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On multilinear fractional strong maximal operator associated with rectangles and multiple weights

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Abstract. In this paper, we introduce the multilinear fractional strong maximal operator $\mathcal{M}_{\mathcal{R},\alpha}$ and the corresponding multiple weights $A_{(\vec{p},q),\mathcal{R}}$ associated with rectangles. Under the dyadic reverse doubling condition, a necessary and sufficient condition for two-weight inequalities is given. As consequences, we first obtain a necessary and sufficient condition for one-weight inequalities. Then, we present a new proof for the weighted estimates of multilinear fractional integral operator and fractional maximal operator associated with cubes, which is quite different from and simpler than the proof that has been presented previously.

1. Introduction

Multilinear Calderón–Zygmund theory originated in the works of Coifman and Meyer on Calderón–Zygmund commutators in the 70s [4], [5]. Later on, the theory was systematically studied by Grafakos and Torres in [11], [12]. In recent years, the theory of multilinear Calderón–Zygmund and related operators, including maximal multilinear singular operators, strong maximal operators, fractional maximal operators, fractional integral operators, has attracted considerable attention as a rapidly developing field in Harmonic analysis and many important results have been achieved. Among such achievements are the celebrated works of Grafakos [9], Grafakos, Liu, Pérez and Torres [10], Grafakos and Torres [12], Kenig and Stein [14], Lerner, Ombrosi, Pérez, Torres and Trujillo-González [16].

The following multiple weight classes $A_{(\vec{p},q)}$ were introduced and studied independently by Moen [19] and Chen and Xue [3].

Definition 1.1 ([3] or [19]). Let $1 < p_1, \dots, p_m < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$, and $q > 0$. Suppose that $\vec{\omega} = (\omega_1, \dots, \omega_m)$ and each ω_i $(i = 1, \dots, m)$ is a nonnegative

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function on $\mathbb{R}^n$. We say that $\vec{\omega} \in A_{(\vec{p},q)}$ if it satisfies

$$\sup_Q \Big(\frac{1}{|Q|}\int_Q \nu_{\vec{\omega}}^q\,dx\Big)^{1/q} \prod_{i=1}^m \Big(\frac{1}{|Q|}\int_Q \omega_i^{-p_i'}\,dx\Big)^{1/p_i'} < \infty,$$

where the supremum is taken over all cubes Q with sides parallel to the coordinate axes and $\nu_{\vec{w}} = \prod_{i=1}^m \omega_i$. If $p_i = 1$, $(\frac{1}{Q}\int_Q \omega_i^{-p_i'})^{1/p_i'}$ is understood as $(\inf_Q \omega_i)^{-1}$.

Based on a characterization of multiple $A_{(\vec{p},q)}$ weights, some weighted estimates for the operators $\mathcal{M}_\alpha$ and $\mathcal{I}_\alpha$ were established in [3] and [19]. These operators are defined as follows.

Definition 1.2 ([3] or [19]). Let $\vec{f} = (f_1, \dots, f_m)$ be an m-dimensional vector of locally integrable functions. For any $x \in \mathbb{R}^n$, we define the multilinear fractional type maximal operator $\mathcal{M}_\alpha$ and the multilinear fractional integral operator $\mathcal{I}_\alpha$ by

$$\mathcal{M}_\alpha(\vec{f})(x) = \sup_{Q \ni x} \prod_{i=1}^m \frac{1}{|Q|^{1-\frac{\alpha}{mn}}} \int_Q |f_i(y_i)|\,dy_i, \quad \text{for} \ \ 0 < \alpha < mn, \tag{1.1}$$

and

$$\mathcal{I}_\alpha(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{\prod_{i=1}^m f_i(x-y_i)}{|(y_1,\dots,y_m)|^{mn-\alpha}}\,d\vec{y}, \quad \text{for} \ \ 0 < \alpha < mn, \tag{1.2}$$

respectively, where the supremum in (1.1) is taken over all cubes Q containing x in $\mathbb{R}^n$ with the sides parallel to the axes, $d\vec{y} = dy_1 \cdots dy_m$ and $|(y_1,\dots,y_m)| = |y_1| + \cdots + |y_m|$.

We summarize now some known results of $\mathcal{M}_\alpha$ and $\mathcal{I}_\alpha$.

Theorem A ([19]). *Let $0 < \alpha < mn$, $1 < p_1, \dots, p_m < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$ and $1/q = 1/p - \alpha/n$. Then, $\vec{\omega} \in A_{(\vec{p},q)}$ if and only if either of the following two inequalities hold:*

$$\|\mathcal{M}_\alpha(\vec{f})\|_{L^q(\nu_{\vec{\omega}}{}^q)} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i{}^{p_i})}; \tag{1.3}$$

$$\|\mathcal{I}_\alpha(\vec{f})\|_{L^q(\nu_{\vec{\omega}}{}^q)} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i{}^{p_i})}. \tag{1.4}$$

Theorem B ([3] or [19]). *Let $0 < \alpha < mn$, $1 \le p_1, \dots, p_m < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$ and $1/q = 1/p - \alpha/n$. Then, for $\vec{\omega} \in A_{(\vec{p},q)}$, there is a constant $C > 0$ independent of $\vec{f}$ such that*

$$\|\mathcal{M}_\alpha(\vec{f})\|_{L^{q,\infty}(\nu_{\vec{\omega}}{}^q)} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i{}^{p_i})}; \tag{1.5}$$

$$\|\mathcal{I}_\alpha(\vec{f})\|_{L^{q,\infty}(\nu_{\vec{\omega}}{}^q)} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i{}^{p_i})}. \tag{1.6}$$

It is well known that the geometry of rectangles in $\mathbb{R}^n$ is more intricate than that of cubes, even when both classes of sets are restricted to have sides parallel to the axes. This makes the investigation of the strong maximal function very complex, but also quite interesting. In 1935, a maximal theorem was given by Jessen, Marcinkiewicz and Zygmund in [13]. They pointed out that unlike the classical Hardy–Littlewood maximal operator, the strong maximal function is not of weak type $(1,1)$. Subsequently, an additional proof of the maximal theorem was presented in 1975 by Córdoba and Fefferman, using an alternative geometric method [6]. Some delicate properties of rectangles in $\mathbb{R}^n$ were also quantified in that study. The basis of the work of Córdoba and Fefferman is a selection theorem for families of rectangles in $\mathbb{R}^n$. Their covering lemma is quite useful in the study of the strong maximal function, as demonstrated in [1], [2], [10], [17], and [18].

Recently, Grafakos, Liu, Pérez and Torres [10] introduced the multilinear strong maximal function $\mathcal{M}_{\mathcal{R}}$ by setting

$$\mathcal{M}_{\mathcal{R}}(\vec{f})(x) = \sup_{\substack{R\ni x\\ R\in\mathcal{R}}} \prod_{i=1}^{m} \frac{1}{|R|}\int_R |f_i(y_i)|\,dy_i,$$

where $\vec{f} = (f_1,\dots,f_m)$ is an m-dimensional vector of locally integrable functions and $\mathcal{R}$ denotes the family of all rectangles in $\mathbb{R}^n$ with sides parallel to the axes. In the same paper, they also defined the corresponding multiple weights $A_{\vec{p},\mathcal{R}}$ associated with $\mathcal{R}$. More precisely, they defined $A_{\vec{p},\mathcal{R}}$ to be the collection of weights $\vec{\omega} = (\omega_1,\dots,\omega_m) \in A_{\vec{p},\mathcal{R}}$ such that

$$\sup_{R\in\mathcal{R}} \Big(\frac{1}{|R|}\int_R \nu_{\vec{\omega}}\,dx\Big) \prod_{i=1}^{m} \Big(\frac{1}{|R|}\int_R \omega_i^{1-p_i'}\,dx\Big)^{p/p_i'} < \infty,$$

where $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{p/p_i'}$. In addition, they demonstrated the weak and strong type boundedness of the multilinear strong maximal operators for the one-weight case $\vec{\omega}$.

Theorem C ([10]). *Let $\vec{p} = (p_1,\dots,p_m)$ with $1 < p_1,\dots,p_m < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$ and let $\vec{\omega}$ be an m-tuple of weights. Then $\vec{\omega} \in A_{\vec{p},\mathcal{R}}$ if and only if one of the following two inequalities holds:*

$$\big\|\mathcal{M}_{\mathcal{R}}(\vec{f})\big\|_{L^{p,\infty}(\nu_{\vec{\omega}})} \le C \prod_{i=1}^{m} \big\|f_i\big\|_{L^{p_i}(\omega_i)}; \tag{1.7}$$

$$\big\|\mathcal{M}_{\mathcal{R}}(\vec{f})\big\|_{L^{p}(\nu_{\vec{\omega}})} \le C \prod_{i=1}^{m} \big\|f_i\big\|_{L^{p_i}(\omega_i)}. \tag{1.8}$$

For the two-weight case $(\vec{\omega},\nu)$, weak type boundedness was established whenever $(\vec{\omega},\nu)$ satisfies a certain power bump variant of the multilinear A_p condition. Moreover, a sharp end-point distributional estimate for the multilinear strong maximal operator was also given.

Motivated by the works in [3], [10] and [19], we first define the multilinear fractional strong maximal operator $\mathcal{M}_{\mathcal{R},\alpha}$ and a class of multiple fractional type weights $A_{(\vec{p},q),\mathcal{R}}$ associated with $\mathcal{R}$. Then, we develop some weighted theory for multilinear fractional strong maximal operators. It is worth noting that all arguments for $\mathcal{M}_{\mathcal{R},\alpha}$ are valid for the multilinear fractional maximal operator $\mathcal{M}_\alpha$ as well. Thus, we can employ a different method to show the above strong type inequalities in Theorem A, which is simple than the method used by Moen in [19].

The article is organized as follows. Necessary definitions and our main results are presented in Section 2. The proof of the two-weight inequality is given in Section 3. In Section 4, we first give a characterization of the weight classes $A_{(\vec{p},q),\mathcal{R}}$, and then we investigate the relationship between the weights $A_{p,\mathcal{R}}$, $A^d_{p,\mathcal{R}}$ and the dyadic reverse doubling condition. Finally, in Section 5, an alternative proof of one-weight estimate of multilinear fractional maximal operator is presented.

2. Definitions and main results

First, we introduce some definitions and notations.

Definition 2.1 (Multilinear fractional strong maximal operator). For $0 < \alpha < mn$, and $\vec{f} = (f_1, \dots, f_m) \in L^1_{\rm loc}(\mathbb{R}^n) \times \cdots \times L^1_{\rm loc}(\mathbb{R}^n)$, we define the multilinear fractional strong maximal operator $\mathcal{M}_{\mathcal{R},\alpha}$ by

$$\mathcal{M}_{\mathcal{R},\alpha}(\vec{f})(x) = \sup_{R \ni x} \prod_{i=1}^m \frac{1}{|R|^{1-\frac{\alpha}{mn}}} \int_R |f_i(y_i)|\, dy_i, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all rectangles R containing x with sides parallel to the coordinate axes. Similarly, we can define the dyadic version of multilinear fractional strong maximal operator $\mathcal{M}^d_{\mathcal{R},\alpha}$.

Definition 2.2 (Class of $A_{(\vec{p},q),\mathcal{R}}$). Let $1 < p_1, \dots, p_m < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$, and $q > 0$. Suppose that $\vec{\omega} = (\omega_1, \dots, \omega_m)$ and each ω_i $(i = 1, \dots, m)$ is a nonnegative function on $\mathbb{R}^n$. We say that $\vec{\omega}$ satisfies the $A_{(\vec{p},q),\mathcal{R}}$ condition, written $\vec{\omega} \in A_{(\vec{p},q),\mathcal{R}}$, if it satisfies

$$\sup_R \left(\frac{1}{|R|} \int_R \nu_{\vec{\omega}}^q \, dx\right)^{1/q} \prod_{i=1}^m \left(\frac{1}{|R|} \int_R \omega_i^{-p_i'} \, dx\right)^{1/p_i'} < \infty,$$

where $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i$. If $p_i = 1$, then $(\frac{1}{R} \int_R \omega_i^{-p_i'})^{1/p_i'}$ is understood as $(\inf_R \omega_i)^{-1}$. By $A^d_{(\vec{p},q),\mathcal{R}}$, we denote the dyadic analog.

Throughout this article, the notation $\mathcal{DR}$ will always denote the family of all dyadic rectangles in $\mathbb{R}^n$ with sides parallel to the axes.

Definition 2.3 (Dyadic reverse doubling condition). We say that a nonnegative measurable function ω satisfies the dyadic reverse doubling condition, or $\omega \in RD^{(d)}$, if ω is locally integrable on $\mathbb{R}^n$ and there is a constant $d > 1$ such that

$$d \int_I \omega(x)\, dx \leq \int_J \omega(x)\, dx$$

for any $I, J \in \mathcal{DR}$, where $I \subset J$ and $|I| = \frac{1}{2^n}|J|$.

Remark 2.4. The definition of dyadic reverse doubling condition and reverse doubling condition associated with cubes can be found in [8], [15] and [24]. In [8], this was introduced to study the boundedness of the dyadic fractional maximal function. In Proposition 4.2, we demonstrate that this condition is very weak.

We now formulate the main results of this paper as follows.

Theorem 2.5 (Two-weighted estimates for $\mathcal{M}_{\mathcal{R},\alpha}$). *Let $0 < \alpha < mn$, $1/p = 1/p_1 + \cdots + 1/p_m$ with $1 < p_1, \dots, p_m < \infty$, and $0 < p < q < \infty$. Assume that ν is an arbitrary weight and that each $\omega_i^{1-p_i'}$ $(i = 1, \dots, m)$ satisfies the dyadic reverse doubling condition. Then $(\vec{\omega}, \nu)$ satisfies*

$$\sup_{R\in\mathcal{R}} |R|^{\alpha/n+1/q-1/p}\Big(\frac{1}{|R|}\int_R \nu\,dx\Big)^{1/q}\prod_{i=1}^m\Big(\frac{1}{|R|}\int_R \omega_i^{1-p_i'}\,dx\Big)^{1/p_i'} < \infty, \tag{2.1}$$

if and only if either of the following two inequalities hold:

$$\|\mathcal{M}_{\mathcal{R},\alpha}(\vec{f})\|_{L^{q,\infty}(\nu)} \le C\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i)}; \tag{2.2}$$

$$\|\mathcal{M}_{\mathcal{R},\alpha}(\vec{f})\|_{L^{q}(\nu)} \le C\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i)}. \tag{2.3}$$

Similar results hold for $\mathcal{M}^d_{\mathcal{R},\alpha}$ with the corresponding dyadic version of two-weight condition (2.1).

Theorem 2.6 (One-weighted estimates for $\mathcal{M}_{\mathcal{R},\alpha}$). *Let $0 < \alpha < mn$ and $1/p = 1/p_1 + \cdots + 1/p_m$, with $1 < p_1, \dots, p_m < \infty$, and $p < q < \infty$ satisfying $1/q = 1/p - \alpha/n$. Then $\vec{\omega} \in A_{(\vec{p},q),\mathcal{R}}$ if and only if either of the following two inequalities hold:*

$$\|\mathcal{M}_{\mathcal{R},\alpha}(\vec{f})\|_{L^{q,\infty}(\nu_{\vec{\omega}}{}^q)} \le C\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i{}^{p_i})}; \tag{2.4}$$

$$\|\mathcal{M}_{\mathcal{R},\alpha}(\vec{f})\|_{L^{q}(\nu_{\vec{\omega}}{}^q)} \le C\prod_{i=1}^m \|f_i\|_{L^{p_i}(\omega_i{}^{p_i})}. \tag{2.5}$$

Similar results hold for $\mathcal{M}^d_{\mathcal{R},\alpha}$ with $\vec{\omega} \in A^d_{(\vec{p},q),\mathcal{R}}$.

3. Proof of Theorem 2.5

In order to prove Theorem 2.5, we need the following lemmas.

Lemma 3.1 ([24]). *Let n be a positive integer, $1 < p < q < \infty$, and $0 < b < 2^n$. Let*

$$D = \big\{(F, f, \nu);\ F \ge 0, \nu > 0, 0 \le f \le F^{1/p}\nu^{1/p'}\big\}.$$

Then, there is a positive constant C such that

$$\Big(F-\frac{f^p}{2\nu^{p/p'}}\Big)^{q/p}\geq C\frac{f^q}{\nu^{q/p'}}+\frac{1}{2^{nq/p}}\sum_{i=1}^{2^n}\Big(F_i-\frac{f_i^p}{2\nu_i{}^{p/p'}}\Big)^{q/p} \tag{3.1}$$

for all (F,f,ν), $(F_i,f_i,\nu_i)\in D$, where

$$F=\frac{1}{2^n}(F_1+\cdots+F_{2^n}),\quad f=\frac{1}{2^n}(f_1+\cdots+f_{2^n}),\quad \nu=\frac{1}{2^n}(\nu_1+\cdots+\nu_{2^n})$$

and $\nu_i\leq b\nu\ \ (i=1,\dots,2^n)$.

Remark 3.2. Inequality (3.1) is clearly not satisfied for $p\geq q$ and $1<p,q<\infty$. Indeed, when $n=1$, this can be verified by taking $F_i=f_i=\nu_i=1$ for $i=1,2$.

We will need to apply the following Carleson embedding theorem at certain points in the proof, with respect to two-weighted estimates.

Lemma 3.3 (Carleson embedding theorem). *Let $1<p<q<\infty$, ω be a nonnegative locally integrable function on $\mathbb{R}^n$. Assume that $\omega^{1-p'}$ satisfies the dyadic reverse doubling condition. Then the inequality*

$$\sum_{I\in\mathcal{DR}}\Big(\int_I\omega^{1-p'}\,dx\Big)^{-q/p'}\Big(\int_I f(x)\,dx\Big)^q\leq C\Big(\int_{\mathbb{R}^n}f(x)^p\omega\,dx\Big)^{q/p}$$

holds for all nonnegative functions $f\in L^p(\omega)$,

Proof. The proof of Lemma 3.3 involves a routine application of the method used in Theorem 1.1 of [24]. For the sake of completeness, we give a detailed proof here.

Given $I\in\mathcal{DR}$, it suffices to show that there exists a positive constant C that does not depend on I, such that the following inequality holds for all nonnegative locally integrable functions f:

$$\sum_{\substack{J\subset I\\ J\in\mathcal{DR}}}\Big(\int_J\omega^{1-p'}\,dx\Big)^{-q/p'}\Big(\int_J f(x)\,dx\Big)^q\leq C\Big(\int_I f(x)^p\omega\,dx\Big)^{q/p}. \tag{3.2}$$

To begin the proof, let $(F,f,\nu)\in D$ and let c be the same constant as in Lemma 3.1. Define

$$B(F,f,\nu)=\frac{1}{c}\Big(F-\frac{f^p}{2\nu^{p/p'}}\Big)^{q/p}.$$

Let f be a nonnegative measurable function such that $\int_I f(x)^p\omega\,dx<\infty$. For a measurable set A in I with $|A|\neq 0$, define

$$F_A=\frac{1}{|A|}\int_A f(x)^p\omega\,dx,\quad f_A=\frac{1}{|A|}\int_A f(x)\,dx\ \text{ and }\ \nu_A=\frac{1}{|A|}\int_A\omega(x)^{1-p'}\,dx.$$

Then, by Hölder's inequality, we have

$$\frac{1}{|I|}\int_I f(x)\,dx\leq\Big(\frac{1}{|I|}\int_I f(x)^p\omega\,dx\Big)^{1/p}\Big(\frac{1}{|I|}\int_I\omega^{-p'/p}\,dx\Big)^{1/p'}.$$

Therefore, $0\leq f_I\leq F_I^{1/p}\nu^{1/p'}$ and we obtain that $(F_I,f_I,\nu_I)\in D$.

Let $I_1, \dots, I_{2^n}$ be the dyadic rectangles that are obtained by splitting I into 2^n equal parts. Then, it follows that

$$(F_{I_i}, f_{I_i}, \nu_{I_i}) \in D, \quad i = 1, \dots, 2^n,$$

$$F_I = \frac{1}{2^n}(F_{I_1} + \dots + F_{I_{2^n}}), \quad f_I = \frac{1}{2^n}(f_{I_1} + \dots + f_{I_{2^n}}) \text{ and } \nu_I = \frac{1}{2^n}(\nu_{I_1} + \dots + \nu_{I_{2^n}}).$$

Applying the dyadic reverse doubling condition to $\omega^{1-p'}$ gives us that $\nu_{I_i} \le b\nu_I$ for $1 \le i \le 2^n$ and $b = 2^n/d < 2^n$. Therefore, by Lemma 3.1,

$$B(F_I, f_I, \nu_I) \ge \frac{f_I^q}{\nu_I^{q/p'}} + \frac{1}{2^{nq/p}} \sum_{i=1}^{2^n} B(F_{I_i}, f_{I_i}, \nu_{I_i}).$$

Thus, we obtain the following inequality:

$$|I|^{q/p} B(F_I, f_I, \nu_I) \ge |I|^{q/p}\, \nu_I^{-q/p'} f_I^q + \sum_1^{2^n} |I_i|^{q/p} B(F_{I_i}, f_{I_i}, \nu_{I_i}).$$

Repeated applications of the same technique to each I_i lead to the inequalities

$$|I|^{q/p} B(F_I, f_I, \nu_I) \ge \sum_{\substack{J \subset I, J \in \mathcal{DR} \\ |J| \ge 2^{-nk}|I|}} |J|^{q/p} \nu_J^{-q/p'} f_J^q + \sum_{\substack{J \subset I, J \in \mathcal{DR} \\ |J| = 2^{-n(k+1)}|I|}} |J|^{q/p} B(F_J, f_J, \nu_J)$$

$$\ge \sum_{\substack{J \subset I, J \in \mathcal{DR} \\ |J| \ge 2^{-nk}|I|}} |J|^{q/p} \nu_J^{-q/p'} f_J^q = \sum_{\substack{J \subset I, J \in \mathcal{DR} \\ |J| \ge 2^{-nk}|I|}} \Big(\int_J \omega^{1-p'}\, dx\Big)^{-q/p'} \Big(\int_J f(x)\, dx\Big)^q.$$

Taking the limit as $k \to \infty$, we obtain

$$\sum_{\substack{J \subset I \\ J \in \mathcal{DR}}} \Big(\int_J \omega^{1-p'}\, dx\Big)^{-q/p'} \Big(\int_J f(x)\, dx\Big)^q \le |I|^{q/p} B(F_I, f_I, \nu_I)$$

$$= |I|^{q/p} \frac{1}{c} \Big(F_I - \frac{f_I^p}{2\nu_I^{p/p'}}\Big)^{q/p} \le C' \, |I|^{q/p}\, F_I^{q/p} = C' \Big(\int_I f(x)^p \omega \, dx\Big)^{q/p}.$$

Thus, the proof of Lemma 3.3 is complete. □

The following key lemma provides a foundation for our analysis. It determines the relationship between the maximal operator and the corresponding dyadic version of the maximal operator.

Lemma 3.4. *Let $x, t \in \mathbb{R}^n$, $0 \le \alpha < mn$ and $0 < q < \infty$. For any $\vec{f} \ge 0$, $k \ge 0$, let $\mathcal{M}_{\mathcal{R},\alpha}^{(k)}$ be the following truncated version of strong maximal operator:*

$$\mathcal{M}_{\mathcal{R},\alpha}^{(k)}(\vec{f})(x) := \sup_{\substack{R \ni x, R \in \mathcal{R}, \\ \text{every side length of } R \le 2^k}} \prod_{i=1}^m \frac{1}{|R|^{1-\frac{\alpha}{mn}}} \int_R f_i(y_i)\, dy_i.$$

Then,

$$\mathcal{M}^{(k)}_{\mathcal{R},\alpha}(\vec{f})(x)^q \le \frac{C_{n,\alpha}}{|B_k|}\int_{B_k}\left[\tau_{-t}\circ\mathcal{M}^d_{\mathcal{R},\alpha}\circ\vec{\tau}_t(\vec{f})(x)\right]^q dt, \quad \textit{for any } k\ge 0,$$

where $B_k=[-2^{k+2},2^{k+2}]^n$, $\tau_t g(x)=g(x-t)$ *and* $\vec{\tau}_t\vec{f}=(\tau_t f_1,\dots,\tau_t f_m)$.

The above inequality was established by Fefferman and Stein (see [7], p. 431) for the Hardy–Littlewood maximal operator associated with cubes. For the fractional maximal operator, the result was been given by Sawyer (see [21] and [22]).

Proof. Our method is similar to that employed in [15]. Let j be an integer and I be an interval satisfying $2^{j-1}<|I|\le 2^j$. For $k\in\mathbb{Z}$, $j\le k$, we introduce the notation

$$E:=\{t\in(-2^{k+2},2^{k+2});\,\exists\, J \text{ such that } |J|=2^{j+1}, I\subset J, J+t \text{ dyadic}\}.$$

Then, E enjoys the property $|E|\ge 2^{k+2}$ (see [7], p. 431).

Given $\vec{f}\ge 0$ and $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, we take intervals $I_j\ni x_j$, $|I_j|\le 2^k$, for $1\le j\le n$, such that

$$\mathcal{M}^{(k)}_{\mathcal{R},\alpha}(\vec{f})(x)<2\prod_{i=1}^m\frac{1}{|R|^{1-\frac{\alpha}{mn}}}\int_R f_i(y_i)\,dy_i,$$

where the rectangle $R=I_1\times\cdots\times I_n$.

For each $j=1,\dots,n$, let n_j be the integers satisfying $2^{n_j-1}<|I_j|\le 2^{n_j}$. Then, it is obvious that $n_j\le k$. Now, we introduce one more notation, E_j:

$$E_j:=\{t\in(-2^{k+2},2^{k+2});\,\exists\, I'_j \text{ such that } |I'_j|=2^{j+1}, I_j\subset I'_j, I'_j+t \text{ dyadic}\}.$$

We write $E=E_1\times\cdots\times E_n$. Then, for each $t=(t_1,\dots,t_n)\in E$, there exist intervals $\{I'_j\}_{j=1}^n$ such that $I_j\subset I'_j$ and I'_j+t_j are all dyadic. Let $R''=(I'_1+t_1)\times\cdots\times(I'_n+t_n)$. Then, R'' is a dyadic rectangle and $R\subset R''-t$. It now follows that

$$\begin{aligned}\mathcal{M}^{(k)}_{\mathcal{R},\alpha}(\vec{f})(x)&<2\prod_{i=1}^m\frac{1}{|R|^{1-\frac{\alpha}{mn}}}\int_R f_i(y_i)\,dy_i\\ &\le 2\prod_{i=1}^m\left(\frac{|R''|}{|R|}\right)^{1-\frac{\alpha}{mn}}\frac{1}{|R''|^{1-\frac{\alpha}{mn}}}\int_{R''-t}f_i(y_i)\,dy_i\\ &\le 2\times 4^{mn-\alpha}\sup_{\substack{R-t\ni x\\ R\in\mathcal{DR}}}\prod_{i=1}^m\frac{1}{|R|^{1-\frac{\alpha}{mn}}}\int_{R-t}f_i(y_i)\,dy_i,\end{aligned}\tag{3.3}$$

where $R-t=(I_1-t_1)\times\cdots\times(I_n-t_n)$.

Therefore, if we take a power on both sides of the inequality (3.3) by q and integrate on E with respect to t, it yields that

$$\mathcal{M}^{(k)}_{\mathcal{R},\alpha}(\vec{f})(x)^q \leq \frac{C_{n,\alpha}}{|E|}\int_E \left[\tau_{-t}\circ \mathcal{M}^d_{\mathcal{R},\alpha}\circ \vec{\tau}_t(\vec{f})(x)\right]^q dt.$$

Since $E\subset B_k$ and $|E|\geq \frac{1}{2^n}|B_k|$, we obtain

$$\mathcal{M}^{(k)}_{\mathcal{R},\alpha}(\vec{f})(x)^q \leq 2^n \times \frac{C_{n,\alpha}}{|B_k|}\int_{B_k} \left[\tau_{-t}\circ \mathcal{M}^d_{\mathcal{R},\alpha}\circ \vec{\tau}_t(\vec{f})(x)\right]^q dt. \qquad \square$$

Proof of Theorem 2.5. Our goal here is to prove that (2.3) ⇒ (2.2) ⇒ (2.1) ⇒ (2.3).

It is obvious that (2.3) ⇒ (2.2). Then, it suffices to show that (2.2) ⇒ (2.1) and (2.1) ⇒ (2.3). We will divide the proof in two steps, as follows.

Step 1. (2.2) ⇒ (2.1).

We may assume that $\vec{f}\geq 0$ and that for a fixed rectangle R, $\vec{f}$ satisfies the following property:

$$\prod_{i=1}^m \frac{1}{|R|^{1-\frac{\alpha}{mn}}}\int_R f_i(y_i)\,dy_i > 0.$$

For $x\in R$, we have

$$\prod_{i=1}^m \frac{1}{|R|^{1-\frac{\alpha}{mn}}}\int_R f_i(y_i)\,dy_i \leq \mathcal{M}_{\mathcal{R},\alpha}(f_1\chi_R,\dots,f_m\chi_R)(x).$$

Therefore, for any λ satisfying

$$0<\lambda<\prod_{i=1}^m \frac{1}{|R|^{1-\frac{\alpha}{mn}}}\int_R f_i(y_i)\,dy_i,$$

we can obtain that

$$R\subset \{x\in\mathbb{R}^n; \mathcal{M}_{\mathcal{R},\alpha}(f_1\chi_R,\dots,f_m\chi_R)(x)>\lambda\}.$$

Hence, by (2.2), we get

$$\nu(R)\leq \nu(\{x\in\mathbb{R}^n; \mathcal{M}^d_{\mathcal{R},\alpha}(f_1\chi_R,\dots,f_m\chi_R)(x)>\lambda\}) \leq \Big(\frac{C}{\lambda}\prod_{i=1}^m \|f_i\chi_R\|_{L^{p_i}(\omega_i)}\Big)^q.$$

Letting $\lambda\to\prod_{i=1}^m \frac{1}{|R|^{1-\frac{\alpha}{mn}}}\int_R f_i(y_i)\,dy_i$, we obtain

$$|R|^{\alpha/n-m}\,\nu(R)^{1/q}\prod_{i=1}^m\int_R f_i(y_i)\,dy_i \leq C\prod_{i=1}^m \|f_i\chi_R\|_{L^{p_i}(\omega_i)}.$$

Taking $f_i=\omega_i^{1-p_i'}$, we get

$$|R|^{\alpha/n-m}\,\nu(R)^{1/q}\prod_{i=1}^m\int_R \omega_i^{1-p_i'}\,dx \leq C\prod_{i=1}^m \Big(\int_R \omega_i^{1-p_i'}\,dx\Big)^{1/p_i}.$$

Therefore, the above inequality implies that $(\vec{\omega},\nu)$ satisfies the condition (2.1).

Step 2. (2.1) $\Rightarrow$ (2.3).

We will divide this step into two subcases.

• *Case* 1. *Estimate for* $\mathcal{M}^d_{\mathcal{R},\alpha}$. Consider first the estimates for the dyadic version of the maximal operator,

$$\mathcal{M}^d_{\mathcal{R},\alpha}(\vec{f})(x) = \sup_{\substack{R\ni x\\ R\in\mathcal{DR}}} \prod_{i=1}^m \frac{1}{|R|^{1-\frac{\alpha}{mn}}} \int_R |f_i(y_i)|\,dy_i, \quad x\in\mathbb{R}^n.$$

Without loss of generality, we can assume that $\vec{f}$ is bounded, $\vec{f}\geq 0$ and has a compact support. Therefore, $\mathcal{M}^d_{\mathcal{R},\alpha}(\vec{f})(x)<\infty$ for all $x\in\mathbb{R}^n$.

According to the definition of $\mathcal{M}^d_{\mathcal{R},\alpha}(\vec{f})(x)$, we have that for any $x\in\mathbb{R}^n$, there exists a dyadic rectangle R such that $x\in R$ and

$$\mathcal{M}^d_{\mathcal{R},\alpha}(\vec{f})(x) \leq 2\prod_{i=1}^m \frac{1}{|R|^{1-\frac{\alpha}{mn}}} \int_R f_i(y_i)\,dy_i. \tag{3.4}$$

For any dyadic rectangle R, define

$$E(R) := \{x\in\mathbb{R}^n;\ x\in R \text{ and } R \text{ is minimal for which (3.4) holds}\}.$$

From the definition of maximal operator and the inequality (3.4), it is obvious that

$$\mathbb{R}^n = \bigcup_{R\in\mathcal{DR}} E(R).$$

Since $(\vec{\omega},\nu)$ satisfies the condition (2.1), it follows that

$$\begin{aligned}
\int_{\mathbb{R}^n} \big(\mathcal{M}^d_{\mathcal{R},\alpha}(\vec{f})(x)\big)^q \nu\,dx &\leq \sum_{R\in\mathcal{DR}} \int_{E(R)} \big(\mathcal{M}^d_{\mathcal{R},\alpha}(\vec{f})(x)\big)^q \nu\,dx \\
&\lesssim \sum_{R\in\mathcal{DR}} \int_R \Big(\prod_{i=1}^m \frac{1}{|R|^{1-\frac{\alpha}{mn}}} \int_R f_i(y_i)\,dy_i\Big)^q \nu\,dx \\
&= \sum_{R\in\mathcal{DR}} \Big(\prod_{i=1}^m \Big(\int_R \omega_i^{1-p_i'}\,dx\Big)^{-q/p_i'} \Big(\int_R f_i(y_i)\,dy_i\Big)^q\Big) \\
&\qquad \times \Big(|R|^{\alpha/n+1/q-1/p}\Big(\frac{1}{|R|}\int_R \nu\,dx\Big)^{1/q} \prod_{i=1}^m \Big(\frac{1}{|R|}\int_R \omega_i^{1-p_i'}\,dx\Big)^{1/p_i'}\Big)^q \\
&\lesssim \sum_{R\in\mathcal{DR}} \prod_{i=1}^m \Big(\int_R \omega_i^{1-p_i'}\,dx\Big)^{-q/p_i'} \Big(\int_R f_i(y_i)\,dy_i\Big)^q.
\end{aligned}$$

Therefor, by Hölder's inequality $\sum_{j=1}^\infty \prod_{i=1}^m |a_{ij}| \leq \prod_{i=1}^m (\sum_{j=1}^\infty |a_{ij}|^{p_i/p})^{p/p_i}$ and Lemma 3.3, we further deduce that

$$\begin{aligned}
\int_{\mathbb{R}^n} \big(\mathcal{M}^d_{\mathcal{R},\alpha}(\vec{f})(x)\big)^q \nu\,dx &\leq \prod_{i=1}^m \Big[\sum_{R\in\mathcal{DR}} \Big(\int_R \omega_i^{1-p_i'}\,dx\Big)^{(-qp_i)/(p_i'p)} \\
&\qquad \times \Big(\int_R f_i(y_i)\,dy_i\Big)^{qp_i/p}\Big]^{p/p_i} \lesssim \prod_{i=1}^m \|f_i\|^q_{L^{p_i}(\omega_i)}.
\end{aligned}$$

• *Case* 2. *Estimate for* $\mathcal{M}_{\mathcal{R},\alpha}$. By Lemma 3.4 and the Fubini–Tonelli theorem, it follows that

$$\begin{aligned}\|\mathcal{M}^{(k)}_{\mathcal{R},\alpha}(\vec{f})\|^q_{L^q(\nu)} &\lesssim \frac{1}{|B_k|}\Big\| \int_{B_k} \big[\tau_{-t}\circ \mathcal{M}^d_{\mathcal{R},\alpha}\circ \vec{\tau}_t(\vec{f})\big]^q\, dt\Big\|_{L^1(\nu)} \\ &= \frac{1}{|B_k|}\int_{B_k} \|\tau_{-t}\circ \mathcal{M}^d_{\mathcal{R},\alpha}\circ \vec{\tau}_t(\vec{f})\|^q_{L^q(\nu)}dt = \frac{1}{|B_k|}\int_{B_k} \|\mathcal{M}^d_{\mathcal{R},\alpha}\vec{\tau}_t(\vec{f})\|^q_{L^q(\tau_t\nu)}dt.\end{aligned}$$

Since $(\vec{\omega},\nu)$ satisfies the condition (2.1), we can further verify that $(\vec{\tau}_t\vec{\omega},\tau_t\nu)$ also satisfies the condition (2.1), and is independent of t. Therefore, from the estimate in the previous step for $\mathcal{M}^d_{\mathcal{R},\alpha}$, we obtain

$$\begin{aligned}\|\mathcal{M}^{(k)}_{\mathcal{R},\alpha}(\vec{f})\|^q_{L^q(\nu)} &\lesssim \frac{1}{|B_k|}\int_{B_k}\prod_{i=1}^m \|\tau_t f_i\|^q_{L^{p_i}(\tau_t\omega_i)}dt \\ &= \frac{1}{|B_k|}\int_{B_k}\prod_{i=1}^m \|f_i\|^q_{L^{p_i}(\omega_i)}dt = \prod_{i=1}^m \|f_i\|^q_{L^{p_i}(\omega_i)}.\end{aligned}$$

Finally, we finish the proof by letting k tend to infinity. □

4. Proof of Theorem 2.6

To complete the proof of Theorem 2.6, we need the following characterizations of $A_{(\vec{p},q),\mathcal{R}}$ classes, and the connection between the weights $A^d_{p,\mathcal{R}}$ and the dyadic reverse doubling condition.

Proposition 4.1 (Characterization of $A_{(\vec{p},q),\mathcal{R}}$ class). *Let* $0<\alpha<mn$, $1<p_1,\dots,p_m<\infty$, $1/p = 1/p_1+\cdots+1/p_m$ *and* $1/q = 1/p-\alpha/n$. *Suppose that* $\vec{\omega}\in A_{(\vec{p},q),\mathcal{R}}$. *Then,*

(a) $\nu_{\vec{\omega}}{}^q \in A_{r,\mathcal{R}} \subset A_{mq,\mathcal{R}}$;

(b) $\omega_i^{-p_i'} \in A_{mp_i',\mathcal{R}}$;

(c) $\omega_i^{-p_i'} \in A_{r_i,\mathcal{R}}$, *if* $\alpha/n < (m-2)+1/p_i+1/p_j$, *for any* $1\le i,j\le m$, *where* $r = 1+q(m-1/p)$ *and* $r_i = 1+\frac{p_i'}{q}[1+(m-1)q-q/p+q/p_i]$.

Proof. The argument used in [3], Theorem 2.2, relies only on the use of Hölder's inequality, and does not involve any geometric property of cubes or rectangles. Hence we may also use the method in [3] to complete our proof. Since the main ideas are almost the same, we omit the proof here. □

Proposition 4.2. *Let* $1<p<\infty$, ω *is a nonnegative weight. Then, it holds that*

(i) $A_{\infty,\mathcal{R}}\subset RD^{(d)}$;

(ii) *if* $\omega\in A^d_{p,\mathcal{R}}$, *then* $\omega^{1-p'}\in RD^{(d)}$.

Proof of Theorem 2.6. By Proposition 4.1 (b) and Proposition 4.2 (i), we have that if $\vec{\omega} \in A_{(\vec{p},q),\mathcal{R}}$, then $\omega_i^{-p_i'}$ $(i = 1, \dots, m)$ satisfies the dyadic reverse doubling condition.

Now, by the similar arguments and substituting $\omega_i^{p_i}$ and $\nu_{\vec{\omega}}^q$ with ω_i and ν, respectively in Theorem 2.5, we can complete the proof of Theorem 2.6. □

Proof of Proposition 4.2. First, let us prove (ii). Let I be any dyadic rectangle. Dividing I into 2^n equal parts, we obtain dyadic sub-rectangles of I, $I_1, \dots, I_{2^n}$. For a measurable set $A \subset I$ and $|A| \neq 0$, denote

$$u_A = \frac{1}{|A|}\int_A \omega(x)\,dx, \quad \nu_A = \frac{1}{|A|}\int_A \omega(x)^{1-p'}\,dx.$$

Then, we have

$$u_I = \frac{1}{2^n}(u_{I_1} + \cdots + u_{I_{2^n}}), \quad \nu_I = \frac{1}{2^n}(\nu_{I_1} + \cdots + \nu_{I_{2^n}}).$$

Notice that $u_{I_i} \leq 2^n u_I$, for each $i = 1, \dots, 2^n$. Since $\omega \in A^d_{p,\mathcal{R}}$, we have

$$1 \leq u_I^{1/p}\,\nu_I^{1/p'} \leq K, \quad 1 \leq u_{I_i}^{1/p}\,\nu_{I_i}^{1/p'} \leq K, \quad \text{for each } i = 1, \dots, 2^n,$$

where K is a positive constant that does not depend on I.

Hence, for all $i = 1, \dots, 2^n$, we obtain the following inequalities:

$$\nu_{I_i} \geq \frac{1}{u_{I_i}^{p'/p}} \geq \frac{1}{(2^n u_I)^{p'/p}} \geq \frac{\nu_I}{2^{np'/p}\cdot K^{p'}},$$

$$\nu_{I_i} = 2^n \nu_I - \sum_{j\neq i} \nu_{I_j} \leq \Big(2^n - \frac{2^n - 1}{2^{np'/p}\cdot K^{p'}}\Big)\nu_I = 2^n\Big(1 - \frac{1 - 2^{-n}}{2^{np'/p}\cdot K^{p'}}\Big)\nu_I.$$

Because $K \geq 1$, $p > 1$, if we take $1/d = 1 - \frac{1-2^{-n}}{2^{np'/p}\cdot K^{p'}}$, then $d > 1$. Hence,

$$d\int_{I_i} \omega(x)^{1-p'}\,dx = d\,|I_i|\,\nu_{I_i} \leq 2^n\,|I_i|\,\nu_I = |I|\,\nu_I = \int_I \omega(x)^{1-p'}\,dx.$$

Therefore, $\omega^{1-p'} \in RD^{(d)}$.

We are now in a position to demonstrate (i). Let $\omega \in A_{\infty,\mathcal{R}}$. Then, there exists some $1 < p < \infty$ such that $\omega \in A_{p,\mathcal{R}}$, where we refer the reader to [7], p. 458, for further details. Therefore, the fact that $A_{p,\mathcal{R}} \subset A^d_{p,\mathcal{R}}$ gives that $\omega \in A^d_{p,\mathcal{R}}$. By the definition of $A^d_{p,\mathcal{R}}$, it can be easily seen that $\omega^{-1/(p-1)} \in A^d_{p',\mathcal{R}}$. Hence, by (ii), we have

$$\omega = \big(\omega^{-1/(p-1)}\big)^{-1/(p'-1)} \in RD^{(d)}.$$

This completes the proof of Proposition 4.2. □

Corollary 4.3. *Let $1 < p < q < \infty$ and $\omega \in A^d_{p,\mathcal{R}}$. Let $\{\mu_I\}_{I\in\mathcal{DR}}$ be a collection of nonnegative numbers indexed by $I \in \mathcal{DR}$. Then, the following two statements are equivalent:*

(i) *There is a positive constant C_1 such that*

$$\sum_{I\in\mathcal{DR}} \mu_I \Big(\frac{1}{|I|}\int_I f(x)\,dx\Big)^q \le C_1 \Big(\int_{\mathbb{R}^n} f(x)^p \omega(x)\,dx\Big)^{q/p}$$

for all nonnegative locally integrable function f.

(ii) *There is a positive constant C_2 such that*

$$\mu_I \le C_2 \Big(\int_I \omega(x)\,dx\Big)^{q/p}$$

for all $I\in\mathcal{DR}$.

Proof. Taking $f=\chi_I$, one can check easily that (i) implies (ii). Conversely, if $\omega\in A^d_{p,\mathcal{R}}$, by Hölder's inequality, we get

$$1=\frac{1}{|I|}\int_I \omega(x)^{1/p}\,\omega(x)^{-1/p}\,dx \le \Big(\frac{1}{|I|}\int_I \omega(x)\,dx\Big)^{1/p}\Big(\frac{1}{|I|}\int_I \omega(x)^{-\frac{1}{p-1}}\,dx\Big)^{1/p'} \le C$$

for all $I\in\mathcal{DR}$, where the constant C is independent of I. By combining Lemma 3.3 and Proposition 4.2 with the above inequality, it is easy to see that (ii) implies (i). □

5. A new proof of multilinear fractional integral operators and maximal operators

In this section, we aim to present a new proof for Theorem A. The method that we used in the proof of Theorem 2.6 can also be applied to the multilinear fractional maximal operator $\mathcal{M}_\alpha$ in Definition 1.2. This means that the conclusions of Theorem 2.6 are also true for $\mathcal{M}_\alpha$, as well as $\mathcal{M}^d_\alpha$. To obtain the weighted estimate for $\mathcal{I}_\alpha$, it is sufficient to prove the following proposition.

Proposition 5.1. *Let $0<q<\infty$, and $0<\alpha<mn$. If $\omega\in A_\infty$, then there exists a positive constant C independent of f, such that*

$$\int_{\mathbb{R}^n} |\mathcal{I}_\alpha(\vec{f})(x)|^q\,\omega(x)\,dx \le C\int_{\mathbb{R}^n} [\mathcal{M}_\alpha(\vec{f})(x)]^q\,\omega(x)\,dx \tag{5.1}$$

and

$$\sup_{\lambda>0}\lambda^q\,\omega(x\in\mathbb{R}^n;\,|\mathcal{I}_\alpha(\vec{f})(x)|>\lambda) \le C\sup_{\lambda>0}\lambda^q\,\omega(\{x\in\mathbb{R}^n;\,\mathcal{M}_\alpha(\vec{f})(x)>\lambda\}). \tag{5.2}$$

Remark 5.2. When $m=1$, Theorem 1 of [20] is the linear result of Proposition 5.1. Furthermore, the inequality (5.1) has been proved by Moen in Theorem 3.1 of [19], where an extrapolation theorem was used. The method we present here is entirely different from that one.

In order to prove Proposition 5.1, we need the following lemma.

Lemma 5.3. *Let $0 < \alpha < mn$, $\lambda > 0$ and $d > 0$. For any m-vector of non-negative locally integrable functions $\vec{f}$ and cube Q in $\mathbb{R}^n$. we assume that $\mathcal{I}_\alpha(\vec{f}) \le \lambda$ at some point of Q. Define $E = \{x \in Q; \mathcal{I}_\alpha(\vec{f})(x) \ge \lambda b, \mathcal{M}_\alpha(\vec{f})(x) \le \lambda d\}$. Then, there exist constants B and K, depending only on α, m and n, such that for $b \ge B$, it holds that $|E| \le K|Q|(d/b)^{n/(mn-\alpha)}$.*

Proof. Here we need to use the end-point unweighted estimate for $\mathcal{I}_\alpha$ in Lemma 7 of [14]. Lemma 5.3 is the multilinear version of Lemma 1 in [20]. Because the main ideas are almost the same as those presented in [20], we omit the proof here. □

Remark 5.4. Lemma 5.3 does not hold for the dyadic maximal operator $\mathcal{M}^d_\alpha$.

We give an example in the case $m = 1$. Let $0 < \alpha < n$, $B > 0$, $Q_1 = [0, 1]^n$ and $Q_{-1} = [-1, 0]^n$. Set

$$f(x) = \frac{\chi_{Q_{-1}}(x)}{|x|^\alpha} \quad \text{and} \quad \lambda = I_\alpha f((1, 1, \dots, 1)).$$

Then, $I_\alpha f(0) = \int_{Q_{-1}} dy/|y|^n = +\infty$, and

$$|\{x \in Q_1; I_\alpha f(x) > b\lambda\}| > 0 \quad \text{for any } b > 0. \tag{5.3}$$

If a dyadic cube Q contains $x \in Q_1$, then it must be contained in $[0, \infty)^n$. Since $\operatorname{supp} f \subset (-\infty, 0]^n$, we get

$$\mathcal{M}^d_\alpha(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)|\, dy = 0 \quad \text{for } x \in Q_1.$$

Therefore, for each $d > 0$, $b \ge B$, by (5.3), we have

$$|\{x \in Q_1; I_\alpha f(x) > b\lambda,\, \mathcal{M}^d_\alpha(f)(x) \le d\lambda\}| = |\{x \in Q_1; I_\alpha f(x) > b\lambda\}| > 0. \tag{5.4}$$

Thus, there exists no $K > 0$ such that

$$|\{x \in Q_1; I_\alpha f(x) > b\lambda,\, \mathcal{M}^d_\alpha(f)(x) \le d\lambda\}| \le K\, |Q_1|\, (d/b)^{n/(n-\alpha)},$$

even though $\lambda = I_\alpha f((1, 1, \dots, 1))$.

Proof of Proposition 5.1. The idea of the following arguments is essentially taken from [20].

Without loss of generality, we can assume that $\vec{f}$ is nonnegative and has compact support. For a given $\lambda > 0$, applying a Whitney decomposition (Theorem 1, p. 167, in [23]), there are cubes $\{Q_j\}$ with disjoint interiors such that

$$\{x \in \mathbb{R}^n; \mathcal{I}_\alpha(\vec{f}) > \lambda\} = \bigcup_{j=1}^\infty Q_j,$$

and for each j, $\mathcal{I}_\alpha(\vec{f}\,) \le \lambda$ at some point of $4Q_j$. Let B and K be the same as in Lemma 5.3 and $b = \max(1, B)$. As a property of A_∞ weight class, it is well known that for any $0 < \varepsilon < 1$ there exists $\delta > 0$, such that $|S| < \delta|Q|$ implies $\omega(S) < \varepsilon\omega(Q)$, for any cube Q and its measurable subset S. Let δ be chosen in this way corresponding to $\varepsilon = \frac{1}{2}b^{-q}$ for $\omega(x)$. Choose D so that $\delta = K4^n(D/b)^{n/(mn-\alpha)}$. Let $0 < d \le D$, and

$$E_j = \{x \in Q_j; \mathcal{I}_\alpha(\vec{f}\,) > \lambda b, \mathcal{M}_\alpha(\vec{f}\,) \le \lambda d\}.$$

According to Lemma 5.3, we have that $|E_j| \le K|4Q_j|(d/b)^{n/(mn-\alpha)} < \delta|Q_j|$. Hence, $\omega(E_j) \le \frac{1}{2}b^{-q}\omega(Q_j)$. Thus, it holds that

$$\begin{aligned}
&\omega(\{x \in \mathbb{R}^n; \mathcal{I}_\alpha(\vec{f}\,)(x) > \lambda b, \mathcal{M}_\alpha(\vec{f}\,)(x) \le \lambda d\}) \\
&\quad= \sum_{j=1}^{\infty} \omega(\{x \in \mathbb{R}^n; \mathcal{I}_\alpha(\vec{f}\,)(x) > \lambda b, \mathcal{M}_\alpha(\vec{f}\,)(x) \le \lambda d\} \cap Q_j) \\
&\quad= \sum_{j=1}^{\infty} \omega(E_j) \le \frac{1}{2}b^{-q}\omega(\{x \in \mathbb{R}^n; \mathcal{I}_\alpha(\vec{f}\,)(x) > \lambda\}).
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
\omega(\{x \in \mathbb{R}^n; \mathcal{I}_\alpha(\vec{f}\,)(x) > \lambda b\}) &\le \omega(\{x \in \mathbb{R}^n; \mathcal{M}_\alpha(\vec{f}\,)(x) > \lambda d\}) \\
&\quad + \frac{1}{2}b^{-q}\omega(\{x \in \mathbb{R}^n; \mathcal{I}_\alpha(\vec{f}\,)(x) > \lambda\}).
\end{aligned} \tag{5.5}$$

Because $\vec{f}$ has compact support, there exists a cube Q such that $\vec{f} = 0$ for any x outside Q. For fixed x outside $3Q$, let x_0 be the point in Q that is closest to x, and let P be the smallest cube with center at x and sides parallel to Q that contains Q. Then, there is a constant $L = L(n) > 1$ such that $|P| \le L|x - x_0|^n$. Moreover,

$$\mathcal{I}_\alpha(\vec{f}\,)(x) \le \frac{1}{(m|x - x_0|)^{mn-\alpha}} \prod_{i=1}^{m} \int_Q f_i(y_i)\,dy_i \le \Big(\frac{L}{m^n}\Big)^{m-\alpha/n} \mathcal{M}_\alpha(\vec{f}\,)(x).$$

Now, taking $d = \min\{D, (L/m^n)^{\alpha/n-m}\}$, we get

$$\big\{x \in \mathbb{R}^n; \mathcal{I}_\alpha(\vec{f}\,)(x) > \lambda\big\} \cap (3Q)^c \subset \big\{x \in \mathbb{R}^n; \mathcal{M}_\alpha(\vec{f}\,)(x) > \lambda d\big\}. \tag{5.6}$$

From (5.5) and (5.6), it follows that

$$\begin{aligned}
&\omega(\{x \in \mathbb{R}^n; \mathcal{I}_\alpha(\vec{f}\,)(x) > \lambda b\}) \\
&\le \omega(\{x\in\mathbb{R}^n; \mathcal{M}_\alpha(\vec{f}\,)(x) > \lambda d\}) + \frac{1}{2}b^{-q}\omega(\{x\in\mathbb{R}^n; \mathcal{I}_\alpha(\vec{f}\,)(x) > \lambda\} \cap (3Q)^c) \\
&\quad + \frac{1}{2}b^{-q}\omega(\{x \in \mathbb{R}^n; \mathcal{I}_\alpha(\vec{f}\,)(x) > \lambda\} \cap (3Q)) \\
&\le 2\omega(\{x \in \mathbb{R}^n; \mathcal{M}_\alpha(\vec{f}\,)(x) > \lambda d\}) + \frac{1}{2}b^{-q}\omega(\{x \in 3Q; \mathcal{I}_\alpha(\vec{f}\,)(x) > \lambda\}). \qquad (5.7)
\end{aligned}$$

Now, we give the proof of inequality (5.1). Let N be any positive number. We multiply both sides of (5.7) by λ^{q-1} and integrate with respect to λ from 0 to N, and then make a change of variables to obtain

$$
\begin{aligned}
& b^{-q} \int_0^{bN} \lambda^{q-1}\omega(\{\mathcal{I}_\alpha(\vec{f})(x) > \lambda\})d\lambda \\
& \leq 2\int_0^{N} \lambda^{q-1}\omega(\{\mathcal{M}_\alpha(\vec{f})(x) > \lambda d\})d\lambda + \frac{1}{2}b^{-q}\int_0^{N} \lambda^{q-1}\omega(\{\mathcal{I}_\alpha(\vec{f})(x) > \lambda\} \cap 3Q)d\lambda \\
& \leq 2d^{-q}\int_0^{dN} \lambda^{q-1}\omega(\{\mathcal{M}_\alpha(\vec{f})(x) > \lambda\})d\lambda + \frac{1}{2}b^{-q}\int_0^{bN} \lambda^{q-1}\omega(\{\mathcal{I}_\alpha(\vec{f})(x) > \lambda\})d\lambda.
\end{aligned}
$$

Therefore,

$$
b^{-q} \int_0^{bN} \lambda^{q-1}\omega(\{\mathcal{I}_\alpha(\vec{f})(x) > \lambda\})\,d\lambda \leq 4d^{-q}\int_0^{dN} \lambda^{q-1}\omega(\{\mathcal{M}_\alpha(\vec{f})(x) > \lambda\})\,d\lambda.
$$

Observing that $\|f\|_{L^q(\omega)} = q\int_0^\infty \lambda^{q-1}\omega(\{x \in \mathbb{R}^n; |f(x)| > \lambda\})d\lambda$ for $0 < q < \infty$, and letting N approach ∞, we deduce that

$$
\int_{\mathbb{R}^n} |\mathcal{I}_\alpha(\vec{f})(x)|^q\omega(x)\,dx \leq 4\left(\frac{b}{d}\right)^q \int_{\mathbb{R}^n} [\mathcal{M}_\alpha(\vec{f})(x)]^q\,\omega(x)\,dx.
$$

Thus, we have finished the proof of the inequality (5.1).

Next, we shall prove inequality (5.2). Let N be any positive number. We multiply both sides of (5.7) by λ^q, and then take the supremum of both sides for $0 < \lambda < N$, and note the fact that $\sup(u + v) \leq \sup u + \sup v$ to obtain

$$
\begin{aligned}
& b^{-q} \sup_{0<\lambda<bN} \lambda^q\omega(\{\mathcal{I}_\alpha(\vec{f})(x) > \lambda\}) \\
& \leq 2d^{-q} \sup_{0<\lambda<dN} \lambda^q\omega(\{\mathcal{M}_\alpha(\vec{f})(x) > \lambda\}) + \frac{1}{2}b^{-q} \sup_{0<\lambda<N} \lambda^q\omega(\{\mathcal{I}_\alpha(\vec{f})(x) > \lambda\} \cap 3Q) \\
& \leq 2d^{-q} \sup_{0<\lambda<dN} \lambda^q\omega(\{\mathcal{M}_\alpha(\vec{f})(x) > \lambda\}) + \frac{1}{2}b^{-q} \sup_{0<\lambda<bN} \lambda^q\omega(\{\mathcal{I}_\alpha(\vec{f})(x) > \lambda\}).
\end{aligned}
$$

Thus

$$
b^{-q} \sup_{0<\lambda<bN} \lambda^q\omega(\{\mathcal{I}_\alpha(\vec{f})(x) > \lambda\}) \leq 4d^{-q} \sup_{0<\lambda<dN} \lambda^q\omega(\{\mathcal{M}_\alpha(\vec{f})(x) > \lambda\}).
$$

Now letting N tend to ∞, we obtain

$$
\sup_{\lambda>0} \lambda^q\omega(\{x \in \mathbb{R}^n; |\mathcal{I}_\alpha(\vec{f})(x)| > \lambda\}) \leq 4\Big(\frac{b}{d}\Big)^q \sup_{\lambda>0} \lambda^q\omega(\{x \in \mathbb{R}^n; \mathcal{M}_\alpha(\vec{f})(x) > \lambda\}).
$$

This shows that the inequality (5.2) is true. Thus, the proof of Proposition 5.1 is finished. □

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