Rev. Mat. Iberoam. **32** (2016), no. 4, 1353–1392
DOI 10.4171/RMI/921

Global Hölder regularity for the fractional p-Laplacian

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Abstract. By virtue of barrier arguments we prove C^α-regularity up to the boundary for the weak solutions of a non-local, non-linear problem driven by the fractional p-Laplacian operator. The equation is boundedly inhomogeneous and the boundary conditions are of Dirichlet type. We employ different methods according to the singular ($p < 2$) of degenerate ($p > 2$) case.

1. Introduction and main result

We study Hölder regularity up to the boundary for the weak solutions of the Dirichlet problem

$$\begin{cases} (-\Delta)_p^s u = f & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c. \end{cases} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^N$ ($N > 1$) is a bounded domain with a $C^{1,1}$ boundary $\partial\Omega$, $\Omega^c = \mathbb{R}^N \setminus \Omega$, $s \in (0,1)$ and $p \in (1,\infty)$ are real numbers and $f \in L^\infty(\Omega)$. The s-fractional p-Laplacian operator is the gradient of the functional

$$J(u) := \frac{1}{p} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}}\, dx\, dy,$$

defined on

$$W_0^{s,p}(\Omega) := \{u \in L^p(\mathbb{R}^N) : J(u) < \infty,\ u = 0 \text{ in } \Omega^c\},$$

which is a Banach space with respect to the norm $J(u)^{1/p}$. Under suitable smoothness conditions on u, the operator can be written as

$$(-\Delta)_p^s\, u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{B_\varepsilon^c(x)} \frac{|u(x) - u(y)|^{p-2}\, (u(x) - u(y))}{|x - y|^{N+sp}}\, dy, \quad x \in \mathbb{R}^N.$$

Mathematics Subject Classification (2010): 35D10, 35R11, 47G20.
Keywords: Fractional p-Laplacian, fractional Sobolev spaces, global Hölder regularity.

A weak solution $u \in W_0^{s,p}(\Omega)$ of problem (1.1) satisfies, for every $\varphi \in W_0^{s,p}(\Omega)$,

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dx\,dy = \int_\Omega f(x)\,\varphi(x)\,dx.$$

Problem (1.1) is thus well posed and, in the case $p = 2$, it corresponds to an inhomogeneous fractional Laplacian equation with Dirichlet boundary condition. For the sake of completeness we recall that in the literature the fractional Laplacian is often defined by

$$\langle(-\Delta)^s u, \varphi\rangle = \frac{c(N,s)}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy, \quad \varphi \in W_0^{s,2}(\Omega),$$

where $c(N,s) = s2^{2s}\,\Gamma((N+2s)/2)/(\pi^{N/2}\Gamma(1-s))$, in order to be coherent with the Fourier definition of $(-\Delta)^s$ (see Remark 3.11 of [4]). We point out that, in the current literature, there are several notions of fractional Laplacian, all of which agree when the problems are set on the whole $\mathbb{R}^N$, but some of them disagree in a bounded domain. We refer the reader to [22] for a discussion on the comparison between the integral fractional Laplacian and the regional (or spectral) notion obtained by taking the s-powers of the Laplacian operator $-\Delta$ with zero Dirichlet boundary conditions.

In the case $p \neq 2$, problem (1.1) is a non-local and non-linear one. Its leading term $(-\Delta)_p^s$ is furthermore degenerate when $p > 2$ and singular when $1 < p < 2$. Determining sufficiently good regularity estimates *up to the boundary* is not only relevant by itself, but it also has useful applications in obtaining multiplicity results for more general non-linear and non-local equations, such as those investigated in [10] in the framework of topological methods and Morse theory. To this regard, this contribution provides a first step in order to obtain the results of [11] in the general case $p \neq 2$.

The regularity up to the boundary of fractional problems in the case $p = 2$ is now rather well understood, even when more general kernels and nonlinearities are considered. Using a viscosity solution approach, the model linear case gives regularity for fully non-linear equations which are "uniformly elliptic" in a suitable sense. Regarding the viscosity approach to fully non-linear, elliptic non-local equation, see [5] and [6] for interior regularity theory with smooth kernels, and [21] for rough kernels; regarding boundary regularity, see [19] for nearly optimal results and a detailed discussion on the delicate role that the kernel's regularity class plays in such problems.

Equation (1.1), however, does not fall in the category of non-local non-linear equations treated in the aforementioned works. This is not surprising, due to the degenerate/singular nature of the nonlinearity, and the s-fractional p-Laplacian is the non-local analogue of a degenerate/singular non-linear *divergence form* equation, rather than of a uniformly elliptic fully non-linear one. Local Hölder continuity has been addressed in [7], [8] using methods *á la* De Giorgi, and in [15] with a Krylov–Safanov approach for $p > 1/(1-s)$. In [3] the fully non-linear approach is used to study the non-local analogue of the p-Laplacian equation in non-divergence form

$$\Delta u + (p-2)\frac{\nabla u}{|\nabla u|}D^2 u\frac{\nabla u}{|\nabla u|} = 0,$$

arising from non-local 'tug of war' games. Interior $C^{1,\alpha}$ estimates and Hölder continuity up to the boundary is proved under rather general assumptions.

Our main result is the following:

Theorem 1.1. *There exist $\alpha \in (0, s]$ and $C_\Omega > 0$, depending only on N, p, and s, with C_Ω also depending on Ω, such that, for all weak solution $u \in W_0^{s,p}(\Omega)$ of problem (1.1), $u \in C^\alpha(\overline{\Omega})$ and*

$$\|u\|_{C^\alpha(\overline{\Omega})} \le C_\Omega \|f\|_{L^\infty(\Omega)}^{1/(p-1)}. \tag{1.2}$$

Notice that, regarding regularity *up to the boundary*, one cannot expect more than s-Hölder continuity due to explicit examples (see Section 3 below). On the other hand, the optimal Hölder exponent up to the boundary seems to be s for any $p > 1$, while we prove C^α regularity for an unspecified small α, the issue being a lack of higher (at least C^s) regularity results in the interior of the domain.

Let us describe the strategy to prove Theorem 1.1. We choose to use the notion of weak rather than viscosity solution, since we feel that the equation is more naturally seen as a variational one. However, we will frequently use barrier arguments, rather than De Giorgi–Nash–Moser techniques. Indeed, the proof of Theorem 1.1 is performed in the spirit of Krylov's approach to boundary regularity, see [13], and uses two main ingredients:

(a) a uniform Hölder control (see Theorem 4.4) on how u reaches its boundary values, which amounts to

$$|u(x)| \le C\,\|f\|_\infty^{1/(p-1)}\,\mathrm{dist}^s(x, \Omega^c); \tag{1.3}$$

(b) a local regularity estimate (see Theorem 5.4) in terms of quantities which may blow up in general when reaching the boundary, but remain bounded for functions satisfying (1.3).

Point (a) is obtained through a barrier argument, and stems from the fact that $(-\Delta)_p^s (x_+)^s = 0$ in the half line $\mathbb{R}_+$. Notice that for $p \ne 2$ we do not have at our disposal the fractional Kelvin transform, and the concrete calculus of the s-fractional p-Laplacian even on smooth functions is a prohibitive task, in general. Thus constructing upper barriers can be quite technical, and it is done as follows:

- Consider $u_N(x) = (x_N)_+^s$: explicit calculus shows that $(-\Delta)_p^s u_N = 0$ in the half-space $\mathbb{R}^N_+$. We locally deform the half-space to Ω by a diffeomorphism Φ close to the identity, and obtain a function $u_N \circ \Phi$ with small s-fractional p-Laplacian in a small ball $\hat{B}$ centered at a point of $\partial\Omega$.
- The resulting function $u_N \circ \Phi$ can be controlled in $\hat{B} \cap \Omega$ by distance-like functions from the boundary, and we can modify it to globalize the controls, while keeping the smallness of $(-\Delta)_p^s(u_N \circ \Phi)$ in $\hat{B} \cap \Omega$.

- We exploit the non-local nature of the equation to add a fixed *positive* quantity to $(-\Delta)_p^s(u_N \circ \Phi)$ in $\hat{B} \cap \Omega$, by truncation away from $\hat{B}$. Since $(-\Delta)_p^s(u_N \circ \Phi)$ is arbitrarily small, its truncation has therefore s-fractional p-Laplacian bounded from below by a positive constant in $\hat{B} \cap \Omega$, and provides the local upper barrier.

Point (b) is a generalization, in the whole range $p > 1$, to non-homogeneous equations of Theorem 1.2 from [7], and it could be deduced in the case $p > 2 - s/N$ using the results of [14], and in the case $p > 1/(1-s)$ using [15]. However we choose to prove it with a different approach. Much in the spirit of [20], rather than considering the non-locality of the equation as an additional technical difficulty to the implementation of the De Giorgi–Moser regularity theory, we use it at our advantage to construct a more elementary proof. It should be noted that we do not employ Caccioppoli-like inequalities, or estimates on $\log u$ (which are the elementary counterpart of John–Nirenberg's lemma). Actually we don't even need a Poincaré or Sobolev inequality, which are usually looked at as basic tools for (variational) regularity theory. This feature seems typical of the non-local framework and it should be noted that the proof doesn't seem to immediately "pass to the limit to local equations" as the obtained estimates blow up for $s \to 1$.

Regarding possible developments and generalizations, a first remark regards the choice of the kernel in the non-local operator

$$L(u) = \text{p.v.} \int_{\mathbb{R}^N} |u(x) - u(y)|^{p-2} (u(x) - u(y)) \, K(x,y) \, dy.$$

Regarding interior regularity, a bound from above and below in terms of the model kernel $|x-y|^{-N-ps}$ seems to suffice to obtain Hölder regularity, due to the results of [7], [14]. For non-local, fully non-linear, uniformly elliptic equation, higher interior regularity (up to $C^{2,\alpha}$) is proved in [5], [6], [21] when the kernel satisfies additional structural and regularity assumption, but no such result is known for the s-fractional p-Laplacian. Regarding regularity up to the boundary things are more subtle. In the uniformly elliptic case ($p = 2$), the optimal regularity is $C^s(\overline{\Omega})$ due to the results of [19], but only for a subclass of rough symmetric kernels arising from stable Lévy processes, of the form

$$K(x,y) = H(x-y), \quad H(z) = \frac{a\big(z/|z|\big)}{|z|^{N+2s}}, \quad 0 < \lambda \leq a \leq \Lambda.$$

Counterexamples show that this is the largest kernel's class where to expect such regularity up to the boundary. However, for any $p > 1$, one still expects $C^\alpha(\overline{\Omega})$ regularity for arbitrarily rough symmetric kernels, for a small $\alpha < s$.

An additional point of interest is the Hölder regularity, up to the boundary, of $u/\text{dist}^s(x, \Omega^c)$, when $(-\Delta)_p^s u$ is bounded in $\overline{\Omega}$. This is proven in [18] for the fractional Laplacian, and in [19] for the Lévy stable fully non-linear, uniformly elliptic non-local equations. While undoubtedly being relevant in light of the applications depicted in [10], we do not treat this problem here.

The structure of the paper is as follows. In Section 2 we mainly discuss the relationship between weak and strong (i.e., in a suitable principal value sense) solutions of (1.1). In doing so we clarify how barrier arguments (which are more suited to viscosity solutions) can be applied in the framework of weak solutions of non-linear non-local problems.

In Section 3 we study the s-fractional p-Laplacian of distance-related functions, and consider their stability with respect to local diffeomorphisms of the domain.

In Section 4 we construct some upper barriers, derive L^∞-bounds for solutions of (1.1) and prove estimate (1.3).

In Section 5 we tackle the local regularity through a weak Harnack inequality. Then we couple it with (1.3) to prove Theorem 1.1.

A short description of the result obtained in the present paper can be found in [12].

2. Preliminaries

2.1. Notations and function spaces

Given a subset $A \subseteq \mathbb{R}^N$ we will set $A^c = \mathbb{R}^N \setminus A$ and for $A, B \subseteq \mathbb{R}^N$,

$$\mathrm{dist}(A,B) = \inf_{x\in A,\, y\in B} |x-y|, \quad \delta_A(x) = \mathrm{dist}(x, A^c),$$
$$\mathrm{dist}_H(A,B) = \max\Big\{ \sup_{x\in A} \mathrm{dist}(x,B), \sup_{y\in B} \mathrm{dist}(y,A) \Big\}.$$

For all $x \in \mathbb{R}^N$, $r > 0$ we denote by $B_r(x)$, $\overline{B}_r(x)$, and $\partial B_r(x)$, respectively, the open ball, the closed ball and the sphere centered at x with radius r. When the center is not specified, we will understand that it's the origin, e.g. $B_1 = B_1(0)$. For all measurable $A \subset \mathbb{R}^N$ we denote by $|A|$ the N-dimensional Lebesgue measure of A. If u is a measurable function and A is a measurable subset of $\mathbb{R}^N$, we will set for brevity

$$\inf_A u = \operatorname*{ess\,inf}_A u, \quad \sup_A u = \operatorname*{ess\,inf}_A u.$$

For all measurable $u\colon \mathbb{R}^N \to \mathbb{R}$ we define

$$[u]_{s,p} = \Big(\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dx\, dy \Big)^{1/p},$$
$$\|u\|_{W^{s,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \Big(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dx\, dy \Big)^{1/p},$$

and we will consider the following spaces (see [9] for details):

$$W^{s,p}(\Omega) = \big\{ u \in L^p(\Omega) : \|u\|_{W^{s,p}(\Omega)} < \infty \big\},$$
$$W_0^{s,p}(\Omega) = \big\{ u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ in } \Omega^c \big\},$$
$$W^{-s,p'}(\Omega) = (W_0^{s,p}(\Omega))^*,$$

where the last one is the Banach dual, whose pairing with $W_0^{s,p}(\Omega)$ will be denoted by $\langle \cdot, \cdot \rangle_{s,p,\Omega}$. We will extensively make use of the following space:

Definition 2.1. Let $\Omega \subseteq \mathbb{R}^N$ be bounded. We set

$$\widetilde{W}^{s,p}(\Omega) := \Big\{u \in L^p_{\rm loc}(\mathbb{R}^N) : \exists U \Supset \Omega \text{ s.t. } \|u\|_{W^{s,p}(U)} + \int_{\mathbb{R}^N} \frac{|u(x)|^{p-1}}{(1+|x|)^{N+ps}}\, dx < \infty\Big\}.$$

If Ω is unbounded, we set

$$\widetilde{W}^{s,p}_{\rm loc}(\Omega) := \big\{u \in L^p_{\rm loc}(\mathbb{R}^N) : u \in \widetilde{W}^{s,p}(\Omega') \text{ for any bounded } \Omega' \subseteq \Omega\big\}.$$

We notice that the condition

$$\int_{\mathbb{R}^N} \frac{|u(x)|^{p-1}}{(1+|x|)^{N+ps}}\, dx < \infty$$

holds if $u \in L^\infty(\mathbb{R}^N)$ or $[u]_{C^s(\mathbb{R}^N)} < \infty$. The spaces $\widetilde{W}^{s,p}(\Omega)$, $\widetilde{W}^{s,p}_{\rm loc}(\Omega)$ can be endowed with a topological vector space structure as inductive limit, but we will not use it. For all $\alpha \in (0,1]$ and all measurable $u\colon \overline{\Omega} \to \mathbb{R}$ we set

$$[u]_{C^\alpha(\overline{\Omega})} = \sup_{x,y\in\overline{\Omega},\, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha},$$
$$C^\alpha(\overline{\Omega}) = \big\{u \in C(\overline{\Omega}) : [u]_{C^\alpha(\overline{\Omega})} < \infty\big\},$$

the latter being a Banach space under the norm $\|u\|_{C^\alpha(\overline{\Omega})} = \|u\|_{L^\infty(\overline{\Omega})} + [u]_{C^\alpha(\overline{\Omega})}$. A similar definition is given for $C^{1,\alpha}(\overline{\Omega})$. When no misunderstanding is possible, we set for all measurable $D \subset \mathbb{R}^N$, $x \in D$, and all measurable $\psi\colon D \times D \to \mathbb{R}$,

$$\text{p.v.} \int_D \psi(x,y)\, dy = \lim_{\varepsilon\to 0^+} \int_{D\setminus B_\varepsilon(x)} \psi(x,y)\, dy.$$

For all measurable $u\colon \mathbb{R}^N \to \mathbb{R}$, we recall that the *non-local tail* centered at $x \in \mathbb{R}^N$ with radius $R > 0$, introduced in [7], is defined as

$$\text{Tail}(u;x,R) = \Big(R^{ps} \int_{B^c_R(x)} \frac{|u(y)|^{p-1}}{|x-y|^{N+ps}}\, dy\Big)^{1/(p-1)}. \tag{2.1}$$

We will also set $\text{Tail}(u;0,R) = \text{Tail}(u;R)$. Unless otherwise stated, the numbers $p > 1$ and $s \in (0,1)$ will be fixed as the order of summability and the order of differentiability. By a *universal* constant we mean a constant $C = C(N,p,s)$. This dependence will always be omitted, even when other dependencies are present, in which case they are the only ones explicitly stated: for example C_Ω will denote a constant depending on N, p, s, and Ω. During chains of inequalities, universal constants will be denoted by the same letter C even if their numerical value may change from line to line. The same treatment will be used for constants which retain their dependencies from line to line. When needed, we will denote a specific universal constant with a number, e.g. C_1, C_2 *et cetera*.

2.2. Some elementary inequalities

For all $a \in \mathbb{R}$, $q > 0$, we set

$$a^q = |a|^{q-1}a.$$

This notation has great advantages in readability and, for future reference, we recall here some more or less known elementary inequalities about the function $a \mapsto a^q$. We will provide a sketch of proof for the less frequent ones.

We begin with the well-known inequalities

$$(a+b)^q \le 2^{q-1}(a^q + b^q) \quad a, b \ge 0,\ q \ge 1; \tag{2.2}$$

$$(a+b)^q \le a^q + b^q \quad a, b \ge 0,\ q \in (0,1]; \tag{2.3}$$

$$|a^q - b^q| \le q\,(|a|^{q-1} + |b|^{q-1})|a-b| \quad a, b \in \mathbb{R},\ q \ge 1, \tag{2.4}$$

the last one being a trivial consequence of Taylor's formula. We will also use

$$a^q - (a-b)^q \le C_M \max\{b, b^q\} \quad |a| \le M,\ b \ge 0,\ q > 0, \tag{2.5}$$

which follows immediately from (2.3) if $q \in (0,1]$. If $q > 1$ we can prove it distinguishing the cases $b \le M$, where we use (2.4), and the case $b \ge M$, where we use $a^q - (a-b)^q \le M^q + 2M^q \le 3b^q$. We now prove

$$(a+b)^q - a^q \le \theta a^q + C_\theta b^q \quad a, b \ge 0,\ q \ge 1,\ C_\theta \to \infty \text{ as } \theta \to 0^+. \tag{2.6}$$

Letting $C_q = 1$ if $q \le 1$ and $C_q = 2^{q-1}$ if $q \ge 1$, (2.2) and (2.3) can be written as

$$(a+b)^q \le C_q(a^q + b^q) \quad a, b \ge 0,\ q > 0.$$

Now (2.6) can be proved using Taylor's formula and Young's inequality:

$$\begin{aligned}(a+b)^q - a^q &\le C_q\,(a^{q-1} + b^{q-1})b = (\theta q' a)^{q-1}\frac{C_q\, b}{(\theta q')^{q-1}} + C_q\, b^q \\ &\le \theta a^q + \frac{1}{q}\Big(\frac{C_q}{(\theta q')^{q-1}}\Big)^q b^q + C_q\, b^q.\end{aligned}$$

We prove the following inequality:

$$a^q - (a-b)^q \ge 2^{1-q}\, b^q \quad a \in \mathbb{R},\ b \ge 0,\ q \ge 1. \tag{2.7}$$

We can suppose $b > 0$ and consider the function

$$f(t) = t^q - (t-b)^q, \quad f'(t) = q\,(|t|^{q-1} - |t-b|^{q-1}).$$

Therefore f is positive, increasing for $t > b$ and decreasing for $t < -b$ and thus it's coercive. Since $f'(t) = 0$ if and only if $t = b/2$, its global minimum is $f(b/2) = 2^{1-q}\, b^q$.

Finally, we will use the following inequality, holding for all $A, B \subset \mathbb{R}^N$ with A bounded and $\operatorname{dist}(A, B^c) = d > 0$:

$$|x-y| \ge C(A,B)(1+|y|), \quad x \in A,\ y \in B^c. \tag{2.8}$$

2.3. Weak and strong solutions

We compare now different notions of solutions for equations driven by $(-\Delta)_p^s$.

Definition 2.2. Let Ω be bounded, $u \in \widetilde{W}^{s,p}(\Omega)$ and $f \in W^{-s,p'}(\Omega)$. We say that u is a *weak solution* of $(-\Delta)_p^s u = f$ in Ω if, for all $\varphi \in W_0^{s,p}(\Omega)$,

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))^{p-1}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dx\,dy = \langle f,\varphi\rangle_{s,p,\Omega}.$$

If Ω is unbounded, we say that $u \in \widetilde{W}^{s,p}_{\rm loc}(\Omega)$ solves $(-\Delta)_p^s u = f$ (with $f \in W^{-s,p'}(\Omega)$) weakly in Ω if it does so in any bounded open set $\Omega' \subseteq \Omega$.

The inequality $(-\Delta)_p^s u \le f$ weakly in Ω will mean that

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))^{p-1}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dx\,dy \le \langle f,\varphi\rangle_{s,p,\Omega}$$

for all $\varphi \in W_0^{s,p}(\Omega)$, $\varphi \ge 0$, and similarly for $(-\Delta)_p^s u \ge f$. Noticing that $\pm K \in W^{-s,p'}(\Omega)$ for any $K>0$ and any bounded Ω, by $|(-\Delta)_p^s u| \le K$ weakly in Ω we mean that both $-K \le (-\Delta)_p^s u \le K$ weakly in Ω.

In the following proposition we will prove that $(-\Delta)_p^s u \in W^{-s,p'}(\Omega)$ if $u \in \widetilde{W}^{s,p}(\Omega)$, which implies that the previous definition makes sense.

Lemma 2.3. *Let Ω be bounded and $u \in \widetilde{W}^{s,p}(\Omega)$. Then the functional*

$$W_0^{s,p}(\Omega) \ni \varphi \mapsto (u,\varphi) := \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))^{p-1}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dx\,dy$$

is finite and belongs to $W^{-s,p'}(\Omega)$.

Proof. Let $U \Supset \Omega$ be such that

$$\|u\|_{W^{s,p}(U)} + \int_{\mathbb{R}^N} \frac{|u(x)|^{p-1}}{(1+|x|)^{N+ps}}\,dx < \infty, \tag{2.9}$$

and write

$$\begin{aligned}
(u,\varphi) &= \int_{U\times U} \frac{(u(x)-u(y))^{p-1}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dx\,dy \\
&\quad + \int_{U\times U^c} \frac{(u(x)-u(y))^{p-1}\varphi(x)}{|x-y|^{N+ps}}\,dx\,dy \\
&\quad - \int_{U^c\times U} \frac{(u(x)-u(y))^{p-1}\varphi(y)}{|x-y|^{N+ps}}\,dx\,dy \\
&= \int_{U\times U} \frac{(u(x)-u(y))^{p-1}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dx\,dy \\
&\quad + 2\int_{\Omega\times U^c} \frac{(u(x)-u(y))^{p-1}\varphi(x)}{|x-y|^{N+ps}}\,dx\,dy,
\end{aligned} \tag{2.10}$$

since $\mathrm{supp}(\varphi) \subset \overline{\Omega}$. The integral in $U \times U$ is finite and continuous with respect to strong convergence of $\varphi \in W_0^{s,p}(\Omega)$ since $u \in W^{s,p}(U)$. For the second term, observe that for a.e. $x \in \Omega$ it holds

$$\begin{aligned}
&\int_{U^c} \frac{|u(x)-u(y)|^{p-1}}{|x-y|^{N+ps}}\,dy \\
&\le C\Big(|u(x)|^{p-1}\int_{U^c}\frac{1}{|x-y|^{N+ps}}\,dy + \int_{U^c}\frac{|u(y)|^{p-1}}{(|x-y|)^{N+ps}}\,dy\Big) \\
&\le C\Big(|u(x)|^{p-1} + \int_{\mathbb{R}^N}\frac{|u(y)|^{p-1}}{(1+|y|)^{N+ps}}\,dy\Big),
\end{aligned} \tag{2.11}$$

where we used (2.8) with $A=\Omega$ and $B=U$. The right hand side of (2.11) belongs to $L^{p'}(\Omega)$ since Ω is bounded and $u \in L^p(\Omega)$. Thus the second term in (2.10) is continuous with respect to $L^p(\Omega)$-convergence of φ. Therefore it is also continuous in $W_0^{s,p}(\Omega)$. □

Definition 2.4 (Point-wise and strong solutions). Let $u \in \widetilde{W}^{s,p}_{\mathrm{loc}}(\Omega)$ and $f : \Omega \to \mathbb{R}$ be measurable. We say that u is an *a.e. point-wise* solution of $(-\Delta)_p^s u = f$ in Ω if for a.e. Lebesgue point $x \in \Omega$ of u it holds

$$2\cdot \mathrm{p.v.}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))^{p-1}}{|x-y|^{N+ps}}\,dy = f(x). \tag{2.12}$$

Moreover, for $f \in L^1_{\mathrm{loc}}(\Omega)$ we say that u is a *strong* solution of $(-\Delta)_p^s u = f$ if

$$2\int_{B_\varepsilon^c(x)}\frac{(u(x)-u(y))^{p-1}}{|x-y|^{N+ps}}\,dy \to f \quad \text{strongly in } L^1_{\mathrm{loc}}(\Omega), \text{ as } \varepsilon \to 0^+. \tag{2.13}$$

Similar definitions are given for sub- and supersolutions.

Now we prove that a strong solution is also a weak solution. First, we introduce a more general result, which will be used in the following: we denote by $\mathbf{D}$ the diagonal of $\mathbb{R}^N \times \mathbb{R}^N$.

Lemma 2.5. *Let $u \in \widetilde{W}^{s,p}_{\mathrm{loc}}(\Omega)$. For all $\varepsilon > 0$ let $A_\varepsilon \subset \mathbb{R}^N \times \mathbb{R}^N$ be a neighborhood of $\mathbf{D}$ which satisfies*

(i) *$(x,y) \in A_\varepsilon$ for all $(y,x) \in A_\varepsilon$;*

(ii) *$\mathrm{dist}_H(A_\varepsilon, \mathbf{D}) \to 0$ as $\varepsilon \to 0^+$.*

For all $x \in \mathbb{R}^N$ we set $A_\varepsilon(x) = \{y \in \mathbb{R}^N : (x,y) \in A_\varepsilon\}$ and

$$g_\varepsilon(x) = \int_{A_\varepsilon^c(x)}\frac{(u(x)-u(y))^{p-1}}{|x-y|^{N+ps}}\,dy.$$

If $2g_\varepsilon \to f$ in $L^1_{\mathrm{loc}}(\Omega)$, then u is a weak solution of $(-\Delta)_p^s u = f$ in Ω.

Proof. We can suppose that Ω is bounded and let $U \Supset \Omega$ be such that (2.9) holds for u, fix $\varphi \in C_c^\infty(\Omega)$ and let $K = \mathrm{supp}(\varphi)$. First we prove that $g_\varepsilon \in L^1(K)$. For all $x \in K$ there exists $\rho > 0$ such that $B_\rho(x) \subset A_\varepsilon(x)$, and by a covering argument we may choose ρ independent of x (while ρ depends on ε). Moreover, for all $x \in K$ and $y \in A_\varepsilon^c(x)$ we have $|x-y| \geq C(1+|y|)$ (see (2.8)). So we can compute

$$\begin{aligned}\int_K |g_\varepsilon(x)|\,dx &\leq C\int_K\int_{A_\varepsilon^c(x)} \frac{|u(x)|^{p-1}}{|x-y|^{N+ps}}\,dy\,dx + C\int_K\int_{A_\varepsilon^c(x)} \frac{|u(y)|^{p-1}}{|x-y|^{N+ps}}\,dy\,dx \\ &\leq C\int_K |u(x)|^{p-1}\,dx \int_{B_\rho^c} \frac{1}{|z|^{N+ps}}\,dz + C\int_K\int_{A_\varepsilon^c(x)} \frac{|u(y)|^{p-1}}{(1+|y|)^{N+ps}}\,dy \\ &\leq C_\varepsilon \int_U |u(x)|^{p-1}\,dx + C|K|\int_{\mathbb{R}^N} \frac{|u(y)|^{p-1}}{(1+|y|)^{N+ps}}\,dy < \infty.\end{aligned}$$

Lemma 2.3 shows that

$$\frac{(u(x)-u(y))^{p-1}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}} \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$$

and thus, through (i), (ii), and Fubini's theorem we have

$$\begin{aligned}&\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))^{p-1}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dx\,dy \\ &\overset{\text{(ii)}}{=} \lim_{\varepsilon\to 0^+}\int_{A_\varepsilon^c} \frac{(u(x)-u(y))^{p-1}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dy\,dx \\ &\overset{\text{(i)}}{=} \lim_{\varepsilon\to 0^+} 2\int_K\int_{A_\varepsilon^c(x)} \frac{(u(x)-u(y))^{p-1}}{|x-y|^{N+ps}}\varphi(x)\,dx\,dy = \lim_{\varepsilon\to 0^+} 2\int_K g_\varepsilon(x)\varphi(x)\,dx.\end{aligned}$$

Since $2g_\varepsilon \to f$ in $L^1(K)$, the density of $C_c^\infty(\Omega)$ in $W_0^{s,p}(\Omega)$ and Lemma 2.3 give the assertion. □

Remark 2.6. As the proof shows, it suffices to assume that the convergence in (2.13) be in $L^1_{\mathrm{loc}}(\Omega)$ weakly. We deliberately choose to assume strong L^1_{loc}-convergence since in all subsequent applications this is enough.

Corollary 2.7. *Let $u \in \widetilde{W}^{s,p}_{\mathrm{loc}}(\Omega)$ be a strong solution of $(-\Delta)_p^s u = f$ in Ω, with $f \in L^1_{\mathrm{loc}}(\Omega)$. Then u is a weak solution of $(-\Delta)_p^s u = f$ in Ω.*

Proof. It follows from Lemma 2.5 with $A_\varepsilon = \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : |x-y| < \varepsilon\}$. □

2.4. Some basic properties of $(-\Delta)_p^s$

The following result describes a fundamental non-local feature of $(-\Delta)_p^s$.

Lemma 2.8 (Non-local behavior of $(-\Delta)_p^s$). *Suppose $u \in \widetilde{W}^{s,p}_{\mathrm{loc}}(\Omega)$ solves $(-\Delta)_p^s u = f$ weakly, strongly or point-wisely in Ω for some $f \in L^1_{\mathrm{loc}}(\Omega)$. Let $v \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ be such that*

$$\mathrm{dist}(\mathrm{supp}(v),\Omega) > 0, \qquad \int_{\Omega^c} \frac{|v(x)|^{p-1}}{(1+|x|)^{N+ps}}\,dx < \infty, \tag{2.14}$$

and define for a.e. Lebesgue point $x \in \Omega$ of u

$$h(x) = 2\int_{\mathrm{supp}(v)} \frac{(u(x)-u(y)-v(y))^{p-1}-(u(x)-u(y))^{p-1}}{|x-y|^{N+ps}}\,dy.$$

Then $u+v \in \widetilde{W}^{s,p}_{\mathrm{loc}}(\Omega)$ and it solves $(-\Delta)^s_p(u+v) = f+h$ weakly, strongly or pointwisely respectively in Ω.

Proof. As usual, it suffices to consider the case Ω bounded, and we first prove that $u+v \in \widetilde{W}^{s,p}(\Omega)$. Let $K = \mathrm{supp}(v)$ and U be such that (2.9) holds for u, and suppose without loss of generality that $\Omega \Subset U \Subset K^c$. Clearly $u+v = u$ in U, and thus it belongs to $W^{s,p}(U)$. Moreover,

$$\int_{\mathbb{R}^N} \frac{|u(x)+v(x)|^{p-1}}{(1+|x|)^{N+ps}}\,dx \le C\Big(\int_{\mathbb{R}^N} \frac{|u(x)|^{p-1}}{(1+|x|)^{N+ps}}\,dx + \int_K \frac{|v(x)|^{p-1}}{(1+|x|)^{N+ps}}\,dx\Big),$$

and the last term is finite due to (2.14). With a similar estimate, we see that the integral defining h is finite (due also to (2.14) and (2.8)). Consider now the case where $(-\Delta)^s_p u = f$ weakly. Choose $\varphi \in C^\infty_c(\Omega)$ and compute

$$\begin{aligned}
&\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)+v(x)-u(y)-v(y))^{p-1}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dx\,dy\\
&= \int_{\Omega\times\Omega} \frac{(u(x)-u(y))^{p-1}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dx\,dy\\
&\quad + \int_{\Omega\times\Omega^c} \frac{(u(x)-u(y)-v(y))^{p-1}\varphi(x)}{|x-y|^{N+ps}}\,dx\,dy\\
&\quad - \int_{\Omega^c\times\Omega} \frac{(u(x)+v(x)-u(y))^{p-1}\varphi(y)}{|x-y|^{N+ps}}\,dx\,dy\\
&= \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))^{p-1}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dx\,dy\\
&\quad - \int_{\Omega\times\Omega^c} \frac{(u(x)-u(y))^{p-1}\varphi(x)}{|x-y|^{N+ps}}\,dx\,dy\\
&\quad + \int_{\Omega^c\times\Omega} \frac{(u(x)-u(y))^{p-1}\varphi(y)}{|x-y|^{N+ps}}\,dx\,dy + 2\int_{\Omega\times\Omega^c} \frac{(u(x)-u(y)-v(y))^{p-1}\varphi(x)}{|x-y|^{N+ps}}\,dx\,dy\\
&= \int_\Omega f(x)\varphi(x)\,dx + 2\int_{\Omega\times\Omega^c} \frac{(u(x)-u(y)-v(y))^{p-1}-(u(x)-u(y)))^{p-1}}{|x-y|^{N+ps}}\varphi(x)\,dx\,dy\\
&= \int_\Omega (f(x)+h(x))\varphi\,dx,
\end{aligned}$$

where in the end we have used Fubini's theorem. The density of $C^\infty_c(\Omega)$ in $W^{s,p}_0(\Omega)$ allows to conclude.

Suppose now that $(-\Delta)^s_p u = f$ strongly or pointwisely in Ω. Let, for $x \in V \Subset \Omega$ and $\varepsilon < \mathrm{dist}(V, \Omega^c)$,

$$g_\varepsilon(x) = \int_{B^c_\varepsilon(x)} \frac{(u(x)+v(x)-u(y)-v(y))^{p-1}}{|x-y|^{N+ps}}\,dy.$$

Using (2.14) we get

$$g_\varepsilon(x) = \int_{\Omega\setminus B_\varepsilon(x)} \frac{(u(x)-u(y))^{p-1}}{|x-y|^{N+ps}}\,dy + \int_{\Omega^c} \frac{(u(x)-u(y)-v(y))^{p-1}}{|x-y|^{N+ps}}\,dy$$
$$= \int_{B^c_\varepsilon(x)} \frac{(u(x)-u(y))^{p-1}}{|x-y|^{N+ps}}\,dy + \int_K \frac{(u(x)-u(y)-v(y))^{p-1}-(u(x)-u(y))^{p-1}}{|x-y|^{N+ps}}\,dy.$$

Taking the limit for $\varepsilon\to 0^+$ gives the claim in the pointwise case. To show that $(-\Delta)^s_p(u+v)=f+h$ strongly it suffices to show that

$$x\mapsto \int_K \frac{(u(x)-u(y)-v(y))^{p-1}-(u(x)-u(y))^{p-1}}{|x-y|^{N+ps}}\,dy$$

belongs to $L^1(K)$, which can be done proceeding as in (2.11) and using (2.14) for the term involving v. □

We also recall the well-known homogeneity, scaling, and rotational invariance properties of $(-\Delta)^s_p$. For all $\rho>0$, $M\in O_N$ (the orthogonal group), v measurable, $\Omega\subseteq\mathbb{R}^N$, set

$$v_\rho(x)=v(\rho x),\quad \rho^{-1}\Omega=\{x/\rho : x\in\Omega\},$$
$$v_M(x)=v(Mx),\quad M^{-1}\Omega=\{M^{-1}x : x\in\Omega\}.$$

Lemma 2.9. *Let $u\in\widetilde{W}^{s,p}_{\rm loc}(\Omega)$ satisfy $(-\Delta)^s_p u=f$ weakly in Ω for some $f\in L^1_{\rm loc}(\Omega)$. Then we have:*

(i) *for all $h>0$, $(-\Delta)^s_p(hu)=h^{p-1}f$ weakly in Ω;*

(ii) *for all $\rho>0$, $u_\rho\in\widetilde{W}^{s,p}(\rho^{-1}\Omega)$ and $(-\Delta)^s_p u_\rho=\rho^{ps}f_\rho$ weakly in $\rho^{-1}\Omega$;*

(iii) *for all $M\in O_N$, $u_M\in\widetilde{W}^{s,p}(M^{-1}\Omega)$ and $(-\Delta)^s_p u_M=f_M$ weakly in $M^{-1}\Omega$.*

Finally, from Lemma 9 of [16], we have the following comparison principle for $(-\Delta)^s_p$.

Proposition 2.10 (Comparison principle). *Let Ω be bounded, $u,v\in\widetilde{W}^{s,p}(\Omega)$ satisfy $u\le v$ in Ω^c and, for all $\varphi\in W^{s,p}_0(\Omega)$, $\varphi\ge 0$ in Ω,*

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))^{p-1}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dx\,dy$$
$$\le \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(v(x)-v(y))^{p-1}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dx\,dy.$$

Then $u\le v$ in Ω.

Proof. The proof follows by the arguments of [16]. It is sufficient to know that both sides of the inequality are finite and $(u-v)_+\in W^{s,p}_0(\Omega)$, which is used there as a test function. By Lemma 2.3, both sides are finite. We claim that $w :=$

$(u-v)_+ \in W_0^{s,p}(\Omega)$. Let $U \Supset \Omega$ be as in Definition 2.1 for both u and v. We split the Gagliardo norm in $\mathbb{R}^N$ as

$$\begin{aligned}\int_{\mathbb{R}^N\times\mathbb{R}^N} &\frac{|w(x)-w(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \\ &= \int_{U\times U} \frac{|w(x)-w(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + 2\int_{\Omega\times U^c} \frac{|w(x)|^p}{|x-y|^{N+ps}}\,dx\,dy,\end{aligned}$$

where we used that $w = 0$ in Ω^c by assumption. The first term is bounded since $u, v \in W^{s,p}(U)$, which is a lattice. The second term is non-singular since $\mathrm{dist}(\Omega, U^c) > 0$ and using (2.8) we get

$$\begin{aligned}\int_{\Omega\times U^c} \frac{|w(x)|^p}{|x-y|^{N+ps}}\,dx\,dy &\le C_{\Omega,U}\int_\Omega (|u(x)|^p + |v(x)|^p)\,dx \int_{\mathbb{R}^N} \frac{1}{(1+|y|)^{N+ps}}\,dy \\ &\le C_{\Omega,U}\int_\Omega (|u(x)|^p + |v(x)|^p)\,dx,\end{aligned}$$

which proves the claim. □

2.5. $(-\Delta)_p^s$ on smooth functions

Next we show that in the class of sufficiently smooth functions, the s-fractional p-Laplacian exists strongly (and thus weakly) and is locally bounded. First we recall the following definition of $(-\Delta)_p^s$, equivalent to (2.12) (by a simple change of variable):

$$(-\Delta)_p^s u(x) = \mathrm{p.v.}\int_{\mathbb{R}^N} \frac{(u(x)-u(x+z))^{p-1} + (u(x)-u(x-z))^{p-1}}{|z|^{N+ps}}\,dz. \tag{2.15}$$

Our first lemma displays an estimate which allows us to remove the singularity at 0, when u is smooth enough:

Lemma 2.11. *If $u \in C^{1,\gamma}_{\mathrm{loc}}(\Omega)$, $\gamma \in [0,1]$, and $K \subset \Omega$ is compact, then there exist $C_{K,u}, R_K > 0$ such that for all $x \in K$, $z \in B_{R_K}$*

$$\big|(u(x)-u(x+z))^{p-1} + (u(x)-u(x-z))^{p-1}\big| \le \begin{cases} C_{K,u}\,|z|^{\gamma+p-1} & \text{if } p \ge 2, \\ C_{K,u}\,|z|^{(\gamma+1)(p-1)} & \text{if } p < 2. \end{cases}$$

Proof. Since K is compact, we can find $R_K > 0$ such that

$$\Omega_K := \{x \in \mathbb{R}^N : \mathrm{dist}(x,K) \le R_K\} \subset \Omega.$$

Consider first the case $p \ge 2$. Since $u \in C^{1,\gamma}(\Omega_K)$, for all $x \in K$, $z \in B_{R_K}$ there exist $\tau_1, \tau_2 \in [0,1]$ with

$$\begin{aligned}\big|(u(x) &- u(x+z))^{p-1} + (u(x)-u(x-z))^{p-1}\big| \\ &= \big|(Du(x+\tau_1 z)\cdot z)^{p-1} - (Du(x-\tau_2 z)\cdot z)^{p-1}\big| \\ &\le (p-1)\sup_{B_{R_K}(x)} |Du|^{p-2}|z|^{p-2}\big|\big(Du(x+\tau_1 z) - Du(x-\tau_2 z)\big)\cdot z\big| \\ &\le C\,\|Du\|^{p-1}_{C^{0,\gamma}(\Omega_K)}\,|z|^{\gamma+p-1}.\end{aligned}$$

If $1 < p < 2$ then $t \mapsto t^{p-1}$ is globally $(p-1)$-Hölder continuous and in this case we directly have, with the same notation as before,

$$\begin{aligned}&\big|(u(x)-u(x+z))^{p-1}+(u(x)-u(x-z))^{p-1}\big|\\&\quad\le C\big|Du(x+\tau_1 z)\cdot z-Du(x-\tau_2 z)\cdot z\big|^{p-1}\le C\,\|Du\|^{p-1}_{C^{0,\gamma}(\Omega_K)}\,|z|^{\gamma(p-1)}|z|^{p-1},\end{aligned}$$

which concludes the proof. □

The following result shows sufficient conditions to write $(-\Delta)_p^s u$ as a locally bounded function:

Proposition 2.12 ($(-\Delta)_p^s$ on $C^{1,\gamma}$ functions). *Suppose Ω is bounded, $u \in \widetilde{W}^{s,p}(\Omega) \cap C^{1,\gamma}_{\rm loc}(\Omega)$, with $\gamma \in [0,1]$ such that*

$$\gamma > \begin{cases} 1-p(1-s) & \text{if } p \ge 2,\\ \dfrac{1-p(1-s)}{p-1} & \text{if } p < 2.\end{cases} \tag{2.16}$$

Then $(-\Delta)_p^s u = f$ strongly in Ω for some $f \in L^\infty_{\rm loc}(\Omega)$

Proof. Let U be as in Definition 2.1 for u, fix a compact set $K \subset \Omega$ and let $R_K, C_K > 0$ be as in Lemma 2.11. Define, for $x \in K$, $\varepsilon > 0$,

$$\begin{aligned} g_\varepsilon(x) &:= \int_{B_\varepsilon^c} \frac{(u(x)-u(x+z))^{p-1}+(u(x)-u(x-z))^{p-1}}{|z|^{N+ps}}\,dz\\ &= 2\int_{B_\varepsilon^c} \frac{(u(x)-u(x-z))^{p-1}}{|z|^{N+ps}}\,dz. \end{aligned}$$

We claim that g_ε converges as $\varepsilon \to 0^+$ in a dominated way to some $f \in L^\infty(K)$. We split the integral in one for $z \in B_{R_K}$ and one over $B^c_{R_K}$. For the first one, the previous lemma gives

$$\Big|\frac{(u(x)-u(x+z))^{p-1}+(u(x)-u(x-z))^{p-1}}{|z|^{N+ps}}\Big| \le \frac{C_{K,u}}{|z|^{N+ps-\sigma}},$$

where $\sigma = \gamma + p - 1$ if $p \ge 2$ and $\sigma = (\gamma+1)(p-1)$ if $1 < p < 2$. Notice that, in both cases, we have $ps - \sigma < 0$. Due to assumptions (2.16), the integral is thus non-singular, and it holds

$$\lim_{\varepsilon\to 0}\int_{B_{R_K}\setminus B_\varepsilon} \frac{(u(x)-u(x+z))^{p-1}+(u(x)-u(x-z))^{p-1}}{|z|^{N+ps}}\,dz =: f_1(x),$$

$$\Big|\int_{B_{R_K}\setminus B_\varepsilon} \frac{(u(x)-u(x+z))^{p-1}+(u(x)-u(x-z))^{p-1}}{|z|^{N+ps}}\,dz\Big| \le \int_{B_{R_K}} \frac{C_{K,u}}{|z|^{N+ps-\sigma}}\,dz,$$

which is a bound independent of $x \in K$ and $\varepsilon > 0$. For the integral over $z \in B^c_{R_K}$ we have, as in (2.11),

$$|f_2(x)| := \left|2 \int_{B^c_{R_K}} \frac{(u(x) - u(x+z))^{p-1}}{|z|^{N+ps}}\, dz\right|$$
$$\le C_{K,U}\Big(\|u\|^{p-1}_{L^\infty(K)} + \int_{\mathbb{R}^N} \frac{|u(y)|^{p-1}}{(1+|y|)^{N+ps}}\, dy\Big).$$

Gathering together the two estimates, we get

$$|g_\varepsilon(x)| \le C_{K,u,U} \quad \forall x \in K,\ \varepsilon > 0, \quad \text{and} \quad \lim_{\varepsilon\to 0^+} g_\varepsilon(x) \ = f_1(x) + f_2(x) \quad \forall x \in K,$$

and thus by the dominated convergence theorem $g_\varepsilon \to f_1 + f_2$ in $L^1(K)$. □

Remark 2.13. It is useful to outline the dependence of $\|(-\Delta)^s_p u\|_\infty$ on s in the previous proposition. Suppose, to fix ideas, that $p \ge 2$ and $u \in C^\infty_c(\mathbb{R}^N)$, so that the domain Ω has no role. Then, following the proof, we can find a constant c_N depending only on N such that

$$\|(-\Delta)^s_p u\|_\infty \le c_N \frac{\|u\|^{p-1}_{C^2(\mathbb{R}^N)}}{1-s}.$$

This is in accordance with the well known fact that $(1-s)(-\Delta)^s_p \to -\Delta_p$ as $s \to 1^-$ (see e.g. [17]).

Remark 2.14. Consider the class of functions

$$\mathcal{L}(\Omega) = \{u \in \widetilde{W}^{s,p}(\Omega) : (-\Delta)^s_p u = f \text{ strongly for some } f \in L^\infty_{\mathrm{loc}}(\Omega)\}.$$

The previous theorem asserts that if $p \ge 2$, then $C^2(\Omega) \subseteq \mathcal{L}(\Omega)$. However, if $1 < p < 2$, it may be difficult to find smooth functions (e.g., smooth cut-offs) belonging to $\mathcal{L}(\Omega)$, since the second condition in (2.16) coupled with $\gamma \le 1$ forces $s < 2(p-1)/p$. One may think that this is just a technical limit of the proof, or that requiring higher regularity than C^2 could solve the issue. Unfortunately, due to the singular nature of the operator for $1 < p < 2$, this is not the case: there are smooth functions u such that $(-\Delta)^s_p u$ (in the strong sense) cannot be pointwise bounded. Consider for example $u(x) = x^2\eta(x)$, where $\eta \in C^\infty_c(\mathbb{R})$ and $\eta = 1$ on $[-1, 1]$. Calculating $(-\Delta)^s_p u(0)$ as a principal value gives

$$|(-\Delta)^s_p u(0)| < \infty \quad \Longleftrightarrow \quad s < 2\,\frac{p-1}{p}.$$

3. Distance functions

In this section we produce some explicit solutions for $(-\Delta)^s_p$ in special domains. Then we prove that $(-\Delta)^s_p \delta^s$ is bounded in a neighborhood of $\partial\Omega$ (here we define $\delta = \delta_\Omega$ as in Section 2). We begin by getting a solution of $(-\Delta)^s_p u = 0$ on the half-line $\mathbb{R}_+$.

Lemma 3.1. *Set $u_1(x) = x_+^s$ for all $x \in \mathbb{R}$. Then $u_1 \in \widetilde{W}^{s,p}_{\rm loc}(\mathbb{R})$ and it solves $(-\Delta)_p^s u_1 = 0$ strongly and weakly in $\mathbb{R}_+$.*

Proof. Let $K \subseteq (\rho, \rho^{-1})$ for some $\rho \in (0,1)$. For $x \in K$, $\varepsilon > 0$ consider the function

$$g_\varepsilon^{(1)}(x) = \int_{B_\varepsilon^c(x)} \frac{(x^s - y_+^s)^{p-1}}{|x-y|^{1+ps}}\, dy. \tag{3.1}$$

We claim that $g_\varepsilon^{(1)} \to 0$ uniformly on K, as $\varepsilon \to 0^+$. Note that for any $\varepsilon < x$ it holds

$$0 < x - \varepsilon < x + \varepsilon < \frac{x^2}{x-\varepsilon}.$$

We split the integral accordingly, namely

$$\begin{aligned} g_\varepsilon^{(1)}(x) &= \int_{-\infty}^0 \frac{(x^s - y_+^s)^{p-1}}{|x-y|^{1+ps}}\, dy + \int_{x+\varepsilon}^{\frac{x^2}{x-\varepsilon}} \frac{(x^s - y_+^s)^{p-1}}{|x-y|^{1+ps}}\, dy \\ &\quad + \Big(\int_0^{x-\varepsilon} \frac{(x^s - y_+^s)^{p-1}}{|x-y|^{1+ps}}\, dy + \int_{\frac{x^2}{x-\varepsilon}}^\infty \frac{(x^s - y_+^s)^{p-1}}{|x-y|^{1+ps}}\, dy \Big) \\ &= I_1(x) + I_2(x,\varepsilon) + I_3(x,\varepsilon). \end{aligned}$$

Let us estimate the three terms separately. It holds

$$I_1(x) = \int_{-\infty}^0 \frac{(x^s - y_+^s)^{p-1}}{|x-y|^{1+ps}}\, dy = \frac{x^{-s}}{ps}.$$

Regarding the integral between $x+\varepsilon$ and $\frac{x^2}{x-\varepsilon}$, since $\xi \mapsto \xi^s$ is globally s-Hölder, we have

$$|I_2(x,\varepsilon)| \le C \int_{x+\varepsilon}^{\frac{x^2}{x-\varepsilon}} \frac{|x-y|^{s(p-1)}}{|x-y|^{1+ps}}\, dy = \frac{Cx^{-s}}{s} \frac{x^s - (x-\varepsilon)^s}{\varepsilon^s}.$$

Finally,

$$\begin{aligned} I_3(x,\varepsilon) &= \frac{x^{s(p-1)}}{x^{1+ps}} \Big(\int_0^{x-\varepsilon} \frac{(1-(y/x)^s)^{p-1}}{(1-y/x)^{1+ps}}\, dy - \int_{x^2/(x-\varepsilon)}^\infty \frac{((y/x)^s - 1)^{p-1}}{(y/x-1)^{1+ps}}\, dy \Big) \\ &\overset{t=y/x=\xi}{=} x^{-s} \Big(\int_0^{1-\varepsilon/x} \frac{(1-t^s)^{p-1}}{(1-t)^{1+ps}}\, dt - \int_{(1-\varepsilon/x)^{-1}}^\infty \frac{(\xi^s-1)^{p-1}}{(\xi-1)^{1+ps}}\, d\xi \Big) \\ &\overset{\xi=t^{-1}}{=} x^{-s} \Big(\int_0^{1-\varepsilon/x} \frac{(1-t^s)^{p-1}}{(1-t)^{1+ps}}\, dt - \int_0^{1-\varepsilon/x} \frac{(t^{-s}-1)^{p-1}}{(t^{-1}-1)^{1+ps}} \frac{dt}{t^2} \Big) \\ &= x^{-s} \int_0^{1-\varepsilon/x} \frac{(1-t^s)^{p-1}}{(1-t)^{1+ps}} (1 - t^{s-1})\, dt \\ &= x^{-s} \Big[\frac{1}{ps} \frac{(1-t^s)^p}{(1-t)^{ps}} \Big]_0^{1-\varepsilon/x} = \frac{x^{-s}}{ps} \Big(\Big(\frac{x^s - (x-\varepsilon)^s}{\varepsilon^s} \Big)^p - 1 \Big). \end{aligned}$$

Let for $\varepsilon < x$

$$\psi(x,\varepsilon) = \frac{x^s - (x-\varepsilon)^s}{\varepsilon^s}, \tag{3.2}$$

and notice that from the subadditivity of $x \mapsto x^s$ we get $\psi(x,\varepsilon) \le 1$. Gathering together the three previous estimates we get

$$|g_\varepsilon^{(1)}(x)| \le Cx^{-s}(\psi(x,\varepsilon) + \psi^p(x,\varepsilon)) \le Cx^{-s}\psi(x,\varepsilon) \quad \forall x > \varepsilon > 0, \tag{3.3}$$

where C is a universal constant. Since $\psi(x,\varepsilon) \to 0$ uniformly on $[\rho, \rho^{-1}] \supseteq K$, as $\varepsilon \to 0^+$, the claim follows. Finally we prove that $u_1 \in \widetilde{W}^{s,p}(a,b)$ for any $a < 0 < b$. We have

$$\begin{aligned}
\int_{[a,b]\times[a,b]} &\frac{|u_1(x) - u_1(y)|^p}{|x-y|^{1+ps}}\, dx\, dy \\
&= 2\int_0^b \int_0^x \frac{|x^s - y^s|^p}{|x-y|^{1+ps}}\, dy\, dx + 2\int_0^b x^{sp} \int_a^0 \frac{1}{|x-y|^{1+ps}}\, dy\, dx \\
&\overset{t=y/x}{=} 2\int_0^b \int_0^1 \frac{|1-t^s|^p}{|1-t|^{1+ps}}\, dt\, dx + \frac{2}{ps}\int_0^b x^{sp}\Big(\frac{1}{x^{ps}} - \frac{1}{|x-a|^{ps}}\Big)\, dx,
\end{aligned}$$

which is readily checked to be finite. The assertion follows through Lemma 2.3 and Corollary 2.7. □

Now we study the solution $u(x) = u_1(x_N)$ in the half-space

$$\mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x_N > 0\}.$$

Lemma 3.2. *Set for any $A \in GL_N$ and $x \in \mathbb{R}^N_+$,*

$$g_\varepsilon(x, A) = \int_{B_\varepsilon^c} \frac{(u_1(x_N) - u_1(x_N + z_N))^{p-1}}{|Az|^{N+ps}}\, dz$$

and $u(x) = u_1(x_N)$. Then $g_\varepsilon \to 0$ uniformly in any compact $K \subseteq \mathbb{R}^N_+ \times GL_N$ and $u \in \widetilde{W}^{s,p}_{\rm loc}(\mathbb{R}^N)$ solves $(-\Delta)^s_p u = 0$ strongly and weakly in $\mathbb{R}^N_+$.

Proof. It suffices to prove the statement for $K = H \times H'$, where $H \subseteq \mathbb{R}^N_+$ and $H' \subseteq GL_N$ are compact (recall that GL_N is open in $\mathbb{R}^{N^2}$). To estimate g_ε we use elliptic coordinates. A consequence of the singular value decomposition is that AS^{N-1} is an ellipsoid whose semiaxes are the singular values of A, and thus its diameter is $2\|A\|_2$, where the latter is the spectral norm of A. The corresponding elliptic coordinates are uniquely defined by

$$y = \rho\omega, \quad \omega \in AS^{N-1}, \quad \rho > 0,$$

for $y \in \mathbb{R}^N \setminus \{0\}$. It holds $dy = \rho^{N-1} d\omega\, d\rho$ where $d\omega$ is the surface element of AS^{N-1}. Setting

$$e_A := A^{-1}e_N, \quad E_A := \{x \in \mathbb{R}^N : x \cdot e_A \ge 0\},$$

we compute, through the change of variable $z = A^{-1}y$,

$$\begin{aligned}
&g_\varepsilon(x,A)\\
&= \int_{B_\varepsilon^c} \frac{(u(x)-u(x+z))^{p-1}}{|Az|^{N+ps}}\,dz = |\mathrm{det}A|^{-1}\int_{AB_\varepsilon^c} \frac{(u(x)-u(x+A^{-1}y))^{p-1}}{|y|^{N+ps}}\,dy\\
&= \int_{AS^{N-1}} \frac{1}{|\mathrm{det}A||\omega|^{N+ps}} \int_\varepsilon^\infty \frac{(u_1(x_N)-u_1(x_N+\omega\cdot e_A\rho))^{p-1}}{\rho^{1+ps}}\,d\rho\,d\omega\\
&= \int_{AS^{N-1}\cap E_A} \frac{|\omega\cdot e_A|^{1+ps}}{|\mathrm{det}A||\omega|^{N+ps}} \int_{(-\varepsilon,\varepsilon)^c} \frac{(u_1(x_N)-u_1(x_N+\omega\cdot e_A\rho))^{p-1}}{|\omega\cdot e_A\rho|^{1+ps}}\,d(\omega\cdot e_A\rho)\,d\omega\\
&= \int_{AS^{N-1}\cap E_A} \frac{|\omega\cdot e_A|^{1+ps}}{|\mathrm{det}A||\omega|^{N+ps}}\, g^{(1)}_{\omega\cdot e_A\varepsilon}(x_N)\,d\omega,
\end{aligned}$$

where $g^{(1)}_{\omega\cdot e_A\varepsilon}$ is defined as in (3.1). Since $|\omega\cdot e_A| \le \|A\|_2\|A^{-1}\|_2$, the condition

$$\omega\cdot e_A\varepsilon < x_N$$

holds for $\varepsilon \le \bar\varepsilon$ where $\bar\varepsilon$ depends only on H and H'. For any such ε we can apply (3.3) to obtain

$$|g_\varepsilon(x,A)| \le C\,x_N^{-s} \int_{AS^{N-1}\cap E_A} \frac{|\omega\cdot e_A|^{1+ps}}{|\mathrm{det}A||\omega|^{N+ps}}\,\psi(x_N,\omega\cdot e_A\varepsilon)\,d\omega,$$

where ψ is defined in (3.2). Since $\xi\mapsto\xi^s$ is concave for $0<s<1$, we have

$$s(x_N-t)^{s-1}t \ge x_N^s-(x_N-t)^s,$$

and being $1>s>0$ it follows

$$\frac{\partial\psi(x_N,t)}{\partial t} = s\frac{(x_N-t)^{s-1}t - x_N^s + (x_N-t)^s}{t^{1+s}} \ge 0, \quad \text{for } 0<t\le x_N.$$

Therefore $\psi(x_N,t)$ is non-decreasing in t, thus we get

$$\begin{aligned}
|g_\varepsilon(x,A)| &\le C\,x_N^{-s}\,\psi(x_N,\|A\|_2\|A^{-1}\|_2\varepsilon)\int_{AS^{N-1}} \frac{|\omega\cdot e_A|^{1+ps}}{|\mathrm{det}A||\omega|^{N+ps}}\,d\omega\\
&\le C\,x_N^{-s}\,\psi(x_N,\|A\|_2\|A^{-1}\|_2\varepsilon)\int_{S^{N-1}} \frac{|\omega\cdot e_N|^{1+ps}}{|A\omega|^{N+ps}}\,d\omega\\
&\le C\,x_N^{-s}\,\psi(x_N,\|A\|_2\|A^{-1}\|_2\,\varepsilon)\,\|A^{-1}\|_2^{N+ps}.
\end{aligned}$$

Now $\|A\|_2$ and $\|A^{-1}\|_2$ are bounded on H' from below and above, as well as x_N on H, and the uniform convergence follows. As in the previous proof, it is readily checked that $u\in\widetilde{W}^{s,p}(V)$ for any bounded V, and the second statement follows as before. □

Remark 3.3. Due to rotational invariance, Lemma 3.2 easily extends to any half-space

$$H_e = \{x\in\mathbb{R}^N : x\cdot e\ge 0\} \quad (e\in S^{N-1}),$$

simply considering the solution $u(x) = (x\cdot e)_+^s$.

The following lemma gives a control on the behaviour of $(-\Delta)_p^s(x_N)_+^s$ under a smooth change of variables.

Lemma 3.4 (Change of variables). *Let Φ be a $C^{1,1}$ diffeomorphism of $\mathbb{R}^N$ such that $\Phi = I$ in B_r^c, $r > 0$. Then the function $v(x) = (\Phi^{-1}(x)\cdot e_N)_+^s$ belongs to $\widetilde{W}^{s,p}_{\rm loc}(\mathbb{R}^N)$ and is a weak solution of $(-\Delta)_p^s v = f$ in $\Phi(\mathbb{R}^N_+)$, with*

$$\|f\|_\infty \le C\,(\|D\Phi\|_\infty, \|D\Phi^{-1}\|_\infty, r)\,\|D^2\Phi\|_\infty. \tag{3.4}$$

Proof. First we recall that, since $D\Phi$ is globally Lipschitz in $\mathbb{R}^N$ with constant $L > 0$, then $D^2\Phi(x)$ exists in the classical sense for a.e. $x \in \mathbb{R}^N$, and $\|D^2\Phi\|_{L^\infty(\mathbb{R}^N)} \le L$. Let $J_\Phi(\cdot) = |\det D\Phi(\cdot)|$, $u_1(t) = t_+^s$. Due to Lemma 2.5, applied with $A_\varepsilon = \{|\Phi^{-1}(x) - \Phi^{-1}(y)| < \varepsilon\}$ it suffices to show that

$$g_\varepsilon(x) = \int_{\{|\Phi^{-1}(x)-\Phi^{-1}(y)|\ge\varepsilon\}} \frac{(v(x)-v(y))^{p-1}}{|x-y|^{N+ps}}\,dy$$

converges in $L^1(K)$ for any compact $K \subseteq \Phi(\mathbb{R}^N_+)$. Changing variables $x = \Phi(X)$, this is equivalent to claiming that

$$X \mapsto \int_{B_\varepsilon^c(X)} \frac{(u_1(X_N) - u_1(Y_N))^{p-1}}{|\Phi(X)-\Phi(Y)|^{N+ps}} J_\Phi(Y)\,dY \tag{3.5}$$

converges as $\varepsilon \to 0$ in $L^1_{\rm loc}(\mathbb{R}^N_+)$. To prove this claim, we write

$$\begin{aligned} g_\varepsilon(x) &= \int_{B_\varepsilon^c(X)} \frac{(u_1(X_N) - u_1(Y_N))^{p-1}}{|D\Phi(X)(X-Y)|^{N+ps}} h(X,Y)\,dY \\ &\quad + \int_{B_\varepsilon^c(X)} J_\Phi(X)\frac{(u_1(X_N) - u_1(Y_N))^{p-1}}{|D\Phi(X)(X-Y)|^{N+ps}}\,dY, \end{aligned} \tag{3.6}$$

where

$$h(X,Y) = \frac{|D\Phi(X)(X-Y)|^{N+ps}}{|\Phi(X)-\Phi(Y)|^{N+ps}} J_\Phi(Y) - J_\Phi(X), \quad X \ne Y.$$

We will now prove the following estimate, from which convergence of (3.5) will follow:

$$|h(X,Y)| \le C_\Phi\,\|D^2\Phi\|_\infty \min\{|X-Y|, 1\}, \tag{3.7}$$

where C_Φ depends on N, p, s as well as on $\|D\Phi\|_\infty$, $\|D\Phi^{-1}\|_\infty$ and r. Write

$$\begin{aligned} h(X,Y) &= \frac{|D\Phi(X)(X-Y)|^{N+ps}}{|\Phi(X)-\Phi(Y)|^{N+ps}} (J_\Phi(Y) - J_\Phi(X)) \\ &\quad + J_\Phi(X)\Big(\frac{|D\Phi(X)(X-Y)|^{N+ps}}{|\Phi(X)-\Phi(Y)|^{N+ps}} - 1\Big) \\ &=: J_1 + J_2. \end{aligned}$$

First observe that using Taylor's formula yields

$$\frac{|D\Phi(X)(X-Y)|}{|\Phi(X)-\Phi(Y)|} \le C\,\|D\Phi\|_\infty\,\|D\Phi^{-1}\|_\infty,$$

therefore,

$$|J_1| \le \tilde{C}\,\|D^2\Phi\|_{L^\infty(\mathbb{R}^N)}\,|X-Y|.$$

To estimate J_2, we note that the mapping $t \mapsto t^{(N+ps)/2}$ is smooth in a neighborhood of 1, and that

$$\lim_{Y\to X} \frac{|D\Phi(X)(X-Y)|^2}{|\Phi(X)-\Phi(Y)|^2} = 1,$$

hence

$$|J_2| \le C_\Phi\Big(\frac{|D\Phi(X)(X-Y)|^2}{|\Phi(X)-\Phi(Y)|^2} - 1\Big). \tag{3.8}$$

Besides, for all $Y \in \mathbb{R}^N$ there exist $\tau_1, \dots, \tau_N \in [0,1]$ such that

$$\Phi^i(X) - \Phi^i(Y) = D\Phi^i(\tau_i X + (1-\tau_i)Y)\cdot(X-Y), \quad i = 1, \dots, N,$$

where Φ^i denotes the i-th component of Φ. So we have (still allowing $C_\Phi > 0$ to depend on $\|D\Phi\|_{L^\infty(\mathbb{R}^N)}$)

$$\begin{aligned}
\big|\,|\Phi(X)&-\Phi(Y)|^2 - |D\Phi(X)(X-Y)|^2\big| \\
&= \big|(\Phi(X)-\Phi(Y)+D\Phi(X)(X-Y))\cdot(\Phi(X)-\Phi(Y)-D\Phi(X)(X-Y))\big| \\
&\le C_\Phi\,|X-Y|\sum_{i=1}^N |\Phi^i(X)-\Phi^i(Y)-D\Phi^i(X)(X-Y)| \\
&\le C_\Phi\,|X-Y|^2\sum_{i=1}^N |D\Phi^i(\tau_i X+(1-\tau_i)Y)-D\Phi^i(X)| \\
&\le C_\Phi\,\|D^2\Phi\|_\infty\,|X-Y|^3.
\end{aligned}$$

Inserting into (3.8) we obtain

$$|J_2| \le C_\Phi\,\frac{\big||D\Phi(X)(X-Y)|^2 - |\Phi(X)-\Phi(Y)|^2\big|}{|\Phi(X)-\Phi(Y)|^2} \le C_\Phi\,\|D^2\Phi\|_\infty\,|X-Y|,$$

which yields

$$|h(X,Y)| \le C_\Phi\,\|D^2\Phi\|_{L^\infty(\mathbb{R}^N)}\,|X-Y|, \quad \text{for all } X, Y \in \mathbb{R}^N,$$

and thus (3.7) for $|X-Y| \le 2r$. Assume now $|X-Y| > 2r$, then at least one of X, Y lies in $\overline{B}_r^c$. Clearly, if $X, Y \in \overline{B}_r^c$, then $h(X,Y) = 0$. If $X \in \overline{B}_r$, $Y \in \overline{B}_r^c$, then for any $1 \le i \le N$ we define a mapping $\eta_i \in C^{1,1}([0,1])$ by setting

$$\eta_i(t) = \Phi^i(X + t(Y-X)).$$

It is readily checked that $|\eta_i''| \leq C\|D^2\Phi\|_\infty |X-Y|^2$ for a.e. $t \in (0,1)$. Moreover, if $t \geq 2r/|X-Y|$ then $X + t(Y-X) \in B_r^c$, and since $\Phi = I$ outside B_r it holds

$$\eta_i(t) = (X + t(Y-X)) \cdot e_i \quad \text{for } t \geq \frac{2r}{|X-Y|}.$$

Therefore $\eta_i''(t) \equiv 0$ for $t \geq 2r/|X-Y|$ and applying the Taylor formula with integral remainder we have

$$\begin{aligned}
&|\Phi^i(Y) - \Phi^i(X) + D\Phi^i(X)(X-Y)| = |\eta_i(1) - \eta_i(0) - \eta_i'(0)| \\
&\quad \leq \int_0^1 |\eta_i''(t)|(1-t)\,dt \leq \int_0^{2r/|X-Y|} |\eta_i''(t)|(1-t)\,dt \leq C_\Phi \,\|D^2\Phi\|_\infty \,|X-Y|.
\end{aligned}$$

So we have

$$\begin{aligned}
|h(X,Y)| &\leq \Big|\frac{|D\Phi(X)(X-Y)|^{N+ps}}{|\Phi(X)-\Phi(Y)|^{N+ps}} - 1\Big| + |1 - J_\Phi(X)| \\
&\leq C_\Phi \left|\frac{|D\Phi(X)(X-Y)|^2 - |\Phi(X)-\Phi(Y)|^2}{|\Phi(X)-\Phi(Y)|^2}\right| + C_\Phi \,\|D^2\Phi\|_\infty \\
&\leq C_\Phi \,\frac{\big|D\Phi(X)(X-Y) + \Phi(X) - \Phi(Y)\big|}{|\Phi(X)-\Phi(Y)|^2} \\
&\qquad \cdot \big|D\Phi(X)(X-Y) - \Phi(X) + \Phi(Y)\big| + C_\Phi \,\|D^2\Phi\|_\infty \\
&\leq \frac{C_\Phi}{|X-Y|} \sum_{i=1}^N \big|D\Phi^i(X)(X-Y) - \Phi^i(X) + \Phi^i(Y)\big| + C_\Phi \,\|D^2\Phi\|_\infty \\
&\leq C_\Phi \,\|D^2\Phi\|_\infty.
\end{aligned}$$

If $X \in \overline{B}_r^c$ and $Y \in \overline{B}_r$, we argue in a similar way. Thus (3.7) is achieved for all $X, Y \in \mathbb{R}^N$.

Let us go back to (3.6). The first integral can be estimated as follows:

$$\begin{aligned}
&\int_{B_\varepsilon^c(X)} \Big|\frac{(u_1(X_N) - u_1(Y_N))^{p-1}}{|D\Phi(X)(X-Y)|^{N+ps}} h(X,Y)\Big|\,dY \\
&\qquad \leq C_\Phi \,\|D^2\Phi\|_\infty \int_{B_\varepsilon^c(X)} \frac{\min\{|X-Y|, 1\}}{|X-Y|^{N+s}}\,dY \qquad (3.9)\\
&\qquad \leq C_\Phi \,\|D^2\Phi\|_\infty \Big(\int_\varepsilon^1 \frac{1}{t^s}\,dt + \int_1^\infty \frac{1}{t^{1+s}}\,dt\Big) \leq C_\Phi \,\|D^2\Phi\|_\infty (\varepsilon^{1-s} + 1).
\end{aligned}$$

The second integral in (3.6) vanishes for $\varepsilon \to 0$, and is estimated through Lemma 3.2: since $D\Phi(\mathbb{R}^N)$ is a compact subset of GL_N, the integral vanishes uniformly in any compact $\Phi^{-1}(K) \subseteq \mathbb{R}^N_+$, and therefore uniformly in any compact $K \subseteq \Phi(\mathbb{R}^N_+)$. Lemma 2.5 thus gives that $(-\Delta)_p^s v = f$ weakly in any open bounded $U \subseteq \Phi(\mathbb{R}^N_+)$, where

$$f(x) := 2 \lim_{\varepsilon \to 0} g_\varepsilon(x).$$

Taking the limit for $\varepsilon \to 0$ in estimate (3.9) gives (3.4). □

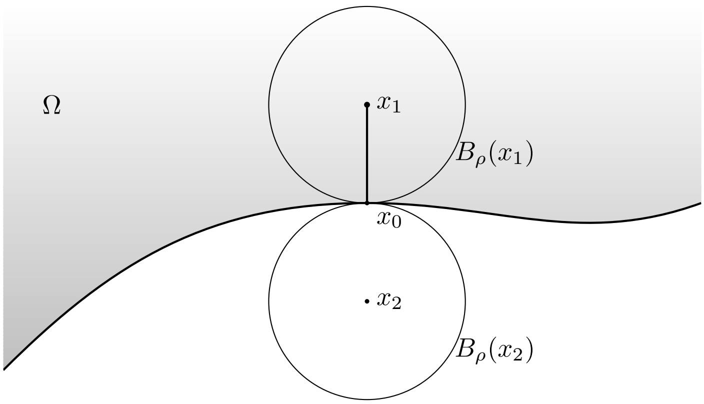

FIGURE 1. The interior and exterior balls at $x_0 \in \partial\Omega$. For all $x \in [x_0, x_1]$ it holds $\delta(x) = |x - x_0|$.

Finally, we consider a general bounded domain Ω with a $C^{1,1}$ boundary. First we recall some geometrical properties, which can be found e.g. in [1] (see figure 1):

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a $C^{1,1}$ boundary $\partial\Omega$. Then, there exists $\rho > 0$ such that for all $x_0 \in \partial\Omega$ there exist $x_1, x_2 \in \mathbb{R}^N$ on the normal line to $\partial\Omega$ at x_0, with the following properties:*

(i) $B_\rho(x_1) \subset \Omega$, $B_\rho(x_2) \subset \Omega^c$;

(ii) $\overline{B}_\rho(x_1) \cap \overline{B}_\rho(x_2) = \{x_0\}$;

(iii) $\delta(x) = |x - x_0|$ *for all* $x \in [x_0, x_1]$.

As a byproduct, we prove that $(-\Delta)_p^s \delta^s$ is bounded in a neighborhood of the boundary.

Theorem 3.6. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a $C^{1,1}$ boundary. There exists $\rho = \rho(N, p, s, \Omega)$ such that $(-\Delta)_p^s \delta^s = f$ weakly in*

$$\Omega_\rho := \{x \in \Omega : \delta(x) < \rho\},$$

for some $f \in L^\infty(\Omega_\rho)$.

Proof. Suppose that ρ is smaller than the one given in Lemma 3.5. We choose a finite covering of Ω_ρ made of balls of radius 2ρ and center $x_i \in \partial\Omega$. Using a partition of unity, it suffices to prove the statement in any set $\Omega \cap B_{2\rho}(x_i)$. To do so, we flatten the boundary near the point x_i, which we can suppose without loss of generality to be the origin. Choosing a smaller ρ (depending only on the geometry of $\partial\Omega$) if necessary, there exists a diffeomorphism $\Phi \in C^{1,1}(\mathbb{R}^N, \mathbb{R}^N)$, $\Phi(X) = x$ such that $\Phi = I$ in $B_{4\rho}^c$ and

$$\Omega \cap B_{2\rho} \Subset \Phi(B_{3\rho} \cap \mathbb{R}^N_+), \quad \delta(\Phi(X)) = (X_N)_+, \quad \forall X \in B_{3\rho}. \tag{3.10}$$

We claim that

$$g_\varepsilon(x) = \int_{\{|\Phi^{-1}(x)-\Phi^{-1}(y)|\geq\varepsilon\}} \frac{(\delta^s(x)-\delta^s(y))^{p-1}}{|x-y|^{N+ps}}\, dy \to f(x) \quad \text{in } L^1_{\rm loc}(\Omega\cap B_{2\rho}).$$

We change variables setting $X = \Phi^{-1}(x)$, noting that $X \in B_{3\rho}\cap\mathbb{R}^N_+$ for any $x\in\Omega\cap B_{2\rho}$, and compute

$$\begin{aligned}
g_\varepsilon(x) &= \int_{\{|X-Y|\geq\varepsilon\}} \frac{(\delta^s(\Phi(X))-\delta^s(\Phi(Y)))^{p-1}}{|\Phi(X)-\Phi(Y)|^{N+ps}}\, J_\Phi(Y)\, dY\\
&= \int_{B^c_\varepsilon(X)\cap B_{3\rho}} \frac{(\delta^s(\Phi(X))-\delta^s(\Phi(Y)))^{p-1}}{|\Phi(X)-\Phi(Y)|^{N+ps}}\, J_\Phi(Y)\, dY\\
&+ \int_{B^c_{3\rho}} \frac{(\delta^s(\Phi(X))-\delta^s(\Phi(Y)))^{p-1}}{|\Phi(X)-\Phi(Y)|^{N+ps}}\, J_\Phi(Y)\, dY\\
&= \int_{B^c_\varepsilon(X)} \frac{(u_1(X_N)-u_1(Y_N))^{p-1}}{|\Phi(X)-\Phi(Y)|^{N+ps}}\, J_\Phi(Y)\, dY\\
&\quad + \int_{B^c_{3\rho}} \frac{(\delta^s(\Phi(X))-\delta^s(\Phi(Y)))^{p-1}-(u_1(X_N)-u_1(Y_N))^{p-1}}{|\Phi(X)-\Phi(Y)|^{N+ps}}\, J_\Phi(Y)\, dY\\
&= f_{1,\varepsilon}(X) + f_2(X),
\end{aligned}$$

for sufficiently small ε, where we used the fact that

$$\delta^s(\Phi(Z)) = u_1(Z_N) \quad \text{for all } Z\in B_{3\rho}$$

thanks to (3.10). Clearly $f_2\circ\Phi^{-1}\in L^1(\Omega\cap B_{2\rho})$, and to estimate its L^∞-norm we observe that, due to (3.10),

$$\mathrm{dist}(\Phi^{-1}(\Omega\cap B_{2\rho}), B^c_{3\rho}) > \theta_{\Phi,\rho} > 0.$$

Then, using the s-Hölder regularity of $\delta^s\circ\Phi$ and u_1, and recalling that $\Phi^{-1}\in \mathrm{Lip}(\mathbb{R}^N)$ and (2.8), we obtain, for all $X\in\Phi^{-1}(\Omega\cap B_{2\rho})$,

$$|f_2(X)| \leq C_{\Phi,\rho}\int_{B^c_{3\rho}} \frac{|X-Y|^{s(p-1)}}{|X-Y|^{N+ps}}\, dY \leq C_{\Phi,\rho}\int_{\mathbb{R}^N}\frac{1}{(1+|Y|)^{N+s}}\, dY \leq C_{\Phi,\rho}.$$

Regarding $f_{1,\varepsilon}$, it coincides with the g_ε of (3.6). Therefore claim (3.5) of Lemma 3.4 shows that the limit

$$f_1(X) := \lim_{\varepsilon\to 0}\int_{B^c_\varepsilon(X)} \frac{(u_1(X_N)-u_1(Y_N))^{p-1}}{|\Phi(X)-\Phi(Y)|^{N+ps}} J_\Phi(Y)\, dY$$

holds in $L^1_{\rm loc}(\mathbb{R}^N_+)$, and $\|f_1\|_\infty \leq C_{\Phi,\rho}$. Therefore $g_\varepsilon \to f_1\circ\Phi^{-1} + f_2\circ\Phi^{-1}$ in $L^1_{\rm loc}(\Omega\cap B_{2\rho})$, and both are bounded. Lemma 2.5 finally gives the conclusion. □

4. Barriers

In this section we provide some barrier-type functions and prove *a priori* bounds for the bounded weak solutions of problem (1.1). We begin by considering the simple problem

$$\begin{cases} (-\Delta)_p^s v = 1 & \text{in } B_1, \\ v = 0 & \text{in } B_1^c. \end{cases} \tag{4.1}$$

The following lemma displays some properties of the solution of (4.1):

Lemma 4.1. *Let $v \in W_0^{s,p}(B_1)$ be a weak solution of (4.1). Then, $v \in L^\infty(\mathbb{R}^N)$ is unique, radially non-increasing, and for all $r \in (0,1)$ it holds $\inf_{B_r} v > 0$.*

Proof. First we prove uniqueness. Let the functional $J\colon W_0^{s,p}(B_1) \to \mathbb{R}$ be defined by

$$J(u) = \frac{1}{p}\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy - \int_{B_1} u(x)\,dx.$$

The functional J is strictly convex and coercive, hence it admits a unique global minimizer $v \in W_0^{s,p}(B_1)$, which is the only weak solution of (4.1). By (iii) in Lemma 2.9, we see that v is radially symmetric, that is, $v(x) = \psi(|x|)$ for all $x \in \mathbb{R}^N$, where $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a mapping vanishing in $[1,\infty)$. Let $v^\#$ be the symmetric non-increasing rearrangement of v. By the fractional Pólya–Szegö inequality (see Theorem 3 of [2]) we have $J(v^\#) \le J(v)$, so by uniqueness $v = v^\#$, that is, ψ is non-increasing and continuous from the right in $\mathbb{R}_+$. Now let

$$r_0 = \inf\{r \in (0,1] : \psi(r) = 0\}.$$

Clearly $r_0 \in (0,1]$. Arguing by contradiction, assume $r_0 \in (0,1)$. Then $v \in W_0^{s,p}(B_{r_0})$ and it solves weakly

$$\begin{cases} (-\Delta)_p^s v = 1 & \text{in } B_{r_0} \\ v = 0 & \text{in } B_{r_0}^c. \end{cases}$$

Reasoning as above and using uniqueness and Lemma 2.9 (ii), we see that $v(x) = r_0^{-ps} v(r_0^{ps} x)$ in B_{r_0}, so

$$\psi(r_0^2) = r_0^{ps}\psi(r_0) = 0,$$

with $r_0^2 < r_0$, against the definition of r_0. So, for all $r \in (0,1)$ we have

$$\inf_{B_r} v = \psi(r) > 0.$$

Finally, we prove that $v \in L^\infty(\mathbb{R}^N)$. Let $w \in C^s(\mathbb{R}^N) \cap \widetilde{W}^{s,p}(B_1)$ be defined by

$$w(x) = \min\{(2 - x_N)_+^s, 5^s\}.$$

Notice that $w(x) = (2-x_N)_+^s = u_1(2-x_N)$ for all $x \in B_2$. Thus we can apply Lemma 2.8 in $B_{3/2}$ to

$$w(x) = u_1(2-x_N) - (u_1(2-x_N) - 5^s)_+$$

to get, by Lemma 3.2

$$\begin{aligned}(-\Delta)_p^s w(x) &= 2\int_{\{y_N \le -3\}} \frac{((2-x_N)_+^s - 5^s)^{p-1} - ((2-x_N)_+^s - (2-y_N)_+^s)^{p-1}}{|x-y|^{N+ps}}\,dy \\ &=: I(x)\end{aligned}$$

weakly in B_1. The function $I : \bar{B}_1 \to \mathbb{R}$ is continuous and positive, so there exists $\alpha > 0$ such that

$$(-\Delta)_p^s w(x) \ge \alpha \quad \text{weakly in } B_1.$$

We set $\tilde{w} = \alpha^{-1/(p-1)} w$, so we have

$$\begin{cases}(-\Delta)_p^s v = 1 \le (-\Delta)_p^s \tilde{w} & \text{weakly in } B_1 \\ v = 0 \le \tilde{w} & \text{in } B_1^c,\end{cases}$$

and Proposition 2.10 yields

$$0 \le v \le \tilde{w} \le \frac{5^s}{\alpha^{1/(p-1)}}, \quad \text{in } \mathbb{R}^N,$$

so $v \in L^\infty(\mathbb{R}^N)$, concluding the proof. □

Next we introduce *a priori* bounds for functions with bounded fractional p-Laplacian.

Corollary 4.2 (L^∞-bound). *Let $u \in W_0^{s,p}(\Omega)$ satisfy $|(-\Delta)_p^s u| \le K$ weakly in Ω for some $K > 0$. Then*

$$\|u\|_\infty \le (C_d K)^{1/(p-1)},$$

for some $C_d = C(N,p,s,d)$, $d = \operatorname{diam}(\Omega)$.

Proof. Let $v \in W_0^{s,p}(B_1)$ be as in Lemma 4.1, x_0 such that $\Omega \Subset B_d(x_0)$, and set

$$\tilde{v}(x) = (Kd^{ps})^{1/(p-1)}\, v\Big(\frac{x-x_0}{d}\Big).$$

By Lemma 2.9 (i), (ii) we have weakly

$$\begin{cases}(-\Delta)_p^s u \le K = (-\Delta)_p^s \tilde{v} & \text{in } \Omega, \\ u = 0 \le \tilde{v} & \text{in } \Omega^c,\end{cases}$$

which, by Proposition 2.10, implies $u \le \tilde{v}$ in $\mathbb{R}^N$. A similar argument, applied to $-u$, gives the lower bound. □

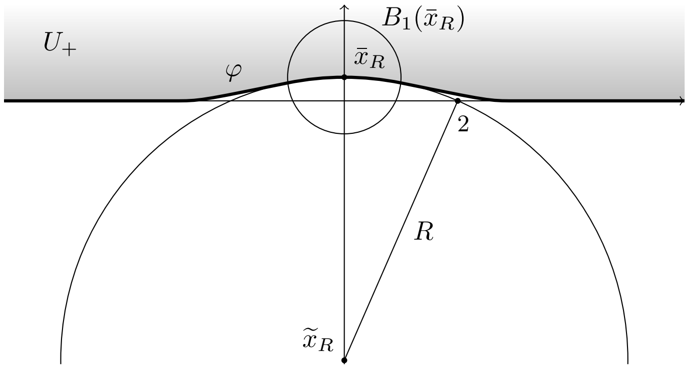

FIGURE 2. The balls $B_R(\widetilde{x}_R)$ and $B_1(\bar{x}_R)$. The thick line is the graph of φ, whose epigraph is U_+.

We can now produce (local) upper barriers on the complements of balls.

Lemma 4.3 (Local upper barrier). *There exist $w \in C^s(\mathbb{R}^N)$, and universal $r > 0$, $a \in (0,1)$, $c > 1$ with*

$$\begin{cases} (-\Delta)_p^s w \geq a & \text{weakly in } B_r(e_N) \setminus \overline{B}_1, \\ c^{-1}(|x|-1)_+^s \leq w(x) \leq c(|x|-1)_+^s & \text{in } \mathbb{R}^N. \end{cases}$$

Proof. By translation, rotation invariance and scaling (Lemma 2.9), it suffices to prove the statement for any fixed ball of radius $R > 2$, at any fixed point $\bar{x}_R$ of its boundary. To fix ideas, we set $\tilde{x}_R = (0, -(R^2-4)^{1/2})$ and $\bar{x}_R = \tilde{x}_R + Re_N$, so that $B_R(\tilde{x}_R)$ intersects the hyperplane $\mathbb{R}^{N-1} \times \{0\}$ in the $(N-1)$-ball $\{|x'| < 2\}$ (we use the notation $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$).

In the following we will choose R large enough, depending only on N, p, s. If $R > 2$, we can find $\varphi \in C^{1,1}(\mathbb{R}^{N-1})$ such that $\|\varphi\|_{C^{1,1}(\mathbb{R}^{N-1})} \leq C/R$ and

$$\varphi(x') = \big((R^2 - |x'|^2)^{1/2} - (R^2-4)^{1/2}\big)_+ \quad \text{for all } |x'| \in [0,1] \cup [3, \infty).$$

We set

$$U_+ = \{x \in \mathbb{R}^N : \varphi(x') < x_N\}$$

(see figure 2). We claim that for any sufficiently large R there exists a diffeomorphism $\Phi \in C^{1,1}(\mathbb{R}^N, \mathbb{R}^N)$ such that $\Phi(0) = \bar{x}_R$, $\Phi = I$ in B_4^c, and

$$\|\Phi - I\|_{C^{1,1}(\mathbb{R}^N,\mathbb{R}^N)} + \|\Phi^{-1} - I\|_{C^{1,1}(\mathbb{R}^N,\mathbb{R}^N)} \leq \frac{C}{R}, \ \Phi(\mathbb{R}^N_+) = U_+. \tag{4.2}$$

Indeed, let $\eta \in C^2(\mathbb{R})$ satisfy $\eta \in [0,1]$, $\eta(0) = 1$, $\operatorname{supp} \eta \subseteq (-1,1)$. Set for all $X = (X', X_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$

$$\Phi(X) = X + \varphi(X')\,\eta(X_N)\,e_N.$$

Then, for sufficiently large R, $\Phi \in C^{1,1}(\mathbb{R}^N, \mathbb{R}^N)$ is a bijection since $\Phi(X_1) = \Phi(X_2)$ implies $X_1' = X_2'$, and the map $t \mapsto t + \varphi(X')\eta(t)$ is increasing whenever

$$\sup_{X \in \mathbb{R}^N} \varphi(X')\,|\eta'(X_N)| = \frac{4\sup_{\mathbb{R}}|\eta'|}{R + \sqrt{R^2 - 4}} < 1.$$

Its inverse mapping satisfies

$$\Phi^{-1}(x) = x - \varphi(x')\,\eta(\Phi^{-1}(x) \cdot e_N)e_N \quad \text{for all } x \in \mathbb{R}^N, \tag{4.3}$$

besides $\Phi(0) = \bar{x}_R$. Moreover, for all $X \in B_4^c$ we have either $|X'| \geq 3$ or $|X_N| \geq 1$, in both cases $\Phi(X) = X$. The $C^{1,1}$-bounds on φ, η and (4.3) yield the required $C^{1,1}$-bounds on $\Phi - I$ and $\Phi^{-1} - I$. Finally, reasoning as above, the monotonicity of $t \mapsto t + \varphi(X')\eta(t)$ implies that $\Phi(\mathbb{R}^N_+) = U_+$, and (4.2) is proved.

Let $v_1(x) = u_1(\Phi^{-1}(x) \cdot e_N)$. Lemma 3.4 ensures that $v_1 \in \widetilde{W}^{s,p}_{\rm loc}(\mathbb{R}^N)$ and

$$(-\Delta)_p^s v_1 = f \quad \text{weakly in } U_+, \text{ with } \|f\|_\infty \leq C/R. \tag{4.4}$$

Define

$$v(x) = \min\{v_1(x), 5^s\},$$

which belongs to $\widetilde{W}^{s,p}(B_4)$. From $\Phi = I$ in B_4^c we infer $\Phi^{-1}(B_4) = B_4$ and thus

$$v_1(x) = u_1(x_N) \quad \text{in } B_4^c, \quad v_1 \leq 4^s \quad \text{in } B_4.$$

Hence

$$v_1(x) - v(x) = (x_N)_+^s - 5^s \quad \text{in } \{x_N \geq 5\}, \quad v_1 - v = 0 \quad \text{in } \{x_N \leq 5\} \Supset B_4.$$

Thus the function $v - v_1$ satisfies conditions (2.14) in B_4, and Lemma 2.8 provides weakly in B_4

$$(-\Delta)_p^s v = (-\Delta)_p^s (v_1 + (v - v_1)) = f + g,$$

where

$$\begin{aligned} g(x) &= 2\int_{B_4^c} \frac{(v_1(x) - v(y))^{p-1} - (v_1(x) - v_1(y))^{p-1}}{|x - y|^{N+ps}}\,dy \\ &\geq 2\int_{\{y_N \geq 5\}} \frac{((x_N)_+^s - 5^s)^{p-1} - ((x_N)_+^s - (y_N)_+^s)^{p-1}}{|x - y|^{N+ps}}\,dy \end{aligned}$$

for any $x \in B_4$. As in the proof of Lemma 4.1, there is a universal $\alpha > 0$ such that $g(x) \geq \alpha$ for all $x \in B_4$, and therefore using (4.4) we have

$$(-\Delta)_p^s v \geq f + g \geq \alpha - \frac{C}{R} \quad \text{weakly in } U_+ \cap B_4.$$

Taking R big enough we thus find $B_2(\bar{x}_R) \Subset B_4$ and

$$(-\Delta)_p^s v \geq \frac{\alpha}{2} > 0, \quad \text{weakly in } U_+ \cap B_2(\bar{x}_R). \tag{4.5}$$

Set, for all $x \in \mathbb{R}^N$,

$$d_R(x) = (|x - \tilde{x}_R| - R)_+.$$

We can estimate v by multiples of d_R^s *globally* from above but only *locally* from below. Indeed, since $v = 0$ in U_+^c, $B_R(\tilde{x}_R) \subset U_+^c$, and $v \in C^s(\mathbb{R}^N)$, there exists $\tilde{c} > 1$ such that

$$v(x) \le \tilde{c}\,\mathrm{dist}(x, U_+^c)^s \le \tilde{c}\, d_R^s(x), \quad \text{for all } x \in \mathbb{R}^N. \tag{4.6}$$

On the other hand, for all $x \in B_1(\bar{x}_R)$ it holds either $x \in B_1(\bar{x}_R) \setminus U_+ \subseteq B_R(\tilde{x}_R)$, in which case $d_R^s(x) = 0 = \tilde{c}v(x)$, or $x \in B_1(\bar{x}_R) \cap U_+ \subseteq B_R^c(\tilde{x}_R)$. In the latter case let $X = (X', X_N)$ be such that $x = \Phi(X)$, $Z = (X', 0)$ and $z = \Phi(Z)$. It holds $|X'| \le 1$ and by the construction of Φ, it follows that $z \in \partial B_R(\tilde{x}_R)$, therefore

$$d_R^s(x) \le |x - z|^s \le \tilde{c}\,|X - Z|^s = \tilde{c}\, X_N^s = \tilde{c}\, v(x).$$

Thus we have (taking $\tilde{c} > 1$ bigger if necessary)

$$v \ge \frac{1}{\tilde{c}}\, d_R^s \quad \text{in } B_1(\bar{x}_R). \tag{4.7}$$

We aim at extending (4.7) to the whole $\mathbb{R}^N$, while retaining (4.5) and (4.6). For any $\varepsilon \in (0, 1/\tilde{c})$, set

$$v_\varepsilon = \max\{v, \varepsilon d_R^s\}.$$

Clearly v_ε satisfies estimates like (4.6) and (4.7) in $\mathbb{R}^N$ with a constant $\tilde{c}_\varepsilon = \max\{\tilde{c} + \varepsilon, \varepsilon^{-1}\}$. Besides $v \le v_\varepsilon \le v + \varepsilon d_R^s$ in $\mathbb{R}^N$, being $\varepsilon < 1/\tilde{c}$, $v_\varepsilon - v = 0$ in $B_1(\bar{x}_R)$. So, by (4.5), Lemma 2.8 and (2.5) (with $M = 5^s$ and $q = p - 1$)

$$\begin{aligned}(-\Delta)_p^s v_\varepsilon(x) &= (-\Delta)_p^s v(x) - 2\int_{B_{1/2}^c(\bar{x}_R)} \frac{(v(x) - v(y))^{p-1} - (v(x) - v_\varepsilon(y))^{p-1}}{|x - y|^{N+ps}}\, dy \\ &\ge \frac{\alpha}{2} - C\int_{B_1^c(\bar{x}_R)} \frac{\max\{\varepsilon d_R^s(y), (\varepsilon d_R^s(y))^{p-1}\}}{|\bar{x}_R - y|^{N+ps}}\, dy \ge \frac{\alpha}{2} - CJ(\varepsilon)\end{aligned}$$

weakly in $B_{1/2}(\bar{x}_R) \cap U_+$ (in the end we have used the inequality $|x-y| \ge 1/2|\bar{x}_R - y|$ for all $x \in B_{1/2}(\bar{x}_R)$, $y \in B_1^c(\bar{x}_R)$). Notice that $J(\varepsilon) \to 0$ as $\varepsilon \to 0^+$ independently of x, thus, for $\varepsilon > 0$ small enough we have

$$(-\Delta)_p^s v_\varepsilon(x) \ge \frac{\alpha}{4} > 0 \text{ weakly in } B_{1/2}(\bar{x}_R) \setminus B_R(\tilde{x}_R).$$

To obtain the barrier of the thesis, we set $w(x) = v_\varepsilon(\tilde{x}_R + Rx)$ and using Lemma 2.9 we reach the conclusion for $r = 1/(2R)$, $a = \alpha/(4R^{ps})$, $c = R^s \max\{\tilde{c} + \varepsilon, \varepsilon^{-1}\}$. □

Finally, we prove that any bounded weak solution of (1.1) can be estimated by means of a multiple of δ^s.

Theorem 4.4. *Let $u \in W_0^{s,p}(\Omega)$ satisfy $|(-\Delta)_p^s u| \le K$ weakly in Ω for some $K > 0$. Then*

$$|u| \le (C_\Omega K)^{1/(p-1)}\, \delta^s \quad a.e. \text{ in } \Omega, \tag{4.8}$$

for some $C_\Omega = C(N, p, s, \Omega)$.

Proof. Considering $u/K^{1/(p-1)}$ and using homogeneity, we can prove (4.8) in the case $K = 1$. Thanks to Corollary 4.2 we may focus on a neighborhood of $\partial\Omega$. Let $\rho > 0$ be as in Lemma 3.5, and let $r \in (0,1)$ be defined in Lemma 4.3. Set

$$U = \Big\{x \in \Omega :\, \delta(x) < r\,\frac{\rho}{2}\Big\},$$

$\bar{x} \in U$ and $x_0 = \Pi(\bar{x}) \in \partial\Omega$ its point of minimal distance from Ω^c. There exists two balls $B_{\rho/2}(x_1)$ and $B_\rho(x_2)$ exteriorly tangent to $\partial\Omega$ at x_0, and (by scaling and translating the supersolution of the previous Lemma 4.3) a function $w \in C^s$ such that

$$(-\Delta)_p^s w \geq a \quad \text{weakly in } B_{r\rho/2}(x_0) \setminus B_{\rho/2}(x_1) \tag{4.9}$$

and

$$c^{-1}\, d^s(x) \leq w(x) \leq c\, d^s(x) \quad \text{in } \mathbb{R}^N. \tag{4.10}$$

where we have set

$$d(x) = \operatorname{dist}(x, B^c_{\rho/2}(x_1)).$$

Notice that the constants in (4.9) (4.10) depend only on ρ, N, p and s, and we will suppose henceforth that $a, r, c^{-1} \in (0,1)$. By Lemma 3.5 it holds

$$d(\bar{x}) = \delta(\bar{x}) = |\bar{x} - x_0|, \tag{4.11}$$

moreover

$$d(x) \geq \theta > 0, \quad \text{in } B^c_\rho(x_2) \setminus B_{r\rho/2}(x_0).$$

for a constant θ which depends only on ρ and r (and thus on Ω alone). Since $\Omega \subseteq B^c_\rho(x_2)$, the latter inequality together with (4.10) provides

$$w \geq c^{-1}\theta^s =: \alpha > 0, \quad \text{in } \Omega \setminus B_{r\rho/2}(x_0). \tag{4.12}$$

We define the open set

$$V = \Omega \cap B_{r\rho/2}(x_0) \subseteq B_{r\rho/2}(x_0) \setminus B_{\rho/2}(x_1),$$

where we will apply the comparison principle. Suppose without loss of generality that in (4.12) $\alpha \in (0,1)$ and let $C_d > 1$ be as in Corollary 4.2. Set

$$M = \frac{1}{\alpha}\Big(\frac{C_d}{a}\Big)^{1/(p-1)}, \quad \bar{w} = Mw.$$

By (4.9) and $C_d/\alpha^{p-1} \geq 1$ we have

$$(-\Delta)_p^s \bar{w} = M^{p-1}(-\Delta)_p^s w \geq \frac{C_d}{\alpha^{p-1}} \geq 1 \geq (-\Delta)_p^s u, \quad \text{weakly in } V.$$

Moreover $u = 0 \leq \bar{w}$ in Ω^c, while (4.12), $a < 1$ and Corollary 4.2 give

$$\bar{w} \geq M\alpha = \Big(\frac{C_d}{a}\Big)^{1/(p-1)} \geq \sup_\Omega u, \quad \text{in } \Omega \setminus B_{r\rho/2}(x_0).$$

Therefore $\bar{w} \geq u$ in the whole V^c, and Proposition 2.10 together with (4.10) yields

$$u(x) \leq \bar{w}(x) \leq cMd^s(x) \quad \text{for a.e. } x \in \mathbb{R}^N.$$

Recalling (4.11) we get

$$u(\bar{x}) \leq c\,M\,d^s(\bar{x}) = c\,M\,\delta^s(\bar{x}) \quad \text{for all } \bar{x} = x_0 - tn_{x_0},\ t \in \Big[0, r\frac{\rho}{2}\Big],$$

where n_{x_0} is the exterior normal to $\partial\Omega$ at x_0, which gives the thesis since cM depends only on $N, p, s,\ \rho,\ r$, and Ω. A similar argument applied to $-u$ yields the lower bound. □

5. Hölder regularity

In this section we will obtain the Hölder regularity of solutions.

5.1. Interior Hölder regularity

We now study the behavior of a weak supersolution in a ball, proving a weak Harnack inequality. Then we will obtain an estimate of the oscillation of a bounded weak solution in a ball (this can be interpreted as a first interior Hölder regularity result). All balls are meant to be centered at 0, as translation invariance of $(-\Delta)_p^s$ allows to extend the results to balls centered at any point.

We begin with a curious Jensen-type inequality:

Lemma 5.1. *Let $E \subset \mathbb{R}^N$ be a set of finite measure and let $u \in L^1(E)$ satisfy*

$$\fint_E u\,dx = 1.$$

Then, for all $r \geq 1$ and $\lambda \geq 0$, it holds

$$\fint_E (u^r - \lambda^r)^{1/r}\,dx \geq 1 - 2^{(r-1)/r}\,\lambda.$$

Proof. Avoiding trivial cases, we assume $r > 1$ and $\lambda > 0$. Set, for all $t \in \mathbb{R}$,

$$g(t) = (t^r - \lambda^r)^{1/r}.$$

Then, for all $t \in \mathbb{R} \setminus \{0, \lambda\}$, we have

$$g'(t) = |t^r - \lambda^r|^{(1-r)/r}\,|t|^{r-1}.$$

In particular, $t_\lambda = 2^{-1/r}\lambda$ is the only solution of $g'(t) = 1$. Elementary calculus shows that for all $t \in \mathbb{R}$

$$g(t) \geq g(t_\lambda) + g'(t_\lambda)(t - t_\lambda) = t - 2t_\lambda.$$

So we have

$$\fint_E (g \circ u)\,dx \geq \fint_E (u - 2t_\lambda)\,dx = 1 - 2^{(r-1)/r}\,\lambda,$$

which concludes the proof. □

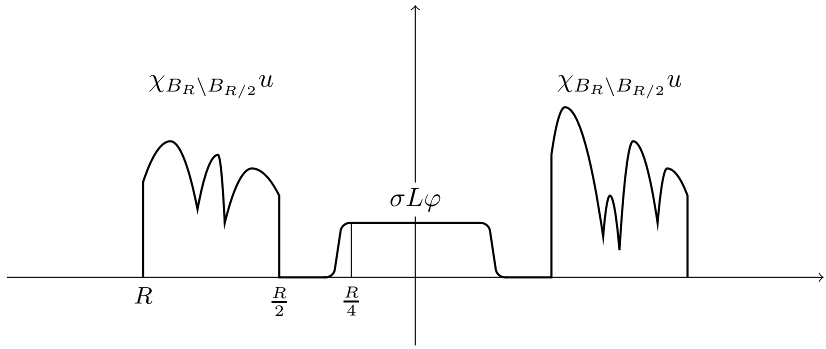

FIGURE 3. The lower barrier w.

Now we prove a weak Harnack-type inequality for non-negative supersolutions:

Theorem 5.2 (Weak Harnack inequality). *There exist universal $\sigma \in (0,1)$, $\bar{C} > 0$ with the following property: if $u \in \widetilde{W}^{s,p}(B_{R/3})$ satisfies weakly*

$$\begin{cases} (-\Delta)_p^s u \geq -K & \text{in } B_{R/3}, \\ u \geq 0 & \text{in } \mathbb{R}^N, \end{cases}$$

for some $K \geq 0$, then

$$\inf_{B_{R/4}} u \geq \sigma\Big(\fint_{B_R\setminus B_{R/2}} u^{p-1}\,dx\Big)^{1/(p-1)} - \bar{C}\,(K\,R^{ps})^{1/(p-1)}.$$

Proof. We first consider the case $p \geq 2$. Let $\varphi \in C^\infty(\mathbb{R}^N)$ be such that $0 \leq \varphi \leq 1$ in $\mathbb{R}^N$, $\varphi = 1$ in $B_{3/4}$, $\varphi = 0$ in B_1^c, and by Proposition 2.12 $|(-\Delta)_p^s\varphi| \leq C_1$ weakly in B_1. We rescale by setting $\varphi_R(x) = \varphi(3x/R)$, so $\varphi_R \in C^\infty(\mathbb{R}^N)$, $0 \leq \varphi_R \leq 1$ in $\mathbb{R}^N$, $\varphi_R = 1$ in $B_{R/4}$, $\varphi_R = 0$ in $B_{R/3}^c$, and $|(-\Delta)_p^s\varphi_R| \leq C_1 R^{-ps}$ weakly in $B_{R/3}$ (taking C_1 bigger).

For all $\sigma \in (0,1)$ we set

$$L = \Big(\fint_{B_R\setminus B_{R/2}} u^{p-1}\,dx\Big)^{1/(p-1)}, \quad w = \sigma L\varphi_R + \chi_{B_R\setminus B_{R/2}}u$$

(see figure 3). So $w \in \widetilde{W}^{s,p}(B_{R/3})$, and by Lemma 2.8 and (2.7) we have, weakly in $B_{R/3}$,

$$\begin{aligned}
&(-\Delta)_p^s w(x) \\
&\quad = (-\Delta)_p^s(\sigma L\varphi_R)(x) + 2\int_{B_R\setminus B_{R/2}} \frac{(\sigma L\varphi_R(x) - u(y))^{p-1} - (\sigma L\varphi_R(x))^{p-1}}{|x-y|^{N+ps}}\,dy \\
&\quad \leq \frac{C_1(\sigma L)^{p-1}}{R^{ps}} - 2^{3-p}\int_{B_R\setminus B_{R/2}} \frac{u^{p-1}(y)}{|x-y|^{N+ps}}\,dy \leq \frac{C_1(\sigma L)^{p-1}}{R^{ps}} - \frac{C_2 L^{p-1}}{R^{ps}}.
\end{aligned}$$

If we assume

$$\sigma < \min\Big\{1, \Big(\frac{C_2}{2C_1}\Big)^{1/(p-1)}\Big\},$$

we get the upper estimate

$$(-\Delta)_p^s w(x) \le -\frac{C_2\, L^{p-1}}{2\, R^{ps}} \quad \text{weakly in } B_{R/3}. \tag{5.1}$$

We set $\bar{C} = (2/C_2)^{1/(p-1)}$ and distinguish two cases:

- if $L \le \bar{C}(KR^{ps})^{1/(p-1)}$, then clearly

$$\inf_{B_{R/4}} u \ge 0 \ge \sigma L - \bar{C}(KR^{ps})^{1/(p-1)};$$

- if $L > \bar{C}(KR^{ps})^{1/(p-1)}$, then we use (5.1) to have

$$\begin{cases} (-\Delta)_p^s w \le -K \le (-\Delta)_p^s u & \text{weakly in } B_{R/3}, \\ w = \chi_{B_R \setminus B_{R/2}} u \le u & \text{in } B_{R/3}^c, \end{cases}$$

  which by Proposition 2.10 implies $w \le u$ in $\mathbb{R}^N$, in particular

$$\inf_{B_{R/4}} u \ge \sigma L.$$

In any case we have

$$\inf_{B_{R/4}} u \ge \sigma L - \bar{C}\,(K\, R^{ps})^{1/(p-1)},$$

which is the conclusion.

Now we consider the case $p \in (1,2)$. Due to Remark 2.14, in this case we cannot use the cut-off function φ as before to construct the barrier w. We use instead the weak solution v of (4.1) introduced in Lemma 4.1, recalling that $\inf_{B_{3/4}} v > 0$, and we set

$$\varphi_R(x) = \Big(\inf_{B_{3/4}} v\Big)^{-1} v\Big(\frac{3x}{R}\Big),$$

so that $0 \le \varphi_R \le \alpha$ (for some universal $\alpha > 0$) in $\mathbb{R}^N$, $\varphi_R \ge 1$ in $B_{R/4}$, $\varphi_R = 0$ in $B_{R/3}^c$, and $(-\Delta)_p^s \varphi_R = C_1 R^{-ps}$ weakly in $B_{R/3}$. Accordingly, to obtain the estimate (5.1) we apply Lemma 5.1 to the function $(u/L)^{p-1}$ with $E = B_R \setminus B_{R/2}$, $r = 1/(p-1)$, and $\lambda = (\sigma\varphi_R(x))^{p-1}$, so that

$$\fint_{B_R \setminus B_{R/2}} \Big(\frac{u(y)}{L} - \sigma\varphi_R(x)\Big)^{p-1} dy \ge 1 - 2^{2-p}(\sigma\varphi_R(x))^{p-1},$$

for a.e. $x \in B_{R/3}$. This, in turn, implies that for a.e. $x \in B_{R/3}$

$$\begin{aligned} &2\int_{B_R \setminus B_{R/2}} \frac{(\sigma L\varphi_R(x) - u(y))^{p-1} - (\sigma L\varphi_R(x))^{p-1}}{|x-y|^{N+ps}}\, dy \\ &\le \frac{C_2}{R^{ps}} \fint_{B_R \setminus B_{R/2}} (\sigma L\varphi_R(x) - u(y))^{p-1}\, dy \\ &\le \frac{C_2}{R^{ps}} \big(2^{2-p}(\sigma L\varphi_R(x))^{p-1} - L^{p-1}\big) \le -\frac{C_2\, L^{p-1}}{2\, R^{ps}}, \end{aligned}$$

where we have chosen $\sigma < 2^{\frac{p-3}{p-1}}\alpha^{-1}$. Then, by taking σ even smaller if necessary, we get (5.1) and the rest of the proof follows *verbatim*. □

We need to extend Theorem 5.2 to supersolutions which are only non-negative in a ball. To do so, we introduce a tail term defined as in (2.1):

Lemma 5.3. *There exist $\sigma \in (0,1)$, $\tilde{C} > 0$, and for all $\varepsilon > 0$ a constant $C_\varepsilon > 0$ with the following property: if $u \in \widetilde{W}^{s,p}(B_{R/3})$ satisfies weakly*

$$\begin{cases} (-\Delta)_p^s u \geq -K & \text{in } B_{R/3}, \\ u \geq 0 & \text{in } B_R, \end{cases}$$

for some $K \geq 0$, then

$$\inf_{B_{R/4}} u \geq \sigma\Big(\fint_{B_R\setminus B_{R/2}} u^{p-1}\,dx\Big)^{1/(p-1)} - \tilde{C}(KR^{ps})^{1/(p-1)} - C_\varepsilon \mathrm{Tail}(u_-;R) - \varepsilon \sup_{B_R} u. \tag{5.2}$$

Proof. First we consider the case $p \geq 2$. We apply Lemma 2.8 to the functions u and $v = u_-$, so that $u + v = u_+$, and $\Omega = B_{R/3}$: we have in a weak sense in $B_{R/3}$

$$\begin{aligned} (-\Delta)_p^s\, u_+(x) &= (-\Delta)_p^s u(x) + 2\int_{B^c_{R/3}} \frac{(u(x)-u(y)-u_-(y))^{p-1} - (u(x)-u(y))^{p-1}}{|x-y|^{N+ps}}\,dy \\ &\geq -K + 2\int_{\{u<0\}} \frac{u(x)^{p-1} - (u(x)-u(y))^{p-1}}{|x-y|^{N+ps}}\,dy \\ &\geq -K + C\int_{\{u<0\}} \frac{u(x)^{p-1} - (u(x)-u(y))^{p-1}}{|y|^{N+ps}}\,dy, \end{aligned}$$

where in the end we have used that $|x-y| \geq 2/3|y|$. By (2.6), for any $\theta > 0$ there exists $C_\theta > 0$ such that weakly in $B_{R/3}$

$$\begin{aligned} (-\Delta)_p^s u_+(x) &\geq -K - \theta\big(\sup_{B_R} u\big)^{p-1}\int_{B^c_R} \frac{1}{|y|^{N+ps}}\,dy - \frac{C_\theta}{R^{ps}}\mathrm{Tail}(u_-;R)^{p-1} \\ &\geq -K - \frac{C\theta}{R^{ps}}\big(\sup_{B_R} u\big)^{p-1} - \frac{C_\theta}{R^{ps}}\mathrm{Tail}(u_-;R)^{p-1} =: -\tilde{K}. \end{aligned}$$

Now, by applying Theorem 5.2 to u_+ we have for any $\varepsilon > 0$ and $\theta < (\varepsilon/\bar{C})^{p-1}$,

$$\begin{aligned} \inf_{B_{R/4}} u &\geq \sigma\Big(\fint_{B_R\setminus B_{R/2}} u^{p-1}\,dx\Big)^{1/(p-1)} - \bar{C}(\tilde{K}R^{ps})^{1/(p-1)} \\ &\geq \sigma\Big(\fint_{B_R\setminus B_{R/2}} u^{p-1}\,dx\Big)^{1/(p-1)} - \tilde{C}(KR^{ps})^{1/(p-1)} - C_\varepsilon \mathrm{Tail}(u_-;R) - \varepsilon \sup_{B_R} u \end{aligned}$$

for some universal constant $\tilde{C} > 0$ and a convenient $C_\varepsilon > 0$ depending also on ε.

Now we turn to the case $p \in (1,2)$. The argument in this case is in fact easier, as by (2.3) we have

$$\int_{\{u<0\}} \frac{u(x)^{p-1} - (u(x)-u(y))^{p-1}}{|y|^{N+ps}}\,dy \le \frac{1}{R^{ps}}\mathrm{Tail}(u_-;R)^{p-1} \quad \text{for a.e. } x \in B_{R/3},$$

then we argue as above using (2.2) instead of (2.7) when required. □

Clearly, symmetric versions of Theorem 5.2 and Lemma 5.3 also hold. Now we use the above results to produce an estimate of the oscillation of a bounded function u such that $(-\Delta)_p^s u$ is locally bounded. We set for all $R > 0$, $x_0 \in \mathbb{R}^N$

$$Q(u;x_0,R) = \|u\|_{L^\infty(B_R(x_0))} + \mathrm{Tail}(u;x_0,R), \quad Q(u;R) = Q(u;0,R).$$

Our result is as follows:

Theorem 5.4. *There exist universal $\alpha \in (0,1)$, $C > 0$ with the following property: if $u \in \widetilde{W}^{s,p}(B_{R_0}) \cap L^\infty(B_{R_0})$ satisfies $|(-\Delta)_p^s u| \le K$ weakly in B_{R_0} for some $R_0 > 0$, then for all $r \in (0,R_0)$,*

$$\operatorname*{osc}_{B_r} u \le C\big[(KR_0^{ps})^{1/(p-1)} + Q(u;R_0)\big]\,\frac{r^\alpha}{R_0^\alpha}.$$

Proof. First we consider the case $p \ge 2$. For all integer $j \ge 0$ we set $R_j = R_0/4^j$, $B_j = B_{R_j}$, and $\frac{1}{2}B_j = B_{R_j/2}$. We put forward the following.

Claim. There exist a universal $\alpha \in (0,1)$ and a real $\lambda > 0$ (depending on all the data), a non-decreasing sequence (m_j), and a non-increasing sequence (M_j), such that, for all $j \ge 0$,

$$m_j \le \inf_{B_j} u \le \sup_{B_j} u \le M_j, \quad M_j - m_j = \lambda R_j^\alpha.$$

We argue by induction on j. Step zero: we set $M_0 = \sup_{B_0} u$ and $m_0 = M_0 - \lambda R_0^\alpha$, where $\lambda > 0$ satisfies

$$\lambda \ge \frac{2\,\|u\|_{L^\infty(B_0)}}{R_0^\alpha}, \tag{5.3}$$

which clearly implies

$$\inf_{B_0} u \ge m_0.$$

Inductive step: assume that sequences (m_j), (M_j) are constructed up to the index j. Then

$$\begin{aligned} M_j - m_j &= \fint_{B_j\setminus\frac{1}{2}B_j} (M_j - u)\,dx + \fint_{B_j\setminus\frac{1}{2}B_j} (u - m_j)\,dx \\ &\le \Big(\fint_{B_j\setminus\frac{1}{2}B_j} (M_j-u)^{p-1}\,dx\Big)^{1/(p-1)} + \Big(\fint_{B_j\setminus\frac{1}{2}B_j} (u-m_j)^{p-1}\,dx\Big)^{1/(p-1)}. \end{aligned} \tag{5.4}$$

Let $\sigma \in (0,1)$, $\tilde{C} > 0$ be as in Lemma 5.3, and multiply the previous inequality by σ to obtain, via (5.2),

$$
\begin{aligned}
\sigma(M_j - m_j) \le \inf_{B_{j+1}} (M_j - u) + \inf_{B_{j+1}} (u - m_j) + 2\tilde{C}\,(K\,R_0^{ps})^{1/(p-1)} \\
+ C_\varepsilon\big[\mathrm{Tail}((M_j - u)_-; R_j) + \mathrm{Tail}((u - m_j)_-; R_j)\big] \\
+ \varepsilon\Big[\sup_{B_j}(M_j - u) + \sup_{B_j}(u - m_j)\Big].
\end{aligned} \tag{5.5}
$$

Setting universally $\varepsilon = \sigma/4$, $C = \max\{2\tilde{C}, C_\varepsilon\}$ and rearranging, we have

$$
\begin{aligned}
\operatorname*{osc}_{B_{j+1}} u \le \Big(1 - \frac{\sigma}{2}\Big)(M_j - m_j) \\
+ C\,\big[(KR_0^{ps})^{1/(p-1)} + \mathrm{Tail}((M_j - u)_-; R_j) + \mathrm{Tail}((u - m_j)_-; R_j)\big].
\end{aligned} \tag{5.6}
$$

Now we provide an estimate of both non-local tails, firstly noting that

$$
\begin{aligned}
\mathrm{Tail}((u - m_j)_-; R_j)^{p-1} = R_j^{ps} \sum_{k=0}^{j-1} \int_{B_k \setminus B_{k+1}} \frac{(u(y) - m_j)_-^{p-1}}{|y|^{N+ps}}\,dy \\
+ R_j^{ps} \int_{B_0^c} \frac{(u(y) - m_j)_-^{p-1}}{|y|^{N+ps}}\,dy.
\end{aligned} \tag{5.7}
$$

We consider the first term: by the inductive hypothesis, for all $0 \le k \le j-1$ we have, in $B_k \setminus B_{k+1}$,

$$
(u - m_j)_- \le m_j - m_k \le (m_j - M_j) + (M_k - m_k) = \lambda(R_k^\alpha - R_j^\alpha),
$$

hence

$$
\begin{aligned}
\sum_{k=0}^{j-1} \int_{B_k \setminus B_{k+1}} \frac{(u(y) - m_j)_-^{p-1}}{|y|^{N+ps}}\,dy &\le \lambda^{p-1} R_j^{\alpha(p-1)} \sum_{k=0}^{j-1} \int_{B_k \setminus B_{k+1}} \frac{(4^{\alpha(j-k)} - 1)^{p-1}}{|y|^{N+ps}}\,dy \\
&\le C\,\lambda^{p-1}\,S(\alpha)\,R_j^{\alpha(p-1)-ps},
\end{aligned}
$$

where we have set, for all $\alpha \in (0,1)$,

$$
S(\alpha) = \sum_{h=1}^{\infty} \frac{(4^{\alpha h} - 1)^{p-1}}{4^{psh}},
$$

noting that $S(\alpha) \to 0$ as $\alpha \to 0^+$. Regarding the second term, by the inductive hypothesis we have

$$
m_j \le \inf_{B_j} u \le \sup_{B_j} u \le \|u\|_{L^\infty(B_0)},
$$

hence

$$
\int_{B_0^c} \frac{(u(y) - m_j)_-^{p-1}}{|y|^{N+ps}}\,dy \le \int_{B_0^c} \frac{(\|u\|_{L^\infty(B_0)} + |u(y)|)^{p-1}}{|y|^{N+ps}}\,dy \le \frac{C\,Q(u; R_0)^{p-1}}{R_0^{ps}}.
$$

Choosing $\alpha < ps/(p-1)$ and plugging the above inequalities in (5.7), we get

$$\begin{aligned}\mathrm{Tail}((u-m_j)_-;R_j) &\le C\left[\lambda^{p-1}\,S(\alpha)\,R_j^{\alpha(p-1)} + \frac{Q(u;R_0)^{p-1}R_j^{ps}}{R_0^{ps}}\right]^{1/(p-1)}\\ &\le C\left[\lambda\,S(\alpha)^{1/(p-1)} + \frac{Q(u;R_0)}{R_0^{\alpha}}\right]R_j^{\alpha}.\end{aligned}$$

An analogous estimate holds for $\mathrm{Tail}((M_j-u)_-;R_j)$, so from (5.6) we have

$$\begin{aligned}\operatorname*{osc}_{B_{j+1}} u &\le \Big(1-\frac{\sigma}{2}\Big)\lambda R_j^{\alpha} + C\Big[(KR_0^{ps})^{1/(p-1)} + \lambda S(\alpha)^{1/(p-1)}R_j^{\alpha} + \frac{Q(u;R_0)R_j^{\alpha}}{R_0^{\alpha}}\Big]\\ &\le 4^{\alpha}\Big[\Big(1-\frac{\sigma}{2}\Big) + CS(\alpha)^{\frac{1}{p-1}}\Big]\lambda R_{j+1}^{\alpha} + 4^{\alpha}C\Big[K^{\frac{1}{p-1}}R_0^{\frac{ps}{p-1}-\alpha} + \frac{Q(u;R_0)}{R_0^{\alpha}}\Big]R_{j+1}^{\alpha}.\end{aligned} \tag{5.8}$$

Now we choose $\alpha \in (0, ps/(p-1))$ universally such that

$$4^{\alpha}\Big[\Big(1-\frac{\sigma}{2}\Big) + CS(\alpha)^{1/(p-1)}\Big] \le 1-\frac{\sigma}{4},$$

and we set

$$\lambda = \frac{4^{\alpha+1}}{\sigma}\,C\left[K^{1/(p-1)}\,R_0^{ps/(p-1)-\alpha} + \frac{Q(u;R_0)}{R_0^{\alpha}}\right], \tag{5.9}$$

which implies (5.3) as $4^{\alpha+1}C/\sigma > 2$. So, (5.8) forces

$$\operatorname*{osc}_{B_{j+1}} u \le \lambda\,R_{j+1}^{\alpha}.$$

We may pick m_{j+1}, M_{j+1} such that

$$m_j \le m_{j+1} \le \inf_{B_{j+1}} u \le \sup_{B_{j+1}} u \le M_{j+1} \le M_j,\ \ M_{j+1}-m_{j+1} = \lambda R_{j+1}^{\alpha},$$

which completes the induction and proves the claim.

Now fix $r \in (0, R_0)$ and find an integer $j \ge 0$ such that $R_{j+1} \le r < R_j$, thus $R_j \le 4r$. Hence, by the claim and (5.9), we have

$$\operatorname*{osc}_{B_r} u \le \operatorname*{osc}_{B_j} u \le \lambda\,R_j^{\alpha} \le C\,[(K\,R_0^{ps})^{1/(p-1)} + Q(u;R_0)]\,\frac{r^{\alpha}}{R_0^{\alpha}},$$

which concludes the argument.

Now we consider the case $p \in (1,2)$. The only major difference is in (5.6): instead of (5.4) we use the inductive hypothesis to see that

$$M_j - u \le (M_j-m_j)^{2-p}(M_j-u)^{p-1}, \quad \text{in } B_j,$$

and similarly for $u - m_j$. Hence

$$M_j - m_j \le (M_j-m_j)^{2-p}\Big[\fint_{B_j\setminus\frac12 B_j}(M_j-u)^{p-1}\,dx + \fint_{B_j\setminus\frac12 B_j}(u-m_j)^{p-1}\,dx\Big],$$

which in turn implies through (2.2)

$$M_j - m_j \leq \Big[\fint_{B_j\setminus\frac{1}{2}B_j} (M_j-u)^{p-1}\,dx + \fint_{B_j\setminus\frac{1}{2}B_j} (u-m_j)^{p-1}\,dx\Big]^{1/(p-1)}$$
$$\leq 2^{\frac{2-p}{p-1}}\Big[\Big(\fint_{B_j\setminus\frac{1}{2}B_j} (M_j-u)^{p-1}\,dx\Big)^{1/(p-1)} + \Big(\fint_{B_j\setminus\frac{1}{2}B_j} (u-m_j)^{p-1}\,dx\Big)^{1/(p-1)}\Big].$$

Multiplying by $\sigma/2^{(2-p)/(p-1)}$ and applying Lemma 5.3 we obtain (5.5) with $\tilde{\sigma} = \sigma/2^{(2-p)/(p-1)}$, and the proof follows *verbatim*. □

The next corollary of Theorem 5.4 follows from standard arguments.

Corollary 5.5. *There exists universal $C > 0$ and $\alpha \in (0,1)$ with the following property: for all $u \in \widetilde{W}^{s,p}(B_{2R_0}(x_0)) \cap L^\infty(B_{2R_0}(x_0))$ such that $|(-\Delta)_p^s u| \leq K$ weakly in $B_{2R_0}(x_0)$,*

$$[u]_{C^\alpha(B_{R_0}(x_0))} \leq C\left[(K\,R_0^{ps})^{1/(p-1)} + Q(u;x_0,2R_0)\right] R_0^{-\alpha}. \tag{5.10}$$

Proof. Given x, y in $B_{R_0}(x_0)$, let $r = |x-y|$. It suffices to apply Theorem 5.4 to the ball $B_{R_0}(x) \subseteq B_{2R_0}(x_0)$. Clearly $\|u\|_{L^\infty(B_{R_0}(x))} \leq \|u\|_{L^\infty(B_{2R_0}(x_0))}$ and

$$\mathrm{Tail}(u;x,R_0)^{p-1} = R_0^{ps}\int_{B^c_{R_0}(x)} \frac{|u(y)|^{p-1}}{|x-y|^{N+ps}}\,dy$$
$$\leq CR_0^{ps}\Big[\int_{B_{2R_0}(x_0)\setminus B_{R_0}(x)} \frac{\|u\|^{p-1}_{L^\infty(B_{2R_0}(x_0))}}{|x-y|^{N+ps}}\,dy + \int_{B^c_{2R_0}(x_0)} \frac{|u(y)|^{p-1}}{|x-y|^{N+ps}}\,dy\Big]$$
$$\leq C\|u\|^{p-1}_{L^\infty(B_{2R_0}(x_0))} + CR_0^{ps}\int_{B^c_{2R_0}(x_0)} \frac{|u(y)|^{p-1}}{|x_0-y|^{N+ps}}\,dy$$

for a universal C, where as usual we used $|x-y| \geq |x_0-y|/2$ for $y \in B^c_{2R_0}(x_0)$, $x \in B_{R_0}(x_0)$. This implies that

$$Q(u;x,R_0) \leq CQ(u;x_0,2R_0),$$

and thus the desired estimate on the Hölder seminorm. □

5.2. Global Hölder regularity

We finally prove the stated Hölder regularity result up to the boundary.

Proof of Theorem 1.1. We set $K = \|f\|_{L^\infty(\Omega)}$. Corollary 4.2 already provides the desired estimate for the sup-norm, namely

$$\|u\|_{L^\infty(\Omega)} \leq C\,K^{1/(p-1)},$$

so we can focus on the Hölder seminorm.

Let α be the one given in Corollary 5.5. We can assume $\alpha \in (0, s]$. Through a covering argument, (5.10) implies that $u \in C^{\alpha}_{\rm loc}(\overline{\Omega}')$ for all $\Omega' \Subset \Omega$, with a bound of the form

$$\|u\|_{C^\alpha(\overline{\Omega}')} \leq C_{\Omega'}\, K^{1/(p-1)}, \quad C_{\Omega'} = C(N, p, s, \Omega, \Omega').$$

Hence it suffices to prove (1.2) in the closure of a fixed ρ-neighbourhood of $\partial\Omega$. We will suppose that $\rho > 0$ is so small (depending only on Ω) that Lemma 3.5 holds, and thus the metric projection

$$\Pi : V \to \partial\Omega, \quad \Pi(x) = \underset{y \in \Omega^c}{\mathrm{Argmin}} |x - y|$$

is well defined on $V := \{x \in \overline{\Omega} : \delta(x) \leq \rho\}$. We claim that

$$[u]_{C^\alpha(B_{r/2}(x))} \leq C_\Omega\, K^{1/(p-1)}, \quad \text{for all } x \in V \text{ and } r = \delta(x) \tag{5.11}$$

for some constant $C_\Omega = C(N, p, s, \Omega)$, independent on $x \in V$. We recall (5.10), which in the present case rephrases (up to a universal constant) as

$$[u]_{C^\alpha(B_{r/2}(x))} \leq C\left[(K\, r^{ps})^{1/(p-1)} + \|u\|_{L^\infty(B_r(x))} + \mathrm{Tail}(u; x, r)\right] r^{-\alpha}.$$

To prove (5.11), it suffices to bound the three terms on the right hand side of the above inequality. The first one it trivially dealt with since $\alpha \leq s \leq ps/(p-1)$, and thus

$$r^{-\alpha}\,(K r^{ps})^{1/(p-1)} \leq K^{1/(p-1)}\, \rho^{ps/(p-1)-\alpha}.$$

For the second one we use Theorem 4.4 and $\alpha \leq s$ to get

$$\|u\|_{L^\infty(B_r(x))} \leq C\, K^{1/(p-1)}\,(\delta(x) + r)^s \leq C\, K^{1/(p-1)}\, \rho^{s-\alpha}\, r^\alpha,$$

and thus the claimed bound. Similarly for the last term we employ again (4.8), together with

$$\delta(y) \leq |y - \Pi(x)| \leq |y - x| + |x - \Pi(x)| \leq |y - x| + r \leq 2|x - y|, \quad \forall y \in B_r^c(x),$$

to get

$$\begin{aligned}\mathrm{Tail}(u; x, r)^{p-1} &\leq C\, K\, r^{ps} \int_{B_r^c(x)} \frac{\delta^{s(p-1)}(y)}{|x - y|^{N+ps}}\, dy \\ &\leq C\, K\, r^{ps} \int_{B_r^c(x)} \frac{|x - y|^{s(p-1)}}{|x - y|^{N+ps}}\, dy \leq C\, K\, r^{s(p-1)}.\end{aligned}$$

Again due to $\alpha \leq s$ we obtain the claimed bound, and the proof of (5.11) is completed. To prove the theorem, pick $x, y \in V$ and suppose without loss of generality that $|x - \Pi(x)| \geq |y - \Pi(y)|$. Two cases may occur:

- either $2|x - y| < |x - \Pi(x)|$, in which case we set $r = \delta(x)$ and apply (5.11) in $B_{r/2}(x)$, to obtain

$$|u(x) - u(y)| \leq C\, K^{1/(p-1)}\, |x - y|^\alpha;$$

- or $2|x-y| \geq |x-\Pi(x)| \geq |y-\Pi(y)|$, in which case (4.8) ensures

$$\begin{aligned} |u(x)-u(y)| &\leq |u(x)|+|u(y)| \leq C\, K^{1/(p-1)}\,(\delta^s(x)+\delta^s(y)) \\ &= C\, K^{1/(p-1)}\,(|x-\Pi(x)|^s+|y-\Pi(y)|^s) \\ &\leq C\, K^{1/(p-1)}\,|x-y|^s \leq C\, K^{1/(p-1)}\,\rho^{s-\alpha}\,|x-y|^\alpha. \end{aligned}$$

Thus in both cases the α-Hölder seminorm is bounded in V and the proof is completed. □

Remark 5.6. As the proofs above show, interior regularity (Theorem 5.4) forces in particular $\alpha < ps/(p-1)$, while in order to control the behavior of weak solutions near the boundary we need the more restrictive bound $\alpha \leq s$. Anyway, our Hölder exponent remains not explicitly determined.

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Received December 11, 2014; revised September 14, 2015.

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The first and second authors were supported by GNAMPA project “Problemi al contorno per operatori non locali non lineari”.