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Multiplicity theorems for nonlinear nonhomogeneous Robin problems

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Abstract. We study a nonlinear Robin boundary value driven by a nonhomogeneous differential operator with a Carathéodory reaction and we look for multiple nontrivial solutions with sign information. We prove four such multiplicity theorems producing three nontrivial solutions, for resonant problems and for problems in which no global growth restriction is assumed on the reaction. Also, in the semilinear case, we show that we can have four nontrivial solutions, by producing a second nodal solution.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear nonhomogeneous Robin problem:

$$\text{(1.1)} \qquad \begin{cases} -\operatorname{div} a(Du(z)) = f(z,u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_a} + \beta(z)|u(z)|^{p-2}u(z) = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Here $a\colon \mathbb{R}^N \to \mathbb{R}^N$ is a strictly monotone, continuous map, which satisfies certain regularity and growth hypotheses. The precise conditions on $a(\cdot)$ are listed in hypotheses $H(a)$ in Section 2. These conditions are general enough to incorporate in our framework many differential operators of interest such as the p-Laplacian. In the boundary condition, $\partial u/\partial n_a$ denotes the generalized normal derivative defined by

$$\frac{\partial u}{\partial n_a} = (a(Du), n)_{\mathbb{R}^N} \quad \text{for all } u \in W^{1,p}(\Omega)$$

(see Lieberman [14]). Here $n(\cdot)$ denotes the outward unit normal at $\partial\Omega$. We should point out that this type of normal derivative is dictated by the nonlinear Green's identity (see Gasinski and Papageorgiou [8], p. 210). The reaction $f(z,x)$ is a

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Carathéodory function (that is, for all $x \in \mathbb{R}$, the mapping $z \mapsto f(z,x)$ is measurable and for almost all $z \in \Omega$, $x \mapsto f(z,x)$ is continuous).

Our aim is to prove multiplicity theorems for problem (1.1) providing precise sign information for all the solutions, under different growth conditions on the reaction $f(z,x)$. We prove four such multiplicity theorems producing three nontrivial solutions. In the first multiplicity theorem, we assume that $f(z,\cdot)$ is $(p-1)$-sublinear near $\pm\infty$ and in the particular case of the p-Laplacian, resonance is allowed with respect to the principal eigenvalue of the negative Robin p-Laplacian (see Papageorgiou and Rădulescu [21]). In the second and third multiplicity theorems, no global growth restriction is imposed on $f(z,\cdot)$. Instead it is assumed that $f(z,\cdot)$ has z-dependent zeros of constant sign and so the reaction $f(z,\cdot)$ exhibits a kind of oscillatory behavior near zero. In all three multiplicity theorems, the geometry near the origin is similar and implies the presence of a "concave" term (that is, a term which is $(p-1)$-superlinear as $x \to 0$). In the particular case of equations driven by the p-Laplacian, we can change this condition near zero and deal also with reactions that are $(p-1)$-sublinear as $x \to 0$. This is our fourth multiplicity theorem. Moreover, in the particular case of semilinear equations (driven by the Laplace operator), we show that we can produce a second nodal solution for a total of four nontrivial solutions.

This paper continues the recent works of Papageorgiou and Rădulescu ([21], [22], [24]), where certain parametric equations driven by the p-Laplacian were studied and multiplicity results were proved for certain values of the parameter.

We refer to the books by Ambrosetti and Arcoya [2] and Ambrosetti and Malchiodi [3] for the basic abstract results used in this paper.

2. Mathematical preliminaries

In this section we review the main mathematical tools which will be used in this work. Also, we introduce the hypotheses on the map $y \mapsto a(y)$ and determine their consequences.

Let X be a Banach space and let X^* be its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X)$, we say that φ satisfies the Palais–Smale condition (PS-condition), if the following holds:

"Every sequence $\{u_n\}_{n \geqslant 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n \geqslant 1} \subseteq \mathbb{R}$ is bounded and

$$\varphi'(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty,$$

admits a strongly convergent subsequence".

This is a compactness type condition on the functional φ. We need such a condition since the ambient space X need not be locally compact (being in general infinite dimensional). Using the PS-condition, we can prove a deformation theorem which is the key to the minimax theory for the critical values of φ. A main result in that theory, is the so-called "mountain pass theorem" of Ambrosetti and Rabinowitz [4].

Theorem 2.1. *Assume that $\varphi \in C^1(X)$ satisfies the PS-condition, $u_0, u_1 \in X$, $||u_1 - u_0|| > \rho > 0$,*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf[\varphi(u) : ||u - u_0|| = \rho] = \eta_\rho$$

and

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leqslant t \leqslant 1} \varphi(\gamma(t)) \quad \text{with} \quad \Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}.$$

Then $c \geqslant \eta_\rho$ and c is a critical value of φ.

The analysis of problem (1.1) will involve the Sobolev space $W^{1,p}(\Omega)$, for $1 < p < \infty$, the Banach space $C^1(\overline{\Omega})$ and the "boundary" spaces $L^r(\partial\Omega)$ $(1 \leqslant r \leqslant \infty)$.

In what follows, by $|\cdot|$ we denote the norm of $\mathbb{R}^N$, by $(\cdot,\cdot)_{\mathbb{R}^N}$ we denote the inner product of $\mathbb{R}^N$ and by $||\cdot||$ we denote the norm of the Sobolev space $W^{1,p}(\Omega)$ defined by

$$||u|| = [\,||u||_p^p + ||Du||_p^p\,]^{1/p} \quad \text{for all } u \in W^{1,p}(\Omega).$$

The space $C^1(\overline{\Omega})$ is an ordered Banach space, with order cone

$$C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geqslant 0 \quad \text{for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \quad \text{for all } z \in \overline{\Omega}\}.$$

On $\partial\Omega$ we use the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure, we can define the Lebesgue spaces $L^r(\partial\Omega)$ $(1 \leqslant r \leqslant \infty)$. We denote the norm of these spaces by $||\cdot||_{r,\partial\Omega}$. We know that there exists a unique linear continuous map $\gamma_0 \colon W^{1,p}(\Omega) \to L^p(\partial\Omega)$, known as the "trace map", such that $\gamma_0(u) = u|_{\partial\Omega}$ for all $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$. The trace map is compact into $L^r(\partial\Omega)$ for all $r \in \big[1, \frac{p(N-1)}{N-p}\big)$ if $1 < p < N$, and into $L^r(\partial\Omega)$ for all $r \in [1, \infty)$ if $p \geqslant N$. We know that

$$\operatorname{im} \gamma_0 = W^{1/p',p}(\partial\Omega) \quad \text{and} \quad \ker \gamma_0 = W_0^{1,p}(\Omega)$$

with $1/p + 1/p' = 1$.

In the sequel, for notational simplicity, we drop the use of the trace map γ_0. The restrictions of all Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

Let $\eta \in C^1(0,\infty)$ with $\eta(t) > 0$ for all $t > 0$ and assume that

$$0 < \hat{c} \leqslant \frac{t\eta'(t)}{\eta(t)} \leqslant c_0 \quad \text{and} \quad c_1\, t^{p-1} \leqslant \eta(t) \leqslant c_2\,(1 + t^{p-1}) \tag{2.1}$$

for all $t > 0$, some $c_1, c_2 > 0$.

Now we are ready to introduce our hypotheses on the map $a(\cdot)$ involved in the differential operator of (1.1):

$(H(a))$ $\quad a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$, with $a_0(t) > 0$ for all $t > 0$, and

(i) $a_0 \in C^1(0,\infty)$, the function $t \mapsto t a_0(t)$ is strictly increasing in $(0,\infty)$,

$$t\, a_0(t) \to 0^+ \text{ as } t \to 0^+ \quad \text{and} \quad \lim_{t \to 0^+} \frac{t a_0'(t)}{a_0(t)} > -1;$$

(ii) there exists $c_3 > 0$ such that $|\nabla a(y)| \leqslant c_3 \frac{\eta(|y|)}{|y|}$ for all $y \in \mathbb{R}^N \setminus \{0\}$;

(iii) $(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \geqslant \frac{\eta(|y|)}{|y|}|\xi|^2$ for all $y \in \mathbb{R}^N \setminus \{0\}$, all $\xi \in \mathbb{R}^N$;

(iv) if $G_0(t) = \int_0^t s a_0(s)\, ds$ for all $t > 0$, then there exists $q \in (1, p]$ such that

$$t \mapsto G_0(t^{1/q}) \text{ is convex in } (0, \infty) \quad \text{and} \quad \lim_{t \to 0^+} \frac{q\, G_0(t)}{t^q} = \tilde{c} > 0.$$

Remark 2.2. Conditions $H(a)$ (i), (ii), (iii) are motivated from the regularity theory of Lieberman [14] and from the nonlinear maximum principle of Pucci and Serrin [27]. Condition $H(a)$ (iv) is particular for our problem, but as we will see below is satisfied in many cases of interest. Hypotheses $H(a)$ imply that the primitive $G_0(\cdot)$ is strictly convex and strictly increasing. Let us see how these hypotheses are satisfied in the case of the p-Laplacian. Additional examples are given below. In the case of the p-Laplace operator, $a(y) = |y|^{p-2}y$ for all $y \in \mathbb{R}^N$, with $1 < p < \infty$. Then $G_0(t) = \frac{1}{p}t^p$ for all $t \geqslant 0$. So $G_0(\cdot)$ is strictly convex and strictly increasing. Also in this case $q = p$ (see hypothesis $H(a)$ (iv)). So, the function

$$t \mapsto G_0(t^{1/p}) = \frac{1}{p}(t^{1/p})^p = \frac{1}{p}t$$

is linear and of course $\tilde{c} = 1$.

We set $G(y) = G_0(|y|)$ for all $y \in \mathbb{R}^N$. Clearly $G(\cdot)$ is convex and $G(0) = 0$. Also, we have

$$\nabla G(y) = G_0'(|y|)\frac{y}{|y|} = a_0(|y|)y = a(y) \quad \text{for all } y \in \mathbb{R}^N \setminus \{0\}, \quad \nabla G(0) = 0.$$

Hence, $G(\cdot)$ is the primitive of the map $a(\cdot)$. The convexity of $G(\cdot)$ and the fact that $G(0) = 0$, imply

$$G(y) \leqslant (a(y), y)_{\mathbb{R}^N} \quad \text{for all } y \in \mathbb{R}^N. \tag{2.2}$$

Hypotheses $H(a)$ (i), (ii), (iii), and (2.1), (2.2), lead to the following lemma summarizing the main properties of the map $a(\cdot)$.

Lemma 2.3. *If hypotheses $H(a)$ (i), (ii), (iii) hold, then*

(a) *the map $y \mapsto a(y)$ is continuous, strictly monotone, hence maximal monotone too;*

(b) *$|a(y)| \leqslant c_4(1 + |y|^{p-1})$ for all $y \in \mathbb{R}^N$ and some $c_4 > 0$;*

(c) *$(a(y), y)_{\mathbb{R}^N} \geqslant \frac{c_1}{p-1}|y|^p$ for all $y \in \mathbb{R}^N$.*

This lemma and (2.2) lead to the following growth estimates for the primitive $G(\cdot)$.

Corollary 2.4. *If hypotheses* $H(a)$ (i), (ii), (iii) *hold, then*

$$\frac{c_1}{p(p-1)}\,|y|^p \leqslant G(y) \leqslant c_5\,(1+|y|^p) \quad \text{for all } y \in \mathbb{R}^N \text{ and some } c_5 > 0.$$

Example 2.5. The following maps satisfy hypotheses $H(a)$:

(a) $a(y) = |y|^{p-2}y$ with $1 < p < \infty$.

This map corresponds to the p-Laplace differential operator

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) \quad \text{for all } u \in W^{1,p}(\Omega).$$

(b) $a(y) = |y|^{p-2}y + |y|^{q-2}y$ with $1 < q < p < \infty$.

This map corresponds to the (p,q)-differential operator defined by

$$\Delta_p u + \Delta_q u \quad \text{for all } u \in W^{1,p}(\Omega).$$

Such operator arise in problems of mathematical physics (Papageorgiou and Rădulescu [23]) and were studied in the context of Dirichlet problems by Mugnai and Papageorgiou [18], Papageorgiou and Rădulescu [20], [25] and Papageorgiou and Winkert [26].

(c) $a(y) = (1+|y|^2)^{(p-2)/2}y$, with $1 < p < \infty$.

This map corresponds to the generalized p-mean curvature differential operator defined by

$$\operatorname{div}\left[(1+|Du|^2)^{(p-2)/2}\,Du\right] \quad \text{for all } u \in W^{1,p}(\Omega).$$

(d) $a(y) = |y|^{p-2}y + \frac{|y|^{p-2}y}{1+|y|^p}$ with $1 < p < \infty$.

The hypotheses on the boundary term $\beta(\cdot)$ are the following:

$(H(\beta))$ $\qquad \beta \in C^{1,\alpha}(\partial\Omega)$, with $\alpha \in (0,1)$, $\quad \beta(z) \geqslant 0$ for all $z \in \partial\Omega$.

Consider a Carathéodory function $f_0\colon \Omega \times \mathbb{R} \to \mathbb{R}$ which exhibits subcritical growth in the $x \in \mathbb{R}$ variable, that is,

$$|f_0(z,x)| \leqslant \hat{a}(z)(1+|x|^{r-1}) \quad \text{for almost all } z \in \Omega, \text{ all } x \in \mathbb{R},$$

with $\hat{a} \in L^\infty(\Omega)_+$ and

$$1 < r < p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } N \leqslant p. \end{cases}$$

Let $F_0(z,x) = \int_0^x f_0(z,s)\,ds$ and consider the C^1-functional $\varphi_0\colon W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \int_\Omega G(Du)\,dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p d\sigma - \int_\Omega F_0(z,u)\,dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

The next theorem can be proved as the corresponding result of Papageorgiou and Rădulescu [21], using the regularity results of Lieberman [14].

Theorem 2.6. *Assume that $u_0 \in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of φ_0, that is, there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leqslant \varphi(u_0 + h) \quad \text{for all } h \in C^1(\overline{\Omega}) \text{ with } ||h||_{C^1(\overline{\Omega})} \leqslant \rho_0 .$$

Then $u_0 \in C^{1,s}(\overline{\Omega})$ with $s \in (0,1)$ and u_0 is also a local $W^{1,p}(\Omega)$-minimizer of φ_0, that is, there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leqslant \varphi_0(u_0 + h) \quad \text{for all } h \in W^{1,p}(\Omega) \text{ with } ||h|| \leqslant \rho_1 .$$

We will also need some facts concerning the spectrum of $-\Delta_q$ $(1 < q < \infty)$ with Robin boundary condition (see Le [12] and Papageorgiou and Rădulescu [21]). So, we consider the following nonlinear eigenvalue problem:

$$\begin{cases} -\Delta_q u(z) = \hat{\lambda}\, |u(z)|^{q-2}\, u(z) & \text{in } \Omega , \\ \frac{\partial u}{\partial n_q} + \beta(z)\, |u(z)|^{q-2}\, u(z) & \text{on } \partial\Omega . \end{cases} \tag{2.3}$$

Here,

$$\frac{\partial u}{\partial n_q} = |Du|^{q-2}(Du, n)_{\mathbb{R}^N} = |Du|^{q-2}\, \frac{\partial u}{\partial n} \quad \text{for all } u \in W^{1,p}(\Omega).$$

We say that $\hat{\lambda} \in \mathbb{R}$ is an eigenvalue of the negative Robin q-Laplacian (denoted for notational economy by $-\Delta_q^R$), if problem (2.3) admits a nontrivial solution $\hat{u} \in W^{1,p}(\Omega)$ known as an eigenfunction corresponding to $\hat{\lambda}$. We know that there is a smallest eigenvalue $\hat{\lambda}_1(q,\beta)$ having the following properties:

- $\hat{\lambda}_1(q,\beta) \geqslant 0$ and $\hat{\lambda}_1(q,\beta) > 0$ if $\beta \not\equiv 0$.
- $\hat{\lambda}_1(q,\beta)$ is isolated in the spectrum $\hat{\sigma}(q,\beta)$ of $-\Delta_q^R$.

and

$$\hat{\lambda}_1(q,\beta) = \inf\Big[\frac{||Du||_q^q + \int_{\partial\Omega} \beta(z)\, |u|^q\, d\sigma}{||u||_q^q} : u \in W^{1,p}(\Omega), u \neq 0\Big]. \tag{2.4}$$

Note that the infimum is realized on the corresponding one dimensional eigenspace. Moreover, from (2.4) it is clear that the elements of this eigenspace do not change sign. In the sequel by $\hat{u}_1(q,\beta)$ we denote the positive L^q-normalized (that is, $||\hat{u}_1(q,\beta)||_q = 1$) eigenfunction corresponding to $\hat{\lambda}_1(q,\beta)$. From the nonlinear regularity theory of Lieberman [14] we have that $\hat{u}_1(q,\beta) \in C_+ \backslash \{0\}$. In fact, using also the nonlinear maximum principle of Pucci and Serrin [27], pp. 111, 120, we conclude that $\hat{u}_1(q,\beta) \in \operatorname{int} C_+$.

The Ljusternik–Schnirelmann minimax scheme gives, in addition to $\hat{\lambda}_1(q,\beta)$, a whole strictly increasing sequence $\{\hat{\lambda}_k(q,\beta)\}_{k\geqslant 1}$ of eigenvalues such that $\hat{\lambda}_k(q,\beta) \to +\infty$ as $k \to \infty$. These are known as the "LS-eigenvalues" of $-\Delta_q^R$.

Since $\hat{\lambda}_1(q,\beta) \geqslant 0$ is isolated and the spectrum $\hat{\sigma}(q,\beta)$ is closed, the second eigenvalue of $-\Delta_q^R$ is defined by

$$\hat{\lambda}_2^*(q,\beta) = \inf[\hat{\lambda} \in \hat{\sigma}(q,\beta) : \hat{\lambda} > \hat{\lambda}_1(q,\beta)].$$

We have that $\hat{\lambda}_2^*(q,\beta) = \hat{\lambda}_2(q,\beta)$, that is, the second eigenvalue of $-\Delta_q^R$ and the second LS-eigenvalue of $-\Delta_q^R$ coincide.

Let $\partial B_1^{L^q} = \{u \in L^q(\Omega) : ||u||_q = 1\}$, $M = W^{1,q}(\Omega) \cap \partial B_1^{L^q}$ and

$$\vartheta(u) = ||Du||_q^q + \int_{\partial\Omega} \beta(z)|u|^q d\sigma \quad \text{for all } u \in W^{1,q}(\Omega)\,.$$

From Papageorgiou and Rădulescu [21], we have the following minimax characterization of $\hat{\lambda}_2(q,\beta)$.

Proposition 2.7. *We have* $\hat{\lambda}_2(q,\beta) = \inf_{\hat{\gamma}\in\hat{\Gamma}} \max_{-1\leqslant t\leqslant 1} \vartheta(\hat{\gamma}(t))$, *where*

$$\hat{\Gamma} = \{\hat{\gamma} \in C([-1,1], M) : \hat{\gamma}(-1) = -\hat{u}_1(q,\beta), \hat{\gamma}(1) = \hat{u}_1(q,\beta)\}.$$

Let $A\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u), v\rangle = \int_\Omega (a(Du), Dv)_{\mathbb{R}^N} dz \quad \text{for all } u, v \in W^{1,p}(\Omega). \tag{2.5}$$

From Gasinski and Papageorgiou [9], we have the following property.

Proposition 2.8. *Assume that hypotheses* $H(a)$ (i), (ii), (iii) *hold. Then the map* $A\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ *defined by* (2.5) *is demicontinuous, monotone, hence maximal monotone too and of type* $(S)_+$, *that is,*

if $u_n \overset{w}{\to} u$ *in* $W^{1,p}(\Omega)$ *and* $\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \leqslant 0$, *then* $u_n \to u$ *in* $W^{1,p}(\Omega)$.

Our approach will also use tools from Morse theory (critical groups). So, let us recall some basic definitions and facts from this theory.

Given $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, we introduce the following sets:

$$\varphi^c = \{u \in X : \varphi(u) \leqslant c\},\ K_\varphi = \{u \in X : \varphi'(u) = 0\} \text{ and } K_\varphi^c = \{u \in K_\varphi : \varphi(u) = c\}.$$

For every topological pair (Y_1, Y_2) with $Y_2 \subseteq Y_1 \subseteq X$ and every integer $k \geqslant 0$, by $H_k(Y_1, Y_2)$ we denote the kth relative singular homology group with integer coefficients. Given an isolated $u \in K_\varphi^c$, the critical groups of φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \backslash \{u\}) \quad \text{for all integers } k \geqslant 0\,,$$

where U is a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$ (recall that $u \in K_\varphi$ is isolated). The excision property of singular homology theory implies that this definition of critical groups is independent of the particular choice of the neighborhood U.

If $\varphi \in C^1(X)$ satisfies the PS-condition and $\inf \varphi(K_\varphi) > -\infty$, then the critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \geqslant 0\,,$$

where $c < \inf \varphi(K_\varphi)$. The second deformation theorem (see Gasinski and Papageorgiou [8], p. 628), implies that this definition is independent of the particular choice of the level $c < \inf \varphi(K_\varphi)$.

Assuming that K_φ is finite, we define

$$M(t,u) = \sum_{k \geqslant 0} \operatorname{rank} C_k(\varphi, u)\, t^k \quad \text{for all } t \in \mathbb{R}, \text{ all } u \in K_\varphi,$$
$$P(t,\infty) = \sum_{k \geqslant 0} \operatorname{rank} C_k(\varphi, \infty)\, t^k \quad \text{for all } t \in \mathbb{R}\,.$$

Then the Morse relation says

$$\sum_{u \in K_\varphi} M(t,u) = P(t,\infty) + (1+t)\,Q(t) \quad \text{for all } t \in \mathbb{R}\,, \tag{2.6}$$

where $Q(t) = \sum_{k \geqslant 0} \beta_k\, t^k$ is a formal series in $t \in \mathbb{R}$, with nonnegative integer coefficients β_k.

Finally we fix our notation. So, given $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Then for $u \in W^{1,p}(\Omega)$, we define $u^\pm(\cdot) = u(\cdot)^\pm$. We know that $u^\pm \in W^{1,p}(\Omega)$, $u = u^+ - u^-$, $|u| = u^+ + u^-$. For a Carathéodory function $g(z,x)$, we define $N_g(u)(\cdot) = g(\cdot, u(\cdot))$ for all $u \in W^{1,p}(\Omega)$.

By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. Also, recall that $W^{1,p}(\Omega)$ is an ordered Banach space with order cone

$$W_+ = \{u \in W^{1,p}(\Omega) : u(z) \geqslant 0 \text{ for almost all } z \in \Omega\}\,.$$

Then given $u, v \in W^{1,p}(\Omega)$ with $u \leqslant v$ (that is, $v - u \in W_+$), by $[u,v]$ we denote the order interval defined by $[u,v] = \{y \in W^{1,p}(\Omega)\colon u(z) \leqslant y(z) \leqslant v(z)$ for almost all $z \in \Omega\}$.

3. Resonant problems

In this section, we consider a reaction which exhibits $(p-1)$-sublinear growth near $\pm\infty$, and in the particular case of a p-Laplacian equation, it can be resonant with respect to the principal eigenvalue $\hat{\lambda}_1(p,\beta)$.

So, the hypotheses on the reaction $f(z,x)$ are the following:

(H_1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for almost all $z \in \Omega$, and

(i) $|f(z,x)| \leqslant \tilde{a}(z)(1 + |x|^{p-1})$ for almost all $z \in \Omega$, all $x \in \mathbb{R}$ with $\tilde{a} \in L^\infty(\Omega)_+$;

(ii) For $\hat{\beta} = \frac{p-1}{c_1}\beta \in L^\infty(\Omega)_+$ (with $c_1 > 0$ as in (3.14)),

$$\limsup_{x \to \pm\infty} \frac{f(z,x)}{|x|^{p-2}\,x} \leqslant \frac{c_1}{p-1}\hat{\lambda}_1(p,\hat{\beta}), \quad \text{uniformly for almost all } z \in \Omega;$$

(iii) if $F(z,x) = \int_0^x f(z,s)\,ds$, then $\lim_{x \to \pm\infty}[f(z,x)x - pF(z,x)] = +\infty$ uniformly for almost all $z \in \Omega$;

(iv) there exists $\eta_0 \in L^\infty(\Omega)_+$ such that $\tilde{c}\,\hat{\lambda}_1(q,\tilde{\beta}) \leqslant \eta_0(z)$ for almost all $z \in \Omega$, $\eta_0 \not\equiv \tilde{c}\,\hat{\lambda}_1(q,\tilde{\beta})$ and

$$\eta_0(z) \leqslant \liminf_{x\to 0} \frac{f(z,x)}{|x|^{q-2}x} \quad \text{uniformly for almost all } z \in \Omega,$$

with $\tilde{\beta} = \frac{1}{\tilde{c}}\beta$ (here $\tilde{c} > 0$ and $q \in (1,p]$ are as in hypothesis $H(a)$ (iv)).

Remark 3.1. Hypothesis $H(a)$ (i) dictates a $(p-1)$-sublinear growth for $f(z,\cdot)$. If $a(y) = |y|^{p-2}y$ for all $y \in \mathbb{R}^N$, then the differential operator is the p-Laplacian and $c_1 = p-1$ (see (2.1)). Hence $\hat{\beta} = \beta$ (see hypothesis H_1 (ii)). So, hypothesis H_1 (ii) says that the reaction can be resonant with respect to the principal eigenvalue of $-\Delta_p^R$ (resonance equation). This possibility of resonance at $\pm\infty$ dictates hypothesis H_1 (iii) which is needed in order for the energy functional of the problem to satisfy the compactness condition. Hypothesis H_1 (iv) which regulates the behavior of $f(z,\cdot)$ near zero, is quite general and allows also for the presence of concave terms (terms which are $(p-1)$-superlinear near zero). Some additional remarks are motivated by several interesting observations of the referee. Note that hypothesis H_1 (iii) excludes some natural examples like the functions

$$f_1(x) = |x|^{p-2}x + x \quad \text{for big } |x| \ (q < p) \quad \text{or} \quad f_2(x) = |x|^{q-2} \quad \text{with } 1 < q < p.$$

However, our emphasis in the work is to treat the resonant problem. To make things more transparent, consider the case of the p-Laplacian. Then hypothesis H_1 (ii) permits for resonance to occur at $\pm\infty$. Hypothesis H_1 (iii) implies that the resonance takes place from the "left" of $\hat{\lambda}_1(p)$ (see the asymptotic condition (3.12) in the proof of Proposition 3.3 below). This makes the energy functional of the problem coercive and permits the use of the direct method of the calculus of variations. So, the use of hypothesis H_1 (iv) leads to the existence of nontrivial solutions of constant sign. Note that in general, since we want to incorporate also the resonant case, an extra condition near $\pm\infty$ is necessary, in order to guarantee that the energy functional of the problem satisfies the compactness condition used in the minimax methods (Palais–Smale condition or Cerami condition). If instead we assume that

$$f(z,x)x - pF(z,x) \to -\infty \quad \text{as } x \to \pm\infty \text{ uniformly for a.a. } z \in \Omega$$

(this is satisfied by the functions f_1 and f_2 mentioned earlier), then we have resonance from the "right" of $\hat{\lambda}_1(p)$, and the coercivity of the energy functional fails. So, we have to proceed in a different way. We use either the mountain pass theorem (but then we need to change the condition near zero, see hypothesis H_1 (iv), and so we fail to have extremal constant sign solutions, and consequently we cannot produce a nodal solution) or we use critical groups (Morse theory). This second approach is more promising, but not at all straightforward, since the computation of critical groups in that case (resonant case) is difficult. This can be an interesting separate project.

Example 3.2. The following function satisfies hypotheses H_1. For the sake of simplicity, we drop the z-dependence:

$$f(x) = \begin{cases} \eta\, x^{q-1} - c\, x^{r-1} & \text{if } |x| \leqslant 1, \\ \vartheta\, x^{p-1} - x^{q-1} & \text{if } |x| > 1, \end{cases}$$

with $1 < q < p < r < \infty$, $\eta > \hat{\lambda}_1(p, \hat{\beta}) \geqslant \vartheta > 0$ and $c = \eta + 1 - \vartheta > 0$.

We introduce the following truncations-perturbations of the reaction:

$$\hat{f}_+(z,x) = \begin{cases} 0 & \text{if } x \leqslant 0, \\ f(z,x) + x^{p-1} & \text{if } x > 0, \end{cases}$$

$$\text{and} \quad \hat{f}_-(z,x) = \begin{cases} f(z,x) + |x|^{p-2}x & \text{if } x < 0, \\ 0 & \text{if } x \geqslant 0. \end{cases} \tag{3.1}$$

Both are Carathéodory functions. We set $\hat{F}_\pm(z,x) = \int_0^x \hat{f}_\pm(z,s)\, ds$ and consider the C^1-functionals $\hat{\varphi}_\pm : W^{1,p}(\Omega) \to \mathbb{R}$ defined, for all $u \in W^{1,p}(\Omega)$, by

$$\hat{\varphi}_\pm(u) = \int_\Omega G(Du)\, dz + \frac{1}{p}||u||_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)(u^\pm)^p\, d\sigma - \int_\Omega \hat{F}_\pm(z,u)\, dz.$$

Also, let $\varphi\colon W^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional for problem (1.1) defined by

$$\varphi(u) = \int_\Omega G(Du)\, dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)\, |u|^p\, d\sigma - \int_\Omega F(z,u)\, dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Evidently, $\varphi \in C^1(W^{1,p}(\Omega))$.

Proposition 3.3. *Assume that hypotheses $H(a)$ (i), (ii), (iii), $H(\beta)$ and H_1 hold. Then the functionals φ and $\hat{\varphi}_\pm$ are coercive.*

Proof. We do the proof for the functional φ, the proofs for $\hat{\varphi}_\pm$ being similar.

We argue indirectly. So, suppose that the functional φ is not coercive. Then we can find $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ and $M_1 > 0$ such that

$$||u_n|| \to \infty \ \text{ as } n \to \infty, \quad \text{and} \quad \varphi(u_n) \leqslant M_1 \ \text{ for all } n \geqslant 1\,. \tag{3.2}$$

We have, for all $n \geqslant 1$,

$$\varphi(u_n) = \int_\Omega G(Du_n)\, dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u_n|^p\, d\sigma - \int_\Omega F(z,u_n)\, dz \leqslant M_1. \tag{3.3}$$

Let $y_n = u_n/||u_n||$, $n \geqslant 1$. Then $||y_n|| = 1$ for all $n \geqslant 1$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(\Omega), \quad \text{and} \quad y_n \to y \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega) \text{ as } n \to \infty. \tag{3.4}$$

From (3.3) and Corollary 2.4, we have that, for all $n \geqslant 1$,

$$\frac{c_1}{p(p-1)}\, ||Dy_n||_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)\, |y_n|^p\, d\sigma - \int_\Omega \frac{F(z,u_n)}{||u_n||^p}\, dz \leqslant \frac{M_1}{||u_n||^p}. \tag{3.5}$$

Hypothesis H_1 (i) implies that

$$|F(z,x)| \leqslant \hat{a}(z)(1+|x|^p) \text{ for almost all } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ with } \hat{a} \in L^\infty(\Omega)_+$$
$$\Longrightarrow \left\{\frac{N_F(u_n)}{||u_n||^p}\right\}_{n\geqslant 1} \subseteq L^1(\Omega) \text{ is uniformly integrable.}$$

From the Dunford–Pettis theorem, and passing to a suitable subsequence if necessary, we may assume that

$$\frac{N_F(u_n)}{||u_n||^p} \xrightarrow{w} k \quad \text{in } L^1(\Omega) \text{ as } n \to \infty. \tag{3.6}$$

From hypothesis H_1 (ii), we have

$$\limsup_{x\to\pm\infty} \frac{F(z,x)}{|x|^p} \leqslant \frac{c_1}{p(p-1)}\,\hat{\lambda}_1(p,\hat{\beta}) \quad \text{uniformly for almost all } z \in \Omega\,. \tag{3.7}$$

Then (3.7) implies that

$$k = \frac{c_1}{p(p-1)} h|y|^p \text{ with } h \in L^\infty(\Omega), \quad h(z) \leqslant \hat{\lambda}_1(p,\hat{\beta}) \text{ for almost all } z \in \Omega \tag{3.8}$$

(see Aizicovici, Papageorgiou and Staicu [1], proof of Proposition 14). We return to (3.5), pass to the limit as $n \to \infty$ and use (3.4), (3.6), (3.8). Then

$$\frac{c_1}{p(p-1)}\left[||Dy||_p^p + \int_{\partial\Omega} \hat{\beta}(z)\,|y|^p d\sigma\right] \leqslant \frac{c_1}{p(p-1)} \int_\Omega h(z)\,|y|^p\,dz$$
$$\Longrightarrow ||Dy||_p^p + \int_{\partial\Omega} \hat{\beta}(z)\,|y|^p d\sigma \leqslant \int_\Omega h(z)\,|y|^p\,dz. \tag{3.9}$$

First suppose that $h \not\equiv \hat{\lambda}_1(p,\hat{\beta})$ (see (3.8)). Then from (3.9) and Proposition 4 of Papageorgiou and Rădulescu [21], we have

$$c_6\,||y||^p \leqslant 0 \text{ for some } c_6 > 0 \quad \Longrightarrow \quad y = 0.$$

Then from (3.4), (3.5), (3.6) and (3.8), we see that

$$y_n \to 0 \quad \text{in } W^{1,p}(\Omega) \text{ as } n \to \infty,$$

which contradicts the fact that $||y_n|| = 1$ for all $n \geqslant 1$.

Next we assume that $h(z) = \hat{\lambda}_1(p,\hat{\beta})$ for almost all $z \in \Omega$ (see (3.8)). Then from (3.9) and (2.4), we have

$$||Dy||_p^p + \int_{\partial\Omega} \hat{\beta}(z)|y|^p\,d\sigma = \hat{\lambda}_1(p,\hat{\beta})||y||_p^p, \quad \Longrightarrow \quad y = \xi\,\hat{u}_1(p,\hat{\beta}), \quad \text{with } \xi \in \mathbb{R}.$$

If $\xi = 0$, then $y = 0$ and as above, using (3.4), (3.6) and (3.8), we obtain

$$y_n \to 0 \quad \text{in } W^{1,p}(\Omega) \text{ as } n \to \infty,$$

contradicting the fact that $||y_n|| = 1$ for all $n \geqslant 1$.

So, we have $\xi \neq 0$ and without any loss of generality, we may assume that $\xi > 0$ (the reasoning is similar if $\xi < 0$). Since $\hat{u}_1(p, \hat{\beta}) \in \operatorname{int} C_+$, we have

$$u_n(z) \to +\infty \quad \text{for almost all } z \in \Omega, \text{ as } n \to \infty. \tag{3.10}$$

Hypothesis H_1 (iii) implies that given $\xi > 0$, we can find $M_2 = M_2(\xi) > 0$ such that

$$f(z,x)x - pF(z,x) \geqslant \xi \quad \text{for almost all } z \in \Omega, \text{ all } |x| \geqslant M_2. \tag{3.11}$$

Then, for almost all $z \in \Omega$ and all $s \geqslant M_2$, we have

$$\frac{d}{ds}\frac{F(z,s)}{s^p} = \frac{f(z,s)s^p - ps^{p-1}F(z,s)}{s^{2p}} = \frac{f(z,s)s - pF(z,s)}{s^{p+1}} \geqslant \frac{\xi}{s^{p+1}} \text{ (see (3.11))}$$

$$\Longrightarrow \quad \frac{F(z,y)}{y^p} - \frac{F(z,x)}{x^p} \geqslant -\frac{\xi}{p}\left[\frac{1}{y^p} - \frac{1}{x^p}\right] \text{ for almost all } z \in \Omega, \text{ all } y \geqslant x \geqslant M_2.$$

We pass to the limit as $y \to +\infty$ and use (3.7). Then

$$\frac{c_1}{p(p-1)}\hat{\lambda}_1(p,\hat{\beta}) - \frac{F(z,x)}{x^p} \geqslant \frac{\xi}{p}\frac{1}{x^p} \quad \text{for almost all } z \in \Omega, \text{ all } x \geqslant M_2$$

$$\Longrightarrow \quad \frac{c_1}{p-1}\hat{\lambda}_1(p,\hat{\beta})x^p - pF(z,x) \geqslant \xi \text{ for almost all } z \in \Omega, \text{ all } x \geqslant M_2.$$

Since $\xi > 0$ is arbitrary, it follows that

$$\lim_{x \to +\infty}\left[\frac{c_1}{p-1}\hat{\lambda}_1(p,\hat{\beta})x^p - pF(z,x)\right] = +\infty \tag{3.12}$$

uniformly for almost all $z \in \Omega$. From (3.10), (3.12) and Fatou's lemma, we obtain

$$\lim_{n \to \infty}\int_\Omega \left[\frac{c_1}{p-1}\hat{\lambda}_1(p,\hat{\beta})\, u_n(z)^p - p\, F(z,u_n(z))\right] dz = +\infty\,. \tag{3.13}$$

On the other hand, from (3.3) and Corollary 2.4, we have

$$\frac{c_1}{p(p-1)}\left[||Du_n||_p^p + \int_{\partial\Omega}\hat{\beta}(z)\,|u_n|^p\, d\sigma\right] - \int_\Omega F(z,u_n)\, dz \leqslant M_1 \quad \text{for all } n \geqslant 1$$

$$\Longrightarrow \quad \int_\Omega \left[\frac{c_1}{p-1}\,\hat{\lambda}_1(p,\hat{\beta})\, u_n^p - p\, F(z,u_n)\right] dz \leqslant p\, M_1 \quad \text{for all } n \geqslant 1. \tag{3.14}$$

Comparing (3.13) and (3.14), we reach a contradiction. Similarly for the functionals $\hat{\varphi}_\pm$. □

From this proposition we have the following additional property (see Papageorgiou and Winkert [26]).

Corollary 3.4. *If hypotheses $H(a)$ (i), (ii), (iii), $H(\beta)$ and H_1 hold, then the functionals φ and $\hat{\varphi}_\pm$ satisfy the PS-condition.*

Now using the direct method, we can produce two solutions of constant sign.

Proposition 3.5. *Assume that hypotheses $H(a)$, $H(\beta)$ and H_1 hold. Then problem (1.1) admits at least two constant sign solutions, $u_0 \in \operatorname{int} C_+$ and $v_0 \in -\operatorname{int} C_+$, both local minimizers of the energy functional φ.*

Proof. First we produce the positive solution. From Proposition 3.3 we know that $\hat{\varphi}_+$ is coercive. Also, using the Sobolev embedding theorem and the trace theorem (which guarantee the compactness of the corresponding embedding and trace maps), we see that $\hat{\varphi}_+$ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\hat{\varphi}_+(u_0) = \inf[\hat{\varphi}_+(u) : u \in W^{1,p}(\Omega)]. \tag{3.15}$$

By virtue of hypotheses $H(a)$ (iv) and H_1 (iv), given $\epsilon > 0$, we can find $\delta = \delta(\epsilon) \in (0,1)$ such that

$$G(y) \leqslant \frac{1}{q}[\tilde{c} + \epsilon]|y|^q \quad \text{for all } y \in \mathbb{R}^N \text{ with } |y| \leqslant \delta\,, \tag{3.16}$$

$$f(z,x) \geqslant (\eta_0(z) - \epsilon)x^{q-1} \quad \text{for almost all } z \in \Omega, \text{ all } x \in [0,\delta]. \tag{3.17}$$

Since $\hat{u}_1(q,\tilde{\beta}) \in \operatorname{int} C_+$, we can choose $t \in (0,1)$ small such that

$$t\hat{u}_1(q,\tilde{\beta})(z) \in (0,\delta] \quad \text{and} \quad t|D\hat{u}_1(q,\tilde{\beta})(z)| \leqslant \delta \quad \text{for all } z \in \overline{\Omega}\,. \tag{3.18}$$

Then we have

$$\begin{aligned}
&\hat{\varphi}_+\left(t\hat{u}_1(q,\tilde{\beta})\right)\\
&\quad\leqslant \frac{t^q}{q}\left[\tilde{c}+\epsilon\right]\,||D\hat{u}_1(q,\tilde{\beta})||_q^q + \frac{1}{p}\int_{\partial\Omega}\beta(z)(t\hat{u}_1(q,\tilde{\beta}))^p\,d\sigma - \int_\Omega F(z,t\hat{u}_1(q,\tilde{\beta}))\,dz\\
&\qquad\text{(see (3.1), (3.16), (3.18))}\\
&\quad\leqslant \frac{t^q}{q}\left[\tilde{c}+\epsilon\right]||D\hat{u}_1(q,\tilde{\beta})||_q^q + \frac{t^q}{q}\int_{\partial\Omega}\beta(z)\hat{u}_1(q,\tilde{\beta})^q d\sigma - \frac{t^q}{q}\int_\Omega(\eta_0(z)-\epsilon)\hat{u}_1(q,\tilde{\beta})^q dz\\
&\qquad\text{(see (3.17), (3.18) and recall that } \delta \in (0,1), q < p)\\
&\quad\leqslant \frac{t^q}{q}\,\tilde{c}\left[||D\hat{u}_1(q,\tilde{\beta})||_q^q + \int_{\partial\Omega}\tilde{\beta}(z)\,\tilde{u}_1(q,\tilde{\beta})^q\,d\sigma\right] + \frac{t^q}{q}\,\epsilon\,\hat{\lambda}_1(q,\tilde{\beta})\\
&\qquad - \frac{t^q}{q}\int_\Omega \eta_0(z)\,\hat{u}_1(q,\tilde{\beta})^q\,dz + \frac{t^q}{q}\,\epsilon \qquad \text{(recall that } ||\hat{u}_1(q,\tilde{\beta})||_q = 1)\\
&\quad= \frac{t^q}{q}\left[\int_\Omega(\tilde{c}\,\hat{\lambda}_1(q,\tilde{\beta}) - \eta_0(z))\,\hat{u}_1(q,\tilde{\beta})^q\,dz + \epsilon(\hat{\lambda}_1(q,\tilde{\beta})+1)\right].
\end{aligned}$$

Note that

$$\int_\Omega(\eta_0(z) - \tilde{c}\,\hat{\lambda}_1(q,\tilde{\beta}))\,\hat{u}_1(q,\tilde{\beta})^q\,dz = \xi^* > 0 \quad \text{(see hypothesis } H_1\text{ (iv))}.$$

Then

$$\hat{\varphi}_+(t\hat{u}_1(q,\tilde{\beta})) \leqslant \frac{t^q}{q}\,[-\xi^* + \epsilon(\hat{\lambda}_1(q,\tilde{\beta})+1)].$$

Choosing $\epsilon \in \big(0, \frac{\xi^*}{\hat{\lambda}_1(q,\tilde{\beta})+1}\big)$, we see that

$$\hat{\varphi}_+(t\hat{u}_1(q,\tilde{\beta})) < 0\,, \quad \Longrightarrow \quad \hat{\varphi}_+(u_0) < 0 = \hat{\varphi}_+(0) \text{ (see (3.15))}, \text{ hence } u_0 \neq 0\,.$$

From (3.15), we have

$$\begin{aligned} &\hat{\varphi}'_+(u_0) = 0 \\ &\Longrightarrow \quad \langle A(u_0), h\rangle + \int_\Omega |u_0|^{p-2}u_0 h\,dz + \int_{\partial\Omega} \beta(z)(u_0^+)^{p-1}h\,d\sigma \\ &\qquad\qquad = \int_\Omega \hat{f}_+(z,u_0)h\,dz \quad \text{for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{3.19}$$

In (3.19) we choose $h = -u_0^- \in W^{1,p}(\Omega)$. Using Lemma 2.3 (c), (3.1) and hypothesis $H(\beta)$, we obtain

$$\frac{c_1}{p-1}||Du_0^-||_p^p + ||u_0^-||_p^p \leqslant 0 \quad \Longrightarrow \quad u_0 \geqslant 0,\ u_0 \neq 0\,.$$

Therefore (3.19) becomes

$$\begin{aligned} &\langle A(u_0), h\rangle + \int_{\partial\Omega} \beta(z)u_0^{p-1}h\,d\sigma = \int_\Omega f(z,u_0)h\,dz \quad \text{for all } h \in W^{1,p}(\Omega) \text{ (see (3.1))}, \\ &\Longrightarrow \quad u_0 \text{ is a positive solution of (1.1) (see Papageorgiou and Rădulescu [21])}. \end{aligned}$$

From Winkert [29], we have that $u_0 \in L^\infty(\Omega)$. So, we can apply the regularity result of Lieberman [14] and infer that

$$u_0 \in C_+\backslash\{0\}\,.$$

Hypotheses H_1 (i), (iv) imply that given any $\rho > 0$, we can find $\xi_\rho > 0$ such that

$$f(z,x) + \xi_\rho\, x^{p-1} \geqslant 0 \quad \text{for almost all } z \in \Omega,\ \text{all } x \in [0,\rho]\,. \tag{3.20}$$

Let $\rho = ||u_0||_\infty$ and let $\xi_\rho > 0$ be as postulated by (3.20). We have

$$-\operatorname{div} a(Du_0(z)) + \xi_\rho u_0(z)^{p-1} = f(z,u_0(z)) + \xi_\rho u_0(z)^{p-1} \geqslant 0 \quad \text{for almost all } z \in \Omega$$

(see (3.20) and Papageorgiou and Rădulescu [21]), and so

$$\operatorname{div} a(Du_0(z)) \leqslant \xi_\rho u_0(z)^{p-1} \quad \text{for almost all } z \in \Omega\,. \tag{3.21}$$

Let $\xi_0(t) = ta_0(t)$ for all $t > 0$. From hypothesis $H(a)$ (iii) and (2.1), we have the following one-dimensional estimate:

$$t\,\xi_0'(t) = t^2 a_0'(t) + t\,a_0(t) \geqslant c_7\,t^{p-1} \quad \text{for all } t > 0 \text{ and some } c_7 > 0.$$

Integrating by parts leads to

$$\int_0^t s\,\xi_0'(s)\,ds = t\,\xi_0(t) - \int_0^t \xi_0(s)\,ds = t^2 a_0(t) - G_0(t) \geqslant \frac{c_7}{p}\,t^p \quad \text{for all } t > 0\,. \tag{3.22}$$

We set $H(t) = t^2 a_0(t) - G_0(t)$ and $H_0(t) = c_2\,t^p/p$ for all $t > 0$. For $\delta \in (0,1)$ and $s > 0$, we introduce the sets

$$C_1 = \{t \in (0,1) : H(t) \geqslant s\} \quad \text{and} \quad C_2 = \{t \in (0,1) : H_0(t) \geqslant s\}.$$

From (3.22) we see that $C_2 \subseteq C_1$ and so $\inf C_1 \leqslant \inf C_2$. Then, from Leoni [13] (see p. 6), we have

$$H^{-1}(s) \leqslant H_0^{-1}(s).$$

Hence

$$\int_0^\delta \frac{1}{H^{-1}\left(\frac{\xi_p}{p}s^p\right)}ds \geqslant \int_0^\delta \frac{1}{H_0^{-1}\left(\frac{\xi_p}{p}s^p\right)}ds = \frac{\xi_p}{C_1}\int_0^\delta \frac{ds}{s} = +\infty\,.$$

Then because of (3.21) we can apply the strong maximum principle of Pucci and Serrin ([27], p. 111) and have $u(z) > 0$ for all $z \in \Omega$. Subsequently, using the boundary point lemma of Pucci and Serrin ([27], p. 120) we have $u_0 \in \operatorname{int} C_+$. Since

$$\hat{\varphi}_+|_{C_+} = \varphi|_{C_+} \quad \text{(see (3.1))},$$

we infer that $u_0 \in \operatorname{int} C_+$ is local $C^1(\overline{\Omega})$-minimizer of φ, hence using Theorem 2.6, we have that u_0 is also a local $W^{1,p}(\Omega)$-minimizer of φ.

In a similar fashion, working this time with the functional $\hat{\varphi}_-$, we produce $v_0 \in -\operatorname{int} C_+$ a negative solution of problem (1.1), which is a local minimizer of the energy functional φ. □

In fact we can produce extremal constant sign solutions of (1.1), that is, the smallest positive solution and the biggest negative solution. To reach that point, we need some preliminary work.

Hypotheses H_1 (i),(iii) imply that given $\epsilon > 0$ and $r \in (p, p^*)$, we can find $c_8 = c_8(\epsilon, r) > 0$ such that

$$f(z,x)\,x \geqslant (\eta_0(z) - \epsilon)\,|x|^q - c_8\,|x|^r \quad \text{for almost all } z \in \Omega \text{ all } x \in \mathbb{R}\,. \tag{3.23}$$

This unilateral growth estimate on the reaction, leads to the following auxiliary Robin problem:

$$\begin{cases} -\operatorname{div} a(Du(z)) = (\eta_0(z) - \epsilon)|u(z)|^{q-2}u(z) - c_8|u(z)|^{r-2}u(z) & \text{in } \Omega\,, \\ \dfrac{\partial u}{\partial n_a}(z) + \beta(z)|u(z)|^{p-2}u(z) = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{3.24}$$

Proposition 3.6. *Assume that hypotheses $H(a)$ and $H(\beta)$ hold, $\eta_0 \in L^\infty(\Omega)$ is as in hypothesis H_1 (iv), and $p < r < p^*$. Then for all $\epsilon > 0$ small enough, problem (3.24) has a unique positive solution $\tilde{u} \in \operatorname{int} C_+$, and since (3.18) is odd, then $\tilde{v} = -\tilde{u} \in -\operatorname{int} C_+$ is the unique negative solution.*

Proof. First we establish the existence of a positive solution for problem (3.24).

Let $\mu_+ : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined, for all $u \in W^{1,p}(\Omega)$, by

$$\mu_+(u) = \int_\Omega G(Du)\,dz + \frac{1}{p}||u^-||_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)(u^+)^p\,d\sigma - \frac{1}{q}\int_\Omega (\eta_0(z)-\epsilon)(u^+)^q\,dz \\ + \frac{c_8}{r}||u^+||_r^r.$$

Using Corollary 2.4, hypothesis $H(\beta)$ and recalling that $\eta_0 \in L^\infty(\Omega)_+$, we have

$$\begin{aligned} \mu_+(u) &\geqslant \frac{c_1}{p(p-1)}||Du||_p^p + \frac{1}{p}||u^-||_p^p - c_9(||u^+||_q^q - ||u^+||_p^r) \text{ for some } c_9 > 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{(recall that } q \leqslant p < r) \\ &= \frac{c_1}{p(p-1)}||Du||_p^p + \frac{1}{p}||u^-||_p^p + c_9(||u^+||_p^{r-q} - 1)||u^+||_p^q, \\ \implies &\quad \mu_+ \text{ is coercive.} \end{aligned}$$

Also, via the Sobolev embedding theorem and the trace theorem, we can check that μ_+ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $\tilde{u} \in W^{1,p}(\Omega)$ such that

$$\mu_+(\tilde{u}) = \inf[\mu_+(u) : u \in W^{1,p}(\Omega)]. \tag{3.25}$$

Reasoning as in the proof of Proposition 3.5, using the hypothesis on $\eta_0 \in L^\infty(\Omega)$ (see hypothesis H_1 (iv)), for $\epsilon > 0$ small we have

$$\mu_+(\tilde{u}) < 0 = \mu_+(0) \quad \implies \quad \tilde{u} \neq 0\,.$$

From (3.25) we have $\mu_+'(\tilde{u}) = 0$, and hence, for all $h \in W^{1,p}(\Omega)$,

$$\begin{aligned} &\langle A(\tilde{u}), h\rangle - \int_\Omega (\tilde{u}^-)^{p-1} h\,dz + \int_{\partial\Omega} \beta(z)(\tilde{u}^+)^{p-1} h\,d\sigma \\ &\qquad = \int_\Omega (\eta_0(z)-\epsilon)(\tilde{u}^+)^{q-1} h\,dz - c_8\int_\Omega (\tilde{u}^+)^{r-1} h\,dz. \end{aligned} \tag{3.26}$$

In (3.26) we choose $h = -\tilde{u}^- \in W^{1,p}(\Omega)$, and using Corollary 2.4, we have

$$\frac{c_1}{p-1}||D\tilde{u}^-||_p^p + ||\tilde{u}^-||_p^p \leqslant 0 \quad \implies \quad \tilde{u} \geqslant 0,\ \tilde{u} \neq 0\,.$$

Therefore (3.26) becomes

$$\langle A(\tilde{u}), h\rangle + \int_{\partial\Omega} \beta(z)\tilde{u}^{p-1} h\,d\sigma = \int_\Omega (\eta_0(z)-\epsilon)\tilde{u}^{q-1} h\,dz - c_8\int_\Omega \tilde{u}^{r-1} h\,dz \tag{3.27}$$

for all $h \in W^{1,p}(\Omega)$ and for $\epsilon > 0$ small.

From (3.27) it follows that $\tilde{u}$ is a positive solution of (3.24) (see Papageorgiou and Rădulescu [21]). As before the nonlinear regularity theory of Lieberman [14], p. 320, and the nonlinear maximum principle of Pucci and Serrin [27], pp. 111, 120, imply that $\tilde{u} \in \operatorname{int} C_+$.

Next we show the uniqueness of this positive solution.

To this end, let $j\colon L^q(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be the integral functional defined by

$$j(u) = \begin{cases} \displaystyle\int_\Omega G(Du^{1/q})\,dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)u^{p/q}d\sigma & \text{if } u \geqslant 0, u^{1/q} \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Let $u_1, u_2 \in \operatorname{dom} j = \{u \in L^q(\Omega) : j(u) < +\infty\}$ (the effective domain of j). Let

$$v_1 = u_1^{1/q} \quad \text{and} \quad v_2 = u_2^{1/q}.$$

We have $v_1, v_2 \in W^{1,p}(\Omega)$. We set

$$v = (tu_1 + (1-t)u_2)^{1/q}, \quad \text{with } t \in [0,1].$$

Using Lemma 1 of Díaz and Saá [6], we have

$$\begin{aligned} &|Dv(z)| \leqslant (t|Dv_1(z)|^q + (1-t)|Dv_2(z)|^q)^{1/q} \\ \Longrightarrow\quad & G_0(|Dv(z)|) \leqslant G_0((t\,|Dv_1(z)|^q + (1-t)|Dv_2(z)|^q)^{1/q}) \\ &\qquad\qquad \text{(since } G_0(\cdot) \text{ is increasing)} \\ &\qquad \leqslant t\,G_0(|Du_1(z)^{1/q}|) + (1-t)\,G_0(|Du_2(z)^{1/q}|) \text{ for almost all } z\in\Omega \\ &\qquad\qquad \text{(see hypothesis } H(a)\,\text{(iv))} \\ \Longrightarrow\quad & G(Dv(z)) \leqslant t\,G(Du_1(z)^{1/q}) + (1-t)G(Du_2(z)^{1/q}) \text{ for almost all } z \in \Omega, \\ \Longrightarrow\quad & u \mapsto \int_\Omega G(Du^{1/q})\,dz \text{ is convex.} \end{aligned}$$

Similarly, since $\beta \geqslant 0$ and $q \leqslant p$, we have that $u \mapsto \int_{\partial\Omega} \beta(z)u^{p/q}d\sigma$ is convex. Therefore it follows that the integral functional $j(\cdot)$ is convex. Also, using Fatou's lemma, we see that $j(\cdot)$ is lower semicontinuous.

Suppose that u_1 and u_2 are positive solutions of problem (3.24). From the first part of the proof, we have

$$u_1, u_2 \in \operatorname{int} C_+ .$$

So, for every $h \in C^1(\overline{\Omega})$ and for $|t| > 0$ small, we have

$$u_1^q + th, u_2^q + th \in \operatorname{dom} j.$$

We can see that $j(\cdot)$ is Gâteaux differentiable at u_1^q and u_2^q in the direction h. Moreover, using the chain rule and the nonlinear Green's identity (see, for example, Gasinski and Papageorgiou [8], p. 120), we have

$$\begin{aligned} j'(u_1^q)(h) &= \frac{1}{q}\int_\Omega \frac{-\operatorname{div} a(Du_1)}{u_1^{q-1}} h\,dz, \\ j'(u_2^q)(h) &= \frac{1}{q}\int_\Omega \frac{-\operatorname{div} a(Du_2)}{u_2^{q-1}} h\,d\sigma \quad \text{for all } h \in C^1(\overline{\Omega}). \end{aligned}$$

The convexity of $j(\cdot)$ implies the monotony of j'. Hence

$$0 \leqslant \int_\Omega \Big(\frac{-\operatorname{div} a(Du_1)}{u_1^{q-1}} - \frac{-\operatorname{div} a(Du_2)}{u_2^{q-1}} \Big)(u_1^q - u_2^q)\, dz$$
$$= \int_\Omega c_8 \,(u_2^{r-q} - u_1^{r-q})(u_1^q - u_2^q)\, dz \quad \Longrightarrow \quad u_1 = u_2 \ \ (\text{since } q < r).$$

This proves the uniqueness of the positive solution $\tilde{u} \in \operatorname{int} C_+$.

The oddness of (3.24) implies that $\tilde{v} = -\tilde{u} \in -\operatorname{int} C_+$ is the unique negative solution of (3.24). □

The functions $\tilde{u} \in \operatorname{int} C_+$ and $\tilde{v} \in -\operatorname{int} C_+$ from Proposition 3.6, provide bounds for the constant sign solutions of problem (1.1).

Let S_+ (resp. S_-) be the set of positive (resp. negative) solutions of (1.1). From Proposition 3.5 and its proof, we know that

$$S_+ \neq \varnothing \quad \text{and} \quad S_+ \subseteq \operatorname{int} C_+$$
$$S_- \neq \varnothing \quad \text{and} \quad S_- \subseteq \operatorname{int} C_+ .$$

Also, as in Filippakis, Kristaly and Papageorgiou [7], exploiting the monotonicity of the map $A(\cdot)$, we have that

$$S_+ \text{ is downward directed,}$$

that is, if $u_1, u_2 \in S_+$, then there exists $u \in S_+$ such that $u \leqslant u_1, u \leqslant u_2$. Also,

$$S_- \text{ is upward directed,}$$

that is, if $v_1, v_2 \in S_-$, then there exists $v \in S_-$ such that $v_1 \leqslant v, v_2 \leqslant v$.

Proposition 3.7. *Assume that hypotheses $H(a)$, H_1 and $H(\beta)$ hold. Then $\tilde{u} \leqslant u$ for all $u \in S_+$ and $v \leqslant \tilde{v}$ for all $v \in S_-$.*

Proof. We do the proof for the elements of S_+, the proof for the elements of S_- being similar.

So, let $u \in S_+$ and let $\vartheta_u : \Omega \times \mathbb{R} \to \mathbb{R}$ be the Carathéodory function defined by

$$\vartheta_u(z,x) = \begin{cases} 0 & \text{if } x < 0, \\ (\eta_0(z) - \epsilon)x^{q-1} - c_8 x^{r-1} + x^{p-1} & \text{if } 0 \leqslant x \leqslant u(z), \\ (\eta_0(z) - \epsilon)u(z)^{q-1} - c_8 u(z)^{r-1} + u(z)^{p-1} & \text{if } u(z) < x\,. \end{cases} \tag{3.28}$$

Let $\ominus_u(z,x) = \int_0^x \vartheta_u(z,s)\, ds$ and consider the C^1-functional $\hat{\gamma}_u^+ : W^{1,p}(\Omega) \to \mathbb{R}$ defined, for all $w \in W^{1,p}(\Omega)$, by

$$\hat{\gamma}_u^+(w) = \int_\Omega G(Dw)\, dz + \frac{1}{p}||w||_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)(w^+)^p\, d\sigma - \int_\Omega \ominus_u(z,w)\, dz.$$

From (3.28) it is clear that $\hat{\gamma}^1_+$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\bar{u} \in W^{1,p}(\Omega)$ such that

$$\hat{\gamma}^+_u(\bar{u}) = \inf[\hat{\gamma}^+_u(w) : w \in W^{1,p}(\Omega)]. \tag{3.29}$$

As before (see the proof of Proposition 3.5), for $t \in (0,1)$ small such that $t\hat{u}_1(q,\tilde{\beta}) \leqslant u$ (recall that $u \in \operatorname{int} C_+$ and use Lemma 3.3. of Filippakis, Kristaly and Papageorgiou [7]), we have

$$\hat{\gamma}^+_u(t\hat{u}_1(q,\tilde{\beta})) < 0 = \hat{\gamma}^+_u(0) \quad \Longrightarrow \quad \hat{\gamma}^+_u(\bar{u}) < 0 = \hat{\gamma}^+_u(0) \text{ (see (3.29))}, \text{ hence } \bar{u} \neq 0\,.$$

From (3.29), we have $(\hat{\gamma}^+_u)'(\bar{u}) = 0$, hence, for all $h \in W^{1,p}(\Omega)$,

$$\langle A(\bar{u}), h\rangle + \int_\Omega |\bar{u}|^{p-2}\bar{u}h\,dz + \int_{\partial\Omega} \beta(z)(\bar{u}^+)^{p-1}h\,d\sigma = \int_\Omega \vartheta_u(z,\bar{u})h\,dz \tag{3.30}$$

In (3.30), first we choose $h = -u^- \in W^{1,p}(\Omega)$. Using Corollary 2.4 and (3.28), we obtain

$$\frac{c_1}{p-1}||D\bar{u}^-||^p_p + ||\bar{u}^-||^p_p \leqslant 0 \quad \Longrightarrow \quad \bar{u} \geqslant 0,\ \bar{u} \neq 0\,.$$

Also, on (3.30) we act with $(\bar{u}-u)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned}
&\langle A(\bar{u}), (\bar{u}-u)^+\rangle + \int_\Omega \bar{u}^{p-1}(\bar{u}-u)^+dz + \int_{\partial\Omega}\beta(z)\bar{u}^{p-1}(\bar{u}-u)^+dz\\
&= \int_\Omega [(\eta_0(z)-\epsilon)u^{q-1} - c_8u^{r-1} + u^{p-1}](\bar{u}-u)^+dz \text{ (see (3.28))}\\
&\leqslant \int_\Omega f(z,u)(\bar{u}-u)^+dz + \int_\Omega u^{p-1}(\bar{u}-u)^+dz \text{ (see (3.23) and recall } u \in \operatorname{int} C_+)\\
&= \langle A(u), (\bar{u}-u)^+\rangle + \int_\Omega u^{p-1}(\bar{u}-u)^+dz + \int_{\partial\Omega}\beta(z)u^{p-1}(\bar{u}-u)^+d\sigma \text{ (since } u \in S_+),\\
\Longrightarrow &\int_{\{\bar{u}>u\}} (a(D\bar{u})-a(Du), D\bar{u}-Du)_{\mathbb{R}^N}\,dz + \int_{\{\bar{u}>u\}} (\bar{u}^{p-1}-u^{p-1})(\bar{u}-u)\,dz +\\
&\qquad + \int_{\partial\Omega}\beta(z)(\bar{u}^{p-1}-u^{p-1})(\bar{u}-u)^+d\sigma \leqslant 0,\\
\Longrightarrow &|\{\bar{u}>u\}|_N = 0 \text{ (see Lemma 2.3 and hypothesis } H(\beta)),\\
\Longrightarrow &\bar{u} \in [0,u] = \{w \in W^{1,p}(\Omega) : 0 \leqslant w(z) \leqslant u(z) \text{ for almost all } z \in \Omega\},\ \bar{u} \neq 0\,.
\end{aligned}$$

Then (3.30) becomes

$$\begin{aligned}
&\langle A(\bar{u}), h\rangle + \int_{\partial\Omega}\beta(z)\bar{u}^{p-1}h\,d\sigma = \int_\Omega ((\eta_0(z)-\epsilon)\bar{u}^{q-1} - c_8\bar{u}^{r-1})h\,dz\\
&\qquad\qquad \text{for all } h \in W^{1,p}(\Omega) \text{ (see (3.28))},\\
\Longrightarrow\ &\bar{u} \text{ is a positive solution of (3.24) (see Papageorgiou and Rădulescu [21])},\\
\Longrightarrow\ &\bar{u} = \tilde{u} \in \operatorname{int} C_+ \text{ (see Proposition 3.6)},\\
\Longrightarrow\ &\bar{u} \leqslant u \quad \text{for all } u \in S_+.
\end{aligned}$$

In a similar fashion we show that $v \leqslant \tilde{v}$ for all $v \in S_-$. □

Now we re ready to produce extremal constant sign solutions for problem (1.1), that is the smallest positive and the biggest negative solutions of (1.1).

Proposition 3.8. *Assume that hypotheses $H(a)$, H_1 and $H(\beta)$ hold. Then problem (1.1) has a smallest positive solution $u_* \in \operatorname{int} C_+$ and a biggest negative solution $v_* \in -\operatorname{int} C_+$.*

Proof. First we produce the smallest positive solution.

Since S_+ is downward directed, without any loss of generality, we may assume that

$$||u||_\infty \leqslant M_3 \quad \text{for some } M_3 > 0 \text{ all } u \in S_+. \tag{3.31}$$

From Hu and Papageorgiou [10], p. 178, we know that we can find $\{u_n\}_{n\geqslant 1} \subseteq S_+$ such that

$$\inf S_+ = \inf_{n\geqslant 1} u_n \,.$$

For every $n \geqslant 1$, we have

$$\begin{cases} \langle A(u_n), h\rangle + \displaystyle\int_{\partial\Omega} \beta(z) u_n^{p-1} h \, d\sigma = \int_\Omega f(z,u_n) h \, dz & \text{for all } h \in W^{1,p}(\Omega), \\ \tilde{u} \leqslant u_n & \text{for all } n \geqslant 1. \end{cases} \tag{3.32}$$

Choosing $h = u_n \in W^{1,p}(\Omega)$ in (3.32) and using Corollary 2.4, hypothesis $H(\beta)$, (3.31) and hypothesis H_1 (i), we see that

$$\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega) \text{ is bounded}\,.$$

So, by passing to a suitable subsequence if necessary, we may assume that

$$u_n \overset{w}{\to} u_* \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u_* \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega) \text{ as } n \to \infty\,. \tag{3.33}$$

In (3.32), we choose $h = u_n - u_* \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.33). Then

$$\begin{aligned} &\lim_{n\to\infty} \langle A(u_n), u_n - u_*\rangle = 0 \\ \implies\quad & u_n \to u_* \text{ in } W^{1,p}(\Omega) \text{ as } n \to \infty \text{ (see Proposition 2.8)},\ \tilde{u} \leqslant u_*\,. \end{aligned} \tag{3.34}$$

So, if in (3.32) we pass to the limit as $n \to \infty$ and use (3.34), then

$$\begin{aligned} &\langle A(u_*), h\rangle + \int_{\partial\Omega} \beta(z) u_*^{p-1} h \, d\sigma = \int_\Omega f(z,u_*) h \, dz \quad \text{for all } h \in W^{1,p}(\Omega),\ \tilde{u} \leqslant u_* \\ \implies\quad & u_* \in S_+ \text{ and } u_* = \inf S_+\,. \end{aligned}$$

Similarly we produce the biggest negative solution $v_* \in -\operatorname{int} C_+$ of problem (1.1). □

Now that we have extremal constant sign solutions, we can produce a nodal (sign changing) solution of problem (1.1). This requires a strengthening of the condition on the reaction $f(z,\cdot)$ near zero. Nevertheless, the new stronger requirement does not alter the overall geometry of the problem.

The new hypotheses on $f(z,x)$, are the following:

(H_2) $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $f(z,0) = 0$ for almost all $z \in \Omega$, hypotheses H_2 (i), (ii), and (iii) are the same as the corresponding hypotheses H_1 (i), (ii), and (iii), and

(iv) $\tilde{c}\,\hat{\lambda}_2(q,\tilde{\beta}) < \liminf\limits_{x\to 0} \frac{f(z,x)}{|x|^{q-2}x}$ uniformly for almost all $z \in \Omega$.

In what follows, $u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$ are the extremal constant sign solutions of problem (1.1) produced in Proposition 3.8. Using them in the next proposition, we produce a nodal solution.

Proposition 3.9. *If hypotheses $H(a)$, H_2 and $H(\beta)$ hold, then problem (1.1) admits a nodal solution $y_0 \in [v_*, u_*] \cap C^1(\overline{\Omega})$.*

Proof. Let $u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$ be the two extremal constant sign solutions produced in Proposition 3.8. We introduce the following modifications of the reaction $f(z,x)$ and the boundary term $\beta(z)|x|^{p-2}x$:

$$k(z,x) = \begin{cases} f(z,v_*(z)) + |v_*(z)|^{p-2}v_*(z) & \text{if } x < v_*(z), \\ f(z,x) + |x|^{p-2}x & \text{if } v_*(z) \leqslant x \leqslant u_*(z), \\ f(z,u_*(z)) + u_*(z)^{p-1} & \text{if } u_*(z) < x \end{cases} \tag{3.35}$$

for all $(z,x) \in \Omega \times \mathbb{R}$;

$$b(z,x) = \begin{cases} \beta(z)|v_*(z)|^{p-2}v_*(z) & \text{if } x < v_*(z), \\ \beta(z)|x|^{p-2}x & \text{if } v_*(z) \leqslant x \leqslant u_*(z), \\ \beta(z)u_*(z)^{p-1} & \text{if } u_*(z) < x \end{cases} \tag{3.36}$$

for all $(z,x) \in \partial\Omega \times \mathbb{R}$.

We also consider the positive and negative truncations of $k(z,\cdot)$ and $b(z,\cdot)$:

$$k_\pm(z,x) = k(z,\pm x^\pm) \quad \text{and} \quad b_\pm(z,x) = b(z,\pm x^\pm).$$

All these functions are Carathéodory. We set

$$K(z,x) = \int_0^x k(z,s)\,ds, \quad K_\pm(z,x) = \int_0^x k_\pm(z,s)\,ds,$$
$$B(z,x) = \int_0^x b(z,s)\,ds, \quad B_\pm(z,x) = \int_0^x b_\pm(z,s)\,ds.$$

We introduce the C^1-functionals $\psi, \psi_\pm\colon W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi(u) = \int_\Omega G(Du)\,dz + \frac{1}{p}||u||_p^p + \int_{\partial\Omega} B(z,u)\,d\sigma - \int_\Omega K(z,u)\,dz,$$
$$\psi_\pm(u) = \int_\Omega G(Du)\,dz + \frac{1}{p}||u||_p^p + \int_{\partial\Omega} B_\pm(z,u)\,d\sigma - \int_\Omega K_\pm(z,u)\,dz,$$

for all $u \in W^{1,p}(\Omega)$.

In what follows, we use the following three order intervals in $W^{1,p}(\Omega)$:

$$I = [v_*, u_*] = \{u \in W^{1,p}(\Omega) : v_*(x) \leqslant u(z) \leqslant u_*(z) \text{ for almost all } z \in \Omega\},$$
$$I_+ = [0, u_*] = \{u \in W^{1,p}(\Omega) : 0 \leqslant u(z) \leqslant u_*(z) \text{ for almost all } z \in \Omega\},$$
$$I_- = [v_*, 0] = \{u \in W^{1,p}(\Omega) : v_*(z) \leqslant u(z) \leqslant 0 \text{ for almost all } z \in \Omega\}.$$

Claim 1.

$$K_\psi \subseteq I, \ K_{\psi_+} = \{0, u_*\}, \ K_{\psi_-} = \{v_*, 0\}.$$

Let $u \in K_\psi$. Then

$$\langle A(u), h\rangle + \int_\Omega |u|^{p-2}uh\,dz + \int_{\partial\Omega} b(z,u)h d\sigma = \int_\Omega k(z,u)h\,dz. \tag{3.37}$$

In (3.37), first we choose $h = (u - u_*)^+ \in W^{1,p}(\Omega)$. Using (3.35) and (3.36), we have

$$\begin{aligned}
&\langle A(u), (u-u_*)^+\rangle + \int_\Omega u^{p-1}(u-u_*)^+ dz + \int_{\partial\Omega} \beta(z)u_*^{p-1}(u-u_*)^+ d\sigma \\
&\qquad = \int_\Omega [f(z,u_*) + u_*^{p-1}](u-u_*)^+ dz \\
&\qquad = \langle A(u_*), (u-u_*)^+\rangle + \int_\Omega u_*^{p-1}(u-u_*)^+ dz \\
&\qquad\quad + \int_{\partial\Omega} \beta(z)u_*^{p-1}(u-u_*)^+ d\sigma \ (\text{since } u_* \in S_+), \\
\Longrightarrow &\int_{\{u>u_*\}} (a(Du)-a(Du_*), Du-Du_*)_{\mathbb{R}^N} dz + \int_{\{u>u_*\}} (u^{p-1}-u_*^{p-1})(u-u_*)\,dz = 0, \\
\Longrightarrow \ &|\{u > u_*\}|_N = 0 \ (\text{ see Lemma 2.3}), \quad \text{hence } u \leqslant u_*.
\end{aligned}$$

If in (3.37) we choose $h = (v_* - v)^+ \in W^{1,p}(\Omega)$, then reasoning in a similar way, we obtain $v_* \leqslant u$. Therefore we conclude that

$$K_\psi \subseteq I = [v_*, u_*].$$

In a similar way, we show that

$$K_{\psi_+} \subseteq I_+ = [0, u_*] \quad \text{and} \quad K_{\psi_-} \subseteq I = [v_*, 0].$$

The extremality of $u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$ (see Proposition 3.8), implies that

$$K_{\psi_+} = \{0, u_*\} \quad \text{and} \quad K_{\psi_-} = \{0, v_*\}.$$

This proves Claim 1.

Claim 2. *$u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$ are local minimizers of ψ.*

Consider the functional ψ_+. From (3.35), (3.36) and Corollary 2.4, it is clear that ψ_+ is coercive. Also, the Sobolev embedding theorem and the trace theorem imply that ψ_+ is sequentially weakly lower semicontinuous.

So, we can find $\bar{u}_* \in W^{1,p}(\Omega)$ such that

$$\psi_+(\bar{u}_*) = \inf[\psi_+(u) : u \in W^{1,p}(\Omega)] = m_+. \tag{3.38}$$

As in the proof of Proposition 3.5, for $t \in (0,1)$ small, we have $t\hat{u}_1(q,\tilde{\beta}) \leqslant u_*$ (see Lemma 3.3 of Filippakis, Kristaly and Papageorgiou [7]) and then

$$\psi_+(t\hat{u}_1(q,\tilde{\beta})) < 0 \quad \Longrightarrow \quad \psi_+(\bar{u}_*) < 0 = \psi_+(0), \text{ hence } \bar{u}_* \neq 0\,.$$

From (3.38) we have

$$\bar{u}_* \in K_{\psi_+}\backslash\{0\} \quad \Longrightarrow \quad \bar{u}_* = u_* \in \operatorname{int} C_+ \text{ (see Claim 1)}\,.$$

Clearly $\psi|_{C_+} = \psi_+|_{C_+}$. So, $u_* \in \operatorname{int} C_+$ is a local $C^1(\overline{\Omega})$-minimizer of ψ. Therefore, we can use Theorem 2.6 and conclude that $u_* \in \operatorname{int} C_+$ is a local $W^{1,p}(\Omega)$-minimizer of ψ.

Similarly for $v_* \in -\operatorname{int} C_+$ using this time the functional ψ_-. This proves Claim 2.

Without any loss of generality, we may assume that

$$\psi(v_*) \leqslant \psi(u_*). \tag{3.39}$$

The analysis is similar if the opposite inequality holds. Also, we may assume that K_ψ is finite. Indeed, if K_ψ is finite, then Claim 1 and the extremality of u_* and v_* imply that we have an infinity of nodal solutions. So, Claim 2 implies that we can find $\rho \in (0,1)$ small such that

$$\psi(v_*) \leqslant \psi(u_*) < \inf[\psi(u) : ||u - u_*|| = \rho] = m_\rho, \quad ||u_* - v_*|| > \rho \tag{3.40}$$

(see Aizicovici, Papageorgiou and Staicu [1], proof of Proposition 29). The functional ψ is coercive (see (3.35),(3.36) and Corollary 2.4). So, we infer that

$$\psi \text{ satisfies the PS-condition} \tag{3.41}$$

(see Corollary 3.4). Then (3.40) and (3.41) above, permit the use of Theorem 2.1 (the mountain pass theorem) and find $y_0 \in W^{1,p}(\Omega)$ such that

$$y_0 \in K_\psi \quad \text{and} \quad m_\rho \leqslant \psi(y_0). \tag{3.42}$$

From (3.42), (3.40) and Claim 1, we have

$$y_0 \in [v_*, u_*]\backslash\{v_*, u_*\} \quad \Longrightarrow \quad y_0 \text{ is a solution of (1.1) (see (3.35), (3.36))}.$$

So, if we show that $y_0 \neq 0$, then because of the extremality of u_* and v_* we have that y_0 is nodal.

Since y_0 is a critical point of ψ of mountain pass type with reference points $u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$ (see (3.40)), we have

$$\psi(y_0) = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} \psi(\gamma(t)) \tag{3.43}$$

$$\text{where} \qquad \Gamma = \{\gamma \in C([0,1], W^{1,p}(\Omega)) : \gamma(0) = v_*, \gamma(1) = u_*\}$$

According to (3.43), in order to show the nontriviality of y_0, it suffices to produce $\gamma_* \in \Gamma$ such that $\psi|_{\gamma_*} < 0$. In what follows, we construct such a path.

Let $\partial B_1^{L^q} = \{u \in L^q(\Omega) : ||u||_q = 1\}$ and set

$$M = W^{1,q}(\Omega) \cap \partial B_1^{L^q} \quad \text{and} \quad M_c = M \cap C^1(\overline{\Omega}).$$

We introduce the following two sets of paths:

$$\hat{\Gamma} = \{\hat{\gamma} \in C([-1,1], M) : \hat{\gamma}(-1) = -\hat{u}_1(q,\tilde{\beta}), \hat{\gamma}(1) = \hat{u}_1(q,\tilde{\beta})\},$$
$$\hat{\Gamma}_c = \{\hat{\gamma} \in C([-1,1], M_c) : \hat{\gamma}(-1) = -\hat{u}_1(q,\tilde{\beta}), \hat{\gamma}(1) = \hat{u}_1(q,\tilde{\beta})\}.$$

From Papageorgiou and Rădulescu [23], we know that $\hat{\Gamma}_c$ is dense in $\hat{\Gamma}$ for the relative $W^{1,q}(\Omega)$-topology. Then using Proposition 2.7, we see that given $\hat{\gamma} > 0$, we can find $\hat{\gamma}_0 \in \hat{\Gamma}_c$ such that

$$\max_{-1 \leqslant t \leqslant 1} \tilde{c}\, \vartheta(\hat{\gamma}_0(t)) \leqslant \tilde{c}\, \hat{\lambda}_2(q,\tilde{\beta}) + \hat{\delta}, \tag{3.44}$$

where we recall that $\vartheta(u) = ||Du||_q^q + \int_{\partial\Omega} \tilde{\beta}(z)|u|^q d\sigma$ for all $u \in W^{1,q}(\Omega)$.

Hypothesis $H(a)$ (iv) implies that given $\epsilon > 0$, we can find $\delta_1 = \delta_1(\epsilon) > 0$ such that

$$G(y) \leqslant \frac{\tilde{c} + \epsilon}{q} |y|^q \quad \text{for all } y \in \mathbb{R}^N \text{ with } |y| \leqslant \delta_1. \tag{3.45}$$

Also, hypothesis H_2 (iv) implies that we can find $\delta_2 > 0$ and $\xi_0 > \tilde{c}\, \hat{\lambda}_2(q,\tilde{\beta})$ such that

$$\frac{1}{q}\, \xi_0 |x|^q \leqslant F(z,x) \quad \text{for almost all } z \in \Omega, \text{ all } |x| \leqslant \delta_2. \tag{3.46}$$

Let $\delta = \min\{\delta_1, \delta_2, 1\}$. Since $\hat{\gamma}_0 \in \hat{\Gamma}_c$, $u_* \in \operatorname{int} C_+$, $v_* \in -\operatorname{int} C_+$, we can find $\tau \in (0,1)$ small such that, for all $t \in [-1,1]$, all $z \in \overline{\Omega}$,

$$\tau\, \hat{\gamma}_0(t) \in [v_*, u_*], \quad \tau\, |\hat{\gamma}_0(t)(z)| \leqslant \delta, \quad \tau\, |D\hat{\gamma}_0(t)(z)| \leqslant \delta. \tag{3.47}$$

Then for all $t \in [-1,1]$, we have

$$\begin{aligned}
\psi(\tau\hat{\gamma}_0(t)) &= \int_\Omega G(\tau D\hat{\gamma}_0(t))\, dz + \frac{1}{p} \int_{\partial\Omega} \beta(z)|\hat{\gamma}_0(t)|^p d\sigma - \int_\Omega F(z, \tau\hat{\gamma}_0(t))\, dz \\
&\qquad \text{(see (3.35), (3.36) and (3.47))} \\
&\leqslant \frac{\tilde{c} + \epsilon}{q}\, \tau^q\, ||D\hat{\gamma}_0(t)||_q^q + \frac{\tilde{c}\, \tau^q}{q} \int_{\partial\Omega} \tilde{\beta}(z)|\hat{\gamma}_0(t)|^q d\sigma - \frac{\tau^q}{q}\, \xi_0\, ||\hat{\gamma}_0(t)||_q^q \\
&\qquad \text{(see (3.45), (3.46) and recall that } q \leqslant p, \delta \leqslant 1) \\
&\leqslant \frac{\tau^q}{q} \left[(\tilde{c}\, \hat{\lambda}_2(q,\tilde{\beta}) + \hat{\delta}) + \epsilon - \xi_0\right] \\
&\qquad \text{(see (3.44) and recall that } ||\hat{\gamma}_0(t)||_q = 1).
\end{aligned}$$

Since $\xi_0 > \tilde{c}\, \hat{\lambda}_2(q,\tilde{\beta})$ and $\epsilon, \hat{\delta} > 0$ are arbitrary, we can choose them small so that

$$\psi(\tau\hat{\gamma}_0(t)) < 0 \quad \text{for all } t \in [-1,1].$$

Let $\gamma_0 = \tau\hat{\gamma}_0$. This is a continuous path in $W^{1,p}(\Omega)$ connecting $-\tau\hat{u}_1(q,\tilde{\beta})$ and $\tau\hat{u}_1(q,\tilde{\beta})$ and have

$$\psi|_{\gamma_0} < 0. \tag{3.48}$$

Let $m_+ \in \mathbb{R}$ be as in (3.38). We have see that

$$\psi_+(u_*) = m_+ < 0 = \psi_+(0). \tag{3.49}$$

Invoking the second deformation theorem (see, for example, Gasinski and Papageorgiou [8], p. 628), we can find a deformation $h\colon [0,1] \times (\psi_+^0 \backslash K_{\psi_+}^0) \to \psi_+^0$ such that

$$h(0,u) = u \quad \text{for all } u \in \psi_+^0 \backslash K_{\psi_+}^0, \tag{3.50}$$

$$h(1, \psi_+^0 \backslash K_{\psi_+}^0) \subseteq \psi_+^{m_+}, \tag{3.51}$$

$$\psi_+(h(t,u)) \leqslant \psi_+(h(s,u)) \quad \text{for all } (t,s) \in [0,1], s \leqslant t, \text{ all } u \in \psi_+^0 \backslash K_{\psi_+}^0. \tag{3.52}$$

Since $u_* \in K_{\psi_+}$, from Claim 1 and (3.49), we se that $\psi_+^{m_+} = \{u_*\}$. Also,

$$\begin{aligned} &\psi_+(\tau\hat{u}_1(q,\tilde{\beta})) = \psi(\tau\hat{u}_1(q,\tilde{\beta})) = \psi(\gamma_0(1)) < 0 \quad (\text{see (3.48)}) \\ \Longrightarrow \quad &\tau\,\hat{u}_1(q,\tilde{\beta}) \in \psi_+^0 \backslash K_{\psi_+}^0 = \psi_+^0 \backslash \{0\}. \end{aligned}$$

Therefore we can define

$$\gamma_+(t) = h(t, \tau\hat{u}_1(q,\tilde{\beta}))^+ \quad \text{for all } t \in [0,1]\,.$$

This is a continuous path in $W^{1,p}(\Omega)$. we have

$$\begin{aligned} &\gamma_+(0) = \tau\hat{u}_1(q,\tilde{\beta}) \text{ (see (3.50) and recall that } \hat{u}_1(q,\tilde{\beta}) \in \operatorname{int} C_+), \\ &\gamma_+(1) = h(1, \tau\hat{u}_1(q,\tilde{\beta}))^+ = u_* \text{ (see (3.51) and recall that } \psi_+^{m_+} = \{u_*\}). \end{aligned}$$

So, the continuous path in $W^{1,p}(\Omega)$ connects $\tau\hat{u}_1(q,\tilde{\beta})$ and u_*. Moreover, for all $t \in [0,1]$, we have

$$\begin{aligned} \psi(\gamma_+(t)) &= \psi(h(t, \tau\hat{u}_1(q,\tilde{\beta}))^+) \\ &= \psi_+(h(t, \tau\hat{u}_1(q,\tilde{\beta}))) \text{ (since } \psi|_{C_+} = \psi|_{C_+}) \\ &\leqslant \psi_+(\tau\hat{u}_1(q,\tilde{\beta})) \text{ (see (3.52))} \\ &= \psi(\tau\hat{u}_1(q,\tilde{\beta})) \text{ (since } \psi_+|_{C_+} = \psi|_{C_+}) \\ &< 0 \text{ (see (3.48))}, \\ \Longrightarrow \quad &\psi|_{\gamma_+} < 0\,. \end{aligned} \tag{3.53}$$

In a similar fashion we produce a continuous path in $W^{1,p}(\Omega)$ connecting $-\tau\hat{u}_1(q,\tilde{\beta})$ and v_* for which we have

$$\psi|_{\gamma_-} < 0. \tag{3.54}$$

We concatenate $\gamma_0, \gamma_+, \gamma_-$ and generate $\gamma_* \in \gamma$ such that

$$\psi|_{\gamma_*} < 0 \text{ (see (3.48), (3.53), (3.54))} \quad \Longrightarrow \quad y_0 \neq 0\,.$$

So, $y_0 \in C^1(\overline{\Omega})$ (nonlinear regularity theory, see Lieberman [14]) is a nodal solution of (1.1). □

Therefore, we can state our first multiplicity theorem for problem (1.1) (resonant problems).

Theorem 3.10. *Assume that hypotheses $H(a)$, H_2 and $H(\beta)$ hold. Then problem (1.1) has at least three nontrivial solutions*

$$u_0 \in \operatorname{int} C_+, \quad v_0 \in -\operatorname{int} C_+ \quad \textit{and} \quad y_0 \in [v_0, u_0] \cap C^1(\overline{\Omega}) \textit{ nodal.}$$

Remark 3.11. Three solutions theorems for coercive problems were also proved by Liu [15] and Liu and Liu [16] (Dirichlet problems driven by the p-Laplacian), and Kyritsi and Papageorgiou [11] (Neumann problems driven by p-Laplacian). However, none of the aforementioned works allows for resonance to occur. Also, they do not obtain nodal solutions, neither extremal constant sign solutions.

4. Semilinear problems

In this section we deal with the semilinear problem (that is, $a(y) = y$ for all $y \in \mathbb{R}^N$). Under stronger regularity conditions on the reaction $f(z,\cdot)$, we can improve Theorem 3.10 and produce a second nodal solution for a total of four nontrivial solutions.

The problem under consideration is the following:

$$-\Delta u(z) = f(z, u(z)) \text{ in } \Omega\,, \quad \frac{\partial u}{\partial n} + \beta(z)u(z) = 0 \text{ on } \partial\Omega\,. \tag{4.1}$$

The hypotheses on the reaction $f(z,x)$, are the following:

(H_3) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for almost all $z \in \Omega$, $f(z,0) = 0$, $f(z,\cdot) \in C^1(\mathbb{R})$ and

(i) $|f'_x(z,x)| \leqslant a(z)(1 + |x|^{r-2})$ for almost all $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$ and $2 < r < 2^*$;

(ii) $\limsup_{x \to \pm\infty} f(z,x)/x \leqslant \hat{\lambda}_1(2,\beta)$ uniformly for almost all $z \in \Omega$;

(iii) $\lim_{x \to \pm\infty} [f(z,x)x - 2F(z,x)] = +\infty$ uniformly for almost all $z \in \Omega$;

(iv) there exist integer $m \geqslant 2$ and $\delta_0 > 0$ such that

$$f'_x(z,0) = \lim_{x \to 0} \frac{f(z,x)}{x} \leqslant \hat{\lambda}_{m+1}(2,\beta) \text{ uniformly for almost all } z \in \Omega,$$
$$f'_x(\cdot,0) \not\equiv \hat{\lambda}_{m+1}(2,\beta),$$
$$F(z,x) \geqslant \frac{\hat{\lambda}_m(2,\beta)}{2}\, x^2 \text{ for almost all } z \in \Omega, \text{ all } |x| \leqslant \delta_0\,.$$

Remark 4.1. Hypothesis H_3 (i) implies that given $\rho > 0$, we can find $\xi_\rho > 0$ such that for almost all $z \in \Omega$, the function $x \mapsto f(z,x) + \xi_\rho x$ is nondecreasing on $[-\rho, \rho]$. Note that now we have weakened a little the condition near zero (see hypothesis H_3 (iv) and compare with hypothesis H_2 (iv), where the inequality is strict with respect to $\hat{\lambda}_2(2,\beta)$). The reason is the extra regularity structure on $f(z,\cdot)$ and the semilinearity of the problem. A careful reading of the proof of Proposition 3.9 reveals that the strict inequality in hypothesis H_2 (iv) was used in order to be able to apply Proposition 2.7 and conclude that $y_0 \neq 0$, therefore y_0 is nodal. In the present semilinear smooth case, this can be avoided and instead use critical groups. Indeed, as we explain in detail in the proof of the next result (Theorem 4.2), the energy functional φ has a local linking at the origin with respect to the orthogonal direct sum

$$H^1(\Omega) = \bar{H}_m \oplus \hat{H}_{m+1}, \quad \text{where } \bar{H}_m = \oplus_{i=1}^m E(\hat{\lambda}_i(2,\beta)), \quad \hat{H}_{m+1} = \bar{H}_m^\perp,$$

and so

$$\begin{aligned} &C_k(\varphi,0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0, d_m = \dim \bar{H}_m \text{ (see [28])} \\ \Longrightarrow \quad &C_k(\psi,0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \text{ and } d_m \geqslant 2 \text{ (see [19])}. \end{aligned} \tag{4.2}$$

On the other hand, y_0 is a critical point of mountain pass-type of ψ, hence

$$C_k(\psi, y_0) \neq 0\,. \tag{4.3}$$

From (4.2) and (4.3) we have that y_0 is nontrivial, hence nodal.

Theorem 4.2. *Assume that hypotheses H_3 and $H(\beta)$ hold. Then problem (4.1) admits at least four nontrivial solutions*

$$u_0 \in \operatorname{int} C_+, \quad v_0 \in -\operatorname{int} C_+ \quad \textit{and} \quad y_0, \hat{y} \in \operatorname{int}_{C^1(\overline{\Omega})}[v_0, u_0] \textit{ nodal.}$$

Proof. From Theorem 3.10, we already have three nontrivial solutions,

$$u_0 \in \operatorname{int} C_+, \quad v_0 \in -\operatorname{int} C_+ \quad \text{and} \quad y_0 \in [v_0, u_0] \cap C^1(\overline{\Omega}) \text{ nodal.}$$

Of course we may assume that u_0 and v_0 are extremal (see Proposition 3.9). Let $\rho = \max\{||u_0||_\infty, ||v_0||_\infty\}$ and let $\xi_\rho > 0$ be as postulated by the previous remark. Then

$$\begin{aligned} \Delta y_0(z) + \xi_\rho y_0(z) &= f(z, y_0(z)) + \xi_\rho y_0(z) \leqslant f(z, u_0(z)) + \xi_\rho u_0(z) \text{ (since } y_0 \leqslant u_0) \\ &= -\Delta u_0(z) + \xi_\rho u_0(z) \text{ for almost all } z \in \Omega, \\ \Longrightarrow \quad &\Delta(u_0 - y_0)(z) \leqslant \xi_\rho (u_0 - y_0)(z) \quad \text{for almost all } z \in \Omega \ (u_0 - y_0 \in C_+ \backslash \{0\}), \\ \Longrightarrow \quad &u_0 - y_0 \in \operatorname{int} C_+ \text{ (by the maximum principle).} \end{aligned}$$

Similarly, we show that

$$y_0 - v_0 \in \operatorname{int} C_+\,.$$

Therefore we have proved that

$$y_0 \in \operatorname{int}_{C^1(\overline{\Omega})}[v_0, u_0]\,. \tag{4.4}$$

As before, $\varphi\colon H^1(\Omega) \to \mathbb{R}$ is the energy functional for problem (4.1). So

$$\varphi(u) = \frac{1}{2}||Du||_2^2 + \frac{1}{2}\int_{\partial\Omega} \beta(z)|u|^2 d\sigma - \int_\Omega F(z,u)\,dz \quad \text{for all } u \in H^1(\Omega).$$

Hypotheses H_3 imply that $\varphi \in C^2(H^1(\Omega))$. Also, let $\psi \in C^1(H^1(\Omega))$ be the truncated functional as in the proof of Proposition 3.9. From (3.35) and (3.36), we have

$$\varphi|_{[v_0,u_0]} = \psi|_{[v_0,u_0]}.$$

So, (4.4) implies that

$$\begin{aligned} &C_k(\varphi|_{C^1(\overline{\Omega})}, y_0) = C_k(\psi|_{C^1(\overline{\Omega})}, y_0) \quad \text{for all } k \geqslant 0 \\ \Longrightarrow \quad &C_k(\varphi, y_0) = C_k(\psi, y_0) \quad \text{for all } k \geqslant 0 \\ &\qquad \text{(since } C^1(\overline{\Omega}) \text{ is dense in } H^1(\Omega), \text{ see Palais [19])}. \end{aligned} \tag{4.5}$$

Recall that y_0 is a critical point of ψ of mountain pass type (see the proof of Proposition 3.9). Hence

$$\begin{aligned} C_1(\psi, y_0) \neq 0 \quad &\Longrightarrow \quad C_1(\varphi, y_0) \neq 0 \text{ (see (4.5))} \\ &\Longrightarrow \quad C_k(\varphi, y_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geqslant 0 \\ &\qquad\qquad \text{(see Proposition 2.5 in Bartsch [5])} \\ &\Longrightarrow \quad C_k(\psi, y_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geqslant 0 \text{ (see (4.5))}. \end{aligned} \tag{4.6}$$

Let $\bar{H}_m = \oplus_{i=1}^m E(\hat{\lambda}_i(2,\beta))$ and $\hat{H}_{m+1} = \bar{H}_m^1 = \overline{\oplus_{i\geqslant m+1} E(\hat{\lambda}_i(2,\beta))}$ (here $E(\hat{\lambda}_i(2,\beta)) \subseteq C^1(\overline{\Omega})$ is the eigenspace corresponding to the eigenvalue $\hat{\lambda}_i(2,\beta)$). We have

$$H^1(\Omega) = \bar{H}_m \oplus \hat{H}_{m+1}. \tag{4.7}$$

From hypothesis H_3 (iv) we see that given $\epsilon > 0$, we can find $c_9 = c_9(\epsilon) > 0$ such that

$$F(z,x) \leqslant \frac{1}{2}(f_x'(z,0) + \epsilon)x^2 + c_9|x|^r \quad \text{for almost all } z \in \Omega, \text{ all } z \in \mathbb{R}.$$

If $u \in \hat{H}_{m+1}$, then

$$\begin{aligned} \varphi(u) &\geqslant \frac{1}{2}||Du||_2^2 + \frac{1}{2}\int_{\partial\Omega} \beta(z)\,u^2\,d\sigma - \frac{1}{2}\int_\Omega f_x'(z,0)\,u^2\,dz - \frac{\epsilon}{2}\,||u||_2^2 - c_{10}\,||u||^r \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for some } c_{10} > 0 \\ &\geqslant c_{11}||u||^2 - c_{10}||u||^r \text{ for some } c_{11} > 0 \text{ (see hypothesis } H_3 \text{ (iv))}. \end{aligned}$$

Since $r > 2$, we can find $\delta_1 \in (0,1)$ such that

$$\varphi(u) > 0 \quad \text{for all } u \in \hat{H}_{m+1} \text{ with } 0 < ||u|| \leqslant \delta_1. \tag{4.8}$$

Note that $\bar{H}_m \subseteq C^1(\overline{\Omega})$ is finite dimensional and so all norms are equivalent. So, we can find $\delta_2 > 0$ such that

$$u \in \bar{H}_m, ||u|| \leqslant \delta_2 \quad \Longrightarrow \quad |u(z)| \leqslant \delta_0 \quad \text{for all } z \in \overline{\Omega}. \tag{4.9}$$

Hence if $u \in \bar{H}_m$, $||u|| \leqslant \delta_2$, then

$$\varphi(u) \leqslant \frac{1}{2}||Du||_2^2 + \frac{1}{2}\int_{\partial\Omega} \beta(z)u^2 d\sigma - \frac{\hat{\lambda}_m(2,\beta)}{2}||u||_2^2 \leqslant 0 \tag{4.10}$$

(see (4.8) and hypothesis H_3 (iv)).

From (4.8) and (4.10) we infer that φ has a local linking with respect to the orthogonal direct sum decomposition (4.7). Since $\varphi \in C^2(H^1(\Omega))$, from Proposition 2.3 of Su [28] we have

$$\begin{aligned} &C_k(\varphi, 0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \geqslant 0, \text{ with } d_m = \dim \bar{H}_m \geqslant 2 \\ \implies \quad &C_k(\psi, 0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \geqslant 0\,. \end{aligned} \tag{4.11}$$

Recall that u_0, v_0 are local minimizers of ψ (see the proof of Proposition 3.9, Claim 2). Hence

$$C_k(\psi, u_0) = C_k(\psi, v_0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geqslant 0\,. \tag{4.12}$$

Finally recall that ψ is coercive (see (3.35), (3.36)). Therefore

$$C_k(\psi, \infty) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geqslant 0\,. \tag{4.13}$$

Suppose that $K_\psi = \{0, u_0, v_0, y_0\}$. Then from (4.6), (4.11), (4.12), (4.13) and the Morse relation (see (2.6)) with $t = -1$, we have

$$(-1)^{dm} + 2(-1)^0 + (-1)^1 = (-1)^0 \quad \Longrightarrow \quad (-1)^{dm} = 0, \text{ a contradiction.}$$

So, there exists $\hat{y} \in K_\psi \backslash \{0, u_0, v_0, y_0\}$. We have

$$\begin{aligned} &\hat{y} \in [v_0, u_0] \text{ (see the proof of Proposition 3.9, Claim 1)} \\ \implies \quad &\hat{y} \in C^1(\overline{\Omega}) \text{ (regularity theory) is a nodal solution of (4.1).} \end{aligned}$$

Moreover, as we did earlier for y_0, we show that

$$\hat{y} \in \operatorname{int}_{C^1(\overline{\Omega})}[v_0, u_0]\,. \qquad \square$$

5. Oscillatory reaction

In this section we return to the study of problem (1.1) and we consider a reaction with no global growth restriction. Instead, we assume a kind of oscillatory behavior for $f(z,\cdot)$ near zero. Also, we weaken the conditions on the map $a(\cdot)$.

The new hypotheses on the map $a(y)$ and the reaction $f(z,x)$ are the following:

$(H(a)')$ $a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$, hypotheses $H(a)'$ (i), (ii), (iii) are the same as hypotheses $H(a)$ (i), (ii), (iii), and

(iv) if $G_0(t) = \int_0^t s a_0(s)\,ds$, then there exists $q \in (1, p]$ such that

$$\limsup_{t \to 0^+} \frac{q\,G_0(t)}{t^q} < +\infty.$$

and

(H_4) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for almost all $z \in \Omega$, and

(i) for every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)_+$ such that $|f(z,x)| \leqslant a_\rho(z)$ for almost all $z \in \Omega$, all $|x| \leqslant \rho$;

(ii) there exist functions $w_\pm \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ such that

$$w_-(z) \leqslant c_- < 0 < c_+ \leqslant w_+(z) \quad \text{for all } z \in \overline{\Omega},$$
$$A(w_-) + \beta(z)|w_-|^{p-2}w_- \leqslant 0 \leqslant A(w_+) + \beta(z)w_+^{p-1} \text{ in } W^{1,p}(\Omega)^*,$$
$$f(z, w_+(z)) \leqslant 0 \leqslant f(z, w_-(z)) \quad \text{for all } z \in \Omega;$$

(iii) there exist $\mu \in (1,q)$ and $\delta > 0$ such that

$$\mu F(z,x) \geqslant f(z,x)x > 0 \text{ for almost all } z \in \Omega, \text{ as } 0 < |x| \leqslant \delta.$$

Remark 5.1. We stress that no global condition is imposed on $f(z,\cdot)$. In hypothesis H_4 (ii), the second inequality means that

$$\langle A(w_-), h\rangle + \int_{\partial\Omega} \beta(z)|w_-|^{p-2}w_- h\, d\sigma \leqslant 0 \leqslant \langle A(w_+), h\rangle + \int_{\partial\Omega} \beta(z)w_+^{p-1} h\, d\sigma$$

for all $h \in W^{1,p}(\Omega)$ with $h \geqslant 0$. Evidently, hypothesis H_4 (ii) is satisfied if there exist $c_- < 0 < c_+$ such that

$$f(z,c_+) \leqslant 0 \leqslant f(z,c_-) \quad \text{for almost all } z \in \Omega.$$

Hypotheses H_4 (ii), (iii) dictate a kind of oscillatory behavior for $f(z,\cdot)$ near zero.

Example 5.2. The following function satisfies hypotheses H_4. For the sake of simplicity, we drop the z-dependence:

$$f(x) = \begin{cases} |x|^{\mu-2}x - |x|^{r-2}x & \text{if } |x| \leqslant 1, \\ \eta(x) & \text{if } |x| > 1, \end{cases}$$

where $\eta \in C^1(\mathbb{R})$ with $\eta(\pm 1) = 0$.

Proposition 5.3. *Assume that hypotheses $H(a)'$, H_4 and $H(\beta)$ hold. Then problem (1.1) admits at least two nontrivial constant sign solutions*

$$u_0 \in [0, w_+] \cap \operatorname{int} C_+ \quad \text{and} \quad v_0 \in [w_-, 0] \cap (-\operatorname{int} C_+).$$

Proof. First we produce the positive solution.

To this end we introduce the following truncation-perturbation of $f(z,x)$:

$$\hat{e}_+(z,x) = \begin{cases} 0 & \text{if } x < 0, \\ f(z,x) + x^{p-1} & \text{if } 0 \leqslant x \leqslant w_+(z), \\ f(z,w_+(z)) + w_+(z)^{p-1} & \text{if } w_+(z) < x. \end{cases} \tag{5.1}$$

This is a Carathéodory function. We set $\hat{E}_+(z,x) = \int_0^x \hat{e}_+(z,s)\,ds$ and consider the C^1-functional $\hat{\tau}_+ : W^{1,p}(\Omega) \to \mathbb{R}$ defined, for all $u \in W^{1,p}(\Omega)$, by

$$\hat{\tau}_+(u) = \int_\Omega G(Du)\,dz + \frac{1}{p}||u||_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)(u^+)^p d\sigma - \int_\Omega \hat{E}_+(z,u)\,dz.$$

From (5.1) it is clear that $\hat{\tau}_+$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\hat{\tau}_+(u_0) = \inf[\hat{\tau}_+(u) : u \in W^{1,p}(\Omega)]\,. \tag{5.2}$$

Hypothesis H_4 (iii) implies that

$$F(z,x) \geqslant c_{12}\,|x|^\mu \quad \text{for almost all } z \in \Omega,\ \text{all } |x| \leqslant \delta \text{ with } c_{12} > 0\,. \tag{5.3}$$

Also, hypothesis $H(a)'$ (iv) implies that we can find $\hat{\delta} \in (0, \min\{\delta, c_+\})$ and $c_{13} > 0$ such that

$$G(y) \leqslant \frac{c_{13}}{q}\,|y|^q \quad \text{for all } y \in \mathbb{R}^N \text{ with } |y| \leqslant \hat{\delta}. \tag{5.4}$$

Let $t \in (0,1)$ small such that $t\hat{u}_1(q,\beta)(z) \leqslant \hat{\delta}$ for all $z \in \overline{\Omega}$ (recall $\hat{u}_1(q,\beta) \in \operatorname{int} C_+$). Then from (5.3) and (5.4) we have

$$\hat{\tau}_+(t\hat{u}_1(q,\beta)) \leqslant \frac{t^q c_{13}}{q}||D\hat{u}_1(q,\beta)||_q^q + \frac{t^q}{q}\int_{\partial\Omega} \beta(z)\hat{u}_1(q,\beta)^p d\sigma - c_{12}t^\mu||\hat{u}_1(q,\beta)||_\mu^\mu$$

(see (5.1)).

Since $\mu < q \leqslant p$ (see H_4 (iv)), choosing $t \in (0,1)$ even smaller if necessary, we will have

$$\hat{\tau}_+(t\hat{u}_1(q,\beta)) < 0 \quad \Longrightarrow \quad \hat{\tau}_+(u_0) < 0 = \hat{\tau}_+(0) \text{ (see (5.2))}, \text{ so } u_0 \neq 0\,.$$

From (5.2) we have $\hat{\tau}'_+(u_0) = 0$, and so, for all $h \in W^{1,p}(\Omega)$,

$$\langle A(u_0), h\rangle + \int_\Omega |u_0|^{p-2}u_0 h\,dz + \int_{\partial\Omega} \beta(z)(u_0^+)^{p-1}h\,d\sigma = \int_\Omega \hat{e}_+(z,u_0)h\,dz. \tag{5.5}$$

In (5.5) we choose $h = -u_0^- \in W^{1,p}(\Omega)$. Then

$$\frac{c_1}{p-1}||Du_0^-||_p^p + ||u_0^-||_p^p \leqslant 0 \text{ (see Corollary 2.4 and (5.1))} \quad \Longrightarrow \quad u_0 \geqslant 0, u_0 \neq 0\,.$$

Also, in (5.5) we choose $h = (u_0 - w_+)^+ \in W^{1,p}(\Omega)$. We have

$$\begin{aligned}
&\langle A(u_0), (u_0 - w_+)^+ \rangle + \int_\Omega u_0^{p-1}(u_0 - w_+)^+ dz + \int_{\partial\Omega} \beta(z) u_0^{p-1}(u_0 - w_+)^+ d\sigma \\
&= \int_\Omega \hat{e}_+(z, u_0)(u_0 - w_+)^+ dz \\
&= \int_\Omega [f(z, w_+) + w_+^{p-1}](u_0 - w_+)^+ dz \text{ (see (5.1))} \\
&\leqslant \langle A(w_+), (u_0 - w_+)^+ \rangle + \int_\Omega w_+^{p-1}(u_0 - w_+)^+ dz + \int_{\partial\Omega} \beta(z) w_+^{p-1}(u_0 - w_+)^+ d\sigma \\
&\qquad\qquad \text{(see hypothesis } H_4 \text{ (ii))} \\
&\Longrightarrow \quad \langle A(u_0) - A(w_+), (u_0 - w_+)^+ \rangle + \int_\Omega (u_0^{p-1} - w_+^{p-1})(u_0 - w_+)^+ dz \\
&\qquad\qquad + \int_{\partial\Omega} \beta(z)(u_0^{p-1} - w_+^{p-1})(u_0 - w_+)^+ d\sigma \leqslant 0 \\
&\Longrightarrow \quad |\{u_0 > w_+\}|_N = 0 \text{ (see Lemma 2.3), hence } u_0 \leqslant w_+.
\end{aligned}$$

So, we have proved that

$$u_0 \in [0, w_+] = \{u \in W^{1,p}(\Omega) : 0 \leqslant u(z) \leqslant w_+(z) \text{ for almost all } z \in \Omega\} \tag{5.6}$$

From (5.1) and (5.6), equation (5.5) becomes

$$\langle A(u_0), h \rangle + \int_{\partial\Omega} \beta(z)\, u_0^{p-1}\, h\, d\sigma = \int_\Omega f(z, u_0)\, h\, dz \quad \text{for all } h \in W^{1,p}(\Omega),$$

and this implies

$$\begin{aligned}
&-\operatorname{div} a(Du_0(z)) = f(z, u_0(z)) \quad \text{for almost all } z \in \Omega, \\
&\frac{\partial u_0}{\partial n_a} + \beta(z)\, u_0^{p-1} = 0 \quad \text{on } \partial\Omega \text{ (see Papageorgiou and Rădulescu [21])}.
\end{aligned} \tag{5.7}$$

Hypotheses H_4 (i), (iii) imply that given $\rho > 0$, we can find $\xi_\rho > 0$ such that

$$f(z, x)x + \xi_\rho |x|^p \geqslant 0 \quad \text{for almost all} z \in \Omega, \text{ all } |x| \leqslant \rho. \tag{5.8}$$

From (5.7) and the nonlinear regularity theory of Lieberman [14], p. 320, we have $u_0 \in C_+ \backslash \{0\}$. Let $\rho = ||u_0||_\infty$ and let $\xi_\rho > 0$ be as in (5.8). Then from (5.7), we have

$$\begin{aligned}
&\operatorname{div} a(Du_0(z)) \leqslant \xi_\rho u_0(z)^{p-1} \text{ for almost all } z \in \Omega \\
&\Longrightarrow \quad u_0 \in \operatorname{int} C_+ \text{ (see Pucci and Serrin [27], pp. 111, 120)}.
\end{aligned}$$

For the negative solution, we introduce the Carathéodory function

$$\hat{e}_-(z, x) = \begin{cases} f(z, w_-(z)) + |w_-(z)|^{p-2} w_-(z) & \text{if } x < w_-(z), \\ f(z, x) + |x|^{p-2}x & \text{if } w_-(z) \leqslant x \leqslant 0, \\ 0 & \text{if } 0 < x. \end{cases} \tag{5.9}$$

We set $\hat{E}_-(z,x) = \int_0^x \hat{e}_-(z,s)\,ds$, and consider the C^1-functional $\hat{\tau}_- : W^{1,p}(\Omega) \to \mathbb{R}$ defined, for all $u \in W^{1,p}(\Omega)$, by

$$\hat{\tau}_-(u) = \int_\Omega G(Du)\,dz + \frac{1}{p}||u||_p^p - \frac{1}{p}\int_{\partial\Omega} \beta(z)(u^-)^p d\sigma - \int_\Omega \hat{E}_-(z,u)\,dz.$$

Working as above with the functional $\hat{\tau}_-$ and suing (5.9), we produce a negative solution $v_0 \in -\operatorname{int} C_+$. □

We introduce the following Carathéodory function:

$$\hat{e}(z,x) = \begin{cases} f(z,w_-(z)) + |w_-(z)|^{p-2}w_-(z) & \text{if } x < w_-(z), \\ f(z,x) + |x|^{p-2}x & \text{if } w_-(z) \leqslant x \leqslant w_+(z), \\ f(z,w_+(z)) + w_+(z)^{p-1} & \text{if } w_+(z) < x\,. \end{cases} \tag{5.10}$$

We set $\hat{E}(z,x) = \int_0^x \hat{e}(z,s)\,ds$ and consider the C^1-functional $\hat{\tau} : W^{1,p}(\Omega) \to \mathbb{R}$ defined, for all $u \in W^{1,p}(\Omega)$, by

$$\hat{\tau}(u) = \int_\Omega G(Du)\,dz + \frac{1}{p}||u||_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p d\sigma - \int_\Omega \hat{E}(z,u)\,dz.$$

As in the proof of Proposition 3.9 (see Claim 1), using (5.10), we show that

$$K_{\hat{\tau}} \subseteq [w_-, w_+]. \tag{5.11}$$

Hypothesis H_4 (iii) implies the presence of a concave (that is a p-superlinear as $x \to 0$) term (see 5.3) and this leads to the following result due to Papageorgiou and Rădulescu [23] (the first result in this direction for a more restricted class of functionals, goes back to Moroz [17]).

Proposition 5.4. *If hypotheses $H(a)'$, H_4 and $H(\beta)$ hold, then $C_k(\hat{\tau}, 0) = 0$ for all $k \geqslant 0$.*

Now we are ready for our second multiplicity theorem for problem (1.1). Note that this theorem, we do not provide information concerning the sign of the third solution.

Theorem 5.5. *Assume that hypotheses $H(a)'$, H_4 and $H(\beta)$ hold. Then problem (1.1) admits at least three nontrivial solutions:*

$$u_0 \in [0, w_+] \cap \operatorname{int} C_+, \quad v_0 \in [w_-, 0] \cap (-\operatorname{int} C_+), \quad y_0 \in [w_-, w_+] \cap C^1(\overline{\Omega}).$$

Proof. From Proposition 5.3 we already have two constant sign solutions

$$u_0 \in \operatorname{int}[0, w_+] \cap \operatorname{int} C_+ \quad \text{and} \quad v_0 \in [w_-, 0] \cap (-\operatorname{int} C_+).$$

From the proof of Proposition 5.3 we know that

- u_0 is a minimizer of the functional $\hat{\tau}_+$.
- v_0 is a minimizer of the functional $\hat{\tau}_-$.

From (5.1), (5.9) and (5.10), we see that

$$\hat{\tau}_+|_{C_+} = \hat{\tau}|_{C_+} \quad \text{and} \quad \hat{\tau}_-|_{-C_+} = \hat{\tau}|_{-C_+}.$$

So, $u_0 \in \operatorname{int} C_+$ and $v_0 \in -\operatorname{int} C_+$ are also local $C^1(\overline{\Omega})$-minimizers of $\hat{\tau}$. Theorem 2.6 implies that they are also local $W^{1,p}(\Omega)$-minimizers of $\hat{\tau}$. Without any loss of generality we may assume that $\hat{\tau}(v_0) \leqslant \hat{\tau}(u_0)$ (the analysis is similar if the opposite inequality holds). Also, we assume that $K_{\hat{\tau}}$ is finite or otherwise we already have an infinity of distinct nontrivial solutions for problem (1.1) (see (5.11)). Since u_0 is a local minimizer of $\hat{\tau}$, we can find $\rho \in (0,1)$ small such that

$$\hat{\tau}(v_0) \leqslant \hat{\tau}(u_0) < \inf[\hat{\tau}(u) : ||u - u_0|| = \rho] = m_\rho, ||v_0 - u_0|| > \rho \text{ (see [1])}. \tag{5.12}$$

The functional $\hat{\tau}$ is coercive (see (5.10)) and so we know that it satisfies the PS-condition (see Corollary 3.4). Using this fact and (5.12), we see that we can apply Theorem 2.1 (the mountain pass theorem) and produce $y_0 \in W^{1,p}(\Omega)$ such that

$$y_0 \in K_{\hat{\tau}} \subseteq [w_-, w_+] \text{ (see (5.11)) and } m_\rho \leqslant \hat{\tau}(y_0). \tag{5.13}$$

From (5.12) and (5.13), we see that

$$y_0 \notin \{v_0, u_0\}.$$

Also since y_0 is a critical point of $\hat{\tau}$ of mountain pass type, we have

$$C_1(\hat{\tau}, y_0) \neq 0\,. \tag{5.14}$$

From Proposition 5.4 we know that

$$C_k(\hat{\tau}, 0) = 0 \quad \text{for all } k \geqslant 0\,. \tag{5.15}$$

Comparing (5.14) and (5.15), we infer that $y_0 \neq 0$. Finally the nonlinear regularity theory (see [14]), implies that $y_0 \in [w_-, w_+] \cap C^1(\overline{\Omega})$ (see also (5.13)). □

If we return to the stronger conditions $H(a)$ for the map $a(\cdot)$ and we impose a unilateral growth condition $f(z,\cdot)$, we can improve Theorem 5.5 and provide sign information for the third solution.

The new hypotheses on the reaction $f(z,x)$ are the following:

(H_5) $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,x)$

(i) for every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)_+$ such that $|f(z,x)| \leqslant a_\rho(z)$ for almost all $z \in \Omega$, all $|x| \leqslant \rho$;

(ii) there exist functions $w_\pm \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{aligned} &w_-(z) \leqslant c_- < 0 < c_+ \leqslant w_+(z) \quad \text{for all } z \in \overline{\Omega}\,,\\ &A(w_-) + \beta(z)|w_-|^{p-2}w_- \leqslant 0 \leqslant A(w_+) + \beta(z)w_+^{p-1} \text{ in } W^{1,p}(\Omega)^*,\\ &f(z, w_+(z)) \leqslant 0 \leqslant f(z, w_-(z)) \quad \text{for almost all } z \in \Omega\,; \end{aligned}$$

(iii) there exist $c_{13} > 0$ and $r \in (p, p^*)$ such that

$$f(z,x)x \geqslant -c_{13}\,|x|^r \quad \text{for almost all } z \in \Omega \text{ all } x \in \mathbb{R}\,;$$

(iv) with $\tilde{\beta} = \frac{1}{\tilde{c}}\beta \in L^\infty(\Omega)_+$, we have

$$\tilde{c}\,\hat{\lambda}_2(q,\tilde{\beta}) < \liminf_{x\to 0} \frac{f(z,x)}{|x|^{p-2}x} \quad \text{uniformly for almost all } z \in \Omega\,.$$

Remark 5.6. Hypotheses H_5 (iii) is the extra unilateral growth condition imposed on $f(z,\cdot)$. Note that hypothesis H_5 (iv) permits also $(q-1)$-linear growth near zero $f(z,\cdot)$. This is more general than hypothesis H_4 (iii), where $\mu < q$.

Hypotheses H_5 (iii), (iv) imply that we can find $\xi_0 > \tilde{c}\,\hat{\lambda}_2(q,\tilde{\beta})$ and $c_{14} > 0$ such that

$$f(z,x)x \geqslant \xi_0\,|x|^q - c_{14}\,|x|^r \quad \text{for almost all } z \in \Omega, \text{ all } x \in \mathbb{R}\,.$$

This leads to the following auxiliary Robin problem:

$$\text{(5.16)} \qquad \begin{cases} -\operatorname{div} a(Du(z)) = \xi_0|u(z)|^{q-2}u(z) - c_{14}|u(z)|^{r-2}u(z) & \text{in } \Omega\,, \\ \dfrac{\partial u}{\partial n_a} + \beta(z)|u(z)|^{p-2}u(z) = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Proposition 3.6 implies that problem (5.16) has a unique positive solution $u_+ \in \operatorname{int} C_+$ and since (5.16) is odd, we have that $v_- = -u_+ \in \operatorname{int} C_+$ is the unique negative solution of (5.16). Also, we have

$$u_+ \leqslant u \text{ for all } u \in S_+ \quad \text{and} \quad v \leqslant v_- \text{ for all } v \in S_- \text{ (see Proposition 3.7).}$$

Having these bounds and reasoning as in the proof of Proposition 3.8, we produce extremal constant sign solutions.

Proposition 5.7. *If hypotheses $H(a)$, H_5 and $H(\beta)$ hold, then problem (1.1) has a smallest positive solution $u_* \in \operatorname{int} C_+$ and a biggest negative solution $v_* \in -\operatorname{int} C_+$.*

These extremal constant sign solutions, leads to a nodal solution (see the proof of Proposition 3.9).

Proposition 5.8. *If hypotheses $H(a), H_5$ and $H(\beta)$ hold, then problem (1.1) admits a nodal solution $y_0 \in [v_*, u_*] \cap C^1(\overline{\Omega})$.*

So, we can state the third multiplicity theorem for problem (1.1).

Theorem 5.9. *Assume that hypotheses $H(a)$, H_5 and $H(\beta)$ hold. Then problem (1.1) admits at least three nontrivial solutions:*

$$u_0 \in [0, w_+] \cap \operatorname{int} C_+, \quad v_0 \in [w_-, 0] \cap (-\operatorname{int} C_+), \quad y_0 \in [w_-, w_+] \cap C^1(\overline{\Omega}).$$

6. p-Laplacian equations

In this section, we deal with equations driven by the p-Laplacian, that is $a(y) = |y|^{p-2}y$ for all $y \in \mathbb{R}^N$. So, now the differential operator is $(p-1)$-homogeneous and we can exploit this fact to drop the unilateral growth condition on $f(z,\cdot)$ (see H_5 (iii)) and return to the case of a reaction with no global growth restriction.

So, the problem under consideration is the following:

$$-\Delta_p u(z) = f(z,u(z)) \text{ in } \Omega, \frac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 \quad \text{on } \partial\Omega\,. \tag{6.1}$$

Recall

$$\frac{\partial u}{\partial n_p} = |Du|^{p-2}\frac{\partial u}{\partial n} \quad \text{for all } u \in W^{1,p}(\Omega).$$

The new hypotheses on the reaction $f(z,x)$ are the following:

(H_6) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,x) = 0$ for almost all $z \in \Omega$, and

(i) for every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)_+$ such that $|f(z,x)| \leqslant a_\rho(z)$ for almost all $z \in \Omega$, all $|x| \leqslant \rho$;

(ii) there exist functions $w_\pm \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{aligned} &w_-(z) \leqslant c_- < 0 \leqslant w_+(z) \quad \text{for all } z \in \overline{\Omega}\,,\\ &A(w_-) + \beta(z)|w_-|^{p-2}w_- \leqslant 0 \leqslant A(w_+) + \beta(z)w_+^{p-1} \text{ in } W^{1,p}(\Omega)^*,\\ &f(z,w_+(z)) \leqslant 0 \leqslant f(z,w_-(z)) \text{ for almost all } z \in \Omega\,; \end{aligned}$$

(iii) we have

$$\hat{\lambda}_2(p,\beta) < \liminf_{x\to 0} \frac{f(z,x)}{|x|^{p-2}x} \leqslant \limsup_{x\to 0} \frac{f(z,x)}{|x|^{p-2}x} \leqslant \eta_0$$

uniformly for almost all $z \in \Omega$.

In this case to produce extremal constant sign solutions, we do not pass through an auxiliary problem (see (5.16)), but instead we argue directly.

Proposition 6.1. *Assume that hypotheses H_6 and $H(\beta)$ hold. Then problem (6.1) admits a smallest positive solution $u_* \in \operatorname{int} C_+$ and a biggest negative solution $v_* \in -\operatorname{int} C_+$.*

Proof. As in the proof of Proposition 3.8, we can find $\{u_n\}_{n\leqslant 1} \subseteq S_+$ such that

$$\inf S_+ = \inf_{n\geqslant 1} u_n\,.$$

Thanks to (3.31), $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ is bounded and so we may assume that

$$u_n \xrightarrow{w} u_* \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{6.2}$$

As in the proof of Proposition 3.8, using (6.2), we show that $u_* \in S_+ \cup \{0\}$. We need to show that $u_* \neq 0$. Arguing by contradiction, suppose that $u_* = 0$. Let $y_n = u_n/||u_n||$ $n \geqslant 1$. Then $||y_n|| = 1$, $y_n \geqslant 0$ for all $n \geqslant 1$. So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega), y \geqslant 0\,. \tag{6.3}$$

We have, for all $n \geqslant 1$, and all $h \in W^{1,p}(\Omega)$,

$$\langle A(y_n), h\rangle + \int_{\partial\Omega} \beta(z)\, y_n^{p-1}\, h\, d\sigma = \int_\Omega \frac{N_f(u_n)}{||u_n||^{p-1}}\, h\, dz. \tag{6.4}$$

In (6.4) we choose $h = y_n - y \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (6.3). Then

$$\begin{aligned} &\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0, \\ &\Longrightarrow \quad y_n \to y \text{ in } W^{1,p}(\Omega) \text{ (see Proposition 2.8), hence } ||y|| = 1,\ y \geqslant 0\,. \end{aligned} \tag{6.5}$$

Hypotheses H_6 (i), (iii) imply that we can find $c_{15} > 0$ such that

$$\begin{aligned} &|f(z,x)| \leqslant c_{15}\,|x|^{p-1} \quad \text{for almost all } z \in \Omega, \text{ all } |x| \leqslant M_3 \text{ (see (3.31))} \\ &\Longrightarrow \quad \left\{\frac{N_f(u_n)}{||u_n||^{p-1}}\right\}_{n\geqslant 1} \subseteq L^{p'}(\Omega) \text{ is bounded } \ (1/p + 1/p' = 1). \end{aligned}$$

Passing to a subsequence if necessary and using hypothesis H_6 (iii) (recall we assume $u_* = 0$), we obtain

$$\frac{N_f(u_n)}{||u_n||^{p-1}} \xrightarrow{w} \vartheta y^{p-1} \text{ in } L^{p'}(\Omega) \text{ with } \hat{\lambda}_2(p,\beta) < \vartheta(z) \leqslant \eta_0 \text{ for almost all } z \in \Omega \tag{6.6}$$

(see Aizicovici, Papageorgiou and Staicu [1], proof of Proposition 14). Returning to (6.4), passing to the limit as $n \to \infty$ and using (6.5) and (6.6), we obtain

$$\langle A(y), h\rangle + \int_{\partial\Omega} \beta(z) y^{p-1} h\, d\sigma = \int_\Omega \vartheta y^{p-1} h\, dz \quad \text{for all } h \in W^{1,p}(\Omega),$$

and thus

$$-\Delta_p y(z) = \vartheta(z) y(z)^{p-1} \text{ for almost all } z \in \Omega, \quad \frac{\partial u}{\partial n_p} + \beta(z) y^{p-1} = 0 \text{ on } \partial\Omega\,. \tag{6.7}$$

From (6.6) and (6.7) it follows that y must be nodal, which contradicts (6.5). Therefore $u_* \in S_+$ and $u_* = \operatorname{int} S_+$.

Similarly we produce $v_* \in -\operatorname{int} C_+$ the biggest negative solution of (6.1). □

Using these extremal constant sign solutions and reasoning as in the proof of Proposition 3.9, we have the final multiplicity theorem.

Theorem 6.2. *Assume that hypotheses H_6 and $H(\beta)$ hold. Then problem (6.1) admits at least three nontrivial solutions:*

$$u_0 \in [0, w_+] \cap \operatorname{int} C_+, \quad v_0 \in [w_-, 0] \cap (-\operatorname{int} C_+), \quad y_0 \in [w_-, w_+] \cap C^1(\overline{\Omega}) \text{ nodal.}$$

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