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A comparison principle for the porous medium equation and its consequences

Benny Avelin and Teemu Lukkari

Abstract. We prove a comparison principle for the porous medium equation in more general open sets in $\mathbb{R}^{n+1}$ than space-time cylinders. We apply this result in two related contexts: we establish a connection between a potential theoretic notion of the obstacle problem and a notion based on a variational inequality. We also prove the basic properties of the PME capacity, in particular that there exists a capacitary extremal which gives the capacity for compact sets.

1. Introduction

We study the porous medium equation (PME for short)

$$\frac{\partial u}{\partial t} - \Delta u^m = 0\,, \tag{1.1}$$

where $m > 1$. This equation is an important prototype of a nonlinear parabolic equation. The equation is degenerate, meaning that the modulus of ellipticity vanishes when the solution is zero. The name stems from modeling the flow of a gas in a porous medium: the continuity equation, Darcy's law, and an equation of state for the gas lead to (1.1) for the density of the gas, after scaling out various physical constants. For more information about this equation, including numerous further references, we refer to the monographs [8] and [17].

The comparison principle is a fundamental tool in the theory of elliptic and parabolic equations. In particular, it can be used to define a class of supersolutions which is the counterpart for superharmonic functions in classical potential theory: we call a function a semicontinuous supersolution, if it satisfies the comparison principle with respect to continuous solutions. The definition is due to F. Riesz [15], and it makes the development of a nonlinear potential theory feasible.

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The comparison principle for parabolic equations is usually formulated for space-time cylinders, meaning sets of the form $\Omega_T = \Omega \times (0, T)$. The boundary values are then taken over *the parabolic boundary*, where only the initial and lateral boundaries are taken into account. However, one often encounters situations where one would like to apply the comparison principle in sets which are not space-time cylinders. Thus our main objective is to establish a comparison principle for the PME in more general open sets in $\mathbb{R}^{n+1}$. Such a result is occasionally called the elliptic comparison principle, in reference to the fact that the time variable no longer has a special role. Moreover, the elliptic comparison principle can be used to develop the Perron method in general space-time domains, see [3], [11]. We also present two applications where such a comparison principle is indispensable.

For the heat equation, when $m = 1$, one may add constants to solutions. A comparison principle for general open sets then follows from the space-time cylinder case by a straightforward exhaustion argument. For the PME, there is a comparison principle over cylindrical domains, but adding constants is no longer possible. Our idea for circumventing this difficulty is to multiply one of the functions being compared by a constant close to one. The modified function is no longer a solution, but it still satisfies the PME with an error-term on the right-hand side. The error-term vanishes as the multiplicative constant tends to one. The comparison principle for the original functions then follows by the usual duality proof, modified to account for the error-term. Our argument yields a comparison principle for open sets of the form $\Omega_T \setminus K$, where K is a compact set.

As the first application, we consider the obstacle problem. Roughly speaking, this amounts to finding a solution to a PDE subject to the constraint that the solution stays above a given function, the obstacle. Here we use a potential theoretic method for solving the problem: we define the solution to the obstacle problem to be the infimum of all supersolutions lying above the obstacle (réduite). For smooth enough obstacles the réduite is the smallest supersolution above the obstacle. The concept of réduite is standard in classical potential theory, and it has been utilized in a nonlinear parabolic context in [14]. Existence and uniqueness follow in a straightforward manner, at least for continuous obstacles. However, the relation between the smallest supersolution and the variational solutions to obstacle problems constructed in [4] is not obvious. In this direction, we prove that the smallest supersolution is also a variational solution for sufficiently smooth obstacles. This follows from two facts. First, we prove that the smallest supersolution can always be approximated by variational solutions. Second, the notion of variational solution is stable with respect to the convergence of the obstacles in certain norms, see [4]. The converse of this, i.e., whether a variational solution agrees with the smallest supersolution, remains a very interesting open problem.

The second application is a notion of parabolic capacity for the PME. This concept is defined via a measure data problem, as in [12] for the parabolic p-Laplacian. See also [18], [19] and the references therein for the capacity for the heat equation. We prove the basic properties of the capacity related to the PME, such as countable subadditivity and the existence of the capacitary extremal of a compact set. Our comparison principle plays a key role in the latter argument.

The paper is organized as follows. In Section 2, we recall the necessary background material, in particular various notions of supersolutions. Section 3 contains the proof of the comparison principle, and Section 4 is concerned with the obstacle problem. Finally, the basic properties of capacity are proved in Section 5.

2. Weak supersolutions and semicontinuous supersolutions

Let Ω be an open and bounded subset of $\mathbb{R}^n$, and let $0 < t_1 < t_2 < T$. We use the notation $\Omega_T = \Omega \times (0, T)$ and $U_{t_1,t_2} = U \times (t_1, t_2)$, where $U \subset \Omega$ is open. The parabolic boundary $\partial_p U_{t_1,t_2}$ of a space-time cylinder U_{t_1,t_2} consists of the initial and lateral boundaries, i.e.,

$$\partial_p U_{t_1,t_2} = (\overline{U} \times \{t_1\}) \cup (\partial U \times [t_1, t_2]) .$$

The notation $U_{t_1,t_2} \Subset \Omega_T$ means that the closure $\overline{U_{t_1,t_2}}$ is compact and $\overline{U_{t_1,t_2}} \subset \Omega_T$.

We use $H^1(\Omega)$ to denote the usual Sobolev space, the space of functions u in $L^2(\Omega)$ such that the weak gradient exists and also belongs to $L^2(\Omega)$. The norm of $H^1(\Omega)$ is defined by

$$\|u\|^2_{H^1(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} .$$

The Sobolev space with zero boundary values, denoted by $H^1_0(\Omega)$, is the completion of $C_0^\infty(\Omega)$ with respect to the norm of $H^1(\Omega)$.

The parabolic Sobolev space $L^2(0, T; H^1(\Omega))$ consists of measurable functions $u\colon \Omega_T \to [-\infty, \infty]$ such that $x \mapsto u(x, t)$ belongs to $H^1(\Omega)$ for almost all $t \in (0, T)$, and

$$\int_{\Omega_T} |u|^2 + |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t < \infty .$$

The definition of $L^2(0, T; H^1_0(\Omega))$ is identical, apart from the requirement that $x \mapsto u(x, t)$ belongs to $H^1_0(\Omega)$. We say that u belongs to $L^2_{\mathrm{loc}}(0, T; H^1_{\mathrm{loc}}(\Omega))$ if $u \in L^2(t_1, t_2; H^1(U))$ for all $U_{t_1,t_2} \Subset \Omega_T$.

Supersolutions to the porous medium equation are defined in the weak sense in the parabolic Sobolev space.

Definition 2.1. A nonnegative function $u\colon \Omega_T \to \mathbb{R}$ is a *weak supersolution* of the equation

$$\frac{\partial u}{\partial t} - \Delta u^m = 0 \quad \text{in } \Omega_T \tag{2.1}$$

if $u^m \in L^2_{\mathrm{loc}}(0, T; H^1_{\mathrm{loc}}(\Omega))$ and

$$\int_{\Omega_T} -u \frac{\partial \varphi}{\partial t} + \nabla u^m \cdot \nabla \varphi \,\mathrm{d}x\,\mathrm{d}t \geq 0 ,$$

for all positive, smooth test functions φ compactly supported in Ω_T. The definition of *weak subsolutions* is similar; the inequality is simply reversed. *Weak solutions* are defined as functions that are both super- and subsolutions.

Weak solutions are locally Hölder continuous, after a possible redefinition on a set of measure zero. See [7], [8], [9], [17], or [20].

We have also the following class of supersolutions.

Definition 2.2. A function $u\colon \Omega_T \to [0,\infty]$ is a *semicontinuous supersolution* if

(1) u is lower semicontinuous,

(2) u is finite in a dense subset of Ω_T, and

(3) the following parabolic comparison principle holds: Let $U_{t_1,t_2} \Subset \Omega$, and let h be a solution to (2.1) which is continuous in $\overline{U_{t_1,t_2}}$. Then, if $h \le u$ on $\partial_p U_{t_1,t_2}$, $h \le u$ also in U_{t_1,t_2}.

Note that a semicontinuous supersolution is defined in every point. Every weak supersolution is a semicontinuous supersolution provided that a proper pointwise representative is chosen. This is a consequence of the following lemma.

Lemma 2.3 ([2]). *Let u be a nonnegative weak supersolution to the porous medium equation in $\Omega \times (t_1, t_2)$. Then u has a lower semicontinuous representative.*

In the other direction, a *bounded* semicontinuous supersolution is also a weak supersolution, as shown in [13]. If unbounded functions are allowed, then the class of semicontinuous supersolutions is strictly larger, since the Barenblatt solution is a semicontinuous supersolution, but is not a weak supersolution, see [13].

Lemma 2.4 ([13]). *Let u be a weak supersolution such that $|u| \le M < \infty$. Then*

$$\iint_{\Omega_T} \eta^2 \, |\nabla u^m|^2 \, dx \, dt \le 16M^{2m} T \int_\Omega |\nabla \eta|^2 \, dx + 6M^{m+1} \int_\Omega \eta^2 \, dx\,,$$

for all nonnegative functions $\eta \in C_0^\infty(\Omega)$.

An application of the Riesz representation theorem shows that for each weak supersolution u, there exists a positive Radon measure μ_u such that

$$\iint_{\Omega_\infty} -u \frac{\partial \varphi}{\partial t} + \nabla u^m \cdot \nabla \varphi \, dx \, dt = \int_{\Omega_\infty} \varphi \, d\mu_u$$

for all smooth compactly supported functions φ. This is the *Riesz measure* of u. The integrals on the left hand side do not depend on the particular pointwise representative of a supersolution. Thus a weak supersolution u and its lower semicontinuous regularization $\widehat{u}$ have the same Riesz measures.

Lemma 2.5. *If u and v are weak supersolutions in Ω_∞, $u, v = 0$ on $\partial_p \Omega_\infty$, $u^m, v^m \in L^2(0,\infty; H_0^1(\Omega))$, and $\mu_v \le \mu_u$, then $v \le u$ a.e. in Ω_∞.*

Proof. Let $\varphi \in C_0^\infty(\Omega_\infty)$ be nonnegative. By subtracting the equations satisfied by u and v and using the assumption about the measures, we have

$$\int_{\Omega_\infty} -(u-v)\varphi_t + \nabla(u^m - v^m) \cdot \nabla \varphi \, dx \, dt = \int_{\Omega_\infty} \varphi \, d\mu_u - \int_{\Omega_\infty} \varphi \, d\mu_v \ge 0\,.$$

By a standard approximation argument using the fact that u^m and v^m belong to $L^2(0,\infty;H_0^1(\Omega))$, we may also take the test functions $\varphi \in C^\infty(\Omega_\infty)$ so that $\varphi = 0$ on the lateral boundary $\partial\Omega \times (0,\infty)$. We apply Green's formula to get

$$\int_{\Omega_\infty} -(u-v)\varphi_t - (u^m - v^m)\Delta\varphi \,\mathrm{d}x\,\mathrm{d}t \geq 0\,.$$

The fact that $v \leq u$ follows from this inequality by repeating the standard duality proof for the comparison principle for the PME, see e.g. Lemma 5 in [7], Theorem 1.1.1 in [8], or Theorem 6.5 in [17]. □

Lemma 2.6. *Let u_i, $i = 1, 2, \ldots$, be a uniformly bounded sequence of weak supersolutions in Ω_∞ such that $u_i \to u$ a.e. in Ω_∞. Then u is a weak supersolution in Ω_∞ and*

$$\lim_{i\to\infty} \int_{\Omega_\infty} \phi \,\mathrm{d}\mu_{u_i} = \int_{\Omega_\infty} \phi \,\mathrm{d}\mu_u\,,$$

for every $\phi \in C_0^\infty(\Omega_\infty)$.

Proof. Due to the uniform bound on the functions u_i, it easily follows that

$$\int_{\Omega_\infty} -u\frac{\partial\phi}{\partial t} - u^m\Delta\phi \,\mathrm{d}x\,\mathrm{d}t \geq 0\,. \tag{2.2}$$

An application of Lemma 2.4 on each u_i implies that $\nabla u^m \in L^2_{\mathrm{loc}}(\Omega_\infty)$. This, together with (2.2), yields that u is a weak supersolution. The claim about the measures follows from the computation

$$\begin{aligned}\lim_{i\to\infty} \int_{\Omega_\infty} \phi \,\mathrm{d}\mu_{u_i} &= \lim_{i\to\infty} \int_{\Omega_\infty} -u_i\frac{\partial\phi}{\partial t} + \nabla u_i^m \cdot \nabla\phi \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\Omega_\infty} -u\frac{\partial\phi}{\partial t} + \nabla u^m \cdot \nabla\phi \,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_\infty} \phi \,\mathrm{d}\mu_u.\end{aligned}$$

□

We will frequently use the following characterization of the weak convergence of measures. See Theorem 1 on p. 54 in [10] for the proof.

Theorem 2.7. *Let μ and μ_k, $k = 1, 2, 3, \ldots$, be Radon measures on $\mathbb{R}^n$. Then the following statements are equivalent.*

(1) *For all compactly supported smooth functions ϕ, one has*

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \phi \,\mathrm{d}\mu_k = \int_{\mathbb{R}^n} \phi \,\mathrm{d}\mu\,.$$

(2) *For all compact sets K, one has*

$$\limsup_{k\to\infty} \mu_k(K) \leq \mu(K)\,.$$

(3) *For all open sets U, one has*

$$\mu(U) \leq \liminf_{k\to\infty} \mu_k(U)\,.$$

3. A comparison principle

The core of our arguments is a suitable form of the comparison principle, which we will prove in this section. We will work extensively with finite unions of space-time cylinders, so we begin by introducing some notation for such sets. For space-time cylinders $U_{t_1,t_2} = U \times (t_1, t_2)$, we denote the lateral boundary by

$$\mathcal{S}(U_{t_1,t_2}) = \partial U \times (t_1, t_2)\,.$$

For a cylinder the definition of the parabolic boundary is standard, but for finite unions of space time cylinders we will recall the definitions. The lateral boundary of a finite union of space time cylinders $U^i_{t^i_1,t^i_2}$ is then given by

$$\mathcal{S}(\cup U^i_{t^i_1,t^i_2}) := (\cup \mathcal{S}(U^i_{t^i_1,t^i_2})) \setminus (\cup U^i_{t^i_1,t^i_2})\,.$$

We also denote the tops of $\cup U^i_{t^i_1,t^i_2}$ by

$$\mathcal{T}(\cup U^i_{t^i_1,t^i_2}) := (\cup \overline{U^i} \times \{t^i_2\}) \setminus (\cup U^i_{t^i_1,t^i_2})\,,$$

and the bottoms similarly as

$$\mathcal{B}(\cup U^i_{t^i_1,t^i_2}) := (\cup \overline{U^i} \times \{t^i_1\}) \setminus (\cup U^i_{t^i_1,t^i_2})\,.$$

Thus the parabolic boundary of $Q = \cup U^i_{t^i_1,t^i_2}$ is $\mathcal{S}(Q) \cup \mathcal{B}(Q)$, and the parabolic boundary of backwards in time equations becomes $\mathcal{S}(Q) \cup \mathcal{T}(Q)$.

We want to use the very weak (i.e., distributional) formulation of the porous medium equation, so we consider smooth test functions $\phi \in C^\infty(Q)$ where $Q = \cup U^i_{t^i_1,t^i_2}$, such that $\phi = 0$ on $\mathcal{S}(Q)$. Note that the gradient of ϕ does not necessarily vanish on $\mathcal{S}(Q)$. In the following we will always work with Ω a smooth domain. Let us now write the PME in terms of the above class of test functions. Assume at first that ϕ has compact support in space. Then a standard approximation argument shows that we may write the definition of weak solutions as

$$\int_Q \big[-u\phi_t + \nabla u^m \cdot \nabla\phi\big]\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathcal{T}(Q)} u\phi\,\mathrm{d}x - \int_{\mathcal{B}(Q)} u\phi\,\mathrm{d}x = 0\,.$$

After this, we may pass from compactly supported test functions to test functions vanishing on the sides $\mathcal{S}(Q)$, since u and ∇u^m are in $L^2(Q)$. Now, apply Green's formula, which is justified by the usual trace theorem, to get

$$\int_{\mathcal{T}(Q)} u\phi\,\mathrm{d}x - \int_{\mathcal{B}(Q)} u\phi\,\mathrm{d}x + \int_Q \big[-u\phi_t - u^m\Delta\phi\big]\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathcal{S}(Q)} u^m\partial_n\phi\,d\sigma\,\mathrm{d}t = 0\,. \tag{3.1}$$

A similar argument can be carried out for weak supersolutions and subsolutions. In these cases, we get the appropriate inequalities in the final form. This formulation will be our starting point in the proof of the comparison principle.

Theorem 3.1. *Let K be a compact set in Ω_T where Ω is a smooth domain, let u be a nonnegative upper semicontinuous function which is a continuous weak supersolution in in $\Omega_T \setminus K$ and satisfies $u^m \in L^2(0,T;H^1(\Omega))$. Let v be a non-negative lower semicontinuous function which is a weak supersolution in $\Omega_T \setminus K$ and satisfies $v^m \in L^2(0,T;H^1(\Omega))$, $v > 0$ on K and $u \leq v$ on $K \cup \partial_p \Omega_T$. Then $u \leq v$ in Ω_T.*

Proof. We let $\epsilon > 0$, and denote

$$D_\epsilon = \left\{(x,t) \in \Omega_T : \frac{u}{1+\epsilon} \geq v\right\}.$$

The function $u/(1+\epsilon) - v$ is upper semicontinuous, so that the set D_ϵ is closed in Ω_T. Moreover, the set D_ϵ does not intersect the set K, since $u \leq v$ on K and $\inf_K v > 0$. Since K is compact, there is a positive distance between D_ϵ and K. Thus we can cover D_ϵ with a finite collection of space time cylinders not intersecting K. Denote the covering set by $\hat{D}^F_\epsilon$, and note that since $D_{\epsilon_1} \subset D_{\epsilon_2}$ for $\epsilon_1 > \epsilon_2$ we may choose the coverings for different values of ϵ so that $\hat{D}^F_{\epsilon_1} \subset \hat{D}^F_{\epsilon_2}$. Let then $D^F_\epsilon = \Omega_T \cap \hat{D}^F_\epsilon$. The set D^F_ϵ is still a finite union of space-time cylinders, and the function u is a weak solution in D^F_ϵ.

Let $u_\epsilon = u/(1+\epsilon)$. We want to compare u_ϵ with v in D^F_ϵ. To this end, note first that $u_\epsilon < v$ on ∂D^F_ϵ. Further, the function u_ϵ is a solution to

$$(u_\epsilon)_t - \Delta(u_\epsilon)^m = f := \frac{(1+\epsilon)^{m-1} - 1}{(1+\epsilon)^m} \Delta u^m ,$$

interpreted in the sense of distributions. To see this, we compute

$$\begin{aligned} [u_\epsilon]_t - \Delta[u_\epsilon]^m &= [u_\epsilon - u]_t - \Delta[u_\epsilon^m - u^m] \\ &= \left(\frac{1}{1+\epsilon} - 1\right) u_t - \left(\frac{1}{(1+\epsilon)^m} - 1\right) \Delta u^m \\ &= \left(\frac{1}{1+\epsilon} - \frac{1}{(1+\epsilon)^m}\right) \Delta u^m = \frac{(1+\epsilon)^{m-1} - 1}{(1+\epsilon)^m} \Delta u^m . \end{aligned} \tag{3.2}$$

We aim at adapting the proof of the comparison principle for the PME, see e.g. [7]. To proceed, let

$$D^F_{\epsilon,s} = \{(x,t) \in D^F_\epsilon : t \leq s\} ,$$

and take positive functions $\phi \in C_0^\infty(D^F_{\epsilon,s})$, and $\psi \in C^\infty(D^F_{\epsilon,s})$ which vanishes on $\mathcal{S}(D^F_{\epsilon,s})$, $\partial_n \psi \leq 0$ on $\mathcal{S}(D^F_{\epsilon,s})$ and so that ψ equals ϕ on $\mathcal{T}(D^F_{\epsilon,s})$. Denote also $b = u_\epsilon - v$ for brevity. Since b is negative and consequently also $u_\epsilon^m - v^m$ on ∂D^F_ϵ, we have from (3.1) and (3.2), since $\partial_n\psi \le 0$ on $\mathcal{S}(D^F_\epsilon)$,

$$\begin{aligned}\int_{D^F_{\epsilon,s}} f\psi \,\mathrm{d}x\,\mathrm{d}t &= \int_{\mathcal{T}(D^F_{\epsilon,s})} b\,\phi\,\mathrm{d}x - \int_{\mathcal{B}(D^F_{\epsilon,s})} b\,\psi\,\mathrm{d}x - \int_{D^F_{\epsilon,s}} b\psi_t\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \int_{D^F_{\epsilon,s}} [u^m_\epsilon - v^m]\,\Delta\psi\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathcal{S}(D^F_{\epsilon,s})} [u^m_\epsilon - v^m]\,\partial_n\psi\,d\sigma\,\mathrm{d}t \\ &\ge \int_{\mathcal{T}(D^F_{\epsilon,s})} b\,\phi\,\mathrm{d}x - \int_{D^F_{\epsilon,s}} b\,\psi_t\,\mathrm{d}x\,\mathrm{d}t - \int_{D^F_{\epsilon,s}} [u^m_\epsilon - v^m]\,\Delta\psi\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}$$

We can rewrite this as

$$\int_{\mathcal{T}(D^F_{\epsilon,s})} b\,\phi\,\mathrm{d}x \le \int_{D^F_{\epsilon,s}} b\,(\psi_t + a\Delta\psi)\,\mathrm{d}x\,\mathrm{d}t + \int_{D^F_{\epsilon,s}} f\psi\,\mathrm{d}x\,\mathrm{d}t\,, \tag{3.3}$$

where

$$a = \begin{cases} \frac{u^m_\epsilon - v^m}{u_\epsilon - v}, & \text{if } u_\epsilon \ne v\,,\\ 0, & \text{if } u_\epsilon = v\,.\end{cases}$$

Next we use a regularization to make the term $\psi_t + a\Delta\psi$ small in the above inequality. To do this let a_k, $k = 1, 2, \dots$, be smooth functions in $\overline{D^F_{\epsilon,s}}$ such that

$$\frac{1}{k} \le a_k \le k\,,$$

and

$$\int_{\overline{D^F_{\epsilon,s}}} \frac{(a_k - a)^2}{a_k}\,\mathrm{d}x\,\mathrm{d}t \to 0 \quad \text{as} \quad k\to\infty\,. \tag{3.4}$$

We replace the function ψ in (3.3) by the solution ψ_k to the following boundary value problem:

$$\begin{cases} u_t + a_k\Delta u = 0\,, & \text{in } D^F_{\epsilon,s}\,,\\ u(x,s) = \phi(x,s)\,, & \text{on } \mathcal{T}(D^F_{\epsilon,s})\,,\\ u = 0\,, & \text{on } \mathcal{S}(D^F_{\epsilon,s})\,,\end{cases} \tag{3.5}$$

and get, by Hölder's inequality,

$$\begin{aligned}\int_{\mathcal{T}(D^F_{\epsilon,s})} (u_\epsilon - v)\phi\,\mathrm{d}x &\le \int_{D^F_{\epsilon,s}} b(a - a_k)\Delta\psi_k\,\mathrm{d}x\,\mathrm{d}t - \int_{D^F_{\epsilon,s}} f\psi_k\,\mathrm{d}x\,\mathrm{d}t \\ &\le \Big[\int_{D^F_{\epsilon,s}} b^2\frac{(a-a_k)^2}{a_k}\,\mathrm{d}x\,\mathrm{d}t\Big]^{1/2}\Big[\int_{D^F_{\epsilon,s}} a_k(\Delta\psi_k)^2\,\mathrm{d}x\,\mathrm{d}t\Big]^{1/2} + \int_{D^F_{\epsilon,s}} f\psi_k\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned} \tag{3.6}$$

To continue, we need to estimate the term on the right-hand side of (3.6) containing the quantity $a_k(\Delta\psi_k)^2$ independently of k. To do this we follow the calculations of [7]. We use the equation (3.5) for ψ_k and integrate by parts, first in time and then in space, which gives

$$\begin{aligned}
\int_{D^F_{\epsilon,s}} [a_k\Delta\psi_k]\Delta\psi_k \,\mathrm{d}x\,\mathrm{d}t &= -\int_{D^F_{\epsilon,s}} [\psi_k]_t\Delta\psi_k \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{D^F_{\epsilon,s}} \psi_k[\Delta\psi_k]_t \,\mathrm{d}x\,\mathrm{d}t - \int_{\mathcal{T}(D^F_{\epsilon,s})} \phi\Delta\phi\,\mathrm{d}x + \int_{\mathcal{B}(D^F_{\epsilon,s})} \psi_k\Delta\psi_k\,\mathrm{d}x \\
&= \int_{D^F_{\epsilon,s}} \psi_k\Delta[\psi_k]_t \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathcal{T}(D^F_{\epsilon,s})} |\nabla\phi|^2\,\mathrm{d}x - \int_{\mathcal{B}(D^F_{\epsilon,s})} |\nabla\psi_k|^2\,\mathrm{d}x \\
&\le -\int_{\mathcal{S}(D^F_{\epsilon,s})} \partial_n\psi_k[\psi_k]_t\,\mathrm{d}\sigma\,\mathrm{d}t + \int_{D^F_{\epsilon,s}} \Delta\psi_k[\psi_k]_t\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathcal{T}(D^F_{\epsilon,s})} |\nabla\phi|^2\,\mathrm{d}x\,. \qquad (3.7)
\end{aligned}$$

Note now that the first term on the right-hand side in (3.7) vanishes, since for almost every $t \le s$ we have $[\psi_k]_t = 0$ on $\mathcal{S}[D^F_{\epsilon,s}]$ due to the fact that ψ_k vanishes smoothly on the boundary. For the second term on the right-hand side in (3.7), we use the first line in (3.7). This implies that

$$\int_{D^F_{\epsilon,s}} a_k(\Delta\psi_k)^2\,\mathrm{d}x\,\mathrm{d}t \le \frac{1}{2}\int_{\mathcal{T}(D^F_{\epsilon,s})} |\nabla\phi|^2\,\mathrm{d}x\,. \qquad (3.8)$$

With the estimate (3.8) in hand, we see from (3.4) and the fact that b is bounded that

$$\Big(\int_{D^F_{\epsilon,s}} b^2\frac{(a-a_k)^2}{a_k}\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2}\Big(\int_{D^F_{\epsilon,s}} a_k(\Delta\psi_k)^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2} \to 0 \quad \text{as } k\to\infty. \qquad (3.9)$$

To proceed we need to take care of the term involving f on the right-hand side in (3.6). Recall that, as a distribution,

$$f = \frac{(1+\epsilon)^{m-1}-1}{(1+\epsilon)^m}\,\Delta u^m\,.$$

Since the function ψ_k vanishes on the lateral boundary $\mathcal{S}(D^F_{\epsilon,s})$ of $D^F_{\epsilon,s}$, we have

$$\begin{aligned}
\int_{D^F_{\epsilon,s}} f\psi_k\,\mathrm{d}x\,\mathrm{d}t &= \frac{(1+\epsilon)^{m-1}-1}{(1+\epsilon)^m}\int_{D^F_{\epsilon,s}} \nabla u^m\cdot\nabla\psi_k\,\mathrm{d}x\,\mathrm{d}t \\
&\le \frac{(1+\epsilon)^{m-1}-1}{(1+\epsilon)^m}\Big(\int_{D^F_{\epsilon,s}} |\nabla u^m|^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2}\Big(\int_{D^F_{\epsilon,s}} |\nabla\psi_k|^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2}. \qquad (3.10)
\end{aligned}$$

By the assumption $u^m \in L^2(0,T;H^1_0(\Omega))$, we see that the first integral is bounded independent of k and ϵ.

Next we need to estimate the L^2-norm of $|\nabla\psi_k|$ independently of k, which we do as in p. 133 in [17]. Multiply the equation (3.5) for ψ_k by the test function $\theta = \Delta\psi_k\chi(t)$, where $\chi(0) = 1/2$ and $\chi(s) = 1$; thus $\chi_t \approx 1/s$. Next we integrate by parts, first in space and then in time, and get

$$
\begin{aligned}
0 &= \int_{D^F_{\epsilon,s}} [\psi_k]_t \, \Delta\psi_k \chi \, \mathrm{d}x \, \mathrm{d}t + \int_{D^F_{\epsilon,s}} a_k (\Delta\psi_k)^2 \, \chi \, \mathrm{d}x \, \mathrm{d}t \\
&= -\int_{D^F_{\epsilon,s}} \nabla(\psi_k)_t \cdot \nabla\psi_k \chi \, \mathrm{d}x \, \mathrm{d}t + \int_{D^F_{\epsilon,s}} a_k (\Delta\psi_k)^2 \, \chi \, \mathrm{d}x \, \mathrm{d}t \\
&= -\frac{1}{2} \int_{D^F_{\epsilon,s}} [|\nabla\psi_k|^2]_t \, \chi \, \mathrm{d}x \, \mathrm{d}t + \int_{D^F_{\epsilon,s}} a_k (\Delta\psi_k)^2 \, \chi \, \mathrm{d}x \, \mathrm{d}t \\
&= \frac{1}{2} \int_{D^F_{\epsilon,s}} (|\nabla\psi_k|^2) \, \chi_t \, \mathrm{d}x \, \mathrm{d}t - \frac{1}{2} \int_{\mathcal{T}(D^F_{\epsilon,s})} |\nabla\psi_k|^2 \, \chi \\
&\quad + \frac{1}{2} \int_{\mathcal{B}(D^F_{\epsilon,s})} |\nabla\psi_k|^2 \, \chi + \int_{D^F_{\epsilon,s}} a_k (\Delta\psi_k)^2 \, \chi \, \mathrm{d}x \, \mathrm{d}t \, .
\end{aligned} \tag{3.11}
$$

Using (3.5) and (3.11) we get

$$
\begin{aligned}
&\frac{1}{s} \int_{D^F_{\epsilon,s}} (|\nabla\psi_k|^2) \, \mathrm{d}x \, \mathrm{d}t + \int_{D^F_{\epsilon,s}} a_k (\Delta\psi_k)^2 \, \chi \, \mathrm{d}x \, \mathrm{d}t \\
&\leq C \Big(\int_{\mathcal{T}(D^F_{\epsilon,s})} |\nabla\psi_k|^2 \, \chi \, \mathrm{d}x - \int_{\mathcal{B}(D^F_{\epsilon,s})} |\nabla\psi_k|^2 \, \chi \, \mathrm{d}x \Big) \leq C \int_{\mathcal{T}(D^F_{\epsilon,s})} |\nabla\phi|^2 \, \mathrm{d}x \, .
\end{aligned} \tag{3.12}
$$

Combining (3.6), (3.9), (3.10) and (3.12), we have so far established

$$
\int_{\mathcal{T}(D^F_{\epsilon,s})} (u_\epsilon - v)\phi \, \mathrm{d}x \leq C \frac{(1+\epsilon)^{m-1} - 1}{(1+\epsilon)^m} \int_{\mathcal{T}(D^F_{\epsilon,s})} |\nabla\phi|^2 \, \mathrm{d}x \, . \tag{3.13}
$$

Before letting $\epsilon \to 0$, we still need to check that

$$
\int_{\mathcal{T}(D^F_{\epsilon,s})} |\nabla\phi|^2 \, \mathrm{d}x \leq C \, ,
$$

for some constant C not depending on $\epsilon > 0$. We are free to assume that $\phi \in C_0^\infty(D_{\epsilon_0,s})$ for some ϵ_0. Then, since $D_{\epsilon_0,s} \subset D^F_{\epsilon,s}$ for $\epsilon < \epsilon_0$, we have $\mathcal{T}(D^F_{\epsilon,s}) \cap \overline{D_{\epsilon_0,s}} \subset \mathcal{T}(D_{\epsilon_0,s})$ which proves the desired bound. Thus, letting $\epsilon \to 0$ in (3.13), we get that

$$
\int_{\overline{D_{\epsilon_0,s}} \cap [\mathbb{R}^n \times \{s\}]} (u - v)\phi \, \mathrm{d}x \leq 0 \, .
$$

Since this holds for any positive ϕ, we obtain that $u \leq v$ a.e. in $\Omega_T \cap [\mathbb{R}^n \times \{s\}]$ for any s, and then also in Ω_T. □

The crucial point in the proof above is that we can approximate the set

$$
D_\epsilon = \Big\{ (x,t) \in \Omega_T : \frac{u}{1+\epsilon} \geq v \Big\} \, ,
$$

by finite unions of space time boxes while staying inside the set where u is a weak solution. Thus we can also deduce the following theorem.

Theorem 3.2. *Let E be an open set in $\mathbb{R}^{n+1}$, let u be a non-negative continuous weak solution in E such that*

$$\int_E [|u^m|^2 + |\nabla u^m|^2]\,\mathrm{d}x\,\mathrm{d}t < \infty\,.$$

Let v be a non-negative lower semicontinuous weak supersolution such that

$$\int_E [|v^m|^2 + |\nabla v^m|^2]\,\mathrm{d}x\,\mathrm{d}t < \infty\,,$$

$v > 0$ on ∂E and $u \le v$ on ∂E. Then $u \le v$ in E. Furthermore if a connected component of ∂E is the boundary of a finite union of space-time cylinders then we can remove the assumption $v > 0$ on that component.

4. The obstacle problem

In this section, we construct solutions to the obstacle problem by a potential theoretic method. More specifically, we call a function u a solution to the obstacle problem if it is the smallest supersolution above the given obstacle function ψ.

Existence and uniqueness are fairly easily established for this notion of solution to the obstacle problem. However, the relationship between the variational solutions studied in [4] and the smallest supersolution is not immediately clear. In this direction, we apply the comparison principle established earlier to prove that the smallest supersolution is also a variational solution, provided that the obstacle is sufficiently regular. This is a consequence of two facts: first, we prove that the smallest supersolution is a pointwise limit of variational solutions. Second, variational solutions are stable with respect to convergence of the obstacles in a suitable norm.

We expect that the converse is also true, i.e., that a variational solution is the smallest supersolution. However, our version of the comparison principle in general domains is not strong enough to prove this.

First we describe the notion of smallest supersolution in more detail.

Definition 4.1. Let ψ be a positive, bounded measurable function in Ω_∞, and denote

$$\mathcal{U}_\psi = \{v \text{ is a semicontinuous supersolution in } \Omega_\infty : v \ge \psi \text{ in } \Omega_\infty\}\,.$$

We define the *réduite* (or reduced function) of ψ as

$$R_\psi = \inf\{v : v \in \mathcal{U}_\psi\}\,.$$

For a measurable set E, we abbreviate $R_E = R_{\chi_E}$. We denote by $\widehat{R}_\psi$ (lower semicontinuous) ess lim inf-regularization of R_ψ. The function $\widehat{R}_\psi$ is usually called the *balayage* of ψ.

The terms réduite and balayage come from classical potential theory. The notion is due to Poincaré. We will need the following basic theorem, for which the proof is standard, but we reproduce it here for the reader's convenience.

Theorem 4.2. *The balayage $\widehat{R}_\psi$ is a semicontinuous supersolution in Ω_T.*

Proof. Pick a space-time cylinder $U_{t_1,t_2} \Subset \Omega_T$ and a weak solution u which is continuous in $\overline{U}_{t_1,t_2}$ with $u \leq \widehat{R}_\psi$ on $\partial_p U_{t_1,t_2}$. Then also $u \leq v$ on $\partial_p U_{t_1,t_2}$ for $v \in \mathcal{U}_\psi$, and by comparison the same holds in U_{t_1,t_2}. We take the infimum over v to get that $u \leq R_\psi$. Since $\widehat{u} = u$ by the continuity of u, we conclude that $u \leq \widehat{R}_\psi$. □

Note that, in general, R_ψ *might not be lower semicontinuous, and* $\widehat{R}_\psi$ *might not be above* ψ *in every point.* However, for *continuous* ψ it holds that $\widehat{R}_\psi \geq \psi$ *everywhere.* This together with Theorem 4.2 implies that $\widehat{R}_\psi$ is the unique smallest semicontinuous supersolution above the obstacle ψ. By the smallest supersolution, we mean a function $u \in \mathcal{U}_\psi$ with the property that

$$u \leq v \quad \text{for all} \;\; v \in \mathcal{U}_\psi \,. \tag{4.1}$$

A semicontinuous supersolution with the property (4.1) is unique, if it exists; indeed, if there are two functions $u_1, u_2 \in \mathcal{U}_\psi$ satisfying (4.1), then two applications of (4.1) give the inequalities $u_1 \leq u_2$ and $u_2 \leq u_1$, so that $u_1 = u_2$.

The next aim is to relate the smallest supersolution to the variational solutions to the obstacle problem constructed in [4]. We first recall some facts from [4].

We consider nonnegative obstacle functions ψ defined on Ω_T, with compact support and satisfying

$$\psi^m \in L^2\big(0,T;H_0^1(\Omega)\big), \quad \partial_t(\psi^m) \in L^{\frac{m+1}{m}}(\Omega_T)\,. \tag{4.2}$$

The class of admissible functions for the obstacle problem is defined by

$$K_\psi(\Omega_T) := \big\{ v \,:\, \Omega_T \to [0,\infty] \,:\, v^m \in L^2(0,T;H_0^1(\Omega))\,, v \geq \psi \text{ a.e. on } \Omega_T \big\}\,.$$

Note that $\psi \in K_\psi$, and therefore $K_\psi \neq \emptyset$.

With the above classes, we can state the definition of a strong solution to the obstacle problem.

Definition 4.3. A nonnegative function $u \in K_\psi(\Omega_T)$ is a *strong solution to the obstacle problem for the porous medium equation* if $\partial_t u \in L^2(0,T;H^{-1}(\Omega))$ and

$$\int_0^T \langle \partial_t u, \alpha(v^m - u^m)\rangle \,\mathrm{d}t + \int_{\Omega_T} \alpha \nabla u^m \cdot \nabla(v^m - u^m)\,\mathrm{d}z \geq 0\,,$$

holds for all comparison maps $v \in K_\psi(\Omega_T)$ and every Lipschitz continuous cut-off function $\alpha \colon [0,T] \to [0,\infty]$ with $\alpha(T) = 0$.

The cutoff function α is needed for making this definition consistent with the definition of weak solutions to the obstacle problem, which we will recall later.

For the existence of strong solutions, we still need the assumption

$$\Psi := \partial_t \psi - \Delta \psi^m \in L^\infty(\Omega_T)\,. \tag{4.3}$$

The following result can be extracted from Theorem 2.6 in [4].

Theorem 4.4. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a smooth boundary. Assume that the obstacle ψ satisfies the regularity conditions (4.2) and (4.3). Then there exists a strong solution u to the obstacle problem for the PME in the sense of Definition 4.3 satisfying $u^m \in L^2(0,T;H_0^1(\Omega))$ and $u(\,\cdot\,,0)=0$.*

The function u is also locally Hölder continuous, and satisfies $u \geq \psi$ everywhere in Ω_T. Further, u is a weak supersolution to the porous medium equation in Ω_T, and a weak solution in the open set $\{z \in \Omega_T : u(z) > \psi(z)\}$.

We now wish to show that u in Theorem 4.4 is a weak solution in the larger set

$$[\Omega_T \setminus \operatorname{supp}(\psi)] \cup \{z \in \Omega_T : u(z) > \psi(z)\}\,.$$

With this in mind we recall the following form of a partition of unity.

Lemma 4.5 (Partition of unity). *Let $U_1, U_2, \ldots, U_n$ be open sets, and let K be a compact set such that $K \subset U_1 \cup U_2 \cup \cdots \cup U_n$. Then there exist functions $\eta_i \in C_0^\infty(U_i)$ such that*

$$\sum_{i=1}^{n} \eta_i = 1 \quad \text{on } K.$$

Proof. For a version where the functions η_i are continuous, see Theorem 2.13, p. 40, in [16]. The fact that one may also choose smooth functions follows easily from the continuous version by applying a suitable mollification. □

Lemma 4.6. *The strong solution to the obstacle problem given by Theorem 4.4 is also a weak solution to the PME in the set $\Omega_T \setminus \operatorname{supp}(\psi)$.*

Proof. Let $\delta > 0$ be a number, and let $\eta_\delta : \mathbb{R} \to [0,1]$ be a Lipschitz function with $\eta_\delta(s) = 0$ for $s \leq -\delta$, $\eta_\delta(s) = 1$ for $s \geq 0$, and $|\eta_\delta'(s)| \leq 1/\delta$. The solution u is constructed in [4] as the uniform limit as $\delta \to 0$ of solutions to

$$\partial_t u_\delta - \Delta u_\delta^m = \eta_\delta(\psi^m - u_\delta^m)(\partial_t \psi - \Delta \psi^m)_+\,.$$

The claim now follows from the fact that $(\partial_t \psi - \Delta \psi^m)_+ = 0$ in $\Omega_T \setminus \operatorname{supp}(\psi)$. □

Theorem 4.7. *Let ψ be a nonnegative, compactly supported function satisfying the regularity assumptions (4.2) and (4.3), let u be the strong solution to the obstacle problem given by Theorem 4.4 with obstacle ψ, and denote $K = \operatorname{supp}(\psi) \cap \{u = \psi\}$. Then u is a weak solution in $\Omega_\infty \setminus K$.*

Proof. Denote $U_1 = \Omega_\infty \setminus \operatorname{supp}(\psi)$ and $U_2 = \{u > \psi\}$. These sets are open, and

$$\Omega_\infty \setminus K = U_1 \cup U_2\,.$$

Further, u is a weak solution in U_1 and in U_2. The claim concerning the set U_1 is Lemma 4.6, and the claim about U_2 is a part of Theorem 4.4. To show that u is a solution also in $U_1 \cup U_2$, let $\varphi \in C_0^\infty(U_1 \cup U_2)$. An application of Lemma 4.5 shows that there are functions $\eta_i \in C_0^\infty(U_i)$, $i = 1, 2$, such that $\eta_1 + \eta_2 = 1$ on the support of φ. By applying the fact that u is a weak solution in U_1 and in U_2, we get

$$\int_{\Omega_\infty} -u\partial_t\varphi + \nabla u^m \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t = \sum_{i=1}^{2}\int_{\Omega_\infty} -u\partial_t(\varphi\eta_i) + \nabla u^m \cdot \nabla(\varphi\eta_i)\,\mathrm{d}x\,\mathrm{d}t = 0\,.$$

Since this holds for any test function φ, u is a weak solution in $U_1 \cup U_2$. □

We are now ready to proceed with the approximation result.

Theorem 4.8. *Let ψ be continuous and compactly supported in Ω_T. Then the smallest supersolution $\widehat{R}_\psi$ is an increasing limit of strong solutions w_j to the obstacle problem with smooth compactly supported obstacles ϕ_j increasing to ψ.*

Proof. Let $U_j = \{\psi > 1/j\}$ for $j = 1, 2, \ldots$, $K_j = \overline{U}_j$. First note that K_j and K_{j+1} have a positive distance between them. Thus if we let $h_j = (\sqrt{\psi} - 1/\sqrt{j})_+$, we see that $h_{j+1} - h_j$ is strictly positive in K_j. By a mollification argument it can easily be seen that for each $j \geq 1$ there exists a function $f_j \in C_0^\infty(K_{j+1})$ such that

$$h_j \leq f_j \leq h_{j+1}\,.$$

Taking $\phi_j = f_j^2$, $j \geq 1$, we immediately see that $\phi_j^m \in C_0^2(K_{j+1})$, thus it satisfies (4.2) and (4.3). Moreover by construction we get

$$\phi_1 < \phi_2 < \cdots < \psi, \quad \phi_j \to \psi \text{ as } j \to \infty\,.$$

Let w_j be the strong solutions to the ϕ_j-obstacle problems. Since $\widehat{R}_\psi \geq \psi$ by the continuity of ψ, we have $w_j < \widehat{R}_\psi$ on $K = \partial(\{w_j = \phi_j\} \cap \operatorname{supp}\phi_j)$. Also note that $K \subset U_{j+1}$, whence $\widehat{R}_\psi > 1/(j+1) > 0$ on K. This allows us to use the comparison principle of Theorem 3.1 together with Theorem 4.7 to get that $w_j \leq \widehat{R}_\psi$. A similar argument shows that $w_j \leq w_{j+1}$. Thus $w = \lim_{j\to\infty} w_j$ is a semicontinuous supersolution as an increasing limit of continuous supersolutions, and $w \leq \widehat{R}_\psi$. To finish the proof, we have that $w \geq \psi$ everywhere in Ω_T, whence $R_\psi \leq w$. Thus

$$w \leq \widehat{R}_\psi \leq R_\psi \leq w\,,$$

and the proof is complete. □

The final step is to combine the approximation result with a stability result for variational solutions to conclude that the smallest supersolution is also a variational solution. We recall some more facts from [4], in particular the notion of a weak variational solution, for which stability with respect to the obstacles can be established.

For the notion of weak solutions, we use the class of admissible comparison functions

$$K'_\psi(\Omega_T) = \left\{v \in K_\psi(\Omega_T) : \partial_t(v^m) \in L^{\frac{m+1}{m}}(\Omega_T)\right\}.$$

We need to make sense of the time term in the variational inequality when we do not know that $\partial_t u$ belongs to the dual of the parabolic Sobolev space. We do this as in [1] and [4]. We recall the notation

$$\langle\langle \partial_t u, \alpha\eta(v^m - u^m)\rangle\rangle_{u_0} = \int_{\Omega_T} \eta\Big[\alpha'\Big[\frac{1}{m+1}u^{m+1} - uv^m\Big] - \alpha u \partial_t v^m\Big]\,\mathrm{d}x\,\mathrm{d}t + \alpha(0)\int_{\Omega} \eta\Big[\frac{1}{m+1}u_0^{m+1} - u_0 v^m(\,\cdot\,,0)\Big]\,\mathrm{d}x\,,$$

where $u_0 \in L^{m+1}(\Omega)$ is a function giving the initial values of the solution, and α is a nonnegative Lipschitz continuous cutoff function depending only on the time variable with $\alpha(T) = 0$. The role of the function α is to eliminate the final time term, as we do not know in general whether u is continuous in time. Observe that if $\partial_t u \in L^2(0,T;H^{-1}(\Omega))$, we have

$$\int_0^T \langle \partial_t u, \alpha\eta(v^m - u^m)\rangle\,\mathrm{d}t = \langle\langle \partial_t u, \alpha\eta(v^m - u^m)\rangle\rangle_{u_0}\,.$$

This follows formally from integration by parts, and the rigorous justification is given in Lemma 3.2 in [4]. This makes the following definition consistent with the definition of strong solutions in the previous section, i.e., strong solutions are also weak solutions.

Definition 4.9. A nonnegative function $u \in K_\psi(\Omega_T)$ is a *weak solution to the obstacle problem for the porous medium equation* if the inequality

$$\langle\langle \partial_t u, \alpha\eta(v^m - u^m)\rangle\rangle_{u_0} + \int_{\Omega_T} \alpha\nabla u^m \cdot \nabla\big(\eta(v^m - u^m)\big)\,\mathrm{d}z \geq 0$$

holds true for all comparison maps $v \in K'_\psi(\Omega_T)$ and every nonnegative, Lipschitz continuous cut-off function depending only on the time variable with $\alpha(T) = 0$.

Theorem 4.10. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a smooth boundary. Assume that the obstacle ψ satisfies the regularity condition (4.2). Then there exists a weak solution u to the obstacle problem for the porous medium equation in the sense of Definition 4.9 satisfying $u^m \in L^2(0,T;H_0^1(\Omega))$. Again, u is a weak supersolution to the porous medium equation in Ω_T.*

The following theorem may be extracted from the proof of Theorem 2.7 in [4].

Theorem 4.11. *Let ψ_i be a sequence of obstacles satisfying (4.2) with compact support in Ω_T such that*

$$\psi_i^m \to \psi^m \text{ in } L^2(0,T;H_0^1(\Omega))\,, \quad \text{and} \quad \partial_t(\psi_i^m) \to \partial_t(\psi^m) \text{ in } L^{\frac{m+1}{m}}(\Omega_T)\,,$$

furthermore let u_i be the respective variational weak solutions to the obstacle problem with obstacle ψ_i, see Theorem 4.10.

Then there is a function $u \in L^\infty(0,T;L^{m+1}(\Omega))$ with $u^m \in L^2(0,T;H_0^1(\Omega))$ and $u(\,\cdot\,,0) = 0$ such that, up to subsequences,

$$u_i \to u \text{ a.e.}, \quad u_i^m \to u^m \text{ in } L^2(\Omega_T), \quad \text{and } \nabla u_i^m \to \nabla u^m \text{ weakly in } L^2(\Omega_T)\,.$$

Furthermore, u is a variational weak solution to the obstacle problem with obstacle ψ and initial values zero.

Since strong variational solutions are also weak variational solutions, we get the following theorem as an immediate consequence of Theorems 4.8 and 4.11.

Theorem 4.12. *Let ψ be a continuous function with compact support in Ω_T satisfying the regularity assumptions (4.2). Then the smallest supersolution $\widehat{R}_\psi$ is also a variational weak solution.*

A general converse for Theorem 4.12 remains open. We record the following partial result for use in Section 5.

Theorem 4.13. *Let ψ be a smooth obstacle with $\psi > 0$ in Ω_T. Then any variational strong solution u to the obstacle problem coming from Theorem 4.4 satisfies $u = \widehat{R}_\psi$.*

Proof. Since $\psi > 0$, any strong solution is strictly positive inside Ω_T. Thus, given a semicontinuous supersolution $v \in \mathcal{U}_\psi$, we may apply Theorem 3.1 on the set $\{u > \psi\}$ to conclude that $u \leq v$. Since $u \in \mathcal{U}_\psi$, we get $u = \widehat{R}_\psi$. □

5. Parabolic capacity for the porous medium equation

In this section, we define the parabolic capacity for the porous medium equation and establish its basic properties.

Definition 5.1. The PME capacity of an arbitrary subset E of Ω_∞ is

$$\operatorname{cap}(E) = \sup\{\mu(\Omega_\infty) : 0 \leq u_\mu \leq 1, \operatorname{supp}(\mu) \subset E\},$$

where μ is a positive Radon measure, and u_μ is a weak supersolution with $u_\mu = 0$ on $\partial_p \Omega_\infty$, and a weak solution to the measure data problem

$$(u_\mu)_t - \Delta u_\mu^m = \mu .$$

Our next result is that there exists a *capacitary extremal* for the PME capacity of a compact set K, i.e., a semicontinuous supersolution u such that $\operatorname{cap}(K) = \mu_u(K)$. We need the following two lemmas.

Lemma 5.2. *Let ψ be a smooth, positive compactly supported function, and set*

$$\psi_\varepsilon = (\psi^m + \varepsilon^m)^{1/m} \quad \textit{and} \quad v_\varepsilon = \widehat{R}_{\psi_\varepsilon} .$$

Then the limit function

$$v = \lim_{\varepsilon \to 0} v_\varepsilon$$

is a continuous weak supersolution and a weak variational solution to the obstacle problem with obstacle ψ in Ω_T. Furthermore, v is a weak solution in the open set $\{v > \psi\}$.

Proof. The existence of the pointwise limit as $\varepsilon \to 0$ follows from the fact that $\widehat{R}_{\psi_{\varepsilon_1}} \le \widehat{R}_{\psi_{\varepsilon_2}}$ if $\varepsilon_1 \le \varepsilon_2$. The limit v is an upper semicontinuous weak supersolution as a decreasing limit of continuous weak supersolutions, and $v \ge \psi$ since $v_\varepsilon \ge \psi_\varepsilon$. By Theorem 4.13, we may take v_ε to be a strong variational solution to the obstacle problem. Hence v is a weak variational solution to the obstacle problem by [4]. The continuity follows from [5].

Since v_ε is a variational strong solution to the obstacle problem, it is a weak solution in the set $\{v_\varepsilon > \psi_\varepsilon\}$. If K is now a compact set contained in $\{v > \psi\}$, we have that K is also contained in $\{v_\varepsilon > \psi_\varepsilon\}$ for all sufficiently small ε, since

$$v_\varepsilon - \psi_\varepsilon \ge \inf_K (v - \psi) - \varepsilon$$

by the inequalities $v_\varepsilon \ge v$, $-\psi_\varepsilon = -(\psi^m + \varepsilon^m)^{1/m} \ge -\psi - \varepsilon$, and the fact that $\inf_K (v - \psi) > 0$. Thus

$$\int_{\Omega_T} -v\, \partial_t \varphi + \nabla v^m \cdot \nabla \varphi \,\mathrm{d}x\,\mathrm{d}t = \lim_{\varepsilon \to 0} \int_{\Omega_T} -v_\varepsilon\, \partial_t \varphi + \nabla v_\varepsilon^m \cdot \nabla \varphi \,\mathrm{d}x\,\mathrm{d}t = 0$$

for all smooth test functions φ with support in K. Since K was arbitrary, v is a weak solution in $\{v > \psi\}$. □

The next lemma is the key step in constructing the capacitary extremal. For the proof, we record the following estimate. Let u be a positive weak supersolution in Ω_∞, u vanishing on the lateral boundary of Ω_∞. Suppose in addition that there exists a time $t_0 \ge 0$ so that u is a weak solution in $\Omega_{t_0,\infty}$. Then

$$u(x,t) \le c\,(t - t_0)^{-1/(m-1)} \tag{5.1}$$

for all $t > t_0$ with a constant depending only on n, m, and the diameter of Ω. This is the so-called universal estimate. See Proposition 5.17 in [17] for the proof.

Lemma 5.3. *Let K be a compact subset of Ω_∞. Assume that u and v are lower semicontinuous weak supersolutions in Ω_∞ and that u is continuous in $\overline{\Omega}_\infty$, and a weak solution after a time T such that $K \Subset \Omega_T$. Moreover, assume that $u > 1$ in K, $u = 0$ on $\partial_p \Omega_\infty$, $0 \le v \le 1$ in Ω_∞, and $v = 0$ on $\partial_p \Omega_\infty$. Then*

$$\mu_v(K) \le \mu_u(\Omega_\infty)\,.$$

Proof. Let $0 \le \psi_i \in C_0^\infty(\Omega_\infty)$ be an increasing sequence of smooth obstacles converging to v such that $\psi_i < v$ and $\psi_i < \psi_{i+1}$. Denote the perturbed obstacles $\psi_i^\epsilon = (\psi_i^m + \epsilon^m)^{1/m}$, and the corresponding solutions to the obstacle problem by v_i^ϵ. Further, let $v_i = \lim_{\epsilon \to 0} v_i^\epsilon$ be the weak supersolutions in Ω_∞ constructed in Lemma 5.2. We argue as in the proof of Theorem 3.2 on p. 148–149 in [13] to see that $v_i \le v_{i+1} \le v$ and $v_i \to v$ as $i \to \infty$. Finally, we have that

$$\sup_{\Omega_\infty} v_i^\epsilon = \sup_{\Omega_\infty} \psi_i^\epsilon \le 1 + \epsilon \quad \text{and} \quad \sup_{\Omega_\infty} v_i = \sup_{\Omega_\infty} \psi_i \le 1\,.$$

We define the supersolution

$$w_i^\epsilon = \min(v_i^\epsilon, u)\,.$$

By lower semicontinuity, the set $\{u > 1\}$ is open, and K is compactly contained in it. This allows us to construct a compact set K' and an open set U such that $K \subset U \subset K' \subset \{u > 1\}$. If ϵ is small enough we know that $1 + \epsilon < u$ in K', so that $w_i^\epsilon = v_i^\epsilon$ in K'. Hence for such ϵ we have that, for $\phi' = 1$ on U and $\phi' = 0$ outside K',

$$\mu_{v_i^\epsilon}(U) \leq \int_U \phi' \,\mathrm{d}\mu_{v_i^\epsilon} = \int_U \phi' \,\mathrm{d}\mu_{w_i^\epsilon} \leq \mu_{w_i^\epsilon}(K')\,. \tag{5.2}$$

Next note that since $u = 0$ on $\partial_p\Omega$ and $u \leq \epsilon$ for sufficiently large times by (5.1), there is a compact set $K'' \supset K'$ in Ω_∞ such that in $\Omega_\infty \setminus K''$ we have $w_i^\epsilon = u$ since $v_i^\epsilon \geq \epsilon$. Hence we obtain for $\phi'' \in C_0^\infty(\Omega_\infty)$ such that $\phi'' = 1$ on K'' that

$$\begin{aligned}\int_{\Omega_\infty} \phi'' \,\mathrm{d}\mu_{w_i^\epsilon} &= \int_{\Omega_\infty} -w_i^\epsilon \frac{\partial \phi''}{\partial t} + \nabla (w_i^\epsilon)^m \cdot \nabla \phi'' \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\Omega_\infty} -u^\epsilon \frac{\partial \phi''}{\partial t} + \nabla u^m \cdot \nabla \phi'' \,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_\infty} \phi'' \,\mathrm{d}\mu_u\,.\end{aligned}$$

Thus we obtain the estimate

$$\mu_{w_i^\epsilon}(K') \leq \int_{\Omega_\infty} \phi'' \,\mathrm{d}\mu_{w_i^\epsilon} = \int_{\Omega_\infty} \phi'' \,\mathrm{d}\mu_u \leq \mu_u(\Omega_\infty)\,. \tag{5.3}$$

We combine (5.2) and (5.3) to get the inequality

$$\mu_{v_i^\epsilon}(U) \leq \mu_u(\Omega_\infty)\,.$$

By construction $v_i^\epsilon \to v_i$ pointwise, thus from Lemma 2.6 we get that $\mu_{v_i^\epsilon} \to \mu_{v_i}$ weakly. By the standard properties of weak convergence of measures, see Theorem 2.7, we get that

$$\mu_{v_i}(U) \leq \liminf_{\epsilon \to 0} \mu_{v_i^\epsilon}(U) \leq \mu_u(\Omega_\infty)\,.$$

The sequence (v_i) is increasing, and converges pointwise to the original supersolution v. Again from Lemma 2.6 we get the weak convergence of the corresponding measures. Another application of Theorem 2.7 now shows that

$$\mu_v(K) \leq \mu_v(U) \leq \liminf_{i \to \infty} \mu_{v_i}(U) \leq \mu_u(\Omega_\infty)\,,$$

and the proof is complete. □

A consequence of Theorem 3.1, is that in the special case that we have a decreasing sequence of smooth obstacles converging to a characteristic function of a compact set, the obstacle problem is stable. If we had a full elliptic comparison principle, this lemma would hold for a decreasing sequence of smooth obstacles converging to an upper semi-continuous obstacle.

Lemma 5.4. *Let $K \subset \Omega_\infty$ be a compact set. Let $E_i \Subset \Omega_\infty$, $i = 1, 2, \ldots$ be a shrinking sequence of open sets such that $E_{i+1} \Subset E_i$*

$$\bigcap_{i=1}^{\infty} \overline{E}_i = K\,.$$

Assume that the non-negative functions $\psi_i : \Omega_\infty \to \mathbb{R}$ are supported in $\overline{E}_i$, satisfy (4.2) and (4.3), and $\psi_i \geq \chi_K$, $i = 1, 2, \ldots$, is a decreasing sequence such that $\psi_i \to \chi_K$ pointwise in Ω_∞ as $i \to \infty$. Then $R_{\psi_i} \to R_K$ pointwise in Ω_∞ and $\mu_{R_{\psi_i}} \to \mu_{R_K}$ weakly as $i \to \infty$.

Proof. By Theorem 4.7, the functions R_{ψ_i} are continuous. Thus an application of Lemma 2.6 shows that $u = \lim_{i\to\infty} R_{\psi_i}$ is an upper semicontinuous weak supersolution, and the respective measures also converge weakly. Further,

$$u \geq R_K\,,$$

since $R_{\psi_i} \geq R_K$ for each i.

The lemma now follows if we prove the opposite inequality. To this end, note first and that from Theorem 4.7, R_{ψ_i} is a weak solution in $\{R_{\psi_i} > \psi_i\} \cup (\Omega_\infty \setminus \operatorname{supp}(\psi_i))$, so that the support of the measure $\mu_{R_{\psi_i}}$ is contained in $\operatorname{supp}(\psi_i) \subset \overline{E}_i$. These sets shrink to K, and the measures $\mu_{R_{\psi_i}}$ converge weakly to μ_u. Thus $\operatorname{supp}(\mu_u) \subset K$, which implies that u is a weak solution in $\Omega_\infty \setminus K$.

If now $v \geq \chi_K$ is an arbitrary semicontinuous supersolution with $v = 0$ on $\partial_p\Omega_\infty$, it follows from Theorem 3.1 that $u \leq v$. We take the infimum over v to get that

$$u \leq R_K\,,$$

and the proof is complete. □

A consequence of the stability Lemma 5.4 is that we have stability of the *balayage* with respect to decreasing sequences of compact sets.

Lemma 5.5. *Let $K_i \subset \Omega_\infty$, $i = 1, 2, \ldots$, be a decreasing sequence of compact sets and denote $K = \cap_{i=1}^{\infty} K_i$. Then $\hat{R}_{K_i}$ is a decreasing sequence converging to $\hat{R}_K$, moreover $\mu_{\hat{R}_{K_i}}$ converges to $\mu_{\hat{R}_K}$, weakly as $i \to \infty$.*

Proof. Let us construct $E_i = \{d((x,t); K_i) < c/i\}$, $i = 1, \ldots$, for a small constant $c < 1$ such that $\overline{E}_1 \subset \Omega_\infty$, then the sequence E_i satisfies the requirements of Lemma 5.4.

Let us now construct smooth functions $\hat{\psi}_i \in C_0^\infty(\overline{E}_i)$ such that $\hat{\psi}_i = 1$ on K_i, then let $\psi_i = [\hat{\psi}_i]^2$, and we have that $R_{\psi_i} \geq R_{K_i}$ by construction. As in the proof of Theorem 4.8, the sequence ψ_i will satisfy (4.2) and (4.3). It is now clear that the sequence ψ_i satisfies all requirements of Lemma 5.4 and thus we get that $R_{\psi_i} \to R_K$ and consequently also $R_{K_i} \to R_K$, furthermore using Lemma 2.6 we see that the measures $\mu_{R_{K_i}}$ converge weakly to μ_{R_K} as $i \to \infty$. □

Theorem 5.6. *Let K be a compact subset of Ω_∞. Then*

$$\operatorname{cap}(K) = \mu_{\hat{R}_K}(K)\,.$$

Proof. Since $\hat{R}_K$ is a semicontinuous supersolution such that $0 \le \hat{R}_K \le 1$, it follows immediately from the definition of the PME capacity that

$$\mu_{\hat{R}_K}(K) \le \operatorname{cap}(K)\,,$$

since $\hat{R}_K$ is a solution outside K.

To prove the opposite inequality, let first $K' \Subset \Omega_\infty$ be a compact set such that $K \Subset K'$. To be able to use Lemma 5.4 we will let $E_i \subset K'$, $i = 1, 2, \dots$ be a shrinking sequence of open sets such that

$$\bigcap_{i=1}^{\infty} \overline{E}_i = K\,.$$

Let $\hat{\psi}_i \in C_0^\infty(\overline{E}_i)$, $i = 1, \dots,$ be a decreasing sequence of smooth functions converging to χ_K pointwise in Ω_∞ as $i \to \infty$, and such that

$$\hat{\psi}_i = \sqrt{1 + \frac{1}{2^i}} \quad \text{on } K\,.$$

Consider now the functions $\psi_i = [\hat{\psi}_i]^2$, then $\psi_i^m \in C_0^2(\overline{E}_i)$, and it is a decreasing sequence of functions converging to χ_K pointwise in Ω_∞ as $i \to \infty$, such that

$$\psi_i = 1 + \frac{1}{2^i} \quad \text{on } K\,,$$

moreover ψ_i satisfies (4.2) and (4.3) for all $m > 1$. Denote by u_i the corresponding solutions to the obstacle problems with obstacle ψ_i. Let now v be a weak supersolution in Ω_∞ such that $0 \le v \le 1$ and $v = 0$ on $\partial_p \Omega_\infty$. Then it follows from Lemma 5.3 that

$$\mu_v(K) \le \mu_{u_i}(\Omega_\infty) = \mu_{u_i}(K')\,.$$

We use Lemma 5.4 to see that $\mu_{u_i} \to \mu_{\hat{R}_K}$ weakly. The claim now follows from the above estimate, since

$$\limsup_{i\to\infty} \mu_{u_i}(K') \le \mu_{\hat{R}_K}(K') = \mu_{\hat{R}_K}(K)$$

by Theorem 2.7. □

We have now developed all the technical tools needed to establish the basic properties of the PME capacity, including that it is a regular, subadditive capacity.

Theorem 5.7. *The PME capacity has the following properties.*

(1) Countable subadditivity. *In other words if E_i, $i = 1, 2, \dots,$ are arbitrary subsets of Ω_∞ and $E = \cup_{i=1}^\infty E_i$, one has*

$$\operatorname{cap}(E) \le \sum_{i=1}^{\infty} \operatorname{cap}(E_i)\,.$$

(2) Stability with respect to increasing sequences of sets. *Let E_i, $i = 1, 2, \dots,$ be arbitrary subsets of Ω_∞ with the property $E_1 \subset E_2 \subset \cdots$, and denote $E = \cup_{i=1}^{\infty} E_i$. Then*
$$\lim_{i\to\infty} \operatorname{cap}(E_i) = \operatorname{cap}(E)\,.$$

(3) Stability with respect to decreasing sequences of compact sets. *Let $K_i \subset \Omega_\infty$, $i = 1, 2, \dots,$ be a decreasing sequence of compact sets and denote $K = \cap_{i=1}^{\infty} K_i$. Then*
$$\lim_{i\to\infty} \operatorname{cap}(K_i) = \operatorname{cap}(K)\,.$$

(4) *Let $U \Subset \Omega_\infty$ be an open set. Then*
$$\operatorname{cap}(U) = \mu_{R_U}(\Omega_\infty)\,.$$

Proof. From the methods developed in [12] we see that (1) and (2) follow from Lemma 2.5. Property (3) is a consequence of Theorem 5.6 and Lemma 5.5. Property (4) follows from (2), Theorem 5.6, and Lemma 2.6 as in Lemma 5.9 in [12]. □

In conclusion we have established more than enough to say that Borel sets are Choquet capacitable:

Theorem 5.8. *The PME capacity is Choquet capacitable* (*inner regular*). *This means that for all Borel sets $E \subset \Omega_\infty$ it holds that*
$$\operatorname{cap}(E) = \sup\{\operatorname{cap}(K) : K \subset E, K \text{ compact}\}\,.$$

Proof. Since the capacity is monotone, stable with respect to increasing sequences of sets (Theorem 5.7 (2)) and stable with respect to decreasing sequences of compact sets (Theorem 5.7 (3)), it is a regular capacity and hence the claim follows from Choquet's capacitability theorem, see Theorem 9.3 on p. 155 in [6]. □

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Benny Avelin: Aalto University, Institute of Mathematics, P.O. Box 11100, 00076 Aalto, Finland; *and* Department of Mathematics, Uppsala University, 751 06 Uppsala, Sweden.
E-mail: benny.avelin@math.uu.se

Teemu Lukkari: Aalto University, Institute of Mathematics, P.O. Box 11100, 00076, Aalto, Finland.
E-mail: teemu.lukkari@aalto.fi

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