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Interpolation of data by smooth nonnegative functions

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Abstract. We prove a finiteness principle for interpolation of data by nonnegative C^m and $C^{m-1,1}$ functions. Our result raises the hope that one can start to understand constrained interpolation problems in which, e.g., the interpolating function F is required to be nonnegative.

Introduction

Continuing from [18], we prove a finiteness principle for interpolation of data by nonnegative smooth functions.

Let us recall some notation used in [18].

We fix positive integers m, n. We write $C^m(\mathbb{R}^n)$ to denote the Banach space of all real valued locally C^m functions F on $\mathbb{R}^n$, for which the norm

$$\|F\|_{C^m(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n} \max_{|\alpha|\le m} |\partial^\alpha F(x)|$$

is finite.

We will also work with the function space $C^{m-1,1}(\mathbb{R}^n)$. A given continuous function $F\colon \mathbb{R}^n \to \mathbb{R}$ belongs to $C^{m-1,1}(\mathbb{R}^n)$ if and only if its distribution derivatives $\partial^\beta F$ belong to $L^\infty(\mathbb{R}^n)$ for $|\beta| \le m$. We may take the norm on $C^{m-1,1}(\mathbb{R}^n)$ to be

$$\|F\|_{C^{m-1,1}(\mathbb{R}^n)} = \max_{|\beta|\le m} \operatorname{ess.\,sup}_{x\in\mathbb{R}^n} \left|\partial^\beta F(x)\right|.$$

Expressions $c(m,n)$, $C(m,n)$, $k(m,n)$, etc. denote constants depending only on m, n; these expressions may denote different constants in different occurrences. Similar conventions apply to constants denoted by $C(m,n,D)$, $k(D)$, etc.

If X is any finite set, then $\#(X)$ denotes the number of elements in X.

We are now ready to state our main theorem.

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Theorem 1. *For large enough $k^{\#} = k(m,n)$ and $C^{\#} = C(m,n)$, the following hold.*

(a) (C^m flavor). *Let $f\colon E \to [0,\infty)$ with $E \subset \mathbb{R}^n$ finite. Suppose that for each $S \subset E$ with $\#(S) \leq k^{\#}$, there exists $F^S \in C^m(\mathbb{R}^n)$ with norm $\|F^S\|_{C^m(\mathbb{R}^n)} \leq 1$, such that $F^S = f$ on S and $F^S \geq 0$ on $\mathbb{R}^n$.*

Then there exists $F \in C^m(\mathbb{R}^n)$ with norm $\|F\|_{C^m(\mathbb{R}^n)} \leq C^{\#}$, such that $F = f$ on E and $F \geq 0$ on $\mathbb{R}^n$.

(b) ($C^{m-1,1}$ flavor). *Let $f\colon E \to [0,\infty)$ with $E \subset \mathbb{R}^n$ arbitrary. Suppose that for each $S \subset E$ with $\#(S) \leq k^{\#}$, there exists $F^S \in C^{m-1,1}(\mathbb{R}^n)$ with norm $\|F^S\|_{C^{m-1,1}(\mathbb{R}^n)} \leq 1$, such that $F^S = f$ on S and $F^S \geq 0$ on $\mathbb{R}^n$.*

Then there exists $F \in C^{m-1,1}(\mathbb{R}^n)$ with norm $\|F\|_{C^{m-1,1}(\mathbb{R}^n)} \leq C^{\#}$, such that $F = f$ on E and $F \geq 0$ on $\mathbb{R}^n$.

Our interest in Theorem 1 arises in part from its possible connection to the interpolation algorithm of Fefferman–Klartag [15], [16]. Given a function $f\colon E \to \mathbb{R}$ with $E \subset \mathbb{R}^n$ finite, the goal of [15], [16] is to compute a function $F \in C^m(\mathbb{R}^n)$ such that $F = f$ on E, with $||F||_{C^m(\mathbb{R}^n)}$ as small as possible up to a factor $C(m,n)$. Roughly speaking, the algorithm in [15], [16] computes such an F using $O(N \log N)$ computer operations, where $N = \#(E)$. The algorithm is based on an easier version [10] of Theorem 1. Our present result differs from the easier version in that we have added the hypothesis $F^S \geq 0$ and the conclusion $F \geq 0$. Accordingly, Theorem 1 raises the hope that we can start to understand constrained interpolation problems, in which e.g. the interpolant F is required to be nonnegative everywhere on $\mathbb{R}^n$.

For results related to Theorem 1, we refer the reader to our paper [18] and references therein.

In the following sections, we will set up the notation; then we will recall a main theorem in [18] and use it to prove Theorem 1.

This paper is part of a literature on extension, interpolation, and selection of functions, going back to H. Whitney's seminal work [33], and including fundamental contributions by G. Glaeser [19], Y. Brudnyi and P. Shvartsman [4], [6]–[9], and [23]–[31], J. Wells [32], E. Le Gruyer [21], and E. Bierstone, P. Milman, and W. Pawłucki [1]–[3], as well as our own papers [10]–[17]. See e.g. [14] for the history of the problem, as well as Zobin [34], [35] for a related problem.

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1. Notation and preliminaries

1.1. Background notation

Fix m, $n \geq 1$. We will work with cubes in $\mathbb{R}^n$; all our cubes have sides parallel to the coordinate axes. If Q is a cube, then δ_Q denotes the sidelength of Q. For real numbers $A > 0$, AQ denotes the cube whose center is that of Q, and whose sidelength is $A\delta_Q$.

A *dyadic* cube is a cube of the form $I_1 \times I_2 \times \cdots \times I_n \subset \mathbb{R}^n$, where each I_ν has the form $[2^k \cdot i_\nu, 2^k \cdot (i_\nu + 1))$ for integers $i_1, \ldots, i_n$, k. Each dyadic cube Q is contained in one and only one dyadic cube with sidelength $2\delta_Q$; that cube is denoted by Q^+.

We write $B_n(x, r)$ to denote the open ball in $\mathbb{R}^n$ with center x and radius r, with respect to the Euclidean metric.

We write $\mathcal{P}$ to denote the vector space of all real-valued polynomials of degree at most $(m-1)$ on $\mathbb{R}^n$. If $x \in \mathbb{R}^n$ and F is a real-valued C^{m-1} function on a neighborhood of x, then $J_x(F)$ (the "jet" of F at x) denotes the $(m-1)^{\text{rst}}$ order Taylor polynomial of F at x, i.e.,

$$J_x(F)(y) = \sum_{|\alpha| \leq m-1} \frac{1}{\alpha!} \partial^\alpha F(x) \cdot (y - x)^\alpha .$$

Thus, $J_x(F) \in \mathcal{P}$.

For each $x \in \mathbb{R}^n$, there is a natural multiplication $\odot_x$ on $\mathcal{P}$ ("multiplication of jets at x") defined by setting

$$P \odot_x Q = J_x(PQ) \quad \text{for } P, Q \in \mathcal{P}.$$

If F is a real-valued function on a cube Q, then we write $F \in C^m(Q)$ to denote that F and its derivatives up to m-th order extend continuously to the closure of Q. For $F \in C^m(Q)$, we define

$$\|F\|_{C^m(Q)} = \sup_{x \in Q} \max_{|\alpha| \leq m} |\partial^\alpha F(x)| .$$

The function space $C^{m-1,1}(Q)$ and the norm $\|\cdot\|_{C^{m-1,1}(Q)}$ are defined analogously.

If $F \in C^m(Q)$ and x belongs to the boundary of Q, then we still write $J_x(F)$ to denote the $(m-1)^{rst}$ degree Taylor polynomial of F at x, even though F isn't defined on a full neighborhood of $x \in \mathbb{R}^n$.

Let $S \subset \mathbb{R}^n$ be non-empty and finite. A *Whitney field* on S is a family of polynomials

$$\vec{P} = (P^y)_{y \in S} \quad (\text{each } P^y \in \mathcal{P}),$$

parametrized by the points of S.

We write $\mathrm{Wh}(S)$ to denote the vector space of all Whitney fields on S. For $\vec{P} = (P^y)_{y \in S} \in \mathrm{Wh}(S)$, we define the seminorm

$$\|\vec{P}\|_{\dot{C}^m(S)} = \max_{x,y \in S, (x \neq y), |\alpha| \leq m} \frac{|\partial^\alpha (P^x - P^y)(x)|}{|x - y|^{m - |\alpha|}}.$$

(If S consists of a single point, then $\|\vec{P}\|_{\dot{C}^m(S)} = 0$.)

We also need an elementary fact about convex sets. See [22].

Helly's theorem. *Let $K_1, \dots, K_N \subset \mathbb{R}^D$ be convex. Suppose that $K_{i_1} \cap \dots \cap K_{i_{D+1}}$ is nonempty for any $i_1, \dots, i_{D+1} \in \{1, \dots, N\}$. Then $K_1 \cap \dots \cap K_N$ is nonempty.*

1.2. Shape fields

Let $E \subset \mathbb{R}^n$ be finite. For each $x \in E$, $M \in (0, \infty)$, let $\Gamma(x, M) \subseteq \mathcal{P}$ be a (possibly empty) convex set. We say that $\vec{\Gamma} = (\Gamma(x, M))_{x \in E, M > 0}$ is a *shape field* if for all $x \in E$ and $0 < M' \leq M < \infty$, we have

$$\Gamma(x, M') \subseteq \Gamma(x, M).$$

Let $\vec{\Gamma} = (\Gamma(x, M))_{x \in E, M > 0}$ be a shape field and let $C_w, \delta_{\max}$ be positive real numbers. We say that $\vec{\Gamma}$ is $(C_w, \delta_{\max})$-*convex* if the following condition holds:

Let $0 < \delta \leq \delta_{\max}$, $x \in E$, $M \in (0, \infty)$, P_1, P_2, Q_1, $Q_2 \in \mathcal{P}$. Assume that

(1.1) $P_1, P_2 \in \Gamma(x, M)$;

(1.2) $|\partial^\beta (P_1 - P_2)(x)| \leq M \delta^{m - |\beta|}$ for $|\beta| \leq m - 1$;

(1.3) $|\partial^\beta Q_i(x)| \leq \delta^{-|\beta|}$ for $|\beta| \leq m - 1$ for $i = 1, 2$;

(1.4) $Q_1 \odot_x Q_1 + Q_2 \odot_x Q_2 = 1$.

Then

$$P := Q_1 \odot_x Q_1 \odot_x P_1 + Q_2 \odot_x Q_2 \odot_x P_2 \in \Gamma(x, C_w M). \tag{1.5}$$

1.3. Finiteness principle for shape fields

We recall a main result proven in [18].

Theorem 2. *For a large enough $k^\#$ determined by m, n, the following holds. Let $\vec{\Gamma}_0 = (\Gamma_0(x, M))_{x \in E, M > 0}$ be a $(C_w, \delta_{\max})$-convex shape field and let $Q_0 \subset \mathbb{R}^n$ be a cube of sidelength $\delta_{Q_0} \leq \delta_{\max}$. Also, let $x_0 \in E \cap 5Q_0$ and $M_0 > 0$ be given. Assume that for each $S \subset E$ with $\#(S) \leq k^\#$ there exists a Whitney field $\vec{P}^S = (P^z)_{z \in S}$ such that*

$$\|\vec{P}^S\|_{\dot{C}^m(S)} \leq M_0,$$

and

$$P^z \in \Gamma_0(z, M_0) \quad \text{for all } z \in S.$$

Then there exist $P^0 \in \Gamma_0(x_0, M_0)$ and $F \in C^m(Q_0)$ such that the following hold, with a constant C_* determined by C_w, m, n:

- *$J_z(F) \in \Gamma_0(z, C_* M_0)$ for all $z \in E \cap Q_0$.*
- *$|\partial^\beta (F - P^0)(x)| \leq C_* M_0 \delta_{Q_0}^{m - |\beta|}$ for all $x \in Q_0$, $|\beta| \leq m$.*
- *In particular, $|\partial^\beta F(x)| \leq C_* M_0$ for all $x \in Q_0$, $|\beta| = m$.*

2. C^m interpolation by nonnegative functions

In this section, c, C, C', etc. denote constants determined by m and n. These symbols may denote different constants in different occurrences. For $x \in \mathbb{R}^n$ and $M > 0$, define

$$\Gamma_*(x,M) = \left\{ \begin{array}{c} P \in \mathcal{P} : \text{ There exists } F \in C^m(\mathbb{R}^n) \text{ with } \|F\|_{C^m(\mathbb{R}^n)} \leq M, \\ F \geq 0 \text{ on } \mathbb{R}^n,\ J_x(F) = P. \end{array} \right\} \tag{2.1}$$

It is not immediately clear how to compute Γ_*; we will return to this issue in a later section. Let $E \subset \mathbb{R}^n$ be finite, and let $f\colon E \to [0,\infty)$. Define $\vec{\Gamma}_f = (\Gamma_f(x,M))_{x\in E, M>0}$, where

$$\Gamma_f(x,M) = \{P \in \Gamma_*(x,M) : P(x) = f(x)\}. \tag{2.2}$$

Lemma 1. *$\vec{\Gamma}_f$ is a $(C,1)$-convex shape field.*

Proof. It is clear that $\vec{\Gamma}_f$ is a shape field, i.e., each $\Gamma_f(x,M)$ is convex, and $M' \leq M$ implies $\Gamma_f(x,M') \subseteq \Gamma_f(x,M)$. To establish $(C,1)$-convexity, suppose we are given the following:

(2.3) $0 < \delta \leq 1$, $x \in E$, $M > 0$;

(2.4) $P_1, P_2 \in \Gamma_f(x,M)$ satisfying

(2.5) $\left|\partial^\beta (P_1 - P_2)(x)\right| \leq M\delta^{m-|\beta|}$ for $|\beta| \leq m-1$;

(2.6) $Q_1, Q_2 \in \mathcal{P}$ satisfying

(2.7) $\left|\partial^\beta Q_i(x)\right| \leq \delta^{-|\beta|}$ for $|\beta| \leq m-1$, $i = 1,2$, and

(2.8) $Q_1 \odot_x Q_1 + Q_2 \odot_x Q_2 = 1$.

Set

$$P = Q_1 \odot_x Q_1 \odot_x P_1 + Q_2 \odot_x Q_2 \odot_x P_2. \tag{2.9}$$

We must prove that

$$P \in \Gamma_f(x, CM). \tag{2.10}$$

Thanks to (2.4), we have

$$P_1(x) = f(x) \quad \text{and} \quad P_2(x) = f(x), \tag{2.11}$$

and there exist functions $F_1, F_2 \in C^m(\mathbb{R}^n)$ such that

(2.12) $\|F_i\|_{C^m(\mathbb{R}^n)} \leq M$ $(i = 1,2)$,

(2.13) $F_i \geq 0$ on $\mathbb{R}^n$ $(i = 1,2)$, and

(2.14) $J_x(F_i) = P_i$ $(i = 1,2)$.

We fix F_1, F_2 as above. By (2.8), we have $|Q_i(x)| \geq 1/\sqrt{2}$ for $i = 1$ or for $i = 2$. By possibly interchanging Q_1 and Q_2, and then possibly changing Q_1 to $-Q_1$, we may suppose that

$$Q_1(x) \geq \frac{1}{\sqrt{2}}. \tag{2.15}$$

For small enough c_0, (2.7) and (2.15) yield

$$Q_1(y) \geq \frac{1}{10} \quad \text{for } |y - x| \leq c_0\, \delta. \tag{2.16}$$

Fix c_0 as in (2.16). We introduce a C^m cutoff function χ on $\mathbb{R}^n$ with the following properties.

(2.17) $0 \leq \chi \leq 1$ on $\mathbb{R}^n$; $\chi = 0$ outside $B_n(x, c_0\delta)$; $\chi = 1$ in a neighborhood of x;

(2.18) $\left|\partial^\beta \chi\right| \leq C\delta^{-|\beta|}$ on $\mathbb{R}^n$, for $|\beta| \leq m$.

We then define

$$\tilde{\theta}_1 = \chi \cdot Q_1 + (1 - \chi) \quad \text{and} \quad \tilde{\theta}_2 = \chi \cdot Q_2.$$

These functions satisfy the following: $\tilde{\theta}_i \in C^m(\mathbb{R}^n)$ and $|\partial^\beta \tilde{\theta}_i| \leq C\delta^{-|\beta|}$ on $\mathbb{R}^n$ for $|\beta| \leq m$, $i = 1, 2$; $\tilde{\theta}_1 \geq 1/10$ on $\mathbb{R}^n$; $J_x(\tilde{\theta}_i) = Q_i$ for $i = 1, 2$; outside $B_n(x, c_0\delta)$ we have $\tilde{\theta}_1 = 1$ and $\tilde{\theta}_2 = 0$. Setting

$$\theta_i = \tilde{\theta}_i \cdot (\tilde{\theta}_1^2 + \tilde{\theta}_2^2)^{-1/2}$$

for $i = 1$, 2, we find that

(2.19) $\theta_i \in C^m(\mathbb{R}^n)$ and $\left|\partial^\beta \theta_i\right| \leq C\delta^{-|\beta|}$ on $\mathbb{R}^n$ for $|\beta| \leq m$, $i = 1, 2$;

(2.20) $\theta_1^2 + \theta_2^2 = 1$ on $\mathbb{R}^n$;

(2.21) $J_x(\theta_i) = Q_i$ for $i = 1, 2$ (here we use (2.8)); and

(2.22) outside $B_n(x, c_0\delta)$ we have $\theta_1 = 1$ and $\theta_2 = 0$.

Now set

$$F = \theta_1^2 F_1 + \theta_2^2 F_2 = F_1 + \theta_2^2 (F_2 - F_1) \quad \text{(see (2.20))}. \tag{2.23}$$

Clearly $F \in C^m(\mathbb{R}^n)$. By (2.14), we have

$$J_x(F_2 - F_1) = P_2 - P_1;$$

hence (2.5) yields the estimate

$$\left|\partial^\beta (F_2 - F_1)(x)\right| \leq CM\delta^{m-|\beta|} \quad \text{for } |\beta| \leq m - 1.$$

Together with (2.12), this tells us that

$$\left|\partial^\beta (F_2 - F_1)\right| \leq CM\delta^{m-|\beta|} \quad \text{on } B_n(x, c_0\delta) \text{ for } |\beta| \leq m.$$

Recalling (2.19), we deduce that

$$\left|\partial^\beta\left(\theta_2^2\cdot(F_2-F_1)\right)\right|\le CM\delta^{m-|\beta|}\quad\text{on } B_n(x,c_0\delta)\quad\text{for } |\beta|\le m.$$

Together with (2.12) and (2.23), this implies that

$$|\partial^\beta F|\le CM\quad\text{on } B_n(x,c_0\delta),$$

since $0<\delta\le 1$ (see (2.3)). On the other hand, outside $B_n(x,c_0\delta)$ we have $F=F_1$ by (2.22), (2.23); hence $|\partial^\beta F|\le CM$ outside $B_n(x,c_0\delta)$ for $|\beta|\le m$, by (2.12). Thus, $|\partial^\beta F|\le CM$ on all of $\mathbb{R}^n$ for $|\beta|\le m$, i.e.,

$$\|F\|_{C^m(\mathbb{R}^n)}\le CM. \tag{2.24}$$

Also, from (2.13) and (2.23) we have

$$F\ge 0\quad\text{on } \mathbb{R}^n; \tag{2.25}$$

and (2.9), (2.14), (2.21), (2.23) imply that

$$J_x(F)=Q_1\odot_x Q_1\odot_x P_1+Q_2\odot_x Q_2\odot_x P_2=P. \tag{2.26}$$

Since $F\in C^m(\mathbb{R}^n)$ satisfies (2.24), (2.25), (2.26), we have

$$P\in\Gamma_*\left(x,CM\right). \tag{2.27}$$

Moreover,

$$P(x)=\left(Q_1(x)\right)^2 f(x)+\left(Q_2(x)\right)^2 f(x)=f(x), \tag{2.28}$$

thanks to (2.8), (2.9), (2.11).

From (2.27), (2.28) we conclude that $P\in\Gamma_f(x,CM)$, completing the proof of Lemma 1. □

Lemma 2. *Let $(P^x)_{x\in E}$ be a Whitney field on the finite set E, and let $M>0$. Suppose that*

$$P^x\in\Gamma_*(x,M)\qquad\textit{for each } x\in E, \tag{2.29}$$

and that

$$\left|\partial^\beta(P^x-P^{x'})(x)\right|\le M\left|x-x'\right|^{m-|\beta|}\qquad\textit{for } x,x'\in E \textit{ and } |\beta|\le m-1. \tag{2.30}$$

Then there exists $F\in C^m(\mathbb{R}^n)$ such that

(2.31) $\|F\|_{C^m(\mathbb{R}^n)}\le CM$,

(2.32) $F\ge 0$ *on* $\mathbb{R}^n$, *and*

(2.33) $J_x(F)=P^x$ *for all* $x\in E$.

Proof. We modify slightly Whitney's proof [33] of the Whitney extension theorem. We say that a dyadic cube $Q \subset \mathbb{R}^n$ is "OK" if $\#(E \cap 5Q) \leq 1$ and $\delta_Q \leq 1$. Then every small enough Q is OK (because E is finite), and no Q of sidelength $\delta_Q > 1$ is OK. Also, let Q, Q' be dyadic cubes with $5Q \subset 5Q'$. If Q' is OK, then also Q is OK. We define a Calderón–Zygmund (or CZ) cube to be an OK cube Q such that no Q' that strictly contains Q is OK. The above remarks imply that the CZ cubes form a partition of $\mathbb{R}^n$; that the sidelengths of the CZ cubes are bounded above by 1 and below by some positive number; and that the following condition holds:

(2.34) "Good geometry": if $Q, Q' \in \mathrm{CZ}$ and $\frac{65}{64}Q \cap \frac{65}{64}Q' \neq \emptyset$, then $\frac{1}{2}\delta_Q \leq \delta_{Q'} \leq 2\delta_Q$.

We classify CZ cubes into three types as follows. $Q \in CZ$ is of

Type 1. If $E \cap 5Q \neq \emptyset$.

Type 2. If $E \cap 5Q = \emptyset$ and $\delta_Q < 1$.

Type 3. If $E \cap 5Q = \emptyset$ and $\delta_Q = 1$.

Let $Q \in CZ$ be of Type 1. Since Q is OK, we have $\#(E \cap 5Q) \leq 1$. Hence $E \cap 5Q$ is a singleton, $E \cap 5Q = \{x_Q\}$. Since $P^{x_Q} \in \Gamma_*(x_Q, M)$, there exists $F_Q \in C^m(\mathbb{R}^n)$ such that

$$\|F_Q\|_{C^m(\mathbb{R}^n)} \leq M, \quad F_Q \geq 0 \text{ on } \mathbb{R}^n, \quad J_{x_Q}(F_Q) = P^{x_Q}. \tag{2.35}$$

We fix F_Q as in (2.35).

Let $Q \in CZ$ be of Type 2. Then $\delta_{Q^+} \leq 1$ but Q^+ is not OK; hence $\#(E \cap 5Q^+) \geq 2$. We pick $x_Q \in E \cap 5Q^+$. Since $P^{x_Q} \in \Gamma_*(x_Q, M)$, there exists $F_Q \in C^m(\mathbb{R}^n)$ satisfying (2.35). We fix such an F_Q.

Let $Q \in CZ$ be of Type 3. Then we set $F_Q = 0$. In place of (2.35), we have the trivial results

$$\|F_Q\|_{C^m(\mathbb{R}^n)} = 0 \quad \text{and} \quad F_Q \geq 0 \text{ on } \mathbb{R}^n. \tag{2.36}$$

Thus, we have defined F_Q for all $Q \in \mathrm{CZ}$, and we have defined $x_Q \in E \cap 5Q^+$ for all Q of Type 1 or Type 2. Note that

$$J_x(F_Q) = P^x \quad \text{for all } x \in E \cap 5Q. \tag{2.37}$$

Indeed, if Q is of Type 1, then (2.37) follows from (2.35) since $E \cap 5Q = \{x_Q\}$. If Q is of Type 2 or Type 3, then (2.37) holds vacuously since $E \cap 5Q = \emptyset$. Now suppose $Q, Q' \in \mathrm{CZ}$ and $\frac{65}{64}Q \cap \frac{65}{64}Q' \neq \emptyset$. We will show that

$$\left|\partial^\beta (F_Q - F_{Q'})\right| \leq CM\delta_Q^{m-|\beta|} \quad \text{on } \tfrac{65}{64}Q \cap \tfrac{65}{64}Q' \text{ for } |\beta| \leq m. \tag{2.38}$$

To see this, suppose first that Q or Q' is of Type 3. Then δ_Q or $\delta_{Q'}$ is equal to 1, hence $\delta_Q \geq 1/2$ by (2.34). Consequently, (2.38) asserts simply that

$$\left|\partial^\beta (F_Q - F_{Q'})\right| \leq CM \quad \text{on } \tfrac{65}{64}Q \cap \tfrac{65}{64}Q' \text{ for } |\beta| \leq m, \tag{2.39}$$

and (2.39) follows at once from (2.35), (2.36). Thus, (2.38) holds if Q or Q' is of Type 3.

Suppose that neither Q nor Q' is of Type 3. Then $x_Q \in E \cap 5Q^+$, $x_{Q'} \in E \cap 5(Q'^+)$, $\frac{65}{64}Q \cap \frac{65}{64}Q' \neq \emptyset$, $\frac{1}{2}\delta_Q \leq \delta_{Q'} \leq 2\delta_Q$. Consequently,

(2.40) $|x_Q - x_{Q'}| \leq C\delta_Q$, and

(2.41) $|x - x_Q|, |x - x_{Q'}| \leq C\delta_Q$ for all $x \in \frac{65}{64}Q \cap \frac{65}{64}Q'$.

Applying (2.35) to Q and to Q', we find that, for $x \in \frac{65}{64}Q \cap \frac{65}{64}Q'$, $|\beta| \leq m$,

(2.42) $\left|\partial^\beta \left(F_Q - P^{x_Q}\right)(x)\right| \leq CM\left|x - x_Q\right|^{m-|\beta|} \leq CM\delta_Q^{m-|\beta|}$, and

(2.43) $\left|\partial^\beta \left(F_{Q'} - P^{x_{Q'}}\right)(x)\right| \leq CM\left|x - x_{Q'}\right|^{m-|\beta|} \leq CM\delta_Q^{m-|\beta|}$,

Also, (2.30), (2.40), (2.41) imply that

$$\left|\partial^\beta \left(P^{x_Q} - P^{x_{Q'}}\right)(x)\right| \leq CM\delta_Q^{m-|\beta|} \quad \text{for } x \in \frac{65}{64}Q \cap \frac{65}{64}Q',\ |\beta| \leq m. \tag{2.44}$$

(Recall, $P^{x_Q} - P^{x_{Q'}}$ is a polynomial of degree at most $m-1$.)

Estimates (2.42), (2.43), (2.44) together imply (2.38) in case neither Q nor Q' is of Type 3. Thus, (2.38) holds in all cases.

Next, as in Whitney [33], we introduce a partition of unity

$$1 = \sum_{Q \in CZ} \theta_Q \quad \text{on } \mathbb{R}^n, \tag{2.45}$$

where each $\theta_Q \in C^m(\mathbb{R}^n)$, and

$$\text{supp}\theta_Q \subset \tfrac{65}{64}Q, \quad |\partial^\beta \theta_Q| \leq C\delta_Q^{-|\beta|} \text{ for } |\beta| \leq m, \quad \theta_Q \geq 0 \text{ on } \mathbb{R}^n. \tag{2.46}$$

We define

$$F = \sum_{Q \in CZ} \theta_Q F_Q \quad \text{on } \mathbb{R}^n. \tag{2.47}$$

Thus, $F \in C^m_{\text{loc}}(\mathbb{R}^n)$ since CZ is a locally finite partition of $\mathbb{R}^n$, and $F \geq 0$ on $\mathbb{R}^n$ since $\theta_Q \geq 0$ and $F_Q \geq 0$ for each Q. Let $\hat{x} \in \mathbb{R}^n$, and let $\hat{Q}$ be the one and only CZ cube containing $\hat{x}$. Then for $|\beta| \leq m$, we have

$$\partial^\beta F(\hat{x}) = \partial^\beta F_{\hat{Q}}(\hat{x}) + \sum_{Q \in CZ} \partial^\beta\big(\theta_Q \cdot (F_Q - F_{\hat{Q}})\big)(\hat{x}). \tag{2.48}$$

A given $Q \in$ CZ enters into the sum in (2.48) only if $\hat{x} \in \frac{65}{64}Q$; there are at most C such cubes Q, thanks to (2.34). Moreover, for each $Q \in$ CZ with $\hat{x} \in \frac{65}{64}Q$, we learn from (2.38) and (2.46) that

$$\left|\partial^\beta(\theta_Q \cdot (F_Q - F_{\hat{Q}}))(\hat{x})\right| \leq CM\delta_Q^{m-|\beta|} \leq CM \quad \text{for } |\beta| \leq m, \text{ since } \delta_Q \leq 1.$$

Since also $|\partial^\beta F_{\hat{Q}}(\hat{x})| \leq CM$ for $|\beta| \leq m$ by (2.35), (2.36), it now follows from (2.48) that $|\partial^\beta F(\hat{x})| \leq CM$ for all $|\beta| \leq m$. Here, $\hat{x} \in \mathbb{R}^n$ is arbitrary. Thus, $F \in C^m(\mathbb{R}^n)$ and $||F||_{C^m(\mathbb{R}^n)} \leq CM$.

Next, let $x \in E$. For any $Q \in \mathrm{CZ}$ such that $x \in \frac{65}{64}Q$, we have $J_x(F_Q) = P^x$, by (2.37). Since support $\theta_Q \subset \frac{65}{64}Q$ for each $Q \in \mathrm{CZ}$, it follows that $J_x(\theta_Q F_Q) = J_x(\theta_Q) \odot_x P^x$ for each $Q \in \mathrm{CZ}$, and consequently,

$$J_x(F) = \sum_{Q \in CZ} J_x(\theta_Q F_Q) = \Big[\sum_{Q \in CZ} J_x(\theta_Q) \Big] \odot_x P^x = P^x, \quad \text{by (2.45)}.$$

Thus, $F \in C^m(\mathbb{R}^n)$, $\|F\|_{C^m(\mathbb{R}^n)} \le CM$, $F \ge 0$ on $\mathbb{R}^n$, and $J_x(F) = P^x$ for each $x \in E$.

The proof of Lemma 2 is complete. □

Theorem 3 (Finiteness principle for nonnegative C^m interpolation). *There exist constants $k^\#$, C, depending only on m, n, such that the following holds.*

Let $E \subset \mathbb{R}^n$ be finite, and let $f: E \to [0, \infty)$. Let $M_0 > 0$. Suppose that for each $S \subset E$ with $\#(S) \le k^\#$, there exists $\vec{P}^S = (P^x)_{x \in S} \in \mathrm{Wh}(S)$ such that

- *$P^x \in \Gamma_f(x, M_0)$ for each $x \in S$, and*
- *$|\partial^\beta (P^x - P^y)(x)| \le M_0 |x-y|^{m-|\beta|}$ for $x, y \in S$, $|\beta| \le m-1$.*

Then there exists $F \in C^m(\mathbb{R}^n)$ such that

- *$\|F\|_{C^m(\mathbb{R}^n)} \le CM_0$,*
- *$F \ge 0$ on $\mathbb{R}^n$, and*
- *$F = f$ on E.*

Proof. Suppose first that $E \subset \frac{1}{2}Q_0$ for a cube Q_0 of sidelength $\delta_{Q_0} = 1$. Pick any $x_0 \in E$. (If E is empty, our theorem holds trivially.)

Let $S \subset E$ with $\#(S) \le k^\#$.

Our present hypotheses supply the Whitney field $\vec{P}^S$ required in the hypotheses of Theorem 2.

Hence, recalling Lemma 1 and applying Theorem 2, we obtain

$$P^0 \in \Gamma_f(x_0, CM_0) \quad \text{and} \quad F^0 \in C^m(Q_0) \tag{2.49}$$

such that

$$J_x(F^0) \in \Gamma_f(x, CM_0) \quad \text{for all } x \in E \cap Q_0 = E \tag{2.50}$$

and

$$|\partial^\beta (P^0 - F^0)| \le CM_0 \quad \text{on } Q_0, \text{ for } |\beta| \le m. \tag{2.51}$$

From (2.1), (2.2), (2.49), we have $|\partial^\beta P^0(x_0)| \le CM_0$ for $|\beta| \le m-1$.

Since P^0 is a polynomial of degree at most $m-1$, and since $x_0 \in E \subset Q_0$ with $\delta_{Q_0} = 1$, it follows that $|\partial^\beta P^0| \le CM_0$ on Q_0 for $|\beta| \le m$.

Together with (2.51), this tells us that

$$|\partial^\beta F^0| \le CM_0 \quad \text{on } Q_0 \text{ for } |\beta| \le m. \tag{2.52}$$

Note that F^0 need not be nonnegative.

Set $P^x = J_x(F^0)$ for $x \in E$. Then

(2.53) $P^x \in \Gamma_f(x, CM_0)$ for $x \in E$, and

(2.54) $\left|\partial^\beta (P^x - P^y)(x)\right| \leq CM_0 |x-y|^{m-|\beta|}$ for $x, y \in E$, $|\beta| \leq m-1$.

By Lemma 2, there exists $F \in C^m(\mathbb{R}^n)$ such that

(2.55) $\|F\|_{C^m(\mathbb{R}^n)} \leq CM_0$,

(2.56) $F \geq 0$ on $\mathbb{R}^n$, and

(2.57) $J_x(F) = P^x$ for each $x \in E$.

From (2.53) and (2.2), we have $P^x(x) = f(x)$ for each $x \in E$; hence, (2.57) implies that

(2.58)
$$F(x) = f(x) \quad \text{for each } x \in E.$$

Our results (2.55), (2.56), (2.58) are the conclusions of our theorem. Thus, we have proven Theorem 3 in the case in which $E \subset \frac{1}{2}Q_0$ with $\delta_{Q_0} = 1$.

To pass to the general case (arbitrary finite $E \subset \mathbb{R}^n$), we set up a partition of unity $1 = \sum_\nu \chi_\nu$ on $\mathbb{R}^n$, where each $\chi_\nu \in C^m(\mathbb{R}^n)$ and $\chi_\nu \geq 0$ on $\mathbb{R}^n$, $\|\chi_\nu\|_{C^m(\mathbb{R}^n)} \leq C$, support $\chi_\nu \subset \frac{1}{2}Q_\nu$, with $\delta_{Q_\nu} = 1$, and with any given point of $\mathbb{R}^n$ belonging to at most C of the Q_ν.

For each ν, we apply the known special case of our theorem to the set $E_\nu = E \cap \frac{1}{2}Q_\nu$ and the function $f_\nu = f|_{E_\nu}$. Thus, we obtain $F_\nu \in C^m(\mathbb{R}^n)$, with $\|F_\nu\|_{C^m(\mathbb{R}^n)} \leq CM_0$, $F_\nu \geq 0$ on $\mathbb{R}^n$, and $F_\nu = f$ on $E \cap \frac{1}{2}Q_\nu$.

Setting $F = \sum_\nu \chi_\nu F_\nu \in C^m_{\mathrm{loc}}(\mathbb{R}^n)$, we verify easily that

$$F \in C^m(\mathbb{R}^n), \quad \|F\|_{C^m(\mathbb{R}^n)} \leq CM_0, \quad F \geq 0 \text{ on } \mathbb{R}^n, \quad \text{and } F = f \text{ on } E.$$

This completes the proof of Theorem 3. □

Remark. Conversely, we make the following trivial observation: let $E \subset \mathbb{R}^n$ be finite, let $f\colon E \to [0, \infty)$, and let $M_0 > 0$. Suppose $F \in C^m(\mathbb{R}^n)$ satisfies $\|F\|_{C^m(\mathbb{R}^n)} \leq M_0$, $F \geq 0$ on $\mathbb{R}^n$, $F = f$ on E. Then for each $x \in E$, we have

- $P^x = J_x(F) \in \Gamma_f(x, M_0)$ by (2.1), (2.2); and
- $|\partial^\beta(P^x - P^y)(x)| \leq CM_0|x-y|^{m-|\beta|}$ for $x, y \in E$, $|\beta| \leq m-1$.

Therefore, for any $S \subset E$, the Whitney field $\vec{P}^S = (P^x)_{x \in S} \in \mathrm{Wh}(S)$ satisfies

- $P^x \in \Gamma_f(x, CM_0)$ for $x \in S$, and
- $|\partial^\beta(P^x - P^y)(x)| \leq CM_0|x-y|^{m-|\beta|}$ for $x, y \in S$, $|\beta| \leq m-1$.

Note that Theorem 1 (a) follows easily from Theorem 3.

3. Computable convex sets

In this section, we discuss computational issues regarding the convex set

$$\Gamma_*(x, M) = \left\{J_x(F) : F \in C^m(\mathbb{R}^n),\ \|F\|_{C^m(\mathbb{R}^n)} \le M,\ F \ge 0 \quad \text{on } \mathbb{R}^n\right\}. \tag{3.1}$$

We write c, C, C', etc., to denote constants determined by m and n. These symbols may denote different constants in different occurrences.

We will define convex sets $\tilde{\Gamma}_*(x, M) \subset \mathcal{P}$, prove that

$$\tilde{\Gamma}_*(x, cM) \subset \Gamma_*(x, M) \subset \tilde{\Gamma}_*(x, CM) \quad \text{for all } x \in \mathbb{R}^n,\ M > 0, \tag{3.2}$$

and explain how (in principle) one can compute $\tilde{\Gamma}_*(x, M)$.

We may then use

$$\tilde{\Gamma}_f(x, M) = \left\{P \in \tilde{\Gamma}_*(x, M) : P(x) = f(x)\right\} \tag{3.3}$$

in place of $\Gamma_f(x, M)$ in the statement of Theorem 3. (The assertion in terms of $\tilde{\Gamma}_f$ follows trivially from (3.2) and the original assertion in terms of Γ_f.)

To achieve (3.2), we will define

$$\tilde{\Gamma}_*(x, M) = \left\{MP(\cdot + x)) : P \in \tilde{\Gamma}_0\right\}, \quad \text{for a convex set } \tilde{\Gamma}_0. \tag{3.4}$$

We will prove that

$$\Gamma_*(0, c) \subset \tilde{\Gamma}_0 \subset \Gamma_*(0, C). \tag{3.5}$$

Property (3.2) then follows at once from (3.1), (3.4), and (3.5).

Thus, our task is to define a convex set $\tilde{\Gamma}_0$ satisfying (3.5), and explain how (in principle) one can compute $\tilde{\Gamma}_0$.

Recall that $\mathcal{P}$ is the vector space of $(m-1)$-jets. We will work in the space of m-jets. In this section, we let $\mathcal{P}^+$ denote the vector space of real-valued polynomials of degree at most m on $\mathbb{R}^n$, and we write $J_x^+(F)$ to denote the m^{th}-degree Taylor polynomial of F at x, i.e.,

$$J_x^+(F)(y) = \sum_{|\alpha| \le m} \frac{1}{\alpha!} \left(\partial^\alpha F(x)\right) \cdot (y - x)^\alpha.$$

We define

$$\Gamma_0^+ = \left\{ \begin{array}{c} P \in \mathcal{P}^+ : |\partial^\beta P(0)| \le 1 \text{ for } |\beta| \le m;\ P(x) + |x|^m \ge 0 \\ \text{for all } x \in \mathbb{R}^n; \text{ and for every } \epsilon > 0, \text{ there exists } \delta > 0 \\ \text{such that } P(x) + \epsilon |x|^m \ge 0 \text{ for } |x| \le \delta. \end{array} \right\}. \tag{3.6}$$

Later, we will discuss how Γ_0^+ may be computed in principle.

We next establish the following result.

Lemma 3. *For small enough c and large enough C, the following hold.*

(a) *If $F \in C^m(\mathbb{R}^n)$, $\|F\|_{C^m(\mathbb{R}^n)} \le c$, $F \ge 0$ on $\mathbb{R}^n$, then $J_0^+(F) \in \Gamma_0^+$.*

(b) *If $P \in \Gamma_0^+$, then there exists $F \in C^m(\mathbb{R}^n)$ such that $\|F\|_{C^m(\mathbb{R}^n)} \le C$, $F \ge 0$ on $\mathbb{R}^n$, and $J_0^+(F) = P$.*

Proof. (a) follows trivially from Taylor's theorem. We prove (b).

Let $P \in \Gamma_0^+$ be given. We introduce cutoff functions φ, $\chi \in C^m(\mathbb{R}^n)$ with the following properties:

(3.7) $\|\chi\|_{C^m(\mathbb{R}^n)} \leq C, \chi = 1$ in a neighborhood of 0, $\quad \chi = 0$ outside $B_n\,(0, 1/2)$, and $0 \leq \chi \leq 1$ on $\mathbb{R}^n$.

and

(3.8) $\|\varphi\|_{C^m(\mathbb{R}^n)} \leq C, \quad \varphi = 1$ for $1/2 \leq |x| \leq 2, \quad \varphi \geq 0$ on $\mathbb{R}^n$, and $\varphi(x) = 0$ unless $1/4 < |x| < 4$.

For $k \geq 0$, let

$$\varphi_k(x) = \varphi\left(2^k x\right) \quad (x \in \mathbb{R}^n). \tag{3.9}$$

Thus,

(3.10) $\|\varphi_k\|_{C^m(\mathbb{R}^n)} \leq C2^{mk}, \quad \varphi_k \geq 0$ on $\mathbb{R}^n, \quad \varphi_k(x) = 1$ for $2^{-1-k} \leq |x| \leq 2^{1-k}$, $\varphi_k(x) = 0$ unless $2^{-2-k} \leq |x| \leq 2^{2-k}$.

Also, for $k \geq 0$, we define a real number b_k as follows.

(3.11) $b_k = 0$ if $P(x) \geq 0$ for $|x| \leq 2^{-k}$; $\quad b_k = -\min\{P(x) : |x| \leq 2^{-k}\}$ otherwise.

Since $P \in \Gamma_0^+$, the b_k satisfy the following:

(3.12) $0 \leq b_k \leq 2^{-mk}$ for all $k \geq 0$.

(3.13) $b_k \cdot 2^{mk} \to 0$ as $k \to \infty$.

By definition of the b_k, we have also for each $k \geq 0$ that

$$P(x) + b_k \geq 0 \quad \text{for } |x| \leq 2^{-k}. \tag{3.14}$$

We define a function $\tilde{F}$ on the closed unit ball $\overline{B_n(0,1)}$ by setting

$$\tilde{F}(x) = P(x) + \sum_{k=0}^{\infty} b_k\,\varphi_k(x) \quad \text{for } x \in \overline{B_n(0,1)}. \tag{3.15}$$

(The sum contains at most C nonzero terms for any given x.)

We will check that

$$\tilde{F} \geq 0 \quad \text{on } \overline{B_n(0,1)}. \tag{3.16}$$

Indeed, $\tilde{F}(0) = P(0) \geq 0$ since each $\varphi_k(0) = 0$ and $P \in \Gamma_0^+$. For $\hat{x} \in \overline{B_n(0,1)} \setminus \{0\}$ we have $2^{-1-\hat{k}} \leq |\hat{x}| \leq 2^{-\hat{k}}$ for some $\hat{k} \geq 0$.

We then have $\varphi_{\hat{k}}(\hat{x}) = 1$ by (3.10), hence $P(\hat{x}) + b_{\hat{k}}\varphi_{\hat{k}}(\hat{x}) \geq 0$ by (3.14). Since also $b_k\,\varphi_k(\hat{x}) \geq 0$ for all k, it follows that

$$\tilde{F}(\hat{x}) = \left[P(\hat{x}) + b_{\hat{k}}\varphi_{\hat{k}}(\hat{x})\right] + \sum_{k \neq \hat{k}} b_k\,\varphi_k(x) \geq 0,$$

completing the proof of (3.16).

Next, we check that

$$\tilde{F} \in C^m\left(\overline{B_n(0,1)}\right), \quad \|\tilde{F}\|_{C^m\left(\overline{B_n(0,1)}\right)} \leq C, \quad J_0^+\left(\tilde{F}\right) = P. \tag{3.17}$$

To see this, let

$$\tilde{F}_K = P + \sum_{k=0}^{K} b_k\, \varphi_k \quad \text{for } K \geq 0. \tag{3.18}$$

Since $P \in \Gamma_0^+$, we have $\left|\partial^\beta P\,(0)\right| \leq 1$ for $|\beta| \leq m$, hence

$$\|P\|_{C^m(\overline{B_n(0,1)})} \leq C. \tag{3.19}$$

Also, (3.10) and (3.12) give

$$\|b_k\, \varphi_k\|_{C^m\left(\overline{B_n(0,1)}\right)} \leq C \quad \text{for each } k.$$

Since any given $x \in \overline{B_n(0,1)}$ belongs to at most C of the supports of the φ_k, it follows that

$$\left\| \sum_{k=0}^{K} b_k\, \varphi_k \right\|_{C^m(\overline{B_n(0,1)})} \leq C. \tag{3.20}$$

From (3.18), (3.19), (3.20), we see that

$$\tilde{F}_K \in C^m\left(\overline{B_n(0,1)}\right) \quad \text{and} \quad \left\|\tilde{F}\right\|_{C^m\left(\overline{B_n(0,1)}\right)} \leq C. \tag{3.21}$$

Also, (3.10) and (3.18) tell us that

$$J_0^+(\tilde{F}_K) = P \quad \text{for each } K. \tag{3.22}$$

Furthermore for $K_1 < K_2$, (3.18) gives $\tilde{F}_{K_2} - \tilde{F}_{K_1} = \sum_{K_1<k\leq K_2} b_k\, \varphi_k$. Let $\epsilon > 0$. From (3.10) and (3.13) we see that

$$\max_{K_1<k\leq K_2} \|b_k\, \varphi_k\|_{C^m\left(\overline{B_n(0,1)}\right)} < \epsilon \quad \text{if } K_1 \text{ is large enough.}$$

Since any given point lies in support φ_k for at most C distinct k, it follows that

$$\left\| \sum_{K_1<k\leq K_2} b_k\, \varphi_k \right\|_{C^m\left(\overline{B_n(0,1)}\right)} \leq C\epsilon \quad \text{if } K_2 > K_1 \text{ and } K_1 \text{ is large enough.}$$

Thus, $(\tilde{F}_K)_{K\geq 0}$ is a Cauchy sequence in $C^m(\overline{B_n(0,1)})$. Consequently, $\tilde{F}_K \to \tilde{F}_\infty$ in $C^m(\overline{B_n(0,1)})$-norm for some $\tilde{F}_\infty \in C^m(\overline{B_n(0,1)})$. From (3.21) and (3.22), we have

$$\left\|\tilde{F}_\infty\right\|_{C^m(\overline{B_n(0,1)})} \leq C \quad \text{and} \quad J_0^+(\tilde{F}_\infty) = P.$$

On the other hand, comparing (3.15) to (3.18), and recalling that any given x belongs to support θ_k for at most C distinct k, we conclude that $\tilde{F}_K \to \tilde{F}$ pointwise as $K \to \infty$.

Since also $\tilde{F}_K \to \tilde{F}_\infty$ pointwise as $K \to \infty$, we have $\tilde{F}_\infty = \tilde{F}$. Thus,

$$\tilde{F} \in C^m\left(\overline{B_n(0,1)}\right), \quad \|\tilde{F}\|_{C^m(\overline{B_n(0,1)})} \le C, \quad \text{and} \quad J_0^+\left(\tilde{F}\right) = P,$$

completing the proof of (3.17).

Finally, we recall the cutoff function χ from (3.7), and define $F = \chi\tilde{F}$ on $\mathbb{R}^n$. From (3.16), (3.17), and the properties (3.7) of χ, we conclude that

$$F \in C^m(\mathbb{R}^n), \quad \|F\|_{C^m(\mathbb{R}^n)} \le C, \quad F \ge 0 \text{ on } \mathbb{R}^n, \quad \text{and} \quad J_0^+(F) = P.$$

Thus, we have established (b). The proof of Lemma 3 is complete. □

Now let $\pi : \mathcal{P}^+ \to \mathcal{P}$ denote the natural projection from m-jets at 0 to $(m-1)$-jets at 0, namely,

$$\pi P = J_0(P)$$

for $P \in \mathcal{P}^+$.

We then set

$$\tilde{\Gamma}_0 = \pi\Gamma_0^+.$$

From the above lemma, we learn the following.

(A') Let $F \in C^m(\mathbb{R}^n)$ with $\|F\|_{C^m(\mathbb{R}^n)} \le c$, $F \ge 0$ on $\mathbb{R}^n$. Then $J_0(F) \in \tilde{\Gamma}_0$.

(B') Let $P \in \tilde{\Gamma}_0$. Then there exists $F \in C^m(\mathbb{R}^n)$ such that $\|F\|_{C^m(\mathbb{R}^n)} \le C$, $F \ge 0$ on $\mathbb{R}^n$, and $J_0(F) = P$.

Recalling the definition (3.1), we conclude from (A') and (B') that

$$\Gamma_*(0,c) \subset \tilde{\Gamma}_0 \subset \Gamma_*(0,C).$$

Thus, our $\tilde{\Gamma}_0$ satisfies the key condition (3.5).

We discuss briefly how the convex set $\tilde{\Gamma}_0$ may be computed in principle. Recall (see [20]) that a semialgebraic set is a subset of a vector space obtained by taking finitely many unions, intersections, and complements of sets of the form $\{P > 0\}$ for polynomials P. Any subset of a vector space V defined by $E = \{x \in V : \Phi(x) \text{ is true}\}$, where Φ is a formula of first-order predicate calculus (for the theory of real-closed fields) is semialgebraic; moreover, there is an algorithm that accepts Φ as input and exhibits E as a Boolean combination of sets of the form $\{P > 0\}$ for polynomials P. For any given m, n, we see, by inspection of the definitions of Γ_0^+ and $\tilde{\Gamma}_0$, that $\Gamma_0^+ \subset \mathcal{P}^+$ is defined by a formula of first-order predicate calculus; hence, the same holds for $\tilde{\Gamma}_0 \subset \mathcal{P}$.

Therefore, in principle, we can compute $\tilde{\Gamma}_0$ as a Boolean combination of sets of the form $\{P \in \mathcal{P} : \Pi(P) > 0\}$, where Π is a polynomial on $\mathcal{P}$.

In practice, we make no claim that we know how to compute $\tilde{\Gamma}_0$.

It would be interesting to give a more practical method to compute a convex set satisfying (3.5).

4. $C^{m-1,1}$ interpolation by nonnegative functions

In this section we will establish Theorem 1 (b) and discuss computational issues for $C^{m-1,1}$ interpolation by nonnegative functions.

We note that the derivatives $\partial^\beta F$ of $F \in C^{m-1,1}(\mathbb{R}^n)$ of order $|\beta| \leq m-1$ are continuous. Also, Taylor's theorem holds in the form

$$\left|\partial^\beta F(y) - \sum_{|\beta|+|\gamma|\leq m-1} \frac{1}{\gamma!}\left[\partial^{\gamma+\beta}F(x)\right]\cdot(y-x)^\gamma\right| \leq C\,\|F\|_{C^{m-1,1}(\mathbb{R}^n)}\cdot|y-x|^{m-|\beta|}$$

for $x, y \in \mathbb{R}^n$.

Similar remarks apply to $C^{m-1,1}(Q)$ and $C^m(Q)$ for cubes $Q \subset \mathbb{R}^n$.

Therefore, we may repeat the proofs of Lemmas 1 and 2 in Section 2, to derive the following results.

Lemma 4. *For $x \in \mathbb{R}^n$, $M > 0$, let*

$$\Gamma'_*(x,M) = \left\{ \begin{array}{c} P \in \mathcal{P} : \exists F \in C^{m-1,1}(\mathbb{R}^n) \text{ such that} \\ \|F\|_{C^{m-1,1}(\mathbb{R}^n)} \leq M, F \geq 0 \text{ on } \mathbb{R}^n,\ J_x(F) = P \end{array} \right\}.$$

Let $f\colon E \to [0,\infty)$, where $E \subset \mathbb{R}^n$ is finite. For $x \in E$, $M > 0$, let

$$\Gamma'_f(x,M) = \left\{P \in \Gamma'_*(x,M) : P(x) = f(x)\right\}.$$

Then $\vec{\Gamma}'_f := (\Gamma'_f(x,M))_{x\in E, M>0}$ is a $(C,1)$-convex shape field, where C depends only on m, n.

Lemma 5. *Let E, f, $\Gamma'_*(x,M)$ be as in Lemma 4, and let $M > 0$, $\vec{P} = (P^x)_{x\in E} \in Wh(E)$. Suppose we have $P^x \in \Gamma'_*(x,M)$ for all $x \in E$, and $\left|\partial^\beta(P^x - P^y)(x)\right| \leq M\,|x-y|^{m-|\beta|}$ for $x, y \in E$, $|\beta| \leq m-1$. Then there exists $F \in C^{m-1,1}(\mathbb{R}^n)$ such that $J_x(F) = P^x$ for all $x \in E$, and $\|F\|_{C^{m-1,1}(\mathbb{R}^n)} \leq CM$, where C depends only on m, n.*

Similarly, by making small changes in the proof of Theorem 3, we obtain the following result.

Lemma 6. *There exist $k^\#$, C, depending only on m, n for which the following holds.*

Let $E \subset \mathbb{R}^n$ be finite, let $f\colon E \to [0,\infty)$, and let $M_0 > 0$. Suppose that for each $S \subset E$ with $\#(S) \leq k^\#$ there exists $\vec{P}^S = (P^x)_{x\in S} \in \mathrm{Wh}(S)$ such that $P^x \in \Gamma'_f(x, M_0)$ for all $x \in S$, and $\left|\partial^\beta(P^x - P^y)\right| \leq M_0\,|x-y|^{m-|\beta|}$ for $x, y \in S$, $|\beta| \leq m-1$.

Then there exists $F \in C^{m-1,1}(\mathbb{R}^n)$ such that $\|F\|_{C^{m-1,1}(\mathbb{R}^n)} \leq CM_0$, $F \geq 0$ on $\mathbb{R}^n$, and $F = f$ on E.

Now we can easily deduce the following result.

Theorem 4 (Finiteness principle for nonnegative $C^{m-1,1}$-interpolation). *There exists constants $k^{\#}$, C, depending only on m, n for which the following holds.*

Let $f\colon E \to [0,\infty)$, with $E \subset \mathbb{R}^n$ arbitrary (not necessarily finite). Let $M_0 > 0$. Suppose that for each $S \subset E$ with $\#(S) \leq k^{\#}$ there exists $\vec{P} = (P^x)_{x\in S} \in \mathrm{Wh}(S)$ such that

- *$P^x \in \Gamma'_f(x, M_0)$ for all $x \in S$,*
- *$\left|\partial^\beta (P^x - P^y)(x)\right| \leq M_0 |x-y|^{m-|\beta|}$ for $x, y \in S$, $|\beta| \leq m-1$.*

Then there exists $F \in C^{m-1,1}(\mathbb{R}^n)$ such that

- *$||F||_{C^{m-1,1}(\mathbb{R}^n)} \leq CM_0$,*
- *$F \geq 0$, and*
- *$F = f$ on E.*

Proof. Suppose first that $E \subset Q$ for some cube $Q \subset \mathbb{R}^n$. Then by Ascoli's theorem,

$$\left\{F \in C^{m-1,1}(Q) : \|F\|_{C^{m-1,1}(Q)} \leq CM_0,\ F \geq 0 \text{ on } Q\right\} \equiv X$$

is compact in the $C^{m-1}(Q)$-norm topology.

For each finite $E_0 \subset E$, Lemma 6 tells us that there exists $F \in X$ such that $F = f$ on E_0.

Consequently, there exists $F \in X$ such that $F = f$ on E. That is,

$$F \in C^{m-1,1}(Q), \quad \|F\|_{C^{m-1,1}(Q)} \leq CM_0, \quad F \geq 0 \text{ on } Q, \quad F = f \text{ on } E. \tag{4.1}$$

We have achieved (4.1), assuming that $E \subset Q$.

Now suppose $E \subset \mathbb{R}^n$ is arbitrary.

We introduce a partition of unity $1 = \sum_\nu \theta_\nu$ on $\mathbb{R}^n$, with $\theta_\nu \geq 0$ on $\mathbb{R}^n$, $\theta_\nu \in C^m(\mathbb{R}^n)$, $\|\theta_\nu\|_{C^m(\mathbb{R}^n)} \leq C$, support $\theta_\nu \subset Q_\nu$ for a cube $Q_\nu \subset \mathbb{R}^n$, with (say) $\delta_{Q_\nu} = 1$, and such that any given $x \in \mathbb{R}^n$ has a neighborhood that intersects at most C of the Q_ν. (Here C depends only on m, n.)

Applying our result (4.1) to $f|_{E\cap Q_\nu} : E \cap Q_\nu \to [0,\infty)$ for each ν, we obtain functions $F_\nu \in C^{m-1,1}(Q_\nu)$ such that $\|F_\nu\|_{C^{m-1,1}(Q_\nu)} \leq CM_0$, $F_\nu \geq 0$ on Q_ν, $F_\nu = f$ on $E \cap Q_\nu$.

(Here C depends only on m, n.)

We define $F = \sum_\nu \theta_\nu F_\nu$ on $\mathbb{R}^n$. One checks easily that $\|F\|_{C^{m-1,1}(\mathbb{R}^n)} \leq C' M_0$ with C' determined by m, n; $F \geq 0$ on $\mathbb{R}^n$; and $F = f$ on E.

This completes the proof of Theorem 4. □

Note that Theorem 4 easily implies Theorem 1 (b).

As in the case of nonnegative C^m-interpolation, we want to replace $\Gamma'_f(x, M)$ by something easier to calculate. In the $C^{m-1,1}$-setting, it is enough to make the following observation.

Define

$$\tilde{\Gamma}'_0 = \left\{ \begin{array}{c} P \in \mathcal{P} : \left|\partial^\beta P(0)\right| \leq 1 \text{ for } |\beta| \leq m-1 \text{ and} \\ P(x) + |x|^m \geq 0 \text{ for all } x \in \mathbb{R}^n \end{array} \right\}.$$

Then

$$\Gamma'_*(0,c) \subset \tilde{\Gamma}'_0 \subset \tilde{\Gamma}'_*(0,C), \quad \text{with } c, C \text{ depending only on } m, n. \tag{4.2}$$

Indeed, the first inclusion in (4.2) is immediate from the definitions and Taylor's theorem. To prove the second inclusion, we let $P \in \tilde{\Gamma}'_0$ be given, and set $F(x) = \chi(x)(P(x) + |x|^m)$, where χ is a nonnegative C^m function with norm at most C_* (depending only on m, n), satisfying $J_0(\chi) = 1$ and support $\chi \subset B_n(0,1)$.

We then have $F \in C^{m-1,1}(\mathbb{R}^n)$, $\|F\|_{C^{m-1,1}(\mathbb{R}^n)} \leq C$ (depending only on m, n), $F \geq 0$ on $\mathbb{R}^n$, $J_0(F) = P$. Hence, $P \in \Gamma'_*(0,C)$, completing the proof of (4.2).

This concludes our discussion of interpolation by nonnegative $C^{m-1,1}$ functions.

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