The double of the doubles of Klein surfaces

Antonio F. Costa, Paola Cristofori and Ana M. Porto

Abstract. A Klein surface is a surface with a dianalytic structure. A double of a Klein surface X is a Klein surface Y such that there is a degree two morphism (of Klein surfaces) $Y \to X$. There are many doubles of a given Klein surface and among them the so-called natural doubles which are: the complex double, the Schottky double and the orienting double. We prove that if X is a non-orientable Klein surface with non-empty boundary, the three natural doubles, although distinct Klein surfaces, share a common double: "the double of doubles" denoted by DX. We describe how to use the double of doubles in the study of both moduli spaces and automorphisms of Klein surfaces. Furthermore, we show that the morphism from DX to X is not given by the action of an isometry group on classical surfaces.

1. Introduction

A (compact) Klein surface is a surface with a dianalytic structure, i.e., a surface where the charts are defined on open sets of the upper-half complex plane $\mathcal{U}$ and the transition functions are analytic or anti-analytic (see [1], [6] or [16]). Topologically compact Klein surfaces may be non-orientable and with boundary. The folding map $\phi \colon \mathbb{C} \to \mathcal{U}$, is defined by $\phi(x+iy) = x+i|y|$, and a smooth morphism of Klein surfaces is a map which is either locally complex smooth or locally the folding map, the latter occurring over the boundary of the image (for a more precise definition see [1]).

A double of a Klein surface X is a Klein surface Y such that there is a degree two morphism $Y \to X$. Three types of doubles, the so-called natural doubles, turn out to be historically interesting: the complex double, the Schottky double and the orienting double; they are defined in [1] in terms of equivalence classes of dianalytic atlases. In [9] the doubles of Klein surfaces are studied by using subgroups of uniformizing Euclidean and non-Euclidean crystallographic groups.

If X is a non-orientable Klein surface with non-empty boundary, we prove that the three natural doubles, although distinct Klein surfaces, share a common

double: "the double of doubles". The main purpose of this paper is the study of this Riemann surface. We first establish the relations between each natural double with the double of doubles (section 5) and we apply this concept to the study of automorphisms of Klein surfaces (section 7) and to the theory of real algebraic curves (section 8). In section 6 it is shown that the morphism from the double of doubles to the given Klein surface cannot be visualized as the natural projection on the space of orbits produced by action of an isometry group on classical surfaces.

This article has been motivated by a question of Gareth Jones after a talk by David Singerman on [9] in the Conference in honour of Emilio Bujalance in Linköping (2013).

2. Klein surfaces and NEC groups

The algebraic genus of a Klein surface X of genus g with k boundary components is, by definition, $2g+k-1$ if X is orientable and $g+k-1$ if X is non-orientable. The algebraic genus is the topological genus of the complex double of X (see section 4).

Every Klein surface has uniformization $\mathcal{S}/\Gamma$ where $\mathcal{S}$ is a simply-connected Riemann surface and Γ is a crystallographic group without elliptic elements (it might have reflections though). If the algebraic genus of the surface is greater than 1, then $\mathcal{S} = \mathcal{U}$, the upper complex half-plane, and Γ is a (planar) non-Euclidean crystallographic (NEC) group. If the algebraic genus is equal to 1 (for example the Möbius band) then $\mathcal{S} = \mathbb{C}$ and Γ is a (planar) Euclidean crystallographic group. These groups are called surface Euclidean or non-Euclidean crystallographic groups and have assigned a signature of the form (see [6] and [19])

$$(2.1) \qquad\qquad (g; \pm; [-]; \{(-)^k\}).$$

Here, $(-)^k$ means k empty period cycles. If this occurs, $\mathcal{S}/\Gamma$ is a compact surface of genus g with k boundary components; it is orientable when the $+$ sign occurs and non-orientable otherwise. The group Γ has a fundamental region that is a Euclidean or hyperbolic polygon $\mathcal{P}$. If the $+$ sign occurs then the fundamental region for the group is a hyperbolic polygon with surface symbol

$$(2.2) \qquad \alpha_1\,\beta_1\,\alpha_1'\,\beta_1'\ldots,\,\alpha_g\,\beta_g\,\alpha_g'\,\beta_g'\,\epsilon_1\,\gamma_1\,\epsilon_1'\ldots,\,\epsilon_k\,\gamma_k\,\epsilon_k'.$$

If the $-$ sign occurs then the fundamental polygon has surface symbol

$$(2.3) \qquad \alpha_1\,\alpha_1^*\ldots\alpha_g\,\alpha_g^*\,\epsilon_1\,\gamma_1\,\epsilon_1'\ldots\epsilon_k\,\gamma_k\,\epsilon_k'$$

The group has two possible presentations; if the $+$ sign occurs the presentation is

$$\langle a_1, b_1, \ldots, a_g, b_g, e_1, \ldots, e_k, c_1, \ldots, c_k \mid$$
$$\Pi_{i=1}^g [a_1, b_i] e_1 \cdots e_k = 1, \ c_i^2 = 1, \ e_i\, c_i\, e_i^{-1} = c_i \quad (i = 1, \ldots, k)\rangle$$

Here a_i, b_i are translations or hyperbolic, c_i are reflections and e_i are orientation-preserving though usually hyperbolic. Moreover $a_i(\alpha_i') = \alpha_i, b_i(\beta_i') = \beta_i, e_i(\epsilon_i') = \epsilon_i$ and c_i fixes the edge γ_i.

If the $-$ sign occurs the presentation is

$$\langle d_1 \ldots, d_g, e_1, \ldots, e_k, c_1, \ldots, c_k \mid$$
$$d_1^2 \cdots d_g^2 \, e_1 \cdots e_k = 1, \ c_i^2 = 1, \ e_i \, c_i \, e_i^{-1} = c_i \quad (i = 1, \ldots, k)\rangle$$

Here d_i are glide-reflections and $d_i(\alpha_i^*) = \alpha_i$

This type of presentations of Euclidean or NEC groups will be called *canonical presentation* and a generator will be a *canonical generator* and in both presentations the first relation is called the *long* relation.

3. Standard epimorphisms of NEC groups and doubles of Klein surfaces

A double of a Klein surface $X = \mathcal{S}/\Gamma$ has the form $\mathcal{S}/\Lambda = Y$ where Λ is a surface subgroup of index 2 in Γ, then there is an epimorphism $\theta \colon \Gamma \to C_2 = \langle t \mid t^2 = 1\rangle$, with $\ker \theta = \Lambda$, called the monodromy epimorphism.

A Klein surface may have a large number of doubles (see Theorem 1 of [9] and [12]). For this reason we focus our study on the most important ones mentioned in [1] and [9].

Let us gather the canonical generators of Γ in sets and define: $E = \{e_1, \ldots, e_k\}$, $C = \{c_1, \ldots, c_k\}$, $A = \{a_1, b_1, \ldots, a_g, b_g\}$ or $A = \{d_1, \ldots, d_g\}$. We will consider only the doubles whose monodromies $\theta \colon \Gamma \to C_2$ are constant on each set of generators. An epimorphism with this property is called a standard epimorphism.

Theorem 1. *If k is even then there are* 7 *standard epimorphisms* $\theta \colon \Gamma \to C_2$, *while if k is odd there are only* 3 *standard epimorphisms.*

Proof in [9].

In Table 1 we have listed the standard epimorphisms and the corresponding topological type of the double $\mathcal{U}/\ker \theta$, see [9]. We distinguish between the cases where Γ has orientable or non-orientable quotient space. Here, k is the number of boundary components of $\mathcal{U}/\Gamma$, B is the number of boundary components of the double $\mathcal{U}/\ker \theta$ and the orientability of $\mathcal{U}/\ker \theta$ is denoted by $+$ or $-$.

The three first rows describe the monodromies of the so-called *natural doubles*, which are the most important from the historical point of view (see section 4 and [9]).

(Table 1)

	Standard epimorphism θ	Boundary B	Orientability of the double $\mathcal{U}/\ker \theta$	
			$\mathcal{U}/\Gamma$ non-orientable	$\mathcal{U}/\Gamma$ orientable
1.	$E \to \{1\} \ C \to \{t\} \ A \to \{t\}$	0	$+$	$-$
2.	$E \to \{1\} \ C \to \{1\} \ A \to \{t\}$	$2k$	$+$	$+$
3.	$E \to \{1\} \ C \to \{t\} \ A \to \{1\}$	0	$-$	$+$
4.	$E \to \{t\} \ C \to \{1\} \ A \to \{1\}$	k	$-$	$+$
5.	$E \to \{t\} \ C \to \{1\} \ A \to \{t\}$	k	$-$	$+$
6.	$E \to \{t\} \ C \to \{t\} \ A \to \{1\}$	0	$-$	$-$
7.	$E \to \{t\} \ C \to \{t\} \ A \to \{t\}$	0	$-$	$-$

4. The natural doubles

Let $X = \mathcal{U}/\Gamma$, where Γ is a crystallographic surface group.

(1) The complex double.

If X is a Klein surface then its complex double X^+ is the unique double which is a Riemann surface without boundary. The complex double of $X = \mathcal{U}/\Gamma$ is $\mathcal{U}/\Gamma^+$, where Γ^+ is the subgroup of Γ consisting of those transformations preserving orientation. If X is non-orientable then the generators of A are glide reflections and so the complex double is given by epimorphism 1; it corresponds to epimorphism 3 when X is orientable. The genus of the complex double X^+ is the algebraic genus of the Klein surface X. If X is orientable and without boundary X^+ has two connected components.

(2) The orienting double.

Let X be a Klein surface and suppose that ∂X has k components. For each $i = 1, \ldots, k$, fill in each boundary component with a disc D_i. We get a surface $\widetilde{X}$ without boundary with the same orientability as X. Now consider the complex double $\widetilde{X}^+$ of $\widetilde{X}$. Let D_i^1 and D_i^2 be the lifts of D_i to $\widetilde{X}^+$. If we remove these discs from $\widetilde{X}^+$ we end up with an orientable surface OX which has $2k$ boundary components and clearly OX is an unbranched two-sheeted covering of X. We call OX the orienting double of X. Note that if X is orientable then OX has two connected components.

If we consider the epimorphisms of Table 1 we see that we only have a covering with twice as many boundary components as the original surface for epimorphism 2; so this epimorphism corresponds to the orienting double of a non-orientable Klein surface. In the case of orientable Klein surfaces the orienting double consists of two copies of the original surface. If the surface X is non-orientable with empty boundary, the orienting double coincides with the complex double.

(3) The Schottky double.

Let Y be a double of the Klein surface X. Then Y admits an involution h such that $Y/\langle h \rangle = X$. As we are considering unbranched but possibly folded coverings, the fixed-point set of h will include a collection of simple closed curves (see for instance [4]). We define the Schottky double of X to be a Klein surface SX without boundary with the same orientability as X admitting a dianalytic involution h whose fixed curves separate SX and such that $SX/\langle h \rangle = X$.

Theorem 2 ([9]). *Let $X = \mathcal{U}/\Gamma$ be a Klein surface with boundary and $SX = \mathcal{U}/\Lambda$ its Schottky double, where Γ and Λ are crystallographic surface groups. Let $\theta \colon \Gamma \to \Gamma/\Lambda \cong C_2$ be the natural epimorphism. Then θ is the epimorphism 3 of* Table 1.

Note that if $X = \mathcal{U}/\Gamma$ is orientable, the Schottky double coincides with the complex double. If X is non-orientable without boundary, the Schottky double has two connected components both isomorphic to X.

5. The double of the natural doubles

Let X be a non-orientable Klein surface with non-empty boundary. As shown in the above section, X has three different natural doubles: X^+, OX and SX. The surfaces OX and SX are, in general, proper Klein surfaces (i.e., non-orientable or bordered Klein surfaces). Note that $(OX)^+ = S(OX)$ since OX is orientable and $(SX)^+ = O(SX)$ because SX has no boundary. The following results establish that $(OX)^+ = S(OX) = (SX)^+ = O(SX)$ as well; this is the Riemann surface that we shall call the double of (the natural) doubles, and denote by DX.

In next theorem the group of conformal and anticonformal automorphisms of the Riemann surface DX will be denoted by $\mathrm{Aut}(DX)$, i.e., we consider the Riemann surface DX as a closed orientable Klein surface.

Theorem 3. *Let $X = \mathcal{U}/\Gamma$ be a non-orientable Klein surface with non-empty boundary. Let SX be the Schottky double, OX be the orienting double and X^+ be the complex double of X. There exists a Riemann surface DX such that $\mathrm{Aut}(DX)$ contains a group $\langle s, t \rangle$ isomorphic to $C_2 \times C_2$ and such that: $DX/\langle s \rangle = OX$ is the orienting double, $DX/\langle t \rangle = SX$ is the Schottky double of X, and $DX/\langle st \rangle = X^+$ is the complex double.*

Proof. We define $\omega : \Gamma \to C_2 \times C_2 = \langle s, t \rangle$ by

$$A \to \{t\}; E \to \{1\}; C \to \{s\}.$$

Let DX be $\mathcal{U}/\ker \omega$. Then we have the following diagram:

$$DX = \mathcal{U}/\ker \omega$$

$$X^+ = \mathcal{U}/\omega^{-1}(\langle st \rangle) \quad \swarrow \quad \begin{matrix} \downarrow \\ OX = \mathcal{U}/\omega^{-1}(\langle s \rangle) \\ \downarrow \\ X \end{matrix} \quad \searrow \quad SX = \mathcal{U}/\omega^{-1}(\langle t \rangle),$$

that proves the theorem. □

If X is non-orientable, has genus g and k boundary components then (see Table 1 and [9]) the complex double is an (orientable) Riemann surface (without boundary) of genus $g + k - 1$, the orienting double is an orientable Klein surface of genus $g - 1$ with $2k$ boundary components, the Schottky double is a non-orientable Klein surface without boundary of genus $2g + 2k - 2$ and, finally, the double of doubles of X is an (orientable) Riemann surface (without boundary) of genus $2g + 2k - 3$.

Note that st is an orientation preserving element while s and t are orientation reversing.

Corollary 4. *Given a non-orientable Klein surface with non-empty boundary, the complex double $(SX)^+ = DX$ of the Schottky double $SX = DX/\langle t \rangle$ of X coincides with the complex double $(OX)^+ = DX$ of the orienting double $OX = DX/\langle s \rangle$ of X. The anticonformal involution t is fixed point free and the fixed point set of s is separating. The conformal involution st is fixed point free.*

Proof. Remember that if X is a Klein surface then its complex double X^+ is the unique double which is a Riemann surface without boundary. The double of doubles DX is a Riemann surface and there are degree two morphisms from DX to OX or SX thus $(OX)^+ = (SX)^+ = DX$. Since SX has no boundary then t is fixed point free and since OX is orientable $DX - \mathrm{Fix}(s)$ has two connected components. Finally, as $DX \to X^+$ is an order two morphism between Klein surfaces which are in fact Riemann surfaces, it is an unbranched two fold covering as well and st is fixed point free. $\square$

The description of the unbranched covering $DX \to X^+$ is the following:

Theorem 5. *Let σ be the anticonformal involution given by $X^+ \to X$ and $\mathrm{Fix}(\sigma)$ be the fixed point set of σ. Let $\langle .,. \rangle$ be the intersection form in $H_1(X^+, \mathbb{Z}_2)$ and $[\mathrm{Fix}(\sigma)]$ be the cycle in $H_1(X^+, \mathbb{Z}_2)$ represented by the union of the curves in $\mathrm{Fix}(\sigma)$. The covering $DX \to DX/\langle st \rangle = X^+$ is an unbranched covering with monodromy*

$$\langle [\mathrm{Fix}(\sigma)], . \rangle : \pi_1(X^+) \to H_1(X^+, \mathbb{Z}_2) \to C_2$$
$$\gamma \mapsto [\gamma] \mapsto \langle [\mathrm{Fix}(\sigma)], [\gamma] \rangle$$

Proof. Let us restrict our proof to the case where X is a Klein surface with algebraic genus > 1, then Γ is an NEC group. The group uniformizing X^+ is Γ^+ and the monodromy of the covering $DX \to X^+$ is just the restriction of ω to $\Gamma^+ = \omega^{-1}(\langle st \rangle)$, which is an epimorphism on $\langle st \rangle = C_2$. Let $g \in \Gamma^+$ such that $\omega(g) = st$ and let w be an expression of g as an irreducible word in some canonical set of generators of Γ. Then an odd number of reflections appears in the word w. If γ is the curve that is the projection on X^+ of the axis of the hyperbolic element g, then γ cuts $\mathrm{Fix}(\sigma)$ in an odd number of points. Hence $\langle [\mathrm{Fix}(\sigma)], [\gamma] \rangle \neq 0$. In similar way if $\omega(g) = 1$ then $\langle [\mathrm{Fix}(\sigma)], [\gamma] \rangle = 0$. Thus $\langle [\mathrm{Fix}(\sigma)], . \rangle$ is given by the restriction of ω to Γ^+, so it is the monodromy of $DX \to X^+$. $\square$

Note that $DX \to X^+$ is something living completely in the theory of Riemann surfaces and that is naturally given by the non-orientable bordered Klein surface X.

6. The automorphism group of $DX \to X$ cannot be visualized in $\mathbb{R}^3$

Every smooth surface in the Euclidean space can be made into a Riemann surface in a natural way by restriction of the Euclidean metric to it. These surfaces are called classical Riemann surfaces and they are considered by Beltrami and Klein (see the introduction of [11] and chapter II, section 5 of [2]). There are some automorphisms of Riemann surfaces that can be represented by the restriction to classical Riemann surfaces of isometries of the Euclidean space. This is a natural way of visualizing automorphisms of Riemann surfaces.

Note that each one of the three automorphisms s, t, st of the preceding section are representable as restriction of isometries to classical Riemann surfaces (see [8]),

but the complete group action of $C_2 \times C_2$ is not the restriction of a finite group of isometries, so it "cannot be visualized".

Example 6.1. If M is a Möbius band we know that DM is an analytical torus conformally equivalent to a classical torus T_1 embedded in $\mathbb{R}^3$, such that T_1 is invariant by an order two rotation r with axis non-cutting T_1 and that the unbranched covering $DM \to M^+$ is analytically equivalent to $T_1 \to T_1/\langle r \rangle$. Analogously there are embedded tori T_2 and T_3 such that T_2 is invariant by a plane reflection p with $T_2 \to T_2/\langle p \rangle$ equivalent to $DM \to OM$ and T_3 is invariant by a central symmetry c such that $T_3 \to T_3/\langle c \rangle$ is equivalent to $DM \to SM$. But there is no embedded torus T and no group of isometries G, isomorphic to $C_2 \times C_2$, such that $T \to T/G$ is equivalent to $DM \to M$. The obstruction is of topological nature. Assume that we have such classical torus T and a group of isometries G such that $T \to T/G$ is equivalent to $DM \to M$: the group G must be generated by a plane reflection and a central symmetry. Furthermore the order two rotation r of G must not cut the torus T because $T \to T/\langle r \rangle$ is equivalent to $DM \to M^+$. The plane of symmetry must be orthogonal to the axis of r, then T/G is homeomorphic to a cylinder and not a Möbius band. Thus $T \to T/G$ is not equivalent to $DM \to M$.

7. The double of doubles and automorphisms

As in Section 5, we shall denote the groups of conformal and anticonformal automorphisms of X^+ and DX respectively by $\text{Aut}(X^+)$ and $\text{Aut}(DX)$, i.e., we consider X^+ and DX as closed orientable Klein surfaces.

Doubles of Klein surfaces are useful for the study of the automorphism groups of Klein surfaces. Every automorphism of a given Klein surface X lifts to an automorphism of the complex double X^+, and in this way it is possible to study the automorphisms of Klein surfaces by using automorphisms of Riemann surfaces. The difficulty arises when some of the automorphisms in X^+ are not liftings of automorphisms of X and then $\text{Aut}(X^+)$ is not isomorphic to $C_2 \times \text{Aut}(X)$. This difficulty remains, even in case of maximal symmetry, when considering the double of doubles for non-orientable Klein surfaces, as shown in the last example of this section; nevertheless, the information the automorphisms of DX may provide is better than the one given by $\text{Aut}(X^+)$. This claim is supported by the fact that although not every automorphism of X^+ lifts to DX (see first example of this section), this is true for the automorphisms of X (next theorem).

Theorem 6. *Let X be a bordered non-orientable Klein surface and let DX be the double of the doubles of X. Then every automorphism of X lifts to an automorphism of $\text{Aut}(DX)$ and $\text{Aut}(DX)$ contains a group isomorphic to $\text{Aut}(X) \times C_2 \times C_2$.*

Proof. We shall prove the result for the case of surfaces X of genus > 1. Let Γ be a surface NEC group such that $X = \mathcal{U}/\Gamma$ and Δ be such that $\Gamma \lhd \Delta$ and Δ/Γ is isomorphic to $\text{Aut}(X)$. Let $\theta \colon \Delta \to \Delta/\Gamma \simeq \text{Aut}(X)$ be the natural map and $\langle S : R \rangle$ be a canonical presentation of Δ (see, for instance, [6] page 14).

Let us define $\theta' : \Delta \to \operatorname{Aut}(X) \times C_2 \times C_2$, by

$$\theta'(s) = (\theta(s), \theta_2'(s), \theta_3'(s)),$$

where $s \in S$ is a generator of the canonical presentation of Δ, $\theta_2'(s) \neq 1$ if and only if s is orientation reversing and $\theta_3'(s) \neq 1$ if and only if s is a reflection in Γ.

Let us see that $\theta_3' : \Delta \to C_2$ is a homomorphism: the relations in R which contain reflections have either the form $e_i^{-1} c_{i0} e_i = c_{is_i}$, $c_{ij}^2 = 1$ or $(c_{i,j-1} c_{i,j})^{n_{ij}} = 1$. Since C_2 is abelian and $\Gamma \lhd \Delta$, the relations of the two first types are automatically respected by θ_3'. In the third type, if n_{ij} is even, the relations are respected by θ_3' because $\theta_3'(\Delta) = C_2$; if n_{ij} is odd the relation $(c_{i,j-1} c_{i,j})^{n_{ij}} = 1$ tells us that $c_{i,j-1}$ and $c_{i,j}$ are conjugate and thus either both $c_{i,j-1}$ and $c_{i,j}$ belong to Γ or none of them is in Γ; in any case, $\theta_3'(c_{i,j-1} c_{i,j}) = 1$ and the relation is also respected.

Note that $\ker \theta = \Gamma$ uniformizes X, $\ker(\theta, \theta_2')$ uniformizes X^+ and $\ker \theta' = \ker(\theta, \theta_2', \theta_3')$ uniformizes a two fold covering of X^+. The monodromy $\omega : \ker(\theta, \theta_2') = \pi_1(X^+) \to C_2$ of $\mathcal{U}/\ker \theta' \to X^+$ is given by the following rule: if $\gamma \in \ker(\theta, \theta_2')$, $\omega(\gamma) \neq 1$ if and only if γ can be expressed as a word w_S in the system of generators S of the canonical presentation of Δ, such that there is an odd number of reflections conjugate to reflections of Γ. And this is exactly the monodromy of $DX \to X^+$ by Theorem 5.

Since $\ker \theta'$ uniformizes DX, every automorphism of X admits a lifting to DX and $\operatorname{Aut}(DX)$ contains a group isomorphic to $\operatorname{Aut}(X) \times C_2 \times C_2$. $\square$

As a consequence, an automorphism of X^+ not lifting to an automorphism of DX, cannot be itself a lift of an automorphism of X, meaning that the automorphisms of DX provide better information on $\operatorname{Aut}(X)$ than $\operatorname{Aut}(X^+)$ do. Next example illustrates this situation:

Example 7.1. Let Δ be a maximal NEC group with signature

$$(1; +; [3]; \{(3)\})$$

and

$$\langle a, b, x, c_0, c_1, e : xeaba^{-1}b^{-1} = 1, x^3 = 1, c_0^2 = c_1^2 = 1, (c_0 c_1)^3 = 1, e c_0 e^{-1} = c_1 \rangle$$

be a canonical presentation of Δ. Let us consider the epimorphism:

$$\theta : \Delta \to D_3 = \langle s, t : s^2 = t^2 = (st)^3 = 1 \rangle$$

defined by:

$$\theta(a) = \theta(b) = 1; \ \ \theta(x) = st; \ \ \theta(e) = ts;$$
$$\theta(c_0) = s; \ \ \theta(c_1) = tst.$$

The NEC group $\theta^{-1}(\langle s \rangle)$ is a non-orientable surface crystallographic group with signature $(7; -; [-]; \{(-)\})$; so $X = \mathcal{U}/\theta^{-1}(s)$ is a Klein surface. The complex double X^+ is uniformized by $\ker \theta$ and its automorphism group is $\Delta / \ker \theta = D_3$ (note that we have assumed Δ maximal).

The group $\theta^{-1}(s)$ is not normal in Δ, so the automorphism group of X is trivial, but $\mathrm{Aut}(X^+) = D_3$, thus $\mathrm{Aut}(X^+) \gneqq \mathrm{Aut}(X) \times C_2$. The anticonformal involution s of X^+ producing X as quotient has a connected closed curve γ as fixed point set. We will call st an order three conformal automorphism of X^+. The automorphism st does not lift to DX. A reason for that is as follows: the curve ∂X lifts to a closed curve with two connected components in DX and to γ in X^+, but $st(\gamma)$ cuts γ just in the only fixed point of st which projects on the boundary of X. Therefore, $st(\gamma)$ lifts to a connected curve of DX and this fact prevents the existence of a lift of st to DX.

Finally, in the next example we show that, in some cases, the information on $\mathrm{Aut}(X)$ provided by $\mathrm{Aut}(DX)$ is not essentially better than the one obtained by $\mathrm{Aut}(X^+)$. In fact, in [5] and [15], it is established that if a bordered Klein surface X has maximal symmetry or "almost maximal symmetry" (more concretely if $\#\mathrm{Aut}(X) \geq 8(p-1)$, where p is the algebraic genus of X) then there is a finite number of Klein surfaces where $\mathrm{Aut}(X^+)$ contains properly $C_2 \times \mathrm{Aut}(X)$. When X is non-orientable with boundary the first occurrence of such situation is described in the following example.

Example 7.2. We shall describe a Klein surface $P2$ which is topologically a projective plane with two holes and such that $\mathrm{Aut}(P2^+) \neq C_2 \times \mathrm{Aut}(P2)$ and $\mathrm{Aut}(DP2) \neq C_2 \times C_2 \times \mathrm{Aut}(P2)$. The surface $P2$ can be uniformized by a NEC group Γ whose fundamental region is a regular right angled hyperbolic octagon O and the elements of Γ produce a pairwise identification of the sides of O given by the following symbol:

$$\alpha_1\,\gamma_1\,\alpha_2\,\gamma_2\,\alpha_1^*\,\gamma_1'\,\alpha_2^*\,\gamma_2',$$

where α_i is identified with α_i^* by a hyperbolic glide reflection d_i, $i = 1, 2$, and $\gamma_1 \cup \gamma_1'$, $\gamma_2 \cup \gamma_2'$ give rise to the two components of $\partial P2$, i.e., for each $i = 1, 2$, γ_i, γ_i', are in the fixed point set of reflections of Γ. The automorphisms group of $P2$ is D_4 then $\mathrm{Aut}(P2)$ has of order 8.

Now $P2^+$ is uniformized by the surface Fuchsian group Γ^+ and the regular octagon in $P2$ lifts to a regular map $\{8, 4\}$ in $P2^+$. Note that there is only a regular map of type $\{8, 4\}$ in surfaces of genus 2: ($R2.3'$ following the notation in [7]). Then $P2^+$ is the underlying Riemann surface in the regular map $R2.3'$. Since such a map can be obtained as a stellation of the regular map $R2.1$ of type $\{3, 8\}$, the Riemann surface $P2^+$ is also the surface underlying such map. The group of automorphisms of $P2^+$ is the group of automorphisms of $R2.1$, hence the group of automorphisms of $P2^+$ is a C_2-extension of $GL(2,3)$ and has 96 elements (see Theorem B of [5]), so $\mathrm{Aut}(P2^+) \neq C_2 \times \mathrm{Aut}(P2)$.

The double of doubles $DP2$ is a two fold covering of $P2^+$ and the map $R2.1$ lifts to a regular map of type $\{3, 8\}$. Since there is only one regular map on genus three surfaces of type $\{3, 8\}$: $R3.2$ (see [7]), $DP2$ is the genus three Riemann surface underlying $R3.2$ (the dual of the Dyck map). The group $\mathrm{Aut}(DP2)$ is the full symmetry group of the map $R3.2$ and $\#\mathrm{Aut}(DP2) = 192$ (the order of the group of conformal automorphisms of $DP2$ is 96, see for instance [13]). Hence $\mathrm{Aut}(DP2) \neq C_2 \times C_2 \times \mathrm{Aut}(P2)$.

8. An application to the study of the moduli space of non-separating real algebraic curves

The complexification $C_{\mathbb{C}}$ of a (smooth projective) *real* algebraic curve C is a *complex* algebraic curve, thus a compact Riemann surface. The conjugation provides an anticonformal involution σ on $C_{\mathbb{C}}$ and the pair $(C_{\mathbb{C}}, \sigma)$ determines completely the real curve C. The pair $(C_{\mathbb{C}}, \sigma)$ is given by the Klein surface $K_C = C_{\mathbb{C}}/\langle \sigma \rangle$, so K_C represents the real algebraic curve, too. The topological type t of K_C is $(h; \pm; k)$, where h is the topological genus, the sign $\pm$ is given by the orientability and k is the number of connected components of ∂K_C. The complexification $C_{\mathbb{C}}$ is, in fact, the complex double of K_C and the genus of $C_{\mathbb{C}}$ is the algebraic genus of K_C.

Assume that K_C is non-orientable, i.e., the topological type is $(h; -; k)$; the fixed point set $\mathrm{Fix}(\sigma)$ of the involution σ does not separate the Riemann surface $C_{\mathbb{C}}$, which is why C is called a non-separating real algebraic curve.

The spaces of deformations or moduli spaces are important tools in the study of algebraic curves. There is a different moduli space for each topological type of Klein surfaces, i.e., once the genus of the complexification of the real algebraic curve is fixed, the space of deformations for real non-separating algebraic curves with algebraic genus g is the disjoint union

$$\mathcal{M}_g^{\mathbb{R},-} = \bigcup_{h+k-1=g} \mathcal{M}_{(h;-;k)}^K \, ,$$

where $\mathcal{M}_{(h;-;k)}^K$ is the moduli space of Klein surfaces with topological type $(h; -; k)$ (see for instance [18], [6], [16]).

In some situations it is important to have a common space to relate the different topological types of real curves with the same complexification. The set $\mathcal{M}_g^{\mathbb{R},-,\mathbb{C}}$ corresponds to the set of points in $\mathcal{M}_g$ that are Riemann surfaces having a non-separating anticonformal involution. Then

$$\mathcal{M}_g^{\mathbb{R},-,\mathbb{C}} = \bigcup_{0 \le k \le g} \mathcal{M}_g^{(-,k)} \, ,$$

where $\mathcal{M}_g^{(-,k)}$ is the set of points in $\mathcal{M}_g$ corresponding to Riemann surfaces with an anticonformal involution of topological type $t = (-, k)$. This space has been studied by many authors: [17], [16], [3], [10]. Now a real non-separating algebraic curve is a pair (X, σ) where $X \in \mathcal{M}_g^{\mathbb{R},-,\mathbb{C}}$ and σ is an anticonformal involution of the Riemann surface X, and $X/\langle \sigma \rangle$ is non-orientable. The map $\phi \colon \mathcal{M}_g^{\mathbb{R},-} \to \mathcal{M}_g^{\mathbb{R},-,\mathbb{C}}$ given by $\phi(K) = K^+$ is continuous and ϕ restricted to $\mathcal{M}_{(h;-;k)}^K$ is an (orbifold) embedding, for $h + k - 1 = g$ (see Corollary 8.9 of [14] and [4]).

Now by using the results of the above sections we obtain that real non-separating algebraic curves of algebraic genus g are in the intersection of just two connected real analytic spaces in $\mathcal{M}_{2g-1}$.

Theorem 7. *Let $\mathcal{M}^{(+,0)}$ (respectively $\mathcal{M}^{(-,0)}$) be the set in $\mathcal{M}_{2g-1}$ consisting of the surfaces admitting a fixed point free conformal (resp. anticonformal) involution.*

We define

$$\mathcal{N}_g^{\mathbb{R},-} = \mathcal{M}^{(+,0)} \cap \mathcal{M}^{(-,0)}.$$

Then there exists a continuous map $\psi : \mathcal{M}_g^{\mathbb{R},-} \to \mathcal{N}_g^{\mathbb{R},-}$, *and* $\psi \circ \phi^{-1}$ *restricted to* $\mathcal{M}_g^{(-,k)}$ *is an embedding, for each* $k = 0, \dots, g$.

Proof. A point in $\mathcal{M}_g^{\mathbb{R},-}$ may be represented by a non-orientable Klein surface K. Let $\psi(K)$ be the point in $\mathcal{M}_{2g-1}$ given by DK. In the case $g > 1$, K is uniformized by a surface NEC group Γ and then the surface $\psi(K)$ is uniformized by the subgroup $\ker \omega$, where $\omega \colon \Gamma \to C_2 \times C_2$ is the epimorphism defined in the proof of Theorem 3. If $T_{(g;-;k)}$ and T_{2g-1} are respectively the Teichmüller spaces of Klein surfaces with topological type $(g; -; k)$ and of Riemann surfaces of genus $2g - 1$, then the inclusion $\ker \omega \to \Gamma$ produces an isometric embedding from $\varphi \colon T_{(g;-;k)} \to T_{2g-1}$ (Corollary 8.9 of [14]). The map φ produces the continuous map $\psi \colon \mathcal{M}_g^{\mathbb{R},-} \to \mathcal{M}_{2g-1}$ sending each Klein surface to its double of doubles DK. Since the double of doubles admits the action of a fixed point free orientation preserving involution (producing as orbit space K^+) and a fixed point free anticonformal involution (producing SK) we have that $\psi(\mathcal{M}_g^{\mathbb{R},-}) \subset \mathcal{N}_g^{\mathbb{R},-}$. $\square$

Next result follows from the above together with Theorems 3.1 and 3.3 of [10]

Corollary 8. $\psi(\mathcal{M}_g^{\mathbb{R},-})$ *is connected.*

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Antonio F. Costa: Departamento de Matemáticas Fundamentales, Facultad de Ciencias, UNED, Madrid, 28040, Spain.
E-mail: acosta@mat.uned.es

Paola Cristofori: Dipartimento di Scienze Fisiche, Informatiche e Matematiche, Università degli studi di Modena e Reggio Emilia, Via Campi 213/A, 41125 Modena, Italy.
E-mail: paola.cristofori@unimore.it

Ana M. Porto: Departamento de Matemáticas Fundamentales, Facultad de Ciencias, UNED, Madrid, 28040, Spain.
E-mail: asilva@mat.uned.es