Rev. Mat. Iberoam. **33** (2017), no. 1, 1–28
DOI 10.4171/RMI/926

Connectivity by geodesics on globally hyperbolic spacetimes with a lightlike Killing vector field

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Abstract. Taking a globally hyperbolic spacetime endowed with a complete lightlike Killing vector field and a complete Cauchy hypersurface, we characterize the points which can be connected by geodesics. A straightforward consequence is the geodesic connectedness of globally hyperbolic generalized plane waves with a complete Cauchy hypersurface.

1. Introduction

During the past years there has been a considerable amount of research related to the problem of geodesic connectedness of Lorentzian manifolds (cf. the classical books [5] and [27], the updated survey [12], and references therein). This topic has wide applications in Physics, but for mathematicians its interest is essentially due to the peculiar difficulty of this natural problem, which makes it challenging from both an analytical and a geometrical point of view. In particular, a striking difference with the Riemannian realm is that no analogous to the Hopf–Rinow theorem holds (for a counterexample, cf. Remark 1.14 in [29], or also p. 150 and Example 7.16 in [27]). Thus, up to now, sufficient conditions for geodesic connectedness have been established only for a few models of Lorentzian spacetimes.

The ideas in the paper [11] led to the following result (cf. Theorem 1.1 in [11]).

Theorem 1.1 (Candela–Flores–Sánchez). *Let $(\mathcal{L}, \langle\cdot,\cdot\rangle_L)$ be a stationary spacetime with a complete timelike Killing vector field K. If $\mathcal{L}$ is globally hyperbolic with a complete* (*smooth, spacelike*) *Cauchy hypersurface S, then it is geodesically connected.*

The interest of this theorem does not only rely on the intrinsic geometric character and accuracy of its hypotheses (cf. Section 6.3 in [11]), but also on the fact

Mathematics Subject Classification (2010): 53C50, 53C22, 58E10.
Keywords: Lightlike vector field, global hyperbolicity, geodesic connectedness, Killing vector field, Cauchy hypersurface, stationary spacetime, gravitational wave, generalized plane wave.

that it is the top result of a series of works on geodesic connectedness for standard stationary spacetimes (cf. [2], [6], [19], [20], and [30]). If one analyzes the extrinsic hypotheses under which standard stationary spacetimes become globally hyperbolic (cf. Corollary 3.4 in [31]) and the ones under which they become geodesically connected (for instance, Theorem 1.2 in [2]), one realizes that the former imply the latter. So, it was natural to wonder if global hyperbolicity implies geodesic connectedness for stationary spacetimes, as Theorem 1.1 finally confirms.

Now observe that Theorem 1.1 admits a natural limit case, which consists of assuming the existence of a lightlike, instead of timelike, Killing vector field. A remarkable family of spacetimes which falls under this hypothesis is the class of generalized plane waves. The geodesic connectedness and global hyperbolicity of these spacetimes have been also studied. In this case, one also finds that the extrinsic hypotheses which ensure global hyperbolicity (see Theorem 4.1 in [17]) imply geodesic connectedness (see Corollary 4.5 in [10]). So, a natural question is if Theorem 1.1 still holds when the Killing vector field K is lightlike, instead of timelike; i.e.,

> *taking any globally hyperbolic spacetime endowed with a complete lightlike Killing vector field and a complete* (*smooth, spacelike*) *Cauchy hypersurface, is it geodesically connected?*

In general, the answer to this question is negative (see Section 7 (c)); however, we can characterize which points can be connected by geodesics in this class of spacetimes. More precisely, setting

$$C^1_K(p,q) = \{z \in C^1(I,\mathcal{L}) : z(0)=p,\ z(1)=q, \text{ and } \exists\, C_z \in \mathbb{R} \text{ such that } \langle \dot z, K(z)\rangle_L \equiv C_z\}, \tag{1.1}$$

here we prove the following statement:

Theorem 1.2. *Let $(\mathcal{L},\langle\cdot,\cdot\rangle_L)$ be a globally hyperbolic spacetime endowed with a complete lightlike Killing vector field K and a complete* (*smooth, spacelike*) *Cauchy hypersurface S. Given two points $p,q\in\mathcal{L}$, the following statements are equivalent:*

(i) *p and q are geodesically connected in $\mathcal{L}$;*

(ii) $C^1_K(p,q) \neq \emptyset$.

Alike Theorem 1.1, this result is intrinsic, sharp and natural. Moreover, it presents nice consistency with previous results on geodesic connectedness for generalized plane waves. The proof is based on a limit argument. First, one perturbs the metric of the spacetime into a sequence of standard stationary metrics which approach to the original one. Given two points, one uses an adapted version of Theorem 1.1 to ensure that they are geodesically connected for sufficiently advanced metrics of the sequence. Then, one uses property (ii) to provide some estimates on the sequence of connecting geodesics. Finally, a thorough limit argument based on these estimates ensures the existence of a limit connecting geodesic for the original metric.

Besides the geodesic connectedness, other geodesic properties of stationary spacetimes have been studied in the last decades. Theorem 1.2 suggests a line of research consisting of translating geodesic properties, from stationary spacetimes to spacetimes with a lightlike Killing vector field, by using a limit argument similar to the one developed below (see also [14]). The fine estimates needed to overcome this procedure for the geodesic connectedness problem, and the fact that this property is only partially preserved when passing to the limit, suggest that, in general, this line of research will be an interesting mathematical challenge. According to these ideas, here we are able to state also a multiplicity result in the spirit of Theorem 4.27 in [12] (cf. Subsection 8.2), while an application of Theorem 1.2 to the case with boundary can be found in [3].

The remainder of this paper is organized as follows. In Section 2 we recall some notations, definitions and background tools on Lorentzian manifolds, especially on standard stationary spacetimes. In Section 3 we explain the main arguments involved in the intrinsic variational approach to the geodesic connectedness problem in a stationary spacetime, when a global splitting is not given a priori. The machinery developed in Section 3 is used in Section 4 to prove Theorem 4.2, an adapted version of Theorem 1.1. In Section 5 we apply Theorem 4.2 to a sequence of standard stationary spacetimes obtained by perturbing the original metric. As a consequence, fixed two arbitrary points, a sequence of connecting geodesics of the perturbed metrics is obtained (Proposition 5.1). Then, in Section 6 we deduce some estimates for these geodesics (Lemmas 6.1 and 6.2) and apply a limit argument to them (Lemma 6.3) in order to prove Theorem 1.2. The accuracy of the hypotheses of Theorem 1.2 is showed in Section 7. Finally, in Section 8, we provide some straightforward applications of Theorem 1.2, such as the Avez–Seifert result in this ambient (Proposition 8.1), a multiplicity result (Theorem 8.4) and the geodesic connectedness of some generalized plane waves (Theorem 8.6).

2. Notation and background tools

In this section we review some basic notions in Lorentzian geometry used throughout the paper (we refer to [5] and [27] for more details).

A *Lorentzian manifold* $(\mathcal{L}, \langle\cdot,\cdot\rangle_L)$ (henceforth often simply denoted by $\mathcal{L}$) is a smooth (connected) finite dimensional manifold $\mathcal{L}$ equipped with a symmetric non-degenerate tensor field $\langle\cdot,\cdot\rangle_L$ of type $(0,2)$ with index 1. A tangent vector $\zeta \in T_z\mathcal{L}$ is called *timelike* (resp. *lightlike*; *spacelike*; *causal*) if $\langle\zeta,\zeta\rangle_L < 0$ (resp. $\langle\zeta,\zeta\rangle_L = 0$ and $\zeta \neq 0$; $\langle\zeta,\zeta\rangle_L > 0$ or $\zeta = 0$; ζ is either timelike or lightlike). The set of causal vectors at each tangent space has a structure of "double cone" called *causal cones*.

A C^1 curve $\gamma\colon I \to \mathcal{L}$ (I real interval) is called *timelike* (resp. *lightlike*; *spacelike*; *causal*) when so is $\dot\gamma(s)$ for all $s \in I$. For causal curves, the definition is extended to include piecewise C^1 curves: in this case, the two limit tangent vectors on the breaks must belong to the same causal cone.

A smooth curve $\gamma\colon I \to \mathcal{L}$ is a *geodesic* if it satisfies the equation

$$D_s^L \dot\gamma = 0,$$

where D_s^L is the covariant derivative along γ associated to the Levi-Civita connection of metric $\langle\cdot,\cdot\rangle_L$. Any geodesic γ satisfies the conservation law

$$\langle\dot\gamma(s),\dot\gamma(s)\rangle_L \equiv E_\gamma \quad \text{for some constant } E_\gamma \in \mathbb{R} \text{ and all } s \in I.$$

So, its causal character can be directly rewritten in terms of the sign of E_γ. Two points p, $q \in \mathcal{L}$ are *geodesically connected* if there exists a geodesic $\gamma\colon I \to \mathcal{L}$ such that $\gamma(0) = p$ and $\gamma(1) = q$ (hereafter, $I := [0,1]$). This property is equivalent to a variational problem: namely, the existence of a critical point of the *action functional*

$$f(z) = \frac{1}{2}\int_0^1 \langle\dot z,\dot z\rangle_L \,\mathrm{d}s \tag{2.1}$$

in the subset of $C^1(I,\mathcal{L})$ of all the C^1 curves $z\colon I \to \mathcal{L}$ such that $z(0) = p$ and $z(1) = q$.

A vector field K in $\mathcal{L}$ is said *complete* if its integral curves are defined on the whole real line. On the other hand, K is said *Killing* if one of the following equivalent statements holds (cf. Propositions 9.23 and 9.25 in [27]):

(i) the stages of its local flow consist of isometries;

(ii) the Lie derivative of $\langle\cdot,\cdot\rangle_L$ in the direction of K is 0;

(iii) $\langle D_X K, Y\rangle_L = -\langle D_Y K, X\rangle_L$ for all vector fields X, Y on $\mathcal{L}$.

If K is a Killing vector field and $\gamma\colon I \to \mathcal{L}$ is a geodesic, then there exists $C_\gamma \in \mathbb{R}$ such that

$$\langle\dot\gamma(s), K(\gamma(s))\rangle_L \equiv C_\gamma \quad \text{for all } s \in I. \tag{2.2}$$

A *spacetime* is a Lorentzian manifold $\mathcal{L}$ with a prescribed *time-orientation*, that is, a continuous choice of a causal cone at each point of $\mathcal{L}$, called *future cone*, in opposition to the non-chosen one, named *past cone*. A causal curve γ in a spacetime is called *future* or *past directed* depending on the time orientation of the cone determined by $\dot\gamma$ at each point. Given $p, q \in \mathcal{L}$, we say that p is in the causal past of q, and we write $p < q$, if there exists a future-directed causal curve from p to q. Moreover, we denote by $p \le q$ either $p < q$ or $p = q$. For each $p \in \mathcal{L}$, the *causal past* $J^-(p)$ and the *causal future* $J^+(p)$ are defined as

$$J^-(p) = \{q \in \mathcal{L} :\ q \le p\} \quad \text{and} \quad J^+(p) = \{q \in \mathcal{L} :\ p \le q\}.$$

Remark 2.1. The causal relations allow one to extend the space of piecewise C^1 causal curves to the space of (non-necessarily smooth) continuous causal curves, in a way which is appropriate for convergence of curves. Actually, such curves have H^1 regularity (cf. p. 54 in [5], p. 442 in [18], and also Definition 2.1, Remarks 2.2 and A.4 in [11]).

A spacetime is called *stationary* if it admits a timelike Killing vector field. There are several equivalent definitions of global hyperbolicity for a spacetime (cf., e.g., [24]). Here, we adopt the following: a spacetime is *globally hyperbolic* if it contains a *Cauchy surface*, that is, a subset which is crossed exactly once by any inextendible timelike curve. Remarkably, the Cauchy surface can be chosen to be a smooth, spacelike hypersurface (cf. [7]). In general, any inextendible causal curve crosses (possibly, along a segment) a Cauchy surface S; if, in addition, S is spacelike (at least C^1), then it crosses S exactly once (cf. p. 342 in [24]). Another important property of a spacetime $\mathcal{L}$ admitting a Cauchy surface S is that $J^-(p)\cap S$ is compact for every $p \in \mathcal{L}$ (cf. Proposition 6.6.6 in [21]).

Finally, following Chapter 14 in [27], we recall that if A is an *achronal* subset of $\mathcal{L}$ (i.e., no $x, y \in A$ are chronologically related), then the *Cauchy development* $D(A)$ of A is defined as the subset of the points p of $\mathcal{L}$ such that every past/future inextendible causal curve through p meets A.

In this paper we are concerned with globally hyperbolic spacetimes admitting a complete causal Killing vector field. The following proposition, which slightly extends Theorem 2.3 in [11], provides a precise description of the structure of these spacetimes.

Proposition 2.2. *Let $(\mathcal{L}, \langle\cdot,\cdot\rangle_L)$ be a globally hyperbolic spacetime admitting a complete causal Killing vector field K. Then, there exist a Riemannian manifold $(S, \langle\cdot,\cdot\rangle)$, a differentiable vector field δ on S and a differentiable non-negative function β on S such that*

$$\mathcal{L} = S \times \mathbb{R} \quad \textit{and} \quad \langle\zeta,\zeta'\rangle_L = \langle\xi,\xi'\rangle + \langle\delta(x),\xi\rangle\tau' + \langle\delta(x),\xi'\rangle\tau - \beta(x)\tau\tau', \tag{2.3}$$

for all $z = (x,t) \in \mathcal{L}$ and $\zeta = (\xi,\tau), \zeta' = (\xi',\tau') \in T_z\mathcal{L} = T_xS \times \mathbb{R}$.

Furthermore, if K is timelike then β is non-vanishing, i.e., $\beta(x) > 0$ for all $x \in S$; if K is lightlike then $\beta \equiv 0$, δ is non-vanishing and the metric on $\mathcal{L}$ becomes

$$\langle\zeta,\zeta'\rangle_L = \langle\xi,\xi'\rangle + \langle\delta(x),\xi\rangle\tau' + \langle\delta(x),\xi'\rangle\tau, \tag{2.4}$$

for all $z = (x,t) \in \mathcal{L}$ and $\zeta = (\xi,\tau), \zeta' = (\xi',\tau') \in T_z\mathcal{L} = T_xS \times \mathbb{R}$.

Proof. Since $\mathcal{L}$ is a globally hyperbolic spacetime, it admits a spacelike Cauchy hypersurface S which becomes a Riemannian manifold when endowed with the induced metric $\langle\cdot,\cdot\rangle$ from $\langle\cdot,\cdot\rangle_L$. Let us consider the map

$$\Psi : (x,t) \in S \times \mathbb{R} \mapsto \Psi_t(x) \in \mathcal{L},$$

being Ψ the flow of the complete vector field K. Since K is causal, its integral curves are also causal. So, each point of $\mathcal{L}$ is crossed by one integral curve of K, which crosses S at exactly one point. Therefore, Ψ is a diffeomorphism. As K is Killing, the pull-back metric $\Psi^*\langle\cdot,\cdot\rangle_L$ is independent of t. Hence, taking $\beta(x) = -\langle K(z), K(z)\rangle_L$ and denoting by $\delta(x)$ the orthogonal projection of $K(z)$ on T_xS for any $z = (x,t) \in S \times \{t\}$, the metric expression (2.3) follows.

Furthermore, if K is timelike, then β is clearly strictly positive; instead, if K is lightlike, then $\beta \equiv 0$ and δ is non-vanishing (since $K(z)$ cannot be orthogonal to T_xS). □

Remark 2.3. For further use, here we emphasize the following relations, contained in the proof of previous proposition: for any $z = (x, t) \in S \times \mathbb{R}$ we have, up to a diffeomorphism Ψ,

$$\begin{aligned} &K \equiv \partial_t, \quad S \equiv S \times \{0\}, \quad \beta(x) = -\langle K(z), K(z)\rangle_L, \\ &\delta(x) \equiv \text{ orthogonal projection of } K(z) \text{ on } T_xS. \end{aligned}$$

In general, a spacetime as in (2.3) with $\beta(x) > 0$ on S is called *standard stationary*. For this class of spacetimes, $K = \partial_t$ is always a complete timelike Killing vector field. A smooth curve $\gamma = (x, t)$ in a standard stationary spacetime $\mathcal{L}$ is a geodesic if and only if it satisfies the following system of differential equations:

$$\begin{cases} D_s\dot{x} - \dot{t}\, F(x)[\dot{x}] + \ddot{t}\,\delta(x) + \dfrac{1}{2}\,\dot{t}^2\, \nabla\beta(x) = 0, \\ \dfrac{\mathrm{d}}{\mathrm{d}s}\left(\beta(x)\dot{t} - \langle\delta(x), \dot{x}\rangle\right) = 0, \end{cases} \tag{2.5}$$

where D_s denotes the covariant derivative along x associated to the Levi-Civita connection of metric $\langle\cdot,\cdot\rangle$, and $F(x)$ denotes the linear (continuous) operator on T_xS associated to the bilinear form

$$\operatorname{curl}\delta(x)[\xi, \xi'] = \langle(\delta'(x))^T[\xi], \xi'\rangle - \langle\delta'(x)[\xi'], \xi\rangle \quad \text{for all } \xi, \xi' \in T_xS,$$

being $\delta'(x)$ the differential map of $\delta(x)$ and $(\delta'(x))^T$ its transpose (cf., e.g., Appendix A in [4]).

We conclude this section with the following result, which will be used later on in the paper.

Proposition 2.4. *Let $(\mathcal{L}, \langle\cdot,\cdot\rangle_L)$ be a standard stationary spacetime as in (2.3) and $(S, \langle\cdot,\cdot\rangle)$ a complete Riemannian manifold. Given two points $p = (x_p, t_p)$ and $q = (x_q, t_q) \in \mathcal{L}$ satisfying $\Delta_t = t_q - t_p \geq 0$, the following assertions hold:*

(i) *if $J^-(q) \cap (S \times \{t_p\})$ is not closed in $S \times \{t_p\}$ then it is not bounded;*

(ii) *if $J^-(q) \cap (S \times \{t_p\})$ is compact in $S \times \{t_p\}$, then there exists $\varepsilon > 0$ such that, setting $q_\varepsilon = (x_q, t_q + \varepsilon)$, $J^-(q_\varepsilon) \cap (S \times \{t_p\})$ is also compact in $S \times \{t_p\}$.*

Proof. (i) Arguing by contradiction, assume that $J^-(q) \cap (S \times \{t_p\})$ is not closed in $S \times \{t_p\}$ but it is bounded. Then, there exists a sequence $(y_k)_k \subset J^-(q) \cap (S \times \{t_p\})$ converging to some point $y \in S \times \{t_p\}$, but

$$y \notin J^-(q). \tag{2.6}$$

By assumption, for each $k \in \mathbb{N}$ there exists a past inextendible[1] causal curve γ_k departing from q and passing through y_k. Then, Proposition 3.31 in [5] ensures that, up to a subsequence, $(\gamma_k)_k$ converges to a past inextendible causal curve γ

[1] The past inextendible causal curves γ_k can be obtained by prolonging the corresponding causal curves from q to y_k (ensured by condition $y_k \in J^-(q)$) with integral lines of the timelike vector field $-\partial_t$.

departing from q. The projection of γ on S remains in $J^-(q)\cap(S\times\{t_p\})$. In particular, it is contained in a compact set, and so, it admits an exhausting sequence of points which is convergent in $S\times\{t_p\}$. The sequence formed by the corresponding lifted points on γ must be also convergent (note that the temporal components of these points are decreasing). So, the causal (thus, locally Lipschitz) curve γ admits an exhausting sequence of points which is convergent, and thus, γ must be extensible, a contradiction.

(ii) By contradiction, assume the existence of a sequence of points $(q_n)_n$, with $q_n=(x_q,t_q+\varepsilon_n)\in\mathcal{L}$ and $\varepsilon_n\searrow 0$, such that for all $n\in\mathbb{N}$ the set $J^-(q_n)\cap(S\times\{t_p\})$ is not compact in $S\times\{t_p\}$. Since $(S,\langle\cdot,\cdot\rangle)$ is complete, by property (i) the set $J^-(q_n)\cap(S\times\{t_p\})$ cannot be bounded. So, for every $n\in\mathbb{N}$ there exists an unbounded sequence of points $(p^n_k)_k\subset J^-(q_n)\cap(S\times\{t_p\})$, with $p^n_k=(x^n_k,t_p)$. By using a Cantor's diagonal type argument, we construct an unbounded sequence $(p_n)_n$, with $p_n=p^n_{k_n}$, such that $p_n\in J^-(q_n)\cap(S\times\{t_p\})$ for all n. Denote by $\gamma_n=(x_n,t_n)$ a future-directed causal curve joining p_n to q_n, and let $s_n\in I$ be such that $t_n(s_n)=t_p+\varepsilon_n$ for each $n\in\mathbb{N}$. Since the future-directed causal curve $\alpha_n=(x_n,t_n-\varepsilon_n)$ on $[s_n,1]$ joins $z_n=(x_n(s_n),t_p)$ to q, we have that $(z_n)_n$ is contained in the compact set $J^-(q)\cap(S\times\{t_p\})$. Thus, since $(p_n)_n$ is unbounded in $S\times\{t_p\}$, there exists $\overline{s}_n\in[0,s_n]$ such that

$$x_n\,|_{[\overline{s}_n,s_n]}\ \text{ remains bounded and } \operatorname{length}(x_n\,|_{[\overline{s}_n,s_n]})\geq 1\quad\forall n\in\mathbb{N}. \tag{2.7}$$

On the other hand, as $\gamma_n=(x_n,t_n)$ is causal and future-directed, t_n is characterized by $\langle\dot\gamma_n,\dot\gamma_n\rangle_L\leq 0$ and $\dot t_n>0$ on I (recall (2.3)), hence it follows that

$$\dot t_n\geq\frac{\langle\delta(x_n),\dot x_n\rangle}{\beta(x_n)}+\sqrt{\frac{\langle\delta(x_n),\dot x_n\rangle^2}{\beta(x_n)^2}+\frac{\langle\dot x_n,\dot x_n\rangle}{\beta(x_n)}}\quad\text{on } I.$$

By integrating the previous inequality in $[\bar s_n,s_n]$, we deduce

$$\int_{\overline{s}_n}^{s_n}\frac{\langle\delta(x_n),\dot x_n\rangle}{\beta(x_n)}\,\mathrm{d}s+\int_{\overline{s}_n}^{s_n}\sqrt{\frac{\langle\delta(x_n),\dot x_n\rangle^2}{\beta(x_n)^2}+\frac{\langle\dot x_n,\dot x_n\rangle}{\beta(x_n)}}\,\mathrm{d}s\leq\int_{\overline{s}_n}^{s_n}\dot t_n\,\mathrm{d}s\leq\varepsilon_n\to 0,$$

as $n\to+\infty$. However, by virtue of (2.7), the first member of the previous expression remains positive and far from zero, a contradiction. □

Remark 2.5. It is conceivable that $J^-(q)\cap(S\times\{t_p\})$ is not closed under the hypotheses of the previous proposition. In fact, this type of property holds just in causally simple spacetimes. So, in order to find a counterexample, one should take into account the characterization of causally simple spacetimes in [13] plus the examples of Randers metrics in [16], in order to construct a suitable non-causally simple spacetime with a complete slice S (this is a remarkable difference with the static case, where the completeness of S implies global hyperbolicity).

About the part (ii) of the theorem, notice that this is intuitively related with the causal continuity of the spacetime. This property holds for every standard stationary spacetime (see [22]).

3. Stationary intrinsic functional framework

A considerable contribution to the study of the geodesic connectedness of spacetimes was given in [19]. In that paper the authors introduced a variational principle for geodesics, based on the natural constraint (2.2), and proved the geodesic connectedness of standard stationary spacetimes $\mathcal{L}$, under some boundedness assumptions for the metric coefficients $|\delta|$ and β (recall (2.3)). Under the hypotheses of Theorem 1.1, the spacetime $\mathcal{L}$ globally splits into (2.3), and previous result can be applied. However, this splitting is neither unique nor canonically associated to $\mathcal{L}$, and the conclusion may depend on it. In order to avoid this arbitrariness, an intrinsic approach to the problem of geodesic connectedness was developed in [20]. There, the variational principle in [19] is translated into a splitting independent form, and a compactness assumption on the infinite dimensional manifold of the paths between two points is introduced, called *pseudo-coercivity* (see from Theorem 3.1 till the end of this section). This condition implies global hyperbolicity, but, in the practice, it is quite difficult to verify. Motivated by this deficiency, in [11] the authors worked under intrinsic geometric assumptions, which involve the causal structure of the spacetime and are shown to be equivalent to pseudo-coercivity. For a given complete spacelike smooth Cauchy hypersurface S and a given complete timelike Killing vector field K, Proposition 2.2 is applied to obtain the corresponding global splitting. But, even if this splitting is neither unique nor canonically associated to $\mathcal{L}$, the result obtained in [11] is independent of the chosen K and S, and no growth hypotheses on the coefficients of the metric $\langle\cdot,\cdot\rangle_L$ are involved.

As we will see later on, the proof of Theorem 1.2 makes use of Theorem 4.2, a refinement of Theorem 1.1. So, in the rest of this section we are going to recall the intrinsic variational functional framework associated to a stationary spacetime, as developed in [11] and [20].

Let $(\mathcal{L},\langle\cdot,\cdot\rangle_L)$ be a stationary spacetime. As shown in [20], by taking into account the constraint (2.2), the geodesics in $\mathcal{L}$ connecting two fixed points $p,q\in\mathcal{L}$ correspond to critical points of functional f in (2.1) restricted to the set of curves in (1.1).

Since our approach will require dealing with H^1 curves on $\mathcal{L}$, we also introduce the infinite dimensional manifold

$$\Omega(p,q) = \Big\{z : I\to\mathcal{L} :\ z \ \text{ is absolutely continuous,} \\ z(0)=p,\ z(1)=q,\ \int_0^1\langle\dot z,\dot z\rangle_R\,\mathrm{d}s<+\infty\Big\},$$

where $\langle\cdot,\cdot\rangle_R$ is the Riemannian metric canonically associated to K and $\langle\cdot,\cdot\rangle_L$, i.e.,

$$\langle\zeta,\zeta'\rangle_R = \langle\zeta,\zeta'\rangle_L - 2\,\frac{\langle\zeta,K(z)\rangle_L\ \langle\zeta',K(z)\rangle_L}{\langle K(z),K(z)\rangle_L} \quad \text{for all } z\in\mathcal{L},\ \ \zeta,\zeta'\in T_z\mathcal{L}.$$

For each $z\in\Omega(p,q)$ the tangent space $T_z\Omega(p,q)$ is given by the H^1 vector fields $\zeta\colon I\to T\mathcal{L}$ along z such that $\zeta(0)=0=\zeta(1)$. Moreover, the functional f in (2.1) is well defined and finite on the whole manifold $\Omega(p,q)$. Standard arguments ensure that f is smooth, with differential given by

$$\mathrm{d}f(z)[\zeta] = \int_0^1 \langle \dot z, \nabla_s^L \zeta\rangle_L \,\mathrm{d}s \quad \text{for all } z \in \Omega(p,q),\ \zeta \in T_z\Omega(p,q),$$

and its critical points are the geodesics in $(\mathcal{L}, \langle\cdot,\cdot\rangle_L)$ connecting p to q.

The set $C^1_K(p,q)$ can be also extended to a subset of $\Omega(p,q)$ defined as

$$\Omega_K(p,q) = \{z \in \Omega(p,q) :\ \exists\, C_z \in \mathbb{R} \text{ such that } \langle \dot z, K(z)\rangle_L \equiv C_z \text{ a.e. on } I\} \tag{3.1}$$

and definitions and theorems below hold on both of them.

The following result reduces the geodesic connectedness problem between p and q to the search of critical points of f on $\Omega_K(p,q)$ (cf. Theorem 3.3 in [20]).

Theorem 3.1. *A curve $\gamma \in \Omega(p,q)$ is a geodesic on $\mathcal{L}$ connecting p to q if and only if $\gamma \in \Omega_K(p,q)$ and γ is a critical point of f in (2.1) restricted to $\Omega_K(p,q)$.*

The following definitions are given in [20]:

(i) given $c \in \mathbb{R}$, the set $\Omega_K(p,q)$ is *c-precompact* for f if every sequence $(z_m)_m$ in $\Omega_K(p,q)$ such that $f(z_m) \le c$ has a subsequence which converges weakly in $\Omega_K(p,q)$ (hence, uniformly in $\mathcal{L}$);

(ii) the restriction of f to $\Omega_K(p,q)$ is *pseudo-coercive* if $\Omega_K(p,q)$ is c-precompact for all $c \ge \inf f(\Omega_K(p,q))$.

Then, the following theorem holds (cf. Theorem 1.2 in [20]).

Theorem 3.2 (Giannoni–Piccione). *If $\Omega_K(p,q)$ is not empty and there exists $c > \inf f(\Omega_K(p,q))$ such that $\Omega_K(p,q)$ is c-precompact, then there exists at least one geodesic in $(\mathcal{L}, \langle\cdot,\cdot\rangle_L)$ joining p to q.*

Remark 3.3. In the hypotheses of Theorem 1.1, the completeness of K guarantees that $\Omega_K(p,q) \neq \emptyset$ for any $p, q \in \mathcal{L}$ (cf. Lemma 5.7 in [20] and Proposition 3.6 in [11]); moreover, the technical condition of pseudo-coercivity holds (cf. Theorem 5.1 in [11]). At the end, Theorem 1.1 will follow from Theorem 3.2.

4. The stationary non-canonical global splitting

Let $(\mathcal{L}, \langle\cdot,\cdot\rangle_L)$ be a standard stationary spacetime as in (2.3) with $\beta(x) > 0$ for all $x \in S$. Given two points $p = (x_p, t_p), q = (x_q, t_q) \in \mathcal{L}$, the space $\Omega(p,q)$ can be rewritten as

$$\Omega(p,q) \;=\; \Omega(x_p, x_q; S) \times W(t_p, t_q),$$

where

$$\begin{aligned}\Omega(x_p, x_q; S) = \Big\{ x : I \to S :\ & x \text{ is absolutely continuous},\\ & x(0) = x_p,\ x(1) = x_q,\ \int_0^1 \langle \dot x, \dot x\rangle \,\mathrm{d}s < +\infty \Big\},\end{aligned}$$

$$W(t_p, t_q) \;=\; \big\{ t \in H^1(I, \mathbb{R}) :\ t(0) = t_p,\ t(1) = t_q \big\} = H^1_0(I, \mathbb{R}) + T^*,$$

being $H^1(I,\mathbb{R})$ the classical Sobolev space,

$$H^1_0(I,\mathbb{R}) = \left\{t \in H^1(I,\mathbb{R}) :\, t(0) = 0 = t(1)\right\}$$

and

$$T^* : s \in I \longmapsto t_p + s\Delta_t \in \mathbb{R}, \qquad \Delta_t = t_q - t_p. \tag{4.1}$$

For every $x \in \Omega(x_p, x_q; S)$, one has

$$T_x\Omega(x_p, x_q; S) \;=\; \Big\{\xi : I \to T_x S : \xi \;\text{ is absolutely continuous,} \\ \xi(0) = 0 = \xi(1),\ \int_0^1 \langle D_s\xi, D_s\xi\rangle\, \mathrm{d}s < +\infty\Big\}.$$

Furthermore, $W(t_p,t_q)$ is a closed affine submanifold of $H^1(I,\mathbb{R})$ having tangent space

$$T_t W(t_p,t_q) = H^1_0(I,\mathbb{R}) \quad \text{for all } t \in W(t_p,t_q).$$

So, for every $z = (x,t) \in \Omega(p,q)$ we have

$$T_z\Omega(p,q) = T_x\Omega(x_p,x_q;S) \times T_t W(t_p,t_q) = T_x\Omega(x_p,x_q;S) \times H^1_0(I,\mathbb{R})$$

and $\Omega(p,q)$ can be equipped with the Riemannian structure

$$\langle\zeta,\zeta\rangle_H \;=\; \langle(\xi,\tau),(\xi,\tau)\rangle_H \;=\; \int_0^1 \langle D_s\xi, D_s\xi\rangle\, \mathrm{d}s + \int_0^1 \dot\tau^2\, \mathrm{d}s,$$

for all $z = (x,t) \in \Omega(p,q)$ and $\zeta = (\xi,\tau) \in T_z\Omega(p,q)$.

Next, assume that $(S, \langle\cdot,\cdot\rangle)$ is complete. Then, $\Omega(x_p,x_q;S)$ is a complete infinite dimensional manifold (cf. [23]). By the Nash embedding theorem, the complete manifold S can be seen as a closed submanifold of an Euclidean space $\mathbb{R}^N$ (cf. [26] for the existence of a closed isometric embedding). Hence, $\Omega(x_p,x_q;S)$ is an embedded submanifold of the classical Sobolev space $H^1(I,\mathbb{R}^N)$. As usual, let us set

$$\|y\|^2 = \|y\|_2^2 + \|\dot y\|_2^2 \quad \text{for all } y \in H^1(I,\mathbb{R}^N),$$

where $\|\cdot\|_2$ denotes the standard L^2-norm. It is well known that the following inequalities hold:

$$\|y\|_2 \;\le\; \|y\|_\infty \;\le\; \|\dot y\|_2 \quad \text{for all } y \in H^1_0(I,\mathbb{R}^N), \tag{4.2}$$

where $\|\cdot\|_\infty$ denotes the norm of the uniform convergence (cf., e.g., Proposition 8.13 in [9]). Moreover, the Ascoli–Arzelá theorem implies that any bounded sequence in $H^1(I,\mathbb{R}^N)$ has a uniformly converging subsequence in $C(I,\mathbb{R}^N)$.

For any absolutely continuous curve $z = (x,t) : I \to \mathcal{L}$, one has

$$\langle\dot z, K(z)\rangle_L = \langle\dot z, \partial_t\rangle_L = \langle\delta(x),\dot x\rangle - \beta(x)\,\dot t \quad \text{(recall that } K = \partial_t). \tag{4.3}$$

Taking into account (4.3), if $z \in \Omega_K(p,q)$ (recall (3.1)) then there exists a constant C_z such that

$$\dot t = \frac{\langle \delta(x), \dot x\rangle - C_z}{\beta(x)} \quad \text{a.e. on } I. \tag{4.4}$$

Thus, integrating both hand sides of (4.4) on I, and isolating C_z, we get

$$C_z = \Big(\int_0^1 \frac{\langle \delta(x), \dot x\rangle}{\beta(x)}\, ds \;-\; \Delta_t\Big)\Big(\int_0^1 \frac{ds}{\beta(x)}\Big)^{-1}. \tag{4.5}$$

Denoting by $\mathcal{J}$ the restriction to $\Omega_K(p,q)$ of the functional f in (2.1) with metric (2.3), and substituting (4.5) in (4.4), $\mathcal{J}$ can expressed as a functional depending only on Δ_t (cf. (4.1)) and the component x of the curve $z = (x,t) \in \Omega_K(p,q)$:

$$\begin{aligned}\mathcal{J}(x) = \frac{1}{2}\|\dot x\|_2^2 + \frac{1}{2}\Big[\int_0^1 \frac{\langle \delta(x), \dot x\rangle^2}{\beta(x)}\, ds - \Big(\int_0^1 \frac{\langle \delta(x), \dot x\rangle}{\beta(x)}\, ds\Big)^2\Big(\int_0^1 \frac{1}{\beta(x)}\, ds\Big)^{-1}\Big]\\ - \frac{\Delta_t}{2}\Big(\Delta_t - 2\int_0^1 \frac{\langle \delta(x), \dot x\rangle}{\beta(x)}\, ds\Big)\Big(\int_0^1 \frac{1}{\beta(x)}\, ds\Big)^{-1}.\end{aligned} \tag{4.6}$$

By construction, $f(z) = \mathcal{J}(x)$ if $z = (x,t) \in \Omega_K(p,q)$; furthermore, by applying the Cauchy–Schwarz inequality to the middle term of (4.6), we get

$$2\,\mathcal{J}(x) \;\geq\; \|\dot x\|_2^2 \;-\; \Delta_t\Big(\Delta_t - 2\int_0^1 \frac{\langle \delta(x), \dot x\rangle}{\beta(x)}\, ds\Big)\Big(\int_0^1 \frac{1}{\beta(x)}\, ds\Big)^{-1}. \tag{4.7}$$

Now, we are ready to establish an adapted version of Theorem 1.1, needed in Section 5. But, first, we recall the following result (cf. Lemma 5.4 in [11]):

Lemma 4.1. *Fixed any $x \in \Omega(x_p, x_q; S) \cap C^1(I,S)$ (x non-constant if $x_p = x_q$) there exists a unique future directed lightlike curve $\gamma^l = (x^l, t^l) : [0,1] \to \mathcal{L}$ joining (x_p, t_p) to $\{x_q\} \times \mathbb{R}$ in a time $T(x) = t^l(1) - t^l(0) > 0$ such that $x^l = x$. Moreover, $T(x)$ satisfies*

$$T(x) = \int_0^1 \frac{\langle \delta(x), \dot x\rangle}{\beta(x)}\, ds \;+\; \int_0^1 \frac{\sqrt{\langle \delta(x), \dot x\rangle^2 + \langle \dot x, \dot x\rangle \beta(x)}}{\beta(x)}\, ds. \tag{4.8}$$

Theorem 4.2. *Let $(\mathcal{L}, \langle\cdot,\cdot\rangle_L)$ be a standard stationary spacetime as in (2.3) and let $(S, \langle\cdot,\cdot\rangle)$ be a complete Riemannian manifold. If two points $p = (x_p, t_p)$, $q = (x_q, t_q) \in \mathcal{L}$ satisfy*

$$\Delta_t = t_q - t_p \geq 0 \quad \textit{and} \quad J^-(q) \cap (S \times \{t_p\}) \textit{ is compact},$$

then they are connected by a geodesic in $\mathcal{L}$.

Proof. [2] From Theorem 3.2 and Remark 3.3, it suffices to show that f restricted to $C^1_K(p,q)$ is c-precompact for some $c > \inf f(C^1_K(p,q))$, i.e. every sequence $(z_m)_m$

[2] Even if the core of this proof is essentially contained in Section 5 in [11], here we rearrange it for reader's convenience. Although the functional f is defined in $C^1_K(p,q)$, it is natural to consider limits in $\Omega_K(p,q)$ (cf. p. 522 and Remark 3.3 in [11]).

in $C^1_K(p,q)$ such that $(f(z_m))_m$ is upper bounded, has a uniformly convergent subsequence. So, let us consider any $c > \inf f(C^1_K(p,q))$ and a sequence of curves $(z_m)_m$ in $C^1_K(p,q)$, with $z_m = (x_m, t_m)$, satisfying

$$(f(z_m))_m \text{ (and thus } (\mathcal{J}(x_m))_m) \text{ is upper bounded by } c. \tag{4.9}$$

Setting

$$C^1(x_p, x_q) = \Omega(x_p, x_q; S) \cap C^1(I, S),$$

we have that

$$(x_m)_m \subset C^1(x_p, x_q).$$

It suffices to prove that

$$(\|\dot{x}_m\|_2)_m \text{ is bounded, up to a subsequence;} \tag{4.10}$$

indeed, by (4.2) it follows that $(x_m)_m$ is bounded in $\Omega(x_p, x_q; S)$ and the supports of these curves are contained in a compact subset of S. Hence, the Ascoli–Arzelá theorem applies.

As we will see later, (4.10) will be a direct consequence of the following three claims.

Claim 1. If (4.10) does not hold, i.e.,

$$\|\dot{x}_m\|_2 \to +\infty, \tag{4.11}$$

then no compact subset of S contains all the elements of the sequence $(x_m)_m$.

Proof of Claim 1. Otherwise, being $(\beta(x_m))_m$ and $(|\delta(x_m)|)_m$ bounded (with $|\delta(x_m)|^2 = \langle \delta(x_m), \delta(x_m) \rangle$), by (4.7) and the Cauchy–Schwarz inequality it follows

$$2\,\mathcal{J}(x_m) \geq \|\dot{x}_m\|_2^2 - C_1\,\|\dot{x}_m\|_2 - C_2$$

for some $C_1, C_2 > 0$ independent of $m \in \mathbb{N}$. Hence, (4.11) implies

$$\mathcal{J}(x_m) \to +\infty, \tag{4.12}$$

in contradiction with (4.9).

Claim 2. If no compact subset of S contains all the elements of the sequence $(x_m)_m$, then there exists some $\varepsilon > 0$ such that (recall (4.8))

$$T_m := T(x_m) > \Delta_t + \varepsilon \quad \text{for infinitely many } m \in \mathbb{N}. \tag{4.13}$$

Proof of Claim 2. Taking $\varepsilon > 0$ provided by Proposition 2.4 (ii), let us assume by contradiction that statement (4.13) does not hold. This means that

$$T_m \leq \Delta_t + \varepsilon \quad \text{for all } m \text{ big enough.} \tag{4.14}$$

From Lemma 4.1, there exist future directed lightlike curves $\gamma^l_m = (x_m, t^l_m)$ joining p to $(x_q, t_p + T_m)$. Then, from (4.14), these curves can be prolonged with the integral curves of ∂_t to get future directed causal curves from p to $q_\varepsilon =$

$(x_q, t_p + \Delta_t + \varepsilon) = (x_q, t_q + \varepsilon)$. These curves have support in $J^-(q_\varepsilon)$, so the curves (x_m, t_p) lie in the compact set $J^-(q_\varepsilon) \cap (S \times \{t_p\})$ (recall Proposition 2.4 (ii)), in contradiction with the hypothesis.

Claim 3. Conditions (4.11) and (4.13) imply (4.12), up to a subsequence.

Proof of Claim 3. If there exists a constant $c_1 > 0$ such that

$$\Big(\Delta_t - 2\int_0^1 \frac{\langle \delta(x_m), \dot{x}_m\rangle}{\beta(x_m)}\,\mathrm{d}s\Big)\Big(\int_0^1 \frac{\mathrm{d}s}{\beta(x_m)}\Big)^{-1} \le c_1 \quad \text{for infinitely many } m \in \mathbb{N},$$

then the desired limit (4.12) follows from (4.7) and (4.11).

Otherwise, assume that

$$\Big(\Delta_t - 2\int_0^1 \frac{\langle \delta(x_m), \dot{x}_m\rangle}{\beta(x_m)}\,\mathrm{d}s\Big)\Big(\int_0^1 \frac{\mathrm{d}s}{\beta(x_m)}\Big)^{-1} \longrightarrow +\infty \quad \text{as } m \to +\infty. \tag{4.15}$$

Setting

$$\tilde{T}_m = \int_0^1 \frac{\langle \delta(x_m), \dot{x}_m\rangle}{\beta(x_m)}\,\mathrm{d}s + \sqrt{\Big(\int_0^1 \frac{\langle \delta(x_m), \dot{x}_m\rangle^2}{\beta(x_m)}\,\mathrm{d}s + \|\dot{x}_m\|^2\Big)\int_0^1 \frac{\mathrm{d}s}{\beta(x_m)}},$$

the Cauchy–Schwarz inequality implies

$$T_m \le \tilde{T}_m \quad \forall m \in \mathbb{N}. \tag{4.16}$$

Moreover,

$$\int_0^1 \frac{\langle \delta(x_m), \dot{x}_m\rangle^2}{\beta(x_m)}\,\mathrm{d}s + \|\dot{x}_m\|_2^2 = \Big(\tilde{T}_m - \int_0^1 \frac{\langle \delta(x_m), \dot{x}_m\rangle}{\beta(x_m)}\,\mathrm{d}s\Big)^2\Big(\int_0^1 \frac{\mathrm{d}s}{\beta(x_m)}\Big)^{-1}.$$

For infinitely many $m \in \mathbb{N}$, inequality (4.13) holds and

$$\Delta_t - 2\int_0^1 \frac{\langle \delta(x_m), \dot{x}_m\rangle}{\beta(x_m)}\,\mathrm{d}s \quad \text{is positive} \tag{4.17}$$

(recall (4.15)). Hence,

$$\begin{aligned}
2\,\mathcal{J}(x_m) &= \Big(\tilde{T}_m - \int_0^1 \frac{\langle \delta(x_m), \dot{x}_m\rangle}{\beta(x_m)}\,\mathrm{d}s\Big)^2\Big(\int_0^1 \frac{\mathrm{d}s}{\beta(x_m)}\Big)^{-1} \\
&\quad - \Big(\int_0^1 \frac{\langle \delta(x_m), \dot{x}_m\rangle}{\beta(x_m)}\,\mathrm{d}s - \Delta_t\Big)^2\Big(\int_0^1 \frac{\mathrm{d}s}{\beta(x_m)}\Big)^{-1} \\
&= \Big(\tilde{T}_m^2 - \Delta_t^2 - 2(\tilde{T}_m - \Delta_t)\int_0^1 \frac{\langle \delta(x_m), \dot{x}_m\rangle}{\beta(x_m)}\,\mathrm{d}s\Big)\Big(\int_0^1 \frac{\mathrm{d}s}{\beta(x_m)}\Big)^{-1} \\
&= (\tilde{T}_m - \Delta_t)\Big(\tilde{T}_m + \Delta_t - 2\int_0^1 \frac{\langle \delta(x_m), \dot{x}_m\rangle}{\beta(x_m)}\,\mathrm{d}s\Big)\Big(\int_0^1 \frac{\mathrm{d}s}{\beta(x_m)}\Big)^{-1} \\
&\ge \varepsilon\Big[\tilde{T}_m + \Big(\Delta_t - 2\int_0^1 \frac{\langle \delta(x_m), \dot{x}_m\rangle}{\beta(x_m)}\,\mathrm{d}s\Big)\Big]\Big(\int_0^1 \frac{\mathrm{d}s}{\beta(x_m)}\Big)^{-1} \\
&\ge \varepsilon\Big(\Delta_t - 2\int_0^1 \frac{\langle \delta(x_m), \dot{x}_m\rangle}{\beta(x_m)}\,\mathrm{d}s\Big)\Big(\int_0^1 \frac{\mathrm{d}s}{\beta(x_m)}\Big)^{-1},
\end{aligned}$$

where, in the first inequality, we have taken into account (4.13), (4.16) and (4.17). So, the limit (4.15) clearly implies the limit (4.12), up to a subsequence.

Summing up, if (4.10) does not hold, Claim 1 ensures that no compact subset of S contains all the elements of the sequence $(x_m)_m$. Then, Claims 2 and 3 imply (4.12), up to a subsequence, in contradiction with (4.9). □

Notice that, under the hypotheses of the previous theorem, the Cauchy development of $S \times \{t_p\}$ must be globally hyperbolic and $S \times \{t_p\}$ is a Cauchy surface of it. So, we deduce the following immediate consequence of Theorem 4.2.

Corollary 4.3. *Given a standard stationary spacetime $(\mathcal{L}, \langle\cdot,\cdot\rangle_L)$ as in (2.3) and $(S, \langle\cdot,\cdot\rangle)$ a complete Riemannian manifold, then any point in $S_t = S \times \{t\}$, $t \in \mathbb{R}$, can be geodesically connected with any point in its Cauchy development $D(S_t)$. In particular, if S_t is a Cauchy hypersurface then $D(S_t) = \mathcal{L}$ and all the spacetime is geodesically connected.*

5. Connecting geodesics in auxiliary stationary spacetimes

Throughout this section, $(\mathcal{L}, \langle\cdot,\cdot\rangle_L)$ will be a spacetime which satisfies the hypotheses of Theorem 1.2. From Proposition 2.2, $\mathcal{L} = S \times \mathbb{R}$ and $\langle\cdot,\cdot\rangle_L$ is as in (2.4), with metric coefficients given by Remark 2.3.

For each $n \in \mathbb{N}$, let us consider the standard stationary spacetime $(\mathcal{L}_n, \langle\cdot,\cdot\rangle_n)$ (often simply denoted by $\mathcal{L}_n$), where $\mathcal{L}_n = \mathcal{L}$ and

$$\langle\zeta,\zeta'\rangle_n = \langle\zeta,\zeta'\rangle_L - \frac{1}{n}\tau\tau' = \langle\xi,\xi'\rangle + \langle\delta(x),\xi\rangle\,\tau' + \langle\delta(x),\xi'\rangle\,\tau - \frac{1}{n}\tau\tau' \tag{5.1}$$

for any $z = (x,t) \in \mathcal{L}$, $\zeta = (\xi,\tau), \zeta' = (\xi',\tau') \in T_z\mathcal{L} = T_xS \times \mathbb{R}$. In particular note that, with this definition, there is a strict inclusion of cones, namely $\langle\cdot,\cdot\rangle_L \prec \langle\cdot,\cdot\rangle_m \prec \langle\cdot,\cdot\rangle_n$ if $m > n$.

In the present section we are going to take advantage of Theorem 4.2 to prove that each two points of $\mathcal{L}$ are geodesically connected in $\mathcal{L}_n$, for n large enough. To avoid misunderstandings, the objects associated to each spacetime $\mathcal{L}_n$ will be denoted by a subindex n. So, the functional f in (2.1) associated to $\mathcal{L}_n$ translates into

$$f_n(z) = \frac{1}{2}\int_0^1 \langle\dot z,\dot z\rangle_n \mathrm{d}s = \frac{1}{2}\,\|\dot x\|_2^2 + \int_0^1 \langle\delta(x),\dot x\rangle\,\dot t\,\mathrm{d}s - \frac{1}{2n}\|\dot t\|_2^2. \tag{5.2}$$

Analogously, the functional $\mathcal{J}$ in (4.6) becomes

$$\begin{aligned} \mathcal{J}_n(x) = &\frac{1}{2}\|\dot x\|_2^2 + \frac{n}{2}\Big[\int_0^1 \langle\delta(x),\dot x\rangle^2\,\mathrm{d}s - \Big(\int_0^1 \langle\delta(x),\dot x\rangle\,\mathrm{d}s\Big)^2\Big] \\ &- \Delta_t\Big(\frac{\Delta_t}{2n} - \int_0^1 \langle\delta(x),\dot x\rangle\,\mathrm{d}s\Big). \end{aligned} \tag{5.3}$$

Furthermore, the geodesic equations (2.5), particularized to $\mathcal{L}_n$ in (5.1), translate into

$$\tag{5.4} \begin{cases} D_s\dot{x} - \dot{t}\,F(x)[\dot{x}] + \ddot{t}\,\delta(x) = 0, \\ \dfrac{\mathrm{d}}{\mathrm{d}s}\Big(\dfrac{1}{n}\dot{t} - \langle\delta(x),\dot{x}\rangle\Big) = 0. \end{cases}$$

With these ingredients, now we can establish the announced result.

Proposition 5.1. *Let $(\mathcal{L},\langle\cdot,\cdot\rangle_L)$ be a spacetime as in Theorem 1.2. Given two points $p=(x_p,t_p)$, $q=(x_q,t_q)\in\mathcal{L}$ with $\Delta_t = t_q - t_p \geq 0$, there exists $n_0\in\mathbb{N}$ such that p and q are connected by a geodesic $\gamma_n = (x_n,t_n)$ in $(\mathcal{L}_n,\langle\cdot,\cdot\rangle_n)$ for every $n\geq n_0$.*

Proof. From Theorem 4.2 applied to each $\mathcal{L}_n$, it suffices to prove the existence of some $n_0\in\mathbb{N}$ such that

$$\tag{5.5} J_n^-(q)\cap(S\times\{t_p\}) \quad \text{is compact in } S\times\{t_p\} \text{ for all } n\geq n_0.$$

Arguing by contradiction, assume that condition (5.5) is false for infinitely many $(\mathcal{L}_m,\langle\cdot,\cdot\rangle_m)$. Since $(S,\langle\cdot,\cdot\rangle)$ is complete, by Proposition 2.4 (i), the set $J_m^-(q)\cap(S\times\{t_p\})$ cannot be bounded. Hence, for each m, there exists an unbounded sequence of points $(y_k^m)_k$ in $J_m^-(q)\cap(S\times\{t_p\})$. Then, by using a Cantor's diagonal type argument applied to the family of these sequences, for each m there exists $k_m\in\mathbb{N}$ such that, denoting $y_m = y_{k_m}^m$ with $y_m\in J_m^-(q)\cap(S\times\{t_p\})$, the sequence $(y_m)_m$ is still unbounded and does not admit any convergent subsequence. Let $(\gamma_m)_m$ be a sequence of past inextendible $\langle\cdot,\cdot\rangle_m$-causal curves departing from q and passing through y_m (recall Footnote 1). Taking any $n_0\in\mathbb{N}$, if $m\geq n_0$ then γ_m is not only causal for $\langle\cdot,\cdot\rangle_m$, but also for $\langle\cdot,\cdot\rangle_{n_0}$ (by the metric expression (5.1)). From Proposition 3.31 in [5] applied to the sequence of curves $(\gamma_m)_m$ in $(\mathcal{L},\langle\cdot,\cdot\rangle_L)$, we obtain an inextendible limit curve $\gamma=(x,t)$ departing from q, which is $\langle\cdot,\cdot\rangle_n$-causal for all n, and thus, $\langle\cdot,\cdot\rangle_L$-causal. Since $(\gamma_m)_m$ intersects $S\times\{t_p\}$ in a sequence of points without convergent subsequences, the limit curve γ cannot intersect $S\times\{t_p\}$, in contradiction with the Cauchy character of the hypersurface $S\times\{t_p\}$ in $(\mathcal{L},\langle\cdot,\cdot\rangle_L)$. □

Remark 5.2. We recall that a C^1 functional $J\colon\Omega\to\mathbb{R}$, defined on a Hilbert manifold Ω, satisfies the *Palais–Smale condition* if each sequence $(x_n)_n\subset\Omega$, such that $(J(x_n))_n$ is bounded and $\mathrm{d}J(x_n)\to 0$ admits a converging subsequence.

The spatial components x_n of the connecting geodesics $\gamma_n=(x_n,t_n)$ provided by Proposition 5.1 are minimum of the functionals $\mathcal{J}_n$ in (5.3): indeed, the c-precompactness of $\Omega_K(p,q)$ for $\mathcal{J}_n$ for $n\geq n_0$ (cf. Theorem 4.2), implies that the functionals $\mathcal{J}_n$ are bounded from below, satisfy the Palais–Smale condition and have complete sublevels, so that they attain their infimum (see Propositions 4.3 and 5.5, Theorem 5.3 in [20] and also Theorem 3.3 in [1]).

6. Proof of Theorem 1.2

Let $(\mathcal{L}, \langle\cdot,\cdot\rangle_L)$ be a spacetime as in Theorem 1.2. In particular, by Proposition 2.2 $\mathcal{L} = S\times\mathbb{R}$ and $\langle\cdot,\cdot\rangle_L$ is as in (2.4), with metric coefficients given by Remark 2.3. Consider two points $p=(x_p,t_p)$, $q=(x_q,t_q)\in\mathcal{L}$ with $\Delta_t = t_q - t_p \geq 0$ and assume that $C^1_K(p,q)\neq\emptyset$. Then, there exists a C^1 curve $\varphi=(y,t): I\to\mathcal{L}$ connecting p with q such that $\langle\dot\varphi, K(\varphi)\rangle_L = \langle\delta(y),\dot y\rangle$ is constant.

Let $(\gamma_n = (x_n,t_n))_{n\geq n_0}$ be the sequence of curves connecting p to q, each γ_n geodesic in $\mathcal{L}_n$, as stated in Proposition 5.1. Then, the following technical results hold.

Lemma 6.1. *The sequence $(\|\dot x_n\|_2)_{n\geq n_0}$ is bounded.*

Proof. Arguing by contradiction, assume that $(\|\dot x_n\|_2)_{n\geq n_0}$ is not bounded. Taking any $\bar n\geq n_0$, the three claims in the proof of Theorem 4.2 imply that $(\mathcal{J}_{\bar n}(x_n))_{n\geq n_0}$ is not upper bounded either. By the expression of the functionals in (5.3) and the Cauchy–Schwarz inequality, it follows that

$$\mathcal{J}_n(x_n)\geq\mathcal{J}_{\bar n}(x_n)\quad\text{for all } n\geq\bar n.$$

Whence, also $(\mathcal{J}_n(x_n))_{n\geq n_0}$ is not bounded from above.

Since $\langle\delta(y),\dot y\rangle$ is constant, one has

$$\begin{aligned}\mathcal{J}_n(y) &= \frac{1}{2}\|\dot y\|_2^2 - \Delta_t\Big(\frac{\Delta_t}{2n} - \int_0^1\langle\delta(y),\dot y\rangle\,\mathrm{d}s\Big)\\ &\leq \frac{1}{2}\|\dot y\|_2^2 + \Delta_t\int_0^1\langle\delta(y),\dot y\rangle\,\mathrm{d}s\quad\text{for all } n\in\mathbb{N}.\end{aligned}$$

Therefore, $\mathcal{J}_n(y)$ admits an upper bound independent of n, and thus

$$\mathcal{J}_n(y) < \mathcal{J}_n(x_n)\quad\text{for infinitely many } n,$$

in contradiction with the minimum character of x_n, as stated in Remark 5.2. □

Lemma 6.2. *The sequence $\big(\|\dot t_n\|_2\big)_{n\geq n_0}$ is bounded.*

Proof. [3] Taking the scalar product of the first equation in (5.4) applied to $\gamma_n = (x_n,t_n)$, $n\geq n_0$, by the vector field $\delta(x_n)$, we get

$$\langle D_s\dot x_n,\delta(x_n)\rangle - \dot t_n\langle F(x_n)[\dot x_n],\delta(x_n)\rangle + \ddot t_n\langle\delta(x_n),\delta(x_n)\rangle\equiv 0\qquad\text{on } I.$$

So, $\tau_n = \dot t_n$ satisfies the first order linear ODE

$$\dot\tau_n = a_n(s)\,\tau_n + b_n(s)\qquad\text{on } I, \tag{6.1}$$

[3] Along this proof, for any integer $j\geq 1$ the constant c_j will always denote a strictly positive real number which does not depend on $s\in I$ and $n\geq n_0$.

where

$$a_n(s) = \frac{\langle F(x_n(s))[\dot{x}_n(s)], \delta(x_n(s))\rangle}{\langle \delta(x_n(s)), \delta(x_n(s))\rangle}, \qquad b_n(s) = -\frac{\langle D_s\dot{x}_n(s), \delta(x_n(s))\rangle}{\langle \delta(x_n(s)), \delta(x_n(s))\rangle} \tag{6.2}$$

(δ is non-vanishing, recall Proposition 2.2). Since

$$\int_0^1 \dot{t}_n\, \mathrm{d}s = t_q - t_p = \Delta_t \quad \text{for all } n \geq n_0, \tag{6.3}$$

necessarily

$$\dot{t}_n(s_n) = \Delta_t \quad \text{for some } s_n \in I. \tag{6.4}$$

So, $\dot{t}_n(s)$ is the unique solution to (6.1) which satisfies condition (6.4), i.e.,

$$\dot{t}_n(s) = \tau_n(s) = e^{A_n(s)}\left(g_n(s) + \Delta_t\right), \tag{6.5}$$

where $A_n(s)$ is the primitive of $a_n(s)$ satisfying $A_n(s_n) = 0$ and, for simplicity, we have set

$$g_n(s) \;=\; \int_{s_n}^{s} b_n(r)\, e^{-A_n(r)}\, \mathrm{d}r. \tag{6.6}$$

Now, in order to prove the boundedness of $(\|\dot{t}_n\|_2)_{n\geq n_0}$, firstly we claim that

$$c_1 \leq e^{A_n(s)} \leq c_2 \quad \text{on } I, \text{ for all } n \geq n_0. \tag{6.7}$$

In fact, by applying inequality (4.2) to x_n, Lemma 6.1 implies that the sequence

$$(\|x_n\|_\infty)_{n\geq n_0} \text{ is bounded;} \tag{6.8}$$

thus,

$$c_3 \leq \langle \delta(x_n(s)), \delta(x_n(s))\rangle \leq c_4 \quad \text{on } I, \text{ for all } n \geq n_0. \tag{6.9}$$

Then, by the Cauchy–Schwarz inequality, (6.2), (6.8) and (6.9) we obtain

$$|a_n(s)| \leq c_5\, |\dot{x}_n(s)| \quad \text{on } I, \tag{6.10}$$

with $|\dot{x}_n(s)|^2 = \langle \dot{x}_n(s), \dot{x}_n(s)\rangle$. Hence, Lemma 6.1 implies

$$|A_n(s)| \leq c_6 \quad \text{on } I, \text{ for all } n \geq n_0,$$

which implies (6.7).

So, in order to conclude the proof, from (6.5) and (6.7) it suffices to show that

$$(\|g_n\|_2)_{n\geq n_0} \text{ is bounded.} \tag{6.11}$$

To this aim, let us note that

$$\langle D_s\dot{x}_n, \delta(x_n)\rangle = -\langle \dot{x}_n, \frac{\mathrm{d}}{\mathrm{d}s}\delta(x_n)\rangle + \frac{\mathrm{d}}{\mathrm{d}s}\langle \dot{x}_n, \delta(x_n)\rangle;$$

thus, by (6.2) and (6.6), integrating by parts we have

$$
\begin{aligned}
g_n(s) &= \int_{s_n}^{s} \langle \dot{x}_n, \frac{\mathrm{d}}{\mathrm{d}r}\delta(x_n)\rangle \, \frac{e^{-A_n(r)}}{\langle\delta(x_n),\delta(x_n)\rangle}\, \mathrm{d}r \\
&\quad - \int_{s_n}^{s} \frac{\mathrm{d}}{\mathrm{d}r}\big(\langle \dot{x}_n, \delta(x_n)\rangle\big)\, \frac{e^{-A_n(r)}}{\langle\delta(x_n),\delta(x_n)\rangle}\, \mathrm{d}r \\
&= \int_{s_n}^{s} \langle \dot{x}_n, \frac{\mathrm{d}}{\mathrm{d}r}\delta(x_n)\rangle \, \frac{e^{-A_n(r)}}{\langle\delta(x_n),\delta(x_n)\rangle}\, \mathrm{d}r \\
&\quad - \frac{e^{-A_n(s)}\langle \dot{x}_n(s), \delta(x_n(s))\rangle}{\langle\delta(x_n(s)),\delta(x_n(s))\rangle} + \frac{e^{-A_n(s_n)}\langle \dot{x}_n(s_n), \delta(x_n(s_n))\rangle}{\langle\delta(x_n(s_n)),\delta(x_n(s_n))\rangle} \\
&\quad + \int_{s_n}^{s} \langle \dot{x}_n, \delta(x_n)\rangle\, \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{e^{-A_n(r)}}{\langle\delta(x_n),\delta(x_n)\rangle}\right) \mathrm{d}r.
\end{aligned}
\tag{6.12}
$$

The smoothness of δ, (6.7)–(6.10), the Cauchy–Schwarz inequality, direct computations and Lemma 6.1 imply that for all $n \geq n_0$ the following bounds hold:

$$
\left| \int_{s_n}^{s} \langle \dot{x}_n, \frac{\mathrm{d}}{\mathrm{d}r}\delta(x_n)\rangle \, \frac{e^{-A_n(r)}}{\langle\delta(x_n),\delta(x_n)\rangle}\, \mathrm{d}r \right| \leq c_7\, \|\dot{x}_n\|_2^2 \leq c_8, \tag{6.13}
$$

$$
\left| \frac{\langle \dot{x}_n(s), \delta(x_n(s))\rangle}{\langle\delta(x_n(s)),\delta(x_n(s))\rangle} \right| \leq c_9\, |\dot{x}_n(s)| \quad \text{on } I, \tag{6.14}
$$

and

$$
\begin{aligned}
&\left| \int_{s_n}^{s} \langle \dot{x}_n, \delta(x_n)\rangle\, \frac{\mathrm{d}}{\mathrm{d}r}\Big(\frac{e^{-A_n(r)}}{\langle\delta(x_n),\delta(x_n)\rangle}\Big)\, \mathrm{d}r \right| \\
&\qquad \leq \int_0^1 |\langle \dot{x}_n, \delta(x_n)\rangle|\, \frac{|a_n(r)| e^{-A_n(r)}}{\langle\delta(x_n),\delta(x_n)\rangle}\, \mathrm{d}r \\
&\qquad\quad + 2\int_0^1 |\langle \dot{x}_n, \delta(x_n)\rangle|\, e^{-A_n(r)}\, \frac{\left|\langle \delta(x_n), \frac{\mathrm{d}}{\mathrm{d}r}\delta(x_n)\rangle\right|}{\langle\delta(x_n),\delta(x_n)\rangle^2}\, \mathrm{d}r \\
&\qquad \leq c_{10}\, \|\dot{x}_n\|_2^2 \leq c_{11}.
\end{aligned}
\tag{6.15}
$$

Moreover, we claim that

$$
\left| \frac{\langle \dot{x}_n(s_n), \delta(x_n(s_n))\rangle}{\langle\delta(x_n(s_n)),\delta(x_n(s_n))\rangle} \right| \leq c_{12}\, \|\dot{x}_n\|_2 \leq c_{13} \quad \text{on } I. \tag{6.16}
$$

In fact, from the second equality in (5.4) we have

$$
\frac{1}{n}\, \dot{t}_n - \langle \delta(x_n), \dot{x}_n\rangle \equiv k_n \quad \text{on } I;
$$

thus, from one hand (6.4) implies

$$
k_n = \frac{1}{n}\, \dot{t}_n(s_n) - \langle \delta(x_n(s_n)), \dot{x}_n(s_n)\rangle = \frac{\Delta_t}{n} - \langle \delta(x_n(s_n)), \dot{x}_n(s_n)\rangle,
$$

while, from the other hand, (6.3) gives

$$k_n = \int_0^1 \Big(\frac{1}{n}\,\dot t_n(s) - \langle \delta(x_n(s)), \dot x_n(s)\rangle\Big)\,\mathrm{d}s = \frac{\Delta_t}{n} - \int_0^1 \langle \delta(x_n(s)), \dot x_n(s)\rangle \mathrm{d}s.$$

Whence,

$$\langle \delta(x_n(s_n)), \dot x_n(s_n)\rangle = \int_0^1 \langle \delta(x_n(s)), \dot x_n(s)\rangle\,\mathrm{d}s,$$

and (6.16) follows from (6.9) and, again, Lemma 6.1.

At last, by using (6.13)–(6.16) in (6.12), we have that

$$|g_n(s)| \le c_{14}\,|\dot x_n(s)| + c_{15} \quad \text{on } I, \text{ for all } n \ge n_0;$$

whence, Lemma 6.1 implies (6.11). □

Lemma 6.3. *There exists $\gamma = (x,t) \in \Omega(x_p, x_q; S) \times W(t_p, t_q)$ such that, up to subsequences, $(\gamma_n)_{n\ge n_0}$ strongly converges to γ on $\Omega(x_p, x_q; S) \times W(t_p, t_q)$.*

Proof. From (4.2) and Lemmas 6.1, 6.2, the sequences $(\|x_n\|)_{n\ge n_0}$ and $(\|t_n\|)_{n\ge n_0}$ are bounded, thus there exists $\gamma = (x,t) \in H^1(I, \mathbb{R}^N) \times H^1(I, \mathbb{R})$ such that, up to subsequences,

$$x_n \rightharpoonup x \quad \text{weakly in } H^1(I, \mathbb{R}^N) \text{ (and also uniformly in } I) \tag{6.17}$$

and

$$t_n \rightharpoonup t \quad \text{weakly in } H^1(I, \mathbb{R}).$$

Furthermore, as S is complete, by (6.17) it follows that $x \in \Omega(x_p, x_q; S)$ and there exist two sequences $(\xi_n)_{n\ge n_0}$, $(\nu_n)_{n\ge n_0}$ in $H^1(I, \mathbb{R}^N)$ such that

$$\begin{aligned} &\xi_n \in T_{x_n}\Omega(x_p, x_q; S), \quad x_n - x = \xi_n + \nu_n \quad \text{for all } n \ge n_0,\\ &\xi_n \rightharpoonup 0 \quad \text{weakly} \quad \text{and} \quad \nu_n \to 0 \quad \text{strongly in } H^1(I, \mathbb{R}^N) \end{aligned} \tag{6.18}$$

(cf. Lemma 2.1 in [6]). Taking any $n \ge n_0$, by Proposition 5.1 and (5.2) we have $\mathrm{d}f_n(\gamma_n)[\zeta] = 0$ for all $\zeta \in T_{\gamma_n}\Omega_n(p,q)$; thus, in particular we have

$$\begin{aligned} \int_0^1 \langle \dot x_n, \dot\xi_n\rangle\,\mathrm{d}s + \int_0^1 \langle \delta'(x_n)\xi_n, \dot x_n\rangle\,\dot t_n\,\mathrm{d}s + \int_0^1 \langle \delta(x_n), \dot\xi_n\rangle\,\dot t_n\,\mathrm{d}s&\\ - \int_0^1 \langle \delta(x_n), \dot x_n\rangle\,\dot\tau_n\,\mathrm{d}s + \int_0^1 \frac{1}{n}\dot t_n\dot\tau_n\,\mathrm{d}s &= 0 \end{aligned} \tag{6.19}$$

for $\zeta = (\xi_n, -\tau_n) \in T_{\gamma_n}\Omega_n(p,q)$ with $\tau_n = t_n - t \in H_0^1(I, \mathbb{R})$. On the other hand, by Lemmas 6.1 and 6.2, and (6.18), it results

$$\int_0^1 \langle \delta'(x_n)\xi_n, \dot x_n\rangle\,\dot t_n\,\mathrm{d}s = o(1),$$

where $o(1)$ denotes an infinitesimal sequence. Whence, (6.19) implies

$$\int_0^1 \langle \dot{x}_n, \dot{\xi}_n\rangle \,\mathrm{d}s + \int_0^1 \frac{1}{n}\, \dot{t}_n \dot{\tau}_n \,\mathrm{d}s = -\int_0^1 \langle \delta(x_n), \dot{\xi}_n\rangle\, \dot{t}_n \,\mathrm{d}s + \int_0^1 \langle \delta(x_n), \dot{x}_n\rangle\, \dot{\tau}_n \,\mathrm{d}s + o(1).$$

Reasoning as in Theorem 3.3 in [19], the strong convergence of $(\gamma_n)_{n\ge n_0}$ to γ, up to a subsequence, is deduced. □

Proof of Theorem 1.2. The implication (i) $\Longrightarrow$ (ii) is a direct consequence of (2.2).

For the implication (ii) $\Longrightarrow$ (i), let $(\gamma_n = (x_n, t_n))_{n\ge n_0}$ be the sequence of curves connecting p to q, with each γ_n geodesic in $\mathcal{L}_n$, provided by Proposition 5.1. From Lemma 6.3 there exists a curve $\gamma = (x,t) \in \Omega(x_p, x_q; S) \times W(t_p, t_q)$ such that, up to subsequences,

$$x_n \to x \text{ strongly in } \Omega(x_p, x_q; S) \quad \text{and} \quad t_n \to t \text{ strongly in } W(t_p, t_q). \tag{6.20}$$

It suffices to prove that γ satisfies equations (2.5) with $\beta \equiv 0$, i.e.,

$$\begin{cases} D_s\dot{x} - \dot{t}\, F(x)[\dot{x}] + \ddot{t}\,\delta(x) = 0,\\ \dfrac{\mathrm{d}}{\mathrm{d}s}\langle \delta(x), \dot{x}\rangle = 0. \end{cases} \tag{6.21}$$

To this aim, let us remark that if $n \ge n_0$, by Theorem 3.1 applied to f_n in (5.2), we have

$$\mathrm{d}f_n(\gamma_n)[\zeta] = 0 \qquad \text{for all } \zeta \in T_{\gamma_n}\Omega(p,q). \tag{6.22}$$

Then, in particular, taking any $\tau \in H_0^1(I,\mathbb{R})$ and $\zeta = (0,\tau)$ in (6.22), it follows that

$$\int_0^1 \langle \delta(x_n), \dot{x}_n\rangle\, \dot{\tau} \,\mathrm{d}s - \frac{1}{n}\int_0^1 \dot{t}_n \dot{\tau} \,\mathrm{d}s \;=\; 0;$$

hence, passing to the limit, by (6.20) we get

$$\int_0^1 \langle \delta(x), \dot{x}\rangle \dot{\tau} \,\mathrm{d}s = 0.$$

Thus, for the arbitrariness of $\tau \in H_0^1(I,\mathbb{R})$ the second equality in (6.21) holds.

On the other hand, taking any $\eta \in T_x\Omega(x_p, x_q; S)$, by (6.20) and Lemma 2.2 in [6] there exists a sequence $(\eta_n)_{n\ge n_0}$, with $\eta_n \in T_{x_n}\Omega(x_p, x_q; S)$, converging weakly to η. Then, choosing $\zeta = (\eta_n, 0)$ in (6.22) for $n \ge n_0$, by passing to the limit and taking into account (6.20), we obtain

$$\int_0^1 \langle \dot{x}, \dot{\eta}\rangle \,\mathrm{d}s + \int_0^1 \langle \delta'(x)\eta, \dot{x}\rangle \dot{t} \,\mathrm{d}s + \int_0^1 \langle \delta(x), \dot{\eta}\rangle \dot{t} \,\mathrm{d}s \;=\; 0.$$

Therefore, integrating by parts and for the arbitrariness of $\eta \in T_x\Omega(x_p, x_q; S)$, we deduce that $\gamma = (x,t)$ is smooth and verifies the first equation in (6.21). Hence, the proof is complete. □

The proof of Theorem 1.2 requires global hyperbolicity only in two points: for ensuring the decomposition (2.4) and for proving the following property:

($\star$) *Any past inextendible causal curve departing from* $q = (x_q, t_q)$, $t_q \geq t_p$, *must intersect* $S \times \{t_p\}$.

Therefore, if we are dealing with a spacetime which already splits globally as in (2.4), the global hyperbolicity assumption can be replaced by property ($\star$). More precisely, the same arguments performed in the proof of Theorem 1.2 allow us to state the following generalization:

Theorem 6.4. *Let* $(\mathcal{L}, \langle\cdot,\cdot\rangle_L)$ *be a spacetime with* $\mathcal{L} = S \times \mathbb{R}$ *and* $\langle\cdot,\cdot\rangle_L$ *as in* (2.4). *Assume that* $(S, \langle\cdot,\cdot\rangle)$ *is a complete Riemannian manifold. Given two points* $p = (x_p, t_p)$, $q = (x_q, t_q)$, *with* $\Delta_t = t_q - t_p \geq 0$, *satisfying property* ($\star$), *the following statements are equivalent:*

(i) p *and* q *are geodesically connected in* $\mathcal{L}$*;*

(ii) p *and* q *can be connected by a* C^1 *curve* $\varphi = (y, t)$ *on* $\mathcal{L}$ *such that* $\langle \delta(y), \dot{y} \rangle$ *is constant.*

7. Accuracy of the hypotheses of Theorem 1.2.

(a) *Counterexample if the lightlike Killing vector field is not complete.*

Consider the spacetime obtained by removing from the Minkowski 2-space $\mathbb{L}^2$ the region $\{(x,t) : x \geq 0,\, t \geq 0\}$. This spacetime admits the hyperplane $t \equiv -1$ as a complete Cauchy hypersurface, and $K = \partial_x + \partial_t$ as a non-complete lightlike Killing vector field. However, the points $p = (1, -1)$, $q = (-1, 1)$, which can be connected with a C^1 curve φ with $\langle \dot{\varphi}, K(\varphi) \rangle_L$ having constant negative sign, cannot be connected by a geodesic.

(b) *Counterexample if the Cauchy hypersurface is not complete.*

Consider $\mathcal{L} = S \times \mathbb{R}$, $S = \mathbb{R}^2 \setminus \{(x_1, 0) : -1 \leq x_1 \leq 1\}$ equipped with the Lorentzian metric

$$\langle \zeta, \zeta' \rangle_L = \langle \xi, \xi' \rangle_0 + \langle \delta(x), \xi \rangle_0 \, \tau' + \langle \delta(x), \xi' \rangle_0 \, \tau,$$

for all $\zeta = (\xi, \tau), \zeta' = (\xi', \tau') \in \mathbb{R}^3$, where $\langle\cdot,\cdot\rangle_0$ is the canonical scalar product on $S \subset \mathbb{R}^2$ and $\delta : x = (x_1, x_2) \in S \mapsto \lambda(x)(1, 0) \in \mathbb{R}^2$, with λ a positive smooth function on S such that $\langle\cdot,\cdot\rangle_0/\lambda^2$ is complete on S. Note that $K = \partial_t$ is a complete lightlike Killing vector field and $S \times \{t\}$ is a non-complete Cauchy hypersurface for every $t \in \mathbb{R}$ (apply Proposition 3.1 in [31] with $F_n(x) \equiv 2\lambda(x)$ for all n). However, this manifold is not geodesically connected. In fact, consider two points $p = (x_p, 0)$, $q = (x_q, 0)$ with $x_p = (0, -1)$, $x_q = (0, 1)$. By the second equation in (6.21), any geodesic $\gamma = (x, t)$ joining p to q must satisfy

$$\frac{\mathrm{d}}{\mathrm{d}s} \langle \delta(x), \dot{x} \rangle_0 = 0,$$

but the sign of $\langle \delta(x), \dot{x} \rangle_0$ must change for any curve $x = x(s)$ departing from x_p and arriving to x_q. Hence, there is no geodesic connecting p to q.

(c) *The existence of a complete lightlike Killing vector field and a complete Cauchy hypersurface do not imply geodesic connectedness.*

Consider $\mathcal{L} = \mathbb{R}^2 \times \mathbb{R}$ equipped with the Lorentzian metric[4]

$$\langle \zeta, \zeta' \rangle_L = \langle \xi, \xi' \rangle_0 + \langle \delta(x), \xi \rangle_0 \, \tau' + \langle \delta(x), \xi' \rangle_0 \, \tau,$$

for all $\zeta = (\xi, \tau), \zeta' = (\xi', \tau') \in \mathbb{R}^3$, where $\langle \cdot, \cdot \rangle_0$ is the canonical scalar product on $\mathbb{R}^2$ and $\delta : x = (x_1, x_2) \in \mathbb{R}^2 \mapsto \delta(x_1) \in \mathbb{R}^2$ satisfies

$$\delta(x_1) = \begin{cases} (-\cos^3 x_1, \sin^3 x_1) & \text{if } x_1 < \pi, \\ (1, 0) & \text{if } x_1 \geq \pi. \end{cases}$$

In this spacetime ∂_t is a complete lightlike Killing vector field and $\mathbb{R}^2 \times \{t\}$ is a complete Cauchy hypersurface for every $t \in \mathbb{R}$ (apply Proposition 3.1 in [31] with $F_n \equiv 2$ for all n). However, this spacetime is not geodesically connected. In fact, for any C^1 curve $x = x(s)$ departing from a point in $\mathbb{R}^2$ with $x_1 = 0$ and arriving to a point in the region $x_1 > \pi$, the quantity $\langle \delta(x), \dot{x} \rangle_0$ cannot be constant as its sign must change. Hence, there is no geodesic which connects the points $p = (x_p, 0)$ and $q = (x_q, 0)$, when, for example, it is $x_p = (0, 0)$ and $x_q = (3\pi/2, 0)$.

Remark 7.1. In previous examples (b) and (c) we have used the result Proposition 3.1 in [31], where an ambient hypothesis says that $\beta = -\langle K, K \rangle > 0$, i.e., K is timelike. However, it is clear that this type of results can be applied when K is lightlike, as here. In fact, they can be applied even when K becomes spacelike, because the function t remains still as a temporal function (see Section 3 in [14]). Nevertheless, when K is allowed to be non-timelike, one assumes implicitly that $\beta + \langle \delta, \delta \rangle_0 > 0$ (true in our setting as by Proposition 2.2 it is $\beta \equiv 0$ but δ is non-vanishing), which is equivalent to saying that the full metric of the spacetime is Lorentzian (see Proposition 3.3 in [14]).

8. Some applications

8.1. An Avez–Seifert result

A first consequence of Theorem 1.2 is that it provides an alternative proof of the classical Avez–Seifert result in our ambient (cf., e.g., Theorem 3.18 in [5]):

Proposition 8.1. *Let $(\mathcal{L}, \langle \cdot, \cdot \rangle_L)$ be a globally hyperbolic spacetime endowed with a complete lightlike Killing vector field K and a complete Cauchy hypersurface S. Then, two points of $\mathcal{L}$ can be connected by a causal geodesic if and only if they are causally related.*

Proof. We will focus on the implication to the left, as the converse is trivial. So, assume that two points $p, q \in \mathcal{L}$ are causally related. Then, they are connectable

[4] This metric is only C^1. Even though, this is enough for the problem of the geodesics (especially from the developed variational viewpoint), it is clear that it could be modified in a C^∞ one.

by a C^1 causal curve $\varphi = (y, \tau)$, which, up to a reparameterization, satisfies that $\langle \dot\varphi, K(\varphi)\rangle_L$ is constant. Thus, from Theorem 1.2 the points p and q are connectable by a geodesic $\gamma = (x, t)$.

In order to prove that $\gamma = (x, t)$ is causal, it suffices to show that $f(\gamma) \le 0$. To this aim, recall that $\gamma = (x, t)$ can be approached by a sequence of geodesics $\gamma_n = (x_n, t_n)$, $n \ge n_0$, of $(\mathcal{L}_n, \langle\cdot,\cdot\rangle_n)$, where each x_n is a minimum of the functional $\mathcal{J}_n$ (recall Remark 5.2 and Lemma 6.3). So, from one hand, $\gamma_n \to \gamma$ strongly in $\Omega(x_p, x_q; S) \times W(t_p, t_q)$ (and also uniformly in I) and the boundedness of $(\|\dot x_n\|_2)_{n\ge n_0}$ and $\left(\|\dot t_n\|_2\right)_{n\ge n_0}$ imply

$$\mathcal{J}_n(x_n) = f_n(\gamma_n) \to f(\gamma) \quad \text{as } n \to \infty$$

(cf. also Theorem 3.3 in [19]). On the other hand,

$$\mathcal{J}_n(x_n) \le \mathcal{J}_n(y) = f_n(\varphi) \to f(\varphi) \le 0 \quad \text{as } n \to \infty.$$

In conclusion, $f(\gamma) \le 0$ and, thus, γ is causal. □

8.2. A multiplicity result

In order to state a multiplicity result, the following abstract tools are required (cf., e.g., [28]).

Definition 8.2. Let X be a topological space. Given $A \subset X$, the *Ljusternik–Schnirelmann category* of A in X, briefly $\mathrm{cat}_X(A)$, is the least number of closed and contractible subsets of X covering A. If it is not possible to cover A with a finite number of such sets, then $\mathrm{cat}_X(A) = +\infty$.

Theorem 8.3. *Let M be a Riemannian manifold and I a C^1 functional on M which satisfies the Palais–Smale condition in $\mathbb{R}$. Given any $k \in \mathbb{N}$, $k > 0$, let us define*

$$c_k = \inf_{A\in\Gamma_k} \sup_{x\in A} I(x) \quad \textit{with } \Gamma_k = \{A \subset M : \mathrm{cat}_M(A) \ge k\}. \tag{8.1}$$

If M is complete or each sublevel of I in M is complete, then c_k is a critical value of I for each k such that $\Gamma_k \ne \emptyset$ and $c_k \in \mathbb{R}$. Moreover, if I is bounded from below but not from above and Ω contains a sequence of compact subsets with arbitrary high category, then a sequence $(x_k)_k$ of critical points exists such that $\lim_{k\to+\infty} I(x_k) = +\infty$.

Then, we can point out that, in the assumptions of Theorem 1.1, Theorem 8.3 allows one to prove the existence of infinitely many geodesics joining two events $p, q \in \mathcal{L}$, if $\Omega(p, q)$ contains a sequence of compact subsets with arbitrary high category[5] (cf. Theorem 4.27 in [12]). Starting from this, we can state a multiplicity result also in the setting of Theorem 1.2.

[5]This is true for any couple of points if $\mathcal{L}$ is non contractible in itself (cf. [15], once applied the Nash embedding theorem to the Cauchy hypersurface).

Theorem 8.4. *Assume that the hypotheses of Theorem 1.2 hold and $p, q \in \mathcal{L}$ are such that $\Omega_K(p,q)$ contains a sequence of compact subsets of arbitrary high category. Then, p and q can be connected by infinitely many geodesics $(\gamma_k)_k$ on $\mathcal{L}$ with diverging action f in (2.1).*

Proof. By Proposition 2.2, $\mathcal{L} = S \times \mathbb{R}$ and the metric is described by (2.4). We assume $\Delta_t = t_q - t_p \geq 0$ and for any $n \in \mathbb{N}$ consider $(\mathcal{L}_n, \langle\cdot,\cdot\rangle_n)$, f_n and $\mathcal{J}_n$ as in (5.1)–(5.3). By Proposition 5.1 and the arguments in Remark 5.2, we have that n_0 exists such that for all $n \geq n_0$ the functionals $\mathcal{J}_n$ are bounded from below, satisfy the Palais–Smale condition in $\mathbb{R}$ and have complete sublevels. Hence, by the hypothesis on $\Omega_K(p,q)$ and Theorem 8.3 for each $n \geq n_0$, a sequence of geodesics $\{\gamma_k^n = (x_k^n, t_k^n)\}_{k\geq 1}$ on $(\mathcal{L}_n, \langle\cdot,\cdot\rangle_n)$ exists and by (8.1) their critical values $c_k^n = \mathcal{J}_n(x_k^n) = f_n(\gamma_k^n)$ can be written as

$$c_k^n = \inf_{A\in\Gamma_k} \sup_{x\in A} \mathcal{J}_n(x)$$

and satisfy $c_k^n \to +\infty$ as $k \to +\infty$.

Now, fixing any $k \geq 1$, in order to ensure the existence of some connecting limit curve (thus, geodesic) γ_k in the given manifold $\mathcal{L}$, we can repeat the procedure in Section 6, once we prove that $(\|\dot{x}_k^n\|_2)_{n\geq n_0}$ is bounded. To this aim, by the first part of Section 4 and (5.2),

$$\mathcal{J}_n(x) = f_n(\gamma) \leq f(\gamma) \quad \text{for all } \gamma = (x,t) \in \Omega_K(p,q); \tag{8.2}$$

moreover, by assumption there exists a compact subset A_k of $\Omega_K(p,q)$ such that $\mathrm{cat}_{\Omega_K}(A_k) \geq k$, and thus

$$\mathcal{J}_n(x_k^n) = c_k^n \leq \max_{x\in A_k} \mathcal{J}_n(x) < +\infty. \tag{8.3}$$

Arguing by contradiction, assume that $(\|\dot{x}_k^n\|_2)_n$ is not bounded. Thus, reasoning as in the first part of the proof of Lemma 6.1, and taking into account that the second term in the expression of $\mathcal{J}_n(x)$ is zero if $x \in A_k$, we prove that

$$\mathcal{J}_n(x_n^k) > \max_{x\in A_k} \mathcal{J}_n(x) \quad \forall n \text{ big enough}, \tag{8.4}$$

which is in contradiction with (8.3). So, for each $k \geq 1$ the sequence $(\gamma_k^n)_{n\geq n_0}$ converges to a connecting geodesic γ_k.

Furthermore, note that

$$c_k^1 \leq c_k^n = \mathcal{J}_n(x_k^n) = f_n(\gamma_k^n) \overset{n\to\infty}{\longrightarrow} f(\gamma_k), \tag{8.5}$$

where the first inequality follows from

$$\mathcal{J}_n(x) - \mathcal{J}_1(x) = \frac{n-1}{2}\left(\int_0^1 \langle\delta(x),\dot{x}\rangle^2\, ds - \left(\int_0^1 \langle\delta(x),\dot{x}\rangle\, ds\right)^2\right) + \frac{n\Delta_t^2 - \Delta_t^2}{2n} \geq 0.$$

So, taking into account that $c_k^1 \to +\infty$ in (8.5), we deduce $f(\gamma_k) \to +\infty$, too. □

8.3. Generalized plane waves

Theorem 1.2 becomes also useful for studying the geodesic connectedness of a family of Lorentzian manifolds which generalizes the gravitational waves, the so-called generalized plane waves (see [25]).[6]

Definition 8.5. A Lorentzian manifold $(\mathcal{L}, \langle\cdot,\cdot\rangle_L)$ is called *generalized plane wave*, briefly *GPW*, if there exists a (connected) finite dimensional Riemannian manifold $(\mathcal{M}, \langle\cdot,\cdot\rangle)$ such that $\mathcal{L} = \mathcal{M} \times \mathbb{R}^2$ and

$$\langle\cdot,\cdot\rangle_L \;=\; \langle\cdot,\cdot\rangle + 2\,du\,dv + \mathcal{H}(x,u)\,du^2,$$

where $x \in \mathcal{M}$, the variables (u,v) are the natural coordinates of $\mathbb{R}^2$ and the smooth function $\mathcal{H} : \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ is not identically zero.

A GPW becomes a gravitational wave if $\mathcal{M} = \mathbb{R}^2$ is equipped with the classical Euclidean metric and $\mathcal{H}(x,u) = g_1(u)(x_1^2 - x_2^2) + 2g_2(u)x_1x_2$, $x = (x_1,x_2) \in \mathbb{R}^2$, for some smooth real functions g_1 and g_2 such that $g_1^2 + g_2^2 \not\equiv 0$ (for more details, cf., e.g., [5]).

The geodesic connectedness and the global hyperbolicity of GPWs have been investigated in [10, 17]. In particular, if the Riemannian manifold $(\mathcal{M}, \langle\cdot,\cdot\rangle)$ is complete with respect to its canonical distance $d(\cdot,\cdot)$ and $\mathcal{H}$ behaves subquadratically at spatial infinity, i.e., there exist $\bar{x} \in \mathcal{M}$ and (positive) continuous functions $R_1(u)$, $R_2(u)$, $p(u)$, with $p(u) < 2$, such that

$$-\mathcal{H}(x,u) \le R_1(u)\,d^{p(u)}(x,\bar{x}) + R_2(u) \quad \text{for all } (x,u) \in \mathcal{M} \times \mathbb{R},$$

then the spacetime is not only geodesically connected (cf. Corollary 4.5 in [10]) but also globally hyperbolic (cf. Theorem 4.1 in [17]). This suggests an intrinsic connection between these two properties, as the following simple consequence of our approach confirms.

Theorem 8.6. *Any globally hyperbolic GPW with a complete Cauchy hypersurface is geodesically connected.*

Proof. Let $(\mathcal{L}, \langle\cdot,\cdot\rangle_L)$ be a GPW. Clearly, $K = \partial_v$ is a complete lightlike Killing vector field on $\mathcal{L}$. Take any $p = (x_p, u_p, v_p)$, $q = (x_q, u_q, v_q) \in \mathcal{L}$, any curve $x = x(s)$ in $\mathcal{M}$ connecting x_p to x_q, and denote $\Delta_u = u_q - u_p$ and $\Delta_v = v_q - v_p$. The curve $\varphi(s) = (x(s), \Delta_u\, s, \Delta_v\, s)$ connects p to q, and the scalar product

$$\langle\dot{\varphi}, K(\varphi)\rangle_L = \dot{u} = \Delta_u$$

is constant. Therefore, the existence of a geodesic connecting p to q follows from Theorem 1.2. □

[6] Indeed, these results are also applicable to the much bigger class of *Brinkmann spacetimes* (for more details about such spacetimes see, e.g., [8] and references therein).

Remark 8.7. To the authors it is not clear if any globally hyperbolic GPW $(\mathcal{L}, \langle\cdot,\cdot\rangle_L)$, with $(\mathcal{M}, \langle\cdot,\cdot\rangle)$ complete, necessarily admits some complete Cauchy hypersurfaces. If this was true, in the hypotheses of Theorem 8.6 this last condition could be replaced by the completeness of $(\mathcal{M}, \langle\cdot,\cdot\rangle)$. This question is in connection with the following more general problem, which goes beyond the scope of the present article: finding general conditions on a globally hyperbolic spacetime which ensure that it admits some complete Cauchy hypersurfaces.

Acknowledgements. We warmly thank the referee for his/her useful suggestions. We also thank Prof. Miguel Sánchez for his careful reading of this paper, and Dr. Bill Karr for pointing out a small gap in a previous version of the example in Section 7 (c).

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Received March 17, 2015; revised May 18, 2015.

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R. Bartolo and A. M. Candela are partially supported by G.N.A.M.P.A. Research Project 2012 "Analisi Geometrica sulle varietà di Lorentz e applicazioni alla Relatività Generale".

A. M. Candela and J. L. Flores are partially supported by the Spanish MICINN Grant FEDER funds MTM2010-18099.

J. L. Flores is partially supported by the Spanish Grant MTM2013-47828-C2-2-P (MINECO and FEDER funds) and the Regional J. Andalucía Grant P09-FQM-4496, with FEDER funds.