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Combining Riesz bases in $\mathbb{R}^d$

Gady Kozma and Shahaf Nitzan

Abstract. We prove that every finite union of rectangles with edges parallel to the axes in $\mathbb{R}^d$ admits a Riesz basis of exponentials.

1. Introduction

Orthogonal bases are used throughout mathematics and its applications. However, in many settings such bases are not easy to come by. For example, even the union of as few as two disjoint intervals in $\mathbb{R}$ may not admit an orthogonal basis of exponentials, $e(\Lambda) := \{e^{i\langle \lambda,t\rangle}\}_{\lambda\in\Lambda}$. This example should be contrasted with the case of a single interval, where the exponential orthogonal basis plays a fundamental role.

Among the systems which may be considered as replacements for orthogonal bases, Riesz bases are the best possible: They are the image of orthogonal bases under a bounded invertible operator and therefore *preserve most of their qualities.* In particular, if $e(\Lambda)$ is a Riesz basis over some set $S \subset \mathbb{R}$ then every $f \in L^2(S)$ can be decomposed into a series $f = \sum a_\lambda e^{2\pi i\lambda t}$ in a unique and stable way.

Our understanding of the existence of Riesz bases of exponentials is still lacking. On the one hand, there are relatively few examples in which it is known how to construct a Riesz basis of exponentials. For example, in two dimensions, we do not know how to construct such a basis for either a ball or a triangle, nor even have a reasonable candidate to be such a basis (it is known that neither set supports an orthogonal basis of exponentials, [2], [4]). Some constructions of Riesz bases (e.g., for polytopes with some arithmetic constraints) can be found in [9], [3], and [1], and references within. On the other hand, we know of no example of a set S of positive measure for which a Riesz basis of exponentials can be shown not to exist.

In [6] we proved the following.

Theorem 1. *Let $S \subset \mathbb{R}$ be a finite union of intervals. Then there exists a set $\Lambda \subset \mathbb{R}$ such that the family $e(\Lambda) := \{e^{2\pi i\lambda t}\}_{\lambda\in\Lambda}$ is a Riesz basis in $L^2(S)$. Moreover, if $S \subset [0,1]$ then Λ may be chosen to satisfy $\Lambda \subset \mathbb{Z}$.*

In this paper we extend this result to higher dimensions in the following way.

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Theorem 2. *Let $S \subset \mathbb{R}^d$ be a finite union of rectangles with edges parallel to the axes. Then there exists a set $\Sigma \subset \mathbb{R}^d$ such that the family $e(\Sigma)$ is a Riesz basis in $L^2(S)$. Moreover, if $S \subset [0,1]^d$ then Σ may be chosen to satisfy $\Sigma \subset \mathbb{Z}^d$.*

We take from [6] the following basic principle. Suppose you try to construct a Riesz basis by combining Riesz bases of simpler sets. If the most natural candidate for a construction of a Riesz basis does not work, try instead a construction that involves first taking unions and intersections of your simpler sets, and then combining their Riesz bases. Take, for example, in the setting of Theorem 1, the case where $S = I \cup J$ with $I \subset [0,1/2]$ and $J \subset [1/2,1]$. Then the most natural candidate for a Riesz basis might be to take a Riesz basis for I and a Riesz basis for J and hope that their union would be a Riesz basis for S. This does not work, but it turns out that taking Riesz bases for $I \cup (J - 1/2)$ and for $I \cap (J - 1/2)$ and taking the union of them works (under certain conditions). Here we need to construct a Riesz basis for a union of products, say $\bigcup X_i \times Y_i$. The natural candidate is to take Riesz bases Ξ_i for X_i and Ψ_i for Y_i, and hope that $\bigcup \Xi_i \times \Psi_i$ would be a Riesz basis for $\bigcup X_i \times Y_i$. This does not work. The correct "union and intersection version" is the following lemma. Denote $Y_{\ge n} = \bigcup_{k=n}^{L} Y_n$.

Lemma 3. *Let $X_1, \dots, X_L \subset [0,1]^a$ be some sets and let $Y_1, \dots, Y_L \subset [0,1]^b$ be pairwise disjoint sets. Assume $\Xi_1 \subset \cdots \subset \Xi_L \subset \mathbb{Z}^a$ satisfy that $e(\Xi_n)$ is a Riesz basis for X_n. Assume further that $\Psi_{\ge 1} \supset \cdots \supset \Psi_{\ge L}$ are subsets of $\mathbb{Z}^b$ such that that $e(\Psi_{\ge n})$ is a Riesz basis for $Y_{\ge n}$. Define*

$$\Sigma := \bigcup_{n=1}^{L} \Xi_n \times \Psi_{\ge n}.$$

Then $e(\Sigma)$ is a Riesz basis for $\bigcup_{n=1}^{L} X_n \times Y_n$.

To get a feeling for the condition $\Xi_1 \subset \cdots \subset \Xi_L$ (which in particular means that the X_n must have increasing sizes for the lemma to have any hope of being applicable) one should first note that without this condition Σ might not even have the right density to be a Riesz basis (see [7], [8], [11] for Landau's theorem, explaining the role of density). The definition of Σ can be reorganized in two other ways which emphasize the issue of density (in particular as a union of disjoint sets). The first is

$$\Sigma = \bigcup_{n=1}^{L} \Xi_n \times (\Psi_{\ge n} \setminus \Psi_{\ge n+1})$$

(where $\Psi_{\ge L+1} := \emptyset$). This version has the mnemonic property of being almost a "union of products of Riesz bases" except, of course, we are not requiring from $\Psi_{\ge j} \setminus \Psi_{\ge j+1}$ to be a Riesz basis for Y_j. The other version is

$$\Sigma = \bigcup_{n=1}^{L} (\Xi_n \setminus \Xi_{n-1}) \times (\Psi_{\ge n})$$

(where $\Xi_0 := \emptyset$). This version will be used in the proof (§3 below). More remarks on the relation with [6] will be given after the proof, in Section 6.

2. Preliminaries

2.1. Systems of vectors in Hilbert spaces

Let H be a separable Hilbert space. A system of vectors $\{f_n\} \subseteq H$ is called a *Riesz basis* if it is the image, under a bounded invertible operator, of an orthonormal basis. This means that $\{f_n\}$ is a Riesz basis if and only if it is complete in H and satisfies the following inequality for all sequences $\{a_n\} \in l^2$,

$$c \sum |a_n|^2 \leq \left\| \sum a_n f_n \right\|^2 \leq C \sum |a_n|^2, \tag{2.1}$$

where c and C are some positive constants which depend on the system f_n but not on the a_n. A system $\{f_n\} \subseteq H$ which satisfies condition (2.1), but is not necessarily complete, is called a *Riesz sequence*.

A simple duality argument shows that $\{f_n\}$ is a Riesz basis if and only if it is minimal (i.e., no vector from the system lies in the closed span of the rest) and satisfies the following inequality for every $f \in H$,

$$c \|f\|^2 \leq \sum |\langle f, f_n \rangle|^2 \leq C \|f\|^2, \tag{2.2}$$

where c and C are some positive constants (in fact, the same constants as in (2.1)). A system $\{f_n\} \subseteq H$ which satisfies condition (2.2), but is not necessarily minimal, is called a *frame*.

In particular, this discussion implies the following:

Lemma 4. *A system of vectors in a Hilbert space is a Riesz basis if and only if it is both a Riesz sequence and a frame.*

In this paper we are interested in frames, Riesz sequences and Riesz bases for $L^2(X)$ of the form $e(\Xi)$. Often we will be lax and simply say that Ξ is a frame, Riesz sequence or Riesz basis for X. An important property of such sets is the complementation property:

Lemma 5. *A $\Xi \subset \mathbb{Z}^d$ is a frame over an $X \subset [0,1]^d$ if and only if $\mathbb{Z}^d \setminus \Xi$ is a Riesz sequence over $[0,1]^d \setminus X$.*

Lemma 5 follows from the following general fact:

Lemma 6. *Let H be a separable Hilbert space and let $\{e_n\}_{n \in I}$ be an orthonormal basis in H. Let $L \subset H$ be a closed subspace of H and let $L^\perp$ be its orthogonal complement. Denote by P the orthogonal projection to L and by $P^\perp$ the orthogonal projection to $L^\perp$. Then for a subset $\Xi \subset I$ we have that $\{Pe_n\}_{n \in \Xi}$ is a frame in L if and only if $\{P^\perp e_n\}_{n \in I \setminus \Xi}$ is a Riesz sequence in $L^\perp$.*

See Proposition 2.1 in Matei and Meyer [10] for a proof.

3. Proof of the main lemma

In this section we prove Lemma 3, which is the main new component in the proof of Theorem 2. Recall that we are given sets $X_j \subset \mathbb{R}^a$ and $Y_j \subset \mathbb{R}^b$ and corresponding $\Xi_j \subset \mathbb{Z}^a, \Psi_{\geq j} \subset \mathbb{Z}^b$ and we wish to show for $\Sigma = \bigcup(\Xi_j \setminus \Xi_{j+1}) \times \Psi_{\geq j}$ that $e(\Sigma)$ is a Riesz basis for $S = \bigcup X_i \times Y_i$. We will use Lemma 4 and show that $e(\Sigma)$ is both a frame and a Riesz sequence for $L^2(S)$.

Throughout the proof we denote by $(x,y) := (x_1,\dots,x_a,y_1,...,y_b)$ a point in $[0,1]^{a+b}$ and by $(\xi,\psi) := (\xi_1,\dots,\xi_a,\psi_1,...,\psi_b)$ a point in $\mathbb{Z}^{a+b}$. We denote by $e_{(\xi,\psi)}$ the function $e^{2\pi i\langle(\xi,\psi),(x,y)\rangle}$.

Frame. To show that $e(\Sigma)$ is a frame in $L^2(S)$ we need to show that, for any $f \in L^2(S)$,

$$\sum_{(\xi,\psi)\in\Sigma} |\langle f, e_{(\xi,\psi)}\rangle|^2 > c_1 \|f\|^2$$

(the right inequality in the definition of a frame, (2.2), is satisfied because $S \subset [0,1]^{a+b}$ and $\Sigma \subset \mathbb{Z}^{a+b}$). For $k \in \{1,\dots,L\}$, denote by f_k the restriction of f to $X_k \times Y_k$. It is enough to show that for every $n = 1,\dots,L$ we have

$$\sum_{(\xi,\psi)\in\Sigma} |\langle f, e_{(\xi,\psi)}\rangle|^2 \geq c_2 \|f_n\|^2 - \sum_{k=1}^{n-1} \|f_k\|^2, \tag{3.1}$$

where c_2 is a positive constant, not depending on f. Indeed, the inequalities in (3.1) imply that for any sequence of positive numbers $\{\delta_n\}_{n=1}^N$ with $\sum \delta_n = 1$ we have

$$\begin{aligned}\sum_{(\xi,\psi)\in\Sigma} |\langle f, e_{(\xi,\psi)}\rangle|^2 &= \sum_{n=1}^{L} \delta_n \sum_{(\xi,\psi)\in\Sigma} |\langle f, e_{(\xi,\psi)}\rangle|^2 \\ &\overset{(3.1)}{\geq} \sum_{n=1}^{L} \delta_n \Big(c_2 \|f_n\|^2 - \sum_{k=1}^{n-1} \|f_k\|^2\Big) = \sum_{n=1}^{L} \Big(c_2\,\delta_n - \sum_{k=n+1}^{L} \delta_k\Big) \|f_n\|^2.\end{aligned}$$

We get that, if the sequence $\{\delta_n\}$ satisfies

$$\delta_n > \frac{2}{c_2} \sum_{k=n+1}^{L} \delta_k, \quad \forall n \in \{1,\dots,L\}$$

(essentially it needs to decrease exponentially), then for $c_1 = \frac{1}{2}c_2 \min \delta_n$,

$$\sum_{(\xi,\psi)\in\Sigma} |\langle f, e_{(\xi,\psi)}\rangle|^2 \geq c_1 \sum_{n=1}^{L} \|f_n\|^2 = c_1 \|f\|^2.$$

as needed.

Hence we need to show (3.1). Fix therefore some $n \in \{1, \ldots, L\}$ until the end of the proof. Now, for any $x, y \in \mathbb{C}$, $|x+y|^2 \geq \frac{1}{2}|x|^2 - |y|^2$. So,

$$|\langle f, e_{(\xi,\psi)}\rangle|^2 \geq \frac{1}{2}\Big|\Big\langle \sum_{k=n}^{L} f_k, e_{(\xi,\psi)}\Big\rangle\Big|^2 - \Big|\Big\langle \sum_{k=1}^{n-1} f_k, e_{(\xi,\psi)}\Big\rangle\Big|^2.$$

For brevity denote

$$f_{\geq n} = \sum_{k=n}^{L} f_k.$$

Summing over all (ξ, ψ) in Σ gives

$$\begin{aligned}
\sum_{(\xi,\psi)\in\Sigma} |\langle f, e_{(\xi,\psi)}\rangle|^2 &\geq \frac{1}{2}\sum_{(\xi,\psi)\in\Sigma} |\langle f_{\geq n}, e_{(\xi,\psi)}\rangle|^2 - \sum_{(\xi,\psi)\in\Sigma}\Big|\Big\langle \sum_{k=1}^{n-1} f_k, e_{(\xi,\psi)}\Big\rangle\Big|^2 \\
&\overset{(*)}{\geq} \frac{1}{2}\sum_{(\xi,\psi)\in\Sigma} |\langle f_{\geq n}, e_{(\xi,\psi)}\rangle|^2 - \Big\|\sum_{k=1}^{n-1} f_k\Big\|^2 \\
&\overset{(**)}{=} \frac{1}{2}\sum_{(\xi,\psi)\in\Sigma} |\langle f_{\geq n}, e_{(\xi,\psi)}\rangle|^2 - \sum_{k=1}^{n-1} \|f_k\|^2
\end{aligned}$$

where $(*)$ is because $\Sigma \subset \mathbb{Z}^d$ and $(**)$ since f_k have disjoint supports. Hence, to obtain (3.1) it remains to show that

$$\sum_{(\xi,\psi)\in\Sigma} |\langle f_{\geq n}, e_{(\xi,\psi)}\rangle|^2 \geq c\,\|f_n\|^2, \tag{3.2}$$

where c is a positive constant not depending on f.

Fix some $\xi \in \Xi_n$ and consider the function of b variables

$$F(y) = \int_{[0,1]^a} f_{\geq n}(x,y)\, e^{-2\pi i\langle \xi, x\rangle}\, dx$$

and note that it is supported on $Y_{\geq n}$. Since $\Psi_{\geq n}$ is a Riesz basis for this set, we have

$$\begin{aligned}
\sum_{\psi\in\Psi_{\geq n}} |\langle f_{\geq n}, e_{(\xi,\psi)}\rangle|^2 &= \sum_{\psi\in\Psi_{\geq n}} \Big|\int_{[0,1]^{a+b}} f_{\geq n}(x,y)\,\overline{e_{(\xi,\psi)}(x,y)}\, dx\, dy\Big|^2 \\
&= \sum_{\psi\in\Psi_{\geq n}} \Big|\int_{[0,1]^{b}} F(y)\,\overline{e_{\psi}(y)}\, dy\Big|^2 \\
&\geq c\int_{Y_{\geq n}} |F(y)|^2\, dy \geq c\int_{Y_n} |F(y)|^2\, dy \\
&= c\int_{Y_n}\Big|\int_{X_n} f_n(x,y)\, e^{-2\pi i\langle \xi, x\rangle}\, dx\Big|^2, \qquad (3.3)
\end{aligned}$$

where the last equality follows from the fact that when $y \in Y_n$ we have $f_{\geq n}(x,y) = f_n(x,y)$ and this function, as a function of x, is supported on X_n.

We now sum this over $\xi \in \Xi_n$. Recall that $\Xi_n \times \Psi_{\geq n} \subset \Sigma$ and that $e(\Xi_n)$ is a Riesz basis for $L^2(X_n)$. We get

$$\begin{aligned}\sum_{(\xi,\psi)\in\Sigma} |\langle f_{\geq n}, e_{(\xi,\psi)}\rangle|^2 &\geq \sum_{\xi\in\Xi_n,\psi\in\Psi_{\geq n}} |\langle f_{\geq n}, e_{(\xi,\psi)}\rangle|^2 \\ &\overset{(3.3)}{\geq} c \sum_{\xi\in\Xi_n} \int_{Y_n} \left| \int_{X_n} f_n(x,y)\, e^{-2\pi i\langle\xi,x\rangle}\, dx \right|^2 dy \\ &= c \int_{Y_n} \sum_{\xi\in\Xi_n} \left| \int_{X_n} f_n(x,y)\, e^{-2\pi i\langle\xi,x\rangle}\, dx \right|^2 dy \\ &\geq c \int_{Y_n} \int_{X_n} |f_n(x,y)|^2\, dx\, dy = c\, \|f_n\|^2.\end{aligned}$$

Hence, (3.2) holds and the system is a frame.

Riesz sequence. We now show that $e(\Sigma)$ is a Riesz sequence in $L^2(S)$, i.e., that for any *finitely supported* sequence $a_{(\xi,\psi)} \in l^2(\Sigma)$,

$$\Big\| \sum_{(\xi,\psi)\in\Sigma} a_{(\xi,\psi)}\, e_{(\xi,\psi)} \Big\|^2_{L^2(S)} \geq c \sum_{(\xi,\psi)\in\Sigma} |a_{(\xi,\psi)}|^2$$

(again, the other inequality in (2.1) follows from $S \subset [0,1]^{a+b}$ and $\Sigma \subset \mathbb{Z}^{a+b}$). We apply a strategy similar to the one we used in the first ("frame") part, but we decompose Σ rather than S. Define therefore $\Sigma_n = (\Xi_n \setminus \Xi_{n-1}) \times \Psi_{\geq n}$. With this definition a similar argument to the one used in the first part shows that it is enough to show that for every $n = 1, \ldots, L$ we have

$$\int_S \Big| \sum_{(\xi,\psi)\in\Sigma} a_{(\xi,\psi)} e_{(\xi,\psi)} \Big|^2 \geq c \sum_{(\xi,\psi)\in\Sigma_n} |a_{(\xi,\psi)}|^2 - \sum_{j=n+1}^{L} \sum_{(\xi,\psi)\in\Sigma_j} |a_{(\xi,\psi)}|^2. \tag{3.4}$$

To this end choose $n \in \{1, \ldots, L\}$. We have,

$$\begin{aligned}&\int_S \Big| \sum_{(\xi,\psi)\in\Sigma} a_{(\xi,\psi)}\, e_{(\xi,\psi)} \Big|^2 dx\, dy \\ &\quad\geq \frac{1}{2} \int_S \Big| \sum_{j=1}^{n} \sum_{(\xi,\psi)\in\Sigma_j} a_{(\xi,\psi)}\, e_{(\xi,\psi)} \Big|^2 - \int_S \Big| \sum_{j=n+1}^{L} \sum_{(\xi,\psi)\in\Sigma_j} a_{(\xi,\psi)}\, e_{(\xi,\psi)} \Big|^2 \\ &\quad\geq \frac{1}{2} \int_S \Big| \sum_{j=1}^{n} \sum_{(\xi,\psi)\in\Sigma_j} a_{(\xi,\psi)}\, e_{(\xi,\psi)} \Big|^2 - \sum_{j=n+1}^{L} \sum_{(\xi,\psi)\in\Sigma_j} |a_{(\xi,\psi)}|^2,\end{aligned}$$

where the second inequality is due to $S \subset [0,1]^{a+b}$ and $\Sigma \subset \mathbb{Z}^{a+b}$. Denote for brevity

$$f = \mathbb{1}_S \cdot \sum_{j=1}^{n} \sum_{(\xi,\psi)\in\Sigma_j} a_{(\xi,\psi)}\, e_{(\xi,\psi)},$$

and get that to prove (3.4) it remains to show that

$$\int_S |f(x,y)|^2\,dx\,dy \geq c \sum_{(\xi,\psi)\in\Sigma_n} |a_{(\xi,\psi)}|^2. \tag{3.5}$$

Here is where the fact that Ξ_n and $\Psi_{\geq n}$ are Riesz bases will enter.

We first apply that Ξ_n is a Riesz sequence over X_k for all $k \geq n$, specifically the left inequality in (2.1), and get, for any y,

$$\begin{aligned}
\int_{X_k} |f(x,y)|^2\,dx &= \int_{X_k} \Big| \sum_{j=1}^{n} \sum_{\xi\in\Xi_j\setminus\Xi_{j-1}} \Big(\sum_{\psi\in\Psi_{\geq j}} a_{(\xi,\psi)} e^{2\pi i\langle\psi,y\rangle}\Big) e^{2\pi i\langle\xi,x\rangle}\Big|^2 dx \\
\text{(by (2.1))} \qquad &\geq c \sum_{j=1}^{n} \sum_{\xi\in\Xi_j\setminus\Xi_{j-1}} \Big| \sum_{\psi\in\Psi_{\geq j}} a_{(\xi,\psi)}\, e^{2\pi i\langle\psi,y\rangle}\Big|^2 \\
\text{(dropping terms)} \qquad &\geq c \sum_{\xi\in\Xi_n\setminus\Xi_{n-1}} \Big| \sum_{\psi\in\Psi_{\geq n}} a_{(\xi,\psi)}\, e^{2\pi i\langle\psi,y\rangle}\Big|^2.
\end{aligned} \tag{3.6}$$

Integrating over y and using the fact that $\Psi_{\geq n}$ is a Riesz basis over $Y_{\geq n}$ we get

$$\begin{aligned}
\int_S |f(x,y)|^2\,dx\,dy &\geq \sum_{k=n}^{L} \int_{Y_k}\int_{X_k} |f(x,y)|^2\,dx\,dy \\
&\overset{(3.6)}{\geq} c \sum_{k=n}^{L} \int_{Y_k} \sum_{\xi\in\Xi_n\setminus\Xi_{n-1}} \Big| \sum_{\psi\in\Psi_{\geq n}} a_{(\xi,\psi)}\, e^{2\pi i\langle\psi,y\rangle}\Big|^2 dy \\
&= c \sum_{\xi\in\Xi_n\setminus\Xi_{n-1}} \int_{Y_{\geq n}} \Big| \sum_{\psi\in\Psi_{\geq n}} a_{(\xi,\psi)}\, e^{2\pi i\langle\psi,y\rangle}\Big|^2 dy \\
\text{(since } \Psi_{\geq n} \text{ is a Riesz basis)} \qquad &\geq c \sum_{\xi\in\Xi_n\setminus\Xi_{n-1}} \sum_{\psi\in\Psi_{\geq n}} |a_{(\xi,\psi)}|^2,
\end{aligned}$$

which asserts (3.5) and completes the proof. □

Remark. The "frame" and "Riesz sequence" parts are in fact independent in the following sense. If $\Xi_1 \subset \cdots \subset \Xi_L$ and $\Psi_{\geq 1} \supset \cdots \supset \Psi_{\geq L}$ are only assumed to be frames, then Σ will be a frame; while if they are assumed to be Riesz sequences then Σ will be a Riesz sequence. In the next section we will see that in another setting this remark allows to shorten the proof.

4. Folding

In this section we prove a version of the main lemma of [6]. That lemma stated that if certain "foldings" of a set have Riesz bases, then one may construct a Riesz basis for the original set too. The result here, while stated in d-dimensions, is essentially one dimensional and we will perform the same transformations performed in [6] on the first coordinate only. The details are below. The proof is also similar to the proof there, but with a simplification suggested by A. Olevskiĭ. Throughout this section we will denote either by $(t,s) := (t, s_1, ..., s_{d-1})$ or by x a point in $[0,1]^d$; and by $(\lambda, \delta) := (\lambda, \delta_1, ..., \delta_{d-1})$ or ξ a point in $\mathbb{Z}^d$.

Fix a positive integer N. Given a set $X \subset [0,1]^d$, define

$$X_n = \Big\{ t \in \Big[0, \frac{1}{N}\Big] \times [0,1]^{d-1} : \Big(t + \frac{j}{N}, s\Big) \in X \quad \text{for exactly } n \text{ values of } j \in \{0, \dots, N-1\} \Big\} \tag{4.1}$$

$$X_{\geq n} = \bigcup_{k=n}^{N} X_k \tag{4.2}$$

Lemma 7. *If there exist $\Xi_1, \dots, \Xi_N \subseteq N\mathbb{Z} \times \mathbb{Z}^{d-1}$ such that the system $e(\Xi_n)$ is a Riesz basis in $L^2(X_{\geq n})$, then the system $e(\Xi)$, where*

$$\Xi = \bigcup_{n=1}^{N} (\Xi_n + (n, 0, \dots, 0)),$$

is a Riesz basis in $L^2(X)$.

Clearly, it is equivalent to prove the lemma under the assumptions that

$$\Xi_n \subset (N\mathbb{Z} + n) \times \mathbb{Z}^{d-1}, \quad \Xi = \bigcup_{n=1}^{N} \Xi_n \tag{4.3}$$

(but still requiring that Ξ_n is a Riesz basis for $X_{\geq n}$, recall that the property of being a Riesz basis is invariant to translations) which will make the notations a little shorter.

We will show that $e(\Xi)$ is a Riesz basis by showing that it is both a frame and a Riesz sequence (recall Lemma 4). It turns out that to show that Ξ is a frame it is enough that all Ξ_j are frames. Let us state this as a lemma.

Lemma 8. *If $\Xi_n \subset (N\mathbb{Z} + n) \times \mathbb{Z}^{d-1}$ satisfy that $e(\Xi_n)$ is a frame in $L^2(X_{\geq n})$ for all $n \in \{1, \dots, N\}$, then the system $e(\Xi)$ is a frame in $L^2(X)$, where Ξ is given by (4.3).*

Furthermore, the same holds if $\Xi_n \subset (N\mathbb{Z} - n) \times \mathbb{Z}^{d-1}$.

Proof. To show that $e(\Xi)$ is a frame in $L^2(X)$ we need to show that for any $f \in L^2(X)$

$$\sum_{\xi \in \Xi} |\langle f, e_\xi \rangle|^2 > c_1 \|f\|^2$$

(the right inequality in the definition of a frame, (2.2), is satisfied because $X \subset [0,1]^d$ and $\Xi \subset \mathbb{Z}^d$). For $n \in \{1, \dots, N\}$, denote by f_n the restriction of f to

$$B_n = \Big\{ (t,s) \in X : \Big(t + \frac{j}{N}, s\Big) \in X \text{ for exactly } n \text{ integers } j \Big\}. \tag{4.4}$$

(X_n is the "folding" of B_n to $[0,1/N]\times[0,1]^{d-1}$, i.e., cutting to N pieces, translating each one to $[0,1/N]\times[0,1]^{d-1}$ and taking a union). For brevity denote $f_{\geq n}=\sum_{k=n}^{N} f_k$. As in the proof of Lemma 3, it is enough to show, for every $n=1,\dots,N$, that $\sum_\xi |\langle f,e_\xi\rangle|^2 \geq c\,\|f_n\|^2 - C\sum_{k=1}^{n-1}\|f_k\|^2$. And, again as in the proof of Lemma 3, this can be reduced further to showing that

$$\sum_{\xi\in\Xi} |\langle f_{\geq n}, e_\xi\rangle|^2 \geq c\,\|f_n\|^2, \tag{4.5}$$

where c is a positive constant not depending on f. The rest of the proof only examines one n at a time, so let us fix n now.

For any $(\lambda,\delta)\in(N\mathbb{Z}+j)\times[0,1]^{d-1}$, we have

$$\begin{aligned}\langle f_{\geq n}, e_{(\lambda,\delta)}\rangle &= \int_{[0,1]^{d-1}}\int_0^1 f_{\geq n}(t,s)\,\overline{e_{(\lambda,\delta)}(t,s)}\,dt\,ds\\ &= \int_{[0,1]^{d-1}}\int_0^{1/N}\sum_{l=0}^{N-1} f_{\geq n}\Big(t+\frac{l}{N},s\Big)\,e_{(-\lambda,-\delta)}\Big(t+\frac{l}{N},s\Big)\,dt\,ds\\ &= \int_{[0,1]^{d-1}}\int_0^{1/N} h_j(t,s)\,e(-\lambda t-\langle\delta,s\rangle)\,dt\,ds = \langle h_j, e_{(\lambda,\delta)}\rangle, \end{aligned}\tag{4.6}$$

where

$$h_j(t,s) = \mathbb{1}_{X_{\geq n}}(t,s)\cdot\sum_{l=0}^{N-1} f_{\geq n}\Big(t+\frac{l}{N},s\Big)\,q_j^l,\quad q_j = e\Big(-\frac{j}{N}\Big).$$

Fix $j\leq n$. Since $e(\Xi_j)$ is a frame for $X_{\geq j}$ and since h_j is supported on $X_{\geq n}\subset X_{\geq j}$ we have

$$\sum_{(\lambda,\delta)\in\Xi_j} |\langle f_{\geq n}, e_{(\lambda,\delta)}\rangle|^2 \overset{(4.6)}{=} \sum_{(\lambda,\delta)\in\Xi_j} |\langle h_j, e_{(\lambda,\delta)}\rangle|^2 \geq c\,\|h_j\|^2, \tag{4.7}$$

where c is the frame constant of Ξ_j. In the "furthermore" clause of the lemma (where $\Xi_j\subset N\mathbb{Z}-j$) we define $q_j=e(j/N)$ instead of $e(-j/N)$ and the calculation follows identically.

Summing over j gives

$$\begin{aligned}\sum_{\xi\in\Xi}|\langle f_{\geq n},e_\xi\rangle|^2 &\geq \sum_{j=1}^{n}\sum_{\xi\in\Xi_j}|\langle f_{\geq n},e_\xi\rangle|^2\\ &\overset{(4.7)}{\geq} c\sum_{j=1}^{n}\|h_j\|^2 \geq c\sum_{j=1}^{n}\|h_j\cdot\mathbb{1}_{X_n}\|^2.\end{aligned}\tag{4.8}$$

For every particular $(t,s)\in X_n$ the values of $\{h_j(t,s)\}_j$ are given by applying the $n\times N$ matrix $L=\{q_j^l\}_{j,l}$ to the vector $\{f_{\geq n}(t+l/N,s)\}_l$. Now, $(t,s)\in X_n$ so exactly n different values of this vector are non-zero. Considering only these values we may think of L as an $n\times n$ Vandermonde matrix which is invertible because the numbers q_j are different. Let C be a bound for the norm of the inverse over all such $n\times n$ sub-matrices of L. We get

$$\sum_{j=1}^{n}|h_j(t,s)|^2\ge\frac{1}{C}\sum_{l=0}^{N-1}\Big|f_{\ge n}\Big(t+\frac{l}{N},s\Big)\Big|^2,$$

which we integrate over $(t,s)\in X_n$ to get

$$\sum_{j=1}^{n}\|h_j\cdot\mathbb{1}_{X_n}\|^2\ge c\sum_{l=0}^{N-1}\int_{X_n}\Big|f_{\ge n}\Big(t+\frac{l}{N},s\Big)\Big|^2\,dt\,ds=c\,\|f_n\|^2.$$

With this we get (4.5) and therefore that Ξ is a frame. □

Proof of Lemma 7. We apply Lemma 8 twice. The first application is straightforward with the same X and Ξ_n and we get that Ξ is a frame. For the second application, let $Y=[0,1]^d\setminus X$ and note that $Y_{\ge n}=[0,1/N]\times[0,1]^{d-1}\setminus X_{\ge N+1-n}$ (for Y_n the correspondence is not as nice as it is for $Y_{\ge n}$). Since Ξ_n is a Riesz basis for $X_{\ge n}$, in particular a Riesz sequence, by Lemma 6 $(N\mathbb{Z}+1-n)\times\mathbb{Z}^{d-1}\setminus\Xi_{N+1-n}$ is a frame for $Y_{\ge n}$. We now apply Lemma 8 for Y and the complements of Ξ_n (we use the "furthermore" clause to rearrange them in decreasing order) and get that

$$\bigcup_{n=1}^{N}(N\mathbb{Z}+1-n)\times\mathbb{Z}^{d-1}\setminus\Xi_{N+1-n}$$

is a frame for Y (we used here that a translation of a frame is also a frame, to solve $+1$ problems). But this set is exactly $\mathbb{Z}^d\setminus\Xi$ and using Lemma 6 again we get that Ξ is a Riesz sequence for X, and we are done. □

We end this section with another lemma from [6]. It is a consequence of Claim 3 and Lemma 4 there.

Lemma 9. *Let $X\subset[0,1]$ be a union of L intervals and N be a positive integer. Then, the sets $X_{\ge n}$ defined before Lemma 7 are all unions of at most L intervals (when considered cyclically). Moreover, there exist infinitely many N for which all these sets are unions of at most $L-1$ intervals (again, when considered cyclically).*

Here and below a "cyclic interval" is either an interval $[a,b]\subset[0,1/N]$ for $a<b$ or a union of intervals $[0,b]\cup[a,1/N]$ for $b<a$.

5. Proof of Theorem 2

The proof follows by induction and, as is quite typical for inductive proofs, we need to prove a stronger claim in order to make the induction tick. We describe it in the following definition.

Definition 10. Let $X_1, X_2, \ldots \subset [0,1]^d$. A *coherent collection of Riesz bases* are $\Xi_i \subset \mathbb{Z}^d$ such that $e(\Xi_i)$ is a Riesz basis for X_i and such that $X_i \subset X_j$ implies that $\Xi_i \subset \Xi_j$.

The "stronger claim" above is now:

Theorem 11. *Any collection of sets, each of which is a union of rectangles with edges parallel to the axes, has a coherent collection of Riesz bases.*

The proof of the $d = 1$ case will follow easily from the following lemma.

Lemma 12. *Let $X \subset [0,1]$ be a union of L intervals and fix $N > 0$. Assume that $m/N \le |X| < (m+1)/N$ where m is a positive integer. Then there exists a Ξ with $e(\Xi)$ a Riesz basis in $L^2(X)$ such that*

$$\bigcup\nolimits_{n=0}^{m-2L-1}(N\mathbb{Z}+n) \subseteq \Lambda \subseteq \bigcup\nolimits_{n=0}^{m+2L}(N\mathbb{Z}+n).$$

Proof. Divide $[0,1]$ into the intervals $[n/N,(n+1)/N]$ and note that, since $m/N \le |X| < (m+1)/N$ and X is a union of L intervals, at least $m-2L$ of the intervals $[n/N,(n+1)/N]$ belong to X and no more then $m+2L+1$ of them intersect X. We wish to apply Lemma 7 with this N, so examine the sets $X_{\ge n}$ from the statement of the lemma. We get that among the $X_{\ge n}$ at least $m-2L$ are equal to $[0,1/N]$ (so the corresponding Ξ_n can, and must be taken to be $N\mathbb{Z}$) and no more then $m+2L$ are non-empty (for which the Ξ_n must be taken empty). The remaining sets are finite unions of intervals so we may apply Theorem 1 to find Riesz bases for them with frequencies from $N\mathbb{Z}$. Applying Lemma 7 the resulting basis Ξ has the necessary property. □

Proof of Theorem 11. As promised, the case $d = 1$ follows directly from Lemma 12. Indeed, let X_i be the unions of rectangles (intervals in our case) for which we need to find a coherent collection of Riesz bases. Let L be the maximum number of intervals in any X_i and take $N > 4L/\min|X_j \setminus X_i|$, where the minimum is taken over all i and j such that $X_i \subset X_j$. Construct Riesz bases Ξ_i for $L^2(X_i)$ using Lemma 12 with this N. We get that the Ξ_i are automatically coherent as $X_i \subset X_j$ implies that, for any k, if $\Xi_i \cap (N\mathbb{Z}+k) \neq \emptyset$ then necessarily $N\mathbb{Z}+k \subset \Xi_j$. This finishes the case $d = 1$.

We now move to the case $d > 1$.

Step 1. First, we prove the induction step in the case where the intersection of each X_i with each line parallel to the first coordinate axis is an interval.

Claim. *In this case, it is possible to find disjoint sets $Y_j \subset [0,1]^{d-1}$ and intervals $I_{i,j} \subset [0,1]$ (possibly empty) such that each X_i can be written as*

$$X_i = \bigcup_j I_{i,j} \times Y_j.$$

Further, all Y_j can be taken to be finite unions of rectangles.

The proof of this claim is simple (take a total refinement of appropriate projections of parts of the X_i) and will be omitted.

Returning to the proof of Theorem 11, we first use the case $d = 1$ already established to find a coherent collection of Riesz bases $\Lambda_{i,j}$ for the intervals $[0, |I_{i,j}|]$, i.e., for translations of $I_{i,j}$ so that their left side is at 0. Since the property of being a Riesz basis is translation invariant we get that each $\Lambda_{i,j}$ is a Riesz basis for $I_{i,j}$. In other words, $\Lambda_{i,j}$ is a collection of Riesz bases for $I_{i,j}$ with the property that if $|I_{i,j}| \le |I_{i',j'}|$ then $\Lambda_{i,j} \subseteq \Lambda_{i',j'}$.

Next we apply the induction assumption for $d-1$ and get a coherent collection of Riesz bases for all *finite unions* of the Y_j. Denote, for each set of indices J, $Y_J = \bigcup_{j\in J} Y_j$ and $\Psi_J \subset \mathbb{Z}^d$ the Riesz basis over Y_J.

For each i let σ be the rearrangement of $I_{i,j}$ by length, i.e., σ is a permutation such that $|I_{i,\sigma(1)}| \le |I_{i,\sigma(2)}| \le \cdots$ and define

$$\Xi_i = \bigcup_j \Lambda_{i,\sigma(j)} \times \Psi_{\{\sigma(j),\sigma(j+1),\dots\}}.$$

By Lemma 3, Ξ_i is a Riesz basis for X_i. To see coherency, let i and i' satisfy that $X_i \subset X_{i'}$ and let σ and σ' be the corresponding permutations. To shorten notations denote the different pieces of Ξ and Ξ' by A_j and A'_j respectively, i.e.,

$$A_j := \Lambda_{i,\sigma(j)} \times \Psi_{\{\sigma(j),\sigma(j+1),\dots\}}, \quad A'_j := \Lambda_{i',\sigma'(j)} \times \Psi_{\{\sigma'(j),\sigma'(j+1),\dots\}}.$$

We need to show that $\Xi_i \subset \Xi_{i'}$, and this will follow once we show that for every j there exists k such that $A_j \subset A'_k$. Fix therefore j and examine the j shortest intervals for $X_{i'}$, i.e., $\sigma'(1), \dots, \sigma'(j)$. They cannot be all in the set $\sigma(1), \dots, \sigma(j-1)$, so let k be the first which is not in it, i.e.,

$$k = \inf\big\{l \le j : \sigma'(l) \notin \{\sigma(1), \dots, \sigma(j-1)\}\big\}.$$

The claim now follows easily. We first note that

$$|I_{i',\sigma'(k)}| \overset{(*)}{\ge} |I_{i,\sigma'(k)}| \overset{(**)}{\ge} |I_{i,\sigma(j)}|$$

where $(*)$ is because $X_i \subset X_{i'}$, and $(**)$ is because $\sigma'(k)$ is not in $\{\sigma(1), \dots, \sigma(j-1)\}$. Hence the coherency of the Λ's gives that $\Lambda_{i',\sigma'(k)} \supset \Lambda_{i,\sigma(j)}$. On the other hand, $\{\sigma'(1), \dots, \sigma'(k-1)\} \subset \{\sigma(1), \dots, \sigma(j-1)\}$ and taking complements gives

$$\{\sigma'(k), \sigma'(k+1), \dots\} \supset \{\sigma(j), \sigma(j+1), \dots\}$$

and the coherency of the Ψ's gives that $\Psi_{\{\sigma'(k),\sigma'(k+1),\dots\}} \supset \Psi_{\{\sigma(j),\sigma(j+1),\dots\}}$. Together we get $A_j \subset A'_k$, as required.

Step 2. As in step 1, we find disjoint sets $Y_j \subset [0,1]^{d-1}$ and $S_{i,j} \subset [0,1]$ (which are no longer necessarily intervals, but are finite unions of intervals) such that

$$X_i = \bigcup_j S_{i,j} \times Y_j.$$

Let $M_{i,j}$ be the number of components of $S_{i,j}$. We argue by induction on the vector $\{M_{i,j}\}$, with the case that all $M_{i,j}$ are either 0 or 1 given by step 1.

Let therefore i_0 and j_0 satisfy that $M_{i_0,j_0} \geq 2$. Recall the notation $X_{\geq n}$ from §4, which was defined with respect to some N which does not appear in the notation. When we apply it to sets which already have a subscript, like $S_{i,j}$, we will write $S_{i,j,\geq n}$. By Lemma 9 we can find some N such that the sets $S_{i_0,j_0,\geq n}$ contain no more than $M_{i_0,j_0} - 1$ intervals, for all n. Since the operation $\bullet_{\geq n}$ examines only the first coordinate, and because the Y_j are disjoint, we have

$$X_{i,\geq n} = \bigcup_j S_{i,j,\geq n} \times Y_j.$$

Again by Lemma 9, $S_{i,j,\geq n}$ has no more than $M_{i,j}$ components, for all i and j. Therefore we may apply our induction hypothesis to $X_{i,\geq n}$ (formally after stretching the first coordinate by N) and get a coherent collection of Riesz bases $\Xi_{i,n}$ in $N\mathbb{Z} \times \mathbb{Z}^{d-1}$. Define

$$\Xi_i = \bigcup_{n=1}^{N} \Xi_{i,n} + (n, 0, \dots, 0).$$

and get from Lemma 7 that $e(\Xi_i)$ is a Riesz basis for $L^2(X_i)$. Since $X_i \subset X_j$ implies that $X_{i,\geq n} \subset X_{j,\geq n}$, we get that the Ξ_i are coherent, finishing step 2 and the proof of the theorem. □

6. Remarks on the proof

The main ingredient in the proof of Theorem 1 from [6] was the one-dimensional case of Lemma 7. Examining its proof it is natural to wonder whether it could have been generalized directly to prove the d-dimensional result by folding in all dimensions simultaneously. As far as we can see, this is not possible. The proof of Lemma 7 relies on the fact that for any choice of n columns in the $N \times N$ Fourier matrix the first n rows will give, universally, an $n \times n$ invertible matrix, as it is a Vandermonde matrix. This is not the case for the analog of the Fourier matrix in higher dimensions, no such “universal” choice of rows exists, as can be checked directly for the 4×4 matrix of the Fourier transform of the group $(\mathbb{Z}/2)^2$.

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Gady Kozma: The Weizmann Institute of Science, P.O.B. 76100, Rehovot, Israel.
E-mail: gady.kozma@weizmann.ac.il

Shahaf Nitzan: School of Mathematics, Georgia Institute of Technology, 686 Cherry Street, Atlanta, GA 30332-0160 USA.
E-mail: shahaf.n.h@gmail.com

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