Rev. Mat. Iberoam. **33** (2017), no. 2, 449–468
DOI 10.4171/RMI/944


# Curvature locus and principal configurations of submanifolds of Euclidean space

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**Abstract.** We study relations between the properties of the curvature loci of a submanifold $M$ in Euclidean space and the behaviour of the principal configurations of $M$, in particular the existence of umbilic and quasiumbilic fields. We pay special attention to the case of submanifolds with vanishing normal curvature. We also characterize local convexity in terms of the curvature locus position in the normal space.

## 1. Introduction

The second order properties of an immersion of a manifold into an ambient space determine a good part of its extrinsic geometry. Remarkable examples of this would be properties such as vanishing of the normal curvature, existence of umbilic normal fields, existence of common principal directions for two or more linearly independent normal fields, local convexity and so on. An interesting and useful geometrical object associated with the second fundamental form is the curvature locus. This is the natural generalization of the curvature ellipse, originally introduced to study the extrinsic geometry of surfaces immersed in $\mathbb{R}^4$ (see [8], [15], [16]) to the case of $n$-submanifolds immersed with any codimension. For submanifolds of higher dimension the curvature locus becomes a more interesting geometrical object. In fact, it is either the image of a Veronese manifold through a convenient linear projection or its projection onto an Euclidean subspace of the normal space at the considered point. A recent work for the particular case of 3-manifolds ([1], [2]) illustrates the rich variety of topological and geometrical types that a curvature locus may present. On the other hand, the study of the curvature locus for submanifolds with codimension 2 of Euclidean space carried out in [17] lead to interesting results concerning the relations among some of the above mentioned properties and the existence of hyperplanes with higher order contact with the

*Mathematics Subject Classification* (2010): Primary 58C25; Secondary 53A05.
*Keywords:* Umbilicity, $\nu$-principal curvature foliation, curvature locus, normal curvature, convexity.

submanifold. In the present paper we extend this last analysis to the case of submanifolds immersed in higher codimension. In sections 2 and 3 we provide the definitions of the notions referred above and prove that at a point where the normal curvature vanishes the curvature locus becomes a polyhedron whose vertices are determined by the principal directions at $p$ (Theorem 3.1). In section 4 we analyze the possible existence of umbilic and preumbilic normal directions in terms of the geometry of the curvature locus at a point. Section 5 is devoted to the study of the connections between the existence of principal directions which are shared by a certain number of linearly independent normal directions and the existence of a $\nu$-umbilic direction at a given point of the manifold. As a consequence, we provide a table displaying the minimal number of shared principal directions, as a function of the dimension and codimension of the manifold, that guarantee the existence of an umbilic direction at a point of the submanifold. We also show the connection between the corank of the singularities of the curvature locus map at a given point $p$ and the number of principal directions shared by normal linearly independent fields at $p$ (Proposition 5.3). In section 6 we analyze local convexity of the manifold including its relation with the existence of higher order contact hyperplanes. This is characterized in terms of the relative position of the curvature locus with respect to origin of the normal space (Corollary 6.1). We conclude by discussing the contributions of the results of the article to the connections between the properties of semiumbilicity, vanishing of the normal curvature, local convexity and existence of higher order tangent hyperplanes for submanifolds of codimension higher than 2.

## 2. Second fundamental form and principal configurations

Let $M$ be an $n$-manifold immersed in $\mathbb{R}^{n+k}$ and let $\bar{\nabla}$ denote the Riemannian connection of $\mathbb{R}^{n+k}$. Given vector fields, $X, Y$, locally defined along $M$, we can choose local extensions $\bar{X}, \bar{Y}$ over $\mathbb{R}^{n+k}$, and define the Riemannian connection on $M$ as $\nabla_X Y = (\bar{\nabla}_{\bar{X}} \bar{Y})^{\top}$, that is, the tangent component of $\bar{\nabla}_{\bar{X}}$ on $M$.

If we denote by $\mathcal{X}(M)$ and $\mathcal{N}(M)$ respectively the spaces of tangent and normal fields on $M$, the *second fundamental form* on $M$ is defined as follows:

$$\begin{array}{rccl} \alpha : & \mathcal{X}(M) \times \mathcal{X}(M) & \longrightarrow & \mathcal{N}(M) \\ & \big(X, Y\big) & \longmapsto & \bar{\nabla}_{\bar{X}} \bar{Y} - \nabla_X Y. \end{array}$$

This bilinear symmetric map induces, for each $p \in M$ and $\nu \in N_pM$, $\nu \neq 0$, a bilinear form on the tangent space $T_pM$ given by $H_\nu(v, w) = \big\langle \alpha_p(\bar{v}(p), \bar{w}(p)), \nu \big\rangle$, where $\bar{v}$ and $\bar{w}$ are tangent vector fields such that $\bar{v}(p) = v$ and $\bar{w}(p) = w$. Since this expression does not depend on the tangent vector fields off $p$, in the sequel we will write $H_\nu(v, w) = \big\langle \alpha_p(v, w), \nu \big\rangle$. The corresponding quadratic form $II_\nu(v) = H_\nu(v, v) = \big\langle \alpha(v, v), \nu \big\rangle$ is known as the *second fundamental form in the direction $\nu$*.

Consider a local coordinate chart in a neighborhood of $p \in M$ defined by $(f, U)$, where $U \subset \mathbb{R}^n$ is an open neighborhood of the origin. Assume that $p = f(0)$

and $\{X_1,\dots,X_n,\nu_1,\dots,\nu_k\}$ is an orthonormal moving frame in $f(U)$, such that $\{X_1,\dots,X_n\}$ is a tangent frame and $\{\nu_1,\dots,\nu_k\}$ is a normal frame. The vector valued quadratic form $\alpha_f$ induces, for each $p\in M$, a linear map $Q_p$ from the normal space, $N_pM$, of $M$ at $p$ to the space $\mathcal{Q}$ of quadratic forms in the variables $\{x_1,\dots,x_n\}$. If we represent a vector $v\in N_pM$ by its coordinates $(v_1,\dots,v_k)$ with respect to the basis $\{\nu_1,\dots,\nu_k\}$, we have

$$Q_p(v_1,\dots,v_k)=v_1\big\langle d^2f,\nu_1\big\rangle+\dots+v_k\big\langle d^2f,\nu_k\big\rangle.$$

By using the natural identifications (through the basis induced by the above frame) of $N_pM$ with $\mathbb{R}^k$ and of $\mathcal{Q}$ with $\mathbb{R}^{\frac{1}{2}n(n+1)}$, we can view this as the linear map $Q_p:\mathbb{R}^k\to\mathbb{R}^{\frac{1}{2}n(n+1)}$, whose matrix is the transpose of that of $\alpha$ at $p$.

If we denote $\langle d^2f,\nu_r\rangle(X_i,X_j)=\alpha^r_{ij}$, we have that the matrix of the map $Q_p$ in the basis $\{\nu_r\}_{r=1}^k$ of $N_pM$ and $\{x_1^2,x_2^2,\dots,x_n^2,x_1x_2,x_1x_3,\dots,x_{n-1}x_n\}$ of $\mathcal{Q}$ is given by

$$\begin{pmatrix}\alpha^1_{11} & \cdots & \alpha^k_{11}\\ \alpha^1_{22} & \cdots & \alpha^k_{22}\\ \vdots & \ddots & \vdots\\ \alpha^1_{nn} & \cdots & \alpha^k_{nn}\\ \alpha^1_{12} & \cdots & \alpha^k_{12}\\ \vdots & \ddots & \vdots\\ \alpha^1_{n-1n} & \cdots & \alpha^k_{n-1n}\end{pmatrix}.$$

The *first normal space* of the immersion $f$ at the point $p$ is defined as the orthogonal complement of the kernel of the linear map $Q_p$ in $N_pM$. We denote it by $N^1_pM$. Clearly $\dim N^1_pM=\operatorname{rank}Q_p$.

Given any normal field $\nu$ on $M$, its *associated shape operator* at a point $p\in M$ is given by

$$\begin{array}{rccl}A_\nu: & T_pM & \longrightarrow & T_pM\\ & X & \longmapsto & A_\nu(X)=-\big(\bar\nabla_{\bar X}\bar\nu\big)^\top,\end{array}$$

where $\bar\nu$ is a local extension of $\nu$ over a neighborhood of $p$ in $\mathbb{R}^{n+k}$ and $\top$ denotes the tangent component of the connection $\bar\nabla$. It satisfies the following equation:

$$\big\langle A_\nu(X),Y\big\rangle=\big\langle\alpha(X,Y),\nu\big\rangle;\forall X,Y\in T_pM.$$

So, we can write

$$II_\nu(X)=\big\langle A_\nu(X),X\big\rangle.$$

For each $p\in M$, there exists an orthonormal basis of eigenvectors of $A_\nu\in T_pM$. The corresponding eigenvalues $\kappa^\nu_1,\dots,\kappa^\nu_n$, will be referred to as the *$\nu$-principal curvatures.* For sake of simplicity we will avoid the superindex $\nu$ when it is not necessary. A point $p$ is said to be *$\nu$-preumbilic* if there is an eigenvalue of multiplicity $r>1$ at $p$. When $r=n$, we say that $p$ is *$\nu$-umbilic* and when $r=n-1$ it is a *$\nu$-quasiumbilic.* If all the points of $M$ are $\nu$-umbilic, we shall say that $\nu$ is an *umbilic* *field* on $M$. Quasiumbilic and preumbilic fields in general are analogously defined. We denote $U_\nu(k^\nu_{i_1},\dots,k^\nu_{i_r}) = \{p \in M : k_{i_1}(p) = \cdots = k_{i_r}(p)\}$, $r = 2,\dots,n$. A point lying in $U_\nu(k_1,\dots,k_n)$ is called *$\nu$-umbilic*. Let us denote by $\mathcal{U}_\nu$ the set of $\nu$-preumbilic points. Given $p \in M - \mathcal{U}_\nu$, there are $n$ $\nu$-principal directions defined by the eigenvectors of $A_\nu$. Provided $M - \mathcal{U}_\nu$ is open, this setting determines fields of directions on $M - \mathcal{U}_\nu$ which are smooth and integrable. The integrals of these fields are $n$ families of orthogonal curves on $M - \mathcal{U}_\nu$, called *$\nu$-principal lines of curvature*. These $n$ orthogonal foliations of $M - \mathcal{U}_\nu$, together with the decomposition $\{U_\nu(k_{i_1},\dots,k_{i_r})\}$ of $\mathcal{U}_\nu$ form the *$\nu$-principal configuration* of $M$. The points of $U_\nu(k_{i_1},\dots,k_{i_r})$ can be seen as the critical points for the $i_j$-th foliation, $j = 1,\dots,r$, whereas the $\nu$-umbilics are critical points for the $n$ foliations. The behavior of these foliations in case $M$ is a surface immersed in $\mathbb{R}^4$ was analyzed in [6], [7] and [19]. Since the self-adjoint operator $A_\nu$ only depends on the value of $\nu$ at the point $p$, the $\nu$-principal directions, $\nu$-preumbilicity and $\nu$-umbilicity are notions that only depend on the normal direction $\nu$ at $p$. We say that $p$ is *umbilic* if it is $\nu$-umbilic for any normal direction $\nu$ and we call it *semiumbilic* if it is $\nu$-umbilic for any normal direction lying in some hyperplane of $N_pM$, i.e., it is an umbilic point of $n-1$ linearly independent normal directions in $N_pM$.

For $X \in \mathcal{X}(M)$ and $\nu \in \mathcal{N}(M)$, we have the Weingarten equation

$$\bar\nabla_X \nu = -S_\nu(X) + (\bar\nabla_X \nu)^\perp.$$

Denote $D_X\nu = (\bar\nabla_X\nu)^\perp$. The *normal curvature* of $M$ at $p$ is defined as

$$\begin{array}{rccl} R_D : & T_pM \times T_pM \times N_pM & \longrightarrow & N_pM \\ & (X, Y, \nu) & \longmapsto & \big(D_{\bar X}(D_{\bar Y}\bar\nu) - D_{\bar Y}(D_{\bar X}\bar\nu) - D_{[\bar X,\bar Y]}\bar\nu\big)_p. \end{array}$$

where bar means, as above, a vector field whose value at $p$ is the corresponding vector. It is well known that the following equivalence holds for submanifolds immersed in any Euclidean space [20].

**Remark 2.1.** $R_D$ vanishes at $p \in M$ if and only if there is an orthonormal basis $\{X_1,\dots,X_n\}$ for $T_pM$ made of eigenvectors of $A_\nu$, for all $\nu \in \mathcal{N}(M)$.

**Definition 2.1.** We say that $k$ normal directions $\nu_i$, $i = 1,\dots,k$ *share a principal direction at a point* $p \in M$ if there is a non-zero vector in $T_pM$ tangent to a $\nu_i$-principal direction for all $i = 1,\dots,k$.

This definition allow us to consider in a natural way unitary normal fields that share their principal lines of curvature.

An immediate result is the following:

**Corollary 2.1.** *$R_D$ vanishes identically on $M$ if and only if all unitary normal fields on $M$ share all their principal lines of curvature at all the points of $M$.*

## 3. Curvature locus

Following Little [8], given $p \in M$, we define *the curvature locus* of $M$ at $p$ as the image of the following map:

$$\begin{array}{cccc} \eta : & S_p^{n-1} & \longrightarrow & N_pM \\ & X & \longmapsto & \alpha(X,X), \end{array}$$

where $S_p^{n-1}$ is the unit sphere in $T_pM$. The normal vector $\eta(X)$ can be interpreted as the curvature vector at $p$ of the normal section of $M$ in the direction $X$ at $p$ (= curve obtained by intersecting $M$ with the $(k+1)$-space given by the direct sum of the line spanned by the tangent direction $X$ with $N_pM$).

The orthonormal frame defined above determine orthonormal basis $\{X_1,\ldots,X_n\}$ and $\{\nu_1,\ldots,\nu_k\}$ for $T_pM$ and $N_pM$, respectively. If $X = \Sigma_{i=1}^n x_i X_i$, we have

$$\eta(X) = (\Sigma_{i,r=1}^n \alpha_{ir}^1\, x_i\, x_r)\, \nu_1 + \cdots + (\Sigma_{i,r=1}^n \alpha_{ir}^k\, x_i\, x_r)\, \nu_k,$$

where we recall that $< d^2 f(X_i, X_r), \nu_j >= \alpha_{ir}^j$. So $\eta$ is the restriction of a homogeneous polynomial map of degree 2 to the $(n-1)$-sphere $S_p^{n-1}$ and its image, is either a Veronese manifold or its projection onto some Euclidean space (depending on the codimension $k$ of $M$ and the rank of $\alpha$ at $p$). This image is called the *curvature locus* of $M$ at $p$ and will be denoted by $\mathcal{V}(p)$. We observe that a conformal map on the ambient space induces a homothety on the curvature locus at every point. The centroid of $\mathcal{V}(p)$ is the *mean curvature vector* of $M$ at $p$, given by

$$H(p) = \frac{1}{n}\big( (\Sigma_{i=1}^n \alpha_{ii}^1)\, \nu_1 + \cdots + (\Sigma_{i=1}^n \alpha_{ii}^k)\, \nu_k \big).$$

Figures 1 and 2 provide examples of curvature loci of 3-manifolds in $\mathbb{R}^6$. The first one corresponds to the immersion

$$f : \mathbb{R}^3 \to \mathbb{R}^6; \quad f(x_1,x_2,x_3) = (x_1, x_2, x_3, x_1^2 + x_2^2 - x_3^3, x_1x_2, x_1x_3),$$

at the point $(0.0126, -0.2652, 0)$, and the curvature locus is a projection of the Veronese surface given by a Steiner's roman surface. The second one, corresponding to the immersion

$$f : \mathbb{R}^3 \to \mathbb{R}^6; \quad f(x_1,x_2,x_3) = (x_1, x_2, x_3, x_1^3 + x_2^3, x_2^2x_3, x_3^3),$$

at the point $(0.1, -0.2, 0)$ illustrates a degenerate case in which the curvature locus is a cone. Both pictures have been obtained with the aid of the program `ImmersionR3ToR6` due to A. Montesinos Amilibia [13].

Denote by $\mathcal{A}\mathrm{ff}(p)$ the affine hull of $\mathcal{V}(p)$ and by $L(p)$ its linear span. Given orthonormal bases $\{X_1,\ldots,X_n\}$ of $T_pM$ and $\{\nu_1,\ldots,\nu_k\}$ of $N_pM$, we have

$$\eta(X_i) = \Sigma_{l=1}^k \alpha_{ii}^l \nu_l,$$

$$\eta(\tfrac{1}{\sqrt{2}}(X_i + X_j)) = \tfrac{1}{2}\Sigma_{l=1}^k (\alpha_{ii}^l + 2\alpha_{ij}^l + \alpha_{jj}^l)\, \nu_l.$$

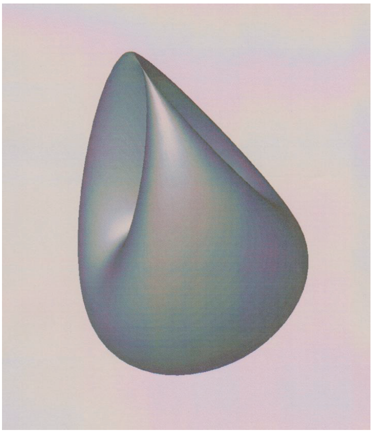

Figure 1. Curvature locus at a generic point of a 3-manifold in $\mathbb{R}^6$.

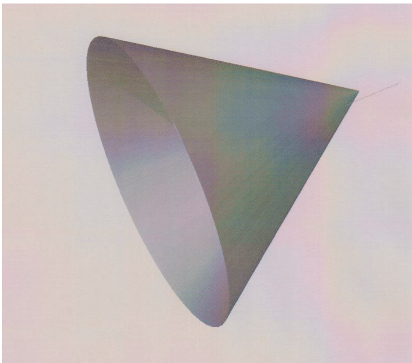

Figure 2. Degenerate curvature locus of a 3-manifold in $\mathbb{R}^6$.

It is not difficult to see that for any $X \in S^{n-1} \subset T_pM$, the point $\eta(X)$ is an affine combination of the points $\eta(X_i)$ and $\eta(\frac{1}{\sqrt{2}}(X_i + X_j))$, $i, j = 1, \dots, n$. So $\mathcal{A}\mathrm{ff}(p)$ is the affine hull of these points. These points are affinely independent if and only if

$$\text{rank } \mathrm{Q_p} = \begin{pmatrix} n+1 \\ n-1 \end{pmatrix}.$$

The vector space $L(p)$ is generated by

$$\Sigma_{l=1}^{k}(\alpha_{ii}^{l} - \alpha_{11}^{l})\,\nu_l, \quad i = 1, \dots, n,$$

$$\Sigma_{l=1}^{k}(2\alpha_{ij}^{l} - \alpha_{11}^{l})\,\nu_l, \quad i \neq j,\ i, j = 1, \dots, n,$$

where all the coefficient functions are evaluated at $p$. We have that $\mathcal{A}\mathrm{ff}(p) = H(p) + L(p)$. Moreover,

$$N_p^1 M = L(p) + \langle H(p) \rangle,$$

where $\langle H(p)\rangle$ means the line defined by $H(p)$ in the normal space $N_pM$. In the case that $\mathrm{rank}(Q_p)$ is maximal, namely the case of $\mathrm{rank}\, Q_p = \binom{n+1}{n-1}$, this is a direct sum and the following holds:

$$\dim \mathcal{A}\mathrm{ff}(p) = \dim L(p) = \mathrm{rank}(Q_p) - 1.$$

If $\mathrm{rank}\,(Q_p) < \binom{n+1}{n-1}$, we may have either

i) $H(p) \in L(p) = \mathcal{A}\mathrm{ff}(p)$, in which case $N_p^1 M = L(p)$, or

ii) $H(p) \notin L(p)$, in which case $N_p^1 M = L(p) \oplus \big\langle H(p)\big\rangle$.

**Theorem 3.1.** *If $R_D = 0$ at $p \in M$, then the curvature locus of $M$ at $p$ is a convex polyhedron, given by the convex hull of the points $\eta(X_i)$ in $N_pM$, where $\{X_i\}_{i=1}^n$ are the (univocally defined) principal directions at $p$.*

*Proof.* Let $\{X_i\}_{i=1}^n$ be an orthonormal basis of $T_pM$ and $\{\nu_i\}_{i=1}^k$ and orthonormal basis of $N_pM$, respectively. The shape operators in this basis have the expressions

$$A_{\nu_j}(X_i) = \Sigma_{r=1}^n \alpha_{ir}^j\, X_r.$$

Since $R_D(p) = 0$, Remark 2.1 implies that we can choose the tangent basis constituted only by $\nu$-principal vectors $\forall \nu \in N_pM$, that is, $A_{\nu_j}(X_i) = \lambda_i^j X_i$ for $i = 1, \ldots, n$ and $j = 1, \ldots, k$. Given any vector $X \in S_p^{n-1} \subset T_pM$, we can write $X = x_1X_1 + \cdots + x_nX_n$, where $x_1^2 + \cdots + x_n^2 = 1$, and we have:

$$\eta(X) = \Sigma_{j=1}^k \big(\Sigma_{i=1}^n \lambda_i^j\, x_i^2\big)\nu_j = x_1^2\big(\Sigma_{j=1}^k \lambda_1^j\, \nu_j\big) + \cdots + x_n^2\big(\Sigma_{j=1}^k \lambda_n^j\, \nu_j\big) = \Sigma_{i=1}^n x_i^2\, P_i,$$

where $P_i = \Sigma_{j=1}^k \lambda_i^j \nu_j$. Since $x_1^2 + \cdots + x_n^2 = 1$, and $0 \le x_i^2 \le 1$ for all $i = 1, \ldots, n$, it follows that $\eta(X)$ lies in the convex hull of the points $P_1, \ldots, P_n$. □

As an immediate consequence of the above theorem we get:

**Corollary 3.1.** *If $R_D(p) = 0$ then $\mathcal{V}(p)$ is a convex polyhedron of dimension less than or equal to* $\min(k, n-1)$ *with at most $n$ vertices.*

**Corollary 3.2.** *If $R_D(p) = 0$ then* $\dim\, \mathcal{A}\mathrm{ff}(p) \le n - 1$.

The following example shows that the converse of Theorem 3.1 is not true.

**Example 3.1.** Consider the embedding $g : \mathbb{R}^5 \to \mathbb{R}^7$ given by

$$g(x, y, z, t, u) = (x, y, z, t, u, 2x^2 - 2z^2 + u^2, -x^2 + 2y^2 - z^2 + t^2 + tu).$$

A simple computation shows that the curvature locus $\mathcal{V}(0)$ is the triangle with vertices $(2, -1)$, $(0, 2)$ and $(-2, -1)$, but $R_D(0) \neq 0$. In fact, the restriction to the $(t, u)$-plane gives a non degenerate ellipse contained in the triangle $\Delta_p$ (see Figure 3).

The following lemma is an easy exercise for quadratic maps in the plane.

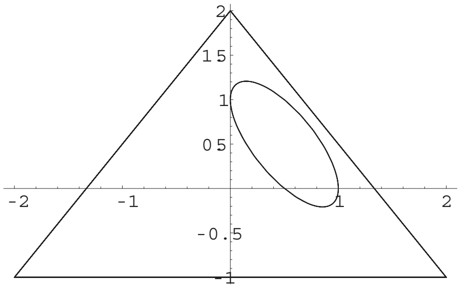


FIGURE 3. Curvature locus at a point with non vanishing normal curvature.

**Lemma 3.1.** *Let $h = (h_1, h_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be a quadratic map. The ellipse $h(S^1)$ degenerates to a segment $\overline{PQ}$ if and only if $P = h(X)$, $Q = h(Y)$, where $\{X, Y\}$ is an orthonormal frame of $\mathbb{R}^2$ which diagonalizes the two quadratic forms $h_1, h_2$ simultaneously.*

The next result can be considered as a kind of "partial converse" of Theorem 3.1.

**Proposition 3.1.** *If $\mathcal{V}(p)$ is a polyhedron with at least $n-1$ vertices, then $R_D(p) = 0$.*

*Proof.* First assume that $\mathcal{V}(p)$ has $n$ vertices, $P_1, \dots, P_n$. If $X_1, \dots, X_n$ are vectors of $S_p^{n-1} \subset T_pM$ such that $\eta(X_i) = P_i$, for $i = 1, \dots, n$, the image of the restriction

$$\eta: \ \mathcal{V}^{ij}(p) := S_p^{n-1} \cap P_{ij} \to N_pM,$$

where $P_{ij}$ is the plane generated by $X_i$ and $X_j$, is the interval $P_iP_j$, since the image of $\mathcal{V}^{ij}(p)$ is a subset of the polygon $\mathcal{V}(p)$ containing the vertices $P_i$ and $P_j$, and the restricted map is also a quadratic map. Therefore, Lemma 3.1 implies that $X_i$ and $X_j$ are orthogonal directions. The restricted quadratic forms are diagonal in this basis. Then, using the expression of the second fundamental form we conclude that all the shape operators $A_{\nu_i}$ are diagonal in this basis. Therefore, $A_{\nu_i}A_{\nu_j} = A_{\nu_j}A_{\nu_i}$. This implies that $R_D(p) = 0$. The same argument can be applied if $\mathcal{V}(p)$ has $n-1$ vertices $P_1, \dots, P_{n-1}$. We only need to observe that a direction normal to $\mathcal{V}(p)$, denoted by $X_n$, determines degenerate ellipses with respect to any other direction $X_i$, $i = 1, \dots, n-1$. □

## 4. Existence of umbilic directions

We have the following characterization of umbilic directions in terms of the curvature locus.

**Proposition 4.1.** *Given $p \in M$ and $\nu \in N_pM$, $p$ is a $\nu$-umbilic point if and only if $\nu \perp L(p)$.*

*Proof.* Observe first that

$$L(p) = \{\lambda(\eta(X) - \eta(Y)) : \forall X, Y \in S^{n-1} \subset T_pM; \forall \lambda \in \mathbb{R}\}.$$

Therefore $\nu \perp L(p)$ if and only if $\langle \eta(X) - \eta(Y), \nu \rangle = 0, \forall X, Y \in S^{n-1} \subset T_pM$. Now, $p$ is $\nu$-umbilic if and only if $\langle A_\nu(X), X \rangle = \lambda, \forall X \in S^{n-1} \subset T_pM$. Considering that $\langle A_\nu(X), X \rangle = \langle \alpha(X, X), \nu \rangle$ we conclude that $p$ is $\nu$-umbilic if and only if $\langle \alpha(X, X), \nu \rangle = \langle \alpha(Y, Y), \nu \rangle, \forall X, Y \in S^{n-1} \subset T_pM$. Which is equivalent to $\langle \eta(X) - \eta(Y), \nu \rangle = 0, \forall X, Y \in S^{n-1} \subset T_pM$. □

**Remark 4.1.** Given $p \in M^n \subset \mathbb{R}^{n+k}$, we have:

1) There exists some umbilic direction $\nu$ at $p$ if and only if $\dim L(p) < k$.
2) Suppose that $n = k + 1$. If $\mathcal{V}(p)$ is an $(n-1)$ simplex then $R_D(p) = 0$. If $R_D(p) = 0$ then, either $\mathcal{V}(p)$ is a $(n-1)$ simplex or $p$ is $\nu$-umbilic for some normal direction $\nu$.
3) Suppose that $n = k$. Then if $R_D(p) = 0$, the point $p$ is $\nu$-umbilic for some normal direction $\nu$.
4) The following example shows that $\nu$ umbilicity does not imply $R_D(p) = 0$. The curvature locus at the origin $O = f(0, 0, 0)$ of the immersion

$$f : \mathbb{R}^3 \to \mathbb{R}^6, \quad f(x, y, z) = (x, y, z, 2x^2 + z^2 + xy, xz, x^2 + y^2 + z^2)$$

is a planar region $\mathcal{R}$ which is not a triangle. It then follows from Corollary 3.1 that $R_D(O) \neq 0$. On the other hand, the orthogonal direction to the plane determined by $\mathcal{R}$ in the normal space at $O$ is an umbilic direction of the 3-manifold at the point $O$.

Let us study the projections of the curvature locus $\mathcal{V}(p)$ on lines normal to $M$ at $p$. For this we consider first the case were $M$ is a hypersurface. In this case the unique line $l_N$ normal to $M$ at $p$ contains the curvature locus. This is an interval denoted by $\mathrm{I}_\nu$ that may degenerate into a point. We can identify $\mathbb{R}$ with this line oriented in such a way that the orientation of the manifold is compatible with that of $\mathbb{R}^{n+1}$. Then, the extreme points of the interval correspond under this identification to the extreme principal curvatures denoted by $k_{\min}$ and $k_{\max}$, respectively. Observe that for a general codimension $k$ there is not a natural orientation of $l_\nu$. Thus, the extreme point of $\mathrm{I}_\nu$ corresponding to the minimal (maximal) curvature will correspond to the maximal (minimal) curvature if we consider the other orientation of $l_\nu$.

**Lemma 4.1.** *Let $p \in M^n \subset \mathbb{R}^{n+k}$. Consider $\nu \in N_pM$ and $l_\nu$ the line of $N_pM$ generated by $\nu$. The orthogonal projection $P_\nu \colon N_pM \to l_\nu$, takes $\mathcal{V}(p)$ onto an interval $\mathrm{I}_\nu$ that may degenerate into a point. Let us identify the oriented line $l_\nu$ with $\mathbb{R}$. Then, the extreme points of $\mathrm{I}_\nu$ correspond under this identification to the minimal $\nu$-principal curvature $k^\nu_{\min}$, and the maximal $\nu$-principal curvature $k^\nu_{\max}$, respectively.*

*Proof.* Consider the projection of $M$ onto the linear space $T_pM \oplus l_\nu$ and denote it by $M^\nu$. It is a hypersurface in this linear space. The curvature locus of $M^\nu$ coincides with the projection of the curvature locus of $M$ onto $l_\nu$. This implies that the extreme values of the principal curvature of $M^\nu$ at $p$ coincide with the extreme values of the $\nu$-principal curvatures of $M$ at $p$. □

A straightforward application of this lemma implies the following.

**Proposition 4.2.** *The $\nu$-principal curvature $\kappa^\nu$ at the $\nu$-umbilic point $p$ satisfies the following: $|\kappa^\nu(p)| = |\langle \nu(p), H(p)\rangle|$. Moreover, we have that $\kappa^\nu(p) \neq 0$ if and only if $H(p) \notin L(p)$, or equivalently, $\dim L(p) < \dim N_p^1 M$.*

**Proposition 4.3.** *Given $p \in M$ such that $R_D(p) = 0$, the normal directions to the faces of the polyhedron $\mathcal{V}(p)$ are preumbilic directions at $p$ with multiplicity greater than or equal to the number of vertices of the given face. In particular, if $\mathcal{V}(p)$ is a simplex of maximal dimension in $N_pM$, the normals to the faces determine quasiumbilic directions.*

*Proof.* It follows immediately by applying the above lemma to the normal directions orthogonal to each face of the polyhedron given by the curvature locus. □

We now discuss the relations between the vanishing of the normal curvature and the existence of umbilic, quasiumbilic and preumbilic normal directions of different multiplicities at a given point.

Let us begin by considering a 3-manifold in $\mathbb{R}^{3+s}$. A direct application of Proposition 4.3 to the possible degenerations of a triangle in $N_pM \equiv \mathbb{R}^s$, $s \geq 2$ implies:

**Corollary 4.1.** 1) *Given a 3-manifold $M$ in $\mathbb{R}^5$ and $p \in M$, if $R_D(p) = 0$ then, either $p$ is a quasiumbilic point of 3 normal directions which are pairwise linearly independent, or $p$ is an umbilic point of some normal direction.*

2) *Given a 3-manifold $M$ in $\mathbb{R}^{3+s}$, $s > 2$ and $p \in M$, if $R_D(p) = 0$ then either there are $s-2$ linearly independent umbilic directions and 3 quasiumbilic directions which are pairwise linearly independent at $p$, or $s-1$ linearly independent umbilic directions at $p$ (i.e., $p$ is a semiumbilic point, or an umbilic point).*

**Remark 4.2.** The existence of two quasiumbilic linearly independent directions at $p$ is not a sufficient condition for $R_D(0) = 0$, as illustrated by the immersion

$$f : \mathbb{R}^3 \to \mathbb{R}^5, \quad f(x, y, z) = (x, y, z, 2y^2 - z^2, xz).$$

The curvature locus of this immersion at the origin is represented in Figure 4. It has a planar cone shape with two linearly independent quasiumbilic normal directions corresponding to the normal directions of the two segments lying on its boundary. On the other hand, since the curvature locus is not a triangle, the normal curvature does not vanish at the origin.

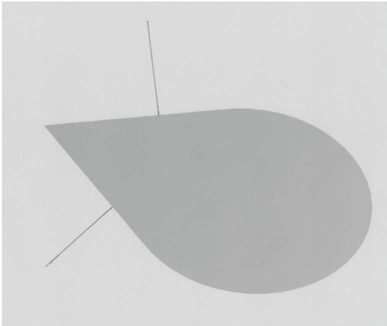

FIGURE 4. Curvature locus with a planar cone shape.

A direct application of Proposition 4.3 to the possible degenerations of an $n$-simplex in $N_pM \equiv \mathbb{R}^s$, $s > 2$, implies the generalization of Corollary 4.1 to higher dimensions.

**Corollary 4.2.** *Let $p$ be a point of an $n$-manifold immersed with codimension $n-1 > 2$ in the Euclidean space. Then, $R_D(p) = 0$ if and only if one of the following conditions hold:*

1) *There are $n$ quasiumbilic directions such that all possible combinations of $n-1$ of them are linearly independent at $p$.*
2) *There exist $r$ umbilic directions for $1 \le r \le n-3$ and the locus is a polyhedron with a number of edges less than or equal to $n$. Thus, if $r = n-2$ the locus is a segment, meanwhile if $r = n-1$ it is a point.*
3) *There are $n-1$ linearly independent umbilic directions at $p$.*

Observe that in the case 2) for $1 \le r \le n-2$, the number of the edges of this polyhedron determine preumbilic directions whose multiplicity increase as this number decrease. We have the following immediate consequence.

**Corollary 4.3.** *Let $p$ be a point of an $n$-manifold immersed with codimension $n-1 > 2$ in the Euclidean space. Any one of the following situations imply that $R_D(p) = 0$:*

1) *There are $n$ quasiumbilic directions such that all possible combinations of $n-1$ of them are linearly independent at $p$.*
2) *There is an umbilic direction and $n-1$ quasiumbilic normal directions such that all possible combinations of $n-2$ of them are linearly independent at $p$.*
3) *There are $n-1$ linearly independent umbilic directions at $p$.*

*Proof.* It is obtained similarly to that of Proposition 3.1 by using Lemma 3.1. □

We can also extend the above results to higher codimension in a straightforward manner.

**Corollary 4.4.** *Let $p$ be a point of an $n$-manifold immersed with codimension $n+s, s \geq 0$ in the Euclidean space. If $R_D(p) = 0$ we have one of the following situations:*

1) *There are $s+1$ umbilic and $n-1$ quasiumbilic directions linearly independent at $p$.*
2) *There are $s+r+1$, $r \leq n-3$ umbilic directions and $n-r-1$ preumbilic directions of multiplicity $n-r$ linearly independent at $p$.*
3) *There are $n+s-1$ linearly independent umbilic directions at $p$ (i.e., $p$ is a semiumbilic or umbilic point).*

**Corollary 4.5.** *Let $p$ be a point of an $n$-manifold immersed with codimension $n+s, s \geq 0$ in the Euclidean space. Any of the following situations imply that $R_D(p) = 0$:*

1) *There are $s+1$ umbilic and $n-1$ quasiumbilic directions linearly independent at $p$.*
2) *There are $n-1$ linearly independent umbilic directions at $p$.*

## 5. Sharing principal curvature directions

We consider in this section the family of principal configurations on an $n$-dimensional manifold $M$ immersed with codimension $k$ in the Euclidean space and study the existence of umbilic directions in terms of the number of principal directions shared by $k$ linearly independent normal fields at a given point $p \in M$. We start with a simple case.

**Theorem 5.1.** *Assume that $k$ linearly independent (i.e., all) unit normal vector fields share all their principal directions at $p$, where $k \geq n$. Then, there exists an umbilic direction at $p$. Moreover, if $H(p) \notin L(p)$ there are $k-n+1$ linearly independent umbilic directions with non-vanishing curvature at $p$.*

The proof of this theorem is obtained by a direct application of Remark 2.1, Corollary 3.1 and Proposition 4.1.

We provide now an upper bound for the dimension of the subspace $L(p)$ in terms of the number of principal directions shared by all the normal directions at the point $p$.

**Proposition 5.1.** *Suppose that there exist $r < n$ common eigenvectors for the shape operators associated to all the normal directions at $p$. Then*

$$\dim L(p) \leq n - 1 + \binom{n-r}{2}.$$

*In particular, if $n-1+\binom{n-r}{2} < k$, then $M$ admits an umbilic normal direction.*

*Proof.* Assume that $X_1, \dots, X_r$ is a frame of common independent eigenvectors for the shape operators associated to all the normal directions. Complete this frame to a basis $\{X_1, \dots, X_r, X_{r+1}, \dots, X_n\}$ of $T_pM$. The first $r$ rows of any shape operator $A_\nu$ in this basis diagonalize in these coordinates. Let $\nu_1, \dots, \nu_k$ be an orthonormal basis of $N_pM$. Since $A_{\nu_l}$, $l = 1, \dots, k$, coincides with the $\nu_l$-second fundamental form at this point, we have that

$$II_{\nu_l}(X) = \sum_{i=1}^{r} \alpha_{ii}^l \, x_i^2 + \sum_{i=r+1}^{n} \alpha_{ij}^l \, x_i \, x_j.$$

This implies that $L(p)$ is generated by

$$\begin{array}{ll} \sum_{l=1}^k (\alpha_{ii}^l - \alpha_{11}^l)\, \nu_l, & i = 2, \dots, n \ \text{ and} \\ \sum_{l=1}^k (2\alpha_{ij}^l - \alpha_{11}^l)\, \nu_l, & i, j = r+1, \dots, n, \ \text{ where } \ i > j. \end{array} \qquad \square$$

By taking the lowest values $r(n,k)$ satisfying $n - 1 + \binom{n-r(n,k)}{2} < k$, we obtain the following table which shows, *for each pair* $(n,k)$*, the minimum number* $r(n,k)$ *of eigenvectors that must be shared by all shape operators at a point* $p$ *of an* $n$*-manifold immersed in* $\mathbb{R}^{n+k}$ *in order to ensure the existence of some umbilic normal direction.*

| $n \backslash k$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | * | 2 | 1 | 1 | 0 | 0 | 0 |
| 4 | * | * | 3 | 2 | 2 | 1 | 1 |
| 5 | * | * | * | 4 | 3 | 3 | 2 |
| 6 | * | * | * | * | 5 | 4 | 4 |
| 7 | * | * | * | * | * | 6 | 5 |
| 8 | * | * | * | * | * | * | 7 |

**Remark 5.1.** This table substitutes a previous one obtained by using alternative arguments in [14], where a mistake in the sign of a term in one the formulae manipulated in the paper lead to wrong entries.

According to this table, for $n = k$ we need to require that $r(n,n) = n-1$. This means that all the normal fields must share all their principal directions which is equivalent to ask that the manifold $M$ have vanishing normal curvature. Observe on the other hand, that the existence of an umbilic field on an $n$-manifold immersed into $\mathbb{R}^{2n}$ does not necessarily imply that the manifold has vanishing curvature as illustrated by the multiple examples of $n$-manifolds immersed with non vanishing normal curvature into a $2n-1$-sphere.

An immediate consequence of the above results is the following.

**Corollary 5.1.** *Suppose that* $R_D(p) = 0$.

a) *If* $\dim L(p) = n - 1$*, there exist* $n$ *normal directions* $\{\nu_i\}$ *such that* $p$ *is a* $\bar{\nu}_i$*-quasiumbilic point.*

b) *If* $\dim L(p) = s < n - 1$*,* $(s > 1)$*, then there exist* $\binom{n}{s}$ *normal directions* $\bar{\nu}_i$ *such that* $p$ *is a* $\nu_i$*-preumbilic with multiplicity* $s$*.*

*Moreover, if $H(p) \notin L(p)$ then, there exist $n-s$ linearly independent umbilic directions with non-vanishing curvature.*

*Proof.* Since $R_D(p)=0$, $\mathcal{V}(p)$ is convex polyhedron of dimension less than or equal to $min(k,n-1)$, with at most $n$ vertices $P_i=\eta(X_i)$, where $X_i$ is a unit vector tangent to a principal direction. In case $a$), $\mathcal{V}(p)$ has $n$ faces of dimension $n-1$. The directions defined by vectors $\nu_i$ normal to these faces are quasiumbilic directions. In case $b$), $\mathcal{V}(p)$ has $s+1<n$ vertices. Therefore, the image by $\eta$ of $n-(s+1)$ principal directions lie inside $\mathcal{V}(p)$. Thus, the faces of dimension $s$ determined by all these points define $\frac{n!}{(n-s)!s!}$ preumbilic normal directions of dimension $s$. □

**Example 5.1.** An example of 3-manifold with everywhere vanishing normal curvature in $\mathbb{R}^6$ is given by the immersion $f(x,y,z)=(x,y,z,x^2+y^2,x^2-y^2,z^2)$. This is the product of a surface contained in a linear 4-space and a curve contained in the complementary plane. It can be seen that the curvature locus at each point of the 3-manifold is a triangle. Therefore, the manifold has an umbilic field and two linearly independent quasiumbilic fields globally defined.

**Lemma 5.1.** ([21]) *Let $(V,\langle,\rangle)$ be a finite dimensional vector space with inner product and $L\colon V\to V$ a self-adjoint operator on $V$. Denote $S=\{v\in V\colon \langle v,v\rangle=1\}$ and consider the function*

$$\begin{array}{rccc} h_L: & S & \longrightarrow & \mathbb{R} \\ & v & \longmapsto & \langle L(v),v\rangle. \end{array}$$

*Then $v_0$ is a critical point of $h_L$ if and only if $v_0$ is an eigenvector of $L$ with eigenvalue $h_L(v_0)$.*

An immediate consequence is the following.

**Proposition 5.2.** *Given a normal field $\nu$ on an $m$-submanifold $M$ of $\mathbb{R}^{m+k}$, $k\geq 1$, the $\nu$-principal directions at a point $p\in M$ are the critical points of the function*

$$\begin{array}{rccc} h_\nu: & S_p^{m-1} & \longrightarrow & \mathbb{R} \\ & X & \longmapsto & \langle \eta(X),\nu\rangle. \end{array}$$

*The corresponding critical values being the principal curvatures.*

This allows us to obtain a sufficient condition, in terms of the curvature locus, for a tangent direction to be a common eigenvector of more than one linearly independent normal fields.

**Proposition 5.3.** *The corank $r$ singularities of the curvature locus map $\eta\colon S_p^{n-1}\to N_pM$ at $p\in M$ are principal directions shared by $r$ normal fields linearly independent at $p$.*

*Proof.* Let $v\in S_p^{n-1}$ be a corank $r$ singularity of $\eta$. The subset $d\eta(p)(T_vS_p^{n-1})$ is a $(k-r)$-dimensional linear subspace of $N_pM$, and we can choose $r$ linearly independent normal directions at $p$, $\nu_1,\dots,\nu_r$, normal to this subspace. This means that the point $v\in S_p^{n-1}$ is a singular point of $h_{\nu_1},\dots,h_{\nu_r}$. Then we get from Proposition 5.2 that $v$ must be a principal direction for the normal fields $\nu_1,\dots,\nu_r$. □

**Remark 5.2.** a) When the curvature locus is a truncated cone, as in Figure 2, we have that the apex of the cone corresponds to a tangent direction which is a principal direction shared by all the normal vector fields at the considered point. On the other hand, the boundary curve of this cone corresponds to a curve of tangent directions, with the property that each one of them is a principal direction of 2 linearly independent normal fields.

b) The vertices of the polyhedron determined by the curvature locus at a point $p \in M$ such that $R_D(p) = 0$ are images of the principal directions shared by all the normal fields at $p$.

c) In the 5-manifold of example 3.1, the 3 vertices of the triangle determined by the curvature locus at $p = g(0)$ (Fig. 3) correspond to the 3 principal curvature directions shared by all the normal fields at $p$. Observe that this manifold has no umbilic directions at this point.

d) Some of the normal fields considered by the above proposition may be umbilic. For instance, we could have that the curvature locus at a point $p$ of a 3-manifold immersed in $\mathbb{R}^7$ is a surface with boundary contained in a normal plane $\Pi \subset N_pM$, which is not a triangle. Then the normal curvature does not vanish at $p$, but there are two linearly independent (umbilic) normal directions at $p$, given by any two linearly independent normal directions to the plane $\Pi$ in $N_pM$, such that all the tangent directions in $T_pM$ can be considered principal directions for this field. Moreover, the curve $\alpha$ determined by the points of $\mathcal{V}(p)$ lying in the convex envelope corresponds to directions of $T_pM$ which are principal curvature directions shared by 3 linearly independent normal fields, two of which are umbilic fields and the third one is the normal direction to the curve $\alpha$ in the plane $\Pi$.

## 6. Strictly locally convex submanifolds

The contact of a submanifold with a hyperplane $\Pi$ of $\mathbb{R}^{n+k}$ at a common point $p$ is determined by the behavior of the height function in the orthogonal direction to the given hyperplane on $M$. That is, if we consider $M$ locally given by an embedding $g\colon \mathbb{R}^n \to \mathbb{R}^{n+k}$ in a neighbourhood of $p$, and $v \in S^{n+k-1}$ is the normal direction to the given hyperplane, the height function $h_v\colon \mathbb{R}^n \to \mathbb{R}$, defined as $h_v(x) = \langle g(x), v\rangle$, provides a *contact map* for $M$ and $\Pi$ at $p$. The singularity type of this map at the origin is independent of the local parameterization $g$ chosen for $M$ (see [12]). Clearly, if $p = g(0)$, we have that 0 is a singular point of $h_v$ if and only if $v \in N_pM$. Then, the singularity type of $h_v$ at 0 will describe the contact class of $M$ with this tangent hyperplane at the point $p$.

**Definition 6.1.** We say that $M$ has a *degenerate contact* with the tangent hyperplane orthogonal to a normal direction $v$ provided the function $h_v$ has a degenerate (non Morse) singularity at 0, that is if the determinant of the Hessian of $h_v$ vanishes at $p$. The normal direction $v$ is called a degenerate direction.

In such case, the Hessian quadratic form has non zero vector in its kernel. These vectors define what we call the *contact directions* of $M$ with the hyperplane at $p$.

Since the Hessian matrix of $h_v$ is equivalent to the matrix of the shape operator $S_v$ at $p$, it follows that $v$ is a degenerate direction if and only if the corresponding shape operator has a principal asymptotic direction at $p$. That is, a principal direction with null eigenvalue.

In the case of surfaces immersed into $\mathbb{R}^4$, it was shown in [9] that there may be either two, one or none degenerate directions (also called binormals) at a point $p \in M$, according it lies outside, on, or inside the curvature ellipse in $N_pM$. Under the first assumption, the two corresponding contact directions happen to be conjugate directions ([8], [4]) and are also known as *asymptotic directions* of $M$, for they correspond to tangent lines with higher order contact with $M$ at $p$ ([10]). For submanifolds immersed in higher codimension, the degenerate directions at a point $p$ form a (possibly degenerate) cone in $N_pM$ ([3], [11]). We shall refer to it as the *cone of degenerate directions* at $p$. We have the following.

**Lemma 6.1.** *The cone of degenerated directions contains all the orthogonal directions to the cone subtended by the curvature locus from the origin $p$ of $N_pM$.*

*Proof.* Given a unit vector $\nu \in N_pM$, let us parameterize $M$ in a neighborhood of $p$ with a Monge coordinate chart. For this, we can consider an orthonormal basis $\{e_1, \dots, e_{n+k}\}$ of $\mathbb{R}^{n+k}$ such that the tangent plane $T_pM$ is the vector subspace of $\mathbb{R}^{n+k}$ generated by $e_1, \dots, e_n$ and the normal vector $\nu$ coincides with $e_{n+k}$. In this coordinate chart the Hessian matrix of the height function $h_\nu$ at $p$ coincides with the matrix of the shape operator $A_\nu$. Then it is easy to see that $A_\nu$ has a zero eigenvalue if the projection segment of the curvature locus onto the line spanned by $\nu$ has one of its end points at the origin of $N_pM$. In other words, the direction $\nu$ is orthogonal to one of the tangent lines of the curvature locus passing through $p$. Then it is a degenerate direction. □

The configuration described in this lemma is illustrated in Figure 5 for the case of a surface in $\mathbb{R}^5$.

**Remark 6.1.** There may be other degenerate directions corresponding to singular points of the curvature locus (see Theorem 2.2 in [5]).

**Definition 6.2.** A tangent hyperplane $\Pi$ is said to be a *locally support hyperplane* for the submanifold $M$ at the point $p$ if $M$ is locally contained at $p$ in one of the two closed half-spaces determined by $\Pi$ in $\mathbb{R}^{n+k}$. We say that $M$ is *locally convex* at $p \in M$ if there is a locally support hyperplane $\Pi$ of $\mathbb{R}^{n+k}$ at $p$. Moreover, $M$ is said to be *strictly locally convex* at $p$, provided there is a locally support hyperplane having non-degenerate contact (i.e., of Morse type) with $M$.

As observed above, the matrix of the shape operator $A_\nu$ and the Hessian matrix of $h_\nu$ at $p$ coincide, therefore we have that the tangent hyperplane orthogonal to $\nu$ at the point $p$ is a support hyperplane if and only if all the $\nu$-principal curvatures are positive (or all of them are negative). Notice that if all of $\nu$-principal curvatures are negative they become positive with respect to $-\nu$. Then we can state the following.

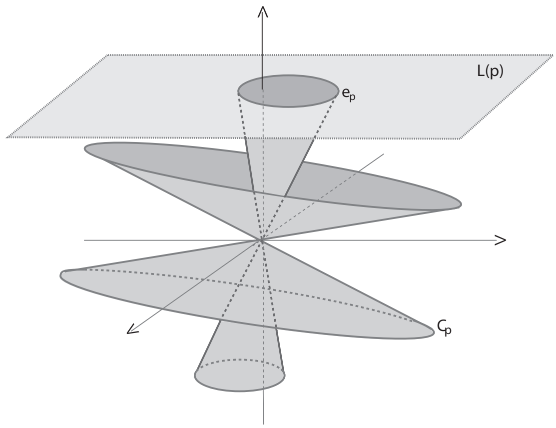


Figure 5. Cone of degenerate directions and curvature locus of a surface in $\mathbb{R}^5$.

**Proposition 6.1.** *A submanifold $M$ is strictly locally convex at $p$ if and only if there exists $\nu \in N_pM$ such that all the $\nu$-principal curvatures are positive.*

Now, as a consequence of Proposition 5.2 we obtain the following geometric characterization of the local convexity in terms of the curvature locus.

**Corollary 6.1.** *Given an $n$-manifold $M$ immersed in $\mathbb{R}^{n+k}$, we have*

a) *$M$ is strictly locally convex at $p$ if and only if the origin of the normal space (identified with $p \in \mathbb{R}^{n+k}$) is not contained in the convex hull of the locus of curvature of $M$ at $p$.*

b) *If $H(p) \notin L(p)$ then $M$ is strictly locally convex at $p$.*

*Proof.* These assertions follow immediately from Propositions 5.2 and 6.1 together with the following observation: if the point $p$ lies in the interior of the convex hull of the curvature locus, any normal direction through the point cuts the curvature locus at points in opposite directions. This implies that the maximal and minimal $\nu$-principal curvatures must have opposite signs for all $\nu$. For a point lying on the boundary of the convex hull the situation is similar but we may also have some normal directions with a vanishing principal curvature (corresponding to a tangency to the boundary of the convex hull) □

**Remark 6.2.** Notice that all the directions contained in the interior of the cone which is orthogonal to the one subtended from the origin by the convex hull of the curvature locus define positive defined height functions that determine locally support hyperplanes at $p$.

**Remark 6.3.** The existence of an umbilic field at a point $p$ of $M$ also implies the existence of a support hyperplane at $p$. Therefore, the conditions stated in Proposition 5.1 guaranteeing the existence of umbilic fields on $M$ guarantee the local (not necessarily strict) convexity.

**Definition 6.3.** A point $p \in M$ is said to be *semiumbilic* if the curvature locus at $p$ is a (non radial) segment. A particular case is given by umbilic points, at which the curvature locus degenerates to a point (which does not coincide with $p$). A submanifold is said to be *totally semiumbilic* if it is exclusively composed of semiumbilic points.

Special examples of semiumbilic submanifolds are provided by codimension 2 submanifolds contained in hyperspheres. On the other hand, not every semiumbilic codimension 2 submanifold is hyperspherical, as illustrated by the Otsuki's spheres [18].

Given a point $p$ of an $n$-manifold $M$ immersed in $\mathbb{R}^{n+2}$, the following relations were shown in [17]:

$$\begin{tikzcd}
 & R_D(p)=0 \arrow[dr, Rightarrow] & \\
p \text{ is semiumbilic} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \exists\, n \text{ degenerate directions at } p. \\
 & M \text{ is strictly locally convex at } p \arrow[ur, Rightarrow] &
\end{tikzcd}$$

In the particular case of surfaces in $\mathbb{R}^4$, we have the following stronger results:

$$\text{semiumbilic} \iff R_D = 0,$$
$$\text{strictly locally convex} \iff \text{there exist degenerate directions.}$$

However, for $n \geq 3$, these equivalences are not true in general. For instance, the 3-manifold $M$ embedded in $\mathbb{R}^5$, given by the parametrization

$$g(x, y, z) = (x, y, z, x^2 - z^2, y^2 - z^2),$$

is not strictly locally convex at $p = 0$, although it has vanishing normal curvature at this point.

We discuss now some extension of these relations to submanifolds of higher codimension. First we observe that from Corollaries 4.4 and 4.5 and Remark 6.3 it follows

$$p \text{ semiumbilic} \implies R_D(p) = 0.$$

On the other hand, Corollary 6.1 together with Lemma 6.1 lead to

$$M \text{ locally convex at } p \implies M \text{ admits degenerate directions at } p.$$

For the case $R_D(p) = 0$, we get from Corollary 3.1 that the curvature locus at $p$ is a convex polyhedron. In such case, we have two possibilities, either $p$ lies outside or inside this polyhedron. In the first case, we get from Corollary 6.1 that $M$ is strictly locally convex at $p$ and from Lemma 6.1 we conclude that $M$ has degenerate directions at $p$. In the case that $p$ lies in the polyhedron, $M$ is not strictly locally convex at $p$, but the vertices and edges of the polyhedron are singular points of the curvature locus and as mentioned in Remark 6.1 they determine degenerate directions at $p$. So we obtain,

$$R_D(p) = 0 \implies \exists \text{ degenerate directions at } p.$$

We finally observe that in high enough codimension (e.g., $k > \frac{1}{2}n(n+1)$) it is possible to show that generically the curvature locus has no singular points and under such assumption we can state: *$M$ is locally convex at $p$ if and only if $M$ admits a cone of degenerate directions at $p$.*

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Received April 14, 2015.

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The work of J. J. Nuño Ballesteros and M. C. Romero Fuster has been partially supported by DGCYT and FEDER grant no. MTM2015-64013-P. The work of F. Sánchez-Bringas has been partially supported by PAPIIT-DGAPA-UNAM grant no. 117714 and PASPA-DGAPA-UNAM.