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## Nonlocal problems with Neumann boundary conditions

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**Abstract.** We introduce a new Neumann problem for the fractional Laplacian arising from a simple probabilistic consideration, and we discuss the basic properties of this model. We can consider both elliptic and parabolic equations in any domain. In addition, we formulate problems with nonhomogeneous Neumann conditions, and also with mixed Dirichlet and Neumann conditions, all of them having a clear probabilistic interpretation.

We prove that solutions to the fractional heat equation with homogeneous Neumann conditions have the following natural properties: conservation of mass inside  $\Omega$ , decreasing energy, and convergence to a constant as  $t \rightarrow \infty$ . Moreover, for the elliptic case we give the variational formulation of the problem, and establish existence of solutions.

We also study the limit properties and the boundary behavior induced by this nonlocal Neumann condition.

For concreteness, one may think that our nonlocal analogue of the classical Neumann condition  $\partial_\nu u = 0$  on  $\partial\Omega$  consists in the nonlocal prescription

$$\int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy = 0 \quad \text{for } x \in \mathbb{R}^n \setminus \overline{\Omega}.$$

We made an effort to keep all the arguments at the simplest possible technical level, in order to clarify the connections between the different scientific fields that are naturally involved in the problem, and make the paper accessible also to a wide, non-specialistic public (for this scope, we also tried to use and compare different concepts and notations in a somehow more unified way).

### 1. Introduction and results

The aim of this paper is to introduce the following Neumann problem for the fractional Laplacian:

$$(1.1) \quad \begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ \mathcal{N}_s u = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{cases}$$

Here,  $\mathcal{N}_s$  is a new “nonlocal normal derivative”, given by

$$(1.2) \quad \mathcal{N}_s u(x) := c_{n,s} \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy, \quad x \in \mathbb{R}^n \setminus \overline{\Omega}.$$

The normalization constant  $c_{n,s}$  is the one appearing in the definition of the fractional Laplacian

$$(1.3) \quad (-\Delta)^s u(x) = c_{n,s} \text{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy.$$

See [12] and [21] for the basic properties of this operator (and for further details on the normalization constant  $c_{n,s}$ , whose explicit value only plays a minor role in this paper).

As we will see below, the corresponding heat equation with homogeneous Neumann conditions

$$(1.4) \quad \begin{cases} u_t + (-\Delta)^s u = 0 & \text{in } \Omega, & t > 0, \\ \mathcal{N}_s u = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, & t > 0, \\ u(x, 0) = u_0(x) & \text{in } \Omega, & t = 0, \end{cases}$$

possesses natural properties like conservation of mass inside  $\Omega$  or convergence to a constant as  $t \rightarrow +\infty$  (see Section 4).

The probabilistic interpretation of the Neumann problem (1.4) may be summarized as follows:

1.  $u(x, t)$  is the probability distribution of the position of a particle moving randomly inside  $\Omega$ .
2. When the particle exits  $\Omega$ , it immediately comes back into  $\Omega$ .
3. The way in which it comes back inside  $\Omega$  is the following: If the particle has gone to  $x \in \mathbb{R}^n \setminus \overline{\Omega}$ , it may come back to any point  $y \in \Omega$ , the probability density of jumping from  $x$  to  $y$  being proportional to  $|x - y|^{-n-2s}$ .

These three properties lead to the equation (1.4), being  $u_0$  the initial probability distribution of the position of the particle.

A variation of formula (1.2) consists in renormalizing  $\mathcal{N}_s u$  according to the underlying probability law induced by the Lévy process. This leads to the definition

$$(1.5) \quad \tilde{\mathcal{N}}_s u(x) := \frac{\mathcal{N}_s u(x)}{c_{n,s} \int_{\Omega} \frac{dy}{|x-y|^{n+2s}}}.$$

Other Neumann problems for the fractional Laplacian (or other nonlocal operators) were introduced in [4, 9], [1, 3], [10, 11, 8], [15], and [18, 22]. All these different Neumann problems for nonlocal operators recover the classical Neumann problem as a limit case, and most of them has clear probabilistic interpretations as well. We postpone to Section 7 a comparison between these different models and ours.

An advantage of our approach is that the problem has a variational structure. In particular, we show that the classical integration by parts formulae

$$\int_{\Omega} \Delta u = \int_{\partial\Omega} \partial_{\nu} u \quad \text{and} \quad \int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} v (-\Delta)u + \int_{\partial\Omega} v \partial_{\nu} u$$

are replaced in our setting by

$$\int_{\Omega} (-\Delta)^s u \, dx = - \int_{\mathbb{R}^n \setminus \Omega} \mathcal{N}_s u \, dx$$

and

$$\frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dx \, dy = \int_{\Omega} v (-\Delta)^s u + \int_{\mathbb{R}^n \setminus \Omega} v \mathcal{N}_s u.$$

Also, the classical Neumann problem

$$(1.6) \quad \begin{cases} -\Delta u = f & \text{in } \Omega, \\ \partial_{\nu} u = g & \text{on } \partial\Omega, \end{cases}$$

comes from critical points of the energy functional

$$\frac{1}{2} \int_{\Omega} |\nabla u|^2 - \int_{\Omega} f u - \int_{\partial\Omega} g u,$$

without trace conditions. In analogy with this, we show that our nonlocal Neumann condition

$$(1.7) \quad \begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ \mathcal{N}_s u = g & \text{in } \mathbb{R}^n \setminus \bar{\Omega}, \end{cases}$$

follows from free critical points of the energy functional

$$(1.8) \quad \frac{c_{n,s}}{4} \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy - \int_{\Omega} f u - \int_{\mathbb{R}^n \setminus \Omega} g u,$$

see Proposition 3.7. Moreover, as is well known, the theory of existence and uniqueness of solutions for the classical Neumann problem (1.6) relies on the compatibility condition

$$\int_{\Omega} f = - \int_{\partial\Omega} g.$$

We provide the analogue of this compatibility condition in our framework, that is

$$\int_{\Omega} f = - \int_{\mathbb{R}^n \setminus \Omega} g,$$

see Theorem 3.9. Also, we give a description of the spectral properties of our nonlocal problem, which are in analogy with the classical case.

The Neumann-type problems studied in [4, 9] and [10, 11, 8], involving the so-called *regional* fractional Laplacian, have a similar variational structure; see Section 7 for more details.

The paper is organized in this way. In Section 2 we give a probabilistic interpretation of our Neumann condition, as a random reflection of a particle inside the domain, according to a Lévy flight. This also allows us to consider mixed Dirichlet and Neumann conditions and to get a suitable heat equation from the stochastic process.

In Section 3 we consider the variational structure of the associated nonlocal elliptic problem, we show an existence and uniqueness result (namely Theorem 3.9), as follows:

*Let  $\Omega \subset \mathbb{R}^n$  be a bounded Lipschitz domain,  $f \in L^2(\Omega)$ , and  $g \in L^1(\mathbb{R}^n \setminus \Omega)$ . Suppose that there exists a  $C^2$  function  $\psi$  such that  $\mathcal{N}_s \psi = g$  in  $\mathbb{R}^n \setminus \overline{\Omega}$ .*

*Then, problem (1.7) admits a weak solution if and only if*

$$\int_{\Omega} f = - \int_{\mathbb{R}^n \setminus \Omega} g.$$

*Moreover, if such a compatibility condition holds, the solution is unique up to an additive constant.*

Also, we give a description of a sort of generalized eigenvalues of  $(-\Delta)^s$  with zero Neumann boundary conditions (see Theorem 3.11):

*Let  $\Omega \subset \mathbb{R}^n$  be a bounded Lipschitz domain. Then, there exist a sequence of nonnegative values*

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots,$$

*and a sequence of functions  $u_i : \mathbb{R}^n \rightarrow \mathbb{R}$  such that*

$$\begin{cases} (-\Delta)^s u_i(x) = \lambda_i u_i(x) & \text{for any } x \in \Omega, \\ \mathcal{N}_s u_i(x) = 0 & \text{for any } x \in \mathbb{R}^n \setminus \overline{\Omega}. \end{cases}$$

*Also, the functions  $u_i$  (when restricted to  $\Omega$ ) provide a complete orthogonal system in  $L^2(\Omega)$ .*

By similarity with the classical case, we are tempted to consider the above  $\lambda_i$  and  $u_i$  as generalized eigenvalues and eigenfunctions. Though the word “generalized” will be omitted from now on for the sake of shortness, we remark that this spectral notion is not completely standard, since our eigenfunctions  $u_i$  are defined in the whole of  $\mathbb{R}^n$  but satisfy the equation  $(-\Delta)^s u_i = \lambda_i u_i$  only in the domain  $\Omega$  (indeed, outside  $\Omega$  they verify our nonlocal Neumann condition). Moreover, the orthogonality and density properties of  $u_i$  also refer to their restriction in  $\Omega$ .

In Section 4 we discuss the associated heat equation. As it happens in the classical case, we show that such equation preserves the mass, it has decreasing energy, and the solutions approach a constant as  $t \rightarrow +\infty$ . In particular, by the results in Propositions 4.1, 4.2 and 4.3 we have:

Assume that  $u(x, t)$  is a classical solution to

$$\begin{cases} u_t + (-\Delta)^s u = 0 & \text{in } \Omega, & t > 0, \\ \mathcal{N}_s u = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, & t > 0, \\ u(x, 0) = u_0(x) & \text{in } \Omega, & t = 0. \end{cases}$$

Then the total mass is conserved, i.e., for all  $t > 0$ ,

$$\int_{\Omega} u(x, t) \, dx = \int_{\Omega} u_0(x) \, dx.$$

Moreover, the energy

$$E(t) = \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{|u(x, t) - u(y, t)|^2}{|x - y|^{n+2s}} \, dx \, dy$$

is decreasing in time  $t > 0$ .

Finally, the solution approaches a constant for large times: more precisely,

$$u \longrightarrow \frac{1}{|\Omega|} \int_{\Omega} u_0 \quad \text{in } L^2(\Omega) \quad \text{as } t \rightarrow +\infty.$$

In Section 5 we compute some limits when  $s \rightarrow 1$ , showing that we can recover the classical case, in the sense that we show in Proposition 5.1:

Let  $\Omega \subset \mathbb{R}^n$  be any bounded Lipschitz domain. Let  $u$  and  $v$  be  $C_0^2(\mathbb{R}^n)$  functions. Then,

$$\lim_{s \rightarrow 1} \int_{\mathbb{R}^n \setminus \Omega} \mathcal{N}_s u \, v = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \, v.$$

Also, we prove that nice functions can be extended continuously outside  $\overline{\Omega}$  in order to satisfy a homogeneous nonlocal Neumann condition, and we characterize the boundary behavior of the nonlocal Neumann function. More precisely, in Proposition 5.2 we show that:

Let  $\Omega \subset \mathbb{R}^n$  be a domain with  $C^1$  boundary. Let  $u$  be continuous in  $\overline{\Omega}$ , with  $\mathcal{N}_s u = 0$  in  $\mathbb{R}^n \setminus \overline{\Omega}$ . Then  $u$  is continuous in the whole of  $\mathbb{R}^n$ .

The boundary behavior of the nonlocal Neumann condition is also addressed in Proposition 5.4:

Let  $\Omega \subset \mathbb{R}^n$  be a  $C^1$  domain, and  $u \in C(\mathbb{R}^n)$ . Then, for all  $s \in (0, 1)$ ,

$$\lim_{\substack{x \rightarrow \partial\Omega \\ x \in \mathbb{R}^n \setminus \overline{\Omega}}} \tilde{\mathcal{N}}_s u(x) = 0,$$

where  $\tilde{\mathcal{N}}$  is defined by (1.5).

Also, if  $s > 1/2$  and  $u \in C^{1,\alpha}(\mathbb{R}^n)$  for some  $\alpha > 0$ , then

$$\partial_{\nu} \tilde{\mathcal{N}}_s u(x) := \lim_{\epsilon \rightarrow 0^+} \frac{\tilde{\mathcal{N}}_s u(x + \epsilon \nu)}{\epsilon} = \kappa \partial_{\nu} u \quad \text{for any } x \in \partial\Omega,$$

for some constant  $\kappa > 0$ .

Later on, in Section 6 we deal with an overdetermined problem and we show that, in general, it is not possible to prescribe both nonlocal Neumann and Dirichlet conditions for a continuous function.

Finally, in Section 7 we recall the various nonlocal Neumann conditions already appeared in the literature, and we compare them with our model.

All the arguments presented are of elementary<sup>1</sup> nature.

## 2. Heuristic probabilistic interpretation

In this section we will give a simple probabilistic interpretation of the nonlocal Neumann condition that we consider in terms of the so-called Lévy flights. Though the possible behavior of a general Lévy process can be more sophisticated than the one we consider, for the sake of clarity we will try to restrict ourselves to the simplest possible scenario and to use the simplest possible language. For this scope, we will not go into all the very rich details of the related probability theory and we will not aim to review all the important, recent results on the topic, but we will rather present an elementary, self-contained exposition, which we hope can serve as an introduction also to a non-specialistic public.

Let us consider the Lévy process in  $\mathbb{R}^n$  whose infinitesimal generator is the fractional Laplacian  $(-\Delta)^s$ . Heuristically, we may think that this process represents the (random) movement of a particle along time  $t > 0$ . As it is well known, the probability density  $u(x, t)$  of the position of the particle solves the fractional heat equation  $u_t + (-\Delta)^s u = 0$  in  $\mathbb{R}^n$ ; see [23] for a simple illustration of this fact.

Recall that when the particle is situated at  $x \in \mathbb{R}^n$ , it may jump to any other point  $y \in \mathbb{R}^n$ , the probability density of jumping to  $y$  being proportional to  $|x - y|^{-n-2s}$ .

In a similar way, one may consider the random movement of a particle inside a bounded domain  $\Omega \subset \mathbb{R}^n$ , but in this case one has to decide what happens when the particle leaves  $\Omega$ .

In the classical case  $s = 1$  (when the Lévy process is the Brownian motion), we have the following:

- (1) If the particle is *killed* when it reaches the boundary  $\partial\Omega$ , then the probability distribution solves the heat equation with homogeneous *Dirichlet* conditions.
- (2) If, instead, when the particle reaches the boundary  $\partial\Omega$  it immediately *comes back into*  $\Omega$  (i.e., it bounces on  $\partial\Omega$ ), then the probability distribution solves the heat equation with homogeneous *Neumann* conditions.

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<sup>1</sup>To keep the notation as simple as possible, given functions  $f$  and  $g$  and an operator  $T$ , we will often write identities like “ $f = g$  in  $\Omega$ ” or “ $Tf = g$  in  $\Omega$ ” to mean “ $f(x) = g(x)$  for every  $x \in \Omega$ ” or “ $Tf = g$  for every  $x \in \Omega$ ”, respectively. Also, if  $u: \mathbb{R}^n \rightarrow \mathbb{R}$ , we often denote the restriction of  $u$  to  $\Omega$  again by  $u$ . We hope that this slight abuse of notation creates no problem to the reader, but for the sake of clarity we also include an appendix at the end of the paper in which Theorems 3.9 and 3.11 are proved using a functional analysis notation that distinguishes between a function and its restriction.

In the nonlocal case  $s \in (0, 1)$ , in which the process has jumps, case (1) corresponds to the following: the particle is killed when it exits  $\Omega$ . In this case, the probability distribution  $u$  of the process solves the heat equation with homogeneous Dirichlet conditions  $u = 0$  in  $\mathbb{R}^n \setminus \Omega$ , and solutions to this problem are well understood; see for example [19], [14], [13], and [2].

The analogue of case (2) is the following: when the particle exits  $\Omega$ , it immediately comes back into  $\Omega$ . Of course, one has to decide how the particle comes back into the domain.

In [1] and [3], the idea was to find a deterministic “reflection” or “projection” which describes the way in which the particle comes back into  $\Omega$ .

The alternative that we propose here is the following: if the particle has gone to  $x \in \mathbb{R}^n \setminus \overline{\Omega}$ , then it may come back to *any* point  $y \in \Omega$ , the probability density of jumping from  $x$  to  $y$  being proportional to  $|x - y|^{-n-2s}$ .

Notice that this is exactly the (random) way as the particle is moving all the time, here we just add the restriction that it has to immediately come back into  $\Omega$  every time it goes outside.

Let us finally illustrate how this random process leads to problems (1.1) or (1.4). In fact, to make the exposition easier, we will explain the case of mixed Neumann and Dirichlet conditions, which, we think, is very natural.

### 2.1. Mixed Dirichlet and Neumann conditions

Assume that we have some domain  $\Omega \subset \mathbb{R}^n$ , and that its complement  $\mathbb{R}^n \setminus \overline{\Omega}$  is split into two parts:  $N$  (with Neumann conditions), and  $D$  (with Dirichlet conditions).

Consider a particle moving randomly, starting inside  $\Omega$ . When the particle reaches  $D$ , it obtains a payoff  $\phi(x)$ , which depends on the point  $x \in D$  where the particle arrived. Instead, when the particle reaches  $N$  it immediately comes back to  $\Omega$  as described before.

If we denote  $u(x)$  the expected payoff, then we clearly have

$$(-\Delta)^s u = 0 \text{ in } \Omega, \quad \text{and} \quad u = \phi \text{ in } D,$$

where  $\phi : D \rightarrow \mathbb{R}$  is a given function.

Moreover, recall that when the particle is in  $x \in N$  then it goes back to some point  $y \in \Omega$ , with probability proportional to  $|x - y|^{-n-2s}$ . Hence, we have that

$$u(x) = \kappa \int_{\Omega} \frac{u(y)}{|x - y|^{n+2s}} dy \quad \text{for } x \in N,$$

for some constant  $\kappa$ , possibly depending on the point  $x \in N$ , that has been fixed. In order to normalize the probability measure, the value of the constant  $\kappa$  is so that

$$\kappa \int_{\Omega} \frac{dy}{|x - y|^{n+2s}} = 1.$$

Finally, the previous identity can be written as

$$\mathcal{N}_s u(x) = c_{n,s} \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy = 0 \quad \text{for } x \in N,$$

and therefore  $u$  solves

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } \Omega, \\ \mathcal{N}_s u = 0 & \text{in } N, \\ u = \phi & \text{in } D, \end{cases}$$

which is a nonlocal problem with mixed Neumann and Dirichlet conditions.

Note that the previous problem is the nonlocal analogue of

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{in } \Gamma_N, \\ u = \phi & \text{in } \Gamma_D, \end{cases}$$

being  $\Gamma_D$  and  $\Gamma_N$  two disjoint subsets of  $\partial\Omega$ , in which classical Dirichlet and Neumann boundary conditions are prescribed.

More generally, the classical Robin condition  $a\partial_\nu u + bu = c$  on some  $\Gamma_R \subseteq \partial\Omega$  may be replaced in our nonlocal framework by  $a\mathcal{N}_s u + bu = c$  on some  $R \subseteq \mathbb{R}^n \setminus \overline{\Omega}$ . Nonlinear boundary conditions may be considered in a similar way.

### 2.2. Fractional heat equation, nonhomogeneous Neumann conditions

Let us consider now the random movement of the particle inside  $\Omega$ , with our new Neumann conditions in  $\mathbb{R}^n \setminus \overline{\Omega}$ .

Denoting  $u(x, t)$  the probability density of the position of the particle at time  $t > 0$ , with a similar discretization argument as in [23], one can see that  $u$  solves the fractional heat equation

$$u_t + (-\Delta)^s u = 0 \quad \text{in } \Omega \quad \text{for } t > 0,$$

with

$$\mathcal{N}_s u = 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega} \quad \text{for } t > 0.$$

Thus, if  $u_0$  is the initial probability density, then  $u$  solves problem (1.4).

Of course, one can now see that with this probabilistic interpretation there is no problem in considering a right hand side  $f$  or nonhomogeneous Neumann conditions

$$\begin{cases} u_t + (-\Delta)^s u = f(x, t, u) & \text{in } \Omega, \\ \mathcal{N}_s u = g(x, t) & \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{cases}$$

In this case,  $g$  represents a “nonlocal flux” of new particles coming from outside  $\Omega$ , and  $f$  would represent a reaction term.

### 3. The elliptic problem

Given  $g \in L^1(\mathbb{R}^n \setminus \Omega)$  and measurable functions  $u, v: \mathbb{R}^n \rightarrow \mathbb{R}$ , we set

$$(3.1) \quad \|u\|_{H_{\Omega, g}^s} := \left( \|u\|_{L^2(\Omega)}^2 + \| |g|^{1/2} u \|_{L^2(\mathbb{R}^n \setminus \Omega)}^2 + \int_{\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy \right)^{1/2}$$



and

$$(3.2) \quad \begin{aligned} (u, v)_{H_{\Omega, g}^s} &:= \int_{\Omega} u v \, dx + \int_{\mathbb{R}^n \setminus \Omega} |g| u v \, dx \\ &+ \int_{\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dx \, dy. \end{aligned}$$

Then, we define the space

$$H_{\Omega, g}^s := \{u : \mathbb{R}^n \rightarrow \mathbb{R} \text{ measurable} : \|u\|_{H_{\Omega, g}^s} < +\infty\}.$$

We will also write  $H_{\Omega, 0}^s$  to mean  $H_{\Omega, g}^s$  with  $g \equiv 0$ .

**Proposition 3.1.**  $H_{\Omega, g}^s$  is a Hilbert space with the scalar product defined in (3.2).

*Proof.* We point out that (3.2) is a bilinear form and  $\|u\|_{H_{\Omega, g}^s} = ((u, u)_{H_{\Omega, g}^s})^{1/2}$ . Also, if  $\|u\|_{H_{\Omega, g}^s} = 0$ , it follows that  $\|u\|_{L^2(\Omega)} = 0$ , hence  $u = 0$  a.e. in  $\Omega$ , and that

$$\int_{\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy = 0,$$

which in turn implies that  $|u(x) - u(y)| = 0$  for any  $(x, y) \in \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2$ . In particular, a.e.  $x \in \mathcal{C}\Omega$  and  $y \in \Omega$  we have that

$$u(x) = u(y) = 0.$$

This shows that  $u = 0$  a.e. in  $\mathbb{R}^n$ , so it remains to prove that  $H_{\Omega, g}^s$  is complete. For this, we take a Cauchy sequence  $u_k$  with respect to the norm in (3.1).

In particular,  $u_k$  is a Cauchy sequence in  $L^2(\Omega)$  and therefore, up to a subsequence, we suppose that  $u_k$  converges to some  $u$  in  $L^2(\Omega)$  and a.e. in  $\Omega$ . More explicitly, there exists  $Z_1 \subset \mathbb{R}^n$  such that

$$(3.3) \quad |Z_1| = 0 \quad \text{and} \quad u_k(x) \rightarrow u(x) \quad \text{for every } x \in \Omega \setminus Z_1.$$

Also, given any  $U : \mathbb{R}^n \rightarrow \mathbb{R}$ , for any  $(x, y) \in \mathbb{R}^{2n}$  we define

$$(3.4) \quad E_U(x, y) := \frac{(U(x) - U(y)) \chi_{\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2}(x, y)}{|x - y|^{(n+2s)/2}}.$$

Notice that

$$E_{u_k}(x, y) - E_{u_h}(x, y) = \frac{(u_k(x) - u_h(x) - u_k(y) + u_h(y)) \chi_{\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2}(x, y)}{|x - y|^{(n+2s)/2}}.$$

Accordingly, since  $u_k$  is a Cauchy sequence in  $H_{\Omega, g}^s$ , for any  $\epsilon > 0$  there exists  $N_\epsilon \in \mathbb{N}$  such that, if  $h, k \geq N_\epsilon$ , then

$$\epsilon^2 \geq \int_{\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2} \frac{|(u_k - u_h)(x) - (u_k - u_h)(y)|^2}{|x - y|^{n+2s}} \, dx \, dy = \|E_{u_k} - E_{u_h}\|_{L^2(\mathbb{R}^{2n})}^2.$$

That is,  $E_{u_k}$  is a Cauchy sequence in  $L^2(\mathbb{R}^{2n})$  and thus, up to a subsequence, we assume that  $E_{u_k}$  converges to some  $E$  in  $L^2(\mathbb{R}^{2n})$  and a.e. in  $\mathbb{R}^{2n}$ . More explicitly, there exists  $Z_2 \subset \mathbb{R}^{2n}$  such that

$$(3.5) \quad |Z_2| = 0 \quad \text{and} \quad E_{u_k}(x, y) \rightarrow E(x, y) \quad \text{for every } (x, y) \in \mathbb{R}^{2n} \setminus Z_2.$$

Now, for any  $x \in \Omega$ , we set

$$\begin{aligned} S_x &:= \{y \in \mathbb{R}^n : (x, y) \in \mathbb{R}^{2n} \setminus Z_2\}, \\ W &:= \{(x, y) \in \mathbb{R}^{2n} : x \in \Omega \text{ and } y \in \mathbb{R}^n \setminus S_x\} \\ \text{and} \quad V &:= \{x \in \Omega : |\mathbb{R}^n \setminus S_x| = 0\}. \end{aligned}$$

We remark that

$$(3.6) \quad W \subseteq Z_2.$$

Indeed, if  $(x, y) \in W$ , then  $y \in \mathbb{R}^n \setminus S_x$ , hence  $(x, y) \notin \mathbb{R}^{2n} \setminus Z_2$ , and so  $(x, y) \in Z_2$ , which gives (3.6).

Using (3.5) and (3.6), we obtain that  $|W| = 0$ , hence by the Fubini theorem we have that

$$0 = |W| = \int_{\Omega} |\mathbb{R}^n \setminus S_x| dx,$$

which implies that  $|\mathbb{R}^n \setminus S_x| = 0$  for a.e.  $x \in \Omega$ .

As a consequence, we conclude that  $|\Omega \setminus V| = 0$ . This and (3.3) imply that

$$|\Omega \setminus (V \setminus Z_1)| = |(\Omega \setminus V) \cup Z_1| \leq |\Omega \setminus V| + |Z_1| = 0.$$

In particular  $V \setminus Z_1 \neq \emptyset$ , so we can fix  $x_0 \in V \setminus Z_1$ .

Since  $x_0 \in \Omega \setminus Z_1$ , equation (3.3) implies

$$\lim_{k \rightarrow +\infty} u_k(x_0) = u(x_0).$$

Furthermore, since  $x_0 \in V$  we have that  $|\mathbb{R}^n \setminus S_{x_0}| = 0$ . As a consequence, a.e.  $y \in \mathbb{R}^n$  (namely, for every  $y \in S_{x_0}$ ), we have that  $(x_0, y) \in \mathbb{R}^{2n} \setminus Z_2$  and so

$$\lim_{k \rightarrow +\infty} E_{u_k}(x_0, y) = E(x_0, y),$$

thanks to (3.5). Notice also that  $\Omega \times (\mathcal{C}\Omega) \subseteq \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2$  and so, recalling (3.4), we get

$$E_{u_k}(x_0, y) := \frac{u_k(x_0) - u_k(y)}{|x_0 - y|^{(n+2s)/2}},$$

for a.e.  $y \in \mathcal{C}\Omega$ . Thus, we obtain

$$\begin{aligned} \lim_{k \rightarrow +\infty} u_k(y) &= \lim_{k \rightarrow +\infty} \{u_k(x_0) - |x_0 - y|^{(n+2s)/2} E_{u_k}(x_0, y)\} \\ &= u(x_0) - |x_0 - y|^{(n+2s)/2} E(x_0, y), \end{aligned}$$

a.e.  $y \in \mathcal{C}\Omega$ .

This and (3.3) say that  $u_k$  converges a.e. in  $\mathbb{R}^n$ . Up to a change of notation, we will say that  $u_k$  converges a.e. in  $\mathbb{R}^n$  to some  $u$ . So, using that  $u_k$  is a Cauchy sequence in  $H_{\Omega,g}^s$ , fixed any  $\epsilon > 0$  there exists  $N_\epsilon \in \mathbb{N}$  such that, for any  $h \geq N_\epsilon$ ,

$$\begin{aligned} \epsilon^2 &\geq \liminf_{k \rightarrow +\infty} \|u_h - u_k\|_{H_{\Omega,g}^s}^2 \\ &\geq \liminf_{k \rightarrow +\infty} \int_{\Omega} (u_h - u_k)^2 + \liminf_{k \rightarrow +\infty} \int_{C\Omega} |g|(u_h - u_k)^2 \\ &\quad + \liminf_{k \rightarrow +\infty} \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{|(u_h - u_k)(x) - (u_h - u_k)(y)|^2}{|x - y|^{n+2s}} dx dy \\ &\geq \int_{\Omega} (u_h - u)^2 + \int_{C\Omega} |g|(u_h - u)^2 \\ &\quad + \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{|(u_h - u)(x) - (u_h - u)(y)|^2}{|x - y|^{n+2s}} dx dy \\ &= \|u_h - u\|_{H_{\Omega,g}^s}^2, \end{aligned}$$

where Fatou’s lemma was used. This says that  $u_h$  converges to  $u$  in  $H_{\Omega,g}^s$ , showing that  $H_{\Omega,g}^s$  is complete.  $\square$

### 3.1. Some integration by parts formulas

The following is a nonlocal analogue of the divergence theorem.

**Lemma 3.2.** *Let  $u$  be any bounded  $C^2$  function in  $\mathbb{R}^n$ . Then,*

$$\int_{\Omega} (-\Delta)^s u = - \int_{\mathbb{R}^n \setminus \Omega} \mathcal{N}_s u.$$

*Proof.* Note that

$$\int_{\Omega} \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dx dy = \int_{\Omega} \int_{\Omega} \frac{u(y) - u(x)}{|x - y|^{n+2s}} dx dy = 0,$$

since the role of  $x$  and  $y$  in the integrals above is symmetric. Hence, we have that

$$\begin{aligned} \int_{\Omega} (-\Delta)^s u dx &= c_{n,s} \int_{\Omega} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy dx = c_{n,s} \int_{\Omega} \int_{\mathbb{R}^n \setminus \Omega} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy dx \\ &= c_{n,s} \int_{\mathbb{R}^n \setminus \Omega} \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dx dy = - \int_{\mathbb{R}^n \setminus \Omega} \mathcal{N}_s u(y) dy, \end{aligned}$$

as desired.  $\square$

More generally, we have the following integration by parts formula.

**Lemma 3.3.** *Let  $u$  and  $v$  be bounded  $C^2$  functions in  $\mathbb{R}^n$ . Then,*

$$\frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy = \int_{\Omega} v (-\Delta)^s u + \int_{\mathbb{R}^n \setminus \Omega} v \mathcal{N}_s u,$$

where  $c_{n,s}$  is the constant in (1.3).

*Proof.* Notice that

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy \\ &= \int_{\Omega} \int_{\mathbb{R}^n} v(x) \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy dx + \int_{\mathbb{R}^n \setminus \Omega} \int_{\Omega} v(x) \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy dx. \end{aligned}$$

Thus, using (1.3) and (1.2), the identity follows. □

**Remark 3.4.** We recall that if one takes  $\partial_\nu u = 1$ , then one can obtain the perimeter of  $\Omega$  by integrating this Neumann condition over  $\partial\Omega$ . Indeed,

$$(3.7) \quad |\partial\Omega| = \int_{\partial\Omega} dx = \int_{\partial\Omega} \partial_\nu u dx.$$

Analogously, we can define  $\tilde{\mathcal{N}}_s u$ , by renormalizing  $\mathcal{N}_s u$  by a factor

$$w_{s,\Omega}(x) := c_{n,s} \int_{\Omega} \frac{dy}{|x - y|^{n+2s}},$$

that is

$$(3.8) \quad \tilde{\mathcal{N}}_s u(x) := \frac{\mathcal{N}_s u(x)}{w_{s,\Omega}(x)} \quad \text{for } x \in \mathbb{R}^n \setminus \overline{\Omega}.$$

Now, we observe that if  $\tilde{\mathcal{N}}_s u(x) = 1$  for any  $x \in \mathbb{R}^n \setminus \overline{\Omega}$ , then we find the fractional perimeter (see [6] where this object was introduced) by integrating such nonlocal Neumann condition over  $\mathbb{R}^n \setminus \Omega$ , that is,

$$\begin{aligned} \text{Per}_s(\Omega) &:= c_{n,s} \int_{\Omega} \int_{\mathbb{R}^n \setminus \Omega} \frac{dx dy}{|x - y|^{n+2s}} = \int_{\mathbb{R}^n \setminus \Omega} w_{s,\Omega}(x) dx \\ &= \int_{\mathbb{R}^n \setminus \Omega} w_{s,\Omega}(x) \tilde{\mathcal{N}}_s u(x) dx = \int_{\mathbb{R}^n \setminus \Omega} \mathcal{N}_s u(x) dx, \end{aligned}$$

that can be seen as the nonlocal counterpart of (3.7).

**Remark 3.5.** The renormalized Neumann condition in (3.8) can also be framed into the probabilistic interpretation of Section 2.

Indeed suppose that  $\mathcal{C}\Omega$  is partitioned into a Dirichlet part  $D$  and a Neumann part  $N$  and that:

- our Lévy process receives a final payoff  $\phi(x)$  when it leaves the domain  $\Omega$  by landing at the point  $x$  in  $D$ ,
- if the Lévy process leaves  $\Omega$  by landing at the point  $x$  in  $N$ , then it receives an additional payoff  $\psi(x)$  and is forced to come back to  $\Omega$  and keep running by following the same probability law (the case discussed in Section 2 is the model situation in which  $\psi \equiv 0$ ).

In this setting, the expected payoff  $u(x)$  obtained by starting the process at the point  $x \in \Omega$  satisfies  $(-\Delta)^s u = 0$  in  $\Omega$  and  $u = \phi$  in  $D$ . Also, for any  $x \in N$ , the expected payoff landing at  $x$  must be equal to the additional payoff  $\psi(x)$  plus the average payoff  $u(y)$  obtained by jumping from  $x$  to  $y \in \Omega$ , that is,

$$\text{for any } x \in N, \quad u(x) = \psi(x) + \frac{\int_{\Omega} \frac{u(y)}{|x-y|^{n+2s}} dy}{\int_{\Omega} \frac{dy}{|x-y|^{n+2s}}},$$

which corresponds to  $\tilde{\mathcal{N}}_s u(x) = \psi(x)$ .

**3.2. Weak solutions with Neumann conditions**

The integration by parts formula from Lemma 3.3 leads to the following.

**Definition 3.6.** Let  $f \in L^2(\Omega)$  and  $g \in L^1(\mathbb{R}^n \setminus \Omega)$ . Let  $u \in H_{\Omega,0}^s$ . We say that  $u$  is a weak solution of

$$(3.9) \quad \begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ \mathcal{N}_s u = g & \text{in } \mathbb{R}^n \setminus \bar{\Omega}, \end{cases}$$

whenever

$$(3.10) \quad \frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy = \int_{\Omega} f v + \int_{\mathbb{R}^n \setminus \Omega} g v$$

for all test functions  $v \in H_{\Omega,g}^s$ .

With this definition, we can prove the following.

**Proposition 3.7.** Let  $f \in L^2(\Omega)$  and  $g \in L^1(\mathbb{R}^n \setminus \Omega)$ . Let  $I: H_{\Omega,g}^s \rightarrow \mathbb{R}$  be the functional defined, for every  $u \in H_{\Omega,g}^s$ , as

$$I[u] := \frac{c_{n,s}}{4} \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy - \int_{\Omega} f u - \int_{\mathbb{R}^n \setminus \Omega} g u.$$

Then any critical point of  $I$  is a weak solution of (3.9).

*Proof.* First of all, we observe that the functional  $I$  is well defined on  $H_{\Omega,g}^s$ . Indeed, if  $u \in H_{\Omega,g}^s$  then

$$\left| \int_{\Omega} f u \right| \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \leq C \|u\|_{H_{\Omega,g}^s},$$

and

$$\left| \int_{\mathbb{R}^n \setminus \Omega} g u \right| \leq \int_{\mathbb{R}^n \setminus \Omega} |g|^{1/2} |g|^{1/2} |u| \leq \|g\|_{L^1(\mathbb{R}^n \setminus \Omega)}^{1/2} \| |g|^{1/2} u \|_{L^2(\mathbb{R}^n \setminus \Omega)} \leq C \|u\|_{H_{\Omega,g}^s}.$$

Therefore, if  $u \in H_{\Omega,g}^s$  we have that

$$|I[u]| \leq C \|u\|_{H_{\Omega,g}^s} < +\infty.$$

Now, we compute the first variation of  $I$ . For this, we take  $|\epsilon| < 1$  and  $v \in H_{\Omega, g}^s$ . Then the function  $u + \epsilon v \in H_{\Omega, g}^s$ , and so we can compute

$$\begin{aligned} I[u + \epsilon v] &= \frac{c_{n,s}}{4} \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{|(u + \epsilon v)(x) - (u + \epsilon v)(y)|^2}{|x - y|^{n+2s}} dx dy \\ &\quad - \int_{\Omega} f(u + \epsilon v) - \int_{\mathbb{R}^n \setminus \Omega} g(u + \epsilon v) \\ &= I(u) \\ &\quad + \epsilon \left( \frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy - \int_{\Omega} f v - \int_{\mathbb{R}^n \setminus \Omega} g v \right) \\ &\quad + \frac{c_{n,s}}{4} \epsilon^2 \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}} dx dy. \end{aligned}$$

Hence,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{I[u + \epsilon v] - I[u]}{\epsilon} &= \frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy - \int_{\Omega} f v - \int_{\mathbb{R}^n \setminus \Omega} g v, \end{aligned}$$

which means that

$$I'[u](v) = \frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy - \int_{\Omega} f v - \int_{\mathbb{R}^n \setminus \Omega} g v.$$

Therefore, if  $u$  is a critical point of  $I$ , then  $u$  is a weak solution to (3.9), according to Definition 3.6.  $\square$

Next result is a sort of maximum principle and it is auxiliary towards the existence and uniqueness theory provided in the subsequent Theorem 3.9.

**Lemma 3.8.** *Let  $f \in L^2(\Omega)$  and  $g \in L^1(\mathbb{R}^n \setminus \Omega)$ . Let  $u$  be any  $H_{\Omega, 0}^s$  function satisfying, in the weak sense,*

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ \mathcal{N}_s u = g & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \end{cases}$$

with  $f \geq 0$  and  $g \geq 0$ .

Then,  $u$  is constant.

*Proof.* First, we observe that the function  $v \equiv 1$  belongs to  $H_{\Omega, g}^s$ , and therefore we can use it as a test function in (3.10), obtaining that

$$0 \leq \int_{\Omega} f = - \int_{\mathbb{R}^n \setminus \Omega} g \leq 0.$$

This implies that

$$f = 0 \quad \text{a.e. in } \Omega, \quad \text{and} \quad g = 0 \quad \text{a.e. in } \mathbb{R}^n \setminus \Omega.$$

Therefore, taking  $v = u$  as a test function in (3.10), we deduce that

$$\int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy = 0,$$

and hence  $u$  must be constant. □

We can now give the following existence and uniqueness result (we observe that its statement is in complete analogy<sup>2</sup> with the classical case, see e.g. [16], p. 294).

**Theorem 3.9.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded Lipschitz domain, let  $f \in L^2(\Omega)$ , and let  $g \in L^1(\mathbb{R}^n \setminus \Omega)$ . Suppose that there exists a  $C^2$  function  $\psi$  such that  $\mathcal{N}_s \psi = g$  in  $\mathbb{R}^n \setminus \bar{\Omega}$ .*

*Then, problem (3.9) admits a weak solution in  $H^s_{\Omega,0}$  if and only if*

$$(3.11) \quad \int_{\Omega} f = - \int_{\mathbb{R}^n \setminus \Omega} g.$$

*Moreover, in case that (3.11) holds, the solution is unique up to an additive constant.*

*Proof.* *Case 1.* We do first the case  $g \equiv 0$ , i.e., with homogeneous nonlocal Neumann conditions. We also assume that  $f \not\equiv 0$ , otherwise there is nothing to prove.

Given  $h \in L^2(\Omega)$ , we look for a solution  $v \in H^s_{\Omega,g}$  of the problem

$$(3.12) \quad \int_{\Omega} v \varphi + \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{(v(x) - v(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} dx dy = \int_{\Omega} h \varphi,$$

for any  $\varphi \in H^s_{\Omega,g}$ , with homogeneous Neumann conditions  $\mathcal{N}_s v = 0$  in  $\mathbb{R}^n \setminus \bar{\Omega}$ .

We consider the functional  $\mathcal{F}: H^s_{\Omega,g} \rightarrow \mathbb{R}$  defined as

$$\mathcal{F}(\varphi) := \int_{\Omega} h \varphi \quad \text{for any } \varphi \in H^s_{\Omega,g}.$$

It is easy to see that  $\mathcal{F}$  is linear. Moreover, it is continuous on  $H^s_{\Omega,g}$ :

$$|\mathcal{F}(\varphi)| \leq \int_{\Omega} |h| |\varphi| \leq \|h\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)} \leq \|h\|_{L^2(\Omega)} \|\varphi\|_{H^s_{\Omega,g}}.$$

Therefore, from the Riesz representation theorem it follows that problem (3.12) admits a unique solution  $v \in H^s_{\Omega,g}$  for any given  $h \in L^2(\Omega)$ .

Furthermore, taking  $\varphi := v$  in (3.12), one obtains that

$$(3.13) \quad \|v\|_{H^s(\Omega)} \leq C \|h\|_{L^2(\Omega)}.$$

---

<sup>2</sup>The only difference with the classical case is that in Theorem 3.9 it is *not* necessary to suppose that the domain is connected in order to obtain the uniqueness result.

Now, we define the operator  $T_o: L^2(\Omega) \rightarrow H^s_{\Omega,g}$  as  $T_o h = v$ . We also define by  $T$  the restriction operator in  $\Omega$ , that is,

$$Th = T_o h|_{\Omega}.$$

That is, the function  $T_o h$  is defined in the whole of  $\mathbb{R}^n$ , then we take  $Th$  to be its restriction in  $\Omega$ . In this way, we see that  $T: L^2(\Omega) \rightarrow L^2(\Omega)$ .

We have that  $T$  is compact. Indeed, we take a sequence  $\{h_k\}_{k \in \mathbb{N}}$  bounded in  $L^2(\Omega)$ . Hence, from (3.13) we deduce that the sequence of  $Th_k$  is bounded in  $H^s(\Omega)$ , which is compactly embedded in  $L^2(\Omega)$  (see e.g. [12]). Therefore, there exists a subsequence that converges in  $L^2(\Omega)$ .

Now, we show that  $T$  is self-adjoint. For this, to avoid any smoothness issue on the test function, we will proceed by approximation. We take  $h_1, h_2 \in C^\infty_0(\Omega)$  and we use the weak formulation in (3.12) to say that, for every  $\varphi, \phi \in H^s_{\Omega,g}$ , we have

$$(3.14) \quad \int_{\Omega} T_o h_1 \varphi + \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{(T_o h_1(x) - T_o h_1(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} dx dy = \int_{\Omega} h_1 \varphi,$$

and

$$(3.15) \quad \int_{\Omega} T_o h_2 \phi + \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{(T_o h_2(x) - T_o h_2(y))(\phi(x) - \phi(y))}{|x - y|^{n+2s}} dx dy = \int_{\Omega} h_2 \phi,$$

Now we take  $\varphi := T_o h_2$  and  $\phi := T_o h_1$  in (3.14) and (3.15) respectively and we obtain that

$$\int_{\Omega} h_1 T_o h_2 = \int_{\Omega} T_o h_1 h_2$$

for any  $h_1, h_2 \in C^\infty_0(\Omega)$ . Accordingly, since  $T_o h_1 = Th_1$  and  $T_o h_2 = Th_2$  in  $\Omega$ , we conclude that

$$(3.16) \quad \int_{\Omega} h_1 Th_2 = \int_{\Omega} Th_1 h_2$$

for any  $h_1, h_2 \in C^\infty_0(\Omega)$ . If  $h_1, h_2 \in L^2(\Omega)$ , there exist sequences of functions in  $C^\infty_0(\Omega)$ , say  $h_{1,k}$  and  $h_{2,k}$ , such that  $h_{1,k} \rightarrow h_1$  and  $h_{2,k} \rightarrow h_2$  in  $L^2(\Omega)$  as  $k \rightarrow +\infty$ . From (3.16) we have that

$$(3.17) \quad \int_{\Omega} h_{1,k} Th_{2,k} = \int_{\Omega} Th_{1,k} h_{2,k}.$$

Moreover, from (3.13) we deduce that  $Th_{1,k} \rightarrow Th_1$  and  $Th_{2,k} \rightarrow Th_2$  in  $H^s(\Omega)$  as  $k \rightarrow +\infty$ , and so

$$\int_{\Omega} h_{1,k} Th_{2,k} \rightarrow \int_{\Omega} h_1 Th_2 \quad \text{as } k \rightarrow +\infty$$

and

$$\int_{\Omega} Th_{1,k} h_{2,k} \rightarrow \int_{\Omega} Th_1 h_2 \quad \text{as } k \rightarrow +\infty.$$



The last two formulas and (3.17) imply that

$$(3.18) \quad \int_{\Omega} h_1 T h_2 = \int_{\Omega} T h_1 h_2 \quad \text{for any } h_1, h_2 \in L^2(\Omega),$$

which says that  $T$  is self-adjoint.

Now we prove that

$$(3.19) \quad \text{Ker}(\text{Id} - T) \text{ consists of constant functions.}$$

First of all, we check that the constants are in  $\text{Ker}(\text{Id} - T)$ . We take a function constantly equal to  $c$  and we observe that  $(-\Delta)^s c = 0$  in  $\Omega$  (hence  $(-\Delta)^s c + c = c$ ) and  $\mathcal{N}_s c = 0$  in  $\mathbb{R}^n \setminus \Omega$ . This shows that  $T_o c = c$  in  $\mathbb{R}^n$ , and so  $Tc = c$  in  $\Omega$ , which implies that  $c \in \text{Ker}(\text{Id} - T)$ . Viceversa, now we show that if  $v \in \text{Ker}(\text{Id} - T) \subseteq L^2(\Omega)$ , then  $v$  is constant. For this, we consider  $T_o v \in H^s_{\Omega, g}$ . By construction,

$$(3.20) \quad (-\Delta)^s(T_o v) + (T_o v) = v \quad \text{in } \Omega,$$

in the weak sense, and

$$(3.21) \quad \mathcal{N}_s(T_o v) = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega.$$

On the other hand, since  $v \in \text{Ker}(\text{Id} - T)$ , we have that

$$(3.22) \quad v = Tv = T_o v \quad \text{in } \Omega.$$

Hence, by (3.20), we have that

$$(-\Delta)^s(T_o v) = 0 \quad \text{in } \Omega.$$

Using this, (3.21) and Lemma 3.8, we obtain that  $T_o v$  is constant. Thus, by (3.22), we obtain that  $v$  is constant in  $\Omega$  and this completes the proof of (3.19).

From (3.19) and the Fredholm alternative, we conclude that

$$\text{Im}(\text{Id} - T) = \text{Ker}(\text{Id} - T)^\perp = \{\text{constant functions}\}^\perp,$$

where the orthogonality notion is in  $L^2(\Omega)$ . More explicitly,

$$(3.23) \quad \text{Im}(\text{Id} - T) = \{f \in L^2(\Omega) \text{ s.t. } \int_{\Omega} f = 0\}.$$

Now, let us take  $f$  such that  $\int_{\Omega} f = 0$ . By (3.23), we know that there exists  $w \in L^2(\Omega)$  such that  $f = w - Tw$ . Let us define  $u := T_o w$ . By construction, we have that  $\mathcal{N}_s u = 0$  in  $\mathbb{R}^n \setminus \Omega$ , and that

$$(-\Delta)^s(T_o w) + (T_o w) = w \quad \text{in } \Omega.$$

Consequently, in  $\Omega$ ,

$$f = w - Tw = w - T_o w = (-\Delta)^s(T_o w) = (-\Delta)^s u,$$

and we found the desired solution in this case.

Viceversa, if we have a solution  $u \in H^s_{\Omega, g}$  of  $(-\Delta)^s u = f$  in  $\Omega$  with  $\mathcal{N}_s u = 0$  in  $\mathbb{R}^n \setminus \Omega$ , we set  $w := f + u$  and we observe that

$$(-\Delta)^s u + u = f + u = w \quad \text{in } \Omega.$$

Accordingly, we have that  $u = T_o w$  in  $\mathbb{R}^n$ , hence  $u = T w$  in  $\Omega$ . This says that

$$(\text{Id} - T)w = w - u = f \quad \text{in } \Omega$$

and so  $f \in \text{Im}(\text{Id} - T)$ . Thus, by (3.23), we obtain that  $\int_{\Omega} f = 0$ .

This establishes the validity of Theorem 3.9 when  $g \equiv 0$ .

*Case 2.* Let us now consider the nonhomogeneous case (3.9). By the hypotheses, there exists a  $C^2$  function  $\psi$  satisfying  $\mathcal{N}_s \psi = g$  in  $\mathbb{R}^n \setminus \overline{\Omega}$ .

Let  $\bar{u} = u - \psi$ . Then,  $\bar{u}$  solves

$$\begin{cases} (-\Delta)^s \bar{u} = \bar{f} & \text{in } \Omega, \\ \mathcal{N}_s \bar{u} = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \end{cases}$$

with

$$\bar{f} = f - (-\Delta)^s \psi.$$

Then, as we already proved, this problem admits a solution if and only if  $\int_{\Omega} \bar{f} = 0$ , i.e., if

$$(3.24) \quad 0 = \int_{\Omega} \bar{f} = \int_{\Omega} f - \int_{\Omega} (-\Delta)^s \psi.$$

But, by Lemma 3.2, we have that

$$\int_{\Omega} (-\Delta)^s \psi = - \int_{\mathbb{R}^n \setminus \Omega} \mathcal{N}_s \psi = - \int_{\mathbb{R}^n \setminus \Omega} g.$$

From this and (3.24) we conclude that a solution exists if and only if (3.11) holds.

Finally, the solution is unique up to an additive constant thanks to Lemma 3.8. □

### 3.3. Eigenvalues and eigenfunctions

Here we discuss the spectral properties of problem (1.1). For it, we will need the following classical tool.

**Lemma 3.10** (Poincaré’s inequality). *Let  $\Omega \subset \mathbb{R}^n$  be any bounded Lipschitz domain, and let  $s \in (0, 1)$ . Then, for all functions  $u \in H^s(\Omega)$ , we have*

$$\int_{\Omega} \left| u - \int_{\Omega} u \right|^2 dx \leq C_{\Omega} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy,$$

where the constant  $C_{\Omega} > 0$  depends only on  $\Omega$  and  $s$ .

*Proof.* We give the details for the facility of the reader. We argue by contradiction and we assume that the inequality does not hold. Then, there exists a sequence of functions  $u_k \in H^s(\Omega)$  satisfying

$$(3.25) \quad \int_{\Omega} u_k = 0, \quad \|u_k\|_{L^2(\Omega)} = 1,$$

and

$$(3.26) \quad \int_{\Omega} \int_{\Omega} \frac{|u_k(x) - u_k(y)|^2}{|x - y|^{n+2s}} dx dy < \frac{1}{k}.$$

In particular, the functions  $\{u_k\}_{k \geq 1}$  are bounded in  $H^s(\Omega)$ .

Using now that the embedding  $H^s(\Omega) \subset L^2(\Omega)$  is compact (see e.g. [12]), it follows that a subsequence  $\{u_{k_j}\}_{j \geq 1}$  converges to a function  $\bar{u} \in L^2(\Omega)$ , i.e.,

$$u_{k_j} \rightarrow \bar{u} \quad \text{in } L^2(\Omega).$$

Moreover, we deduce from (3.25) that

$$(3.27) \quad \int_{\Omega} \bar{u} = 0, \quad \text{and} \quad \|\bar{u}\|_{L^2(\Omega)} = 1.$$

On the other hand, (3.26) implies that

$$\int_{\Omega} \int_{\Omega} \frac{|\bar{u}(x) - \bar{u}(y)|^2}{|x - y|^{n+2s}} dx dy = 0.$$

Thus,  $\bar{u}$  is constant in  $\Omega$ , and this contradicts (3.27). □

We finally give the description of the eigenvalues of  $(-\Delta)^s$  with zero Neumann boundary conditions.

**Theorem 3.11.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded Lipschitz domain. Then, there exist a sequence of nonnegative values*

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots,$$

and a sequence of functions  $u_i : \mathbb{R}^n \rightarrow \mathbb{R}$  such that

$$\begin{cases} (-\Delta)^s u_i(x) = \lambda_i u_i(x) & \text{for any } x \in \Omega, \\ \mathcal{N}_s u_i(x) = 0 & \text{for any } x \in \mathbb{R}^n \setminus \bar{\Omega}. \end{cases}$$

Also, the functions  $u_i$  (when restricted to  $\Omega$ ) provide a complete orthogonal system in  $L^2(\Omega)$ .

*Proof.* We define

$$L_0^2(\Omega) := \{u \in L^2(\Omega) : \int_{\Omega} u = 0\}.$$

Let the operator  $T_o$  be defined by  $T_o f = u$ , where  $u$  is the unique solution of

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ \mathcal{N}_s u = 0 & \text{in } \mathbb{R}^n \setminus \bar{\Omega}, \end{cases}$$

according to Definition 3.6. We remark that the existence and uniqueness of such solution is a consequence of the fact that  $f \in L^2_0(\Omega)$  and Theorem 3.9. We also define  $T$  to be the restriction of  $T_o$  in  $\Omega$ , that is

$$Tf = T_o f|_{\Omega}.$$

In this way,  $T: L^2_0(\Omega) \rightarrow L^2_0(\Omega)$ .

Also, we claim that the operator  $T$  is compact and self-adjoint.

We first show that  $T$  is compact. Indeed, taking  $v = u = T_o f$  in the weak formulation of the problem (3.10), we obtain

$$(3.28) \quad \frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}.$$

Now, using the Poincaré inequality in Lemma 3.10 (recall that  $\int_{\Omega} u = 0$ ), we deduce that

$$(3.29) \quad \|u\|_{L^2(\Omega)} \leq C \left( \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy \right)^{1/2}.$$

This and (3.28) give that

$$(3.30) \quad \left( \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy \right)^{1/2} \leq C \|f\|_{L^2(\Omega)}.$$

Now, we take a sequence  $\{f_k\}_{k \in \mathbb{N}}$  bounded in  $L^2(\Omega)$ . From (3.29) and (3.30) we obtain that  $u_k = T f_k$  is bounded in  $H^s(\Omega)$ . Hence, since the embedding  $H^s(\Omega) \subset L^2(\Omega)$  is compact, there exists a subsequence that converges in  $L^2(\Omega)$ . Therefore,  $T$  is compact.

Now we show that  $T$  is self-adjoint in  $L^2_0(\Omega)$ . The proof is very similar to the one in (3.14)–(3.18), but for the facility of the reader we give it in full detail (the reader who is not interested can jump directly to (3.35)). To show self-adjointness, we take  $f_1$  and  $f_2$  in  $C^\infty_0(\Omega)$ , with  $\int_{\Omega} f_1 = \int_{\Omega} f_2 = 0$ . Then from the weak formulation in (3.10) we have that, for every  $v, w \in H^s_{\Omega, g}$ ,

$$(3.31) \quad \frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{(T_o f_1(x) - T_o f_1(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy = \int_{\Omega} f_1 v$$

and

$$(3.32) \quad \frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{(T_o f_2(x) - T_o f_2(y))(w(x) - w(y))}{|x - y|^{n+2s}} dx dy = \int_{\Omega} f_2 w.$$

We observe that we can take  $v := T_o f_2$  in (3.31) and  $w := T_o f_1$  in (3.32) (and recall that  $T_o f_i = T f_i$  in  $\Omega$ ), obtaining that

$$(3.33) \quad \int_{\Omega} f_1 T f_2 = \int_{\Omega} f_2 T f_1, \quad \text{for any } f_1, f_2 \in C^\infty_0(\Omega).$$

Now, if  $f_1, f_2 \in L^2_0(\Omega)$  we can find sequences of functions  $f_{1,k}, f_{2,k} \in C^\infty_0(\Omega)$  such that  $f_{1,k} \rightarrow f_1$  and  $f_{2,k} \rightarrow f_2$  in  $L^2(\Omega)$  as  $k \rightarrow +\infty$ . Therefore, from (3.33), we have

$$(3.34) \quad \int_{\Omega} f_{1,k} T f_{2,k} = \int_{\Omega} f_{2,k} T f_{1,k}.$$

We notice that, thanks to (3.29) and (3.30),  $T f_{1,k} \rightarrow T f_1$  and  $T f_{2,k} \rightarrow T f_2$  in  $L^2(\Omega)$  as  $k \rightarrow +\infty$ , and therefore, from (3.34), we obtain that

$$\int_{\Omega} f_1 T f_2 = \int_{\Omega} f_2 T f_1,$$

thus proving that  $T$  is self-adjoint in  $L^2_0(\Omega)$ .

Thus, by the spectral theorem there exists a sequence of eigenvalues  $\{\mu_i\}_{i \geq 2}$  of  $T$ , and its corresponding eigenfunctions  $\{e_i\}_{i \geq 2}$  are a complete orthogonal system in  $L^2_0(\Omega)$ .

We remark that

$$(3.35) \quad \mu_i \neq 0.$$

Indeed, suppose by contradiction that  $\mu_i = 0$ . Then

$$(3.36) \quad 0 = \mu_i e_i = T e_i = T_o e_i \quad \text{in } \Omega.$$

By construction,  $\mathcal{N}_s(T_o e_i) = 0$  in  $\mathbb{R}^n \setminus \overline{\Omega}$ . This and (3.36) give that

$$T_o e_i(x) = \frac{\int_{\Omega} \frac{T_o e_i(y)}{|x-y|^{n+2s}} dy}{\int_{\Omega} \frac{dy}{|x-y|^{n+2s}}} = 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}.$$

Using this and (3.36) once again we conclude that  $T_o e_i \equiv 0$  a.e. in  $\mathbb{R}^n$ . Therefore

$$0 = (-\Delta)^s(T_o e_i) = e_i \quad \text{in } \Omega,$$

which gives that  $e_i \equiv 0$  in  $\Omega$ , hence it is not an eigenfunction. This establishes (3.35).

From (3.35), we can define

$$\lambda_i := \mu_i^{-1}.$$

We also define  $u_i := T_o e_i$  and we claim that  $u_2, u_3, \dots$  is the desired system of eigenfunctions, with corresponding eigenvalues  $\lambda_2, \lambda_3, \dots$

Indeed,

$$(3.37) \quad u_i = T_o e_i = T e_i = \mu_i e_i \quad \text{in } \Omega,$$

hence the orthogonality and completeness properties of  $u_2, u_3, \dots$  in  $L^2_0(\Omega)$  follow from those of  $e_2, e_3, \dots$

Furthermore, in the domain  $\Omega$ , we have that  $(-\Delta)^s u_i = (-\Delta)^s(T_o e_i) = e_i = \lambda_i u_i$ , where (3.37) was used in the last step, and this proves the desired spectral property.

Now, we notice that

$$(3.38) \quad \lambda_i > 0 \quad \text{for any } i \geq 2.$$

Indeed, its corresponding eigenfunction  $u_i$  solves

$$(3.39) \quad \begin{cases} (-\Delta)^s u_i = \lambda_i u_i & \text{in } \Omega, \\ \mathcal{N}_s u_i = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{cases}$$

Then, if we take  $u_i$  as a test function in the weak formulation of (3.39), we obtain that

$$\frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{|u_i(x) - u_i(y)|^2}{|x - y|^{n+2s}} dx dy = \lambda_i \int_{\Omega} u_i^2,$$

which implies that  $\lambda_i \geq 0$ . Now, suppose by contradiction that  $\lambda_i = 0$ . Then, from Lemma 3.8 we have that  $u_i$  is constant. On the other hand, we know that  $u_i \in L^2_0(\Omega)$ , and this implies that  $u_i \equiv 0$ , which is a contradiction since  $u_i$  is an eigenfunction. This establishes (3.38).

From (3.38), up to reordering them, we can suppose that  $0 < \lambda_2 \leq \lambda_3 \leq \dots$ ; now, we notice that  $\lambda_1 := 0$  is an eigenvalue, with eigenfunction  $u_1 := 1$ , thanks to Lemma 3.8. Therefore, we have a sequence of eigenvalues  $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$ , and its corresponding eigenfunctions are a complete orthogonal system in  $L^2(\Omega)$ . To check the latter statement, we argue as follows: first of all, the system  $\{e_i\}_{i \geq 1}$  is orthogonal in  $L^2(\Omega)$ , since we already know that the system  $\{e_i\}_{i \geq 2}$  is orthogonal, and each  $e_i$  is orthogonal to  $e_1$  for any  $i \geq 2$ , because  $e_i \in L^2_0(\Omega)$  and  $e_1 \equiv 1$ . To check that the system  $\{e_i\}_{i \geq 1}$  is complete in  $L^2(\Omega)$ , given any  $\gamma \in L^2(\Omega)$ , we set

$$\gamma_1 := \int_{\Omega} \gamma \quad \text{and} \quad \tilde{\gamma} := \gamma - \gamma_1.$$

Then,  $\tilde{\gamma} \in L^2_0(\Omega)$ , and so, since  $\{e_i\}_{i \geq 2}$  is a complete orthogonal system in  $L^2_0(\Omega)$ , there exists a sequence of real numbers  $\{\gamma_i\}_{i \geq 2}$  such that

$$\lim_{N \rightarrow +\infty} \left\| \tilde{\gamma} - \sum_{i=2}^N \gamma_i e_i \right\|_{L^2(\Omega)} = 0.$$

Accordingly, since  $\tilde{\gamma} = \gamma - \gamma_1 e_1$ , we get

$$\lim_{N \rightarrow +\infty} \left\| \gamma - \sum_{i=1}^N \gamma_i e_i \right\|_{L^2(\Omega)} = 0.$$

Since  $\gamma$  is an arbitrary function of  $L^2(\Omega)$ , we have shown that the system  $\{e_i\}_{i \geq 1}$  is complete in  $L^2(\Omega)$ , as desired. This concludes the proof of Theorem 3.11.  $\square$

**Remark 3.12.** We point out that the notion of eigenfunctions in Theorem 3.11 is not completely standard. Indeed, the eigenfunctions  $u_i$  corresponding to the eigenvalues  $\lambda_i$  are defined in the whole of  $\mathbb{R}^n$ , but they satisfy an orthogonality conditions only in  $L^2(\Omega)$ .

Alternatively, one can think that the “natural” domain of definition for  $u_i$  is  $\Omega$  itself, since there the eigenvalue equation  $(-\Delta)^s u_i = \lambda_i u_i$  takes place, together with the orthogonality condition, and then  $u_i$  is “naturally” extended outside  $\Omega$  via the nonlocal Neumann condition. Notice indeed that the condition  $\mathcal{N}_s u_i = 0$  in  $\mathbb{R}^n \setminus \overline{\Omega}$  is equivalent to prescribing  $u_i$  outside  $\Omega$  from the values inside  $\Omega$  according to the formula

$$u_i(x) = \frac{\int_{\Omega} \frac{u_i(y)}{|x-y|^{n+2s}} dy}{\int_{\Omega} \frac{dy}{|x-y|^{n+2s}}} \quad \text{for any } x \in \mathbb{R}^n \setminus \overline{\Omega}.$$

In the following proposition we deal with the behavior of the solution of (1.1) at infinity.

**Proposition 3.13.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain and let  $u \in H_{\Omega,g}^s$  be a weak solution (according to Definition 3.6) of*

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ \mathcal{N}_s u = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{cases}$$

Then

$$\lim_{|x| \rightarrow \infty} u(x) = \frac{1}{|\Omega|} \int_{\Omega} u \quad \text{uniformly in } x.$$

*Proof.* First we observe that, since  $\Omega$  is bounded, there exists  $R > 0$  such that  $\Omega \subset B_R$ . Hence, if  $y \in \Omega$ , we have that

$$|x| - R \leq |x - y| \leq |x| + R,$$

and so

$$1 - \frac{R}{|x|} \leq \frac{|x - y|}{|x|} \leq 1 + \frac{R}{|x|}.$$

Therefore, given  $\epsilon > 0$ , there exists  $\bar{R} > R$  such that, for any  $|x| \geq \bar{R}$ , we have

$$\frac{|x|^{n+2s}}{|x - y|^{n+2s}} = 1 + \gamma(x, y),$$

where  $|\gamma(x, y)| \leq \epsilon$ .

Recalling the definition of  $\mathcal{N}_s u$  given in (1.2) and using the fact that  $\mathcal{N}_s u = 0$  in  $\mathbb{R}^n \setminus \overline{\Omega}$ , we have that for any  $x \in \mathbb{R}^n \setminus \overline{\Omega}$ ,

$$\begin{aligned} u(x) &= \frac{\int_{\Omega} \frac{u(y)}{|x-y|^{n+2s}} dy}{\int_{\Omega} \frac{dy}{|x-y|^{n+2s}}} = \frac{\int_{\Omega} \frac{|x|^{n+2s} u(y)}{|x-y|^{n+2s}} dy}{\int_{\Omega} \frac{|x|^{n+2s}}{|x-y|^{n+2s}} dy} \\ &= \frac{\int_{\Omega} (1 + \gamma(x, y)) u(y) dy}{\int_{\Omega} (1 + \gamma(x, y)) dy} = \frac{\int_{\Omega} u(y) dy + \int_{\Omega} \gamma(x, y) u(y) dy}{|\Omega| + \int_{\Omega} \gamma(x, y) dy}. \end{aligned}$$

We set

$$\gamma_1(x) := \int_{\Omega} \gamma(x, y) u(y) dy \quad \text{and} \quad \gamma_2(x) := \int_{\Omega} \gamma(x, y) dy,$$

and we notice that  $|\gamma_1(x)| \leq C\epsilon$  and  $|\gamma_2(x)| \leq \epsilon$ , for some  $C > 0$ .

Hence, we have that for any  $x \in \mathbb{R}^n \setminus \overline{\Omega}$

$$\begin{aligned} \left| u(x) - \int_{\Omega} u(y) dy \right| &= \left| \frac{\int_{\Omega} u(y) dy + \gamma_1(x)}{1 + \gamma_2(x)} - \int_{\Omega} u(y) dy \right| \\ &= \frac{|\gamma_1(x) - \gamma_2(x) \int_{\Omega} u(y) dy|}{1 + \gamma_2(x)} \leq \frac{C \epsilon}{1 - \epsilon}. \end{aligned}$$

Therefore, sending  $\epsilon \rightarrow 0$  (that is,  $|x| \rightarrow +\infty$ ), we obtain the desired result. □

**Remark 3.14** (Interior regularity of solutions). We notice that, in particular, Proposition 3.13 implies that  $u$  is bounded at infinity. Thus, if solutions are locally bounded, then one could apply interior regularity results for solutions to  $(-\Delta)^s u = f$  in  $\Omega$  (see e.g. [17], [21], [7], and [20]).

### 4. The heat equation

Here we show that solutions of the nonlocal heat equation with zero Neumann datum preserve their mass and have energy that decreases in time.

To avoid technicalities, we assume that  $u$  is a classical solution of problem (1.4), so that we can differentiate under the integral sign.

**Proposition 4.1.** *Assume that  $u(x, t)$  is a classical solution to (1.4), in the sense that  $u$  is bounded and  $|u_t| + |(-\Delta)^s u| \leq K$  for all  $t > 0$ . Then, for all  $t > 0$ ,*

$$\int_{\Omega} u(x, t) dx = \int_{\Omega} u_0(x) dx.$$

*In other words, the total mass is conserved.*

*Proof.* By the dominated convergence theorem, and using Lemma 3.2, we have

$$\frac{d}{dt} \int_{\Omega} u = \int_{\Omega} u_t = - \int_{\Omega} (-\Delta)^s u = \int_{\mathbb{R}^n \setminus \Omega} \mathcal{N}_s u = 0.$$

Thus, the quantity  $\int_{\Omega} u$  does not depend on  $t$ , and the result follows. □

**Proposition 4.2.** *Assume that  $u(x, t)$  is a classical solution to (1.4), in the sense that  $u$  is bounded and  $|u_t| + |(-\Delta)^s u| \leq K$  for all  $t > 0$ . Then, the energy*

$$E(t) = \int_{\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2} \frac{|u(x, t) - u(y, t)|^2}{|x - y|^{n+2s}} dx dy$$

*is decreasing in time  $t > 0$ .*



*Proof.* Let us compute  $E'(t)$ , and we will see that it is negative. Indeed, using Lemma 3.3,

$$\begin{aligned} E'(t) &= \frac{d}{dt} \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{|u(x,t) - u(y,t)|^2}{|x - y|^{n+2s}} dx dy \\ &= \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{2(u(x,t) - u(y,t))(u_t(x,t) - u_t(y,t))}{|x - y|^{n+2s}} dx dy = \frac{4}{c_{n,s}} \int_{\Omega} u_t (-\Delta)^s u dx, \end{aligned}$$

where we have used that  $\mathcal{N}_s u = 0$  in  $\mathbb{R}^n \setminus \bar{\Omega}$ .

Thus, using now the equation  $u_t + (-\Delta)^s u = 0$  in  $\Omega$ , we find

$$E'(t) = -\frac{4}{c_{n,s}} \int_{\Omega} |(-\Delta)^s u|^2 dx \leq 0,$$

with strict inequality unless  $u$  is constant. □

Next we prove that solutions of the nonlocal heat equation with Neumann condition approach a constant as  $t \rightarrow +\infty$ .

**Proposition 4.3.** *Assume that  $u(x, t)$  is a classical solution to (1.4), in the sense that  $u$  is bounded and  $|u_t| + |(-\Delta)^s u| \leq K$  for all  $t > 0$ . Then, as  $t \rightarrow +\infty$ ,*

$$u \rightarrow \frac{1}{|\Omega|} \int_{\Omega} u_0 \quad \text{in } L^2(\Omega).$$

*Proof.* Let

$$m := \frac{1}{|\Omega|} \int_{\Omega} u_0$$

be the total mass of  $u$ . Define also

$$A(t) := \int_{\Omega} |u - m|^2 dx.$$

Notice that, by Proposition 4.1, we have

$$A(t) = \int_{\Omega} (u^2 - 2mu + m^2) dx = \int_{\Omega} u^2 dx - |\Omega|m^2.$$

Then, by Lemma 3.3,

$$\begin{aligned} A'(t) &= 2 \int_{\Omega} u_t u dx = -2 \int_{\Omega} u (-\Delta)^s u dx \\ &= -c_{n,s} \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{|u(x,t) - u(y,t)|^2}{|x - y|^{n+2s}} dx dy. \end{aligned}$$

Hence,  $A$  is decreasing.

Moreover, using the Poincaré inequality in Lemma 3.10 and again Proposition 4.1, we deduce that

$$A'(t) \leq -c \int_{\Omega} |u - m|^2 dx = -cA(t),$$

for some  $c > 0$ . Thus, it follows that

$$A(t) \leq e^{-ct} A(0),$$

and thus

$$\lim_{t \rightarrow +\infty} \int_{\Omega} |u(x, t) - m|^2 dx = 0,$$

i.e.,  $u$  converges to  $m$  in  $L^2(\Omega)$ .

Notice that, in fact, we have proved that the convergence is exponentially fast. □

### 5. Limits

In this section we study the limits as  $s \rightarrow 1$  and the continuity properties induced by the fractional Neumann condition.

#### 5.1. Limit as $s \rightarrow 1$

**Proposition 5.1.** *Let  $\Omega \subset \mathbb{R}^n$  be any bounded Lipschitz domain. Let  $u$  and  $v$  be  $C_0^2(\mathbb{R}^n)$  functions. Then,*

$$\lim_{s \rightarrow 1} \int_{\mathbb{R}^n \setminus \Omega} \mathcal{N}_s u v = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v.$$

*Proof.* By Lemma 3.3, we have that

$$(5.1) \quad \int_{\mathbb{R}^n \setminus \Omega} \mathcal{N}_s u v = \frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy - \int_{\Omega} v(-\Delta)^s u.$$

Now, we claim that

$$(5.2) \quad \lim_{s \rightarrow 1} \frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy = \int_{\Omega} \nabla u \cdot \nabla v.$$

We observe that to show (5.2) it is enough to prove that, for any  $u \in C_0^2(\mathbb{R}^n)$ ,

$$(5.3) \quad \lim_{s \rightarrow 1} \frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy = \int_{\Omega} |\nabla u|^2.$$

Indeed,

$$\begin{aligned} & \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy \\ &= \frac{1}{2} \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{|(u + v)(x) - (u + v)(y)|^2}{|x - y|^{n+2s}} dx dy \\ & \quad - \frac{1}{2} \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy - \frac{1}{2} \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}} dx dy. \end{aligned}$$

Now, we recall that

$$\lim_{s \rightarrow 1} \frac{c_{n,s}}{1-s} = \frac{4n}{\omega_{n-1}},$$

(see Corollary 4.2 in [12]), and so we have to show that

$$(5.4) \quad \lim_{s \rightarrow 1} (1-s) \int_{\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dx dy = \frac{\omega_{n-1}}{2n} \int_{\Omega} |\nabla u|^2.$$

For this, we first show that

$$(5.5) \quad \lim_{s \rightarrow 1} (1-s) \int_{\Omega \times (\mathcal{C}\Omega)} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dx dy = 0.$$

Without loss of generality, we can suppose that  $B_r \subset \Omega \subset B_R$ , for some  $0 < r < R$ . Since  $u \in C_0^2(\mathbb{R}^n)$ , then

$$\begin{aligned} \int_{\Omega \times (\mathcal{C}\Omega)} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dx dy &\leq 4\|u\|_{L^\infty(\mathbb{R}^n)}^2 \int_{\Omega \times (\mathcal{C}\Omega)} \frac{1}{|x-y|^{n+2s}} dx dy \\ &\leq 4\|u\|_{L^\infty(\mathbb{R}^n)}^2 \int_{B_R \times (\mathcal{C}B_r)} \frac{1}{|x-y|^{n+2s}} dx dy \\ &\leq 4\|u\|_{L^\infty(\mathbb{R}^n)}^2 \omega_{n-1} \int_{B_R} dx \int_r^{+\infty} \rho^{n-1} \rho^{-n-2s} d\rho \\ &= 4\|u\|_{L^\infty(\mathbb{R}^n)}^2 \omega_{n-1} \int_{B_R} dx \int_r^{+\infty} \rho^{-1-2s} d\rho \\ &= 4\|u\|_{L^\infty(\mathbb{R}^n)}^2 \frac{\omega_{n-1} r^{-2s}}{2s} \int_{B_R} dx = 4\|u\|_{L^\infty(\mathbb{R}^n)}^2 \frac{\omega_{n-1}^2 R^n r^{-2s}}{2s}, \end{aligned}$$

which implies (5.5). Hence,

$$(5.6) \quad \begin{aligned} \lim_{s \rightarrow 1} (1-s) \int_{\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dx dy \\ = \lim_{s \rightarrow 1} (1-s) \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dx dy = C_n \int_{\Omega} |\nabla u|^2, \end{aligned}$$

where  $C_n > 0$  depends only on the dimension, see [5].

In order to determine the constant  $C_n$ , we take a  $C^2$ -function  $u$  supported in  $\Omega$ . In this case, we have

$$(5.7) \quad \int_{\Omega} |\nabla u|^2 dx = \int_{\mathbb{R}^n} |\nabla u|^2 dx = \int_{\mathbb{R}^n} |\xi|^2 |\hat{u}(\xi)|^2 d\xi,$$

where  $\hat{u}$  is the Fourier transform of  $u$ . Moreover,

$$\begin{aligned} \int_{\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dx dy &= \int_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dx dy \\ &= 2 c_{n,s}^{-1} \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{u}(\xi)|^2 dx, \end{aligned}$$

thanks to Proposition 3.4 in [12]. Therefore, using Corollary 4.2 in [12] and (5.7), we have

$$\begin{aligned} \lim_{s \rightarrow 1} (1-s) \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy &= \lim_{s \rightarrow 1} \frac{2(1-s)}{c_{n,s}} \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{u}(\xi)|^2 dx \\ &= \frac{\omega_{n-1}}{2n} \int_{\mathbb{R}^n} |\xi|^2 |\hat{u}(\xi)|^2 dx = \frac{\omega_{n-1}}{2n} \int_{\Omega} |\nabla u|^2 dx. \end{aligned}$$

Hence, the constant in (5.6) is  $C_n = \omega_{n-1}/(2n)$ . This concludes the proof of (5.4), and in turn of (5.2).

On the other hand,

$$-(-\Delta)^s u \rightarrow \Delta u \quad \text{uniformly in } \mathbb{R}^n,$$

(see Proposition 4.4 in [12]). This, (5.1) and (5.2) give

$$\lim_{s \rightarrow 1} \int_{\mathbb{R}^n \setminus \Omega} \mathcal{N}_s u v = \int_{\Omega} \nabla u \cdot \nabla v + \int_{\Omega} v \Delta u = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v,$$

as desired. □

We remark that the result in Proposition 5.1 holds for a fixed (and “nice”) function and it can be seen as the counterpart of classical limit in the fractional framework (see for instance Proposition 4.4 in [12]). If one aims to prove that solutions of the corresponding elliptic equations of fractional parameter  $s$  correspond to classical solutions as  $s \rightarrow 1$ , an appropriate a-priori regularity theory needs to be developed in the cases under consideration.

### 5.2. Continuity properties

Following is a continuity result for functions satisfying the nonlocal Neumann condition:

**Proposition 5.2.** *Let  $\Omega \subset \mathbb{R}^n$  be a domain with  $C^1$  boundary. Let  $u$  be continuous in  $\bar{\Omega}$ , with  $\mathcal{N}_s u = 0$  in  $\mathbb{R}^n \setminus \bar{\Omega}$ . Then  $u$  is continuous in the whole of  $\mathbb{R}^n$ .*

*Proof.* First, let us fix  $x_0 \in \mathbb{R}^n \setminus \bar{\Omega}$ . Since the latter is an open set, there exists  $\rho > 0$  such that  $|x_0 - y| \geq \rho$  for any  $y \in \Omega$ . Thus, if  $x \in B_{\rho/2}(x_0)$ , we have that  $|x - y| \geq |x_0 - y| - |x_0 - x| \geq \rho/2$ .

Moreover, if  $x \in B_{\rho/2}(x_0)$ , we have that

$$|x - y| \geq |y| - |x_0| - |x_0 - x| \geq \frac{|y|}{2} + \left( \frac{|y|}{4} - |x_0| \right) + \left( \frac{|y|}{4} - \frac{\rho}{2} \right) \geq \frac{|y|}{2},$$

provided that  $|y| \geq R := 4|x_0| + 2\rho$ . As a consequence, for any  $x \in B_{\rho/2}(x_0)$ , we have that

$$\frac{|u(y)| + 1}{|x - y|^{n+2s}} \leq 2^{n+2s} (\|u\|_{L^\infty(\bar{\Omega})} + 1) \left( \frac{\chi_{B_R}(y)}{\rho^{n+2s}} + \frac{\chi_{\mathbb{R}^n \setminus B_R}(y)}{|y|^{n+2s}} \right) =: \psi(y)$$

and the function  $\psi$  belongs to  $L^1(\mathbb{R}^n)$ .

Thus, by the Neumann condition and the dominated convergence theorem, we obtain that

$$\lim_{x \rightarrow x_0} u(x) = \lim_{x \rightarrow x_0} \frac{\int_{\Omega} \frac{u(y)}{|x - y|^{n+2s}} dy}{\int_{\Omega} \frac{dy}{|x - y|^{n+2s}}} = \frac{\int_{\Omega} \frac{u(y)}{|x_0 - y|^{n+2s}} dy}{\int_{\Omega} \frac{dy}{|x_0 - y|^{n+2s}}} = u(x_0).$$

This proves that  $u$  is continuous at any points of  $\mathbb{R}^n \setminus \overline{\Omega}$ .

Now we show the continuity at a point  $p \in \partial\Omega$ . We take a sequence  $p_k \rightarrow p$  as  $k \rightarrow +\infty$ . We let  $q_k$  be the projection of  $p_k$  to  $\overline{\Omega}$ . Since  $p \in \overline{\Omega}$ , we have from the minimizing property of the projection that

$$|p_k - q_k| = \inf_{\xi \in \overline{\Omega}} |p_k - \xi| \leq |p_k - p|,$$

and so

$$|q_k - p| \leq |q_k - p_k| + |p_k - p| \leq 2|p_k - p| \rightarrow 0$$

as  $k \rightarrow +\infty$ . Therefore, since we already know from the assumptions the continuity of  $u$  at  $\overline{\Omega}$ , we obtain that

$$(5.8) \quad \lim_{k \rightarrow +\infty} u(q_k) = u(p).$$

Now we claim that

$$(5.9) \quad \lim_{k \rightarrow +\infty} u(p_k) - u(q_k) = 0.$$

To prove it, it is enough to consider the points of the sequence  $p_k$  that belong to  $\mathbb{R}^n \setminus \overline{\Omega}$  (since, of course, the points  $p_k$  belonging to  $\overline{\Omega}$  satisfy  $p_k = q_k$  and for them (5.9) is obvious). We define  $\nu_k := (p_k - q_k)/|p_k - q_k|$ . Notice that  $\nu_k$  is the exterior normal of  $\Omega$  at  $q_k \in \partial\Omega$ . We consider a rigid motion  $\mathcal{R}_k$  such that  $\mathcal{R}_k q_k = 0$  and  $\mathcal{R}_k \nu_k = e_n = (0, \dots, 0, 1)$ . Let also  $h_k := |p_k - q_k|$ . Notice that

$$(5.10) \quad h_k^{-1} \mathcal{R}_k p_k = h_k^{-1} \mathcal{R}_k (p_k - q_k) = \mathcal{R}_k \nu_k = e_n.$$

Then, the domain

$$\Omega_k := h_k^{-1} \mathcal{R}_k \Omega$$

has vertical exterior normal at 0 and approaches the halfspace  $\Pi := \{x_n < 0\}$  as  $k \rightarrow +\infty$ .

Now, we use the Neumann condition at  $p_k$  and we obtain that

$$u(p_k) - u(q_k) = \frac{\int_{\Omega} \frac{u(y)}{|p_k - y|^{n+2s}} dy}{\int_{\Omega} \frac{dy}{|p_k - y|^{n+2s}}} - u(q_k) = \frac{\int_{\Omega} \frac{u(y) - u(q_k)}{|p_k - y|^{n+2s}} dy}{\int_{\Omega} \frac{dy}{|p_k - y|^{n+2s}}} = I_1 + I_2,$$

with

$$I_1 := \frac{\int_{\Omega \cap B_{\sqrt{h_k}}(q_k)} \frac{u(y) - u(q_k)}{|p_k - y|^{n+2s}} dy}{\int_{\Omega} \frac{dy}{|p_k - y|^{n+2s}}} \quad \text{and} \quad I_2 := \frac{\int_{\Omega \setminus B_{\sqrt{h_k}}(q_k)} \frac{u(y) - u(q_k)}{|p_k - y|^{n+2s}} dy}{\int_{\Omega} \frac{dy}{|p_k - y|^{n+2s}}}.$$

We observe that the uniform continuity of  $u$  in  $\bar{\Omega}$  gives that

$$\lim_{k \rightarrow +\infty} \sup_{y \in \Omega \cap B_{\sqrt{h_k}}(q_k)} |u(y) - u(q_k)| = 0.$$

As a consequence,

$$(5.11) \quad |I_1| \leq \sup_{y \in \Omega \cap B_{\sqrt{h_k}}(q_k)} |u(y) - u(q_k)| \rightarrow 0$$

as  $k \rightarrow +\infty$ . Moreover, exploiting the change of variable  $\eta := h_k^{-1} \mathcal{R}_k y$  and recalling (5.10), we obtain that

$$\begin{aligned} |I_2| &\leq \frac{\int_{\Omega \setminus B_{\sqrt{h_k}}(q_k)} \frac{|u(y) - u(q_k)|}{|p_k - y|^{n+2s}} dy}{\int_{\Omega} \frac{dy}{|p_k - y|^{n+2s}}} \leq 2 \|u\|_{L^\infty(\bar{\Omega})} \frac{\int_{\Omega \setminus B_{\sqrt{h_k}}(q_k)} \frac{dy}{|p_k - y|^{n+2s}}}{\int_{\Omega} \frac{dy}{|p_k - y|^{n+2s}}} \\ &= 2 \|u\|_{L^\infty(\bar{\Omega})} \frac{\int_{\Omega_k \setminus B_{1/\sqrt{h_k}}} \frac{d\eta}{|e_n - \eta|^{n+2s}}}{\int_{\Omega_k} \frac{d\eta}{|e_n - \eta|^{n+2s}}}. \end{aligned}$$

Notice that, if  $\eta \in \Omega_k \setminus B_{1/\sqrt{h_k}}$  then

$$\begin{aligned} |e_n - \eta|^{n+2s} &= |e_n - \eta|^{n+s} |e_n - \eta|^s \geq |e_n - \eta|^{n+s} (|\eta| - 1)^s \\ &\geq |e_n - \eta|^{n+s} (h_k^{-1/2} - 1)^s \geq |e_n - \eta|^{n+s} h_k^{-s/4} \end{aligned}$$

for large  $k$ . Therefore,

$$|I_2| \leq 2h_k^{s/4} \|u\|_{L^\infty(\bar{\Omega})} \frac{\int_{\Omega_k} \frac{d\eta}{|e_n - \eta|^{n+s}}}{\int_{\Omega_k} \frac{d\eta}{|e_n - \eta|^{n+2s}}}.$$

Since

$$\lim_{k \rightarrow +\infty} \frac{\int_{\Omega_k} \frac{d\eta}{|e_n - \eta|^{n+s}}}{\int_{\Omega_k} \frac{d\eta}{|e_n - \eta|^{n+2s}}} = \frac{\int_{\Pi} \frac{d\eta}{|e_n - \eta|^{n+s}}}{\int_{\Pi} \frac{d\eta}{|e_n - \eta|^{n+2s}}},$$

we conclude that  $|I_2| \rightarrow 0$  as  $k \rightarrow +\infty$ . This and (5.11) imply (5.9).

From (5.8) and (5.9), we conclude that

$$\lim_{k \rightarrow +\infty} u(p_k) = u(p),$$

hence  $u$  is continuous at  $p$ . □

As a direct consequence of Proposition 5.2 we obtain:

**Corollary 5.3.** *Let  $\Omega \subset \mathbb{R}^n$  be a domain with  $C^1$  boundary. Let  $v_0 \in C(\mathbb{R}^n)$ . Let*

$$v(x) := \begin{cases} v_0(x) & \text{if } x \in \overline{\Omega}, \\ \frac{\int_{\Omega} \frac{v_0(y)}{|x-y|^{n+2s}} dy}{\int_{\Omega} \frac{dy}{|x-y|^{n+2s}}} & \text{if } x \in \mathbb{R}^n \setminus \overline{\Omega}. \end{cases}$$

Then  $v \in C(\mathbb{R}^n)$  and it satisfies  $v = v_0$  in  $\overline{\Omega}$  and  $\mathcal{N}_s v = 0$  in  $\mathbb{R}^n \setminus \overline{\Omega}$ .

*Proof.* By construction,  $v = v_0$  in  $\overline{\Omega}$  and  $\mathcal{N}_s v = 0$  in  $\mathbb{R}^n \setminus \overline{\Omega}$ . Then we can use Proposition 5.2 and obtain that  $v \in C(\mathbb{R}^n)$ . □

Now we study the boundary behavior of the nonlocal Neumann function  $\tilde{\mathcal{N}}_s u$ .

**Proposition 5.4.** *Let  $\Omega \subset \mathbb{R}^n$  be a  $C^1$  domain, and  $u \in C(\mathbb{R}^n)$ . Then, for all  $s \in (0, 1)$ ,*

$$(5.12) \quad \lim_{\substack{x \rightarrow \partial\Omega \\ x \in \mathbb{R}^n \setminus \overline{\Omega}}} \tilde{\mathcal{N}}_s u(x) = 0.$$

Also, if  $s > 1/2$  and  $u \in C^{1,\alpha}(\mathbb{R}^n)$  for some  $\alpha \in (0, 2s - 1)$ , then

$$(5.13) \quad \partial_\nu \tilde{\mathcal{N}}_s u(x) := \lim_{\epsilon \rightarrow 0^+} \frac{\tilde{\mathcal{N}}_s u(x + \epsilon\nu)}{\epsilon} = \kappa \partial_\nu u \quad \text{for any } x \in \partial\Omega,$$

for some constant  $\kappa > 0$ .

*Proof.* Let  $x_k$  be a sequence in  $\mathbb{R}^n \setminus \overline{\Omega}$  such that  $x_k \rightarrow x_\infty \in \partial\Omega$  as  $k \rightarrow +\infty$ .

By Corollary 5.3 (applied here with  $v_0 := u$ ), there exists  $v \in C(\mathbb{R}^n)$  such that  $v = u$  in  $\overline{\Omega}$  and  $\mathcal{N}_s v = 0$  in  $\mathbb{R}^n \setminus \overline{\Omega}$ . By the continuity of  $u$  and  $v$  we have that

$$(5.14) \quad \lim_{k \rightarrow +\infty} u(x_k) - v(x_k) = u(x_\infty) - v(x_\infty) = 0.$$

Moreover,

$$\begin{aligned} \tilde{\mathcal{N}}_s u(x_k) &= \tilde{\mathcal{N}}_s u(x_k) - \tilde{\mathcal{N}}_s v(x_k) = \frac{\int_{\Omega} \frac{u(x_k) - u(y)}{|x_k - y|^{n+2s}} dy - \int_{\Omega} \frac{v(x_k) - v(y)}{|x_k - y|^{n+2s}} dy}{\int_{\Omega} \frac{dy}{|x_k - y|^{n+2s}}} \\ &= \frac{\int_{\Omega} \frac{u(x_k) - v(x_k)}{|x_k - y|^{n+2s}} dy}{\int_{\Omega} \frac{dy}{|x_k - y|^{n+2s}}} = u(x_k) - v(x_k). \end{aligned}$$

This and (5.14) imply that

$$\lim_{k \rightarrow +\infty} \tilde{\mathcal{N}}_s u(x_k) = 0,$$

that is (5.12).

Now, we prove (5.13). For this, we suppose that  $s > 1/2$ , that  $0 \in \partial\Omega$  and that the exterior normal  $\nu$  coincides with  $e_n = (0, \dots, 0, 1)$ ; then we use (5.12) and the change of variable  $\eta := \epsilon^{-1}y$  in the following computation:

$$\begin{aligned} \epsilon^{-1}(\tilde{\mathcal{N}}_s u(\epsilon e_n) - \tilde{\mathcal{N}}_s u(0)) &= \epsilon^{-1} \tilde{\mathcal{N}}_s u(\epsilon e_n) = \frac{\epsilon^{-1} \int_{\Omega} \frac{u(\epsilon e_n) - u(y)}{|\epsilon e_n - y|^{n+2s}} dy}{\int_{\Omega} \frac{dy}{|\epsilon e_n - y|^{n+2s}}} \\ &= \frac{\epsilon^{-1} \int_{\frac{1}{\epsilon}\Omega} \frac{u(\epsilon e_n) - u(\epsilon \eta)}{|e_n - \eta|^{n+2s}} d\eta}{\int_{\frac{1}{\epsilon}\Omega} \frac{d\eta}{|e_n - \eta|^{n+2s}}} = I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &:= \frac{\int_{\frac{1}{\epsilon}\Omega} \frac{\nabla u(\epsilon e_n) \cdot (e_n - \eta)}{|e_n - \eta|^{n+2s}} d\eta}{\int_{\frac{1}{\epsilon}\Omega} \frac{d\eta}{|e_n - \eta|^{n+2s}}} \\ \text{and } I_2 &:= \frac{\epsilon^{-1} \int_{\frac{1}{\epsilon}\Omega} \frac{u(\epsilon e_n) - u(\epsilon \eta) - \epsilon \nabla u(\epsilon e_n) \cdot (e_n - \eta)}{|e_n - \eta|^{n+2s}} d\eta}{\int_{\frac{1}{\epsilon}\Omega} \frac{d\eta}{|e_n - \eta|^{n+2s}}}. \end{aligned}$$

So, if  $\Pi := \{x_n < 0\}$ , we have that

$$\lim_{\epsilon \rightarrow 0^+} I_1 = \frac{\int_{\Pi} \frac{\nabla u(0) \cdot (e_n - \eta)}{|e_n - \eta|^{n+2s}} d\eta}{\int_{\Pi} \frac{d\eta}{|e_n - \eta|^{n+2s}}} = \frac{\int_{\Pi} \frac{\partial_n u(0)(1 - \eta_n)}{|e_n - \eta|^{n+2s}} d\eta}{\int_{\Pi} \frac{d\eta}{|e_n - \eta|^{n+2s}}},$$

where we have used that, for any  $i \in \{1, \dots, n-1\}$  the map  $\eta \mapsto \frac{\partial_i u(0) \cdot \eta_i}{|e_n - \eta|^{n+2s}}$  is odd and so its integral averages to zero. So, we can write

$$(5.15) \quad \lim_{\epsilon \rightarrow 0^+} I_1 = \kappa \partial_n u(0) \quad \text{with } \kappa := \frac{\int_{\Pi} \frac{(1 - \eta_n)}{|e_n - \eta|^{n+2s}} d\eta}{\int_{\Pi} \frac{d\eta}{|e_n - \eta|^{n+2s}}}.$$



We remark that  $\kappa$  is finite, since  $s > 1/2$ . Moreover,

$$\begin{aligned} &\epsilon^{-1} |u(\epsilon e_n) - u(\epsilon \eta) - \epsilon \nabla u(\epsilon e_n) \cdot (e_n - \eta)| \\ &= \left| \int_0^1 (\nabla u(t\epsilon e_n + (1-t)\epsilon \eta) - \nabla u(\epsilon e_n)) \cdot (e_n - \eta) dt \right| \\ &\leq \|u\|_{C^{1,\alpha}(\mathbb{R}^n)} |e_n - \eta| \int_0^1 |t\epsilon e_n + (1-t)\epsilon \eta - \epsilon e_n|^\alpha dt \\ &\leq \|u\|_{C^{1,\alpha}(\mathbb{R}^n)} \epsilon^\alpha |e_n - \eta|^{1+\alpha}. \end{aligned}$$

As a consequence,

$$\epsilon^{-\alpha} |I_2| \leq \frac{\|u\|_{C^{1,\alpha}(\mathbb{R}^n)} \int_{\frac{1}{\epsilon}\Omega} \frac{d\eta}{|e_n - \eta|^{n+2s-1-\alpha}}}{\int_{\frac{1}{\epsilon}\Omega} \frac{d\eta}{|e_n - \eta|^{n+2s}}} \longrightarrow \frac{\|u\|_{C^{1,\alpha}(\mathbb{R}^n)} \int_{\Pi} \frac{d\eta}{|e_n - \eta|^{n+2s-1-\alpha}}}{\int_{\Pi} \frac{d\eta}{|e_n - \eta|^{n+2s}}}$$

as  $\epsilon \rightarrow 0$ , which is finite, thanks to our assumptions on  $\alpha$ . This shows that  $I_2 \rightarrow 0$  as  $\epsilon \rightarrow 0$ . Hence, recalling (5.15), we get that

$$\lim_{\epsilon \rightarrow 0^+} \epsilon^{-1} (\tilde{\mathcal{N}}_s u(\epsilon e_n) - \tilde{\mathcal{N}}_s u(0)) = \kappa \partial_n u(0),$$

which establishes (5.13). □

### 6. An overdetermined problem

In this section we consider an overdetermined problem. For this, we will use the renormalized nonlocal Neumann condition that has been introduced in Remark 3.4. Indeed, as we pointed out in Remark 3.5, this is natural if one considers nonhomogeneous Neumann conditions.

**Theorem 6.1.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded and Lipschitz domain. Then there exists no function  $u \in C(\mathbb{R}^n)$  satisfying*

$$(6.1) \quad \begin{cases} u(x) = 0 & \text{for any } x \in \mathbb{R}^n \setminus \Omega, \\ \tilde{\mathcal{N}}_s u(x) = 1 & \text{for any } x \in \mathbb{R}^n \setminus \bar{\Omega}. \end{cases}$$

**Remark 6.2.** We notice that  $u = \chi_\Omega$  satisfies (6.1), but it is a discontinuous function.

*Proof.* Without loss of generality, we can suppose that  $0 \in \partial\Omega$ . We argue by contradiction and we assume that there exists a continuous function  $u$  that satisfies (6.1). Therefore, there exists  $\delta > 0$  such that

$$(6.2) \quad |u| \leq 1/2 \quad \text{in } B_\delta.$$

Since  $\Omega$  is Lipschitz, up to choosing  $\delta$  small enough, we have that

$$\Omega \cap B_\delta = \tilde{\Omega} \cap B_\delta, \quad \text{where } 0\tilde{\Omega} := \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t. } x_n < \gamma(x')\}$$

for a suitable Lipschitz function  $\gamma : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$  such that  $\gamma(0) = 0$  and  $\partial_{x'} \gamma(0) = 0$ .

Now we let  $x := \epsilon e_n \in \mathbb{R}^n \setminus \bar{\Omega}$ , for suitable  $\epsilon > 0$  sufficiently small. We observe that

$$(6.3) \quad u(\epsilon e_n) = 0.$$

Moreover we consider the set

$$\frac{1}{\epsilon} \tilde{\Omega} = \left\{ y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t. } y_n < \frac{1}{\epsilon} \gamma(\epsilon y') \right\}.$$

We also define

$$K := \left\{ y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \text{ s.t. } y_n < -L |y'| \right\},$$

where  $L$  is the Lipschitz constant of  $\gamma$ .

We claim that

$$(6.4) \quad K \subseteq \epsilon^{-1} \tilde{\Omega}.$$

Indeed, since  $\gamma$  is Lipschitz and  $0 \in \partial\Omega$ , we have that

$$-\gamma(\epsilon y') = -\gamma(\epsilon y') + \gamma(0) \leq L |\epsilon y'|,$$

and so, if  $y \in K$ ,

$$y_n \leq -L |y'| \leq \frac{1}{\epsilon} \gamma(\epsilon y'),$$

which implies that  $y \in \epsilon^{-1} \tilde{\Omega}$ . This shows (6.4).

Now we define

$$\Sigma_\epsilon := \int_{B_\delta \cap \Omega} \frac{dy}{|\epsilon e_n - y|^{n+2s}},$$

and we observe that

$$(6.5) \quad \int_{B_\delta \cap \Omega} \frac{u(y) - u(\epsilon e_n)}{|\epsilon e_n - y|^{n+2s}} dy \leq \frac{1}{2} \Sigma_\epsilon,$$

thanks to (6.3) and (6.2). Furthermore, if  $y \in \mathbb{R}^n \setminus B_\delta$  and  $\epsilon \leq \delta/2$ , we have

$$|y - \epsilon e_n| \geq |y| - \epsilon \geq \frac{|y|}{2},$$

which implies that

$$(6.6) \quad \begin{aligned} \int_{\Omega \setminus B_\delta} \frac{u(y) - u(\epsilon e_n)}{|\epsilon e_n - y|^{n+2s}} dy &\leq C \int_{\Omega \setminus B_\delta} \frac{dy}{|\epsilon e_n - y|^{n+2s}} \\ &\leq C \int_{\mathbb{R}^n \setminus B_\delta} \frac{dy}{|y|^{n+2s}} dy = C \delta^{-2s}, \end{aligned}$$

up to renaming the constants.

On the other hand, we have that

$$(6.7) \quad \int_{\Omega} \frac{dy}{|\epsilon e_n - y|^{n+2s}} \geq \int_{B_{\delta} \cap \Omega} \frac{dy}{|\epsilon e_n - y|^{n+2s}} = \Sigma_{\epsilon}.$$

Finally, we observe that

$$(6.8) \quad \begin{aligned} \epsilon^{2s} \Sigma_{\epsilon} &= \epsilon^{2s} \int_{B_{\delta} \cap \Omega} \frac{dy}{|\epsilon e_n - y|^{n+2s}} = \int_{B_{\delta/\epsilon} \cap (\epsilon^{-1}\Omega)} \frac{dz}{|e_n - z|^{n+2s}} \\ &\geq \int_{B_{\delta/\epsilon} \cap K} \frac{dz}{|e_n - z|^{n+2s}} =: \kappa, \end{aligned}$$

where we have used the change of variable  $y = \epsilon z$  and (6.4).

Hence, using the second condition in (6.1) and putting together (6.5), (6.6), (6.7) and (6.8), we obtain

$$\begin{aligned} 0 &= \int_{\Omega} \frac{dy}{|\epsilon e_n - y|^{n+2s}} - \int_{\Omega} \frac{u(\epsilon e_n) - u(x)}{|\epsilon e_n - y|^{n+2s}} dy \\ &= \int_{\Omega} \frac{dy}{|\epsilon e_n - y|^{n+2s}} - \int_{\Omega \cap B_{\delta}} \frac{u(\epsilon e_n) - u(x)}{|\epsilon e_n - y|^{n+2s}} dy - \int_{\Omega \setminus B_{\delta}} \frac{u(\epsilon e_n) - u(x)}{|\epsilon e_n - y|^{n+2s}} dy \\ &\geq \Sigma_{\epsilon} - \frac{1}{2} \Sigma_{\epsilon} - C \delta^{-2s} = \frac{1}{2} \Sigma_{\epsilon} - C \delta^{-2s} \\ &= \epsilon^{-2s} \left( \frac{\epsilon^{2s}}{2} \Sigma_{\epsilon} - C \epsilon^{2s} \delta^{-2s} \right) \geq \epsilon^{-2s} \left( \frac{\kappa}{2} - C \epsilon^{2s} \delta^{-2s} \right) > 0 \end{aligned}$$

if  $\epsilon$  is sufficiently small. This gives a contradiction and concludes the proof.  $\square$

The reader may compare the result in Corollary 5.3 with the one in Theorem 6.1. We stress that the two types of result are quite different in spirit, since Corollary 5.3 only takes into account the homogeneous nonlocal condition, while Theorem 6.1 considers the case in which both Dirichlet and nonhomogeneous Neumann data are prescribed, and this explains why the regularity results obtained are so different.

We also point out that both Corollary 5.3 and Theorem 6.1 only take into account the data outside  $\Omega$ , so they leave it open to study under which condition it is possible to develop a regularity theory for the associated equations.

## 7. Comparison with previous works

In this last section we compare our new Neumann nonlocal conditions with the previous works in the literature that also deal with Neumann-type conditions for the fractional Laplacian  $(-\Delta)^s$  (or related operators).

The idea of [4] and [9] (and also [10], [11] and [8]) is to consider the *regional* fractional Laplacian, associated to the Dirichlet form

$$(7.1) \quad c_{n,s} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} dx dy.$$

This operator corresponds to a censored process, i.e., a process whose jumps are restricted to be in  $\Omega$ . The operator can be defined in general domains  $\Omega$ , and seems to give a natural analogue of homogeneous Neumann condition. This problem has a variational formulation and some properties similar to those obtained in the present paper. However, no nonhomogeneous Neumann conditions can be considered with this model.

Indeed, in analogy with (1.8) one may consider an even kernel  $J$  and a censored energy functional of the form

$$\frac{1}{4} \int_{\Omega} \int_{\Omega} J(x-y)(u(x) - u(y))^2 dx dy - \int_{\Omega} f(x) u(x) dx - \int_{\Omega} \left[ \int_{\mathbb{R}^n \setminus \Omega} J(x-y)g(y)u(x) dy \right] dx,$$

which in turn produces free critical points which satisfy, for any  $x \in \Omega$ ,

$$\int_{\Omega} J(x-y)(u(x) - u(y)) dy - f(x) - \int_{\mathbb{R}^n \setminus \Omega} J(x-y)g(y) dy = 0.$$

By integrating over  $\Omega$ , and using the odd symmetry in  $(x, y)$  of the first term, we thus obtain that

$$(7.2) \quad \int_{\Omega} f(x) dx + \int_{\Omega} \left[ \int_{\mathbb{R}^n \setminus \Omega} J(x-y)g(y) dy \right] dx = 0.$$

We remark that such condition is, in the end, a homogeneous condition, since one can define

$$\tilde{f}(x) := f(x) + \int_{\mathbb{R}^n \setminus \Omega} J(x-y)g(y) dy$$

and so write (7.2) simply as

$$\int_{\Omega} \tilde{f}(x) dx = 0.$$

On the other hand, in [1] and [3] the usual diffusion associated to the fractional Laplacian (1.3) was considered inside  $\Omega$ , and thus the “particle” can jump outside  $\Omega$ . When it jumps outside  $\Omega$ , then it is “reflected” or “projected” inside  $\Omega$  in a deterministic way. Of course, different types of reflections or projections lead to different Neumann conditions. To appropriately define these reflections, some assumptions on the domain  $\Omega$  (like smoothness or convexity) need to be done. In contrast with the regional fractional Laplacian, this problem does not have a variational formulation and everything is done in the context of viscosity solutions.

In [15] a different Neumann problem for the fractional Laplacian was considered. Solutions to this type of Neumann problems are “large solutions”, in the sense that they are not bounded in a neighborhood of  $\partial\Omega$ . More precisely, it is proved in [15] that the following problem is well-posed:

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \\ \partial_{\nu}(u/d^{s-1}) = g & \text{on } \partial\Omega, \end{cases}$$

where  $d(x)$  is the distance to  $\partial\Omega$ .

Finally, in [18] and [22] homogeneous Neumann problems for the *spectral* fractional Laplacian were studied. The operator in this case is defined via the eigenfunctions of the Laplacian  $-\Delta$  in  $\Omega$  with Neumann boundary condition  $\partial_\nu u = 0$  on  $\partial\Omega$ .

With respect to the existing literature, the new Neumann problems (1.1) and (1.4) that we present here have the following advantages:

- The equation satisfied inside  $\Omega$  does not depend on anything (domain, right hand side, etc). Notice that the operator in (1.3) does not depend<sup>3</sup> on the domain  $\Omega$ , while for instance the regional fractional Laplacian defined in (7.1) depends on  $\Omega$ .
- The problem can be formulated in general domains, including nonsmooth or even unbounded ones.
- The problem has a variational structure. For instance, solutions to the elliptic problem (1.1) can be found as critical points of the functional

$$\mathcal{E}(u) = \frac{c_{n,s}}{4} \int_{\mathbb{R}^{2n} \setminus (C\Omega)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy - \int_{\Omega} fu.$$

We notice that the variational formulation of the problem is the analogue of the case  $s = 1$ . Also, this allows us to easily prove existence of solutions (whenever the compatibility condition  $\int_{\Omega} f = 0$  is satisfied).

- Solutions to the fractional heat equation (1.4) possess natural properties like conservation of mass inside  $\Omega$  or convergence to a constant as  $t \rightarrow +\infty$ .
- Our probabilistic interpretation allows us to formulate problems with nonhomogeneous Neumann conditions  $\mathcal{N}_s u = g$  in  $\mathbb{R}^n \setminus \bar{\Omega}$ , or with mixed Dirichlet and Neumann conditions.
- The formulation of nonlinear equations like  $(-\Delta)^s u = f(u)$  in  $\Omega$  with Neumann conditions is also clear.

### A. Proof of Theorems 3.9 and 3.11 with a functional analytic notation

As anticipated in the footnote of page 382, we provide this appendix in order to satisfy the reader who wish to prove Theorems 3.9 and 3.11 by keeping the distinction between a function defined in the whole of  $\mathbb{R}^n$  and its restriction to the domain  $\Omega$ . For this scope, we will use the notation of denoting  $r^+u$  and  $r^-u$  the restriction of  $u$  to  $\Omega$  and  $\mathbb{R}^n \setminus \Omega$ , respectively. Notice that, in this notation, we have that  $u: \mathbb{R}^n \rightarrow \mathbb{R}$ , but  $r^+u: \Omega \rightarrow \mathbb{R}$  and  $r^-u: \mathbb{R}^n \setminus \Omega \rightarrow \mathbb{R}$ .

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<sup>3</sup>That is, the equation satisfied in the domain  $\Omega$  does not depend on  $\Omega$  itself, in the sense that the same equation is satisfied in any subdomain  $\Omega' \subset \Omega$ . Of course, from the point of view of operator theory, the operator has to be understood with the boundary conditions, as it happens also for the usual Laplacian in a bounded domain, when it is complemented with either Dirichlet or Neumann boundary conditions (which indeed produce different spectra).

*Proof of Theorem 3.9.* One can reduce to the case  $g \equiv 0$ . By the Riesz representation theorem, given  $h \in L^2(\Omega)$ , one finds  $v := T_o h \in H_{\Omega,g}^s$  that is a weak solution of

$$r^+((-\Delta)^s v + v) = h,$$

with  $r^- \mathcal{N}_s v = 0$ .

Notice that  $T_o: L^2(\Omega) \rightarrow H_{\Omega,g}^s$ . We also define by  $T: L^2(\Omega) \rightarrow L^2(\Omega)$  the restriction operator of  $T_o$ , that is  $Th := r^+ T_o h$ . One sees that  $T$  is compact and self-adjoint. By construction  $r^- \mathcal{N}_s T_o h = 0$  and

$$h = r^+((-\Delta)^s T_o h + T_o h) = r^+(-\Delta)^s T_o h + Th,$$

that is,

$$r^- \mathcal{N}_s T_o = 0 \quad \text{and} \quad \text{Id} - T = r^+(-\Delta)^s T_o.$$

Therefore, by Lemma 3.8,

$$\begin{aligned} \text{Ker}(\text{Id} - T) &= \{h \in L^2(\Omega) \text{ s.t. } r^+(-\Delta)^s T_o h = 0\} \\ &= \{h \in L^2(\Omega) \text{ s.t. } r^+(-\Delta)^s T_o h = 0 \text{ and } r^- \mathcal{N}_s T_o h = 0\} \\ &= \{h \in L^2(\Omega) \text{ s.t. } T_o h \text{ is constant}\} \\ &= \{h \in L^2(\Omega) \text{ s.t. } h \text{ is constant}\}. \end{aligned}$$

From the Fredholm alternative, we conclude that  $\text{Im}(\text{Id} - T)$  is the space of functions in  $L^2(\Omega)$  that are orthogonal to constants.  $\square$

*Proof of Theorem 3.11.* We define

$$L_0^2(\Omega) := \{u \in L^2(\Omega) : \int_{\Omega} u = 0\}.$$

By Theorem 3.9, for any  $f \in L_0^2(\Omega)$  one finds  $v := T_o f \in H_{\Omega,g}^s$  that is a weak solution of  $r^+(-\Delta)^s v = f$ , with  $r^- \mathcal{N}_s v = 0$  and zero average in  $\Omega$ . We also define  $T$  to be the restriction of  $T_o$ , that is  $Tf := r^+ T_o f$ . The operator  $T$  is compact and self-adjoint in  $L_0^2(\Omega)$ . Thus, by the spectral theorem there exists a sequence of eigenvalues  $\{\mu_i\}_{i \geq 2}$  of  $T$ , and its corresponding eigenfunctions  $\{e_i\}_{i \geq 2}$  are a complete orthogonal system in  $L_0^2(\Omega)$ .

Notice that  $r^- \mathcal{N}_s T_o e_i$ , which gives, for every  $x \in \mathbb{R}^n \setminus \Omega$ ,

$$\begin{aligned} T_o e_i(x) \int_{\Omega} \frac{dy}{|x-y|^{n+2s}} &= \int_{\Omega} \frac{r^+ T_o e_i(y)}{|x-y|^{n+2s}} dy \\ &= \int_{\Omega} \frac{T e_i(y)}{|x-y|^{n+2s}} dy = \mu_i \int_{\Omega} \frac{e_i(y)}{|x-y|^{n+2s}} dy. \end{aligned}$$

This gives that

$$\mu_i \neq 0.$$

Indeed, otherwise we would have that  $r^- T_o e_i = 0$ . Since also

$$0 = \mu_i e_i = T e_i = r^+ T_o e_i,$$

we would get that  $T_o e_i = 0$  and thus  $0 = (-\Delta)^s T_o e_i = e_i$ , which is impossible.

As a consequence, we can define  $\lambda_i := \mu_i^{-1}$ , and  $u_i := T_o e_i$ .

Then

$$r^+ u_i = r^+ T_o e_i = T e_i = \mu_i e_i$$

thus  $\{r^+ u_i\}_{i \geq 2}$  are a complete orthogonal system in  $L_0^2(\Omega)$ , since so are  $\{e_i\}_{i \geq 2}$ .

Furthermore,

$$r^+ (-\Delta)^s u_i = r^+ (-\Delta)^s T_o e_i = e_i = \mu_i^{-1} r^+ u_i = r^+ \lambda_i u_i. \quad \square$$

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## References

- [1] BARLES, G., CHASSEIGNE, E., GEORGELIN, C. AND JAKOBSEN, E. R.: On Neumann type problems for nonlocal equations in a half space. *Trans. Amer. Math. Soc.* **366** (2014), no. 9, 4873–4917.
- [2] BARLES, G., CHASSEIGNE, E. AND IMBERT, C.: The Dirichlet problem for second-order elliptic integro-differential equations. *Indiana Univ. Math. J.* **57** (2008), no. 1, 213–246.
- [3] BARLES, G., GEORGELIN, C. AND JAKOBSEN, E. R.: On Neumann and oblique derivatives boundary conditions for nonlocal elliptic equations. *J. Differential Equations* **256** (2014), no. 4, 1368–1394.
- [4] BOGDAN, K., BURDZY, K. AND CHEN Z.-Q.: Censored stable processes. *Probab. Theory Related Fields* **127** (2003), no. 1, 89–152.
- [5] BOURGAIN, J., BREZIS, H. AND MIRONESCU, P.: Another look at Sobolev spaces. In *Optimal control and partial differential equations*, 439–455. IOS Press, Amsterdam, 2001.
- [6] CAFFARELLI, L., ROQUEJOFFRE, J. M. AND SAVIN, O.: Nonlocal minimal surfaces. *Comm. Pure Appl. Math.* **63** (2010), no. 9, 1111–1144.
- [7] CAFFARELLI, L. AND SILVESTRE, L.: Regularity theory for fully nonlinear integro-differential equations. *Comm. Pure Appl. Math.* **62** (2009), no. 5, 597–638.
- [8] CHASSEIGNE, E., CHAVES, M. AND ROSSI, J. D.: Asymptotic behavior for nonlocal diffusion equations. *J. Math. Pures Appl.* **86** (2006), no. 3, 271–291.
- [9] CHEN, Z.-Q. AND KIM, P.: Green function estimate for censored stable processes. *Probab. Theory Related Fields* **124** (2002), no. 4, 595–610.
- [10] CORTAZAR, C., ELGUETA, M., ROSSI, J. D. AND WOLANSKI, N.: Boundary fluxes for nonlocal diffusion. *J. Differential Equations* **234** (2007), no. 2, 360–390.
- [11] CORTAZAR, C., ELGUETA, M., ROSSI, J. D. AND WOLANSKI, N.: How to approximate the heat equation with Neumann boundary conditions by nonlocal diffusion problems. *Arch. Ration. Mech. Anal.* **187** (2008), no. 1, 137–156.
- [12] DI NEZZA, E., PALATUCCI, G. AND VALDINOCI, E.: Hitchhiker’s guide to the fractional Sobolev spaces. *Bull. Sci. Math.* **136** (2012), no. 5, 521–573.
- [13] FELSINGER, M., KASSMANN, M. AND VOIGT, P.: The Dirichlet problem for nonlocal operators. *Math. Z.* **279** (2015), no. 3-4, 779–809.

- [14] GRUBB, G.: Fractional Laplacians on domains, a development of Hörmander's theory of  $\mu$ -transmission pseudodifferential operators. *Adv. Math.* **268** (2015), 478–528.
- [15] GRUBB G.: Local and nonlocal boundary conditions for  $\mu$ -transmission and fractional elliptic pseudodifferential operators. *Anal. PDE* **7** (2014), no. 7, 1649–1682.
- [16] JOST, J.: *Partial differential equations*. Springer, New York, 2013.
- [17] LANDKOF, N. S.: *Foundations of modern potential theory*. Die Grundlehren der mathematischen Wissenschaften 180, Springer-Verlag, New York-Heidelberg, 1972.
- [18] MONTEFUSCO, E., PELLACCI, B. AND VERZINI, G.: Fractional diffusion with Neumann boundary conditions: the logistic equation. *Discrete Contin. Dyn. Syst. Ser. B* **18** (2013), no. 8, 2175–2202.
- [19] ROS-OTON, X. AND SERRA, J.: The Dirichlet problem for the fractional Laplacian: regularity up to the boundary. *J. Math. Pures Appl.* (9) **101** (2014), no. 3, 275–302.
- [20] SERVADEI, R. AND VALDINOCI, E.: Weak and viscosity solutions of the fractional Laplace equation. *Publ. Mat.* **58** (2014), no. 1, 133–154.
- [21] SILVESTRE, L.: Regularity of the obstacle problem for a fractional power of the Laplace operator. *Comm. Pure Appl. Math.* **60** (2007), no. 1, 67–112.
- [22] STINGA, P. R. AND VOLZONE, B.: Fractional semilinear Neumann problems arising from a fractional Keller–Segel model. *Calc. Var. Partial Differential Equations* **54** (2015), no. 1, 1009–1042.
- [23] VALDINOCI, E.: From the long jump random walk to the fractional Laplacian. *Bol. Soc. Esp. Mat. Apl. SĕMA* **49** (2009), 33–44.

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