$, letting $n \rightarrow \infty$, we infer that

$$\partial_{yy} u(x'_0, 0) = 0.$$

This is a contradiction since in this case, recalling that $u = 0$ in $\{y = 0\}$, it follows that

$$-\Delta u(x'_0, 0) = 0 > f(u(x'_0, 0)),$$

and the result is proved. □

As a consequence of Theorem 6.1, we see that

$$(6.3) \quad \lambda^* = \lambda^*(u) := \sup \Lambda^* \in (0, +\infty].$$

We have the following.

Theorem 6.2. *Let $u \in C^2(\overline{\mathbb{R}_+^N})$ be a solution to (6.1) and let $\lambda^* = \lambda^*(u)$ be defined by (6.3). Then, if λ^* is finite, it follows that u is one-dimensional and periodic, i.e.,*

$$u(x, y) = u_0(y) \quad \forall (x, y) \in \mathbb{R}_+^2,$$

where $u_0 \in C^2(\mathbb{R}, [0, \infty))$ is periodic of period λ^* and is the unique solution of

$$(6.4) \quad u_0'' + f(u_0) = 0 \quad \text{in } [0, \infty) \quad u_0'(0) = u_0(0) = 0 = u_0'(\lambda^*) = u_0(\lambda^*).$$

Also, u is symmetric with respect to $\{y = \lambda^*/2\}$ with $\partial_y u > 0$ in $\Sigma_{\lambda^*/2}$.

Proof. It follows by the definition of λ^* that $\{y = \lambda^*\} \subseteq \{u = 0\}$. Since u is nonnegative this implies that

$$\{y = \lambda^*\} \subseteq \{\nabla u = 0\}.$$

To conclude, we argue as at the end of the proof of Theorem 1.5. Indeed, for every $t \in \mathbb{R}^{N-1}$, the function $u^t(x', y) := u(x' + t, y)$ is a nonnegative solution of (1.1) with $u^t = \nabla u^t = 0$ on the set $\{y = \lambda^*\}$ and the unique continuation principle implies that $u \equiv u^t$ on \mathbb{R}_+^N . This immediately gives that u depends only on the variable y , i.e.,

$$u(x', y) = u_0(y) \quad \forall (x', y) \in \mathbb{R}_+^N$$

where $u_0 \in C^2([0, +\infty))$ is the unique solution of $u_0'' + f(u_0) = 0$ with $u_0'(0) = u_0(0) = 0 = u_0'(\lambda^*) = u_0(\lambda^*)$. A simple analysis of (6.4) yields that u_0 is periodic of period λ^* with u_0 even with respect to $\{y = \lambda^*/2\}$, and satisfying $u_0' > 0$ in $(0, \lambda^*/2)$. This concludes the proof. \square

Now we turn to the case of coercive epigraphs and we prove Theorem 1.10.

Proof of Theorem 1.10. We plan to use the classical moving plane procedure [22]. We use the notation $\Sigma_\lambda := \{(x', y) \in \mathbb{R}^N \mid 0 < y < \lambda\}$ and we denote by $R_\lambda(x', y)$ the point symmetric to (x', y) with respect to the hyperplane $\{y = \lambda\}$, namely

$$R_\lambda(x', y) := (x', 2\lambda - y).$$

We set $u_\lambda(x', y) = u(R_\lambda(x', y)) = u(x', 2\lambda - y)$ and $\Omega_\lambda := \Omega \cap \Sigma_\lambda$. Since Ω is a smooth coercive epigraph, we have that Ω_λ is a bounded open set (possibly non connected) satisfying $R_\lambda(\Omega_\lambda) \subset \Omega$, for every $\lambda > 0$.

Given any $\delta > 0$, we can find $\lambda_0 = \lambda_0(\delta) > 0$ such that $\mathcal{L}(\Omega_\lambda) < \delta$ for any $0 < \lambda \leq \lambda_0$. Therefore we can take δ small such that the *weak comparison principle in small domains* (Proposition 2.1) applies. Since $u \leq u_\lambda$ on $\partial\Omega_\lambda$, Proposition 2.1 yields

$$u \leq u_\lambda \quad \text{in } \Omega_\lambda \quad \text{for every } 0 < \lambda \leq \lambda_0.$$

Therefore the set

$$\Lambda := \{\lambda > 0 : u \leq u_\mu \text{ in } \Omega_\mu \text{ for any } 0 < \mu \leq \lambda\}$$

is not empty and

$$\bar{\lambda} := \sup \Lambda \in (0, +\infty].$$

Assume by contradiction that $\bar{\lambda} < +\infty$. By continuity it follows that $u \leq u_{\bar{\lambda}}$ in $\Omega_{\bar{\lambda}}$. Let us now prove that

$$(6.5) \quad u < u_{\bar{\lambda}} \quad \text{in } \Omega_{\bar{\lambda}}.$$

To this end, we observe that u is *positive* in a neighborhood of the boundary. Namely, for any $x \in \partial\Omega$, there exists $\rho = \rho(x) > 0$ such that

$$(6.6) \quad u > 0 \quad \text{in } \Omega \cap B_{\rho}(x).$$

A proof of this fact can be found, for instance, in [1] (cf. Lemma 4.1 on p. 485). If (6.5) were false, there would exist a point $x_0 \in \Omega_{\bar{\lambda}}$ such that $u(x_0) = u_{\bar{\lambda}}(x_0)$. Thence, the strong maximum principle would imply

$$u \equiv u_{\bar{\lambda}} \quad \text{in } \omega_{\bar{\lambda}},$$

where $\omega_{\bar{\lambda}}$ is the connected component of $\Omega_{\bar{\lambda}}$ containing the point x_0 . This clearly contradicts (6.6) (it would imply $u = 0$ on $R_{\bar{\lambda}}(\partial\Omega \cap \omega_{\bar{\lambda}})$). Hence (6.5) is satisfied.

In order to exploit Proposition 2.1 once again, let us fix a bounded domain $D \subset \Omega$ containing the bounded set $\Omega_{\bar{\lambda}+1} \cup R_{\bar{\lambda}+1}(\Omega_{\bar{\lambda}+1})$ and then consider a compact set $\mathcal{K} \subset \Omega_{\bar{\lambda}}$ such that

$$\mathcal{L}(\Omega_{\bar{\lambda}} \setminus \mathcal{K}) \leq \frac{\vartheta}{10}$$

where ϑ is given by Proposition 2.1. It follows by compactness that, for some $\sigma > 0$,

$$(u_{\bar{\lambda}} - u) \geq \sigma > 0 \quad \text{on } \mathcal{K},$$

and therefore, by the uniform continuity of u on compact sets, we can find $\varepsilon_0 \in (0, 1)$ such that

$$(u_{\bar{\lambda}+\varepsilon} - u) \geq \frac{\sigma}{2} > 0 \quad \text{and} \quad \mathcal{L}(\Omega_{\bar{\lambda}+\varepsilon} \setminus \Omega_{\bar{\lambda}}) \leq \frac{\vartheta}{10} \quad \text{for every } 0 < \varepsilon < \varepsilon_0.$$

Then it follows that Proposition 2.1 applies in $\Omega_{\bar{\lambda}+\varepsilon} \setminus \mathcal{K}$, for every $0 < \varepsilon < \varepsilon_0$, since

$$\mathcal{L}(\Omega_{\bar{\lambda}+\varepsilon} \setminus \mathcal{K}) < \vartheta.$$

Therefore $u \leq u_{\bar{\lambda}+\varepsilon}$ in $\Omega_{\bar{\lambda}+\varepsilon} \setminus \mathcal{K}$, for every $0 < \varepsilon < \varepsilon_0$, and consequently

$$u \leq u_{\bar{\lambda}+\varepsilon} \quad \text{in } \Omega_{\bar{\lambda}+\varepsilon} \quad \text{for every } 0 < \varepsilon < \varepsilon_0.$$

The latter contradicts the definition of $\bar{\lambda}$. Thus, we have proved that

$$\bar{\lambda} = +\infty.$$

As a consequence, u is monotone non-decreasing in the y -direction. Actually, arguing exactly as above and using again that u is *positive* in a neighborhood of the boundary, we get

$$(6.7) \quad u < u_{\lambda} \quad \text{in } \Omega_{\lambda} \quad \text{for every } \lambda > 0.$$

Hence the Hopf lemma yields

$$\partial_{x_N} u = \partial_y u > 0 \quad \text{in } \Omega.$$

Furthermore, as a consequence, u is positive in Ω . □

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ALBERTO FARINA: LAMFA, CNRS UMR 7352, Université de Picardie Jules Verne, 33 Rue Saint-Leu, 80039 Amiens Cedex 1, France.

E-mail: alberto.farina@u-picardie.fr

BERARDINO SCIUNZI: Dipartimento di Matematica e Informatica, Università della Calabria, V. P. Bucci 1, 87036 Arcavacata di Rende, Cosenza, Italy.

E-mail: sciunzi@mat.unical.it