Rev. Mat. Iberoam. **33** (2017), no. 1, 139–168
DOI 10.4171/RMI/930

Homogeneous structures of linear type on ϵ-Kähler and ϵ-quaternion Kähler manifolds

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Abstract. We analyze degenerate homogeneous structures of linear type in the pseudo-Kähler and para-Kähler cases. The local form and the holonomy of pseudo-Kähler or para-Kähler manifolds admitting such structure are obtained. In addition the associated homogeneous models are studied exhibiting their relation with the incompleteness of the metric. The same questions are tackled in the pseudo-quaternion Kähler and para-quaternion Kähler cases.

These results complete the study of homogeneous structures of linear type in pseudo-Kähler, para-Kähler, pseudo-quaternion Kähler and para-quaternion Kähler cases.

1. Introduction

Ambrose and Singer [2] generalized Cartan's theorem on symmetric spaces characterizing connected, simply-connected and complete homogeneous Riemannian spaces in terms of a $(1,2)$-tensor field S called *homogeneous structure tensor* (or simply *homogeneous structure*) satisfying a system of geometric PDE's, nowadays called Ambrose–Singer equations. In [12] this result is extended to homogeneous Riemannian manifolds presenting an additional geometric structure (such as Kähler, quaternion Kähler, G_2, etc.), and in [10] the theory is adapted to metrics with signature. Homogeneous structures have proved to be one of the most successful tools in the study of homogeneous spaces, probably due to the combination of their algebraic and geometric nature. The first classification of homogeneous structures was provided in [18] in the purely Riemannian case, and later in [8] the classification of homogeneous structures for all the possible holonomy groups in Berger's list is given using a representation theoretical approach. These techniques have been also used for metrics with signature (see for instance [3]). In many cases (such as Kähler, hyper-Kähler, quaternion Kähler, G_2 or Spin(7), as well as in the

Mathematics Subject Classification (2010): Primary 53C30; Secondary 53C50, 53C55, 53C80.
Keywords: Homogeneous plane waves, para-Kähler, pseudo-Kähler, pseudo-quaternion Kähler, para-quaternion Kähler, reductive homogeneous pseudo-Riemannian spaces.

pseudo-Riemannian analogs) these classifications contain a class such that the corresponding pointwise tensor submodule has dimension growing linearly with the dimension of the manifold. For that reason homogeneous structures belonging to these classes are called of *linear type*. The corresponding tensor fields are defined by a set of vector fields satisfying a system of PDE's equivalent to Ambrose–Singer equations.

For definite metrics, in the purely Riemannian case so as in the case of Kähler and quaternion Kähler manifolds, homogeneous structures of linear type characterize spaces of negative constant sectional (resp. holomorphic sectional, quaternionic sectional) curvature (see [5], [9] and [18]). When metrics with signature are studied, the causal character of the vector fields defining the homogeneous structure tensor needs to be taken into account. In the purely pseudo-Riemannian case, non-degenerate structures of linear type (i.e., given by a non null vector field) characterize spaces of constant sectional curvature [10], while degenerate homogeneous structures of linear type (i.e., given by a null vector field) characterize singular scale-invariant plane waves [16]. Furthermore, in [15] it is shown that homogeneous structures in the composed class $\mathcal{S}_1 + \mathcal{S}_3$ are related to a larger class of singular homogeneous plane waves. In [6] the authors generalize this result to the pseudo-Kähler setting in the strongly degenerate case, i.e., homogeneous pseudo-Kähler structures of linear type characterized by a null vector field ξ and vanishing vector field ζ, resulting that the underlying geometry presents significant analogies with the geometry of a singular homogenous plane wave. The same problem is analyzed in the case of pseudo-hyper-Kähler and pseudo-quaternion Kähler geometry, finding that a metric admitting such a structure must be flat.

In this paper we study degenerate homogeneous structures of linear type in the pseudo-Kähler and the para-Kähler settings, that is, homogeneous pseudo-Kähler and para-Kähler structures of linear type defined by a null vector field ξ and an arbitrary vector field ζ (see [3]). Note that this includes the strongly degenerate case, so that the results obtained in this paper generalize those in [6]. With these results, together with those in [14], we give a complete description of the geometry of homogeneous structures of linear type. Essentially two cases arise in pseudo-Kähler and para-Kähler manifolds. On one hand, non-degenerate structures locally characterize constant curvature spaces. On the other, the degenerate case (structures studied in §3 and §4) provides geometries with an interesting parallelism with homogeneous plane waves. This is analyzed at the end of the paper. Finally the pseudo-quaternion and para-quaternion Kähler framework does not provide any geometry other than spaces of constant curvature, a fact that indicates that the quaternionic realm seems to be too rigid to contain generalizations of plane waves.

The paper is organized as follows. In Section 2, the general framework and the notation is settled. Throughout the manuscript, the notions of pseudo-Kähler and para-Kähler geometry will be unified and treated together via the definition of ϵ-Kähler geometry, $\epsilon = \pm 1$. In Section 3, we obtain the curvature and holonomy of an ϵ-Kähler manifold admitting a degenerate homogeneous ϵ-Kähler structure of linear type. In addition we prove that the vector field ζ must be a multiple of ξ by a factor 0 or $-\epsilon/2$. In Section 4 the local form of a metric admitting these structures is obtained. The corresponding local model is studied, focusing in the singular nature of the metric. In Section 5 the homogeneous model associated to a degenerate homogeneous ϵ-Kähler structure of linear type is computed, showing that it is (geodesically) incomplete. In Section 6 the same problem in the pseudo-quaternion Kähler and para-quaternion Kähler settings is tackled, resulting that the corresponding metrics must be flat. Finally, Section 7 gives a complete view of the geometry of manifolds endowed with a homogeneous structure of linear type.

2. Preliminaries

We shall combine the treatment of complex geometry and para-complex geometry by defining $\epsilon = \pm 1$, so that hereafter ϵ should be substituted by -1 for complex geometry and by 1 for para-complex geometry (for a survey on para-complex geometry see for example [7]).

Definition 2.1. Let (M, g) be a pseudo-Riemannian manifold of dimension $2n$.

(1) An almost ϵ-Hermitian structure on (M, g) is a smooth section J of $\mathfrak{so}(TM)$ such that $J^2 = \epsilon$.

(2) (M, g) is called ϵ-Kähler if it admits a parallel almost ϵ-Hermitian structure J with respect to the Levi-Civita connection.

The first previous definition implies that the signature of g is $(2r, 2s)$, $r+s = n$, for $\epsilon = -1$, and (n, n) for $\epsilon = 1$. The second definition is equivalent to the holonomy being contained in $U(r, s)$ for $\epsilon = -1$, and $GL(n, \mathbb{R})$ for $\epsilon = 1$.

Hereafter (M, g, J) is supposed to be a connected ϵ-Kähler manifold of dimension $\dim M \geq 4$.

Definition 2.2. An ϵ-Kähler manifold (M, g, J) is called a homogeneous ϵ-Kähler manifold if there is a connected Lie group G of isometries acting transitively on M and preserving J. (M, g, J) is called a reductive homogeneous ϵ-Kähler manifold if the Lie algebra $\mathfrak{g}$ of G can be decomposed as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}.$$

Using Kiričenko's theorem, [12] (see also [10]), we have:

Theorem 2.3. *Let (M, g, J) be a connected, simply connected and complete ϵ-Kähler manifold. Then the following are equivalent:*

(1) *(M, g, J) is a reductive homogeneous ϵ-Kähler manifold.*

(2) *(M, g, J) admits a linear connection $\widetilde{\nabla}$ such that*

$$\widetilde{\nabla} g = 0, \quad \widetilde{\nabla} R = 0, \quad \widetilde{\nabla} S = 0, \quad \widetilde{\nabla} J = 0, \tag{2.1}$$

where $S = \nabla - \widetilde{\nabla}$, ∇ is the Levi-Civita connection of g, and R is the curvature tensor of g.

Definition 2.4. A tensor field S of type $(1,2)$ on an ϵ-Kähler manifold (M,g,J) satisfying (2.1) is called a homogeneous ϵ-Kähler structure.

The classification with respect to the action of the maximal holonomy group of homogeneous ϵ-Kähler structures is carried out in [3] and [10], resulting four primitive classes $\mathcal{K}_1^{-1},\mathcal{K}_2^{-1},\mathcal{K}_3^{-1},\mathcal{K}_4^{-1}$ for $\epsilon=-1$, and eight primitive classes $\mathcal{K}_1^1,\dots,\mathcal{K}_8^1$ for $\epsilon=1$. Among them, for $\mathcal{K}_2^{-1}$, $\mathcal{K}_4^{-1}$, $\mathcal{K}_2^1$, $\mathcal{K}_4^1$, $\mathcal{K}_6^1$, and $\mathcal{K}_8^1$ the corresponding pointwise modules have dimension growing linearly with the dimension of M. For this reason we define

Definition 2.5. A homogeneous ϵ-Kähler structure is called of linear type if it belongs to

(1) $\mathcal{K}_2^1\oplus\mathcal{K}_4^1$ for $\epsilon=-1$,

(2) $\mathcal{K}_2^1\oplus\mathcal{K}_4^1\oplus\mathcal{K}_6^1\oplus\mathcal{K}_8^1$ for $\epsilon=1$.

The following characterization can be obtained from [3] and [10].

Proposition 2.6. *A homogeneous ϵ-Kähler structure S is of linear type if and only if*

$$S_XY=g(X,Y)\xi-g(\xi,Y)X+\epsilon g(X,JY)J\xi-\epsilon g(\xi,JY)JX-2g(\zeta,JX)JY, \tag{2.2}$$

for some vector fields ξ and ζ.

Since we are dealing with metrics with signature we further distinguish the following cases.

Definition 2.7. A homogeneous ϵ-Kähler structure of linear type S is called (see [3])

(i) non-degenerate if $g(\xi,\xi)\neq 0$,

(ii) degenerate if $g(\xi,\xi)=0$,

(iii) strongly degenerate if $g(\xi,\xi)=0$ and $\zeta=0$.

Case (i) was studied in [14], and case (iii) was studied in [6]. In this paper we concentrate in case (ii).

3. Degenerate homogeneous ϵ-Kähler structures of linear type

It is a straightforward computation to prove (see [3]):

Proposition 3.1. *A tensor field S on (M,g,J) defined by formula (2.2) is a homogeneous ϵ-Kähler structure if and only if*

$$\widetilde{\nabla}\xi=0,\quad \widetilde{\nabla}\zeta=0\,,\quad \widetilde{\nabla}R=0\,.$$

where $\widetilde{\nabla}=\nabla-S$.

Equation $\widetilde{\nabla} R = 0$ reads

$$(\nabla_X R)_{YZWU} = -R_{S_X YZWU} - R_{Y S_X ZWU} - R_{YZ S_X WU} - R_{YZW S_X U}, \tag{3.1}$$

so applying the second Bianchi identity and substituting (2.2) we have

$$\begin{aligned} 0 = \mathop{\mathfrak{S}}_{XYZ} \{ & 2g(X,\xi)R_{YZWU} + g(X,W)R_{YZ\xi U} + g(X,U)R_{YZW\xi} \\ & +2\epsilon g(X,JY)R_{J\xi ZWU} + \epsilon g(X,JW)R_{YZJ\xi U} + \epsilon g(X,JU)R_{YZWJ\xi}\} . \end{aligned} \tag{3.2}$$

Since $g(\xi,\xi) = 0$, there exists we an orthonormal basis $\{e_k\}$ such that $g(e_1, e_1) = 1$, $g(e_2, e_2) = -1$, and $\xi = g(\xi, e_1)(e_1+e_2)$. Whence, contracting the previous formula with respect to X and W and applying the first Bianchi identity, we obtain

$$\begin{aligned} (2n+2)R_{ZY\xi U} = & -2g(Y,\xi)\, r(Z,U) + 2g(Z,\xi)\, r(Y,U) \\ & -2\epsilon g(Y,JZ)\, r(J\xi,U) - g(Y,U)\, r(Z,\xi) \\ & -\epsilon g(Y,JU)\, r(Z,J\xi) + g(Z,U)\, r(Y,\xi) + \epsilon g(Z,JU)\, r(Y,J\xi)\, , \end{aligned} \tag{3.3}$$

where r denotes the Ricci tensor. With the same orthonormal basis, contracting the previous expression with respect to Y and U we arrive to $r(Z,\xi) = (\mathrm{s}/2n)g(Z,\xi)$, where s stands for the scalar curvature. Setting $a = 1/(2n+2)$ and $\nu = \mathrm{s}/2n$, we can write

$$\frac{1}{a} R_{\xi U} = 2\theta \wedge r(U) - 2\nu\, \epsilon\, \theta(JU)F + \nu\, U^\flat \wedge \theta - \epsilon\, \nu (JU)^\flat \wedge (\theta \circ J)\, , \tag{3.4}$$

where F denotes the symplectic form associated to g and J. From Bianchi's first identity we have $R_{WUJ\xi\cdot} = R_{\xi JWU\cdot} - R_{\xi JUW\cdot}$ so we can write (3.3) as

$$\begin{aligned} 0 = & 2\theta \wedge R_{WU} + W^\flat \wedge R_{\xi U} - U^\flat \wedge R_{\xi W} - 2\epsilon\, F \wedge (R_{\xi JUW} - R_{\xi JWU}) \\ & -\epsilon\, (JW)^\flat \wedge R_{\xi JU} + \epsilon\, (JU)^\flat \wedge R_{\xi JW} . \end{aligned} \tag{3.5}$$

Denoting by $\Xi(U)$ the right-hand side of (3.4) and substituting in (3.5) we obtain

$$\begin{aligned} 0 = & \frac{2}{a}\, \theta \wedge R_{WU} + W^\flat \wedge \Xi(U) - U^\flat \wedge \Xi(W) - 2\epsilon\, F \wedge (i_W\, \Xi(JU) - i_U\, \Xi(JW)) \\ & -\epsilon\, (JW)^\flat \wedge \Xi(JU) + \epsilon\, (JU)^\flat \wedge \Xi(JW). \end{aligned}$$

Then, taking $W = \xi$ in the previous formula,

$$0 = \epsilon(\theta \wedge (\theta \circ J)) \wedge (r(JU) - \nu(JU)^\flat).$$

Now, since U is arbitrary, denoting $\alpha = r - \nu g$, one has

$$\theta \wedge (\theta \circ J) \wedge \alpha(X) = 0,$$

for any vector field X. This implies that

$$\alpha = \lambda\theta + \mu\theta \circ J,$$

for some 1-forms λ and μ.

Note that since (M, g, J) is ϵ-Kähler, $\alpha = r - \nu g$ is symmetric and of type $(1, 1)$. Imposing this to the right-hand side of the previous equality we have that

$$\lambda = f\theta, \quad \mu = -\epsilon f(\theta \circ J),$$

for some function f, so that we obtain

$$r = \nu g + f\left(\theta \otimes \theta - \epsilon(\theta \circ J) \otimes (\theta \circ J)\right). \tag{3.6}$$

Substituting (3.6) in (3.4) we obtain

$$\frac{1}{a} R_{\xi U} = \nu\, R^0_{\xi U} + P_{\xi U}, \tag{3.7}$$

where

$$\begin{aligned} R^0_{XYZW} = {} & g(X,Z)g(Y,W) - g(X,W)g(Y,Z) - \epsilon\, g(X,JZ)g(Y,JW) \\ & + \epsilon\, g(X,JW)g(Y,JZ) - 2\epsilon\, g(X,JY)g(Z,JW), \end{aligned}$$

and

$$P_{\xi U} = -2\epsilon f\theta(JU)\theta \wedge (\theta \circ J).$$

On the other hand, from $\nabla\xi = S \cdot \xi$ and (2.2), the formula

$$R_{XY}Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$$

gives

$$R_{XY}\,\xi = -g(\xi,\xi)\, R^0_{XY}\,\xi + \Theta^\zeta_{XY}\,\xi = \Theta^\zeta_{XY}\,\xi, \tag{3.8}$$

where

$$\begin{aligned} \Theta^\zeta_{XY}\xi = & - 2g(\zeta,JY)g(X,J\xi)\xi + 2g(\zeta,Y)g(X,J\xi)J\xi - 4g(\zeta,JY)g(X,\xi)J\xi \\ & + 2g(\zeta,JX)g(Y,J\xi)\xi - 2g(\zeta,X)g(Y,J\xi)J\xi + 4g(\zeta,JX)g(Y,\xi)J\xi \\ & + 4g(\xi,\zeta)g(Y,JX)J\xi - 4\epsilon g(\zeta,JY)g(\xi,X)J\xi + 4\epsilon g(\zeta,JX)g(\zeta,Y)J\xi. \end{aligned}$$

Taking $Y = JX$ and comparing (3.7) and (3.8) one finds that

$$2a\nu\, g(\xi,JX) = 0, \quad 2a\nu\, g(\xi,X) = 0,$$

for every X, so that $\nu = 0$. Hence the scalar curvature vanish. We now choose at every point $p \in M$ a basis

$$\{\xi, J\xi, q_1, Jq_1, X_i, JX_i\}$$

of T_pM, where $g(\xi, q_1) = 1$, $g(q_1, q_1) \neq 0$, and $\{X_i, JX_i\}$ is an orthonormal basis of $\operatorname{span}\{\xi, J\xi, q_1, Jq_1\}^\perp$. Comparing again (3.7) and (3.8) for $X = \xi$ and $Y = Jq_1$, and for $X = J\xi$ and $Y = Jq_1$ we obtain that $g(\zeta, J\xi) = 0$ and $g(\zeta, \xi) = 0$, so that $\zeta \in \operatorname{span}\{\xi, J\xi\}^\perp$. Taking $X = X_i$ and $Y = Jq_1$, and $X = JX_i$ and $Y = Jq_1$ we also have $g(\zeta, JX_i) = 0$ and $g(\zeta, X_i) = 0$ respectively, so that $\zeta \in \operatorname{span}\{\xi, J\xi\}$. Finally, writing $\zeta = \lambda\xi + \mu J\xi$ for some functions λ and μ, and taking $X = q_1$ and $Y = Jq_1$ one finds $g(\zeta, Jq_1) = 0$ and $2af = -2\epsilon\lambda - 4\lambda^2$, so that

$$\zeta = \lambda\xi, \quad f = -\frac{1}{a}\lambda(\epsilon + 2\lambda).$$

Note that equations $\widetilde{\nabla}\xi = 0$ and $\widetilde{\nabla}\zeta = 0$ imply that λ must be constant. This agrees with the fact that the Ricci form

$$\rho = f\theta \wedge (\theta \circ J)$$

is closed as (M, g, J) is ϵ-Kähler. We have proved:

Proposition 3.2. *Let (M, g, J) be a ϵ-Kähler manifold admitting a degenerate ϵ-Kähler homogeneous structure of linear type S given by (2.2). Then $\zeta = \lambda\xi$ for some $\lambda \in \mathbb{R}$ and the Ricci curvature is*

$$r = -\frac{1}{a}\lambda(\epsilon + 2\lambda)(\theta \otimes \theta - \epsilon(\theta \circ J) \otimes (\theta \circ J)),$$

where $a = 1/(\dim M + 2)$ and $\theta = \xi^\flat$. In particular the scalar curvature vanishes.

Since $\nu = 0$, formula (3.7) becomes

$$R_{ZY\xi U} = aP_{ZY\xi U} = -2a\epsilon f(\theta \wedge (\theta \circ J) \otimes (\theta \circ J))(Z, Y, U).$$

Looking again to formula (3.2) we obtain

$$\begin{aligned} -\mathop{\mathfrak{S}}_{XYZ} 2g(X,\xi)R_{YZWU} = \mathop{\mathfrak{S}}_{XYZ} 2af\Big\{&(\theta \wedge (\theta \circ J)) \otimes (X^\flat \wedge (\theta \circ J))(Y, Z, W, U) \\ &+ \epsilon(\theta \wedge (\theta \circ J)) \otimes (JX^\flat \wedge (\theta))(Y, Z, W, U) \\ &- 2\epsilon\, g(X, JY)\theta \otimes (\theta \wedge (\theta \circ J))(Z, W, U)\Big\}. \end{aligned} \tag{3.9}$$

Substituting this in (3.1), and after a quite long computation, we obtain

$$\nabla_X R = 4\theta(X) \otimes (R - \frac{1}{2}ag \boxtimes r) - 2a\epsilon\big((X^\flat \wedge (\theta \circ J)) \odot \rho + (JX^\flat \wedge (\theta)) \odot \rho\big), \tag{3.10}$$

where ρ is the Ricci form and $\boxtimes$ stands for the ϵ-complex Kulkarni–Nomizu product defined as

$$\begin{aligned} h \boxtimes k(X_1, X_2, X_3, X_4) = {} & h(X_1, X_3)k(X_2, X_4) + h(X_2, X_4)k(X_1, X_3) \\ & - h(X_1, X_4)k(X_2, X_3) - h(X_2, X_3)k(X_1, X_4) \\ & - \epsilon h(X_1, JX_3)k(X_2, JX_4) - \epsilon h(X_2, JX_4)k(X_1, JX_3) \\ & + \epsilon h(X_1, JX_4)k(X_2, JX_3) + \epsilon h(X_2, JX_3)k(X_1, JX_4) \\ & - 2\epsilon h(X_1, JX_2)k(X_3, JX_4) - 2\epsilon h(X_3, JX_4)k(X_1, JX_2), \end{aligned}$$

for h and k symmetric $(0, 2)$-tensors.

With the help of (3.10) we now compute some terms of the curvature tensor of g. We again choose a basis

$$\{\xi, J\xi, q_1, Jq_1, X_i, JX_i\}$$

of T_pM for every $p \in M$.

Taking the symmetric sum with respect to X, Y, Z in (3.10) we have

$$\begin{aligned} 0 = 4\theta(X)\,(R_{YZWU} - 2ag \boxtimes r_{YZWU}) \\ - 2a\epsilon\big((X^\flat \wedge (\theta \circ J)) \odot \rho + (JX^\flat \wedge (\theta)) \odot \rho\big)(Y, Z, W, U) \\ - 2a\epsilon\big((Y^\flat \wedge (\theta \circ J)) \odot \rho + (JY^\flat \wedge (\theta)) \odot \rho\big)(Z, X, W, U) \\ - 2a\epsilon\big((Z^\flat \wedge (\theta \circ J)) \odot \rho + (JZ^\flat \wedge (\theta)) \odot \rho\big)(X, Y, W, U). \end{aligned}$$

Setting $Y, Z \in \operatorname{span}\{\xi, J\xi\}^\perp$ we obtain

$$R_{YZWU} = -8a\epsilon\, g(Y, JZ)\,\rho(W, U), \quad Y, Z \in \operatorname{span}\{\xi, J\xi\}^\perp \tag{3.11}$$

for every W, U. On the other hand, setting $X = q_1$, $Y = Jq_1$ and $Z \in \operatorname{span} X_i, JX_i$ we find

$$\begin{aligned} R_{YZWU} = af\big(g(Z, W)\theta(JU) - g(Z, U)\theta(JW) \\ - g(Z, JW)\theta(U) + g(Z, JU)\theta(W)\big), \end{aligned}$$

for every W, U, so that

$$\begin{aligned} R_{q_1ZWU} = af\big(g(JZ, U)\theta(JW) - g(JZ, W)\theta(JU) + \epsilon\, g(Z, U)\theta(W) \\ - \epsilon\, g(Z, W)\theta(U)\big) \end{aligned} \tag{3.12}$$

for $Z \in \operatorname{span}\{X_i, JX_i\}$ and all W, U.

Proposition 3.3. *(M, g, J) is Ricci-flat.*

Proof. Let $g(q_1, q_1) = b$ and suppose for simplicity that $b > 0$ (the case $b < 0$ is analogous). Denoting $q_2 = Jq_1$,for every $p \in M$ we choose an orthonormal basis

$$\left\{\sqrt{b}\Big(\xi - \frac{q_1}{b}\Big), \sqrt{b}\Big(J\xi - \frac{q_2}{b}\Big), \frac{q_1}{\sqrt{b}}, \frac{q_2}{\sqrt{b}}, X_i, JX_i\right\}$$

of T_pM, which has signature $(-1, \epsilon, 1, -\epsilon, \varepsilon^i, -\epsilon\varepsilon^i)$, where $g(X_i, X_i) = \varepsilon^i \in \{\pm 1\}$. We compute the Ricci curvature by contracting the curvature tensor with respect to this orthonormal basis and using (3.11) and (3.12):

$$\begin{aligned} r(W, U) = & -R\Big(W, \sqrt{b}\Big(\xi - \frac{q_1}{b}\Big), U, \sqrt{b}\Big(\xi - \frac{q_1}{b}\Big)\Big) \\ & + \epsilon R\Big(W, \sqrt{b}\Big(\xi - \frac{q_2}{b}\Big), U, \sqrt{b}\Big(\xi - \frac{q_2}{b}\Big)\Big) \\ & + R\Big(W, \frac{q_1}{\sqrt{b}}, U, \frac{q_1}{\sqrt{b}}\Big) - \epsilon R\Big(W, \frac{q_2}{\sqrt{b}}, U, \frac{q_2}{\sqrt{b}}\Big) \\ & + \varepsilon^i R(W, X_i, U, X_i) - \epsilon\varepsilon^i R(W, JX_i, U, JX_i) \\ = & \; 4af\Big(\theta \otimes \theta - \epsilon(\theta \circ J) \otimes (\theta \circ J)\Big) \\ & + 2af\epsilon \sum_i \varepsilon^i \left(\theta \otimes \theta - \epsilon(\theta \circ J) \otimes (\theta \circ J)\right) \\ = & \left(4a + 2a\epsilon \sum_i \varepsilon^i\right) r(W, U). \end{aligned}$$

We deduce that if $r(W,U) \neq 0$ then $4a + 2a\epsilon \sum_i \varepsilon^i = 1$, therefore

$$\dim M + 2 = 4 + 2a\epsilon \sum_i \varepsilon^i \,,$$

whence

$$\dim M = 2 + 2a\epsilon \sum_i \varepsilon^i < \dim M \,.$$

Since this is impossible we conclude that $r = 0$. □

Corollary 3.4. *The only possible values for λ are $\lambda = 0$ and $\lambda = -\epsilon/2$.*

In the next section we shall study the cases $\lambda = 0$ and $\lambda = -\epsilon/2$ separately.

Proposition 3.5. *The curvature tensor of g is given by*

$$R = k(\theta \wedge (\theta \circ J)) \otimes (\theta \wedge (\theta \circ J)),$$

for some function k. Moreover, if $k \neq 0$, the holonomy algebra of g is given by

$$\mathfrak{hol} \cong \mathbb{R} \begin{pmatrix} i_\epsilon & i_\epsilon & 0 \\ -i_\epsilon & -i_\epsilon & 0 \\ 0 & 0 & 0_n \end{pmatrix},$$

where i_ϵ is the ϵ-complex imaginary unit, which is a one dimensional subalgebra of $\mathfrak{su}(1,1) \subset \mathfrak{su}(r,s)$, $r+s = n+2$, for $\epsilon = -1$, and $\mathfrak{sl}(2,\mathbb{R}) \subset \mathfrak{sl}(n+2,\mathbb{R})$ for $\epsilon = 1$.

Proof. Since (M,g,J) is Ricci-flat, we have that $f = 0$, so that (3.10) becomes

$$\nabla R = 4\theta \otimes R.$$

Taking symmetric sum in the previous formula and applying second Bianchi identity we have that $\theta \wedge R_{WU} = 0$ for every W, U. But from the ϵ-Kähler symmetries of R we also have $(\theta \circ J) \wedge R_{WU} = 0$. These force the curvature to be of the form

$$R = k(\theta \wedge (\theta \circ J)) \otimes (\theta \wedge (\theta \circ J)),$$

for some function k.

On the other hand, since (M,g,J) is real analytic, the infinitesimal holonomy algebra coincides with the holonomy algebra (see [13]). Recall that the infinitesimal holonomy algebra at $p \in M$ is defined as $\mathfrak{hol}' = \bigcup_{k=0}^{\infty} \mathfrak{m}_k$, where

$$\mathfrak{m}_0 = \mathrm{span}\{R_{XY} / X, Y \in T_pM\}$$

and

$$\mathfrak{m}_k = \mathrm{span}\big\{\mathfrak{m}_{k-1} \cup \{(\nabla_{Z_k} \dots \nabla_{Z_1} R)_{XY} / Z_1, \dots, Z_k, X, Y \in T_pM\}\big\}.$$

As a simple computation shows, one has

$$\nabla\theta = \theta \otimes \theta + (2\lambda + \epsilon)(\theta \circ J) \otimes (\theta \circ J).$$

It is easy to see that this together with the recurrent formula $\nabla R = 4\theta \otimes R$ imply that $\mathfrak{m}_0 = m_1 = \cdots = \mathfrak{m}_k$ for every $k \in \mathbb{N}$, so that $\mathfrak{hol}' = \mathfrak{m}_0$. Now, since $R = k(\theta \wedge (\theta \circ J)) \otimes (\theta \wedge (\theta \circ J))$, the space $\mathfrak{m}_0$ is the one dimensional space generated by the endomorphism

$$\begin{aligned} A: \quad T_pM &\to T_pM \\ \xi, J\xi &\mapsto 0 \\ q_1 &\mapsto J\xi \\ q_2 &\mapsto \epsilon\xi \\ X_i, JX_i &\mapsto 0\,. \end{aligned}$$

This endomorphism is expressed as

$$\frac{1}{b}\begin{pmatrix} i_\epsilon & i_\epsilon & 0 \\ -i_\epsilon & -i_\epsilon & 0 \\ 0 & 0 & 0_n \end{pmatrix}$$

with respect to the ϵ-complex orthonormal basis

$$\left\{\frac{1}{\sqrt{|b|}}(q_1 + \epsilon i_\epsilon q_2), \left(\frac{1}{\sqrt{|b|}}q_1 - s\sqrt{|b|}\xi\right) + \epsilon i_\epsilon\left(\frac{1}{\sqrt{|b|}}q_2 - s\sqrt{|b|}J\xi\right), X_i + \epsilon i_\epsilon JX_i\right\},$$

where $g(q_1, q_1) = b$ and s is the sign of b. □

As a consequence of Proposition 3.5 we have that for all the values $\epsilon = \pm 1$ and $\lambda = 0, -\epsilon/2$, (M, g, J) is an *Osserman manifold* with a 2-step nilpotent Jacobi operator. It is also easy to see that (M, g, J) is VSI (vanishing scalar invariants). Finally, it is worth noting that if (M, g, J) is connected and simply-connected, then it is the product of a $2n$-dimensional ϵ-complex flat and totally geodesic manifold which can be thought of as an ϵ-complex wavefront, and a 4-dimensional Walker ϵ-Kähler manifold with a parallel null ϵ-complex vector field, which can be think of as the ϵ-complex time and direction of propagation of the wave.

4. Local form of the metrics

In previous sections (Propositions 3.2 and 3.3) we have seen that an ϵ-Kähler manifold (M, g, J) admitting a degenerate homogeneous ϵ-Kähler structure of linear type satisfies $\zeta = \lambda\xi$ for some constant $\lambda \in \mathbb{R}$ and is Ricci-flat. As stated in Corollary 3.4, this implies that the only possible values for λ are $\lambda = 0$ and $\lambda = -\epsilon/2$. Hereafter M is supposed to be non-flat and of dimension $2n + 4$.

4.1. $\lambda = -\epsilon/2$

We shall obtain the local form of the metric in the case $\lambda = -\epsilon/2$ for both values of ϵ simultaneously.

Substituting the value $\lambda = -\epsilon/2$ in (2.2) we have

$$S_XY = g(X, Y)\xi - g(\xi, Y)X + \epsilon g(X, JY)J\xi - \epsilon g(\xi, JY)JX + \epsilon g(\xi, JX)JY.$$

The condition $\widetilde{\nabla}\xi = 0$ then implies

$$\nabla\xi = \theta \otimes \xi,$$

which gives

$$\nabla\theta = \theta \otimes \theta, \quad \text{and} \quad \nabla\theta \circ J = \theta \otimes (\theta \circ J).$$

In particular $d\theta = 0$, so that fixing a point $p \in M$ there is a neighborhood $\mathcal{U}$ and a function $v\colon \mathcal{U} \to \mathbb{R}$ such that $\theta = dv$. We consider

$$w_1 = e^{-v},$$

whence $dw_1 = -e^{-v} = -w_1\theta$. We now consider

$$dw_1 \circ J = -w_1(\theta \circ J).$$

Differentiating we obtain

$$d(dw_1 \circ J) = -dw_1 \wedge (\theta \circ J) - w_1 d(\theta \circ J) = w_1\theta \wedge (\theta \circ J) - w_1\theta \wedge (\theta \circ J) = 0.$$

Therefore, reducing $\mathcal{U}$ if necessary, there is a function $w_2 : \mathcal{U} \to \mathbb{R}$ such that $dw_2 = \epsilon dw_1 \circ J$. We consider the function $w = w_1 + i_\epsilon w_2$. Then $dw = dw_1 + \epsilon i_\epsilon(dw_1 \circ J)$, so that dw is of type $(1,0)$ with respect to J and $w\colon \mathcal{U} \to \mathbb{C}^\epsilon$ is ϵ-holomorphic. In addition, it is a straightforward computation to see that

$$\nabla dw = -dw_1 \otimes \theta - w_1\nabla\theta - i_\epsilon dw_1 \otimes (\theta \circ J) - i_\epsilon w_1 \nabla(\theta \circ J) = 0,$$

i.e., dw is a nowhere vanishing parallel 1-form.

The function $w\colon \mathcal{U} \to \mathbb{C}^\epsilon$ defines a foliation of $\mathcal{U}$ by ϵ-complex hypersurfaces $\mathcal{H}_\tau = w^{-1}(\tau)$, $\tau \in \mathbb{C}^\epsilon$ (for those τ with non empty $w^{-1}(\tau)$). Note that since the tangent space to $\mathcal{H}_\tau$ is given by the kernel of dw, the hypersurfaces $\mathcal{H}_\tau$ are tangent to the distribution $\operatorname{span}\{\xi, J\xi\}^\perp$. We consider the vector field

$$Z = \operatorname{grad}(w_1) = dw_1^\sharp\,.$$

It is easy to see that, by construction,

$$JZ = -\epsilon\operatorname{grad}(w_2)\,.$$

These vector fields are written as

$$Z = -w_1\xi \quad \text{and} \quad JZ = -w_1 J\xi\,,$$

so that

$$\nabla Z = -dw_1 \otimes \xi - w_1\nabla\xi = w_1\theta \otimes \xi - w_1\theta \otimes \xi = 0\,,$$

and thus also $\nabla JZ = 0$. This implies in particular that Z and JZ are commuting ϵ-holomorphic Killing vector fields.

We now look at the holonomy of g at p, which was computed in Proposition 3.5. Using the same notation as before we denote $E = \operatorname{span}\{\xi, J\xi, q_1, q_2\} \subset T_pM$. This subspace is invariant under the holonomy action and so is $E^\perp$. In fact, the holonomy action on $E^\perp$ is trivial. This implies that, using the parallel transport with respect to ∇, we can extend an orthonormal basis $\{(X_a)|_p, (JX_a)|_p / \, a = 1, \dots, n\}$

of $E^{\perp}$ to an orthonormal reference $\{X_a, JX_a / \, a = 1, \ldots, n\}$ on $\mathcal{U}$ such that $\nabla X_a = 0 = \nabla JX_a$, $a = 1, \ldots, n$. In particular these vector fields are commuting ϵ-holomorphic Killing vector fields. In addition, let γ be any smooth curve on $\mathcal{U}$, we have

$$\left.\frac{d}{dt}\right|_{t=0} \big(dw(X_a)_{\gamma(t)}\big) = \big(\nabla_{\dot\gamma(t)} dw\big)(X_a) + dw\left(\nabla_{\dot\gamma(t)} X_a\right) = 0\,,$$

whence the function $dw(X_a)$ is constant along γ and takes the value 0 at p. This implies that X_a and thus JX_a are tangent to the foliation $\mathcal{H}_\tau$. Finally note that since they are parallel, X_a and JX_a commute with Z and JZ.

We have thus constructed a set of commuting para-holomorphic Killing vector fields $\{Z, JZ, X_a, JX_a\}$ tangent to $\mathcal{H}_\tau$. Therefore, reducing $\mathcal{U}$ if necessary, we can take ϵ-complex coordinates $\{w, z, z^a\}$ on U such that $\partial_z = \frac{1}{2}(Z + \epsilon i_\epsilon JZ)$, $\partial_{z^a} = \frac{1}{2}(X_a + \epsilon i_\epsilon JX_a)$. Note that since the distributions $\mathrm{span}\{\partial_w, \partial_z\}$ and $\mathrm{span}\{\partial_{z^a}, a = 1, \ldots, n\}$ are invariant by holonomy, the vector fields X_a and JX_a are orthogonal to $\mathrm{span}\{\partial_w, \partial_z\}$. Writing $z = z^1 + i_\epsilon z^2$, $z^a = x^a + i_\epsilon y^a$ and $w = w^1 + i_\epsilon w^2$, and rearranging the coordinates as $\{z^1, z^2, w^1, w^2, x^a, y^a\}$, we have that the metric with respect to these coordinates is

$$g = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & -\epsilon & 0 & \ldots & 0 \\ 1 & 0 & b & 0 & 0 & \ldots & 0 \\ 0 & -\epsilon & 0 & -\epsilon b & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & & & \\ \vdots & \vdots & \vdots & \vdots & & \Sigma & \\ 0 & 0 & 0 & 0 & & & \end{pmatrix},$$

for some function b, where

$$\Sigma = \mathrm{diag}\left(\begin{pmatrix} \varepsilon^a & 0 \\ 0 & -\epsilon\varepsilon^a \end{pmatrix}, a = 1, \ldots, n \right),$$

with $\varepsilon^a = g(X_a, X_a) \in \{\pm 1\}$. With respect to these coordinates, the ϵ-complex structure reads

$$J = \begin{pmatrix} 0 & \epsilon & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \epsilon \\ & & & 1 & 0 \end{pmatrix}.$$

Imposing that ∂_{z^1}, ∂_{z^2}, ∂_{x^a} and ∂_y^a are parallel, it is easy to see that b does not depend on z^1, z^2, x^a, y^a.

Finally, computing the curvature tensor with respect to these coordinates we obtain

$$R = \frac{1}{2}\Delta^\epsilon b (dw^1 \wedge dw^2) \otimes (dw^1 \wedge dw^2),$$

where

$$\Delta^\epsilon = -\epsilon \frac{\partial^2}{\partial (w^1)^2} + \frac{\partial^2}{\partial (w^2)^2}.$$

Denoting $F = \Delta^\epsilon b$ and taking into account that dw^1 and dw^2 are parallel, we have that

$$\nabla R = \frac{1}{2} dF \otimes (dw^1 \wedge dw^2) \otimes (dw^1 \wedge dw^2).$$

Recall that the recurrent formula (3.10) together with the Ricci-flatness of g give that

$$\nabla R = 4\theta \otimes R.$$

Comparing these two formulas for ∇R we have that

$$dF = 4F\theta\,,$$

where θ can be written as

$$\theta = -\frac{1}{w^1}\, dw^1\,.$$

Note that by construction $w^1 \neq 0$. The system of partial differential equations is thus

$$\frac{\partial F}{\partial w^1} = -\frac{4}{w_1} F$$
$$\frac{\partial F}{\partial w^2} = 0\,,$$

which has solution

$$F = \frac{R_0}{(w^1)^4}\,,$$

for some constant $R_0 \in \mathbb{R}$. We have thus proved:

Proposition 4.1. *Let (M, g, J) be an ϵ-Kähler manifold of dimension $2n + 4$, $n \geq 0$, admitting a degenerate homogeneous ϵ-Kähler structure of linear type S with $\zeta = -\frac{\epsilon}{2}\xi$. Then each $p \in M$ has a neighborhood ϵ-holomorphically isometric to an open subset of $(\mathbb{C}^\epsilon)^{n+2}$ with the ϵ-Kähler metric*

$$g = dw^1 dz^1 - \epsilon dw^2 dz^2 + b(dw^1 dw^1 - \epsilon dw^2 dw^2) + \sum_{a=1}^{n} \varepsilon^a (dx^a dx^a - \epsilon dy^a dy^a), \tag{4.1}$$

where $\varepsilon^a = \pm 1$, and the function b only depends on the coordinates $\{w^1, w^2\}$ and satisfies

$$\Delta^\epsilon b = \frac{R_0}{(w^1)^4}$$

for $R_0 \in \mathbb{R} - \{0\}$.

4.2. $\lambda = 0$

The case $\epsilon = -1$ and $\lambda = 0$ was studied in [6], where the local form and some properties of the metric g and the complex structure J were obtained. We reproduce below the main lines of the proof in [6], now for both values of ϵ simultaneously and putting special attention to the formulas which depend on ϵ.

Substituting the value $\lambda = 0$ in (2.2) we have that the homogeneous structure S takes the form

$$S_X Y = g(X,Y)\xi - g(\xi,Y)X + \epsilon g(X,JY)J\xi - \epsilon g(\xi,JY)JX\,.$$

In analogy with the complex case we shall call S *strongly degenerate*. We consider the form $\alpha = \theta + \epsilon i_\epsilon(\theta \circ J)$, which is of type $(1,0)$ with respect to the ϵ-complex structure J (see [10]). As a straightforward computation shows, $\nabla\alpha = \alpha \otimes \alpha$ so that $d\alpha = 0$. This implies in particular that α is an ϵ-holomorphic 1-form. Fixing a point $p \in M$, by the closeness of α there is a neighborhood $\mathcal{U}$ of p and an ϵ-holomorphic function $v\colon \mathcal{U} \to \mathbb{C}^\epsilon$ such that $\alpha = dv$. We consider the ϵ-holomorphic function

$$w = e^{-v},$$

where the the exponential must read $e^{x+i_\epsilon y} = e^x(\cos y + i_\epsilon \sin y)$ for $\epsilon = -1$ and $e^{x+i_\epsilon y} = e^x(\cosh y + i_\epsilon \sinh y)$ for $\epsilon = 1$. Differentiating we obtain that $\nabla dw = 0$ so that dw is a nowhere vanishing parallel ϵ-holomorphic 1-form on $\mathcal{U}$. The function $w\colon \mathcal{U} \to \mathbb{C}^\epsilon$ defines a foliation of $\mathcal{U}$ by ϵ-complex hypersurfaces $\mathcal{H}_\tau = w^{-1}(\tau)$, $\tau \in \mathbb{C}^\epsilon$ (if $w^{-1}(\tau)$ is non empty).

It is easy to adapt the construction made in [6] for both values of ϵ simultaneously in order to find a set of coordinates $\{z^1, z^2, w^1, w^2, x^a, y^a\}$ with respect to which the metric takes the form

$$g = dw^1dz^1 - \epsilon dw^2dz^2 + b(dw^1dw^1 - \epsilon dw^2dw^2) + \sum_{a=1}^{n} \varepsilon^a(dx^adx^a - \epsilon dy^ady^a),$$

and J is the standard ϵ-complex structure of $\mathbb{R}^{2n+4}$, where $\varepsilon^a = \pm 1$ and the function b only depends on the coordinates $\{w^1, w^2\}$. In addition, as a simple computation shows,

$$R = \frac{1}{2}\Delta^\epsilon b(dw^1 \wedge dw^2) \otimes (dw^1 \wedge dw^2)\,,$$

and

$$\theta = \frac{-1}{(w^1)^2 - \epsilon(w^2)^2}(w^1dw^1 - \epsilon w_2 dw^2)\,.$$

Finally, imposing the equation $\nabla R = 4\theta \otimes R$ and denoting $F = \Delta^\epsilon b$, we obtain the system of partial differential equations

$$\frac{\partial F}{\partial w^1} = \frac{-4w^1}{(w^1)^2 - \epsilon(w^2)^2}F$$
$$\frac{\partial F}{\partial w^2} = \frac{4\epsilon w^2}{(w^1)^2 - \epsilon(w^2)^2}F\,,$$

which has solution

$$F = \frac{R_0}{((w^1)^2 - \epsilon(w^2)^2)^2}\,,$$

for some constant $R_0 \in \mathbb{R}$. Note that since $w = e^{-v}$ we always have $(w^1)^2 - \epsilon(w^2)^2 \neq 0$. We have thus proved:

Proposition 4.2. *Let (M, g, J) be an ϵ-Kähler manifold of dimension $2n+4$, $n \geq 0$, admitting a strongly degenerate homogeneous ϵ-Kähler structure of linear type S. Then each $p \in M$ has a neighborhood ϵ-holomorphically isometric to an open subset of $(\mathbb{C}^\epsilon)^{n+2}$ with the ϵ-Kähler metric*

$$g = dw^1 dz^1 - \epsilon dw^2 dz^2 + b(dw^1 dw^1 - \epsilon dw^2 dw^2) + \sum_{a=1}^{n} \varepsilon^a (dx^a dx^a - \epsilon dy^a dy^a)\,, \tag{4.2}$$

where $\varepsilon^a = \pm 1$, and the function b only depends on the coordinates $\{w^1, w^2\}$ and satisfies

$$\Delta^\epsilon b = \frac{R_0}{((w^1)^2 - \epsilon (w^2)^2)^2}$$

for $R_0 \in \mathbb{R} - \{0\}$.

4.3. The manifold $((\mathbb{C}^\epsilon)^{n+2}, g)$

Propositions 4.2 and 4.1 give the local forms (4.2) and (4.1) of the metric of a manifold with a degenerate homogeneous ϵ-Kähler structure of linear type. This motivates the study of the space $(\mathbb{C}^\epsilon)^{2+n}$ endowed with this particular ϵ-Kähler metric, which can thus be understood as the simplest instance of this type of manifolds. In particular, the goal of this section is to study the singular nature of this spaces, and their analogies with homogeneous plane waves. We shall restricts ourselves to the Lorentz ϵ-Kähler case, i.e., metrics of index 2. Throughout this section $\|w\|_\lambda$ must be understood as

$$\|w\|_\lambda^2 = \begin{cases} w_1^2 - \epsilon w_2^2 & \text{for } \lambda = 0, \\ w_1^2 & \text{for } \lambda = -\epsilon/2\,. \end{cases} \tag{4.3}$$

In addition, Δ^ϵ shall stand for the differential operator

$$\Delta^\epsilon = -\epsilon \frac{\partial^2}{\partial w_1^2} + \frac{\partial^2}{\partial w_2^2}\,.$$

We thus consider the manifold $(\mathbb{C}^\epsilon)^{2+n} = (\mathbb{R}^{2n+4}, J_0)$, where J_0 is the standard ϵ-complex structure, with real coordinates $\{z^1, z^2, w^1, w^2, x^a, y^a\}$, endowed with the metric

$$g = dw^1 dz^1 - \epsilon dw^2 dz^2 + b(dw^1 dw^1 - \epsilon dw^2 dw^2) + \sum_{a=1}^{n} (dx^a dx^a - \epsilon dy^a dy^a), \tag{4.4}$$

where b is a function of the variables (w^1, w^2) satisfying

$$\Delta^\epsilon b = \frac{R_0}{\|w\|_\lambda^4}\,, \quad R_0 \in \mathbb{R} - \{0\}\,. \tag{4.5}$$

As computed before, the curvature $(1,3)$-tensor field of g is

$$R = \frac{1}{2} \frac{R_0}{\|w\|_\lambda^4} \big((dw^1 \wedge dw^2) \otimes (dw^1 \otimes \partial_{z^2}) + \epsilon (dw^1 \wedge dw^2) \otimes (dw^2 \otimes \partial_{z^1}) \big).$$

As $R_0 \neq 0$, it exhibits a singular behavior at

$$\mathcal{S} = \{\|w\|_\lambda = 0\}.$$

This set can be understood as a singularity of g in the cosmological sense: the geodesic deviation equation is governed by the components of the curvature tensor $R^{z^j}_{w^1 w^2 w^i}$, $i, j = 1, 2$, making the tidal forces infinite at $\mathcal{S}$. Indeed, we can compute a component of the curvature tensor with respect to an orthonormal parallel frame along a geodesic reaching the singular set in finite time, and see that it is singular (see [17]). Let γ be the geodesic with initial value $\gamma(0) = (0, 0, 1, 0, \ldots, 0)$ and $\dot{\gamma} = (0, 0, -1, 0, \ldots, 0)$. It is easy to see that this geodesic is of the form

$$\gamma(t) = (z^1(t), z^2(t), 1 - t, 0, x^a(t), y^a(t))$$

for some functions $z^1(t), z^2(t), x^a(t), y^a(t)$, $a = 1, \ldots, n$, and reaches the singular set $\mathcal{S}$ at $t = 1$. Let

$$E(t) = W^1(t)\partial_{w^1} + W^2(t)\partial_{w^2} + Z^1(t)\partial_{z^1} + Z^2(t)\partial_{z^2} + X^a(t)\partial_{x^a} + Y^a(t)\partial_{y^a}$$

be a vector field along γ. E is parallel if the following equations hold:

$$\begin{gathered} 0 = \dot{W}^1, \quad 0 = \dot{W}^2, \\ 0 = \dot{Z}^1 - W^1\Gamma^{z^1}_{w^1 w^1} - W^2\Gamma^{z^1}_{w^1 w^2}, \quad 0 = \dot{Z}^2 - W^1\Gamma^{z^2}_{w^1 w^1} - W^2\Gamma^{z^2}_{w^1 w^2}, \\ 0 = \dot{X}^a, \quad 0 = \dot{Y}^a. \end{gathered}$$

We can thus obtain an orthonormal parallel frame $\{E_1(t), \ldots, E_{4+2n}(t)\}$ with $E_1(t)$ and $E_2(t)$ of the form

$$\begin{aligned} E_1(t) =& \frac{1}{\sqrt{|b(0)|}}\partial_{w^1} + Z_1^1(t)\partial_{z^1} + Z_1^2(t)\partial_{z^2} + X_1^a\partial_{x^a} + Y_1^a\partial_{y^a}, \\ E_2(t) =& \frac{1}{\sqrt{|b(0)|}}\partial_{w^2} + Z_2^1(t)\partial_{z^1} + Z_2^2(t)\partial_{z^2} + X_2^a\partial_{x^a} + Y_2^a\partial_{y^a}, \end{aligned}$$

where

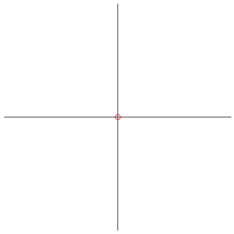
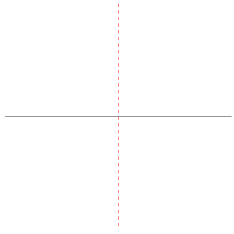
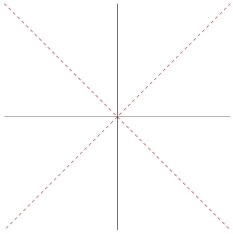
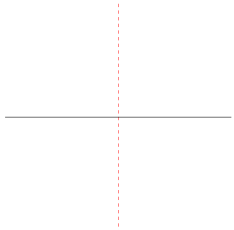
$$E_1(0) = \frac{1}{\sqrt{|b(0)|}}\partial_{w^1}, \quad E_2(0) = \frac{1}{\sqrt{|b(0)|}}\partial_{w^2}, \quad \text{and} \quad b(0) = b(0,0).$$

The value of the curvature tensor applied to $E_1(t), E_2(t)$ is

$$R_{E_1(t)E_2(t)E_1(t)E_2(t)} = \frac{R_0}{2b(0)^2}\frac{1}{||w(t)||_\lambda^4} = \frac{R_0}{2b(0)^2}\frac{1}{(1-t)^4},$$

which is singular at $t = 1$.

Note that $(\mathbb{C}^\epsilon)^{2+n} - \mathcal{S}$ is connected and not simply-connected for $\epsilon = -1$ and $\lambda = 0$ while it is not connected nor simply-connected for the other values. Moreover, $(\mathbb{C}^\epsilon)^{2+n} - \mathcal{S}$ has two connected components for $\lambda = -\epsilon/2$ and $\epsilon = \pm 1$ and four connected components for $\lambda = 0$ and $\epsilon = 1$.

Singular set $\mathcal{S}$	$\lambda = 0$	$\lambda = -\epsilon/2$
$\epsilon = -1$	 $\mathcal{S} : (w^1)^2 + (w^2)^2 = 0$	 $\mathcal{S} : w^1 = 0$
$\epsilon = 1$	 $\mathcal{S} : (w^1)^2 - (w^2)^2 = 0$	 $\mathcal{S} : w^1 = 0$

Finally we show that degenerate homogeneous ϵ-Kähler structures of linear type indeed exist and are realized in the manifold $((\mathbb{C}^\epsilon)^{2+n} - \mathcal{S}, g)$.

Proposition 4.3. *For every data (b, R_0) satisfying (4.5), the ϵ-Kähler manifold $((\mathbb{C}^\epsilon)^{2+n} - \mathcal{S}, g)$ admits a strongly degenerate pseudo-Kähler homogeneous structure of linear type.*

Proof. Let

$$\xi = \begin{cases} \dfrac{-1}{(w^1)^2 - \epsilon (w^2)^2}(w^1 \partial_{z^1} + w^2 \partial_{z^2}) & \lambda = 0\,, \\ -\dfrac{1}{w^1}\partial_{z^1} & \lambda = -\dfrac{\epsilon}{2}\,. \end{cases}$$

We take the tensor field

$$\begin{aligned} S_X Y = {} & g(X,Y)\xi - g(\xi, Y)X + \epsilon g(X, JY)J\xi - \epsilon g(\xi, JY)JX \\ & - 2\lambda g(\xi, JX)JY. \end{aligned}$$

A straightforward computation shows that $\widetilde{\nabla}\xi = 0$ and $\widetilde{\nabla}R = 0$, where $\widetilde{\nabla} = \nabla - S$, so that S satisfies equations (2.1). □

5. The homogeneous model for a degenerate homogeneous ϵ-Kähler structure of linear type

Let (M, g, J) be an ϵ-Kähler manifold admitting a degenerate homogeneous structure of linear type S. From [18] one can construct a Lie algebra of infitesimal isometries associated to S. This algebra is (fixing a point $p \in M$ as the origin)

$$\mathfrak{g} = T_pM \oplus \mathfrak{hol}^{\widetilde{\nabla}}$$

where $\widetilde{\nabla} = \nabla - S$ is the canonical connection associated to the homogeneous structure tensor S. The brackets in $\mathfrak{g}$ are

$$\begin{cases} [A,B] = AB - BA, & A, B \in \mathfrak{hol}^{\widetilde{\nabla}} \\ [A,\eta] = A \cdot \eta, & A \in \mathfrak{hol}^{\widetilde{\nabla}}, \eta \in T_pM \\ [\eta,\zeta] = S_\eta\zeta - S_\zeta\eta - \widetilde{R}_{\eta\zeta}\,, & \eta,\zeta \in T_pM\,, \end{cases}$$

where $\widetilde{R}$ is the curvature tensor of $\widetilde{\nabla}$. This curvature tensor can be computed as

$$\widetilde{R} = R - R^S,$$

where

$$R^S_{XY}Z = [S_X, S_Y]\,Z - S_{S_XY - S_YX}Z\,.$$

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let H be the connected Lie subgroup with Lie algebra $\mathfrak{hol}^{\widetilde{\nabla}}$. If H is closed in G, then the infinitesimal model $(\mathfrak{g}, \mathfrak{hol}^{\widetilde{\nabla}})$ is called *regular*, and we shall call G/H the *homogeneous model* for M associated to S, which means that (M, g, J) is locally ϵ-holomorphically isometric to G/H with the G-invariant metric and ϵ-complex structure given by g and J at T_pM.

In [6] the infinitesimal model for $\epsilon = -1$ and $\lambda = 0$ is computed, proving that it is regular and the corresponding homogeneous model is not complete. We shall obtain the same result for the rest of the values of ϵ and λ.

5.1. The case $\lambda = -\epsilon/2$

Denoting $p_1 = \xi$ and $p_2 = J\xi$, for the sake of simplicity we choose $p \in M$ such that with respect to the basis

$$\{p_1, p_2, q_1, q_2, X_a, JX_a\}$$

and its dual $\{p^1, p^2, q^1, q^2, X^a, JX^a\}$ the curvature is written

$$R_p = R_0 q^1 \wedge q^2 \otimes (q^1 \otimes p_2 + \epsilon q^2 \otimes p_1)\,.$$

Substituting $\lambda = -\epsilon/2$ in (2.2) we obtain by direct calculation that the non-vanishing terms of $\widetilde{R}$ are:

$$\begin{aligned} \widetilde{R}_{p_2q_1}: \quad q_1 &\mapsto 2p_2 & \widetilde{R}_{q_1q_2}: \quad q_1 &\mapsto (R_0 - b(p))p_2 \\ q_2 &\mapsto 2\epsilon p_1 & q_2 &\mapsto (R_0 - b(p))\epsilon p_1 \\ X_a &\mapsto 0 & X_a &\mapsto -JX_a \\ JX_a &\mapsto 0 & JX_a &\mapsto -\epsilon X_a \\ p_1, p_2 &\mapsto 0 & p_1, p_2 &\mapsto 0 \end{aligned}$$

$$\begin{aligned} \widetilde{R}_{q_2X_a}: \quad q_1 &\mapsto -JX_a & \widetilde{R}_{q_2JX_a}: \quad q_1 &\mapsto -\epsilon X_a \\ q_2 &\mapsto -\epsilon X_a & q_2 &\mapsto -\epsilon JX_a \\ X_a &\mapsto p_2 & X_a &\mapsto \epsilon p_1 \\ JX_a &\mapsto \epsilon p_1 & JX_a &\mapsto \epsilon p_2 \\ p_1, p_2 &\mapsto 0 & p_1, p_2 &\mapsto 0 \end{aligned}$$

$$\begin{aligned} \widetilde{R}_{X_aJX_a}: \quad q_1 &\mapsto -2p_2 \\ q_2 &\mapsto -2\epsilon p_1 \\ X_a &\mapsto 0 \\ JX_a &\mapsto 0 \\ p_1, p_2 &\mapsto 0, \end{aligned}$$

so that $\dim(\mathfrak{hol}^{\widetilde{\nabla}}) = 2n + 2$. Choosing endomorphisms

$$A = 2(q^1 \otimes p_2 + \epsilon q^2 \otimes p_1), \quad B_a = \widetilde{R}_{q_2X_a}, \quad C_a = \widetilde{R}_{q_2JX_a},$$

$$K = \frac{1}{2}(R_0 - b(p))A - \sum_a (X^a \otimes JX_a + \epsilon JX^a \otimes X_a)$$

as basis of $\mathfrak{hol}^{\widetilde{\nabla}}$, the Lie algebra $\mathfrak{g}$ has non-vanishing brackets

$$\begin{aligned} &[B_a, C_a] = \epsilon A, \quad [B_a, K] = -C_a, \quad [C_a, K] = -\epsilon B_a, \\ &[A, q_1] = 2p_2, \quad [A, q_2] = 2\epsilon p_1, \\ &[B_a, q_1] = -JX_a, \quad [B_a, q_2] = -\epsilon X_a, \quad [B_a, X_a] = -p_2, \quad [B_a, JX_a] = -\epsilon p_1, \\ &[C_a, q_1] = -\epsilon X_a, \quad [C_a, q_2] = -\epsilon y_a, \quad [C_a, X_a] = \epsilon p_1, \quad [C_a, JX_a] = \epsilon p_2, \\ &[K, X_a] = JX_a, \quad [K, JX_a] = \epsilon X_a, \\ &[p_1, q_1] = -p_1, \quad [p_2, q_1] = -3p_2 - A, \quad [p_2, q_2] = -2\epsilon p_1, \\ &[q_1, q_2] = 2b(p)p_2 - q_2 - \tfrac{1}{2}(R_0 - b(p))A + K, \\ &[q_1, X_a] = X_a, \quad [q_1, JX_a] = JX_a, \\ &[q_2, X_a] = 2JX_a - B_a, \quad [q_2, JX_a]\, 2\epsilon X_a - C_a, \\ &[X_a, JX_a] = 2p_2 + A. \end{aligned}$$

One can check that $\mathfrak{g}$ is a solvable Lie algebra with a 3-step nilradical $\mathfrak{n} = \operatorname{span}\{p_1, p_2, q_2 - K, X_a, JX_a, A, B_a, C_a, a = 1, \ldots, n\}$. Since $\mathfrak{g}$ has trivial center, the adjoint representation is faithful and provides a matrix realization of $\mathfrak{g}$. With respect to this realization it is a straightforward computation that by exponentiation of $\mathfrak{g}$ and $\mathfrak{hol}^{\widetilde{\nabla}}$ we obtain a Lie group G and a closed Lie subgroup H respectively, so that the infinitesimal model $(\mathfrak{g}, \mathfrak{hol}^{\widetilde{\nabla}})$ is regular and G/H is a homogeneous model for (M, g, J). Let $\bar{g}$ and $\bar{J}$ be the G-invariant metric and complex structure on G/H induced from (M, g, J) on G/H.

Proposition 5.1. *The homogeneous model $(G/H, \bar{g}, \bar{J})$ is not geodesically complete.*

Proof. Let σ be the Lie algebra involution of $\mathfrak{g}$ given by

$$\begin{aligned} \sigma: \quad \mathfrak{g} &\to \mathfrak{g} \\ A &\mapsto -A \\ B_a &\mapsto -B_a \\ C_a &\mapsto C_a \\ K &\mapsto -K \\ p_1 &\mapsto p_1 \\ p_2 &\mapsto -p_2 \\ q_1 &\mapsto q_1 \\ q_2 &\mapsto -q_2 \\ X_a &\mapsto X_a \\ JX_a &\mapsto -JX_a\,. \end{aligned}$$

One can check that the restriction of σ to $\mathfrak{m}$ is an isometry with respect to the bilinear form given by $\bar{g}$. The subalgebra of fixed points is

$$\mathfrak{g}^\sigma = \operatorname{span}\{p_1, q_1, X_a, C_a, a = 1, \ldots, n\}\,.$$

Working with the universal cover if necessary we can assume that G is simply-connected so that σ induces an involution in G and therefore an isometric involution in G/H. We will denote all this involutions by σ. Let G^σ be the connected Lie subgroup of G with Lie algebra $\mathfrak{g}^\sigma$, note that $\sigma(\mathfrak{hol}^{\widetilde{\nabla}}) \subset \mathfrak{hol}^{\widetilde{\nabla}}$, so that $(G/H)^\sigma = G^\sigma/H^\sigma$, where the superindex σ stands for the fixed point set by σ. It is a well-known result that $(G/H)^\sigma$ is a closed totally geodesic submanifold of G/H. Let now θ be the Lie algebra involution of $\mathfrak{g}^\sigma$ given by

$$\begin{aligned} \theta: \mathfrak{g}^\sigma &\to \mathfrak{g}^\sigma \\ C_a &\mapsto -C_a \\ p_1 &\mapsto p_1 \\ q_1 &\mapsto q_1 \\ X_a &\mapsto -X_a\,, \end{aligned}$$

which is again an isometry with respect to the bilinear form induced in $\mathfrak{g}^\sigma$ by restriction from $\mathfrak{m}$. The subalgebra of fixed points is $\mathfrak{k} = (\mathfrak{g}^\sigma)^\theta = \operatorname{span}\{p_1, q_1\}$. Note that $\mathfrak{k} \cap \mathfrak{hol}^{\widetilde{\nabla}} = 0$. Let $\widetilde{G^\sigma}$ be the universal cover of G^σ and $\widetilde{H^\sigma}$ the corresponding closed subgroup, $\theta\colon \mathfrak{g}^\sigma \to \mathfrak{g}^\sigma$ induces an isometric involution $\theta\colon \widetilde{G^\sigma}/\widetilde{H^\sigma} \to \widetilde{G^\sigma}/\widetilde{H^\sigma}$. Therefore, let K be the connected Lie subgroup of $\widetilde{G^\sigma}$ with lie algebra $\mathfrak{k}$, K is a totally geodesic submanifold of $\widetilde{G^\sigma}/\widetilde{H^\sigma}$. Let s be the sign of $b(p)$. We define the left-invariant vector fields

$$U = 1/(\sqrt{|b(p)|})\, q_1 \quad \text{and} \quad V = U - s\sqrt{|b(p)|}\, p_1$$

in $\mathfrak{k}$. We have

$$< U, U >= s, \quad < V, V >= -s, \quad < U, V >= 0, \quad \text{and} \quad [U, V] = \frac{1}{\sqrt{|b(p)|}}(V - U),$$

where $<,>$ stands for the bilinear form inherited by $\mathfrak{k}$ from $\mathfrak{g}^\sigma$. It is a straightforward computation to see that K is not geodesically complete. Hence, since we have the following inclusions of totally geodesic submanifolds,

$$K \subset \widetilde{G^\sigma}, \quad G^\sigma = (G/H)^\sigma \subset G/H,$$

the manifold $(G/H, g, J)$ is not geodesically complete. □

Corollary 5.2. *Let (M, g, J) be a connected and simply-connected ϵ-Kähler manifold admitting a degenerate ϵ-Kähler structure of linear type with $\zeta = -\frac{\epsilon}{2}\xi$, then it is not geodesically complete.*

Proof. Suppose that (M, g, J) is geodesically complete, then Ambrose–Singer theorem assures that (M, g, J) is (globally) ϵ-holomorphically isometric to the homogeneous model $(G/H, \bar{g}, \bar{J})$. But this homogeneous model is not geodesically complete. □

5.2. The case $\lambda = 0$

Denoting again $p_1 = \xi$ and $p_2 = J\xi$, we choose $p \in M$ such that with respect to the basis $\{p_1, p_2, q_1, q_2, X_a, JX_a\}$ and its dual $\{p^1, p^2, q^1, q^2, X^a, JX^a\}$ the curvature is written

$$R_p = R_0\, q^1 \wedge q^2 \otimes (q^1 \otimes p_2 + \epsilon q^2 \otimes p_1).$$

Substituting $\lambda = 0$ in (2.2) we obtain by direct calculation that the non-vanishing terms of $\widetilde{R}$ are:

$$\widetilde{R}_{p_1 q_2}: \begin{array}{r} q_1 \mapsto -2p_2 \\ q_2 \mapsto -2\epsilon p_1 \\ X_a \mapsto 0 \\ JX_a \mapsto 0 \\ p_1, p_2 \mapsto 0 \end{array} \qquad \widetilde{R}_{p_2 q_1}: \begin{array}{r} q_1 \mapsto 2p_2 \\ q_2 \mapsto 2\epsilon p_1 \\ X_a \mapsto 0 \\ JX_a \mapsto 0 \\ p_1, p_2 \mapsto 0 \end{array}$$

$$\widetilde{R}_{q_1 q_2}: \begin{array}{r} q_1 \mapsto (R_0 - 2b(p))p_2 \\ q_2 \mapsto (R_0 - 2b(p))\epsilon p_1 \\ X_a \mapsto 0 \\ JX_a \mapsto 0 \\ p_1, p_2 \mapsto 0 \end{array} \qquad \widetilde{R}_{X_a JX_a}: \begin{array}{r} q_1 \mapsto -2p_2 \\ q_2 \mapsto -2\epsilon p_1 \\ X_a \mapsto 0 \\ JX_a \mapsto 0 \\ p_1, p_2 \mapsto 0, \end{array}$$

so that $\dim(\mathfrak{hol}^{\widetilde{\nabla}}) = 1$. Choosing the endomorphism

$$A = 2(q^1 \otimes p_2 + \epsilon q^2 \otimes p_1)$$

as basis of $\mathfrak{hol}^{\widetilde{\nabla}}$, the Lie algebra $\mathfrak{g}$ has non-vanishing brackets

$$\begin{array}{l} [A, q_1] = 2p_2, \quad [A, q_2] = 2\epsilon p_1, \\ [p_1, q_1] = -p_1, \quad [p_1, q_2] = p_2 + A, \\ [p_2, q_1] = -3p_2 - A, \quad [p_2, q_2] = -\epsilon p_1, \\ [q_1, q_2] = 2b(p)p_2 - \frac{1}{2}(R_0 - 2b(p))A, \\ [q_1, X_a] = X_a, \quad [q_1, JX_a] = JX_a, \\ [q_2, X_a] = JX_a, \quad [q_2, JX_a] = \epsilon X_a, \\ [X_a, JX_a] = 2p_2 + A. \end{array}$$

One can check that $\mathfrak{g}$ is a solvable Lie algebra with a 2-step nilradical $\mathfrak{n} = \operatorname{span}\{p_1, p_2, X_a, JX_a, A, a = 1, \dots, n\}$. Since $\mathfrak{g}$ has trivial center, the adjoint representation is faithful and provides a matrix realization of $\mathfrak{g}$. Exponentiating $\mathfrak{g}$ and $\mathfrak{hol}^{\tilde{\nabla}}$ we obtain a Lie group G and a closed Lie subgroup H respectively, so that the infinitesimal model $(\mathfrak{g}, \mathfrak{hol}^{\tilde{\nabla}})$ is regular and G/H is a homogeneous model for (M, g, J). Let $\bar{g}$ and $\bar{J}$ be the G-invariant metric and complex structure on G/H induced from (M, g, J) on G/H.

Proposition 5.3. *The homogeneous model $(G/H, \bar{g}, \bar{J})$ is not geodesically complete.*

Proof. Following the same arguments as in the proof of Proposition 3.1 (see also those in §4 of [6]), we find isometric involutions $\sigma\colon \mathfrak{g} \to \mathfrak{g}$ and $\mathfrak{g}^\sigma \to \mathfrak{g}^\sigma$ so that the connected Lie subgroup with lie algebra $\mathfrak{k} = (\mathfrak{g}^\sigma)^\theta$ is not complete. □

Corollary 5.4. *Let (M, g, J) be a connected and simply-connected ϵ-Kähler manifold admitting a strongly-degenerate ϵ-Kähler structure of linear type. Then it is not geodesically complete.*

6. The ϵ-quaternion Kähler case

Throughout this section $\dim(M) = 4n \geq 8$ is assumed. We shall study degenerate homogeneous structures of linear type in the ϵ-quaternion Kähler case.

Let $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)$, we can combine some definitions of pseudo-quaternion geometry and para-quaternion geometry in the following way. For pseudo-quaternion geometry ϵ must be substituted by $(-1, -1, -1)$, and for para-quaternion geometry ϵ must be substituted by $(-1, 1, 1)$.

Definition 6.1. Let (M, g) be a pseudo-Riemannian manifold.

(1) An ϵ-quaternion Hermitian structure on (M, g) is a 3-rank subbundle $Q \subset \mathfrak{so}(TM)$ with a local basis J_1, J_2, J_3 satisfying

$$J_a^2 = \epsilon_a, \quad J_1 J_2 = J_3.$$

(2) (M, g) is called ϵ-quaternion Kähler if it is strongly-oriented and it admits a parallel ϵ-quaternion Hermitian structure with respect to the Levi-Civita connection.

The first previous definition means that at every point $p \in M$ there is a subalgebra $Q_p \subset \mathfrak{so}(T_pM)$ isomorphic to the imaginary ϵ-quaternions, and in particular g has signature $(4r, 4s)$, $r+s = n$ for $\epsilon = (-1, -1, -1)$ and $(2n, 2n)$ for $\epsilon = (-1, 1, 1)$. We shall denote by $Sp^\epsilon(n)$ the group $Sp(r, s)$, $r+s = n$, when $\epsilon = (-1, -1, -1)$ and $Sp(n, \mathbb{R})$ when $\epsilon = (-1, 1, 1)$. Their Lie algebras are denoted by $\mathfrak{sp}^\epsilon(n)$ respectively. For the proof of the following proposition, see [1].

Proposition 6.2 ([1]). *An ϵ-quaternion Kähler manifold is Einstein and has Riemann curvature tensor*

$$R = \nu_q R_0 + R^{\mathfrak{sp}^\epsilon(n)},$$

where $\nu_q = \frac{s}{16n(n+2)}$ is one quarter the reduced scalar curvature, R_0 is four times the curvature of the ϵ-quaternionic hyperbolic space (of the corresponding signature)

$$(R_0)_{XYZW} = g(X,Z)g(Y,W) - g(Y,Z)g(X,W) - \sum_a \epsilon_a \{g(J_aX,Z)g(J_aY,W) - g(J_aY,Z)g(J_aX,W) + 2g(X,J_aY)g(Z,J_aW)\}, \tag{6.1}$$

and $R^{\mathfrak{sp}^\epsilon(n)}$ is an algebraic curvature tensor of type $\mathfrak{sp}^\epsilon(n)$.

Let J_1, J_2, J_3 be a local basis of Q, and $\omega_a = g(\cdot, J_a\cdot)$, $a = 1,2,3$. The 4-form

$$\Omega = \sum_a -\epsilon_a\, \omega_a \wedge \omega_a$$

is independent of the choice of basis and hence it is globally defined. An ϵ-quaternion Hermitian manifold (M,g,Q) is ϵ-quaternion Kähler if and only if Ω is parallel with respect to the Levi-Civita connection (cf. [1]), or equivalently if the holonomy of the Levi-Civita connection is contained in $Sp^\epsilon(n)Sp^\epsilon(1)$. This is also equivalent to

$$\nabla J_a = \sum_{b=1}^{3} c_{ab}\, J_b, \quad a = 1,2,3, \tag{6.2}$$

where (c_{ab}) is a matrix in $\mathfrak{sp}^\epsilon(1)$.

Definition 6.3. An ϵ-quaternion Kähler manifold (M,g,Q) is called a homogeneous ϵ-quaternion Kähler manifold if there is a connected Lie group G of isometries acting transitively on M and preserving Q. (M,g,Q) is called a reductive homogeneous ϵ-quaternion Kähler manifold if the Lie algebra $\mathfrak{g}$ of G can be decomposed as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with

$$[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m}.$$

As a corollary of Kiričenko's theorem [12], we have

Theorem 6.4. *A connected, simply-connected, (geodesically) complete ϵ-quaternion Kähler manifold (M,g,Q) is reductive homogeneous if and only if it admits a linear connection $\widetilde{\nabla}$ satisfying*

$$\widetilde{\nabla} g = 0, \quad \widetilde{\nabla} R = 0, \quad \widetilde{\nabla} S = 0, \quad \widetilde{\nabla}\Omega = 0, \tag{6.3}$$

where $S = \nabla - \widetilde{\nabla}$, ∇ is the Levi-Civita connection, R is the curvature tensor of ∇, and Ω is the canonical 4-form associated to Q.

Note that the condition $\widetilde{\nabla}\Omega = 0$ is equivalent to

$$\widetilde{\nabla} J_a = \sum_{b=1}^{3} \widetilde{c}_{ab}\, J_b, \quad a = 1,2,3, \tag{6.4}$$

where $(\widetilde{c}_{ab}) \in \mathfrak{sp}^\epsilon(1)$. A tensor field S satisfying the previous equations is called a *homogeneous ϵ-quaternion Kähler structure.* The classification of such structures was obtained in [3] and [14], resulting five primitive classes: $\mathcal{QK}_1^\epsilon$, $\mathcal{QK}_2^\epsilon$, $\mathcal{QK}_3^\epsilon$, $\mathcal{QK}_4^\epsilon$ and $\mathcal{QK}_5^\epsilon$. Among them, $\mathcal{QK}_1^\epsilon$, $\mathcal{QK}_2^\epsilon$ and $\mathcal{QK}_3^\epsilon$ have dimension growing linearly with respect to the dimension of M. Hence:

Definition 6.5. A homogeneous ϵ-quaternion Kähler structure S is called of linear type if it belongs to the class $\mathcal{QK}_1^\epsilon + \mathcal{QK}_2^\epsilon + \mathcal{QK}_3^\epsilon$.

The local expression of $S \in \mathcal{QK}_1^\epsilon + \mathcal{QK}_2^\epsilon + \mathcal{QK}_3^\epsilon$ is

$$(6.5) \qquad S_X Y = g(X,Y)\xi - g(Y,\xi)X - \sum_{a=1}^{3} \epsilon_a \left(g(J_aY,\xi)J_aX - g(X,J_aY)J_a\xi\right) + \sum_{a=1}^{3} g(X,\zeta^a)J_aY,$$

where ξ and ζ^a, $a = 1,2,3$, are vector fields. We then give the following further definition.

Definition 6.6. A homogeneous ϵ-quaternion Kähler structure of linear type is called degenerate if $\xi \neq 0$ and $g(\xi,\xi) = 0$.

Remark 6.7. The case $\zeta^a = 0$ for $a = 1,2,3$ was studied in [6] resulting that the manifold (M,g,Q) must be flat.

Proposition 6.8. *Let (M,g,Q) be a ϵ-quaternion Kähler manifold admitting a degenerate homogeneous ϵ-quaternion Kähler structure of linear type. Then (M,g,Q) is flat.*

Proof. Following Proposition 6.2 we decompose the curvature of (M,g,Q) as $R = \nu_q R_0 + R^{\mathfrak{sp}^\epsilon(n)}$. Recall that the space of algebraic curvature tensors $\mathcal{R}^{\mathfrak{sp}^\epsilon(n)}$ is $[S^4E]$ with $E = \mathbb{C}^{2n}$ for $\epsilon = (-1,-1,-1)$, and S^4E with $E = \mathbb{R}^{2n}$ for $\epsilon = (-1,1,1)$. Since R_0 is $Sp^\epsilon(n)Sp^\epsilon(1)$-invariant and ν_q is constant, the covariant derivative ∇R_0 vanishes. Moreover, for every vector field X, S_X acts as an element of $\mathfrak{sp}^\epsilon(n) + \mathfrak{sp}^\epsilon(1)$, whence $SR_0 = 0$. Using the second equation of (6.3) and $\widetilde{\nabla} = \nabla - S$, we have that

$$0 = \widetilde{\nabla} R = \nu_q \widetilde{\nabla} R_0 + \widetilde{\nabla} R^{\mathfrak{sp}^\epsilon(n)} = \nabla R^{\mathfrak{sp}^\epsilon(n)} - SR^{\mathfrak{sp}^\epsilon(n)}.$$

Writing $T^*M \otimes (\mathfrak{sp}^\epsilon(n) + \mathfrak{sp}^\epsilon(1)) = T^*M \otimes \mathfrak{sp}^\epsilon(n) + T^*M \otimes \mathfrak{sp}^\epsilon(1)$ we can decompose $S = S_E + S_H$, and hence $S_H R^{\mathfrak{sp}^\epsilon(n)} = 0$. We thus obtain

$$\nabla R = \nabla R^{\mathfrak{sp}^\epsilon(n)} = S_E R^{\mathfrak{sp}^\epsilon(n)},$$

which we can write as

$$(6.6) \qquad (\nabla_X R)_{YZWU} = -R^{\mathfrak{sp}^\epsilon(n)}_{S_XYZWU} - R^{\mathfrak{sp}^\epsilon(n)}_{YS_XZWU} - R^{\mathfrak{sp}^\epsilon(n)}_{YZS_XWU} - R^{\mathfrak{sp}^\epsilon(n)}_{YZWS_XU}.$$

Taking the cyclic sum in X, Y, Z and applying the Bianchi identities we obtain

$$0 = \underset{XYZ}{\mathfrak{S}} \Big\{ 2g(X,\xi) R^{\mathfrak{sp}^\epsilon(n)}_{YZWU} + g(X,W) R^{\mathfrak{sp}^\epsilon(n)}_{YZ\xi U} + g(X,U) R^{\mathfrak{sp}^\epsilon(n)}_{YZW\xi}$$
$$+ 2\sum_a \epsilon_a (g(X,J_aY) R^{\mathfrak{sp}^\epsilon(n)}_{J_a\xi ZWU} + g(X,J_aW) R^{\mathfrak{sp}^\epsilon(n)}_{YZJ_a\xi U} + g(X,J_aU) R^{\mathfrak{sp}^\epsilon(n)}_{YZWJ_a\xi}) \Big\}.$$

Contracting the previous formula with respect to X and W, and taking into account that $R^{\mathfrak{sp}^\epsilon(n)}$ is traceless, we obtain

$$(4n+2) R^{\mathfrak{sp}^\epsilon(n)}_{YZ\xi U} = 0\,,$$

for every vector fields Z, Y, U. Expanding the expression of S in (6.6) and using the previous formula, we arrive at

$$0 = \underset{XYZ}{\mathfrak{S}} \theta(X) R^{\mathfrak{sp}^\epsilon(n)}_{YZWU}\,,$$

where $\theta = \xi^\flat$, or equivalently

$$0 = \theta \wedge R^{\mathfrak{sp}^\epsilon(n)}_{WU}. \tag{6.7}$$

Noting that $R^{\mathfrak{sp}^\epsilon(n)}$ satisfies the symmetries $R^{\mathfrak{sp}^\epsilon(n)}_{XJ_aYWU} + R^{\mathfrak{sp}^\epsilon(n)}_{J_aXYWU} = 0$, $a = 1,2,3$, we will also have

$$0 = (\theta \circ J_a) \wedge R^{\mathfrak{sp}^\epsilon(n)}_{WU} = 0\,, \quad a = 1,2,3\,. \tag{6.8}$$

It is easy to prove that a curvature tensor of type $\mathfrak{sp}^\epsilon(n)$ satisfying equations (6.7) and (6.8) must vanish. Therefore we conclude that $R = \nu_q R_0$.

Now, using the third equation of (6.3) together with (6.5), and taking into account (6.4), we have

$$0 = g(X,Y)\widetilde{\nabla}_Z\xi - g(\widetilde{\nabla}_Z\xi, Y)X - \sum_a \epsilon_a (g(\widetilde{\nabla}_Z\xi, J_aY)J_aX + g(X,J_aY)J_a\widetilde{\nabla}_Z\xi)$$
$$+ \sum_a g(X, \widetilde{\nabla}_Z\zeta^a - \sum_b \widetilde{c}_{ba}(Z)\zeta^b) J_aY,$$

whence we deduce that $\widetilde{\nabla}\xi = 0$. On the other hand,

$$\begin{aligned} R_{XY}\xi =& -\nabla_X\nabla_Y\xi + \nabla_Y\nabla_X\xi + \nabla_{[X,Y]}\xi \\ =& -g(Y,\nabla_X\xi)\xi - g(Y,\xi)\nabla_X\xi + g(X,\nabla_Y\xi)\xi + g(X,\xi)\nabla_Y\xi \\ & - \sum_a \epsilon_a\, (g(Y,\nabla_XJ_a\xi)J_a\xi + g(Y,J_a\xi)\nabla_XJ_a\xi \\ & - g(X,\nabla_YJ_a\xi)J_a\xi - g(X,J_a\xi)\nabla_YJ_a\xi) \\ & + \sum_a -g(Y,\nabla_X\zeta^a)J_a\xi - g(Y,\zeta^a)\nabla_XJ_a\xi \\ & + g(X,\nabla_Y\zeta^a)J_a\xi + g(X,\zeta^a)\nabla_YJ_a\xi. \end{aligned}$$

Therefore, applying $\widetilde{\nabla}\xi = 0$ and (6.2) to this formula we see that $R_{XY}\xi \in \operatorname{span}\{\xi, J_1\xi, J_2\xi, J_3\xi\}$. Comparing this fact with $R_{XY}\xi = \nu_q(R_0)_{XY}\xi$ we obtain that $\nu_q = 0$, so that $R = 0$. □

7. Homogeneous structures of linear type

The aim of this section is to bring together all results for homogeneous structures of linear type in the purely pseudo-Riemannian, ϵ-Kähler, and ϵ-quaternion Kähler cases, in order to provide a general picture and a complete study of this kind of structures. It is worth noting how different the non-degenerate and the degenerate cases are, being the former closely related with the well known case of definite metrics and spaces of constant curvature, while the latter is rather related to the geometry of singular homogeneous plane waves.

7.1. The general picture

For the purely pseudo-Riemannian case, the following theorem was subsequently proved in [18], [10], [16]. We recall that in this setting a homogeneous pseudo-Riemannian structure is of linear type if it is of the form $S_X Y = g(X,Y)\xi - g(\xi,Y)X$, for some vector field ξ. S is degenerate if $g(\xi,\xi)=0$ and non-degenerate otherwise.

Theorem 7.1. *Let (M,g) be a pseudo-Riemannian manifold of dimension $m \geq 3$.*

(1) *If (M,g) admits a non-degenerate homogeneous structure of linear type, then it has constant sectional curvature $c=-g(\xi,\xi)$. Conversely, every non-flat simply-connected pseudo-Riemannian space form locally admits a non-degenerate homogenous structure of linear type, being this structure globally defined if and only if g is definite.*

(2) *If (M,g) admits a degenerate homogeneous structure of linear type, then (M,g) is locally isometric to a singular scale-invariant homogeneous plane wave. Conversely, every singular scale-invariant homogeneous plane wave admits a degenerate homogeneous structure of linear type.*

For the ϵ-Kähler and ϵ-quaternion Kähler cases, the non-degenerate case was obtained in [14] (see [5] and [9] for definite metrics), and the degenerate case was partially solved in [6] and completed in the present manuscript.

Theorem 7.2. *Let (M,g,J) be an ϵ-Kähler manifold of dimension $m \geq 4$.*

(1) *If (M,g,J) admits a non-degenerate homogeneous ϵ-Kähler structure of linear type, then it has constant ϵ-holomorphic sectional curvature $c=-4g(\xi,\xi)$ and $\zeta=0$. Conversely, every non-flat simply-connected ϵ-complex space form locally admits a non-degenerate homogenous ϵ-Kähler structure of linear type, being this structure globally defined if and only if g is definite.*

(2) *If (M,g,J) admits a degenerate homogeneous ϵ-Kähler structure of linear type, then $\zeta=\lambda\xi$, with $\lambda\in\{0,-\epsilon/2\}$, and (M,g) is locally ϵ-holomorphically isometric to $(\mathbb{C}^\epsilon)^{n+2}$ with the metric*

$$\begin{aligned}\bar{g} &= dw^1dz^1 - \epsilon dw^2dz^2 + b(dw^1dw^1 - \epsilon dw^2dw^2)\\ &\quad + \sum_{a=1}^{n}\varepsilon^a(dx^adx^a - \epsilon dy^ady^a),\end{aligned} \tag{7.1}$$

where $\varepsilon^a = \pm 1$, and the function b only depends on the coordinates $\{w^1, w^2\}$ and satisfies

$$\Delta^\epsilon b = \frac{R_0}{\|w\|_\lambda^4}$$

for $R_0 \in \mathbb{R} - \{0\}$ and $\|w\|_\lambda$ defined in (4.3). Conversely, $((\mathbb{C}^\epsilon)^{n+2}, \bar{g})$ admits a degenerate homogeneous ϵ-Kähler structure of linear type.

Theorem 7.3. *Let (M, g, Q) be an ϵ-quaternion Kähler manifold of dimension $m \geq 8$.*

(1) *If (M, g, Q) admits a non-degenerate homogeneous ϵ-quaternion Kähler structure of linear type, then it has constant ϵ-quaternionic sectional curvature $c = -4g(\xi, \xi)$ and $\zeta^a = 0$. Conversely, every non-flat simply-connected ϵ-quaternion space form locally admits a non-degenerate homogenous ϵ-quaternion Kähler structure of linear type, being this structure globally defined if and only if g is definite.*

(2) *If (M, g, Q) admits a degenerate homogeneous ϵ-quaternion Kähler structure of linear type, then (M, g, Q) is flat.*

7.2. Relation with homogeneous plane waves

We exhibit the parallelism between certain kind of (Lorentzian) homogeneous plane waves and Lorentz–Kähler spaces admitting degenerate homogeneous structures of linear type (by Lorentz–Kähler we mean pseudo-Kähler of index 2). Although, as far as the authors know, there is no formal definition of a plane wave in complex geometry, this relation could allow us to understand the latter spaces as a complex generalization of the former, at least in the important Lorentz–Kähler case, suggesting a starting point for a possible definition of *complex plane wave.*

A plane wave is a Lorentz manifold $M = \mathbb{R}^{n+2}$ with metric

$$g = dudv + A_{ab}(u)x^a x^b du^2 + \sum_{a=1}^{n} (dx^a)^2,$$

where $(A_{ab})(u)$ is a symmetric matrix called the profile of g. Moreover, a plane wave is called *homogeneous* if the Lie algebra of Killing vector fields acts transitively in the tangent space at every point. Among homogeneous plane waves we will be interested in two types. A *Cahen–Wallach space* is defined as a plane wave with profile a constant symmetric matrix (B_{ab}), which makes it symmetric and geodesically complete. On the other hand, a *singular scale-invariant homogeneous plane wave* is a plane wave with profile $(B_{ab})/u^2$, where (B_{ab}) is a constant symmetric matrix. Singular scale-invariant homogeneous plane waves are homogeneous but not symmetric. In addition they present a singularity in the cosmological sense at $\{u = 0\}$, since the geodesic deviation equation (or Jacobi equation) governed by the curvature is singular at this set (see [17]). Note that these two kind of spaces are composed by the twisted product of a totally geodesic flat wave front and a 2-dimensional manifold containing time and the direction of propagation.

This 2-dimensional space gives the real geometric information of the total manifold and in particular it contains a null parallel vector field. They are all VSI and the curvature information is contained in the profile $A_{ab}(u)$, since the only non-vanishing component of the curvature is given by $R_{uaub} = -A_{ab}(u)$. By Theorem 7.1, singular scale-invariant homogeneous plane waves are characterized by degenerate homogeneous structures of linear type. In addition, any indecomposable simply-connected Lorentzian symmetric space is isometric exactly to one of the following: $(\mathbb{R}, -dt^2)$, the de Sitter or the anti de Sitter space, or a Cahen–Wallach space [4].

In the Lorentz–Kähler case, according to [11], there is only one possibility for a simply-connected, indecomposable (and not irreducible), symmetric space of complex dimension 2 and signature $(2, 2)$ with a null parallel complex vector field, that is a manifold with holonomy

$$\mathfrak{hol}_{n=0}^{\gamma_1=0,\gamma_2=0} = \mathbb{R}\begin{pmatrix} i & i \\ -i & -i \end{pmatrix}$$

in the notation of [11]. Note that this is the holonomy algebra in Proposition 3.5. In order to get a plane wave structure we add a plane wave front by considering a manifold (M, g, J) with holonomy $\mathfrak{hol}_{n=0}^{\gamma_1=0,\gamma_2=0} \oplus \{0_n\}$. The result (see [6]) is, up to local holomorphic isometry, the space $\mathbb{C}^{n+2}$ with

$$\bar{g} = dw^1dz^1 + dw^2dz^2 + b(dw^1dw^1 + dw^2dw^2) + \sum_{a=1}^{n}(dx^adx^a + dy^ady^a)\,, \tag{7.2}$$

where the function b only depends on w^1 and w^2 and satisfies

$$\Delta b = R_0, \quad R_0 \in \mathbb{R} - \{0\}\,.$$

This suggests to consider the manifold $(\mathbb{C}^{2+n}, \bar{g})$ as a natural Lorentz–Kähler analogue to Cahen–Wallach spaces. As Cahen–Wallach spaces are simply-connected, in order to have an actual analogue we only consider non-singular functions b, so that $(\mathbb{C}^{2+n}, \bar{g})$ is complete.

On the other hand, since Lorentzian singular scale-invariant homogeneous plane waves are characterized by degenerate pseudo-Riemannian homogeneous structures of linear type, from Theorem 7.2 the natural analogues to these spaces are Lorentz–Kähler manifolds with degenerate homogeneous pseudo-Kähler structures of linear type. More precisely, the spaces $(\mathbb{C}^{n+2} - \{\|w\|_\lambda = 0\}, \bar{g})$ with $\|w\|_\lambda$ as in (4.3), $\bar{g}$ as in (7.1) with $\epsilon = -1$ and signature $(2, 2 + 2n)$, and $(\mathbb{C}^{n+2}, \bar{g})$ with $\bar{g}$ given in (7.2), are composed by the twisted product of a flat and totally geodesic complex manifold and a 2-dimensional complex manifold containing a null parallel complex vector field. Moreover, the expression of (7.1) and (7.2) are the same except for the function b, which has a different Laplacian in each case. As a straightforward computation shows, the curvature tensor of both metrics is

$$R = \frac{1}{2}\Delta b(dw^1 \wedge dw^2) \otimes (dw^1 \wedge dw^2)\,,$$

whence all the curvature information is contained in the Laplacian of the function b. For this reason, analogously to Lorentz plane waves, we call Δb the profile of the metric.

It is worth noting that in the Lorentz case one goes from Cahen–Wallach spaces to singular scale-invariant homogeneous plane waves by making the profile be singular with a term $1/u^2$. Doing so, the space is no longer geodesically complete and a cosmological singularity at $\{u = 0\}$ is created. In the same way, in the Lorentz–Kähler case one goes from metric (7.2) to (7.1) by making the profile be singular with a term $1/\|w\|_\lambda^4$ and again one transforms a geodesically complete space to a geodesically incomplete space, and a singularity at $\{\|w\|_\lambda = 0\}$ is created. This reinforces the parallelism and exhibits a close relation between this two couples of spaces.

	Symmetric space	Deg. homog. of linear type
Lorentz	Cahen–Wallach spaces Profile: $A(u) = B(const.)$ Geodesically complete	Singular s.-i. homog. plane wave Profile: $A(u) = B/u^2$ Geodesically incomplete
Lorentz–Kähler	$\mathbb{C}^{2+n}$ with metric (7.2) Profile: $\Delta b = R_0(const.)$ Geodesically complete	$\mathbb{C}^{2+n} - \{\|w\|_\lambda = 0\}$ with metric (7.1) Profile: $\Delta b = R_0/\|w\|_\lambda^4$ Geodesically incomplete

Finally, note that a pseudo-quaternion Kähler manifold admitting a degenerate homogeneous pseudo-quaternion Kähler structure of linear type must be flat, suggesting that the notion of homogeneous plane wave cannot be adapted to pseudo-quaternion Kähler geometry.

Acknowledgements. The authors are deeply indebted to Prof. Andrew Swann and Prof. P. M. Gadea for useful conversations about the topics of this paper.

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Received February 8, 2015.

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This work has been partially funded by MINECO (Spain) under project MTM2011-22528.