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Characters of p'-degree
and Thompson's character degree theorem

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Abstract. A classical theorem of John Thompson on character degrees asserts that if the degree of every ordinary irreducible character of a finite group G is 1 or divisible by a prime p, then G has a normal p-complement. We obtain a significant improvement of this result by considering the average of p'-degrees of irreducible characters. We also consider fields of character values and prove several improvements of earlier related results.

1. Introduction

One of the classical results on character degrees is the celebrated theorem of J. G. Thompson, which asserts that if the degree of every ordinary irreducible character of a finite group G is 1 or divisible by a prime p, then G has a normal p-complement, see [25] or Corollary 12.2 of [8]. Let $\mathrm{acd}_{p'}(G)$ denote the average of p'-degrees of irreducible characters of G. Then this result can be reformulated as follows.

Thompson's theorem. *Let G be a finite group. If $\mathrm{acd}_{p'}(G) = 1$, then G has a normal p-complement.*

In this paper, we significantly improve Thompson's theorem in the point of view of $\mathrm{acd}_{p'}$ and investigate further the relation between characters of p'-degree and p-nilpotency.

Theorem 1.1. *Let p be an odd prime and G a finite group. We have*

(i) *if $\mathrm{acd}_{2'}(G) < 3/2$, then G has a normal 2-complement, and*

(ii) *if $\mathrm{acd}_{p'}(G) < 4/3$, then G has a normal p-complement.*

We emphasize that, in contrast to Thompson's theorem, where it is required that G has no nontrivial character degrees coprime to p at all, in Theorem 1.1 we

allow G to have nontrivial character degrees coprime to p, and we still can conclude the p-nilpotency of G as long as the number of linear characters of G is large enough.

A deep part in the proof of Theorem 1.1 is to prove the solvability of the groups in consideration. In fact, we obtain the following.

Theorem 1.2. *Let $p > 5$ be a prime and G a finite group. If one of the following happens*

(i) $\mathrm{acd}_{2'}(G) < 3$,

(ii) $\mathrm{acd}_{3'}(G) < 3$,

(iii) $\mathrm{acd}_{5'}(G) < 11/4$,

(iv) $\mathrm{acd}_{p'}(G) < 16/5$,

then G is solvable.

Given a finite group G, one can always find a prime $5 < p \nmid |G|$ so that $\mathrm{acd}_{p'}(G)$ is simply the average degree of *all* irreducible characters of G. Therefore Theorem 1.2 (iv) refines and improves the main result of [16]. We remark that, as illustrated by the nonsolvable groups A_5 and $\mathrm{SL}(2,5)$, all the bounds in Theorem 1.2 are best possible. Though the bounds in Theorem 1.1 also cannot be improved when $p = 2$ or 3, as shown by A_4 and S_3, we think that the correct bound when $p > 2$ is $(2p+2)/(p+3)$, attained at the dihedral group of order $2p$. We also remark that, by using the classification of finite simple groups with a nontrivial irreducible character degree less than 6, it is clear that the assumptions in the parts of Theorem 1.2 imply that G is not nonabelian simple; but it requires a lot more work to show that G is solvable.

It has been shown in several earlier works that there is a close connection between important characteristics of finite groups such as nilpotency, supersolvability, solvability, or p-solvability and invariants concerning character degrees such as the average character degree, the character degree sum, the largest character degree, or the character degree ratio, see [3]–[6], [9], [11], [13], [14], [16], [23]. Theorems 1.1 and 1.2 reinforce this phenomenon for characters of p'-degree.

Several refinements of Thompson's theorem have been proposed in the literature. One of the remarkable refinements is due to G. Navarro and P. H. Tiep [21]. They weakened the condition that all nonlinear irreducible characters of G have degree divisible by p, and assumed only that those characters with values in $\mathbb{Q}_p$ have this property. (Here $\mathbb{Q}_p$ is the cyclotomic field obtained by adjoining a primitive p-root of unity to $\mathbb{Q}$.) To state their result, we write $\mathrm{acd}_{\mathbb{F},p'}$ to denote the average of p'-degrees of irreducible characters of G with values in a field $\mathbb{F}$.

Theorem (Navarro and Tiep, [21]). *Let G be a finite group and let p be a prime. If $\mathrm{acd}_{\mathbb{Q}_p,p'}(G) = 1$, then G has a normal p-complement.*

We are able to prove the following.

Theorem 1.3. *Let p be an odd prime and G a finite group. We have:*

(i) *if $\mathrm{acd}_{\mathbb{Q},2'}(G) < 3/2$, then G has a normal 2-complement, and*

(ii) *if $\mathrm{acd}_{\mathbb{Q}_p,p'}(G) < 4/3$, then G has a normal p-complement.*

Theorem 1.3 implies several earlier results related to Thompson's theorem and fields of character values, including Theorem A in [18], Theorem A in [19], and Theorems A and C in [21]. Moreover, it has the following consequence.

Corollary 1.4. *Let p be an odd prime and G a finite group. Then we have:*

(i) *If $\operatorname{acd}_{\mathbb{Q}}(G) < 3/2$, then G has a normal 2-complement.*

(ii) *If $\operatorname{acd}_{\mathbb{Q}_p}(G) < 4/3$, then G has a normal p-complement.*

(iii) *If $\operatorname{acd}_{\mathbb{R},2'}(G) < 3/2$, then G has a normal 2-complement.*

(iv) *If $\operatorname{acd}_{\mathbb{R}}(G) < 3/2$, then G has a normal 2-complement.*

Proof. The statements (i) and (ii) are clear from Theorem 1.3. Since every real-valued character of degree 1 is also rational-valued, statement (iii) follows from Theorem 1.3 (i); and finally, (iv) follows from (i) or (iii). □

To prove the solvability, we utilize a character-orbit result on nonabelian simple groups of Navarro and Tiep, Theorem 3.3 in [21], to show that, if G has a non-abelian minimal normal subgroup N, then there exists $\psi \in \operatorname{Irr}(N)$ of large degree with many *good* properties such as $\psi(1)$ is coprime to p, ψ is extendible to the stabilizer $\operatorname{Stab}_G(\psi)$ of ψ in G, and $|G : \operatorname{Stab}_G(\psi)|$ is coprime to p, see Theorem 2.1. This, together with other results on bounding the number of irreducible characters of small degree in Section 3, allow us to control the average of p'-degrees. To go from solvability to p-nilpotency, we reduce the problem to the situation where G is a split extension of an abelian p-group, and then analyze the $\operatorname{acd}_{p'}$ of such a group. We hope that some new techniques in this paper will be further developed to study other problems on the connection between the average character degree, fields of character values, and the local structure of groups, see [7] for instance.

The paper is organized as follows. After some preparation results in Sections 2, 3, and 4, we prove Theorem 1.2 in Sections 5, 6, 7, and 8. Theorem 1.1 is then proved in Section 9. In Section 10 we establish some solvability results on the average of p'-degrees of rational-valued characters and $\mathbb{Q}_p$-valued characters in general. Finally, Theorem 1.3 is proved in Section 11.

2. Extending characters of p'-degree

We begin by setting up some notation. As usual, $\operatorname{Irr}(G)$ denotes the set of irreducible characters of a finite group G, and $\operatorname{Irr}_{p'}(G)$ the set of those characters of degree not divisible by p. If d is a positive integer, then $n_d(G)$ is the number of irreducible characters of G of degree d. If $N \trianglelefteq G$, then

$$\operatorname{Irr}(G|N) := \{\chi \in \operatorname{Irr}(G) \mid N \not\subseteq \operatorname{Ker}(\chi)\},$$
$$\operatorname{Irr}_{p'}(G|N) := \{\chi \in \operatorname{Irr}(G) \mid N \not\subseteq \operatorname{Ker}(\chi), p \nmid \chi(1)\},$$
$$n_d(G|N) := |\{\chi \in \operatorname{Irr}(G|N)) \mid \chi(1) = d\}|.$$

We also write $\operatorname{acd}_{p'}(G|N)$ to denote the average degree of the characters in $\operatorname{Irr}_{p'}(G|N)$. Furthermore, if $\theta \in \operatorname{Irr}(N)$ then $\operatorname{Irr}_{p'}(G|\theta)$ denotes the set of irreducible

characters of degree coprime to p of G that lie over θ, and $\mathrm{acd}_{p'}(G|\theta)$ denotes the average degree of the characters in $\mathrm{Irr}_{p'}(G|\theta)$. Finally, whenever a field $\mathbb{F}$ is put into the subscript of any of these notation, we mean that the characters in consideration have values in $\mathbb{F}$.

The following result plays an important role in the proof of Theorem 1.2. It helps us to bound the number of irreducible characters of small degree in finite groups with a nonabelian minimal normal subgroup.

Theorem 2.1. *Let p be a prime. Let G be a finite group with a nonabelian minimal normal subgroup $N \ncong \mathsf{A}_5$. Then there exists $\psi \in \mathrm{Irr}(N)$ such that*

(i) *$\psi(1) \geq 7$ and $\psi(1)$ is coprime to p,*

(ii) *ψ is extendible to a $\mathbb{Q}_p$-valued character of $\mathrm{Stab}_G(\psi)$, and*

(iii) *$|G : \mathrm{Stab}_G(\psi)|$ is coprime to p.*

To prove this theorem, we need the following character-orbit result for finite simple groups, which is essentially due to Navarro and Tiep.

Lemma 2.2. *Let p be a prime and S be a nonabelian finite simple group. Then there exists an orbit $\mathcal{O}$ of the action of $\mathrm{Aut}(S)$ on $\mathrm{Irr}(S)$ satisfying the following conditions:*

(i) *every $\theta \in \mathcal{O}$ is nontrivial of degree at least 4 and coprime to p,*

(ii) *$|\mathcal{O}|$ is coprime to p, and*

(iii) *every $\theta \in \mathcal{O}$ extends to a $\mathbb{Q}_p$-valued character of $\mathrm{Stab}_{\mathrm{Aut}(S)}(\theta)$.*

Furthermore, if $S \ncong \mathsf{A}_5$ then $\mathcal{O}$ can be chosen so that $\theta(1) \geq 7$ for every $\theta \in \mathcal{O}$.

Proof. The orbit $\mathcal{O}$ has been constructed in [21], Theorem 3.3, but without the condition that $\theta(1) \geq 7$ when $S \ncong \mathsf{A}_5$. The case $S \cong \mathsf{A}_5$ is clear from [2], p. 2.

Though by following the proof in [21] one can show that $\theta(1) \geq 7$ when $S \ncong \mathsf{A}_5$, we propose here another way to verify it. First, by [24] the smallest nontrivial degree of the alternating group A_n is $n - 1$ when $n \geq 6$. Together with results on the low-degree characters of simple groups of Lie type in [12], [22], [26], and the character tables of the sporadic simple groups in [2], we can check that if a nonabelian simple group S is not one of A_5, A_6, A_7, $\mathrm{PSL}(2,7)$, $\mathrm{PSL}(2,11)$, and $\mathrm{PSU}(3,3)$, then the smallest degree of a nontrivial irreducible character of S is at least 7, and thus the condition $\theta(1) \geq 7$ is automatically satisfied. For the exceptional groups, the desired orbit can be found easily from [2]. $\qquad\square$

Now we use the orbit $\mathcal{O}$ to prove Theorem 2.1.

Proof of Theorem 2.1. Since N is a nonabelian minimal normal subgroup of G, it is direct product of r copies of a nonabelian simple group, say S. Replacing G by $G/\mathbf{C}_G(N)$ if necessary, we may assume that $\mathbf{C}_G(N) = 1$. Then we have

$$N \trianglelefteq G \leq \mathrm{Aut}(N) = \mathrm{Aut}(S) \wr \mathsf{S}_r.$$

Let θ be an irreducible character of S in the orbit $\mathcal{O}$ found in Lemma 2.2. Consider the character $\varphi := \theta \times \cdots \times \theta \in \mathrm{Irr}(N)$. The stabilizer of φ in $\mathrm{Aut}(N)$ is

$$\mathrm{Stab}_{\mathrm{Aut}(N)}(\varphi) = \mathrm{Stab}_{\mathrm{Aut}(S)}(\theta) \wr \mathsf{S}_r.$$

By the choice of $\mathcal{O}$, θ extends to an irreducible $\mathbb{Q}_p$-valued character, say α, of $\mathrm{Stab}_{\mathrm{Aut}(S)}(\theta)$. Thus φ extends to the character $\alpha \times \cdots \times \alpha$ of $\mathrm{Stab}_{\mathrm{Aut}(S)}(\theta) \times \cdots \times \mathrm{Stab}_{\mathrm{Aut}(S)}(\theta)$, which is the base group of the wreath product $\mathrm{Stab}_{\mathrm{Aut}(S)}(\theta) \wr \mathsf{S}_r$. Since $\alpha \times \cdots \times \alpha$ is invariant under $\mathrm{Stab}_{\mathrm{Aut}(N)}(\varphi)$, it follows from Lemma 1.3 in [15] that $\alpha \times \cdots \times \alpha$ is extendible to $\mathrm{Stab}_{\mathrm{Aut}(N)}(\varphi)$. We deduce that φ is extendible to $\mathrm{Stab}_{\mathrm{Aut}(N)}(\varphi)$. Let $\phi \in \mathrm{Irr}(\mathrm{Stab}_{\mathrm{Aut}(N)}(\varphi))$ be an extension of φ. By the formula for character values given in [15], Lemma 1.3, we can choose ϕ so that its values are contained in the field of values of α. That is, ϕ is $\mathbb{Q}_p$-valued.

Now we consider the action of G on the set

$$\mathcal{C} := \{\theta_1 \times \cdots \times \theta_r \mid \theta_i \in \mathcal{O}\}$$

of irreducible characters of N. Since the cardinality of this set is $|\mathcal{O}|^r$, which is not divisible by p by the choice of $\mathcal{O}$, there must be a G-orbit of length coprime to p. Let $\psi \in \mathcal{C}$ be a character in such an orbit. We then have that $|G : \mathrm{Stab}_G(\psi)|$ is coprime to p.

Note that $\mathrm{Aut}(N)$ acts transitively on $\mathcal{C}$. Therefore there is some $x \in \mathrm{Aut}(N)$ such that $\psi = \varphi^x$, and hence $\mathrm{Stab}_{\mathrm{Aut}(N)}(\psi) = \mathrm{Stab}_{\mathrm{Aut}(N)}(\varphi)^x$. As $\phi \in \mathrm{Irr}(\mathrm{Stab}_{\mathrm{Aut}(N)}(\varphi))$ is an extension of φ, we deduce that $\phi^x \in \mathrm{Irr}(\mathrm{Stab}_{\mathrm{Aut}(N)}(\psi))$ is an extension of ψ to $\mathrm{Stab}_{\mathrm{Aut}(N)}(\psi)$. In particular, $\phi^x \downarrow_{\mathrm{Stab}_G(\psi)}$ is an extension of ψ to $\mathrm{Stab}_G(\psi) = G \cap \mathrm{Stab}_{\mathrm{Aut}(N)}(\psi)$. We observe that, as ϕ is $\mathbb{Q}_p$-valued, ϕ^x is $\mathbb{Q}_p$-valued as well. Finally, we note that $\psi(1) = \theta(1)^r$ is not divisible by p and that $\psi(1) \geq 7$ if $(S, r) \neq (\mathsf{A}_5, 1)$. $\square$

3. Bounding the number of characters of small degree

We begin the section with the following observation.

Lemma 3.1. *Let G be a finite group and $T \leq G$. Then*

(i) $n_1(G) \leq n_1(T)|G : T|$,

(ii) $n_2(G) \leq n_2(T)|G : T| + \frac{1}{2}n_1(T)|G : T|$, *and*

(iii) $n_3(G) \leq n_3(T)|G : T| + \frac{1}{3}n_1(T)|G : T|$.

Proof. (i) This is clear since $n_1(G) = |G : G'|$ and $n_1(T) = |T : T'|$.

(ii) Let $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = 2$. Take ϕ to be an irreducible constituent of $\chi \downarrow_T$. Frobenius reciprocity then implies that χ in turn is an irreducible constituent of ϕ^G. If $\phi(1) = 2$ then, as $\phi^G(1) = 2|G : T|$, there are at most $|G : T|$ irreducible constituents of degree 2 of ϕ^G. We deduce that there are at most $n_2(T)|G : T|$ irreducible characters of degree 2 of G that arise as constituents of ϕ^G with $\phi(1) = 2$.

On the other hand, if $\phi(1) = 1$ then, as $\phi^G(1) = |G : T|$, there are at most $|G : T|/2$ irreducible constituents of degree 2 of ϕ^G. As above, we deduce that there are at most $n_1(T)|G : T|/2$ irreducible characters of degree 2 of G that arise as constituents of ϕ^G with $\phi(1) = 1$. Now (ii) is proved.

(iii) This can be argued similarly as in (ii). Let $\chi \in \text{Irr}(G)$ with $\chi(1) = 3$. Then $\chi \downarrow_T$ is either irreducible, or a sum of three linear characters of T, or a sum of one linear character and one irreducible character of degree 2 of T. In particular, if $\chi \downarrow_T$ is reducible then there is always a linear constituent in $\chi \downarrow_T$. Now we see that there are at most $n_3(T)|G : T|$ irreducible characters of degree 3 of G that arise as constituents of ϕ^G with $\phi(1) = 3$ and there are at most $n_1(T)|G : T|/3$ irreducible characters of degree 3 of G that arise as constituents of ϕ^G with $\phi(1) = 1$. The proof is complete. $\quad\square$

Lemma 3.1 can help us to bound $n_1(G)$, $n_2(G)$, and $n_3(G)$ in terms of the number of irreducible characters of larger degree, especially in the case G has a nonabelian minimal normal subgroup.

Proposition 3.2. *Let G be a finite group with a nonabelian minimal normal subgroup N. Assume that there is some $\psi \in \text{Irr}(N)$ such that ψ is extendible to $\text{Stab}_G(\psi)$ and let $T := \text{Stab}_G(\psi)$ and $a := \psi(1)|G : T|$. We have:*

(i) $n_1(G) \le n_a(G)|G : T|$, *and*

(ii) $n_2(G) \le n_{2a}(G)|G : T| + \frac{1}{2}n_a(G)|G : T|$,

Moreover, if $G = T$ then $n_2(G) \le n_{2a}(G)$.

Proof. First, by Lemma 3.1 (i) we have $n_1(G) \le n_1(T)|G : T|$. On the other hand, as $N = N' \subseteq T'$, N is contained in the kernel of every linear character of T so that $n_1(T) = n_1(T/N)$. It follows that

$$n_1(G) \le n_1(T/N)|G : T|.$$

Recall that $\psi \in \text{Irr}(N)$ is extendible to T and so we let $\chi \in \text{Irr}(T)$ be an extension of ψ. Using Gallagher's theorem and Clifford's theorem (see Corollary 6.17 and Theorem 6.11 in [8]), we see that each linear character λ of T/N produces the irreducible character $\lambda\chi$ of T of degree $\psi(1)$, and this character in turn produces the irreducible character $(\lambda\chi)^G$ of G of degree $(\lambda\chi)^G(1) = \psi(1)|G : T| = a$. It follows that

$$n_1(T/N) \le n_a(G),$$

and we therefore have

$$n_1(G) \le n_a(G)\,|G : T|,$$

as claimed in (i).

We now prove (ii). By Lemma 3.1 (ii), we have that $n_2(G) \le n_2(T)|G : T| + \frac{1}{2}n_1(T)|G : T|$. Since we have already proved that $n_1(T) = n_1(T/N) \le n_a(G)$, it remains to prove that $n_2(T) \le n_{2a}(G)$.

We claim that $n_2(T) = n_2(T/N)$ or in other words N is contained in the kernel of every irreducible character of degree 2 of T. Let $\phi \in \text{Irr}(T)$ with $\phi(1) = 2$.

Since N has no irreducible character of degree 2 (see Problem 3.3 of [8]) and has only one linear character, which is the trivial one, it follows that $\phi_N = 2 \cdot 1_N$. We then have $N \subseteq \mathrm{Ker}(\phi)$, as claimed.

Recall that $\chi \in \mathrm{Irr}(T)$ is an extension of ψ. Using Gallagher's theorem and Clifford's theorem again, we obtain that each irreducible character $\mu \in \mathrm{Irr}(T/N)$ of degree 2 produces the character $(\mu\chi)^G \in \mathrm{Irr}(G)$ of degree $(\mu\chi)^G(1) = 2\psi(1)|G:T| = 2a$. It follows that

$$n_2(T/N) \leq n_{2a}(G),$$

and thus $n_2(T) \leq n_{2a}(G)$, as wanted. $\square$

Proposition 3.3. *Let G be a finite group with a nonabelian minimal normal subgroup N, which has no direct factor isomorphic to A_5 or $\mathrm{PSL}(2,7)$. Assume that there is some $\psi \in \mathrm{Irr}(G)$ such that ψ is extendible to $\mathrm{Stab}_G(\psi)$ and let $T := \mathrm{Stab}_G(\psi)$ and $a := \psi(1)|G:T|$. Then*

$$n_3(G) \leq n_{3a}(G)|G:T| + \frac{1}{3}n_a(G)|G:T|.$$

Moreover, if $G = T$ then $n_3(G) \leq n_{3a}(G)$.

Proof. By Lemma 3.1 (iii), we have $n_3(G) \leq n_3(T)|G:T| + \frac{1}{3}n_1(T)|G:T|$. As in the proof of Proposition 3.2, we have $n_1(T) \leq n_a(G)$. So it suffices to show that $n_3(T) \leq n_{3a}(G)$.

It is well known that A_5 and $\mathrm{PSL}(2,7)$ are the only finite simple groups having an irreducible character of degree 3. Therefore, every nontrivial irreducible character of N has degree at least 4 and, by using the same arguments as in the proof of Proposition 3.2, we see that N is contained in the kernel of every irreducible character of degree 3 of T. In other words we have $n_3(T) = n_3(T/N)$.

Recall that ψ is extendible to T and let $\chi \in \mathrm{Irr}(T)$ be an extension of ψ. We then obtain an injection $\nu \mapsto (\nu\chi)^G$ from the set of irreducible characters of T/N of degree 3 to the set of irreducible characters of G of degree $(\nu\chi)^G(1) = 3\psi(1)|G:T| = 3a$. It follows that $n_3(T/N) \leq n_{3a}(G)$, and therefore $n_3(T) \leq n_{3a}(G)$, which completes the proof. $\square$

4. Characters of a central product

The proof of Theorem 1.2 requires us to analyze the characters of a particular central product. This central product indeed has already appeared in the study of the average of *all* irreducible character degrees of a finite group, see [9], [16].

Proposition 4.1. *Let $L \cong \mathrm{SL}(2,5)$ and $G = LC$ be a central product with the central amalgamated subgroup $Z := \mathbf{Z}(L) = L \cap C$ such that $L \subseteq G'$. Assume that G has an irreducible character of degree 2 such that $Z \nsubseteq \mathrm{Ker}(\chi)$. Then*

  (i) $n_2(G) = 2n_1(G) + n_2(C/Z)$,

  (ii) $n_3(G) \geq 2n_1(G)$,

(iii) $n_4(G) \geq 2n_1(G)$,

(iv) $n_5(G) \geq n_1(G)$,

 (v) $n_6(G) \geq n_1(G)$, and

(vi) $n_8(G) \geq n_2(C/N)$.

Proof. Since $G = LC$ is a central product with the central amalgamated subgroup Z, there is a bijection $(\alpha, \beta) \mapsto \tau$ from $\mathrm{Irr}(L|Z) \times \mathrm{Irr}(C|Z)$ to $\mathrm{Irr}(G|Z)$ such that $\tau(1) = \alpha(1)\beta(1)$.

By hypothesis, there is $\chi \in \mathrm{Irr}(G|Z)$ such that $\chi(1) = 2$. If $(\alpha, \beta) \mapsto \chi$ under the above bijection, we must have $\beta(1) = 1$ since $L \cong \mathrm{SL}(2,5)$ and there are only three possibilities for $\alpha(1)$, namely 2, 4, and 6. So $\beta \in \mathrm{Irr}(C|Z)$ is an extension of the unique nonprincipal linear character of Z. Using Gallagher's theorem, we then have a degree-preserving bijection from $\mathrm{Irr}(C/Z)$ to $\mathrm{Irr}(C|Z)$. In particular,

$$n_1(C|Z) = n_1(C/Z)$$

Since $G/L \cong C/Z$ and $L \subseteq G'$, we have

$$n_1(C|Z) = n_1(C/Z) = n_1(G).$$

Now we evaluate $n_2(G)$. As $n_1(L|Z) = 0$ and $n_2(L|Z) = 2$, we have

$$n_2(G|Z) = n_1(L|Z)n_2(C|Z) + n_2(L|Z)n_1(C|Z) = 2n_1(C|Z) = 2n_1(G).$$

Note that there is also a bijection $(\alpha, \beta) \mapsto \tau$ from $\mathrm{Irr}(L/Z) \times \mathrm{Irr}(C/Z)$ to $\mathrm{Irr}(G/Z)$ such that $\tau(1) = \alpha(1)\beta(1)$. Therefore, as $n_1(L/Z) = 1$ and $n_2(L/Z) = 0$, we have

$$n_2(G/Z) = n_1(L/Z)n_2(C/Z) + n_2(L/Z)n_1(C/Z) = n_2(C/Z),$$

and it follows that

$$n_2(G) = n_2(G/Z) + n_2(G|Z) = n_2(C/Z) + 2n_1(G).$$

Next we estimate $n_4(G)$, $n_5(G)$, and $n_6(G)$. We have

$$n_4(G|Z) \geq n_4(L|Z)n_1(C|Z) \geq n_1(C|Z) = n_1(G)$$

since $n_4(L|Z) = 1$, and

$$n_4(G/Z) \geq n_4(L/Z)n_1(C/Z) = n_1(C/Z) = n_1(G)$$

since $n_4(L/Z) = 1$. We deduce that

$$n_4(G) = n_4(G|Z) + n_4(G/Z) \geq 2n_1(G).$$

Similarly,

$$n_5(G) \geq n_5(G/Z) \geq n_5(L/Z)n_1(C/Z) = n_1(C/Z) = n_1(G)$$

since $n_5(L/Z) = 1$, and

$$n_6(G) \geq n_6(G|Z) \geq n_6(L|Z)n_1(C|Z) = n_1(G)$$

since $n_6(L|Z) = 1$.

Finally, we estimate $n_8(G)$ by

$$n_8(G) \geq n_8(G/N) \geq n_4(L/N)n_2(C/N) \geq n_2(C/N),$$

and we have completed the proof.   □

5. Characters of odd degree and solvability

In this section we prove Theorem 1.2 (i), which we restate below for the reader's convenience.

Theorem 5.1. *Let G be a finite group. If* $\mathrm{acd}_{2'}(G) < 3$, *then G is solvable.*

Proof. Assume that the theorem is false, and let G be a minimal counterexample. In particular, G is nonsolvable and $\mathrm{acd}_{2'}(G) < 3$. Then we have

$$\frac{\sum_{d \text{ odd}} d\, n_d(G)}{\sum_{d \text{ odd}} n_d(G)} < 3,$$

and hence

$$\sum_{d \geq 5 \text{ odd}} (d-3)\, n_d(G) < 2n_1(G).$$

Since G is nonsolvable, G' is nontrivial and therefore we can choose a minimal normal subgroup N of G such that $N \subseteq G'$. So N is contained in the kernel of every linear character of G so that $n_1(G) = n_1(G/N)$. Therefore

$$\sum_{d \geq 5 \text{ odd}} (d-3)n_d(G/N) \leq \sum_{d \geq 5 \text{ odd}} (d-3)\, n_d(G) < 2n_1(G) = 2n_1(G/N)$$

and it follows that

$$\mathrm{acd}_{2'}(G/N) < 3.$$

By the minimality of G, we deduce that G/N is solvable. But G is nonsolvable, so N is a nonabelian minimal normal subgroup of G. Theorem 2.1 then implies that N has an irreducible character ψ with three properties:

(i) $\psi(1) \geq 5$ is odd,

(ii) ψ is extendible to $\mathrm{Stab}_G(\psi)$, and

(iii) $|G : \mathrm{Stab}_G(\psi)|$ is odd.

(In Theorem 2.1 we in fact assume that $N \not\cong \mathsf{A}_5$. But one easily sees that if $N \cong \mathsf{A}_5$ then the character ψ can be chosen to be the unique irreducible character of degree 5 of N.)

Now applying Proposition 3.2 (i), we have

$$n_1(G) \leq n_a(G)|G : T|,$$

where $T := \mathrm{Stab}_G(\psi)$ and $a := \psi(1)|G : T|$. As $\psi(1) \geq 5$, we have $a \geq 5|G : T|$, and it follows that

$$n_1(G) \leq \frac{1}{2}\,n_a(G)\,(a - 3).$$

Using the fact that $a = \psi(1)|G : T|$ is odd by the choice of ψ, we arrive at

$$n_1(G) \leq \frac{1}{2} \sum_{d \geq 5 \text{ odd}} (d - 3)\,n_d(G)$$

and, equivalently, $\mathrm{acd}_{2'}(G) \geq 3$. This contradiction completes the proof.   □

6. Characters of $3'$-degree and solvability

In this section we prove Theorem 1.2 (ii).

Theorem 6.1. *Let G be a finite group. If $\mathrm{acd}_{3'}(G) < 3$, then G is solvable.*

First we handle the groups with a nonabelian minimal normal subgroup.

Proposition 6.2. *Let G be a finite group with a nonabelian minimal normal subgroup N. Then $\mathrm{acd}_{3'}(G) \geq 3$.*

Proof. Suppose that N is direct product of r copies of S, a nonabelian simple group. First we consider the case $N \cong \mathsf{A}_5$. Then N has an irreducible character of degree 5 that is extendible to G. Applying Proposition 3.2, we have

$$n_1(G) \leq n_5(G) \quad \text{and} \quad n_2(G) \leq n_{10}(G).$$

It follows that

$$2n_1(G) + n_2(G) \leq 2n_5(G) + n_{10}(G)$$

and hence

$$2n_1(G) + n_2(G) \leq \sum_{3 \nmid d,\, d \geq 4} (d - 3)\,n_d(G),$$

which is equivalent to $\mathrm{acd}_{3'}(G) \geq 3$, and we are done.

So we may assume that $N \not\cong \mathsf{A}_5$. By Theorem 2.1, there is some $\psi \in \mathrm{Irr}(N)$ with the conditions:

  (i) $\psi(1) \geq 7$ and $3 \nmid \psi(1)$,

 (ii) ψ is extendible to $\mathrm{Stab}_G(\psi)$, and

(iii) $3 \nmid |G : \mathrm{Stab}_G(\psi)|$.

We then apply Proposition 3.2 to have

$$n_1(G) \leq n_a(G)|G:T| \quad \text{and} \quad n_2(G) \leq n_{2a}(G)|G:T| + \frac{1}{2}n_a(G)|G:T|,$$

where $T := \mathrm{Stab}_G(\psi)$ and $a := \psi(1)|G:T|$. It then follows that

$$2n_1(G) + n_2(G) \leq \frac{5}{2}n_a(G)|G:T| + n_{2a}(G)|G:T|.$$

Note that $\psi(1) \geq 7$, and thus $a \geq 7|G:T|$. So we have $(5/2)|G:T| < a - 3$ and $|G:T| < 2a - 3$. Therefore

$$2n_1(G) + n_2(G) < (a-3)n_a(G) + (2a-3)n_{2a}(G).$$

As a is coprime to 3, we deduce that

$$2n_1(G) + n_2(G) < \sum_{3 \nmid d,\, d \geq 4} (d-3)\, n_d(G),$$

which is equivalent to $\mathrm{acd}_{3'}(G) > 3$, and we are done again. $\qquad\square$

Now we are ready to prove Theorem 6.1. We write $\mathbf{O}_\infty(G)$ to denote the largest solvable normal subgroup of G.

Proof of Theorem 6.1. Assume that the theorem is false and let G be a minimal counterexample. Then G is nonsolvable and $\mathrm{acd}_{3'}(G) < 3$.

Let $L \lhd G$ be minimal such that L is non-solvable. Then clearly L is perfect and contained in the last term of the derived series of G. Let $N \subseteq L$ be a minimal normal subgroup of G. We choose N so that $N \leq [L, \mathbf{O}_\infty(L)]$ if $[L, \mathbf{O}_\infty(L)]$ is nontrivial. We then have $N \subseteq L = L' \subseteq G'$.

If N is nonabelian then $\mathrm{acd}_{3'}(G) \geq 3$ by Proposition 6.2, and this is a contradiction. So we may assume that N is abelian so that G/N is nonsolvable. By the minimality of G, it follows that $\mathrm{acd}_{3'}(G/N) \geq 3$ and hence

$$\mathrm{acd}_{3'}(G) < 3 \leq \mathrm{acd}_{3'}(G/N).$$

Note that $n_d(G) \geq n_d(G/N)$ for every positive integer d and $n_1(G) = n_1(G/N)$ since $N \subseteq G'$. We then deduce that

$$n_2(G) > n_2(G/N).$$

That is, there is some $\chi \in \mathrm{Irr}(G)$ of degree 2 whose kernel does not contain N.

Now let $C/\mathrm{Ker}(\chi) := \mathbf{Z}(G/\mathrm{Ker}(\chi))$. Arguing similarly as in the proof of Theorem 2.2 in [9], we obtain that $G/C \cong \mathsf{A}_5$, $L \cong \mathrm{SL}(2,5)$, and $G = LC$ is a central product with the central amalgamated subgroup $Z := L \cap C = \mathbf{Z}(L)$.

We are now in the situation of Proposition 4.1. Therefore,

$$2n_1(G) + n_2(G) = 4n_1(G) + n_2(C/Z) \leq n_4(G) + 2n_5(G) + n_8(G)$$
$$\leq \sum_{3 \nmid d,\, d \geq 4} (d-3)\, n_d(G).$$

It then follows that $\mathrm{acd}_{3'}(G) \geq 3$ and this is a contradiction. $\qquad\square$

7. Characters of $5'$-degree and solvability

In this section we prove Theorem 1.2 (iii).

Theorem 7.1. *Let G be a finite group. If* $\mathrm{acd}_{5'}(G) < 11/4$, *then G is solvable.*

As in Section 6, we first handle finite groups with a nonabelian minimal normal subgroup.

Proposition 7.2. *Let G be a finite group with a nonabelian minimal normal subgroup N. Then* $\mathrm{acd}_{5'}(G) \geq 11/4$.

Proof. As before, we suppose that N is direct product of r copies of a nonabelian simple group S. First we consider $N \cong \mathsf{A}_5$. Then N has an irreducible character of degree 4 that is extendible to G. Applying Proposition 3.2, we have $n_1(G) \leq n_4(G)$ and $n_2(G) \leq n_8(G)$.

Now we need to estimate $n_3(G)$ and $n_6(G)$. Observe that N has two irreducible characters of degree 3 and let us denote them by ψ_1 and ψ_2. Then it is easy to see that both ψ_1 and ψ_2 are extendible to

$$T := \mathrm{Stab}_G(\psi_1) = \mathrm{Stab}_G(\psi_2) = N \times \mathbf{C}_G(N).$$

Since N has index 2 in $\mathrm{Aut}(N) = \mathsf{S}_5$, we have

$$|G : T| = 1 \text{ or } 2.$$

If $|G : T| = 1$ then each linear character of G, which can be considered as a linear character of G/N, produces two irreducible characters of G of degree 3, one lying above ψ_1 and the other lying above ψ_2. We then obtain that $2n_1(G) \leq n_3(G)$. Now taking $n_1(G) \leq n_4(G)$ and $n_2(G) \leq n_8(G)$ into account, we have

$$7n_1(G) + 3n_2(G) \leq n_3(G) + 5n_4(G) + 3n_8(G).$$

Therefore

$$7n_1(G) + 3n_2(G) \leq \sum_{5 \nmid d,\, d \geq 3} (4d - 11)\, n_d(G),$$

and thus $\mathrm{acd}_{5'}(G) \geq 11/4$, as desired.

If $|G : T| = 2$ then by Proposition 3.2(1) we have $n_1(G) \leq 2n_6(G)$. Similarly we have

$$7n_1(G) + 3n_2(G) \leq 5n_4(G) + 3n_8(G) + 4n_6(G).$$

Therefore

$$7n_1(G) + 3n_2(G) \leq \sum_{5 \nmid d,\, d \geq 3} (4d - 11)\, n_d(G),$$

and we are done again.

From now on to the end of the proof we can assume that $N \not\cong \mathsf{A}_5$ and we will argue as in the proof of Proposition 6.2. By Theorem 2.1, there is some $\psi \in \mathrm{Irr}(N)$ such that $\psi(1) \geq 7$, $5 \nmid \psi(1)$, ψ is extendible to $\mathrm{Stab}_G(\psi)$, and $5 \nmid |G : \mathrm{Stab}_G(\psi)|$.

We then apply Proposition 3.2 to have

$$n_1(G) \leq n_a(G)|G : T| \quad \text{and} \quad n_2(G) \leq n_{2a}(G)|G : T| + \frac{1}{2}n_a(G)|G : T|,$$

where $T := \mathrm{Stab}_G(\psi)$ and $a := \psi(1)|G : T|$. It then follows that

$$7n_1(G) + 3n_2(G) \leq \frac{17}{2}n_a(G)|G : T| + 3n_{2a}(G)|G : T|.$$

Since $\psi(1) \geq 7$, we have $a \geq 7|G : T|$, and therefore $(17/2)|G : T| < 4a - 11$ and $3|G : T| < 8a - 11$. We deduce that

$$7n_1(G) + 3n_2(G) < (4a - 11)n_a(G) + (8a - 11)n_{2a}(G),$$

and it follows that $\mathrm{acd}_{5'}(G) > 11/4$. The proof is complete. □

Proof of Theorem 7.1. Assume that the theorem is false and let G be a minimal counterexample. Then G is nonsolvable and $\mathrm{acd}_{5'}(G) < 11/4$.

By using Proposition 7.2 and choosing the subgroups L, N, and C as in the proof of Theorem 6.1, we have that $G/C \cong \mathsf{A}_5$, $L \cong \mathrm{SL}(2, 5)$, and $G = LC$ is a central product with the central amalgamated subgroup $Z := L \cap C = \mathbf{Z}(L)$.

Applying Proposition 4.1, we deduce that

$$7n_1(G) + 3n_2(G) = 13n_1(G) + 3n_2(C/Z) \leq n_3(G) + 5n_4(G) + n_6(G) + 3n_8(G)$$

$$\leq \sum_{5 \nmid d,\, d \geq 3} (4d - 11)\, n_d(G).$$

From this it follows that $\mathrm{acd}_{5'}(G) \geq 11/4$, violating the assumption. □

8. Characters of p'-degree for $p > 5$ and solvability

We now prove Theorem 1.2(iv) and therefore complete the proof of Theorem 1.2.

Theorem 8.1. *Let $p > 5$ be a prime and G and finite group. If $\mathrm{acd}_{p'}(G) < 16/5$ then G is solvable.*

Unlike the proofs for the smaller primes, the proof of Theorem 8.1 requires upper bounds for not only $n_1(G)$ and $n_2(G)$ but also $n_3(G)$ and this makes things harder. Since the simple groups A_5 and $\mathrm{PSL}(2, 7)$ have some irreducible characters of degree 3, they need special attention.

Lemma 8.2. *Let G be a finite group with a minimal normal subgroup N, which is direct product of r copies of a nonabelian simple group S. We have*

(i) *if $S \cong \mathsf{A}_5$ then N has two irreducible characters of degrees 4^r and 5^r which are both extendible to G; and*

(ii) *if $S \cong \mathrm{PSL}(2, 7)$ then N has three irreducible characters of degrees $6^r, 7^r, 8^r$ which are all extendible to G.*

Proof. This is almost obvious as A_5 has two irreducible characters of degrees 4 and 5 which are both extendible to $\mathrm{Aut}(\mathsf{A}_5) = \mathsf{S}_5$, and $\mathrm{PSL}(2,7)$ has three irreducible characters of degrees $6, 7, 8$, which are all extendible to $\mathrm{Aut}(\mathrm{PSL}(2,7)) = \mathrm{PGL}(2,7)$. □

Using the previous lemma and the techniques in the proofs of Propositions 3.2 and 3.3, we have the following.

Proposition 8.3. *Let G be a finite group with a nonabelian minimal normal subgroup N, which is direct product of r copies of a simple group S. We have:*

(i) *if $S \cong \mathsf{A}_5$ then $n_1(G) \leq \min\{n_{4^r}(G), n_{5^r}(G)\}$, $n_2(G) \leq n_{2 \cdot 5^r}(G)$, and $n_3(G) \leq n_{3 \cdot 5^r}(G) + 2rn_1(G)$; and*

(ii) *if $S \cong \mathrm{PSL}(2,7)$ then $n_1(G) \leq n_{8^r}(G)$, $n_2(G) \leq n_{2 \cdot 8^r}(G)$, and $n_3(G) \leq n_{3 \cdot 8^r}(G) + 2rn_1(G)$.*

Proof. The proofs of (i) and (ii) are fairly similar, so let us prove (i) only. So assume that $S \cong \mathsf{A}_5$. Indeed, the inequalities $n_1(G) \leq \min\{n_{4^r}(G), n_{5^r}(G)\}$ and $n_2(G) \leq n_{2 \cdot 5^r}(G)$ already follows from Proposition 3.2 and hence it remains to prove $n_3(G) \leq n_{3 \cdot 5^r}(G) + 2rn_1(G)$.

Since N has an irreducible character of degree 5^r that is extendible to G, Gallagher's theorem implies that there is an injection from the irreducible characters of degree 3 of G/N to the irreducible characters of degree $3 \cdot 5^r$ of G. That is

$$n_3(G/N) \leq n_{3 \cdot 5^r}(G).$$

Now we need to bound the number of irreducible characters of G of degree 3 whose kernels do not contain N. So let $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) = 3$ and $N \nsubseteq \mathrm{Ker}(\chi)$. Since N has no nonprincipal linear character and no irreducible character of degree 2, the restriction $\chi \downarrow_N$ must be irreducible. By Gallagher's theorem, the number of irreducible characters of G of degree 3 lying over $\chi \downarrow_N$ equals to $n_1(G/N)$, which is the same as $n_1(G)$. Note that $\chi \downarrow_N$ has degree 3 and N has exactly $2r$ irreducible characters of degree 3. We conclude that the number of irreducible characters of G of degree 3 whose kernels do not contain N is at most $2kn_1(G)$. Now we have $n_3(G|N) \leq 2rn_1(G)$ and thus

$$n_3(G) = n_3(G/N) + n_3(G|N) \leq n_{3 \cdot 5^r}(G) + 2rn_1(G),$$

as desired. □

The next result is a refinement of Proposition 3 in [16] for characters of p'-degrees.

Proposition 8.4. *Let $p > 5$ be a prime. Let G be a finite group with a nonabelian minimal normal subgroup N. Then $\mathrm{acd}_{p'}(G) \geq 16/5$.*

Proof. Suppose that N is direct product of r copies of S, a nonabelian simple group.

First we assume that $S \ncong \mathsf{A}_5$ and $S \ncong \mathrm{PSL}(2,7)$. It then follows from Theorem 2.1 that there is some $\psi \in \mathrm{Irr}(N)$ such that $\psi(1) \geq 7$, $\psi(1)$ is coprime to p, ψ is extendible to $\mathrm{Stab}_G(\psi)$, and $|G : \mathrm{Stab}_G(\psi)|$ is coprime to p.

Propositions 3.2 and 3.3 then imply that

$$n_1(G) \leq n_a(G)|G : T|,$$

$$n_2(G) \leq n_{2a}(G)|G : T| + \frac{1}{2}\, n_a(G)|G : T|, \quad \text{and}$$

$$n_3(G) \leq n_{3a}(G)|G : T| + \frac{1}{3}\, n_a(G)|G : T|,$$

where $T := \operatorname{Stab}_G(\psi)$ and $a := \psi(1)|G : T|$. Now we can estimate

$$
\begin{aligned}
11n_1(G) + 6n_2(G) + n_3(G) &\leq \frac{43}{3}\, n_a(G)|G : T| + 6n_{2a}(G)|G : T| + n_{3a}(G)|G : T| \\
&< (5a - 16)n_a(G) + (10a - 16)n_{2a}(G) + (15a - 16)n_{3a}(G) \\
&\leq \sum_{p \nmid d,\, d \geq 4} (5d - 16)\, n_d(G),
\end{aligned}
$$

where the last two inequalities follow from the fact that $a \geq 7|G : T|$ and a is coprime to $p > 5$. Now it follows that $\operatorname{acd}_{p'}(G) > 16/5$ and we are done.

Next we consider the case $S \cong \mathsf{A}_5$. We use Proposition 8.3 (i) to deduce that

$$
\begin{aligned}
11n_1(G) + 6n_2(G) + n_3(G) &\leq 9n_{5^r}(G) + 6n_{2 \cdot 5^r}(G) + n_{3 \cdot 5^r}(G) + (2 + 2r)n_{4^r}(G) \\
&\leq (5 \cdot 5^r - 16)n_{5^r}(G) + (10 \cdot 5^r - 16)n_{2 \cdot 5^r}(G) \\
&\quad + (15 \cdot 5^r - 16)n_{3 \cdot 5^r}(G) + (5 \cdot 4^r - 16)n_{4^r}(G) \\
&\leq \sum_{p \nmid d,\, d \geq 4} (5d - 16)\, n_d(G),
\end{aligned}
$$

and we are done again. The case $S \cong \operatorname{PSL}(2, 7)$ is treated similarly with the help of Proposition 8.3 (ii) and we skip the details. □

We are now able to prove Theorem 8.1.

Proof of Theorem 8.1. Assume, to the contrary, that the theorem is false and let G be a minimal counterexample. Then G is nonsolvable and $\operatorname{acd}_{p'}(G) < 16/5$.

As in the proof of Theorem 6.1, we let $L \lhd G$ be minimal such that L is non-solvable and let $N \subseteq L$ be a minimal normal subgroup of G. We choose N such that $N \leq [L, \mathbf{O}_\infty(L)]$ if $[L, \mathbf{O}_\infty(L)]$ is nontrivial and when possible we choose N to be of order 2. Note that L is perfect and $L \subseteq G'$.

If N is nonabelian then $\operatorname{acd}_{p'}(G) \geq 16/5$ by Proposition 8.4 and so we are done. Therefore we assume from now on that N is abelian. As G is nonsolvable, it follows that so is G/N. By the minimality of G, we then have $\operatorname{acd}_{p'}(G/N) \geq 16/5$ and hence

$$\operatorname{acd}_{p'}(G) < 16/5 \leq \operatorname{acd}_{p'}(G/N).$$

Since $n_1(G) = n_1(G/N)$ as $N \subseteq L \subseteq G'$, we then deduce that

$$\text{either } n_2(G) > n_2(G/N) \ \text{ or } \ n_3(G) > n_3(G/N).$$

That is, there is some irreducible character $\chi \in \operatorname{Irr}(G)$ of degree 2 or 3 such that $\operatorname{Ker}(\chi)$ does not contain N.

Now we can use the classification of the primitive linear groups of degree 2 and 3 in [1], Chapter V, Section 81, and argue similarly as in the proof of Theorem A in [16] to obtain that $G = LC$ is a central product with the central amalgamated subgroup $L \cap C = \mathbf{Z}(L)$, where $\mathbf{Z}(L) \supseteq N > 1$,

$$C/\operatorname{Ker}(\chi) = \mathbf{Z}(G/\operatorname{Ker}(\chi))$$

and

$$L/\mathbf{Z}(L) \cong G/C \cong \mathsf{A}_5, \mathsf{A}_6, \text{ or } \operatorname{PSL}(2,7).$$

Moreover, from the proof of Theorem 6.1 we see that if $\chi(1) = 2$ then $L/\mathbf{Z}(L)$ must be isomorphic to A_5.

Since $G = LC$ is a central product with the central amalgamated subgroup $\mathbf{Z}(L)$, for each $\lambda \in \operatorname{Irr}(\mathbf{Z}(L))$ there is a bijection

$$\operatorname{Irr}(L|\lambda) \times \operatorname{Irr}(C|\lambda) \to \operatorname{Irr}(G|\lambda)$$

such that if $(\alpha, \beta) \mapsto \chi$ then $\chi(1) = \alpha(1)\beta(1)$. It is clear that $\chi(1)$ is coprime to p if and only if both $\alpha(1)$ and $\beta(1)$ are coprime to p. Therefore this bijection produces another bijection

$$\operatorname{Irr}_{p'}(L|\lambda) \times \operatorname{Irr}_{p'}(C|\lambda) \to \operatorname{Irr}_{p'}(G|\lambda),$$

and in particular we have

$$\operatorname{acd}_{p'}(G|\lambda) = \operatorname{acd}_{p'}(L|\lambda)\operatorname{acd}_{p'}(C|\lambda).$$

Therefore

$$\operatorname{acd}_{p'}(G|\lambda) \geq \operatorname{acd}_{p'}(L|\lambda).$$

If $L/\mathbf{Z}(L) \cong \mathsf{A}_5$ then we must have $L \cong \operatorname{SL}(2,5)$ since this is the only nontrivial perfect central cover of A_5. So $\mathbf{Z}(L) \cong C_2$, the cyclic group of order 2. Now, using [2], p. 2, we can check that $\operatorname{acd}_{p'}(L|\lambda) \geq 16/5$ whether λ is trivial or the only nontrivial character of $\mathbf{Z}(L)$. Thus $\operatorname{acd}_{p'}(G) \geq 16/5$ and we are done.

If $L/\mathbf{Z}(L) \cong \operatorname{PSL}(2,7)$ then similarly we have $L \cong \operatorname{SL}(2,7)$ so that $\mathbf{Z}(L) \cong C_2$. Since $N \subseteq L$ and $N \nsubseteq \operatorname{Ker}(\chi)$, it follows that $L \nsubseteq \operatorname{Ker}(\chi)$. As $\chi(1) = 3$ and the smallest degree of a nontrivial irreducible character of L is 3, we deduce that the restriction $\chi \downarrow_L \in \operatorname{Irr}(L)$. But then the character table of $\operatorname{SL}(2,7)$ (see [2], p. 3) implies that $\mathbf{Z}(L) \subseteq \operatorname{Ker}(\chi \downarrow_L)$, which in turns implies that $N \subseteq \operatorname{Ker}(\chi)$ since $N \subseteq \mathbf{Z}(L)$, and this violates the choice of χ.

Finally we consider $L/\mathbf{Z}(L) \cong \mathsf{A}_6$. Then as mentioned above we must have $\chi(1) = 3$. Also, L is one of three perfect central covers of A_6, namely $2 \cdot \mathsf{A}_6$, $3 \cdot \mathsf{A}_6$, and $6 \cdot \mathsf{A}_6$. First assume that $L \cong 2 \cdot \mathsf{A}_6$ or $6 \cdot \mathsf{A}_6$. Then $N \cong C_2$ since we chose N to be of order 2 when possible. Arguing as in the case $L/\mathbf{Z}(L) \cong \operatorname{PSL}(2,7)$, we obtain that N is contained in the kernel of an irreducible character of degree 3 of $6 \cdot \mathsf{A}_6$, and this is a contradiction by [2], p. 5. So it remains to consider $L \cong 3 \cdot \mathsf{A}_6$. But then one can check that $\operatorname{acd}_{p'}(L|\lambda) > 16/5$ whether λ is the trivial character or one of the two nontrivial irreducible characters of $\mathbf{Z}(L)$. It follows that $\operatorname{acd}_{p'}(G|\lambda) > 16/5$ for every $\lambda \in \operatorname{Irr}(\mathbf{Z}(L))$, and hence $\operatorname{acd}_{p'}(G) > 16/5$ in this case. $\square$

9. Characters of p'-degree and p-nilpotency

We recall that, for a finite group G,

$$\mathrm{Irr}_{p'}(G) := \{\chi \in \mathrm{Irr}(G) \mid p \nmid \chi(1)\}.$$

We begin the section with the following easy observation, which can be viewed as a p'-version of Lemma 3.1 in [9].

Lemma 9.1. *Let p be a prime and A be a subgroup of a finite group G. Then*

$$|\mathrm{Irr}_{p'}(G)| \leq |G : A||\mathrm{Irr}_{p'}(A)|.$$

Proof. Let χ be an irreducible character of G such that $\chi(1)$ is not divisible by p. Consider the restriction χ_A. There must be an irreducible constituent $\lambda \in \mathrm{Irr}(A)$ of χ_A such that $\lambda(1)$ is not divisible by p, and moreover, χ in turn is an irreducible constituent of λ^G by Frobenius reciprocity. On the other hand, given any $\lambda \in \mathrm{Irr}(A)$, each irreducible constituent of λ^G has degree at least $\lambda(1)$, and therefore the number of irreducible constituents of λ^G is at most $|G : A|$ since $\lambda^G(1) = |G : A|\lambda(1)$. The lemma now easily follows. $\square$

In the next result, we analyze the average of p'-degrees of irreducible characters in a special situation.

Lemma 9.2. *Let p be a prime, N be an abelian p-group, and G be a split extension of N. Assume that no nonprincipal irreducible character of N is fixed under G. Then*

$$\mathrm{acd}_{p'}(G) \geq \begin{cases} 3/2 & \text{if } p = 2, \\ 4/3 & \text{if } p > 2. \end{cases}$$

Proof. The sum of orbit sizes of the action of G on nontrivial irreducible characters of N is $|N| - 1$. Since N is an abelian p-group, there must be at least one nontrivial orbit of size coprime to p. Let $\{1_N = \alpha_0, \alpha_1, \ldots, \alpha_l\}$ be a set of representatives of p'-size orbits of the action of G on $\mathrm{Irr}(N)$. For each $0 \leq i \leq l$, let I_i be the inertia subgroup of α_i in G. Then $|G : I_i|$ is not divisible by p. Moreover, since no nonprincipal irreducible character of N is invariant under G, we have that I_i is a proper subgroup of G for every $1 \leq i \leq l$.

Since G splits over N, every I_i also splits over N, and thus α_i extends to a linear character, say β_i, of I_i. Gallagher's theorem then implies that the mapping $\lambda \mapsto \lambda\beta_i$ is a bijection from $\mathrm{Irr}(I_i/N)$ to the set of irreducible characters of I_i lying above α_i. Using Clifford correspondence, we then obtain a bijection $\lambda \mapsto (\lambda\beta_i)^G$ from $\mathrm{Irr}(I_i/N)$ to the set of irreducible characters of G lying above α_i. We observe that, since $(\lambda\beta_i)^G(1) = |G : I_i|\lambda(1)$ and $p \nmid |G : I_i|$, $(\lambda\beta_i)^G(1)$ is coprime to p if and only if $\lambda(1)$ is coprime to p.

From the above analysis, we see that $|\mathrm{Irr}_{p'}(G)| = \sum_{i=0}^{l} |\mathrm{Irr}_{p'}(I_i/N)|$, and therefore

$$\sum_{\chi \in \mathrm{Irr}_{p'}(G)} \chi(1) = \mathrm{acd}_{p'}(G) \sum_{i=0}^{l} |\mathrm{Irr}_{p'}(I_i/N)|.$$

On the other hand, since each irreducible character of G lying above α_i has degree at least $|G : I_i|$ and the number of those characters of p'-degree is precisely equal to $|\mathrm{Irr}_{p'}(I_i/N)|$, we have

$$\sum_{\chi \in \mathrm{Irr}_{p'}(G)} \chi(1) \geq \sum_{i=0}^{l} |G : T_i| \, |\mathrm{Irr}_{p'}(I_i/N)|.$$

We therefore deduce that

$$\sum_{i=0}^{l} |G : T_i| \, |\mathrm{Irr}_{p'}(I_i/N)| \leq \mathrm{acd}_{p'}(G) \sum_{i=0}^{l} |\mathrm{Irr}_{p'}(I_i/N)|,$$

which implies that

$$\sum_{i=1}^{l} (|G : I_i| - \mathrm{acd}_{p'}(G)) |\mathrm{Irr}_{p'}(I_i/N)| \leq (\mathrm{acd}_{p'}(G) - 1) \, |\mathrm{Irr}_{p'}(G/N)|$$

since $I_0 = G$. In particular, as $l \geq 1$, it follows that

$$(\mathrm{acd}_{p'}(G) - 1) \, |\mathrm{Irr}_{p'}(G/N)| \geq (|G : I_1| - \mathrm{acd}_{p'}(G)) \, |\mathrm{Irr}_{p'}(I_1/N)|.$$

Since $|\mathrm{Irr}_{p'}(G/N)| \leq |G : I_1| |\mathrm{Irr}_{p'}(I_1/N)|$ by Lemma 9.1, we then deduce that

$$(\mathrm{acd}_{p'}(G) - 1) \, |G : I_1| \geq |G : I_1| - \mathrm{acd}_{p'}(G).$$

Equivalently, we obtain

$$\mathrm{acd}_{p'}(G) \geq \frac{2|G : I_1|}{|G : I_1| + 1}.$$

Recall that $|G : I_1|$ is not equal to 1 and not divisible by p. Now if $p = 2$ then $|G : I_1| \geq 3$ and we have $\mathrm{acd}_{p'}(G) \geq 3/2$. On the other hand, if $p > 2$ then $|G : I_1| \geq 2$ and we have $\mathrm{acd}_{p'}(G) \geq 4/3$. The proof is now complete. $\square$

We are now able to prove Theorem 1.1, which is restated below.

Theorem 9.3. *Let p be an odd prime and G a finite group. We have:*
  (i) *if $\mathrm{acd}_{2'}(G) < 3/2$, then G has a normal 2-complement, and*
  (ii) *if $\mathrm{acd}_{p'}(G) < 4/3$, then G has a normal p-complement.*

Proof. Let $b_p := 3/2$ if $p = 2$ and $b_p := 4/3$ if $p > 2$. Assume that $\mathrm{acd}_{p'}(G) < b_p$, and we wish to show that G has a normal p-complement. If G is abelian then the statement is obvious. So we assume that G is nonabelian. We then can choose a minimal normal subgroup N of G such that $N \subseteq G'$. Since $\mathrm{acd}_{p'}(G) < b_p \leq 3/2$, Theorem 1.2 implies that G is solvable, and hence N is elementary abelian.

Since $N \subseteq G'$, we observe that if χ is a linear character of G, then $N \subseteq \mathrm{Ker}(\chi)$ so that χ can be viewed as a linear character of G/N. It follows that $n_1(G/N) = n_1(G)$, which implies that

$$\mathrm{acd}_{p'}(G/N) \leq \mathrm{acd}_{p'}(G) < b_p.$$

By induction on $|G|$, we have that G/N has a normal p-complement, say H/N. If N is a p'-group, then H is a normal p-complement in G and we would be done. So we assume that N is an elementary abelian p-group. It then follows from the

Schur–Zassenhaus theorem that H splits over N. Let us assume that $H = NH_1$, where H_1 is a Hall p'-subgroup of H (and indeed of G as well).

We now use Frattini's argument to show that $G = N\mathbf{N}_G(H_1)$. Let g be any element of G. Since $H \trianglelefteq G$ and $H_1 < H$, we have $g^{-1}H_1g < H$ so that $g^{-1}H_1g$ is also a Hall p'-subgroup of H. By Hall's theorems, $g^{-1}H_1g$ is H-conjugate to H_1. In other words, there exists $h \in H$ such that $g^{-1}H_1g = h^{-1}H_1h$. Thus $gh^{-1} \in \mathbf{N}_G(H_1)$ so that $g \in \mathbf{N}_G(H_1)H = H\mathbf{N}_G(H_1)$. Since g is arbitrary in G, we deduce that $G = H\mathbf{N}_G(H_1)$, and therefore $G = N\mathbf{N}_G(H_1)$ as $H = NH_1$.

Since $G = N\mathbf{N}_G(H_1)$, if N is contained in the Frattini subgroup of G, we would have $G = \mathbf{N}_G(H_1)$ and we are done. So we assume that N is not contained in the Frattini subgroup of G. Then there exists a maximal subgroup M of G such that $N \nsubseteq M$. We then have $G = NM$ and $N \cap M < N$. As N is abelian, it follows that $N \cap M$ is a normal subgroup of G, and hence $N \cap M = 1$ by the minimality of N. We conclude that $G = N \rtimes M$. In other words, G is a split extension of N.

If $N \subseteq \mathbf{Z}(G)$ then we would have $H = N \times H_1$ and thus $H_1 \triangleleft G$, as desired. So we assume that N is noncentral in G. Thus, by the minimality of N, we have $[N, G] = N$. It follows that no nonprincipal irreducible character of N is invariant under G.

We now have all the hypotheses of Lemma 9.2, and therefore we deduce that $\mathrm{acd}_{p'}(G) \geq 3/2$ if $p = 2$ and $\mathrm{acd}_{p'}(G) \geq 4/3$ if $p > 2$. This contradiction completes the proof of the theorem. $\square$

10. $\mathbb{Q}_p$-valued characters of p'-degree and solvability

We need the following.

Lemma 10.1. *Let G be a finite group with a nonabelian minimal normal subgroup N. Assume that there exists $\psi \in \mathrm{Irr}(N)$ that is extendible to a $\mathbb{Q}_p$-valued character of $\mathrm{Stab}_G(\psi)$. Then $n_{\mathbb{Q}_p,1}(G) \leq n_{\mathbb{Q}_p,a}(G)|G : \mathrm{Stab}_G(\psi)|$, where $a := \psi(1)|G : \mathrm{Stab}_G(\psi)|$. Moreover, if ψ extends to a rational-valued character of $\mathrm{Stab}_G(\psi)$, then $n_{\mathbb{Q},1}(G) \leq n_{\mathbb{Q},a}(G)|G : \mathrm{Stab}_G(\psi)|$.*

Proof. Assume that ψ extends to $\chi \in \mathrm{Irr}_{\mathbb{Q}_p}(\mathrm{Stab}_G(\psi))$. Remark that, if λ is a linear character of $\mathrm{Stab}_G(\psi)/N$ with values in $\mathbb{Q}_p$, then $(\lambda\chi)^G \in \mathrm{Irr}(G)$ has values in $\mathbb{Q}_p$ as well. Now just repeat the arguments in the proof of Proposition 3.2 (i) to obtain the first statement of the lemma. The second statement is argued similarly. $\square$

To prove Theorem 1.3, we first prove a $\mathbb{Q}_p$-analogue of Theorem 1.2, that is an extension of Theorem A (i) in [10], Theorem C (i) in [20], and Theorem 6.3 in [21].

Theorem 10.2. *Let $p > 2$ be a prime and G a finite group. If one of the following happens:*

(i) $\mathrm{acd}_{\mathbb{Q},2'}(G) < 3$,

(ii) $\mathrm{acd}_{\mathbb{Q}_p,p'}(G) \leq 2$,

(iii) $\mathrm{acd}_{\mathbb{Q},p'}(G) \leq 2$ *for $p > 3$,*

then G is solvable.

Proof. We use Theorem 2.1 and Lemma 10.1, and argue as in the proof of Theorem 5.1 to prove (i) and (ii).

Now we assume that $p \neq 3$ and prove (iii). By Theorem 6.2 in [21], the orbit $\mathcal{O}$ in Lemma 2.2 can be chosen so that every $\theta \in \mathcal{O}$ is extendible to a rational-valued character of $\mathrm{Stab}_{\mathrm{Aut}(S)}(\theta)$. Therefore, the character ψ produced in Theorem 2.1 is also extendible to a rational-valued character of $\mathrm{Stab}_S(\psi)$. The proof now follows as before. $\square$

11. $\mathbb{Q}_p$-valued characters of p'-degree and p-nilpotency

We begin with an easy observation, which is recalled to our attention by Mark L. Lewis.

Lemma 11.1. *Let p be a prime, N be an elementary abelian p-group, and G be a split extension of N. Let $\theta \in \mathrm{Irr}(N)$ be invariant under G. Then θ extends to a $\mathbb{Q}_p$-valued character of G.*

Proof. Let $K := \mathrm{Ker}(\theta)$. Since θ is G-invariant, K is normal in G. Note that N/K is cyclic since it is abelian and has a faithful irreducible character, so θ being G-invariant will imply that N/K is central in G/K. Thus, $G/K = N/K \times HK/K$, where H is a complement for N in G. It follows that θ, viewed as a character of N/K, extends to $\theta \times 1_{HK/K} \in \mathrm{Irr}(G/K)$. Now we are done by viewing $\theta \times 1_{HK/K}$ as a character of G and noting that θ has values in $\mathbb{Q}_p$. $\square$

Next result is a $\mathbb{Q}_p$-analogue of Lemma 9.2, but the proof is somewhat different.

Lemma 11.2. *Let p be a prime, N be an elementary abelian p-group, and G be a split extension of N. Assume that no nonprincipal irreducible character of N is fixed under G. Then*

$$\mathrm{acd}_{\mathbb{Q}_p, p'}(G) \geq \left\{ \begin{array}{ll} 3/2 & \text{if } p = 2, \\ 4/3 & \text{if } p > 2. \end{array} \right.$$

Proof. We use the same setup as in the proof of Lemma 9.2. In particular, $\{1_N = \alpha_0, \alpha_1, \ldots, \alpha_l\}$ is a set of representatives of the p'-size orbits of the action of G on $\mathrm{Irr}(N)$, and I_i is the inertia subgroup of α_i in G for every $0 \leq i \leq l$.

By Lemma 11.1, each α_i extends to a $\mathbb{Q}_p$-valued character, say β_i, of I_i. Therefore, each irreducible character of p'-degree of G has the form $(\lambda\beta_i)^G$ where $\lambda \in \mathrm{Irr}_{p'}(I_i/N)$.

Since no nonprincipal irreducible character of N is fixed under G, $(\lambda\beta_i)^G(1) = |G : I_i|\lambda(1) > 1$ for every $1 \leq i \leq l$. Thus, every linear character of G must lie above the trivial character of N. We deduce that

$$n_{\mathbb{Q}_p, 1}(G) = n_{\mathbb{Q}_p, 1}(G/N).$$

Since $n_{\mathbb{Q}_p, 1}(G/N) \leq n_{\mathbb{Q}_p, 1}(I_1/N)|G : I_1|$, we then obtain

$$n_{\mathbb{Q}_p, 1}(G) \leq n_{\mathbb{Q}_p, 1}(I_1/N)|G : I_1|.$$

Recall that β_1 has values in $\mathbb{Q}_p$. Therefore if λ is a $\mathbb{Q}_p$-valued linear character of I_1/N, then so is $(\lambda\beta_1)^G$, whose degree is $|G : I_1|$. We deduce that

$$n_{\mathbb{Q}_p,1}(I_1/N) \leq n_{\mathbb{Q}_p,|G:I_1|}(G).$$

Together with the above inequality, we have

$$n_{\mathbb{Q}_p,1}(G) \leq n_{\mathbb{Q}_p,|G:I_1|}(G)|G : I_1|.$$

When $p = 2$ we have $|G : I_1| \geq 3$ since $|G : I_1|$ is not 1 and coprime to p. It follows that

$$n_{\mathbb{Q}_p,1}(G) \leq n_{\mathbb{Q}_p,|G:I_1|}(G)(2|G : I_1| - 3),$$

and thus $\mathrm{acd}_{\mathbb{Q}_p,2'}(G) \geq 3/2$, as claimed. On the other hand, if $p > 2$ then $|G:I_1| \geq 2$, and hence

$$n_{\mathbb{Q}_p,1}(G) \leq n_{\mathbb{Q}_p,|G:I_1|}(G)(3|G : I_1| - 4),$$

which implies that $\mathrm{acd}_{\mathbb{Q}_p,2'}(G) \geq 4/3$, and we are done. $\qquad\square$

Finally we prove Theorem 1.3.

Theorem 11.3. *Let p be an odd prime and G a finite group. Then*

  (i) *if $\mathrm{acd}_{\mathbb{Q},2'}(G) < 3/2$, then G has a normal 2-complement, and*

  (ii) *if $\mathrm{acd}_{\mathbb{Q}_p,p'}(G) < 4/3$, then G has a normal p-complement.*

Proof. Repeat the arguments in the proof of Theorem 9.3, with the help of Theorem 10.2 and Lemma 11.2. $\qquad\square$

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