



Classes of contractions and Harnack domination

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Abstract. Several properties of the Harnack domination of linear operators acting on Hilbert space with norm less or equal than one are studied. Thus, the maximal elements for this relation are identified as precisely the singular unitary operators, while the minimal elements are shown to be the isometries and the adjoints of isometries. We also show how a large range of properties (e.g., convergence of iterates, peripheral spectrum, ergodic properties) are transferred from a contraction to one that Harnack dominates it.

1. Introduction

The classical Harnack inequality for positive harmonic functions in the unit disc was generalized to some operator inequalities for contractions (linear operators of norm no greater than one) on Hilbert space by Ion Suciu in the 1970s. Using this generalized inequality, a preorder relation for Hilbert space contractions, called the Harnack domination, has been introduced in [21] and [22]. Notice also that different operator theoretical generalizations of the Harnack inequality have been proved by Ky Fan (see [7] and the references therein); we will not consider these generalizations here.

The Harnack preorder condition between two contractions can be expressed in several equivalent forms: majorization of the associated operator Poisson kernels, certain positive-definiteness conditions, or majorization of the semi-spectral measures (cf. Theorem 2.1 below). It has both analytic and geometric consequences. The preorder given by Harnack domination induces an equivalence relation, the corresponding equivalence classes being the Harnack parts. The concept of Harnack parts, as well as the hyperbolic metric defined in [25], are the analogues in the noncommutative case of the Gleason parts and metric defined in the context of function algebras. Different aspects of the Harnack domination of contractions have been studied by several authors ([1], [5], [9], [15], [21], [22], [23], [24], [26],

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and [25]). An extension of Harnack domination to the operators of class C_ρ (that is, ρ -contractions) in the sense of [27] appears in [4], while in [19], [20] the Harnack domination in the non-commutative unit ball, or C_ρ -ball of $\mathcal{B}(\mathcal{H})^n$ for $n > 1$ was studied.

The aim of the present paper is to study several properties of Harnack domination of contractions on a Hilbert space. We identify the maximal elements for this relation as precisely the singular unitary operators. We prove that the minimal elements are the isometries and the coisometries (adjoints of isometries). We also show how a large range of properties are transferred from a contraction to one that Harnack dominates it. A useful tool is the asymptotic limit S_T , defined as the strong limit of the sequence $\{T^{*n}T^n\}_{n \in \mathbb{N}}$.

The plan of the paper is the following. Section 2 is devoted to different preliminary definitions and results. Among others, we include a new characterization of Harnack domination of an isometry by a contraction, which is useful in the sequel. This characterization is in terms of the behaviour of the resolvent of one operator applied to the difference of the two operators, and quickly gives the characterization of minimal elements for the Harnack domination. In Section 3 we find the maximal elements, while Section 4 investigates the effect of Harnack domination on certain ergodic properties as well as on the peripheral spectrum of a contraction. In Section 5 we show how different classes of operators are preserved by Harnack domination. The final section contains several examples, one of them showing some spectral and structural properties which are not preserved by Harnack domination.

2. Notations and preliminaries

In the sequel $T, T' \in \mathcal{B}(\mathcal{H})$ will be linear contractions acting on the complex Hilbert space \mathcal{H} ; V acting on \mathcal{K} and V' acting on \mathcal{K}' will denote the minimal isometric dilations of T and T' respectively. $\mathcal{N}(T)$ and $\mathcal{R}(T)$ stand for the kernel and respectively the range of the operator T . We shall denote by I the identity operator on \mathcal{H} and by

$$T_\lambda = (T - \lambda I)(I - \bar{\lambda}T)^{-1}$$

the Möbius transform of T . Here λ is an element of the open unit disk \mathbb{D} . For a contraction T we denote by $D_T = (I - T^*T)^{1/2}$ the *defect operator*, and by $\mathcal{D}_T = \overline{\mathcal{R}(D_T)}$, the *defect space* of T . The *Poisson kernel* of T is

$$K(T, \lambda) = (I - \bar{\lambda}T)^{-1} + (I - \lambda T^*)^{-1} - I.$$

As

$$K(T, \lambda) = (I - \lambda T^*)^{-1}(I - |\lambda|^2 T^*T)(I - \bar{\lambda}T)^{-1}$$

and $\|T\| \leq 1$, the Poisson kernel is a positive operator in the sense that

$$\langle K(T, \lambda)h, h \rangle \geq 0 \quad (h \in \mathcal{H}, \lambda \in \mathbb{D}).$$

We also consider the operators

$$T^{[k]} = \begin{cases} T^k & : k \geq 0, \\ T^{*|k|} & : k < 0. \end{cases}$$

The *asymptotic limit* $S_T \in \mathcal{B}(\mathcal{H})$ of the contraction T (see, for instance, Chapter 3 in [13]) is the strong limit of the sequence $\{T^{*n}T^n\}_{n \in \mathbb{N}}$. It is a positive contraction with $\|S_T\| = 1$ whenever $S_T \neq 0$. Notice that $\mathcal{N}(I - S_T) = \bigcap_{n \geq 1} \mathcal{N}(I - T^{*n}T^n)$ is the maximal invariant subspace (of \mathcal{H}) for T on which T is an isometry, while $\mathcal{N}(I - S_T) \cap \mathcal{N}(I - S_{T^*})$ is the maximal reducing subspace for T on which T is unitary.

We say that T is *strongly (weakly) stable* if the sequence $\{T^n\}_{n \in \mathbb{N}}$ is strongly (weakly) convergent to 0 in $\mathcal{B}(\mathcal{H})$ (see, for instance, [13]). Also, T is of *class* C_0 . (respectively, $C_{.0}$) in the case that T (T^*) is strongly stable, which means $S_T = 0$ ($S_{T^*} = 0$), while T is of *class* C_{00} if it is of class C_0 . and of class $C_{.0}$. We say that T is of *class* C_1 . (respectively, $C_{.1}$) if $T^n h \rightarrow 0$ (respectively $T^{*n} h \rightarrow 0$) for all $0 \neq h \in \mathcal{H}$. Also, T is of *class* C_{11} if both T and T^* are of class C_1 .. For two subsets M and M' of \mathcal{H} we write $M \vee M'$ for the smallest closed subspace of \mathcal{H} containing $M \cup M'$.

A $\mathcal{B}(\mathcal{H})$ -valued *semi-spectral measure* on \mathbb{T} is a map F from the σ -algebra of Borel subsets of \mathbb{T} into $\mathcal{B}(\mathcal{H})$ with the property that for any $h \in \mathcal{H}$ the map $\sigma \mapsto \langle F(\sigma)h, h \rangle$ is a positive measure on \mathbb{T} . For each contraction $T \in \mathcal{B}(\mathcal{H})$ there exists a unique $\mathcal{B}(\mathcal{H})$ -valued *semi-spectral measure* F_T on \mathbb{T} satisfying

$$\langle p(T)h, k \rangle = \int_{\mathbb{T}} p(\lambda) d\langle F_T(\lambda)h, k \rangle$$

for all $h, k \in \mathcal{H}$ and p a trigonometric polynomial. If T is unitary then F_T is precisely its spectral measure, denoted also by E_T , while for $T = 0$ the corresponding F_0 is mI where m is the normalized Lebesgue measure on \mathbb{T} .

According to [21] we say that T is *Harnack dominated* by T' (notation $T \overset{H}{\succ} T'$) if there exists a positive constant $c \geq 1$ such that for any analytic polynomial p verifying $\operatorname{Re} p(z) \geq 0$ for $|z| \leq 1$ we have

$$(2.1) \quad \operatorname{Re} p(T) \leq c \operatorname{Re} p(T').$$

We say that T is *Harnack dominated* by T' *with constant* c whenever we want to emphasize the constant. We say that T and T' are *Harnack equivalent* if $T \overset{H}{\succ} T'$ and $T' \overset{H}{\succ} T$; we also say in this case that T and T' belong to the same *Harnack part*. It was proved in [15] that the Harnack part of T is formed by $\{T\}$ alone if and only if T is an isometry or a coisometry (the adjoint of an isometry).

The contraction T is said to be *maximal* for the Harnack domination if $T \overset{H}{\succ} T'$ implies $T' = T$, and *minimal* if $T' \overset{H}{\succ} T$ implies $T' = T$. Since maximal and minimal elements are Harnack equivalent only with themselves, it follows that they have to be isometries or coisometries.

Several useful equivalent definitions of the Harnack domination are collected in the following known result.

Theorem 2.1 ([21], [22], [23], [25], [1]). *For two contractions $T, T' \in \mathcal{B}(\mathcal{H})$ and $c \geq 1$ and with the previous notation, the following statements are equivalent:*

- (i) T is Harnack dominated by T' with constant c^2 .
- (ii) $K(T, \lambda) \leq c^2 K(T', \lambda)$ for every $\lambda \in \mathbb{D}$.

(iii) For every finite set of vectors $\{h_k\}$ in \mathcal{H} we have

$$(2.2) \quad \sum_{i,j} \langle T^{[i-j]} h_i, h_j \rangle \leq c^2 \sum_{i,j} \langle T'^{[i-j]} h_i, h_j \rangle.$$

(iv) For every finite set of vectors $\{h_k\}$ in \mathcal{H} we have

$$(2.3) \quad \sum_{i,j} \langle V^i h_i, V^j h_j \rangle \leq c^2 \sum_{i,j} \langle V'^i h_i, V'^j h_j \rangle.$$

- (v) There is an operator $A \in \mathcal{B}(\mathcal{K}', \mathcal{K})$ such that $A(\mathcal{H}) \subseteq \mathcal{H}$, $A|_{\mathcal{H}} = I$, $AV' = VA$ and $\|A\| \leq c$.
- (vi) The semi-spectral measures of T, T' satisfy $F_T \leq c^2 F_{T'}$.

The next lemma gives simple properties of Harnack domination that we will use in the sequel.

Lemma 2.2. *Suppose $T \stackrel{H}{\succ} T'$. Then:*

- (i) T^n is Harnack dominated by T'^n , for any integer $n \geq 2$.
- (ii) $T_1 \stackrel{H}{\succ} T'_1$ and $T_2 \stackrel{H}{\succ} T'_2$ if and only if $T_1 \oplus T_2 \stackrel{H}{\succ} T'_1 \oplus T'_2$.
- (iii) If $\mathcal{H}' \subset \mathcal{H}$ is a closed subspace invariant both to T and T' , then $T|_{\mathcal{H}'} \stackrel{H}{\succ} T'|_{\mathcal{H}'}$.
- (iv) The adjoint T^* of T is Harnack dominated by the adjoint of T' .

Proof. The assertions in (i) and (ii) are immediate. As for (iii), note that (2.1) means that for any $h \in \mathcal{H}$ and polynomial p such that $\text{Re } p \geq 0$ on \mathbb{D} we have

$$\text{Re} \langle p(T)h, h \rangle \leq c \text{Re} \langle p(T')h, h \rangle.$$

The left hand side of the inequality depends on $\langle T^n h, h \rangle$ and $\langle T^{*n} h, h \rangle = \langle h, T^n h \rangle$; and similarly for the right hand side. It is then clear that the inequality is satisfied if we take only $h \in \mathcal{H}'$. The condition (iv) follows easily from Theorem 2.1 (ii) (or (iii)). □

Another domination relation, introduced in [5], has been used in [1]. As in the latter, we say that T is Z-dominated by T' , and we write $T \stackrel{Z}{\succ} T'$, if there exists a bounded operator \tilde{A} from $\mathcal{H} \vee V'\mathcal{H}$ to $\mathcal{H} \vee V\mathcal{H}$ such that for any $h_0, h_1 \in \mathcal{H}$,

$$\tilde{A}(h_0 + V'h_1) = h_0 + Vh_1.$$

In this case, the operator \tilde{A} is the unique bounded operator from $\mathcal{H} \vee V'\mathcal{H}$ to $\mathcal{H} \vee V\mathcal{H}$ which intertwines V' and V and whose restriction to \mathcal{H} is the identity operator. We say that T is Z-dominated by T' with constant $c \geq 1$ if $\|\tilde{A}\| \leq c$.

Theorem 2.3 (cf. Lemma 1 in [1]). *With the previous notation, the following statements are equivalent:*

- (i) $T \stackrel{Z}{\succ} T'$ with constant $c \geq 1$;

(ii) there is $c' \geq 1$ such that, for any $h \in \mathcal{H}$,

$$\|D_T h\| \leq c' \|D_{T'} h\| \quad \text{and} \quad \|(T' - T)h\| \leq c' \|D_{T'} h\|.$$

The next corollary follows easily, and shows in particular that isometries are minimal elements for the Z -domination.

Corollary 2.4. (i) If T' is an isometry, then $T \overset{Z}{\succ} T'$ if and only if $T = T'$.

(ii) If T is an isometry, then $T \overset{Z}{\succ} T'$ if and only if $\|(T' - T)h\| \leq c' \|D_{T'} h\|$.

(iii) If T and T' are orthogonal projections, then $T \overset{Z}{\succ} T'$ if and only if $T' \leq T$.

It is clear from the characterization of Harnack domination given by Theorem 2.1 (v) that $T \overset{H}{\succ} T'$ implies $T \overset{Z}{\succ} T'$ (with the same constant). The relation between them is completed by the following result.

Theorem 2.5 (cf. Theorem 3 in [1]). If $T \overset{H}{\succ} T'$ with constant $c \geq 1$, then $T_\lambda \overset{H}{\succ} T'_\lambda$ with constant c , for each $\lambda \in \mathbb{D}$, and so $T_\lambda \overset{Z}{\succ} T'_\lambda$ with constant c , for each $\lambda \in \mathbb{D}$. Conversely, if $T_\lambda \overset{Z}{\succ} T'_\lambda$ with constant $c' \geq 1$, for each $\lambda \in \mathbb{D}$, then $T \overset{H}{\succ} T'$ with constant $c = \sqrt{3}c'$.

In the case of positive contractions, there is a closer relation between our two domination relations. The next result is a consequence of [15] (more precisely, it follows from Corollary 2.13, Lemma 2.17, and Corollary 3.3 therein).

Lemma 2.6. Suppose $A, A' \geq 0$ are contractions. Then:

(i) $A \overset{Z}{\succ} A'$ if and only if $I - A^2 \leq c(I - A'^2)$ for some constant c .

(ii) A and A' are Z -equivalent if and only if they are Harnack equivalent.

We end this section with a result that shows that Harnack domination implies a useful resolvent estimate. In the case of isometries this necessary condition is also sufficient.

Theorem 2.7. Let T and T' be contractions in $\mathcal{B}(\mathcal{H})$. Suppose that T is Harnack dominated by T' . Then there is $c > 0$ such that for each $h \in \mathcal{H}$ we have

$$(2.4) \quad \|(I - \lambda T)^{-1}(T - T')h\|^2 \leq \frac{c}{1 - |\lambda|^2} \|D_{T'} h\|^2 \quad (\lambda \in \mathbb{D}).$$

If T is an isometry, the converse is also true: if (2.4) is satisfied for all $\lambda \in \mathbb{D}$ and $h \in \mathcal{H}$, then $T \overset{H}{\succ} T'$.

Proof. (i) Suppose that T is Harnack dominated by T' with constant c . Thus $T_\lambda \overset{Z}{\succ} T'_\lambda$ with constant c for every $\lambda \in \mathbb{D}$. Then

$$(2.5) \quad \|(T'_\lambda - T_\lambda)x\|^2 \leq c \|D_{T'_\lambda} x\|^2$$

for each $x \in \mathcal{H}$.

Denoting $h = (I - \bar{\lambda}T')^{-1}x$, we obtain

$$T'_\lambda x = (T' - \lambda I)h \quad \text{and} \quad \|D_{T'_\lambda} x\|^2 = (1 - |\lambda|^2)\|D_{T'} h\|^2.$$

Since

$$(2.6) \quad T_\lambda - T'_\lambda = \frac{1 - |\lambda|^2}{\bar{\lambda}} [(I - \bar{\lambda}T)^{-1} - (I - \bar{\lambda}T')^{-1}],$$

we get, from (2.5),

$$\|[(I - \bar{\lambda}T)^{-1} - (I - \bar{\lambda}T')^{-1}] x\|^2 \leq \frac{c|\lambda|^2}{1 - |\lambda|^2} \|D_{T'_\lambda} h\|^2.$$

But

$$[(I - \bar{\lambda}T)^{-1} - (I - \bar{\lambda}T')^{-1}] x = (I - \bar{\lambda}T)^{-1} [\bar{\lambda}(T - T')] h,$$

and therefore (2.4) is true.

Suppose now that T is an isometry and that (2.4) is satisfied for every $\lambda \in \mathbb{D}$. The above proof can be reversed to get

$$\|(T'_\lambda - T_\lambda)x\|^2 \leq c \|D_{T'_\lambda} x\|^2$$

for each $x \in \mathcal{H}$. Since T is an isometry, the same is true for each Möbius transform T_λ . Therefore $T_\lambda \overset{Z}{\succ} T'_\lambda$ uniformly in $\lambda \in \mathbb{D}$, and thus $T \overset{H}{\succ} T'$. □

Remark 2.8. With similar methods it can be proved that the contraction T is Harnack dominated by T' with constant c if and only if, for each $h \in \mathcal{H}$, and each $\lambda \in \mathbb{D}$ one has

$$\begin{aligned} \|(I - \lambda T)^{-1}(T - T')h\|^2 + \frac{1}{1 - |\lambda|^2} \left(1 - \frac{1}{c^2}\right) (\|z\|^2 - \|Tz\|^2) \\ \leq \frac{c^2 - 1}{1 - |\lambda|^2} (\|h\|^2 - \|T'h\|^2), \end{aligned}$$

where $z = z(\lambda, h) = (I - \lambda T)^{-1}(I - \lambda T')h$. We will not use this more general result in the sequel.

Corollary 2.9. *A contraction is a minimal element for Harnack domination if and only if it is an isometry or a coisometry.*

Proof. We have already noticed above that a minimal element has to be an isometry or a coisometry. Suppose then that T' is an isometry and that T is Harnack dominated by T' . Then the inequality (2.4) implies $(I - \lambda T)^{-1}(T - T')h = 0$ for each $h \in \mathcal{H}$, and so $T = T'$. Thus T' is minimal.

Using Lemma 2.2 (iv), we obtain that T'^* is minimal whenever T' is minimal. Thus coisometries are also minimal elements for Harnack domination. □

3. Maximal elements for Harnack domination

In this section we prove that singular unitary operators are precisely the maximal elements with respect to Harnack domination.

Given a finite measure μ on \mathbb{T} we denote by $D_\mu(x)$ its upper density

$$D_\mu(x) = \limsup_{\epsilon \rightarrow 0} \frac{\mu(x - \epsilon, x + \epsilon)}{2\epsilon}$$

at x . It is known that if μ is singular with respect to Lebesgue measure, then $D_\mu(x) = \infty$ μ a.e.

Theorem 3.1. *Let $U \in \mathcal{B}(\mathcal{H})$ be a unitary operator with spectral measure E_U and let $T \in \mathcal{B}(\mathcal{H})$ be a contraction. Let $h \in \mathcal{H}$. Suppose that U is Harnack dominated by T and that $y = (U - T)h \neq 0$. If $\mu_y = \langle E_U y, y \rangle$, then for any $t \in \mathbb{T}$ we have $D_{\mu_y}(t) < +\infty$.*

Proof. If $U \overset{H}{\succ} T$ with constant c , then the resolvent estimate (2.4) is satisfied with the same constant c , and thus

$$\|(I - \lambda U)^{-1}(U - T)h\|^2 \leq \frac{c}{1 - |\lambda|^2} (\|h\|^2 - \|Th\|^2).$$

By the spectral theorem, we have

$$\|(I - \lambda U)^{-1}(U - T)h\|^2 = \int_0^{2\pi} \frac{1}{|1 - \lambda e^{it}|^2} d\mu_y(t)$$

for every $\lambda \in \mathbb{D}$. Let $\epsilon > 0$ and fix $t_0 \in \mathbb{T}$. For $\lambda = (1 - \epsilon)e^{-it_0}$, we obtain

$$\int_0^{2\pi} \frac{1}{|1 - \lambda e^{it}|^2} d\mu_y(t) \geq \int_{t_0 - \epsilon}^{t_0 + \epsilon} \frac{1}{|e^{-it} - (1 - \epsilon)e^{-it_0}|^2} d\mu_y(t) \geq \frac{1}{2\epsilon^2} \mu_y([t_0 - \epsilon, t_0 + \epsilon]).$$

Therefore,

$$\frac{\mu_y([t_0 - \epsilon, t_0 + \epsilon])}{2\epsilon^2} \leq \int_0^{2\pi} \frac{1}{|1 - \lambda e^{it}|^2} d\mu_y(t) \leq \frac{c}{1 - |\lambda|^2} (\|h\|^2 - \|Th\|^2) \leq \frac{c}{\epsilon} \|h\|^2.$$

We obtain

$$\frac{\mu_y([t_0 - \epsilon, t_0 + \epsilon])}{2\epsilon} \leq c \|h\|^2,$$

which proves the theorem. □

Corollary 3.2. *Any singular unitary operator is a maximal element for Harnack domination.*

Note that the particular case of the maximality of a symmetry (a unitary operator T with $T^2 = I$) follows from Corollary 3.3 and Proposition 3.5 in [15].

The next lemma, which we need here as well as in Section 6, is a simple computation. We use the notation $\xi \otimes \eta$ for the rank one operator $x \mapsto \langle x, \eta \rangle \xi$.

Lemma 3.3. *Suppose $U \in \mathcal{B}(\mathcal{H})$ is an isometry, $\xi \in \mathcal{H}$, $\|\xi\| = 1$, and $\alpha \in \mathbb{C}$. If $T = U - (1 - \alpha)U\xi \otimes \xi$, then*

$$(3.1) \quad I - T^*T = (1 - |\alpha|^2)\xi \otimes \xi.$$

Consequently, $\|T\| \leq 1$ if and only if $|\alpha| \leq 1$, in which case D_T^2 is given by (3.1).

Theorem 3.4. (i) *If $U \in \mathcal{B}(\mathcal{H})$ is an absolutely continuous unitary operator, then U is not maximal with respect to Harnack domination.*

(ii) *A unilateral shift of arbitrary multiplicity is not maximal with respect to Harnack domination.*

Proof. (i) Applying the spectral theorem and Lemma 2.2 (ii), we may assume that U is the operator of multiplication by the variable ζ on $L^2(\omega, d\nu)$, where $\omega \subset [0, 2\pi]$ and $d\nu$ is Lebesgue measure normalized so as to have $\nu(\omega) = 1$. If $\xi = 1$ (the constant function), then taking $\alpha = 0$ in Lemma 3.3 it follows that the operator $T = U - U\xi \otimes \xi$ is a contraction, while (3.1) yields

$$(3.2) \quad \|D_T f\|^2 = \langle D_T^2 f, f \rangle = |\langle f, 1 \rangle|^2$$

for all $f \in L^2(\omega, d\nu)$. Also, since $(U - T)f = \langle f, 1 \rangle e^{it}$, we have for such an f and $|\lambda| < 1$,

$$(3.3) \quad \|(I - \lambda U)^{-1}(U - T)f\|^2 = \|(I - \lambda U)^{-1}\langle f, 1 \rangle 1\|^2 = \int_{\omega} \frac{|\langle f, 1 \rangle|^2}{|1 - \lambda e^{it}|^2} d\nu(t).$$

Since

$$\begin{aligned} \int_{\omega} \frac{1}{|1 - \lambda e^{it}|^2} d\nu(t) &= \frac{1}{m(\omega)} \int_{\omega} \frac{1}{|1 - \lambda e^{it}|^2} dm(t) \\ &\leq \frac{1}{m(\omega)} \int_{[0, 2\pi]} \frac{1}{|1 - \lambda e^{it}|^2} dm(t) = \frac{1}{m(\omega)(1 - |\lambda|^2)}, \end{aligned}$$

we obtain, by (3.2) and (3.3),

$$\|(I - \lambda U)^{-1}(U - T)f\|^2 \leq \frac{1}{m(\omega)(1 - |\lambda|^2)} \|D_T f\|^2.$$

By Theorem 2.7, it follows that U is Harnack dominated by T , and is therefore not maximal.

(ii) By Lemma 2.2 (ii), it is enough to show the non-maximality of the unilateral shift of multiplicity one, which is unitarily equivalent to the restriction to H^2 of the unitary operator U defined as multiplication by the variable ζ acting on $L^2([0, 2\pi], dm)$. In the first part of the proof we have shown that U is Harnack dominated by $T = U - U\xi \otimes \xi$, where ξ is the constant function. Since $U\xi \in H^2$, $TH^2 \subset H^2$. Therefore the assertion follows from Lemma 2.2 (iii). \square

We can give now the promised characterization of elements maximal with respect to Harnack domination.

Theorem 3.5. *A contraction $T \in \mathcal{B}(\mathcal{H})$ is a maximal element with respect to Harnack domination if and only if it is a singular unitary operator.*

Proof. Suppose $T \in \mathcal{B}(\mathcal{H})$ is maximal with respect to Harnack domination. In particular, it follows that the Harnack equivalence class containing T is reduced to $\{T\}$, whence it follows by Corollary 3.4 in [15] that T is an isometry or a coisometry. Since T is maximal if and only if T^* is maximal, we may assume that T is an isometry.

By the Wold decomposition, we can write $T = S \oplus U$, where S is a unilateral shift of some multiplicity and U is unitary. By Theorem 3.4 (ii), S cannot appear, and thus T has to be unitary. Then the assertion follows from Corollary 3.2 and Theorem 3.4 (i). □

4. Ergodic properties and spectrum

An interesting feature of Harnack domination of contractions is the way it implies preservation of certain properties. The results of this section show, in particular, that this is true about the peripheral spectrum. Our development will go through establishing some ergodic properties.

The following lemma is proved in [15], Theorem 3.1.

Lemma 4.1. *Let T and T' be contractions on \mathcal{H} such that $T \overset{H}{\succ} T'$. If C denotes the bounded linear operator defined by*

$$(4.1) \quad CD_{T'}h = (T - T')h, \quad h \in \mathcal{H},$$

then the linear operator $X : l^2_{\mathbb{N}}(\mathcal{D}_{T'}) \rightarrow \mathcal{H}$ having the row matrix representation

$$X = [C, TC, T^2C, \dots]$$

is also bounded.

Note that the boundedness of C is given by Theorem 2.3.

Theorem 4.2. *Let T and T' be contractions on \mathcal{H} such that $T \overset{H}{\succ} T'$. Then:*

- (i) $\mathcal{N}(I - T) = \mathcal{N}(I - T')$ and $\overline{\mathcal{R}(T - T')} \subset \overline{\mathcal{R}(I - T)} = \overline{\mathcal{R}(I - T')}$.
- (ii) *With respect to the decomposition $\mathcal{H} = \mathcal{N}(I - T) \oplus \overline{\mathcal{R}(I - T)}$ we have*

$$(4.2) \quad T = I \oplus T_1, \quad T' = I \oplus T'_1,$$

and $T_1 \overset{H}{\succ} T'_1$.

- (iii) *For every sequence $\{\alpha_n\} \subset l^2_{\mathbb{N}}(\mathbb{C})$ the series*

$$\sum_{n=0}^{\infty} \alpha_n T^n (T - T')h$$

converges in norm, for every $h \in \mathcal{H}$.

Proof. (i) It follows immediately from Theorem 2.3 that $T = T' = I$ on $\mathcal{N}(I - T)$, hence $\mathcal{N}(I - T') \subset \mathcal{N}(I - T)$.

For the opposite inclusion, note that, if C is defined by (4.1), then by Lemma 4.1 it follows, in particular, that for $h \in \mathcal{H}$,

$$\|C^*T^{*n}h\| \rightarrow 0, \quad n \rightarrow \infty.$$

This implies that

$$\mathcal{N}(I - T) = \mathcal{N}(I - T^*) \subset \mathcal{N}(C^*) = \mathcal{N}(T^* - T'^*).$$

If $h \in \mathcal{N}(I - T)$, then $C^*h = 0 = (T^* - T'^*)h$, hence $T'^*h = T^*h = h$, that is $h \in \mathcal{N}(I - T'^*) = \mathcal{N}(I - T')$. Therefore $\mathcal{N}(I - T) \subset \mathcal{N}(I - T')$.

Consequently, $\mathcal{N}(I - T) = \mathcal{N}(I - T')$, which also implies that

$$\overline{\mathcal{R}(T - T')} \subset \overline{\mathcal{D}_{T'^*}} \subset \overline{\mathcal{R}(I - T')} = \overline{\mathcal{R}(I - T)}.$$

Here the first inclusion follows by Theorem 2.3 (ii) from the relation $T^* \overset{H}{\succ} T'^*$, while the second inclusion is true because $\overline{\mathcal{R}(I - T')}^\perp = \mathcal{N}(I - T') \subset \mathcal{N}(I - T'T'^*) = \mathcal{N}(\mathcal{D}_{T'^*})$. The last equality is true since $\mathcal{N}(I - T^*) = \mathcal{N}(I - T'^*)$ and the same is true for their orthogonal complements.

(ii) The decompositions in direct sum are an immediate consequence of the contractivity of T and T' . The Harnack domination $T_1 \overset{H}{\succ} T'_1$ follows then from Lemma 2.2 (ii).

(iii) The boundedness of X means, in particular, that for every $d = \{d_n\}_{n \in \mathbb{N}} \in l^2_{\mathbb{N}}(\mathcal{D}_{T'})$ the series $Xd = \sum_{n=0}^\infty T^n C d_n$ converges in the norm of \mathcal{H} . Thus, if $\{\alpha_n\} \subset l^2_{\mathbb{N}}(\mathbb{C})$, then setting $d_n = \alpha_n D_{T'} h$ for $h \in \mathcal{H}$ one obtains that the series

$$\sum_{n=0}^\infty \alpha_n T^n (T - T') h$$

converges in norm, for every $h \in \mathcal{H}$. □

A first application of Theorem 4.2 is related to functional calculus. Lemma 2.2 of [10] states that if $f(z) = \sum_{n=0}^\infty \alpha_n z^n$ is an analytic function on \mathbb{D} which has no zeroes in \mathbb{D} and such that the function $1/f$ has absolutely summable Taylor coefficients, then, whenever T is a contraction on \mathcal{H} and $x \in \mathcal{H}$ is such that $y := \sum_{n=0}^\infty \alpha_n T^n x$ converges weakly, we have $\frac{1}{f}(T)y = x$. If $T \overset{H}{\succ} T'$, then Theorem 4.2 (iii) produces a whole class of such vectors x , namely those in $\mathcal{R}(T - T')$. Therefore $\mathcal{R}(T - T') \subset \mathcal{R}(\frac{1}{f}(T))$.

More interesting applications of Theorem 4.2 are related to the Cesàro means of a contraction T . These are defined by

$$M_n(T) = \frac{1}{n+1} \sum_{j=0}^n T^j.$$

It is known that $\{M_n(T)\}$ uniformly converges in $\mathcal{B}(\mathcal{H})$ if and only if $\mathcal{R}(I - T)$ is closed (see [17]), and such a contraction is called *uniformly Cesàro ergodic*. It is also known (see [14]) that if the Cesàro means $\{M_n(T)\}$ weakly converge in $\mathcal{B}(\mathcal{H})$, then their limit is the ergodic projection P_T , that is the orthogonal projection onto $\mathcal{N}(I - T)$. So, by the decomposition (4.2) we have $M_n(T) - P_T = 0 \oplus M_n(T_1)$

and $T^n - P_T = 0 \oplus T_1^n$ on $\mathcal{H} = \mathcal{N}(I - T) \oplus \overline{\mathcal{R}(I - T)}$. We have thus proved the following lemma.

Lemma 4.3. *If T is a contraction on \mathcal{H} , then any type (weak, strong or uniform) convergence of $\{M_n(T)\}$, respectively of $\{T^n\}$, is equivalent with the corresponding convergence to 0 of $\{M_n(T_1)\}$, or $\{T_1^n\}$ respectively.*

A related notion is the *one-sided ergodic Hilbert transform* of T , which is given by the formula

$$(4.3) \quad H_T x := \sum_{n=1}^{\infty} \frac{T^n}{n} x,$$

having as domain the subspace $\text{Dom } H_T$ of vectors $x \in \mathcal{H}$ for which the series in (4.3) is norm convergent. We refer the reader to [6], where it is also proved that

$$\mathcal{R}(I - T) \subset \text{Dom } H_T \subset \overline{\mathcal{R}(I - T)}.$$

It was shown in Theorem 4.1 of [10] that if $x \in \text{Dom } H_T$, then $(\log n)M_n(T)x \rightarrow 0$ when $n \rightarrow \infty$.

Using Theorem 4.2, we get the following relationship between the ranges of $I - T$ and $I - T'$ when T is Harnack dominated by T' .

Corollary 4.4. *Suppose that T and T' are contractions on \mathcal{H} and $T \stackrel{H}{\succ} T'$. Then*

$$\mathcal{R}(I - T) = \mathcal{R}(T - T') + \mathcal{R}(I - T') \subset \text{Dom } H_T.$$

In particular, if T and T' are Harnack equivalent then $\mathcal{R}(I - T) = \mathcal{R}(I - T')$.

Proof. We can apply the above remark concerning the functional calculus by choosing the function $f(z) = (1 - z)^{-1}$ for $z \in \mathbb{D}$ to conclude that $\mathcal{R}(T - T') \subset \mathcal{R}(I - T)$. This later implies $\mathcal{R}(I - T') \subset \mathcal{R}(I - T)$, and also $\mathcal{R}(T - T') + \mathcal{R}(I - T') \subset \mathcal{R}(I - T)$, while the reverse inclusion is trivial. We obtain the inclusion quoted in the corollary. When T and T' are Harnack equivalent we have by symmetry $\mathcal{R}(I - T) = \mathcal{R}(I - T')$. \square

These ergodic properties may be used to relate Harnack domination to the spectrum of contractions. The following lemma is implicitly proved in [1], Theorem 1. As usually, $\sigma(T)$ denotes the spectrum of T and $\sigma_p(T)$ its point spectrum.

Lemma 4.5. *If $T \stackrel{Z}{\succ} T'$, then $\sigma(T') \cap \mathbb{T} \subset \sigma(T) \cap \mathbb{T}$.*

Theorem 4.6. *Let T and T' be contractions on \mathcal{H} such that $T \stackrel{H}{\succ} T'$. Then $\sigma(T) \cap \mathbb{T} = \sigma(T') \cap \mathbb{T}$ and $\sigma_p(T) \cap \mathbb{T} = \sigma_p(T') \cap \mathbb{T}$. In particular, $\sigma(T) \subset \mathbb{D}$ if and only if $\sigma(T') \subset \mathbb{D}$.*

Proof. (i) Let $\lambda \in \mathbb{T}$ be such that $\lambda \notin \sigma(T')$. Since $T \stackrel{H}{\succ} T'$ we have also $\bar{\lambda}T \stackrel{H}{\succ} \bar{\lambda}T'$ (by Theorem 2.1 (ii), for instance). Thus, by Corollary 4.4 we have

$$\mathcal{H} = \mathcal{R}(I - \bar{\lambda}T') = \text{Dom } H_{\bar{\lambda}T},$$

and so $(\log n)M_n(\bar{\lambda}T)x \rightarrow 0$ for every $x \in \mathcal{H}$. According to the uniform boundedness principle, $(\log n)M_n(\bar{\lambda}T)$ is bounded in norm, and so $\|M_n(\bar{\lambda}T)\|$ tends to 0

when n tends to infinity. But this implies that $I - \overline{\lambda}T$ is invertible, hence $\lambda \notin \sigma(T)$. Thus, $\sigma(T) \cap \mathbb{T} \subset \sigma(T') \cap \mathbb{T}$. The opposite inclusion follows from Lemma 4.5.

By Theorem 4.2 we have $\mathcal{N}(\lambda I - T) = \mathcal{N}(\lambda I - T')$ for each $\lambda \in \mathbb{T}$, which means $\sigma_p(T) \cap \mathbb{T} = \sigma_p(T') \cap \mathbb{T}$. □

Corollary 4.7. *Let T and T' be contractions on \mathcal{H} such that $T \overset{H}{\rightsquigarrow} T'$. Then T is uniformly Cesàro ergodic if and only if T' is uniformly Cesàro ergodic. Also, $\{T^n\}$ uniformly converges if and only if $\{T'^n\}$ uniformly converges.*

Proof. As noted above, T is uniformly ergodic if and only if $\mathcal{R}(I - T)$ is closed. Using the decompositions (4.2) it is easy to see that $\mathcal{R}(I - T) = \mathcal{R}(I - T_1)$, $\mathcal{R}(I - T') = \mathcal{R}(I - T'_1)$, $I - T_1$ and $I - T'_1$ are injective, and $T_1 \overset{H}{\rightsquigarrow} T'_1$. Therefore $\mathcal{R}(I - T)$ is closed if and only if $I - T_1$ is invertible, and similarly for T' . But from Theorem 4.6 it follows that $1 \in \sigma(T_1)$ if and only if $1 \in \sigma(T'_1)$, which proves the statement.

For the second statement, it follows from Lemma 4.3 that $\{T^n\}$ converges in $\mathcal{B}(\mathcal{H})$ if and only if $\|T_1^n\| \rightarrow 0$, and the last assertion is equivalent to $\sigma(T_1) \subset \mathbb{D}$. The same being true about T' , the proof is finished by applying Theorem 4.6. □

Another consequence of Theorem 4.6 is related to the Katznelson–Tzafriri theorem [16] which implies that for a contraction $T \in \mathcal{B}(\mathcal{H})$ we have $\sigma(T) \subset \mathbb{D} \cup \{1\}$ if and only if $\|T^n(T - I)\| \rightarrow 0$ as $n \rightarrow \infty$. So we obtain the following

Corollary 4.8. *Let T and T' be contractions on \mathcal{H} such that $T \overset{H}{\rightsquigarrow} T'$. Then $\|T^n(T - I)\| \rightarrow 0$ if and only if $\|T'^n(T' - I)\| \rightarrow 0$.*

5. Harnack domination and various classes of contractions

In this section we intend to show that certain classes of contractions are preserved by Harnack domination. This will be used, in particular, to give an alternate proof of Corollary 3.2. The main tool used is the asymptotic limit of contractions.

Lemma 5.1. *Let T and T' be two contractions on \mathcal{H} such that T is Harnack dominated by T' . The following statements hold:*

(i) *There exists a constant $c \geq 1$ such that, for all $h \in \mathcal{H}$,*

$$(5.1) \quad \frac{1}{4} | \langle (S_T - S_{T'})h, h \rangle |^2 + \| (I - S_T)^{1/2} h \|^2 \leq c^2 \| (I - S_{T'})^{1/2} h \|^2,$$

(ii) *We have $\mathcal{N}(I - S_{T'}) \subset \mathcal{N}(I - S_T)$ and $T = T'$ on $\mathcal{N}(I - S_{T'})$.*

(iii) *$S_T^{1/2}$ is Z -dominated by $S_{T'}^{1/2}$.*

(iv) *If, moreover, $S_T^{1/2}$ and $S_{T'}^{1/2}$ are Z -equivalent, then they are Harnack equivalent and $\mathcal{N}(I - S_T) = \mathcal{N}(I - S_{T'})$.*

Proof. Suppose that $T \overset{H}{\rightsquigarrow} T'$ with constant $c \geq 1$. Then $T^n \overset{H}{\rightsquigarrow} T'^n$ with constant c and, in particular, $T^n \overset{Z}{\rightsquigarrow} T'^n$ with the same constant c , for every $n \geq 1$. This implies,

by Theorem 2.3, that for $h \in \mathcal{H}$,

$$\|(T^n - T'^n)h\|^2 + \|D_{T^n}h\|^2 \leq c^2 \|D_{T'^n}h\|^2.$$

Therefore,

$$|\|T^n h\| - \|T'^n h\||^2 + \langle (I - T^{*n}T^n)h, h \rangle \leq c^2 \langle (I - T'^{*n}T'^n)h, h \rangle,$$

and letting $n \rightarrow \infty$ we get

$$(5.2) \quad |\|S_T^{1/2}h\| - \|S_{T'}^{1/2}h\||^2 + \|(I - S_T)^{1/2}h\|^2 \leq c^2 \|(I - S_{T'})^{1/2}h\|^2.$$

Now, if $\|h\| = 1$ we have

$$|\langle (S_T - S_{T'})h, h \rangle| = |\|S_T^{1/2}h\|^2 - \|S_{T'}^{1/2}h\|^2| \leq 2 |\|S_T^{1/2}h\| - \|S_{T'}^{1/2}h\||,$$

which, together with (5.2), yields (5.1).

From (5.1) it follows immediately that $\mathcal{N}(I - S_{T'}) \subset \mathcal{N}(I - S_T)$. Since $\mathcal{N}(I - S_{T'}) \subset \mathcal{N}(D_{T'}) \subset \mathcal{N}(T - T')$ (the last inclusion follows from Theorem 2.3), we conclude that $T = T'$ on $\mathcal{N}(I - S_{T'})$.

Inequality (5.1) also implies $\|D_{S_T^{1/2}}h\| \leq c\|D_{S_{T'}^{1/2}}h\|$ for $h \in \mathcal{H}$. Since $S_T, S_{T'}$ are positive contractions, Lemma 2.6 (i) implies that $S_T^{1/2} \succcurlyeq S_{T'}^{1/2}$.

If $S_T^{1/2}$ and $S_{T'}^{1/2}$ are Z-equivalent, then Lemma 2.6 (ii) implies that they are Harnack equivalent. Therefore their squares S_T and $S_{T'}$ are Harnack equivalent, and so $\mathcal{N}(I - S_T) = \mathcal{N}(I - S_{T'})$. \square

Corollary 5.2. *If T and T' are Harnack equivalent contractions on \mathcal{H} then $S_T^{1/2}$ and $S_{T'}^{1/2}$, as well as S_T and $S_{T'}$, are Harnack equivalent. In this case one has $\mathcal{N}(I - S_T) = \mathcal{N}(I - S_{T'})$ and $\mathcal{N}(I - S_{T^*}) = \mathcal{N}(I - S_{T'^*})$.*

Remark 5.3. If $T \stackrel{H}{\succ} T'$ but they are not Harnack equivalent, then $\mathcal{N}(I - S_{T'}) \subsetneq \mathcal{N}(I - S_T)$, in general. An example may be obtained by taking T to be an absolutely continuous unitary and T' a nonisometric contraction that dominates T , as in the proof of Theorem 3.4.

Lemma 5.4. *Let T and T' be two contractions on \mathcal{H} such that T is Harnack dominated by T' . If $\mathcal{H}_u, \mathcal{H}'_u$ are the maximum subspaces of \mathcal{H} which reduce T, T' to unitary operators, respectively, then $\mathcal{H}'_u \subset \mathcal{H}_u$, \mathcal{H}'_u reduces T and $T|_{\mathcal{H}'_u} = T'|_{\mathcal{H}'_u}$, while $T|_{\mathcal{H} \ominus \mathcal{H}'_u}$ is Harnack dominated by $T'|_{\mathcal{H} \ominus \mathcal{H}'_u}$.*

Proof. Since $T \stackrel{H}{\succ} T'$ and (by Lemma 2.2 (iv)) $T^* \stackrel{H}{\succ} T'^*$, we have by Lemma 5.1 $\mathcal{N}(I - S_{T'}) \subset \mathcal{N}(I - S_T)$ and $\mathcal{N}(I - S_{T'^*}) \subset \mathcal{N}(I - S_{T^*})$. Therefore

$$\mathcal{H}'_u = \mathcal{N}(I - S_{T'}) \cap \mathcal{N}(I - S_{T'^*}) \subset \mathcal{N}(I - S_T) \cap \mathcal{N}(I - S_{T^*}) = \mathcal{H}_u.$$

In addition, as $T = T'$ on $\mathcal{N}(I - S_{T'})$ and $T^* = T'^*$ on $\mathcal{N}(I - S_{T'^*})$, it follows that \mathcal{H}'_u reduces T to a unitary operator. Hence $\mathcal{H} \ominus \mathcal{H}'_u$ also reduces T and T' , while by (2.2) we have $T|_{\mathcal{H} \ominus \mathcal{H}'_u} \stackrel{H}{\succ} T'|_{\mathcal{H} \ominus \mathcal{H}'_u}$. \square

The next theorem gathers a series of results that show how different classes of contractions behave with respect to Harnack domination.

Theorem 5.5. *Let T and T' be two contractions on \mathcal{H} such that T is Harnack dominated by T' .*

- (i) *If T is completely nonunitary then T' is also completely nonunitary.*
- (ii) *T is absolutely continuous if and only if T' is absolutely continuous.*
- (iii) *T belongs to the class C_0 , $C_{.0}$, or C_{00} if and only if T' belongs to the same class, respectively.*
- (iv) *T^n is strongly convergent if and only if T'^n is strongly convergent.*
- (v) *T is weakly stable if and only if T' is weakly stable.*
- (vi) *T^n is weakly convergent if and only if T'^n is weakly convergent.*

Proof. By Lemma 5.4, if T is completely nonunitary then T' is completely nonunitary. If T' is not absolutely continuous, it should have a reducing subspace on which it is a singular unitary operator. But, again by Lemma 5.4, T would have the same property (since it coincides with T' on the space on which the latter is unitary).

On the other hand, if T' is absolutely continuous, then the $\mathcal{B}(\mathcal{H})$ -valued semi-spectral measure of T' is absolutely continuous with respect to Lebesgue measure. By Theorem 2.1 the same is true about the $\mathcal{B}(\mathcal{H})$ -valued semi-spectral measure of T , and thus T is absolutely continuous. We have thus proved (i) and (ii).

It is enough to prove (iii) for the case C_0 . (we may consider adjoints in the other cases). Assume first that T is of class C_0 , that is $S_T = 0$. From Lemma 5.1 (iii), it follows that $0 \stackrel{Z}{\prec} S_{T'}^{1/2}$. By Corollary 2 in [1], we have $\|S_{T'}^{1/2}\| < 1$. This forces $S_{T'} = 0$, that is T' is of class C_0 .

Conversely, suppose T' is of class C_0 , that is $T'^{*n} \rightarrow 0$ strongly on \mathcal{H} . This means (see [27]) that if V' on \mathcal{K}' is the minimal isometric dilation of T' then $V'^{*n} \rightarrow 0$ strongly on \mathcal{K}' . If V on \mathcal{K} is the minimal isometric dilation of T , then $T \stackrel{H}{\prec} T'$ implies that there exists $A \in \mathcal{B}(\mathcal{K}', \mathcal{K})$ satisfying $AV' = VA$ such that A is an extension of $I_{\mathcal{H}}$. Then A^* is a lift of $I_{\mathcal{H}}$, that is $P_{\mathcal{H}}A^*k = P_{\mathcal{H}}k$, for each $k \in \mathcal{K}$. Therefore, for any integer $n \geq 1$ and $h \in \mathcal{H}$ we have (V^* being an extension of T^*)

$$T^{*n}h = V^{*n}h = P_{\mathcal{H}}A^*V^{*n}h = P_{\mathcal{H}}V'^{*n}A^*h \rightarrow 0, \quad n \rightarrow \infty.$$

Hence T is of class C_0 . To close this converse part of (iii), let us remark that if T' is of class C_0 . then as $T^* \stackrel{H}{\prec} T'^*$, we can apply the previous argument for T^* and T'^* to conclude that T is also of class C_0 .

Suppose now that T is weakly stable. By the Foguel decomposition of T' (see 7.2 in [13]) we have $\mathcal{H} = \mathcal{H}'_1 \oplus \mathcal{H}'_0$, where \mathcal{H}'_1 reduces T' to a unitary operator, and $T'|_{\mathcal{H}'_0}$ is weakly stable, \mathcal{H}'_0 being the maximum subspace of \mathcal{H} with this property. So $\mathcal{H}'_1 \subset \mathcal{H}'_u$, and by Lemma 5.4 we have $T = T'$ on \mathcal{H}'_1 , hence \mathcal{H}'_1 is invariant for T . Since T is weakly stable on \mathcal{H} , it follows that T' is weakly stable on \mathcal{H}'_1 , therefore $\mathcal{H}'_1 = \{0\}$. We conclude that T' is weakly stable on $\mathcal{H} = \mathcal{H}'_0$.

Conversely, if we suppose that T' is weakly stable, then its unitary part $T'|_{\mathcal{H}'_u} \stackrel{H}{\prec}$ is weakly stable, and $T = T'$ on \mathcal{H}'_u by Lemma 5.4. In addition, $T|_{\mathcal{H} \ominus \mathcal{H}'_u} \stackrel{H}{\prec} T'|_{\mathcal{H} \ominus \mathcal{H}'_u}$ while the contraction in the right side is completely nonunitary. By

the above statement (ii), both these contractions are absolutely continuous, hence $T = T|_{\mathcal{H}'_u} \oplus T|_{\mathcal{H} \ominus \mathcal{H}'_u}$ is weakly stable.

Finally, (iv) follows from (iii), and (vi) from (v), by applying Lemma 4.3. \square

Remark 5.6. The implication in Theorem 5.5(i) cannot be reversed; indeed, we have seen in Theorem 3.4 that a unitary operator can be Harnack dominated by a completely nonunitary contraction.

Remark 5.7. The weak convergence mentioned in Theorem 5.5(vi) is equivalent to the fact that the contraction T has the *Blum–Hanson property* [3], which means that for every subsequence $\{k_n\}$ of positive integers and each $h \in \mathcal{H}$ the sequence $\{\frac{1}{N} \sum_{n=1}^N T^{k_n} h\}$ converges in the norm topology (see for instance [12]). So (vi) above can be reformulated as: T has the Blum–Hanson property if and only if T' has the same property. Note that for isometries induced by measure-preserving transformations, the Blum–Hanson property is equivalent to the strong mixing property of the transformation (see also [8] for other related results).

Remark 5.8. A consequence is the following alternate proof of Corollary 3.2. Suppose U is a singular unitary operator, T is a contraction on \mathcal{H} , and $U \stackrel{H}{\simeq} T$. If we denote by \mathcal{H}_u the maximal space that reduces T to a unitary, then by Lemma 5.4 we have $U_1 := U|_{\mathcal{H} \ominus \mathcal{H}_u} \stackrel{H}{\simeq} T|_{\mathcal{H} \ominus \mathcal{H}_u} =: T_1$. By Theorem 5.5, U_1 is absolutely continuous, which implies $\mathcal{H} \ominus \mathcal{H}_u = \{0\}$. Therefore T is unitary, whence $U = T$.

6. Examples and counterexamples

We give in this section several examples showing the usefulness of the resolvent estimate and the existence of some spectral and structural properties which are not preserved by Harnack domination.

Example 6.1. In this example S denotes the shift operator of multiplicity $\dim \mathcal{E}$: for $x = (x_0, x_1, x_2, \dots) \in \ell^2_{\mathbb{N}}(\mathcal{E})$ we set

$$S(x_0, x_1, x_2, \dots) = (0, x_0, x_1, x_2, \dots).$$

Let $A \in \mathcal{B}(\mathcal{E})$ and consider the operator T' defined on $\ell^2_{\mathbb{N}}(\mathcal{E})$ by

$$T'(x_0, x_1, x_2, \dots) = (0, Ax_0, x_1, x_2, \dots).$$

Then, S is an isometry with resolvent given by

$$(I - \lambda S)^{-1}(x_0, x_1, \dots) = (x_0, x_1 + \lambda x_0, \dots, x_n + \lambda x_{n-1} + \dots + \lambda^{n-1} x_1 + \lambda^n x_0, \dots)$$

for any $x \in \ell^2_{\mathbb{N}}(\mathcal{E})$ and any $\lambda \in \mathbb{D}$. We have

$$(S - T')x = (0, (I - A)x_0, 0, 0, \dots), \quad \|x\|^2 - \|T'x\|^2 = \|x_0\|^2 - \|Ax_0\|^2$$

and

$$(I - \lambda S)^{-1}(S - T')x = (0, (I - A)x_0, \lambda(I - A)x_0, \lambda^2(I - A)x_0, \dots).$$

Therefore the resolvent condition of Theorem 2.7 implies that S is Harnack dominated by T' if and only if A is a *Halperin contraction*, that is, A verifies the following condition:

$$(6.1) \quad \text{there is } K \geq 0 \text{ such that } \|x_0 - Ax_0\|^2 \leq K(\|x_0\|^2 - \|Ax_0\|^2) \quad (x_0 \in \mathcal{E}).$$

This condition was introduced by I. Halperin in [11]; we refer the reader to [2] and the references therein for more information. In particular, a product of orthogonal projections satisfies (6.1).

In our context, one sees that (6.1) is equivalent to $I \stackrel{Z}{\succ} A$. In particular, any strict contraction A satisfies it, and this yields another proof of the fact that a shift operator (of arbitrary multiplicity) is not a maximal element for the Harnack relation.

We remark that any contraction which is Z -equivalent to, or Z -dominates a Halperin contraction also verifies (6.1). On the other hand, it is clear that an operator T with $\|T\| = 1$ and $\sigma(T) \subset \mathbb{D}$ cannot be a Halperin contraction since $I - T$ is invertible while D_T is not. The latter statement follows from $\|T\| = 1$. But $T \stackrel{H}{\succ} 0$ (see [1], [26]), hence a Halperin contraction can Harnack dominate a contraction which does not necessarily satisfy (6.1). However, by Corollary 4.8, a contraction which Harnack dominates a Halperin contraction certainly satisfies the Katznelson–Tzafriri condition $\sigma(T) \subset \mathbb{D} \cup \{1\}$. Indeed, this spectral condition is satisfied by any Halperin contraction as was proved in [2].

Example 6.2. Suppose $\alpha \in \overline{\mathbb{D}}$, and let Z denote multiplication by the variable $\zeta = e^{it}$ on the space $\mathcal{H} = L^2([0, 2\pi], dm)$ (dm being the normalized Lebesgue measure). Define the operators $T(\alpha) = Z - (1 - \alpha)Z1 \otimes 1$; by Lemma 3.3 they are contractions for all $\alpha \in \mathbb{D}$ and unitary for $|\alpha| = 1$. One can see that $Z = T(1)$, while the proof of Theorem 3.4, in the case $\omega = [0, 2\pi]$, shows that $Z \stackrel{H}{\succ} T(0)$. To discuss in more detail the class of all $T(\alpha)$ s, we need the following well-known result concerning perturbations of unitary operators. The proof, which is a computation, can be found, for instance, in [18], Proposition 1.3.

Lemma 6.3. *Suppose U is unitary and $T = U - b \otimes a$ for some vectors a, b . Let $\lambda \in \mathbb{C}$ be such that $I - \lambda U$ is invertible, and denote $a_\lambda = \bar{\lambda}(I - \bar{\lambda}U^*)^{-1}a$. Then $I - \lambda T$ is invertible if and only if $1 + \langle b, a_\lambda \rangle \neq 0$, in which case we have*

$$(6.2) \quad (I - \lambda T)^{-1} = (I - \lambda U)^{-1} \left(I - \frac{1}{1 + \langle b, a_\lambda \rangle} b \otimes a_\lambda \right).$$

We want to apply this result to obtain $(I - \bar{\lambda}T(\alpha))^{-1}$ for $\lambda \in \mathbb{D}$. We take $U = Z$, $a(\zeta) = 1$, $b(\zeta) = (1 - \alpha)Za = (1 - \alpha)\zeta$. In this case $a_\lambda(\zeta) = \lambda(1 - \lambda\bar{\zeta})^{-1} \in \overline{H^2}$, so $\langle b, a_\lambda \rangle = 0$, $I - \bar{\lambda}T(\alpha)$ is invertible and

$$(6.3) \quad ((I - \bar{\lambda}T(\alpha))^{-1}f)(\zeta) = \frac{1}{1 - \bar{\lambda}\zeta} (f(\zeta) - \langle f, \lambda(1 - \lambda\bar{\zeta})^{-1} \rangle (1 - \alpha)\zeta).$$

Proposition 6.4. *All contractions $T(\alpha)$ with $|\alpha| < 1$ are Harnack equivalent, and they all Harnack dominate the unitary operators $T(\alpha')$ with $|\alpha| = 1$ (in particular, they dominate Z).*

Proof. Take $\alpha, \alpha' \in \overline{\mathbb{D}}$, and denote, to simplify notation, $T = T(\alpha), T' = T(\alpha')$. To discuss Harnack domination, we intend to apply Theorem 2.5, so we have to make some computations related to the Möbius transforms of T and T' . First, by Lemma 3.3 we have $D_T = (1 - |\alpha|^2)1 \otimes 1$ and thus

$$(6.4) \quad \|D_T f\|^2 = (1 - |\alpha|^2)|\langle f, 1 \rangle|^2.$$

Since $\|D_{T_\lambda} f\|^2 = (1 - |\lambda|^2)\|D_T(I - \bar{\lambda}T)^{-1}f\|^2$, while, by (6.3),

$$\langle (I - \bar{\lambda}T)^{-1}f, 1 \rangle = \langle (1 - \bar{\lambda}\zeta)^{-1}f, 1 \rangle,$$

we have

$$(6.5) \quad \|D_{T_\lambda} f\|^2 = (1 - |\lambda|^2)(1 - |\alpha|^2)|\langle (1 - \bar{\lambda}\zeta)^{-1}f, 1 \rangle|^2.$$

Similarly,

$$(6.6) \quad \|D_{T'_\lambda} f\|^2 = (1 - |\lambda|^2)(1 - |\alpha'|^2)|\langle (1 - \bar{\lambda}\zeta)^{-1}f, 1 \rangle|^2.$$

From (2.6) and (6.3) it follows that

$$(6.7) \quad \begin{aligned} \|(T_\lambda - T'_\lambda)f\|^2 &= \frac{(1 - |\lambda|^2)^2}{|\lambda|^2} \|(I - \bar{\lambda}T)^{-1}f - (I - \bar{\lambda}T')^{-1}f\|^2 \\ &= \frac{(1 - |\lambda|^2)^2}{|\lambda|^2} \left\| \frac{\zeta}{1 - \bar{\lambda}\zeta} \langle f, \lambda(1 - \lambda\bar{\zeta})^{-1} \rangle (\alpha' - \alpha) \right\|^2 \\ &= (1 - |\lambda|^2)|\alpha' - \alpha|^2 |\langle (1 - \bar{\lambda}\zeta)^{-1}f, 1 \rangle|^2. \end{aligned}$$

It follows now from (6.5), (6.6), and (6.7) that if $|\alpha| < 1$ and $|\alpha'| \leq 1$, then $T'_\lambda \stackrel{Z}{\sim} T_\lambda$ with constants independent of λ . By Theorem 2.5, this proves the proposition. \square

Theorem 5.5 yields several properties of contractions that are preserved by Harnack domination. We will see below some other that are not necessarily preserved.

As seen in Theorem 5.5, strong stability is preserved by Harnack domination in both senses. This property appears in the canonical triangulation of a contraction T : it is known from [27] that T has on $\mathcal{H} = \mathcal{N}(S_T) \oplus \overline{\mathcal{R}(S_T)}$ a triangulation of the form

$$T = \begin{pmatrix} Q & \star \\ 0 & W \end{pmatrix}$$

where Q is of class C_0 on $\mathcal{N}(S_T)$ and W is of class C_1 on $\overline{\mathcal{R}(S_T)}$.

As we will show below, in contrast to C_0 , the class C_1 and the related ones $C_{.1}$ and C_{11} are not in general preserved by Harnack equivalence.

Example 6.5. We will now look at Example 6.2 from a different perspective. By considering the standard isomorphism between $L^2([0, 2\pi], dm)$ and $\ell^2_{\mathbb{Z}}$, one may describe it in terms of weighted bilateral shifts. Moreover, since Harnack domination is preserved by taking direct sums, one can also consider vector valued sequence spaces $\ell^2_{\mathbb{Z}}(\mathcal{E})$. We define then, for $\alpha \in \overline{\mathbb{D}}$, the contractions $\tau(\alpha)$ by

$$\tau(\alpha)(\dots, h_{-1}, \boxed{h_0}, h_1, \dots) = (\dots, h_{-2}, \boxed{h_{-1}}, \alpha h_0, h_1, \dots)$$

for $\{h_n\}_{n \in \mathbb{Z}} \in \ell^2_{\mathbb{Z}}(\mathcal{E})$. Here the components of a vector in $\ell^2_{\mathbb{Z}}(\mathcal{E})$ are arranged in order of increasing subscripts, the central component (i.e., the one with subscript 0) being framed in a box.

Then $\tau(\alpha)$ is unitarily equivalent to $T(\alpha)$. So all $\tau(\alpha)$ s are Harnack equivalent for $|\alpha| < 1$, and they all dominate the unitary operators $\tau(\alpha)$ with $|\alpha| = 1$ (in particular the multivariate bilateral shift, which corresponds to $\alpha = 1$).

This approach allows us to obtain more properties of $\tau(\alpha)$. Thus, for $|\alpha| < 1$, $\tau(\alpha)$ is completely nonunitary, since one sees easily that for a nonzero element $x \in \ell^2_{\mathbb{Z}}(\mathcal{E})$ we cannot have $\|\tau(\alpha)^n x\| = \|\tau(\alpha)^{*n} x\| = \|x\|$ for all $n \in \mathbb{N}$. For $\alpha \neq 0$ $\tau(\alpha)$ is invertible, while $\tau(0)$ is unitarily equivalent to the partial isometry $S \oplus S^*$. In particular, this shows, in contrast to Theorem 4.6, that the whole spectrum is not preserved by Harnack equivalence, since $0 \in \sigma(\tau(0))$, but $0 \notin \sigma(\tau(\alpha))$ for $\alpha \neq 0$.

According to [27], a contraction T is called a *weak contraction* if $\sigma(T)$ does not fill in the closed unit disc $\overline{\mathbb{D}}$ and its defect operator D_T is of finite trace. If $\dim \mathcal{E} < \infty$, then $\tau(\alpha)$ is a weak contraction only for $\alpha \neq 0$, but not for $\alpha = 0$. So weak contractions are not preserved by Harnack equivalence.

We may also compute the asymptotic limit $S_{\tau(\alpha)}$. Indeed, we have

$$\tau(\alpha)^{*n} \tau(\alpha)^n h = (\dots, h_{-n}, |\alpha|^2 h_{-n+1}, \dots, \boxed{|\alpha|^2 h_0}, h_1, h_2, \dots)$$

and consequently

$$S_{\tau(\alpha)} h = (\dots, |\alpha|^2 h_{-n}, \dots, \boxed{|\alpha|^2 h_0}, h_1, h_2, \dots)$$

for $h = \{h_n\} \in \ell^2_{\mathbb{Z}}(\mathcal{E})$, $\alpha \in \mathbb{D}$. The two operators displayed above are diagonal with respect to the standard basis of $\ell^2_{\mathbb{Z}}(\mathcal{E})$. For $\alpha = 0$ the operator $S_{\tau(\alpha)}$ is thus a nontrivial orthogonal projection, while for $\alpha \neq 0$ it is an invertible positive operator. This is equivalent to saying that $\tau(\alpha)$ is of class C_1 . for $\alpha \neq 0$, but not for $\alpha = 0$. Therefore the class C_1 . is not preserved by Harnack equivalence. One can show similarly that $\tau(\alpha)$ is actually in C_{11} , but $\tau(0)$ is neither in C_1 . nor in C_{11} . Also, the class of operators whose asymptotic limit is an orthogonal projection is not preserved by Harnack equivalence.

Denote now $T = \tau(0)$. With respect to the decomposition $\ell^2_{\mathbb{Z}}(\mathcal{E}) = \mathcal{N}(I - S_T) \oplus \mathcal{N}(S_T)$ we may write $T = S \oplus S^*$, where S is the unilateral shift of multiplicity $\dim \mathcal{E}$. We can then obtain some more information on the Harnack class of T .

Proposition 6.6. *Each contraction T' in the Harnack part of T has the form*

$$T' = \begin{pmatrix} S & W \\ 0 & S^* \end{pmatrix}, \quad \text{with } S^*W = S^*W^* = 0.$$

Proof. Let T' be in the Harnack part of T . Then by Lemma 5.1 one has that $\mathcal{N}(I - S_T) = \mathcal{N}(I - S_{T'})$ and $T' = T = S$ on this kernel. Also, by Lemma 2.2 (iii), $T'^*|_{\mathcal{N}(S_T)}$ is Harnack equivalent to S , hence $T'^* = S$ on $\mathcal{N}(S_T)$. We conclude that T' has the desired matrix representation. For T' to be a contraction, one checks easily that we must have $S^*W = S^*W^* = 0$. □

Remark 6.7. Proposition 6.6 gives the matrix structure of contractions in the Harnack part of $T = \tau(0)$. The condition $S^*W = S^*W^* = 0$ means that with respect to the decomposition $\mathcal{H} = \mathcal{N}(S^*) \oplus \mathcal{N}(S^*)^\perp$ we have $W = W_0 \oplus 0$, with W_0 contractive. It is necessary, but in general not sufficient for T' to be Harnack equivalent to T . The case $T' = T(\alpha)$, with $|\alpha| < 1$, corresponds to $W_0 = \alpha I_{\mathcal{N}(S^*)}$. If $\mathcal{E} = \mathbb{C}$ we obtain then that the Harnack part of $T(0)$ is precisely the set of $T(\alpha)$ with $|\alpha| < 1$. It would be interesting to characterize in the general case $\dim \mathcal{E} > 1$ the class of W_0 for which T' is in the Harnack part of T .

References

- [1] ANDO, T., SUCIU, I. AND TIMOTIN, D.: Characterization of some Harnack parts of contractions. *J. Operator Theory* **2** (1979), no. 2, 233–245.
- [2] BADEA, C. AND LYUBICH, YU. I.: Geometric, spectral and asymptotic properties of averaged products of projections in Banach spaces. *Studia Math.* **201** (2010), no. 1, 21–35.
- [3] BLUM, J. R. AND HANSON, D. L.: On the mean ergodic theorem for subsequences. *Bull. Amer. Math. Soc.* **66** (1960), 308–311.
- [4] CASSIER, G. AND SUCIU, N.: Mapping theorems and Harnack ordering for ρ -contractions. *Indiana Univ. Math. J.* **55** (2006), no. 2, 483–523.
- [5] CEAUȘESCU, Z.: On intertwining dilations. *Acta Sci. Math. (Szeged)* **38** (1976), no. 3–4, 281–290.
- [6] COHEN, G., CUNY, C. AND LIN, M.: The one-sided ergodic Hilbert transform in Banach spaces. *Studia Math.* **196** (2010), no. 3, 251–263.
- [7] FAN, K.: Harnack’s inequalities for operators. In *General inequalities, 2 (Proc. Second Internat. Conf., Oberwolfach, 1978)*, 333–339. Birkhäuser, Basel-Boston, Ma., 1980.
- [8] FARKAS, B., EISNER, T., HAASE, M. AND NAGEL, R.: *Operator theoretic aspects of ergodic theory*. Graduate Texts in Mathematics 272, Springer, Cham, 2015.
- [9] FOIAȘ, C.: On Harnack parts of contractions. *Rev. Roumaine Math. Pures Appl.* **19** (1974), 315–318.
- [10] GOMILKO, A., HAASE, M. AND TOMILOV, Y.: On rates in mean ergodic theorems. *Math. Res. Lett.* **18** (2011), no. 2, 201–213.
- [11] HALPERIN, I.: The product of projection operators. *Acta Sci. Math. (Szeged)* **23** (1962), 96–99.
- [12] JONES, L. K. AND KUFTINEC, V.: A note on the Blum–Hanson theorem. *Proc. Amer. Math. Soc.* **30** (1971), 202–203.
- [13] KUBRUSLY, C. S.: *An introduction to models and decompositions in operator theory*. Birkhäuser, Boston, Ma., 1997.
- [14] KRENGEL, U.: *Ergodic theorems*. Walter de Gruyter Studies in Mathematics 6, Walter de Gruyter, Berlin, 1985.
- [15] KHATSKEVICH, V. A., SHMUL’YAN, YU. L. AND SHUL’MAN V. S.: Pre-orders and equivalences in the operator ball. *Sibirsk. Mat. Zh.* **32** (1991), no. 3, 172–183 (Russian); English transl. in *Siberian Math. J.* **32** (1991), no. 3, 496–506.

- [16] KATZNELSON, Y. AND TZAFRIRI, L.: On power-bounded operators. *J. Funct. Anal.* **68** (1986), no. 3, 313–328.
- [17] LIN, M.: On the uniform ergodic theorem. *Proc. Amer. Math. Soc.* **43** (1974), 337–340.
- [18] NIKOLSKI, N. AND TREIL, S.: Linear resolvent growth of rank one perturbation of a unitary operator does not imply its similarity to a normal operator. *J. Anal. Math.* **87** (2002), 415–431.
- [19] POPESCU, G.: Noncommutative hyperbolic geometry on the unit ball of $B(H)^n$. *J. Funct. Anal.* **256** (2009), no. 12, 4030–4070.
- [20] POPESCU, G.: Hyperbolic geometry on noncommutative balls. *Doc. Math.* **14** (2009), 595–651.
- [21] SUCIU, I.: Harnack inequalities for a functional calculus. In *Hilbert space operators and operator algebras (Proc. Intern. Conf., Tihany, 1970)*, 499–511. Colloq. Proc. Math. Soc. János Bolyai 5, North-Holland, Amsterdam, 1972.
- [22] SUCIU, I.: Analytic relations between functional models for contractions. *Acta Sci Math. (Szeged)* **33** (1973), 359–365.
- [23] SUCIU, I.: Analytic formulas for the hyperbolic distance between two contractions. *Ann. Polon. Math.* **66** (1997), 239–252.
- [24] SUCIU, L. AND SUCIU, N.: Harnack domination for contractions and related intertwining operators. *Acta Sci. Math. (Szeged)* **72** (2006), no. 1-2, 319–343.
- [25] SUCIU, I. AND VALUŞESCU, I.: On the hyperbolic metric on Harnack parts. *Studia Math.* **55** (1975/76), no. 1, 97–109.
- [26] SUCIU, N.: On Harnack ordering of contractions. *Bull. Acad. Polon. Sci. Sér. Sci. Math.* **27** (1979), no. 6, 467–471.
- [27] SZ.-NAGY, B., FOIAS, C., BERCOVICI, H. AND KÉRCHY, L.: *Harmonic analysis of operators on Hilbert space*. Second edition. Universitext, Springer, New York, 2010.

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