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# Riesz transform on manifolds with quadratic curvature decay

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**Abstract.** We investigate the  $L^p$ -boundedness of the Riesz transform on Riemannian manifolds whose Ricci curvature has quadratic decay. Two criterions for the  $L^p$ -unboundedness of the Riesz transform are given. We recover known results about manifolds that are Euclidean or conical at infinity.

## 1. Introduction

Let  $(M^n, g)$  be a complete Riemannian manifold with infinite volume and let  $\Delta$  be its associated Laplacian. The Green formula

$$\forall f \in C_0^\infty(M), \quad \int_M |df|_g^2 \, \text{dvol}_g = \langle \Delta f, f \rangle_{L^2} = \int_M |\Delta^{1/2} f|^2 \, \text{dvol}_g,$$

implies that the Riesz transform

$$R := d\Delta^{-1/2}: L^2(M) \rightarrow L^2(T^*M)$$

is a bounded operator. It is well known [46] that on an Euclidean space, the Riesz transform has a bounded extension  $R: L^p(\mathbb{R}^n) \rightarrow L^p(T^*\mathbb{R}^n)$  for every  $p \in (1, +\infty)$ . In general, it is of interest to figure out the range of  $p$  for which the Riesz transform extends to a bounded operator  $R: L^p(M) \rightarrow L^p(T^*M)$  ([47]). A first remarkable result was obtained by D. Bakry.

**Theorem 1.1** ([4]). *On a manifold with non negative Ricci curvature, the Riesz transform is bounded on  $L^p$ , for all  $p \in (1, \infty)$ .*

Recall that if  $(M, g)$  is a complete Riemannian manifold, its heat kernel is the Schwartz kernel of the heat operator  $e^{-t\Delta}$ :

$$e^{-t\Delta} f(x) = \int_M h_t(x, y) f(y) \, dy.$$

According to a well-known result of P. Li and S-T. Yau ([36]), the non negativity of the Ricci curvature implies that the heat kernel satisfies the upper bound

$$(DUE) \quad \text{for all } t > 0, x, y \in M, \quad h_t(x, y) \leq \frac{C}{\text{vol} B(x, \sqrt{t})},$$

and moreover the Bishop–Gromov inequality implies that the manifold is doubling: there is a constant  $\vartheta$  such that, for any  $x \in M$  and radius  $R > 0$ ,

$$(D) \quad \text{vol} B(x, 2R) \leq \vartheta \text{vol} B(x, R) .$$

Another important result is due to T. Coulhon and X-T. Duong.

**Theorem 1.2** ([14]). *The conditions (DUE) and (D) imply that the Riesz transform is bounded on  $L^p$  for any  $p \in (1, 2]$ .*

Manifolds with conical ends satisfy the above conditions (D) and (DUE), and according to the work of H-Q. Li and C. Guillarmou and A. Hassell, we have a complete understanding of the boundedness of the Riesz transform for these Riemannian manifolds.

**Theorem 1.3** ([35], [27], [28]). *Assume that  $(M^n, g)$  has conical ends: there is a compact set  $K \subset M$  such that  $(M \setminus K, g)$  is isometric to a truncated cone*

$$\mathcal{C}_R(\Sigma) := ((R, \infty) \times \Sigma, (dr)^2 + r^2h) ,$$

where  $(\Sigma, h)$  is a compact Riemannian manifold.

- *If  $\Sigma$  is not connected, then the Riesz transform is bounded on  $L^p$  if and only if  $p \in (1, n) \cup \{2\}$ .*
- *If  $\Sigma$  is connected, let  $\beta(\beta + n - 1)$  with  $\beta > 0$  be the first non zero eigenvalue of the Laplacian on  $(\Sigma, h)$  and  $\alpha = \min\{\beta, 1\}$ . Then the Riesz transform is bounded on  $L^p$  if and only if  $p(1 - \alpha) < n$ .*

In this paper, we study the boundedness of the Riesz transform on Riemannian manifolds whose Ricci curvature satisfies a quadratic decay lower bound, that is to say:

$$(QD) \quad \text{Ricci} \geq -\frac{\kappa^2}{r^2(x)} g ,$$

where  $o \in M$  is a fixed point and  $r(x) := d(o, x)$ .

Manifolds with conical ends satisfy this condition and our results can be applied to prove the above Theorem 1.3 (see subsection 7.1). Moreover, we will also be able to study more general model manifolds (see subsection 7.2).

**Theorem 1.4.** *Let  $(M^n, g)$  be a Riemannian manifold and assume that outside a compact set  $(M, g)$  is isometric to the warped product*

$$([1, \infty) \times \Sigma, (dr)^2 + r^{2\gamma}h) ,$$

where  $(\Sigma, h)$  is a compact manifold with non negative Ricci curvature and  $\gamma \in (0, 1)$ .

- If  $\Sigma$  is connected, then the Riesz transform is bounded on  $L^p$  for every  $p \in (1, +\infty)$ .
- If  $\Sigma$  is not connected, then the Riesz transform is bounded on  $L^p$  if and only if  $1 < p \leq 2$  or  $1 < p < (n - 1)\gamma + 1$ .

Our analysis relies on recent results of A. Grigor’yan and L. Saloff-Coste [26] and V. Minerbe [40], and we will also use some ideas of P. Auscher, T. Coulhon, X-T. Duong and S. Hofmann [3], [2].

We introduce now two conditions:

- the (VC) (volume comparison) condition: for some constant  $C$  and for any  $R \geq 1$  and any  $x \in \partial B(o, R)$ ,

$$(VC) \quad \text{vol } B(o, R) \leq C \text{vol } B(x, R/2) ;$$

- the (RCE) (relatively connected to an end) condition: there is a constant  $\theta \in (0, 1)$  such that for any  $R \geq 1$ , any  $x \in \partial B(o, R)$ , there is a continuous path  $c: [0, 1] \rightarrow B(o, R) \setminus B(o, \theta R)$  and a geodesic ray  $\gamma: [0, +\infty) \rightarrow M \setminus B(o, R)$  satisfying:

1.  $c(0) = x, c(1) = \gamma(0)$ ,
2. the length of  $c$  is not too long:  $\mathcal{L}(c) \leq \theta^{-1} R$ .

The (RCE) condition is an adaption for manifold with several ends of the *relatively connected annuli* (RCA) condition introduced by A. Grigor’yan and L. Saloff-Coste in [26].

According to [26], [40], under the conditions (QD), (VC) and (RCE), we have a good understanding of the behavior of the heat kernel  $\{h_t(x, y)\}$ ; indeed, in this case  $(M, g)$  satisfies the doubling condition (D) and the (DUE) estimates (see the discussion in subsection 2.5). Our first result is the following.

**Theorem A.** *Assume that  $(M^n, g)$  is a complete Riemannian manifold satisfying the conditions (QD), (VC) and (RCE). If some positive constants  $c$  and  $\nu > 2$ , balls anchored at  $o$  satisfy the reverse doubling hypothesis*

$$(RD_\nu) \quad \forall R \geq r \geq 1, \quad c \left(\frac{r}{R}\right)^\nu \text{vol } B(o, r) \leq \text{vol } B(o, R),$$

*then the Riesz transform is bounded on  $L^p$  for any  $p \in (1, \nu)$ .*

According to a beautiful recent result of B. Devyver ([21], Theorem 5.6), our hypothesis on the reverse doubling is equivalent to an isoperimetric inequality for the capacity of anchored balls. Recall that if  $\mathcal{O} \subset M$  is a bounded open subset of a complete Riemannian manifold, then its  $p_0$ -capacity is defined by

$$\text{cap}_{p_0} \mathcal{O} := \inf \left\{ \int_M |d\varphi|^{p_0} \text{dvol}_g, \varphi \in C_0^\infty(M) \text{ and } \varphi \geq 1 \text{ on } \mathcal{O} \right\}.$$

And that  $(M, g)$  is said to be  $p_0$ -hyperbolic if the  $p_0$ -capacity of some/any bounded open subsets is positive. A non- $p_0$ -hyperbolic manifold is called  $p_0$ -parabolic. In fact, a Riemannian manifold  $(M, g)$  is  $p_0$ -parabolic if and only if we can find a sequence of smooth functions with compact support  $(\chi_k)$  such that

$$\begin{cases} 0 \leq \chi_k \leq 1, \\ \lim_{k \rightarrow \infty} \|d\chi_k\|_{L^{p_0}} = 0 \text{ and} \\ \chi_k \rightarrow 1, \text{ uniformly on compact set.} \end{cases}$$

When  $p_0 = 2$ , a 2-hyperbolic manifold is also said to be non-parabolic; moreover a 2-hyperbolic manifold is a manifold carrying a positive Green kernel. A corollary of Theorem A and of B. Devyver’s result is the following.

**Corollary B.** *Assume that  $(M^n, g)$  is a complete Riemannian manifold satisfying the conditions (QD), (VC) and (RCE). If  $(M, g)$  is  $p_0$ -hyperbolic and if the  $p_0$ -capacity of anchored balls satisfy*

$$\frac{\text{vol } B(o, R)}{R^{p_0}} \leq C \text{cap}_{p_0}(B(o, R)),$$

*then the Riesz transform is bounded on  $L^p$  for any  $p \in (1, p_0)$ .*

The proof of Theorem A is based on estimates of the Schwartz kernel of the Riesz transform. Outside the diagonal of  $M \times M$ , this kernel is smooth and given by

$$R(x, y) = \int_0^\infty \nabla_x h_t(x, y) \frac{dt}{\sqrt{\pi t}}.$$

Following the theory of pseudo-differential operators on open manifolds ([39]), which we already used in [8], we separate our analysis in two parts: the part closer to the diagonal  $\{(d(x, y) \leq \kappa r(x))\}$ , where we can use the result of [3], and the off diagonal part, where we get the estimate

$$|R(x, y)| \leq \frac{d(x, y)}{r(x)} \frac{C}{\text{vol } B(o, d(x, y))}.$$

When  $(M, g)$  is a manifold with Euclidean ends, these estimates are sharp when one compares to the results of [8].

We will improve an earlier result of [8] and show that if  $M$  has two ends then there are very general restrictions on the range of  $p$  where the Riesz transform is  $L^p$  bounded.

**Theorem C.** *Let  $p > 2$  and let  $(M, g)$  be a  $p$ -parabolic manifold that is 2-hyperbolic. If the Riesz transform is bounded on  $L^p$  and on  $L^{p/(p-1)}$ , then  $M$  has only one end.*

Remark that in this Theorem C, no assumptions on the curvature or on the heat kernel are done.

For  $q \in (1, 2]$ , Theorem 1.2 provides very general conditions for the  $L^q$  boundedness of the Riesz transform hence, the above criterion is mainly a criterion for the unboundedness of the Riesz transform on  $L^p$  when  $p > 2$ . Moreover, this criterion implies that the gluing result of B. Devyver is optimal ([20]): *assume  $M_1$  and  $M_2$  are two Riemannian manifolds that satisfy a Sobolev inequality and a lower bound on the Ricci curvature. If, on both  $M_1$  and  $M_2$ , the Riesz transform is bounded on  $L^p$  and if the connected sum  $M_1 \# M_2$  is  $p$ -hyperbolic, then the Riesz transform is  $L^p$ -bounded on  $M_1 \# M_2$ .*

We recall now several classical notations:

- if  $B \subset M$  is a ball, we denote  $r(B)$  its radius and for  $\theta > 0$ ,  $\theta B$  is the ball with the same center and radius  $\theta r(B)$ ;
- if  $\mathcal{O} \subset M$  and  $f \in L^1(\mathcal{O})$  we denote  $f_{\mathcal{O}}$  the mean of  $f$  on  $\mathcal{O}$ :

$$f_{\mathcal{O}} = \frac{1}{\text{vol } \mathcal{O}} \int_{\mathcal{O}} f.$$

We recall the following condition about the oscillation of harmonic functions.

**Definition 1.5.** Let  $\alpha \in (0, 1]$ . A complete Riemannian manifold  $(M^n, g)$  is said to satisfy the scale invariant  $\alpha$ -Hölder elliptic  $(HE_{\alpha})$  estimates if there is a constant  $C$  such that for any ball  $B \subset M$  and any harmonic function  $h$  defined on  $3B$ , we have for all  $x, y \in B$ ,

$$|h(x) - h(y)| \leq C \left( \frac{d(x, y)}{r(B)} \right)^{\alpha} \sup_{z \in 2B} |h(z)|.$$

The next result improves Theorem A in the case where the manifold  $M$  has only one end.

**Theorem D.** *Let  $(M^n, g)$  be a complete Riemannian manifold with only one end and assume  $(M^n, g)$  satisfies the conditions (QD), (VC) and (RCE) and the reverse doubling hypothesis  $(RD_{\nu})$  for some exponent  $\nu > 0$ .*

*If  $(M, g)$  satisfies the scale invariant  $\alpha$ -Hölder elliptic estimates, then the Riesz transform is bounded on  $L^p$  for any  $p$  such that  $p > 1$  and  $(1 - \alpha)p < \nu$ .*

Let us explain why this result improves Theorem A when  $M$  has only one end. In the setting of Theorem D, the manifold  $(M, g)$  satisfies the (RCA) condition introduced by A. Grigor'yan and L. Saloff-Coste and we get a scale invariant Poincaré inequality: for any ball  $B \subset M$  and any function  $f \in C^{\infty}(2B)$  we have

$$\|f - f_B\|_{L^2(B)} \leq C r(B) \|df\|_{L^2(2B)}.$$

As  $(M, g)$  satisfies the scale invariant Poincaré inequalities and the doubling condition (D), the parabolic/elliptic Harnack inequalities hold ([23], [42]); in particular there is some  $\varepsilon \in (0, 1]$  such that the property  $(HE_{\varepsilon})$  holds. Hence Theorem D yields that the Riesz transform is bounded on  $L^p$  as soon as  $(1 - \varepsilon)p < \nu$ .

Our proof is based on a result of P. Auscher and T. Coulhon: assume that  $(M, g)$  is a Riemannian manifold satisfying the doubling condition and the scale invariant  $L^1$ -Poincaré inequality. If there is some  $r > p$  such that a  $L^r$ -reverse Hölder inequality holds for the gradient of harmonic functions, then the Riesz transform is bounded on  $L^p$  ([2], see also [44]). Although it was not noticed by the authors, this result and the Cheng–Yau gradient estimate ([11]) provided another proof of Theorem 1.1 of D. Bakry. It should be noted (see Lemma 6.8) that if a manifold  $(M, g)$  carries a non constant sublinear harmonic function  $h$  with

$$h(x) = \mathcal{O}(r^\beta(x)),$$

and if  $\alpha > \beta$ , then  $(M, g)$  can not satisfy the  $\alpha$ -Hölder elliptic estimates. We will show that the existence of such a sublinear harmonic function yields some restrictions on the range of  $p$  where the Riesz transform is  $L^p$ -bounded.

**Proposition E.** *Let  $(M^n, g)$  be a complete Riemannian manifold satisfying the conditions (QD), (VC), (RCE) and the reverse doubling hypothesis  $(RD_\nu)$  for some exponent  $\nu > 0$ . Assume moreover that there are some positive constants  $C$  and  $\mu$ , such that anchored geodesic balls satisfy*

$$\forall R \geq 1, \quad \text{vol} B(o, R) \leq CR^\mu.$$

*If  $(M, g)$  carries a non constant sublinear harmonic function  $h$ ,*

$$h(x) = \mathcal{O}(r^\beta(x)),$$

*then for  $p \geq \mu/(1 - \beta)$  and  $p > \max\{\nu/(\nu - 1), 2\}$ , the Riesz transform can not be bounded on  $L^p$ .*

In [7], we have obtained, as a corollary of Theorem D:

**Corollary F.** *Let  $(M^n, g)$  be a complete Riemannian manifold that satisfies the curvature quadratic decay (QD) condition. If the diameter of geodesic sphere grows slowly,*

$$\text{diam } \partial B(o, R) = \sup_{x, y \in \partial B(o, R)} d(x, y) = o(R),$$

*then the Riesz transform is bounded on  $L^p$  for every  $p \in (1, +\infty)$ .*

In the next section, the condition (RCE) will be introduced and compared to the (RCA) condition of [26]; we will also explain how the analysis of [26], [40] yields the heat kernel estimate (DUE). In the third section, we will explain how the Li and Yau’s gradient estimates for solution of the heat equation imply good estimates for the Schwartz kernel of the Riesz transform. The fourth section is devoted to the proof of Theorem A, and the fifth section to the proof of Theorem D. Negative results about the boundedness of the Riesz transform (Theorem C and Proposition E) are proved in section 6. Theorems 1.3 and 1.4 will be proved in section 7; we also include examples of manifolds with infinite topological type. Eventually we finish with some perspectives.

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## 2. Analysis on manifolds with a quadratic decay of the Ricci curvature

### 2.1. Setting

In this section, we consider a complete Riemannian manifold  $(M^n, g)$  such that for a fixed point  $o \in M$ , the Ricci curvature satisfies

$$(QD) \quad \text{Ricci} \geq -\frac{\kappa^2}{r^2(x)}g,$$

where we have defined  $r(x) = d(o, x)$ . We are going to review geometric conditions that ensure that  $(M, g)$  satisfies the so-called relative Faber–Krahn inequality: there are positive constants  $C, \mu$  such that for any  $x \in M$  and  $R > 0$  and any open domain  $\Omega \subset B(x, R)$ ,

$$\lambda_1^D(\Omega) \geq \frac{C}{R^2} \left( \frac{\text{vol } \Omega}{\text{vol } B(x, R)} \right)^{-2/\mu}.$$

We have denoted  $\lambda_1^D(\Omega)$  the lowest eigenvalue of the Dirichlet Laplacian on  $\Omega$ :

$$\lambda_1^D(\Omega) = \inf_{\varphi \in \mathcal{C}_0^\infty(\Omega)} \frac{\int_\Omega |d\varphi|^2}{\int_\Omega |\varphi|^2}.$$

Our discussion below is based on the results of A. Grigor’yan and L. Saloff-Coste [26], and V. Minerbe [40].

It is well known [24] that the relative Faber–Krahn inequality is equivalent to the conjunction of the following two properties:

- $(M, g)$  is doubling: there is a constant  $\vartheta$  such that, for any  $x \in M$  and radius  $R > 0$ ,

$$\text{vol } B(x, 2R) \leq \vartheta \text{vol } B(x, R).$$

- The heat kernel  $h_t(x, y)$  satisfies the upper bound

$$h_t(x, y) \leq \frac{C}{\text{vol } B(x, \sqrt{t})}$$

for all  $t > 0$  and all  $x, y \in M$ .

According to Proposition 5.2 in [24], the above relative Faber–Krahn inequality implies (for the same exponent  $\mu$ ) that for some constant  $C > 0$ ,

$$\text{for all } x \in M, 0 < r < R, \quad \text{vol } B(x, R) \leq C \left( \frac{R}{r} \right)^\mu \text{vol } B(x, r).$$

**2.2. Remote balls**

A ball  $B(x, \rho) \subset M$  is called remote if its center  $x$  and radius  $\rho$  satisfy

$$\rho \leq \frac{r(x)}{2}.$$

Note that on the remote ball  $B(x, r(x)/2)$  the Ricci curvature satisfies

$$\text{Ricci}_g \geq -\frac{4\kappa^2}{r^2(x)} g_x,$$

hence the Bishop–Gromov comparison theorem implies that all remote balls satisfy the doubling condition: if  $B$  is a remote ball and if  $\theta \in (0, 1)$ , then

$$(2.1) \quad \theta^n \text{vol}(B) \leq C(n, \kappa) \text{vol}(\theta B).$$

Similarly, the condition (QD) implies that remote balls satisfy the Poincaré inequality ([5], inequality (4.5)), and a relative Faber–Krahn inequality (Theorem 2 in [23] or Theorem 3.1 in [41]).

**Lemma 2.1.** *If  $B \subset M$  is a remote ball, then for all  $\varphi \in C^1(B)$ ,*

$$\|\varphi - \varphi_B\|_{L^1(B)}^2 \leq B(n, \kappa) r(B) \|d\varphi\|_{L^1(B)},$$

and for all domain  $\Omega \subset B$ ,

$$\lambda_1^D(\Omega) \geq \frac{C(n, \kappa)}{r(B)^2} \left(\frac{\text{vol } \Omega}{\text{vol } B}\right)^{-2/n}.$$

A ball centered at  $o$  will be called *anchored*.

**2.3. The doubling condition**

According to Proposition 4.7 in [26], this condition is ensured by the doubling of remote balls (2.1) and by the volume comparison (VC) assumption: there is a constant  $C$  such that, for any  $x \in M$ ,

$$(VC) \quad \text{vol } B(o, r(x)) \leq C \text{vol } B(x, r(x)/2).$$

We recall that the doubling condition implies that the volume of balls varies slowly with the center of balls: for any  $\gamma \geq 1$  there is a constant  $C_\gamma$  such that if  $d(x, y) \leq \gamma R$  and  $\gamma^{-1}R \leq r \leq \gamma R$ , then

$$C_\gamma^{-1} \leq \frac{\text{vol } B(x, R)}{\text{vol } B(y, r)} \leq C_\gamma.$$

In particular, if  $R \geq r(x)/\gamma$ , then

$$(2.2) \quad C_\gamma^{-1} \text{vol } B(o, R) \leq \text{vol } B(x, R) \leq C_\gamma \text{vol } B(o, R).$$

Finally, it is well known (see Lemma 2.10 in [26]) that for a connected and non-compact manifold, the doubling condition implies a reverse doubling condition:



there is a constant  $\delta > 0$ , depending only on the doubling constant  $\vartheta$ , such that, for all  $x \in M$  and all  $R > r$ ,

$$\delta \left(\frac{R}{r}\right)^\delta \leq \frac{\text{vol } B(x, R)}{\text{vol } B(x, r)}.$$

**2.4. Geometry of annuli**

**2.4.1. Number of ends.** We remark that the doubling condition implies that  $(M, g)$  has a finite number of ends, i.e., there is an integer  $N$  such that for any  $R$ ,  $M \setminus B(o, R)$  has at most  $N$  unbounded connected components. Indeed if  $\mathcal{O} \subset M \setminus B(o, R)$  is an unbounded connected component, there is a point  $x_{\mathcal{O}} \in \mathcal{O} \cap \partial B(o, 2R)$ , we have the inclusions  $B(x_{\mathcal{O}}, R) \subset \mathcal{O}$  and  $B(x_{\mathcal{O}}, R) \subset B(o, 3R)$ , hence we get

$$\sum_{\mathcal{O}} \text{vol } B(x_{\mathcal{O}}, R) \leq \text{vol } B(o, 3R).$$

However, using the doubling condition we get

$$\text{vol } B(o, 3R) \leq \text{vol } B(x_{\mathcal{O}}, 5R) \leq \vartheta^3 \text{vol } B(x_{\mathcal{O}}, R).$$

Hence  $M \setminus B(o, R)$  has at most  $\vartheta^3$  unbounded connected components.

A slight variation of this argument shows that for any  $\lambda > 1$ , the annulus  $A_{\lambda, R} = B(o, \lambda R) \setminus B(o, R)$  has at most  $N(\lambda, \vartheta)$  connected components that intersects  $\partial B(o, \lambda R)$ . However, these connected components do not necessary intersect an unbounded connected component of  $M \setminus B(o, R)$ .

**2.4.2. The (RCE) condition.** In [26], A. Grigor'yan and L. Saloff-Coste have introduced the *relatively connected annuli* (RCA) condition.

**Definition 2.2.** A manifold  $(M, g)$  is said to satisfy the (RCA) condition if there is a point  $o \in M$  and a constant  $\theta \in (0, 1)$  such that for any  $R > 0$  and any points  $x, y \in \partial B(o, R)$  there is a  $\mathcal{C}^1$  path  $c: [0, 1] \rightarrow M$  satisfying:

- $c(0) = x, c(1) = y,$
- $\mathcal{L}(c) \leq r(x)/\theta,$
- $c([0, 1]) \subset B(o, \theta^{-1}R) \setminus B(o, \theta R).$

It is easy to show that the (RCA) condition implies that  $M$  has only one end, i.e., for any compact set  $K \subset M$ ,  $M \setminus K$  has only one unbounded connected component. The (RCE) condition is an adaptation of the (RCA) condition for manifolds with several ends.

**Definition 2.3.** We say that a complete Riemannian manifold  $(M, g)$  with a finite number of ends satisfies the *relatively connected to an end* (RCE) condition if there is a constant  $\theta \in (0, 1)$  such that for any point  $x$  with  $r(x) \geq 1$  there is a continuous path  $c: [0, 1] \rightarrow M$  satisfying:

- $c(0) = x,$
- the length of  $c$  is bounded by  $\mathcal{L}(c) \leq r(x)/\theta,$

- $c([0, 1]) \subset B(o, \theta^{-1}r(x)) \setminus B(o, \theta r(x))$ ,
- There is a geodesic ray  $\gamma : [1, +\infty) \rightarrow M \setminus B(o, r(x))$  with  $\gamma(0) = c(1)$ .

The (RCE) condition implies that any point can be connected to an end by a path that stays at bounded distance away from the origin. It is easy to see that if  $M$  has only one end, the (RCE) condition is just the (RCA) condition of A. Grigor’yan and L. Saloff-Coste.

If  $M$  has a finite number of ends, then there is a finite number of geodesic rays  $c_1, \dots, c_r : [0, +\infty) \rightarrow M$  with  $c_i(0) = o$  such that for every  $R \gg 1$ ,  $M \setminus B(o, R)$  has exactly  $r$ -unbounded connected component  $\mathcal{O}_1, \dots, \mathcal{O}_r$ , and for all  $i$ ,

$$c_i((R, +\infty)) \subset \mathcal{O}_i .$$

In this setting, we could replace the last condition in Definition 2.3 by

- there is some  $i \in \{1, \dots, r\}$  such that  $c(1) = c_i(r(x))$ .

### 2.5. Relative Faber–Krahn inequality

The results of A. Grigor’yan and L. Saloff-Coste and of V. Minerbe imply:

**Theorem 2.4.** *Assume that  $(M^n, g)$  is a complete Riemannian manifold satisfying the conditions (QD), (VC) and (RCE). Then  $(M, g)$  satisfies the relative Faber–Krahn inequality: for some  $\mu > 0$  and  $C > 0$  and for any ball  $B \subset M$  and any domain  $\Omega \subset B$ ,*

$$\lambda_1^D(\Omega) \geq \frac{C}{r(B)^2} \left( \frac{\text{vol } \Omega}{\text{vol } B} \right)^{-2/\mu} .$$

When  $M$  has only one end, then  $(M, g)$  satisfies the scale invariant  $L^1$  Poincaré inequality: for any ball  $B$  and any function  $f \in C^\infty(2B)$ ,

$$\|f - f_B\|_{L^1(B)} \leq C r(B) \|df\|_{L^1(2B)} .$$

The second assertion is one of the main result of [26] (Theorem 5.2) – a priori, the article deals with the scale invariant  $L^2$  Poincaré inequality, but the argument carries over the case of any  $L^p$  Poincaré inequality. Stricto sensu, the first assertion can not be found in the paper of V. Minerbe; however a quick glimpse on the argumentation shows that the limitation on the exponent  $\nu > 1$  in the reverse doubling condition

$$(RD_\nu) \quad \forall R > r, \quad \text{vol } B(o, R) \geq \varepsilon \left( \frac{R}{r} \right)^\nu \text{vol } B(o, r)$$

is made only to ensure the (RCA) condition. Under the assumptions of the Theorem 2.4, the proof of Theorem 2.19 in [40] implies that for any  $p \geq n$  with  $p > 2$  there is a constant  $C$  such that the weighted Sobolev inequality holds:

$$\forall f \in C_0^\infty(M), \quad \left( \int_M |f(x)|^{2p/(p-2)} dx \right)^{1-2/p} \leq \int_M \frac{Cr(x)^2}{(\text{vol } B(o, r(x)))^{2/p}} |df|^2(x) dx.$$

Let us explain how this inequality implies the relative Faber–Krahn inequality for anchored balls, i.e., balls centered at  $o$ : the doubling condition yields a constant  $\mu$  such that

$$\forall R > r, \quad \text{vol } B(o, R) \leq C \left(\frac{R}{r}\right)^\mu \text{vol } B(o, r).$$

Note that if this inequality is true for some  $\mu$ , then it holds for any  $\mu' \geq \mu$ . Moreover, looking at the limit  $r \rightarrow 0$ , we see that  $\mu \geq n$  and in the following we will assume that  $\mu > 2$ .

In particular, using the above Sobolev inequality for  $p = \mu$ , we get that for any function  $f \in C_0^\infty(B(o, R))$ ,

$$\begin{aligned} \left(\int_{B(o, R)} |f(x)|^{2\mu/(\mu-2)} dx\right)^{1-2/\mu} &\leq C \int_{B(o, R)} \frac{r(x)^2}{(\text{vol } B(o, r(x)))^{-2/\mu}} |df|^2(x) dx \\ &\leq C \frac{R^2}{(\text{vol } B(o, R))^{-2/\mu}} \int_{B(o, R)} |df|^2(x) dx. \end{aligned}$$

Then, with the Hölder inequality, we get for any domain  $\Omega \subset B(o, R)$ ,

$$1 \leq C \frac{R^2}{(\text{vol } B(o, R))^{-2/\mu}} (\text{vol } \Omega)^{-2/\mu} \lambda_1^D(\Omega).$$

It is now easy to show that the relative Faber–Krahn inequality holds for all balls. Indeed it remains to show the relative Faber–Krahn inequality for balls  $B(x, \rho)$  with  $\rho \geq r(x)/2$ . But such a ball satisfies

$$B(x, \rho) \subset B(o, \rho + r(x)) \subset B(o, 3\rho) \quad \text{and} \quad B(o, 3\rho) \subset B(x, 5\rho).$$

Hence, if  $\Omega \subset B(x, \rho)$ , then

$$\begin{aligned} \lambda_1^D(\Omega) &\geq \frac{C}{(3\rho)^2} \left(\frac{\text{vol } B(o, 3\rho)}{\text{vol } \Omega}\right)^{-2/\mu} \geq \frac{C}{(3\rho)^2} \left(\frac{\vartheta^{-2} \text{vol } B(o, 5\rho)}{\text{vol } \Omega}\right)^{-2/\mu} \\ &\geq \frac{C}{(3\rho)^2} \left(\frac{\vartheta^{-2} \text{vol } B(x, 3\rho)}{\text{vol } \Omega}\right)^{-2/\mu} \geq \frac{C}{(3\rho)^2} \left(\frac{\vartheta^{-4} \text{vol } B(x, \rho)}{\text{vol } \Omega}\right)^{-2/\mu}. \end{aligned}$$

In fact, a remarkable result of V. Minerbe (Proposition 2.8 in [40]; see also Proposition 4.5 in [31] for an earlier result) shows that the (RCA) condition is ensured by an anchored Poincaré inequality and a reverse doubling condition:

**Theorem 2.5.** *Assume that  $(M, g)$  is a complete Riemannian manifold that is doubling and such that balls  $B = B(o, R)$  centered at  $o$  satisfy the Poincaré inequalities*

$$\forall f \in C^\infty(2B), \quad \|f - f_B\|_{L^p(B)} \leq CR \|df\|_{L^p(2B)}.$$

*If for positive constants  $C$  and  $\nu > p$ , we have the reverse doubling condition*

$$\forall R > r, \quad \text{vol } B(o, R) \geq C \left(\frac{R}{r}\right)^\nu \text{vol } B(o, r),$$

*then  $(M, g)$  satisfies the (RCA) condition.*

### 3. Estimates on the Riesz kernel

#### 3.1. Assumptions

In this section, we assume that  $(M^n, g)$  is a complete Riemannian manifold with a based point  $o \in M$  satisfying the following conditions:

- i) A quadratic decay on the negative part of the Ricci curvature

$$\text{Ricci}_g \geq -\frac{\kappa^2}{r^2(x)} g.$$

- ii) There are positive constants  $\mu$  and  $C$ , such that the relative Faber–Krahn inequality holds: for any ball  $B \subset M$  and any domain  $\Omega \subset B$ ,

$$\lambda_1^D(\Omega) \geq \frac{C}{r(B)^2} \left( \frac{\text{vol} \Omega}{\text{vol} B} \right)^{-2/\mu}.$$

- iii) For some positive constants  $c$  and  $\nu > 2$ , we have the reverse doubling condition  $(\text{RD}_\nu)$  for anchored balls:

$$\forall R > r, \quad c \left( \frac{R}{r} \right)^\nu \text{vol} B(o, r) \leq \text{vol} B(o, R).$$

**Remark 3.1.** The limitation  $\nu > 2$  is not essential, we can handle the case where  $\nu > 1$  as well, but in the case  $\nu \in (1, 2]$  the estimate on the Riesz kernel is more complicated and the conclusion of the main theorem is interesting only when  $\nu > 2$ . Indeed the relative Faber–Krahn hypothesis implies that if  $p \in (1, 2]$ , then the Riesz transform is bounded on all  $L^p$  (Theorem 1.2 or [14]).

#### 3.2. Li and Yau’s inequality

When  $B \subset M$  is a remote ball, then on  $\frac{3}{2}B$  the Ricci curvature is bounded from below by  $-16\kappa^2 r(B)^{-2}$ , so that according to P. Li and S-T. Yau’s Harnack inequality ([36], Theorem 2.1), there is a constant  $c(n, \kappa)$  such that for any positive solution of the heat equation  $u: [0, 2T] \times \frac{3}{2}B \rightarrow \mathbb{R}_+^*$ , we have on  $[T, 2T] \times B$ ,

$$(3.1) \quad \frac{|\nabla u|^2}{u^2} - 2 \frac{1}{u} \frac{\partial u}{\partial t} \leq c(n, \kappa) \left( \frac{1}{T} + \frac{1}{r^2(B)} \right).$$

#### 3.3. Spatial derivative of the heat kernel

According to A. Grigor’yan [24], [25], in our setting the heat kernel satisfies the following Gaussian upper bound: for all  $t > 0$  and  $x, y \in M$ , we have

$$h_t(x, y) \leq \frac{C}{\text{vol} B(x, \sqrt{t})} e^{-d^2(x, y)/(ct)};$$

$$\left| \frac{\partial}{\partial t} h_t(x, y) \right| \leq \frac{C}{\sqrt{t} \text{vol} B(x, \sqrt{t})} e^{-d^2(x, y)/(ct)}.$$

Let  $t > 0$  and  $x, y \in M$ ; on the parabolic ball  $[0, t/2] \times B(x, \sqrt{t})$ , the function  $u(s, z) := h_{t/2+s}(z, y)$  satisfies

$$(3.2) \quad u(s, z) + \sqrt{t} \left| \frac{\partial}{\partial s} u(s, z) \right| \leq \frac{C}{\text{vol } B(x, \sqrt{t})} e^{-d^2(x,y)/(ct)}.$$

Introduce now  $\rho = \min\{\sqrt{t}, r(x)/2\}$ ; the fact that the ball  $B(o, \rho)$  is remote and Li and Yau’s above estimate (3.1) yield the following:

$$|\nabla u|^2(t/2, x) \leq 2u(t/2, x) \frac{\partial u}{\partial t}(t/2, x) + C \left( \frac{1}{t} + \frac{1}{\rho^2} \right) u^2(t/2, x)$$

and with the estimate (3.2), we get

$$|\nabla_x h_t(x, y)| \leq \left( \frac{1}{\sqrt{t}} + \frac{1}{r(x)} \right) \frac{C}{\text{vol } B(x, \sqrt{t})} e^{-d^2(x,y)/(ct)}.$$

### 3.4. Application to the Schwartz kernel of the Riesz transform

Recall that the Riesz transform is the operator

$$R = d\Delta^{-1/2}: L^2(M) \rightarrow L^2(T^*M).$$

Its Schwartz kernel is smooth on  $M \times M \setminus \text{Diag}$ : if  $x \neq y \in M$ , then  $R(x, y) \in T_x^*M$  is given by

$$R(x, y) = \int_0^{+\infty} \nabla_x h_t(x, y) \frac{dt}{\sqrt{\pi t}}.$$

Let  $\kappa \geq 4$ . We are going to estimate  $|R(x, y)|$  in three different regimes:

- i) First regime:  $d(x, y) \geq \frac{1}{\kappa}r(x)$  and  $\frac{1}{\kappa}r(x) \leq r(y) \leq \kappa r(x)$ .
- ii) The short to long range regime:  $r(x) \geq \kappa r(y)$ .
- iii) The long to short range regime:  $r(y) \geq \kappa r(x)$ .

**3.4.1. First and second regime.** In these regimes, we have  $r(x) \simeq d(x, y)$ , and hence

$$|R(x, y)| \leq C \left[ \int_0^{r^2(x)} \frac{e^{-r^2(x)/(ct)}}{\text{vol } B(x, \sqrt{t})} \frac{dt}{t} + \int_{r^2(x)}^{+\infty} \frac{e^{-r^2(x)/(ct)}}{r(x) \text{vol } B(x, \sqrt{t})} \frac{dt}{\sqrt{t}} \right].$$

Using the doubling assumption, the first integral is bounded by

$$C \frac{r(x)^\mu}{\text{vol } B(x, r(x))} \int_0^{r^2(x)} \frac{e^{-r^2(x)/(ct)}}{t^{\mu/2+1}} dt \leq \frac{C}{\text{vol } B(x, r(x))} \leq \frac{C}{\text{vol } B(o, r(x))}.$$

For the second integral, we use the fact that if  $\sqrt{t} \geq r(x)$  then

$$\text{vol } B(x, \sqrt{t}) \geq C \text{vol } B(o, \sqrt{t}) \geq C \left( \frac{\sqrt{t}}{r(x)} \right)^\nu \text{vol } B(o, r(x)),$$

and because  $\nu > 1$ , we obtain

$$\int_{r^2(x)}^{+\infty} \frac{e^{-r^2(x)/(ct)}}{r(x) \operatorname{vol} B(x, \sqrt{t})} \frac{dt}{\sqrt{t}} \leq C \frac{r^{\nu-1}(x)}{\operatorname{vol} B(x, r(x))} \int_{r^2(x)}^{+\infty} \frac{e^{-r^2(x)/(ct)}}{t^{(\nu+1)/2}} dt \leq \frac{C}{\operatorname{vol} B(o, r(x))}.$$

**3.4.2. The long to short range regime.** In this regime we have  $r(y) \simeq d(x, y)$ , hence we have

$$|R(x, y)| \leq C \int_0^{r^2(x)} \frac{e^{-r^2(y)/(ct)}}{\operatorname{vol} B(x, \sqrt{t})} \frac{dt}{t} + C \int_{r^2(x)}^{r^2(y)} \frac{e^{-r^2(y)/(ct)}}{r(x) \operatorname{vol} B(x, \sqrt{t})} \frac{dt}{\sqrt{t}} + C \int_{r^2(y)}^{+\infty} \frac{e^{-r^2(y)/(ct)}}{r(x) \operatorname{vol} B(x, \sqrt{t})} \frac{dt}{\sqrt{t}}.$$

Using the same techniques, we get

$$\int_0^{r^2(x)} \frac{e^{-r^2(y)/(ct)}}{\operatorname{vol} B(x, \sqrt{t})} \frac{dt}{t} \leq C \frac{r(y)^\mu}{\operatorname{vol} B(x, r(y))} \int_0^{r^2(x)} \frac{e^{-r^2(y)/(ct)}}{t^{\mu/2+1}} dt \leq \frac{C}{\operatorname{vol} B(o, r(y))}$$

and

$$\int_{r^2(x)}^{r^2(y)} \frac{e^{-r^2(y)/(ct)}}{r(x) \operatorname{vol} B(x, \sqrt{t})} \frac{dt}{\sqrt{t}} \leq \frac{r(y)}{r(x)} \frac{C}{\operatorname{vol} B(o, r(y))}.$$

Similarly, the reverse doubling hypothesis  $(RD_\nu)$  and the fact that  $\nu > 2$  yield

$$\int_{r^2(y)}^{+\infty} \frac{e^{-r^2(y)/(ct)}}{r(x) \operatorname{vol} B(x, \sqrt{t})} \frac{dt}{\sqrt{t}} \leq \frac{r(y)}{r(x)} \frac{C}{\operatorname{vol} B(o, r(y))}.$$

As a conclusion, we have obtained:

**Lemma 3.2.** *There is a positive constant  $C$  such that:*

- if  $x, y \in M$  satisfy  $d(x, y) \geq \frac{1}{\kappa}r(x)$  and  $\frac{1}{\kappa}r(x) \leq r(y) \leq \kappa r(x)$  or  $r(x) \geq \kappa r(y)$ , then

$$|R(x, y)| \leq \frac{C}{\operatorname{vol} B(o, r(x))},$$

- if  $x, y \in M$  satisfy  $r(y) \geq \kappa r(x)$ , then

$$|R(x, y)| \leq \frac{r(y)}{r(x)} \frac{C}{\operatorname{vol} B(o, r(y))}.$$

### 4. Boundedness of the Riesz transform

#### 4.1. Decomposition of the Riesz transform

In this section, Theorem A will be proved. Hence, we consider a complete Riemannian manifold  $(M, g)$  satisfying the hypotheses of this theorem.

When  $f \in C_0^\infty(M)$ , we decompose  $R(f)$  in three parts:

$$R(f) = R_d(f) + R_0(f) + R_1(f),$$

where the Schwartz kernels of  $R_0$  and  $R_1$  are locally bounded and given by the restriction of the Schwartz kernel of  $R$  to the sets

$$\begin{aligned} \Omega_0 &:= \{(x, y) \in M \times M, d(x, y) \geq \kappa^{-1}r(x) \text{ and } \kappa r(x) \geq r(y)\}, \\ \Omega_1 &:= \{(x, y) \in M \times M, \kappa r(x) \leq r(y)\}. \end{aligned}$$

That is to say for  $\alpha \in C_0^\infty(T^*M)$  and  $f \in C_0^\infty(M)$ ,

$$\langle \alpha, R_0(f) \rangle_{L^2} = \int_{\Omega_0} \langle \alpha(x), R(x, y) \rangle_g f(y) dy dx,$$

and similarly for  $R_1$ . Note that if  $\kappa r(x) \leq r(y)$  then, recalling  $\kappa \geq 4$ , we get

$$d(x, y) \geq r(y) - r(x) \geq (\kappa - 1)r(x) \geq \kappa^{-1}r(x).$$

Hence on  $\Omega_1$  we are far away from the diagonal. The Schwartz kernel of the Riesz transform has a singularity along the diagonal of  $M \times M$ , and  $R_d$  is the restriction the kernel of the Riesz transform to the following neighborhood of the diagonal:

$$\mathcal{V}(\text{Diag}) := \{(x, y) \in M \times M, d(x, y) \leq \kappa^{-1}r(x)\}.$$

#### 4.2. The short to long range part

This part is now relatively easy to handle.

**Proposition 4.1.** *The operator  $R_0$  is bounded  $L^\infty(M) \rightarrow L^\infty(T^*M)$  and  $L^1 \rightarrow L^1_w$ : that is to say for any  $f \in L^1$  and any  $\lambda > 0$ , we have*

$$\text{vol}\{x \in M, |R_0(f)(x)| > \lambda\} \leq \frac{C}{\lambda} \|f\|_{L^1}.$$

*In particular, by interpolation,  $R_0: L^p(M) \rightarrow L^p(T^*M)$  is bounded for any  $p \in (1, +\infty)$ .*

*Proof.* As a matter of fact, our previous analysis (Lemma 3.2) implies that if  $f \in C_0^\infty(M)$  then

$$|R_0(f)(x)| \leq \frac{C}{\text{vol} B(o, r(x))} \int_{B(o, \kappa r(x))} |f(y)| dy,$$

hence the boundedness  $L^\infty(M) \rightarrow L^\infty(T^*M)$  is a direct consequence of the doubling property. Moreover, this also implies that

$$\{x \in M, |R_0(f)(x)| > \lambda\} \subset B(o, \rho),$$

where  $\rho$  satisfies

$$\text{vol } B(o, \rho) < \frac{C}{\lambda} \|f\|_{L^1},$$

and the boundedness  $L^1 \rightarrow L^1_w$  follows immediately. □

### 4.3. The diagonal part

In this part, we are going to use an idea from [14], section 4, and a result from [3], section 4, in order to prove the following.

**Proposition 4.2.** *The operator  $R_d$  is bounded on  $L^p$  for every  $p \in (2, +\infty)$ .*

*Proof of Proposition 4.2.* We build a cover on  $M$  by remote balls. By induction on  $N \in \mathbb{N}$ :

- $B_{0,1} = B(o, 1)$ .
- We cover  $B(o, 2^N) \setminus \cup_{i < N, j} B_{i,j}$  by a collection of balls  $B_{N,1}, \dots, B_{N,k_N}$  of radius  $2^{N-10}$  that are centered on  $B(o, 2^N) \setminus B(o, 2^{N-1})$  and such that the balls  $\frac{1}{2}B_{N,1}, \dots, \frac{1}{2}B_{N,k_N}$  are disjoint and included in  $B(o, 2^N) \setminus \cup_{i < N, j} B_{i,j}$ .

At each stage  $N$ , the number of balls is bounded independently of  $N$ :

$$k_N \leq m(\vartheta).$$

We obtain in this way a subset  $A \subset \mathbb{N}^2$  and a cover

$$M = \cup_{\alpha \in A} B_\alpha$$

by balls  $B_\alpha = B(x_\alpha, r_\alpha)$ . Note that we have, by construction, that  $2^{-10}r(x_\alpha) \leq r_\alpha \leq 2^{-9}r(x_\alpha)$ . Moreover this cover has a finite multiplicity: there is a constant  $p$  such that for any  $x \in M$ ,

$$\text{card}\{\alpha \in A, x \in B_\alpha\} \leq p.$$

Let  $\chi_\alpha$  be a partition of unity subordinated to this covering; we can assume  $|d\chi_\alpha| \leq C/r_\alpha$ . If  $\kappa$  is chosen large enough ( $\kappa \geq 2^{10}$ ), then we have

$$|R_d(f)(x)| \leq \sum_{\alpha} |\mathbf{1}_{4B_\alpha}(x) R(\chi_\alpha f)(x)|.$$

Let  $R_\alpha = \mathbf{1}_{4B_\alpha} R\chi_\alpha$ . We decompose

$$R_\alpha = R_{\alpha,0} + R_{\alpha,1},$$



where

$$R_{\alpha,0}(f)(x) = \mathbf{1}_{4B_\alpha}(x) \int_0^{r_\alpha^2} \nabla_x e^{-\tau\Delta}(\chi_\alpha f)(x) \frac{d\tau}{\sqrt{\pi\tau}}$$

and

$$R_{\alpha,1}(f)(x) = \mathbf{1}_{4B_\alpha}(x) \int_{r_\alpha^2}^\infty \nabla_x e^{-\tau\Delta}(\chi_\alpha f)(x) \frac{d\tau}{\sqrt{\pi\tau}}.$$

The covering  $M = \cup_{\alpha \in A} B_\alpha$  has finite multiplicity and  $(M, g)$  is doubling, hence we only need to prove that there is a uniform constant  $C$  such that for all  $\alpha$ ,

$$\|R_{\alpha,0}\|_{L^p \rightarrow L^p} \leq C, \quad \|R_{\alpha,1}\|_{L^p \rightarrow L^p} \leq C.$$

**Lemma 4.3.** *There is a constant  $C$ , independent of  $\alpha$ , such that*

$$\|R_{\alpha,0}\|_{L^p \rightarrow L^p} \leq C.$$

*Proof.* We will use the arguments of [3], subsection 3.2 and section 4, together with the following estimates on the gradient of the heat kernel: for all  $x, y \in M$  and all  $t \in (0, Ar^2(x))$ ,

$$(4.1) \quad |\nabla_x h_t(x, y)| \leq \frac{C}{\sqrt{t} \operatorname{vol} B(x, \sqrt{t})} e^{-d^2(x,y)/(ct)}.$$

We will apply Theorem 2.4 of [3]. The setting is the following:

- $(M, g)$  is a complete Riemannian manifold.
- $T: L^2(M) \rightarrow L^2(M)$  is a bounded sublinear operator.
- $\{A_r\}_{r>0}$  is a family of bounded operator on  $L^2$ :

$$\sup_{r>0} \|A_r\|_{L^2 \rightarrow L^2} < \infty.$$

- $U \subset \Omega \subset M$  are two open subsets such that  $\Omega$  satisfies the relative doubling condition: there is a constant  $\bar{\vartheta}$  such that for all balls  $B \subset M$ ,

$$\operatorname{vol}(2B \cap \Omega) \leq \bar{\vartheta} \operatorname{vol}(B \cap \Omega).$$

- $S: L^p(U) \rightarrow L^p(\Omega)$  is a bounded operator for all  $p > 2$ .

The assumptions are

- i) For all  $p > 2$ , the sublinear operator  $\mathcal{M}^\#$  defined by

$$\mathcal{M}^\#(f)(x) = \sup_{B, B \cap \Omega \ni x} \frac{1}{\operatorname{vol}(\Omega \cap B)} \int_{B \cap \Omega} |T(\operatorname{Id} - A_{r(B)})(f)|^2$$

is bounded on  $L^p$ .

- ii) For all  $f \in L^p(U)$  and all balls  $B \subset M$  and  $x, y \in \Omega \cap B$ ,

$$|TA_{r(B)}(f)|^2(y) \leq C \mathcal{M}_\Omega (|T(f)|^2 + |S(f)|^2)(x),$$

where we have denoted  $\mathcal{M}_\Omega$  the maximal operator relative to  $\Omega$ :

$$\mathcal{M}_\Omega(f)(x) = \sup_{B, B \cap \Omega \ni x} \frac{1}{\text{vol}(\Omega \cap B)} \int_{B \cap \Omega} |f|.$$

The conclusion is that the operator

$$T: L^p(U) \rightarrow L^p(\Omega)$$

is bounded and there is an upper bound on operator norm of  $T: L^p \rightarrow L^p$  that depends only on the constants involved in the setting and the hypothesis.

The following result can be deduced from Proposition 2.4 in [22], or [14], page 1159.

**Lemma 4.4.** *If  $(M, g)$  is doubling, then*

$$\int_{M \setminus B(x,r)} e^{-d^2(x,y)/(ct)} |f(y)| dy \leq C \text{vol} B(x, \sqrt{t}) e^{-r^2/(2ct)} \mathcal{M}(f)(x),$$

where

$$\mathcal{M}(f)(x) = \sup_{B, B \ni x} \frac{1}{\text{vol}(B)} \int_B |f|$$

is the maximal operator associated to  $(M, g)$ .

The next lemma is folklore, and a proof can be found in [9], Proposition 4.14, or [34], Lemma 2.3.1.

**Lemma 4.5.** *If  $B(x, r) \subset M$  is a remote ball and if  $M$  satisfies the (QD) condition, then for all  $f \in C^1(B(x, r))$ ,*

$$|f(x) - f_{B(x,r)}| \leq Cr \sup_{0 < s \leq r} \frac{1}{\text{vol}(B(x, s))} \int_{B(x,s)} |df|.$$

Using Lemma 4.4, the gradient estimate (4.1) and the argumentation of [3], subsection 3.2, we easily get

$$\mathcal{M}^\#(f)(x) \leq C \sqrt{\mathcal{M}_\Omega(|f|^2)(x)}.$$

Moreover, if  $f \in C^1_0(M)$ , if  $B$  is a remote ball of radius  $r$  and if  $x, y \in B$ , then the Lemma 4.5 yields

$$(4.2) \quad |\nabla e^{-r^2\Delta} f|(y) \leq C \sup_{0 < s \leq r} \frac{1}{\text{vol}(B(x, s))} \int_{B(x,s)} |df| + \frac{C}{r} \mathcal{M}(|f|)(x).$$

We will use Theorem 2.4 in [3] with  $U = B_\alpha$ ,  $\Omega = 4B_\alpha$ ,  $A_r = e^{-r^2\Delta}$  and

$$T(f)(x) = \left| \int_0^{r_\alpha^2} (de^{-\tau\Delta} f)(x) \frac{d\tau}{\sqrt{\pi\tau}} \right|.$$

Let  $v \in L^p(U)$ . If we apply inequality (4.2) to

$$f = \mathcal{S}(v) = \int_0^{r_\alpha^2} e^{-\tau\Delta} v \frac{d\tau}{\sqrt{\pi\tau}},$$

we get that for all balls  $B$  and all  $x, y \in B \cap \Omega$ ,

$$|TA_{r(B)}v|(y) \leq C \sup_{0 < s \leq r_\alpha} \frac{1}{\text{vol}(B(x, s))} \int_{B(x, s)} |T(v)| + \frac{C}{r_\alpha} \mathcal{M}(\mathcal{S}(v))(x).$$

The estimate of the gradient of the heat kernel implies that

$$\int_{B(x, s)} |\mathcal{S}(v)| + \int_{B(x, s)} |T(v)| \leq \int_{B(x, s) \cap \Omega} |T(v)| + C \frac{\text{vol}(B(x, s) \setminus \Omega)}{\text{vol}U} \int_U |v|.$$

Hence we get

$$|TA_{r(B)}v|(y) \leq C \mathcal{M}_\Omega(T(v))(x) + \frac{C}{r_\alpha} \mathcal{M}_\Omega(\mathcal{S}(v))(x) + c \mathcal{M}_\Omega(v)(x).$$

The fact that

$$\|\mathcal{S}\|_{L^p \rightarrow L^p} \leq C r_\alpha$$

yields the uniform estimate

$$\|R_{\alpha,0}\|_{L^p \rightarrow L^p} \leq \|T\|_{L^p(U) \rightarrow L^p(\Omega)} \leq C. \quad \square$$

**Lemma 4.6.** *There is a constant  $C$ , independent of  $\alpha$ , such that*

$$\|R_{\alpha,1}\|_{L^p \rightarrow L^p} \leq C.$$

*Proof.* We use the fact that, for all  $y \in B_\alpha$ ,  $x \in 4B_\alpha$  and  $t \geq r_\alpha^2$ ,

$$|\nabla_x h_t(x, y)| \leq \frac{C}{r_\alpha \text{vol}B(x, \sqrt{t})} e^{-d^2(x,y)/(ct)},$$

so that the Schwartz kernel of  $R_{\alpha,1}$  is bounded by

$$C \mathbf{1}_{4B_\alpha}(x) \mathbf{1}_{B_\alpha}(y) \int_{r_\alpha}^\infty \frac{e^{-d^2(x,y)/(ct)}}{r_\alpha \text{vol}B(x, \sqrt{t})} e^{-d^2(x,y)/(ct)} \frac{dt}{\sqrt{t}}.$$

Using the slow variation of the volume of balls, we get that for all  $x \in 4B_\alpha$  and all  $t \geq r_\alpha^2$ ,  $\text{vol}B(x, \sqrt{t}) \simeq \text{vol}B(o, \sqrt{t})$ . And with the reverse doubling condition  $(RD_\nu)$ , we obtain

$$|T_{\alpha,1}(f)(x)| \leq \frac{C \mathbf{1}_{4B_\alpha}(x)}{\text{vol}B_\alpha} \int_{B_\alpha} |f|(y) dy.$$

Hence  $T_{\alpha,1}$  is bounded on  $L^1$  and on  $L^\infty$  with an operator norm bounded independently of  $\alpha$ . □

Lemma 4.3 and Lemma 4.6 imply Proposition 4.2. □

**4.4. The long to short range part**

This is the most significant part; according to our previous analysis, the  $L^p$  boundedness of the Riesz transform is equivalent to the  $L^p$  boundedness of  $R_1$ . And with Lemma 3.2, it is sufficient to find conditions under which the operator

$$T(f)(x) := \frac{1}{r(x)} \int_{M \setminus B(o, \kappa r(x))} \frac{r(y)}{\text{vol} B(o, r(y))} |f(y)| dy$$

is bounded  $L^p \rightarrow L^p_w$ .

When  $f \in L^p(M)$ , we have

$$|T(f)(x)| \leq M_p(x) \|f\|_{L^p},$$

where

$$M_p(x) = \frac{1}{r(x)} \left( \int_{M \setminus B(o, \kappa r(x))} \left( \frac{r(y)}{\text{vol} B(o, r(y))} \right)^{p/(p-1)} dy \right)^{1-1/p}.$$

If we introduce the Riemann–Stieltjes measure associated to the non-decreasing function  $V(r) = \text{vol} B(o, r)$ , we get (by integrating by parts)

$$\begin{aligned} \int_{M \setminus B(o, R)} \left( \frac{r(y)}{\text{vol} B(o, r(y))} \right)^{p/(p-1)} dy &= \int_R^\infty \left( \frac{r}{\text{vol} B(o, r)} \right)^{p/(p-1)} dV(r) \\ &= (p-1) \frac{R^{p/(p-1)}}{V(R)^{1/(p-1)}} + p \int_R^\infty \frac{r^{1/(p-1)}}{V(r)^{1/(p-1)}} dr, \end{aligned}$$

provided

- $\lim_{R \rightarrow \infty} \frac{R^{p/(p-1)}}{V(R)^{1/(p-1)}} = 0$  and
- $\int_1^\infty \left( \frac{r}{V(r)} \right)^{1/(p-1)} dr < \infty$ .

The second condition implies the first one, and in our setting, the second condition is equivalent to the  $p$ -hyperbolicity of the manifold  $(M, g)$  ([33]). Recall that if  $\mathcal{O} \subset M$ , then its  $p$ -capacity is defined by

$$\text{cap}_p \mathcal{O} := \inf \left\{ \int_M |d\varphi|^p d\text{vol}_g, \varphi \in C_0^\infty(M) \text{ and } \varphi \geq 1 \text{ on } \mathcal{O} \right\},$$

and that  $(M, g)$  is said to be  $p$ -hyperbolic if the  $p$ -capacity of bounded open subsets is positive. Using the argument of the proof of the Proposition 4.1, we obtain as before that

$$T: L^p \rightarrow L^p_w$$

is bounded provided that, for some constant  $C$  independent of  $R$ , we have

$$(4.3) \quad \int_R^\infty \left( \frac{r}{V(r)} \right)^{1/(p-1)} dr \leq C \left( \frac{r^p}{V(r)} \right)^{1/(p-1)}.$$

Using the reverse doubling condition  $(RD_\nu)$ , we get that this condition is satisfied when  $p < \nu$ ; hence we have proven Theorem A.

According to [17], in our setting the  $p$ -capacity of an anchored ball can be estimated as

$$\text{cap}_p(B(o, R)) \leq C \left( \int_R^\infty \frac{r^{1/(p-1)}}{V(r)^{1/(p-1)}} dr \right)^{1-p}.$$

Hence, when the  $p$ -capacity of anchored balls satisfy the uniform estimate

$$\text{cap}_p(B(o, R)) \geq c \frac{\text{vol } B(o, R)}{R^p},$$

then the condition (4.3) is satisfied. This argumentation provides a direct proof of Corollary B.

### 5. Passing the volume growth exponent

We prove now Theorem D.

#### 5.1. Reverse Hölder inequality

According to [26], when  $(M, g)$  satisfies the conditions (D), (QD) and (RCA), then  $(M, g)$  satisfies the scale invariant  $L^1$  Poincaré inequalities: for any ball  $B \subset M$  and any  $f \in C^\infty(2B)$ ,

$$\|f - f_B\|_{L^1(B)} \leq C r(B) \|df\|_{L^1(2B)}.$$

These Poincaré inequalities and the doubling condition imply ([2]) that for all  $q \in (1, 2]$ , the reverse Riesz transform is bounded in  $L^q$ : there is a constant  $C$  such that for any  $f \in C_0^\infty(M)$ ,

$$\|\sqrt{\Delta}f\|_{L^q} \leq C \|df\|_{L^q}.$$

Hence, when  $p > 2$ , the Riesz transform is bounded on  $L^p$  as soon as the Hodge projector  $\Pi = d\Delta^{-1}d^* : L^2(T^*M) \rightarrow L^2(T^*M)$  has a bounded extension to  $L^p$  (see Lemma 0.1 in [2]).

Now the proof of the implication 1)  $\Rightarrow$  2) of Theorem 2.1 in [2] (see also [44]) shows that, if for some  $\tilde{p} > p$  we have a  $L^{\tilde{p}}$ -reverse Hölder inequality for the gradient of harmonic functions, then the Hodge projector has a bounded extension on  $L^p$ .

**Definition 5.1.** A complete Riemannian manifold  $(M, g)$  is said to satisfy the  $L^p$ -reverse Hölder inequality if for some constants  $C > 0$ ,  $\bar{\alpha} > \underline{\alpha} > 1$  and for any ball  $B \subset M$  and any harmonic function  $h$  defined on  $\bar{\alpha}B$ , one has the reverse Hölder inequality:

$$\left( \frac{1}{\text{vol } B} \int_B |dh|^p \right)^{1/p} \leq C \left( \frac{1}{\text{vol}(\underline{\alpha}B)} \int_{\underline{\alpha}B} |dh|^2 \right)^{1/2}.$$

In our case, the quadratic decay of the negative part of the Ricci curvature and the Cheng and Yau estimate on the gradient of harmonic function ([11]) yield

a  $L^\infty$ -reverse Hölder inequality for remote balls. The following lemma shows that in our setting, we will get the  $L^p$ -reverse Hölder inequality provided it holds for anchored balls.

**Lemma 5.2.** *Let  $(M, g)$  be a complete Riemannian manifold that satisfies the doubling condition. The  $L^p$ -reverse Hölder inequality holds provided it holds for remote and anchored balls.*

*Proof.* Assume that the  $L^p$ -reverse Hölder inequality holds for remote and anchored balls with parameters  $\bar{\alpha} > \underline{\alpha} > 1$ .

Let  $B(x, r)$  be a ball that is not anchored nor remote, i.e.,  $x \neq o$  and  $r \geq r(x)/2$ . Let  $\lambda \geq 1$  be a real parameter.

i) Assume that  $r \geq \lambda r(x)$  and let  $B' = B(o, (1 + \lambda^{-1})r)$ . We get  $B(x, r) \subset B'$  and  $\underline{\alpha}B' \subset \underline{\beta}B(x, r)$  provided  $\beta = (1 + \lambda^{-1})\alpha + \lambda^{-1}$ . Define now  $\bar{\beta} = (1 + \lambda^{-1})\bar{\alpha} + \lambda^{-1}$  and  $\underline{\beta} = (1 + \lambda^{-1})\underline{\alpha} + \lambda^{-1}$ . The six balls

$$B(x, r), \quad \underline{\beta}B(x, r), \quad \bar{\beta}B(x, r), \quad B', \quad \underline{\alpha}B', \quad \bar{\alpha}B'$$

have comparable volumes. The inclusions  $B(x, r) \subset B'$ ,  $\bar{\alpha}B' \subset \bar{\beta}B(x, r)$  and  $\underline{\alpha}B' \subset \underline{\beta}B(x, r)$ , together with the  $L^p$ -reverse Hölder inequality for the ball  $B'$ , imply that if  $h$  is a harmonic function defined on  $\bar{\beta}B(x, r)$ , then

$$\left( \frac{1}{\text{vol } B(x, r)} \int_{B(x, r)} |dh|^p \right)^{1/p} \leq C \left( \frac{1}{\text{vol}(\underline{\beta}B(x, r))} \int_{\underline{\beta}B(x, r)} |dh|^2 \right)^{1/2}.$$

ii) Assume now that  $r(x)/2 \leq r \leq \lambda r(x)$ . Let  $h$  be a harmonic function defined on  $B(x, 4r)$ . Then we have the inclusion  $B(o, (4 - \lambda^{-1})r) \subset B(x, 4r)$ . We consider a minimal covering of  $B(o, (4 - \lambda^{-1})r) \setminus B(o, 4\delta r)$  by balls of radius  $\delta r$ :

$$B(o, (4 - \lambda^{-1})r) \setminus B(o, 4\delta r) = \cup_{i \in I} B_i.$$

All the balls  $B_i$  are remote and for some constant  $N$  depending only on  $\delta$  and on the doubling constant  $\vartheta$ ,

$$\text{card } I \leq N.$$

Moreover, all the balls  $B_i, B(o, 4\delta r)$  have a comparable volume. We choose  $\delta$  so that

$$8\delta < 1 \quad \text{and} \quad (1 + \bar{\alpha})\delta < 1.$$

We introduce the collection of balls:  $\mathcal{B}_* = \{B(o, 4\delta r)\} \cup \{B_i, i \in I\}$  and let

$$\mathcal{B} = \{B \in \mathcal{B}_*, B \cap B(x, r) \neq \emptyset\}.$$

If  $B \in \mathcal{B}$  then  $\underline{\alpha}B \subset B(x, 2r)$  and also  $\bar{\alpha}B \subset B(x, 2r)$ , so that

$$\begin{aligned} \int_{B(x, r)} |dh|^p &\leq \sum_{B \in \mathcal{B}} \int_B |dh|^p \leq C \sum_{B \in \mathcal{B}} (\text{vol } B)^{1-2/p} \left( \int_{\underline{\alpha}B} |dh|^2 \right)^{2/p} \\ &\leq C (\text{card } I + 1) (\text{vol } B(x, r))^{1-2/p} \left( \int_{2B(x, r)} |dh|^2 \right)^{2/p}. \end{aligned}$$

Hence the result. □

**5.2. Hölder elliptic estimates and the Green kernel**

Recall that we say that  $(M^n, g)$  satisfies the scale invariant Hölder elliptic estimates  $(EH_\alpha)$  if there is a constant  $C$  such that, for any ball  $B \subset M$  and any harmonic function  $h$  defined on  $3B$ , we have for all  $x, y \in B$ ,

$$|h(x) - h(y)| \leq C \left( \frac{d(x, y)}{r(B)} \right)^\alpha \sup_{z \in 2B} |h(z)|.$$

**Remark 5.3.** The argument given in the proof of Lemma 5.2 shows that with the (RCA) and the doubling condition, the scale invariant Hölder elliptic estimates  $(EH_\alpha)$  holds for all balls provided its holds for all remote and anchored balls. In our setting, the quadratic decay of the negative part of the Ricci curvature implies a Lipschitz estimates for harmonic function on remote balls; hence the scale invariant  $\alpha$ -Hölder elliptic estimates holds for any ball if and only if it holds for anchored balls.

We assume now that  $(M, g)$  is a complete Riemannian manifold that satisfies the conditions (QD), (VC) and (RCA) and the scale invariant Hölder elliptic estimates  $(EH_\alpha)$ . For  $R \geq 1$  and  $\kappa > 2$ , we consider the anchored balls

$$B^* = B(o, R/\kappa) \subset\subset B^\# = B(o, R).$$

We will denote  $h^\#(t, x, y)$  the heat kernel on the ball  $B^\#$  for the Dirichlet boundary condition, and  $G^\#(x, y)$  the associated Green kernel:

$$G^\#(x, y) = \int_0^{+\infty} h_t^\#(x, y) dt.$$

If  $f \in L^2(B^\#)$ , then

$$u(x) = \int_{B^\#} G^\#(x, y) f(y) dy$$

is the solution of the equation

$$\Delta u = f : u = 0 \text{ on } \partial B^\#.$$

Because our manifold satisfies the Faber–Krahn inequality, we get

$$\lambda_1^D(B^\#) \geq \frac{c}{R^2}.$$

Recall that, for all  $t > 0, x, y \in B^\#$ ,

$$h_t^\#(x, y) \leq h_t(x, y),$$

and for all  $t > s > 0, x, y \in B^\#$ ,

$$h_t^\#(x, y) \leq \sqrt{h_t^\#(x, x) h_t^\#(y, y)} \leq e^{-\lambda_1^D(B^\#)(t-s)} \sqrt{h_s^\#(x, x) h_s^\#(y, y)}.$$

Hence we get the estimate: for  $t \in (0, R^2), x, y \in B^\#$ ,

$$h_t^\#(x, y) \leq \frac{C}{\text{vol } B(x, \sqrt{t})} e^{-d^2(x,y)/(ct)};$$

and for  $t \geq R^2, x, y \in B^\#$ ,

$$h_t^\#(x, y) \leq \frac{C}{\text{vol } B^\#} e^{-ct/R^2}.$$

Hence if  $x \in B^*, y \in B^\# \setminus \frac{1}{2}B^\#$  we get  $d(x, y) \simeq R$  and

$$G^\#(x, y) \leq \int_0^{R^2} \frac{C}{\text{vol } B(x, \sqrt{t})} e^{-R^2/(ct)} dt + \frac{CR^2}{\text{vol } B^\#}.$$

But using the doubling condition we obtain

$$\int_0^{R^2} \frac{C}{\text{vol } B(x, \sqrt{t})} e^{-R^2/(ct)} dt \leq \int_0^{R^2} \frac{C}{\text{vol } B(x, R)} \left(\frac{R}{\sqrt{t}}\right)^\mu e^{-R^2/(ct)} dt \leq \frac{CR^2}{\text{vol } B^\#}.$$

Finally, we obtain that if  $x \in B^*$  and  $y \in B^\# \setminus \frac{1}{2}B^\#$ , then

$$G^\#(x, y) \leq C \frac{R^2}{\text{vol } B^\#}.$$

Using the  $\alpha$ -Hölder regularity estimate, we get the following estimation on the gradient of the Green kernel.

**Lemma 5.4.** *There is a constant  $C > 0$  such that if  $x \in \frac{1}{2}B^*$  and  $y \in B^\# \setminus \frac{1}{2}B^\#$ , then*

$$|\nabla_x G^\#(x, y)| \leq C \left(\frac{R}{r(x) + 1}\right)^{1-\alpha} \frac{R}{\text{vol } B^\#}.$$

*Proof.* Indeed, we apply the above  $\alpha$ -Hölder regularity estimates for the harmonic function  $h(z) = G^\#(z, y) - G^\#(x, y)$ . We have seen that

$$\sup_{z \in B^*} |h(z)| \leq C \frac{R^2}{\text{vol } B^\#}.$$

Hence if  $x \in \frac{1}{2}B^*$  and  $z \in B(x, \frac{1}{2}r(x))$  we obtain

$$|h(z)| = |h(z) - h(x)| \leq C \left(\frac{r(x)}{R}\right)^\alpha \frac{R^2}{\text{vol } B^\#}.$$

And with the Cheng and Yau gradient estimate [11], we get

$$|\nabla h|(x) = |\nabla_x G^\#(x, y)| \leq \frac{C}{r(x)} \left(\frac{r(x)}{R}\right)^\alpha \frac{R^2}{\text{vol } B^\#}.$$

When  $r(x) \leq 1/2$ , the same idea leads to the estimates

$$\forall z \in B(o, 1), \quad |h(z)| = |h(z) - h(o)| \leq C \left(\frac{1}{R}\right)^\alpha \frac{R^2}{\text{vol } B^\#}.$$

The Cheng and Yau gradient estimate implies that

$$x \in B(o, 1/2): |\nabla_x G^\#(x, y)| \leq C \left(\frac{1}{R}\right)^\alpha \frac{R^2}{\text{vol } B^\#}. \quad \square$$



**5.3. From  $\alpha$ -Hölder elliptic regularity to reverse Hölder inequality**

In order to prove the Theorem D, we only need to prove that if  $(M, g)$  satisfies the hypotheses of this theorem and if  $p$  is such that  $(1 - \alpha)p < \nu$ , then anchored balls satisfy the  $L^p$ -reverse Hölder inequality.

With J. Cheeger and T. Colding [10], we can build a smooth function  $\chi$  such that

- $\chi = 0$  on  $M \setminus B(o, 3R/4)$ ,
- $\chi = 1$  on  $B(o, R/2)$ ,
- $R|d\chi| + R^2|\Delta\chi| \leq C$ .

Indeed, according to Theorem 6.33 in [10], if  $B$  is a remote ball then there is a smooth function  $\chi_B$  with compact support in  $B$  such that  $\chi_B = 1$  on  $\frac{1}{2}B$  and such that

$$r(B)|d\chi_B| + r^2(B)|\Delta\chi_B| \leq C(n, \kappa).$$

Using the doubling hypothesis, we can cover  $\partial B(o, R/2)$  by at most  $N$  balls of radius  $R/8$  and centered in  $\partial B(o, R/2)$ :

$$\partial B(o, R/2) \subset \cup_i B_i.$$

Introduce  $\varphi = \sum_i \chi_{2B_i}$  on  $\partial B(o, R/2)$ ; we have  $\varphi \geq 1$  and  $\varphi$  has compact support in  $B(o, \frac{3}{4}R)$ . Moreover, there is a constant (independent of  $R$ ) such that  $R|d\varphi| + R^2|\Delta\varphi| \leq C$ . We let  $\chi = u(\varphi)$ , where  $u: [0, \infty[ \rightarrow [0, 1]$  is a smooth function such that  $u = 1$  on  $[1, \infty)$  and  $u = 0$  on  $[0, 1/2]$ .

We will use that annuli satisfy a scale invariant Poincaré inequality (according to [26]): there are constants  $\kappa > 1$  and  $C > 0$  such that for any  $R > 1$ ,

$$\forall f \in C^\infty(A_R^*), \quad \|f - f_{A_R}\|_{L^2(A_R)} \leq CR \|df\|_{L^2(A_R^*)},$$

where

$$A_R := B(o, R) \setminus B(o, R/2) \quad \text{and} \quad A_R^* := B(o, \kappa R) \setminus B(o, R/(2\kappa)).$$

Let  $B^\# = B(o, R)$  and let  $G^\#(x, y)$  be the Green kernel of the Dirichlet Laplacian on  $B^\#$ .

Let  $h$  be a harmonic function on  $B(o, \kappa R)$ . Using the Green kernel, we have for  $x \in B(o, R/(2\kappa))$  and any constant  $c$ ,

$$h(x) - c = \int_{A_R} G^\#(x, y) \Delta(\chi(h - c))(y) dy.$$

Using the fact that  $h$  is harmonic, we know that

$$\Delta(\chi(h - c)) = (\Delta\chi)(h - c) - 2\langle d\chi, dh \rangle_g.$$

With the Cauchy-Schwarz inequality, we obtain

$$|dh|^2(x) \leq C \int_{A_R} |\nabla_x G^\#(x, y)|^2 dy \times \int_{A_R} [R^{-4}|h(y) - c|^2 + R^{-2}|dh|(y)^2] dy.$$

Choosing

$$c = \frac{1}{\text{vol}(A_R)} \int_{A_R} h(y) \, dy = h_{A_R}$$

and using the Poincaré inequality for the annulus, we get

$$\int_{A_R} |h(y) - c|^2 \, dy \leq CR^2 \int_{A_R^*} |dh|(y)^2 \, dy.$$

Eventually, we obtain

$$|dh|^2(x) \leq CR^{-2} \left( \int_{A_R} |\nabla_x G^\#(x, y)|^2 \, dy \right) \|dh\|_{L^2(B(o, \kappa R))}^2.$$

Recall the estimates of the gradient of the Green kernel: for  $x \in B(o, R/(2\kappa))$  and  $y \in B(o, R) \setminus B(o, R/2)$ , we have

$$|\nabla_x G^\#(x, y)| \leq C \left( \frac{R}{r(x) + 1} \right)^{1-\alpha} \frac{R}{\text{vol } B(o, R)}.$$

So we get that

$$\left( \frac{1}{\text{vol } B(o, R/(2\kappa))} \int_{B(o, R/(2\kappa))} |dh|^p \right)^{1/p} \leq \left( \frac{C(R)^2}{\text{vol}(B(o, \kappa R))} \int_{B(o, \kappa R)} |dh|^2 \right)^{1/2},$$

where

$$C^p(R) = \frac{1}{\text{vol } B(o, R/(2\kappa))} \int_{B(o, R/(2\kappa))} \left( \frac{R}{r(x) + 1} \right)^{p(1-\alpha)} \, d\text{vol}(x).$$

Now it is easy to show that the reverse doubling assumption  $(RD_\nu)$  yields an uniform bound on  $C(R)$  as soon as

$$p(1 - \alpha) < \nu.$$

### 5.4. On the Hölder elliptic regularity estimates

In our setting, the scale invariant  $\alpha$ -Hölder elliptic regularity estimates are equivalent to a “quasi”-monotonicity result for the  $L^2$  norm of the gradient of harmonic function.

**Proposition 5.5.** *Assume that  $(M, g)$  is a complete Riemannian manifold satisfying the hypothesis (QD), (VC) and (RCA). Let  $\alpha \in (0, 1]$ . Then  $(M, g)$  satisfies the  $\alpha$ -Hölder elliptic regularity estimates if and only if there are some constants  $\kappa > 1$ ,  $C > 0$  such that for any harmonic function  $h$  on  $B(o, \kappa R)$  and any  $1 \leq r \leq R$ ,*

$$\frac{r^{2-2\alpha}}{\text{vol } B(o, r)} \int_{B(o, r)} |dh|^2 \leq C \frac{R^{2-2\alpha}}{\text{vol } B(o, R)} \int_{B(o, R)} |dh|^2.$$

*Proof.* If  $f$  is a continuous function on a subset  $\mathcal{O} \subset M$ , we define its oscillation by

$$\text{Osc}_{\mathcal{O}} f = \sup_{x \in \mathcal{O}} f(x) - \inf_{y \in \mathcal{O}} f(y).$$

Let  $R \geq 1$  and let  $h$  be a harmonic function defined on  $B(o, 2\kappa R)$ . We let again  $A_R := B(o, R) \setminus B(o, R/2)$  and  $A_R^* := B(o, \kappa R) \setminus B(o, R/(2\kappa))$ . We have the Poincaré inequality

$$\|h - h_{A_R}\|_{L^2(A_R)}^2 \leq CR^2 \|dh\|_{L^2(A_R^*)}^2.$$

But according to Lemma 6.3 in [26], we have a Harnack inequality on annuli, so

$$(5.1) \quad (\text{Osc}_{A_R} h)^2 \leq \frac{C}{\text{vol } A_R} \|h - h_{A_R}\|_{L^2(A_R)}^2 \leq C \frac{R^2}{\text{vol } B(o, \kappa R)} \|dh\|_{L^2(B(o, \kappa R))}^2.$$

Using the function  $\chi$  defined by

$$\chi(x) = \begin{cases} 1 & \text{on } B(o, R/2), \\ 2 - 2r(x)/R & \text{on } B(o, R) \setminus B(o, R/2), \\ 0 & \text{on } M \setminus B(o, R), \end{cases}$$

we get

$$\begin{aligned} \int_{B(o, R/2)} |dh|^2 &\leq \int_{B(o, R)} |d(\chi(h - h_{A_R}))|^2 \\ &= \int_{A_R} (h - h_{A_R})^2 |d\chi|^2 = \frac{4}{R^2} \int_{A_R} (h - h_{A_R})^2. \end{aligned}$$

In particular we get

$$(5.2) \quad \frac{1}{\text{vol } B(o, R/2)} \int_{B(o, R/2)} |dh|^2 \leq \frac{C}{R^2} (\text{Osc}_{A_R} h)^2.$$

But in our case, the  $\alpha$ -Hölder elliptic regularity estimates is equivalent to a monotonicity inequality for  $\rho \mapsto \rho^{-\alpha} \text{Osc}_{A_\rho} h$ . The result is then a consequence of inequalities (5.1) and (5.2). □

## 6. Non boundedness of the Riesz transform

In this section, we give two criterions for the  $L^p$  unboundedness of the Riesz transform.

### 6.1. Parabolicity and the Riesz transform

Our argument is a slight improvement of some earlier results proved in [8]. The starting point is to understand the  $L^p$  closure of the space of differential of smooth function with compact support; the following lemma is a  $L^p$  adaptation of an idea from [6].

**Lemma 6.1.** *Let  $(M^n, g)$  be a complete Riemannian manifold and let  $f \in W_{\text{loc}}^{1,p}$  be such that  $df \in L^p$ . If  $M(r) = \int_{B(o,r)} |f|^p$  satisfies*

$$\int_1^\infty \left(\frac{r}{M(r)}\right)^{1/(p-1)} dr = \infty,$$

*then there is a sequence  $(\chi_\ell)$  of smooth function with compact support such that*

$$\lim_{\ell \rightarrow \infty} \|df - d(\chi_\ell f)\|_{L^p} = 0.$$

*In particular,*

$$df \in \overline{d\mathcal{C}_0^\infty(M)}^{L^p}.$$

*Proof.* Let  $r < R$ . We define a function  $\chi_{r,R}$  by letting  $\chi_{r,R} = 1$  on  $B(o, r)$ ,  $\chi_{r,R} = 0$  outside  $B(o, R)$ , and for  $x \in B(o, R) \setminus B(o, r)$ ,

$$\chi_{r,R}(x) = \xi_{r,R}(r(x)) = \varepsilon(r, R) \int_{r(x)}^R \left(\frac{s}{M(s)}\right)^{1/(p-1)} ds,$$

where

$$\varepsilon(r, R) = \left(\int_r^R \left(\frac{s}{M(s)}\right)^{1/(p-1)} ds\right)^{-1}.$$

Let  $f$  be a function satisfying the hypotheses of the lemma. Then we get

$$\|df - d(\chi_{r,R}f)\|_{L^p}^p \leq C \left( \int_{M \setminus B(o,r)} |df|^p + \int_{B(o,R) \setminus B(o,r)} |f|^p |d\chi_{r,R}|^p \right).$$

Now we introduce the Riemann–Stieltjes measure associated to the non-decreasing function  $s \mapsto M(s)$ , and we have

$$\begin{aligned} \int_{B(o,R) \setminus B(o,r)} |f|^p |d\chi_{r,R}|^p &= \int_r^R |\xi'_{r,R}(s)|^p dM(s) \\ &= \varepsilon(r, R)^p \int_r^R s^{p/(p-1)} \frac{1}{M(s)^{p/(p-1)}} dM(s) \\ &\leq \varepsilon(r, R)^p \left( r^{p/(p-1)} \frac{p-1}{M(r)^{1/(p-1)}} + \int_r^R \frac{p s^{1/(p-1)}}{M(s)^{1/(p-1)}} ds \right) \\ &\leq (p-1) \varepsilon(r, R)^p r^{p/(p-1)} \frac{1}{M(r)^{1/(p-1)}} + p \varepsilon(r, R)^{p-1}. \end{aligned}$$

With the hypothesis  $\int_1^\infty (r/M(r))^{1/(p-1)} dr = \infty$  it is possible to find two increasing and divergent sequences of  $r_\ell < R_\ell$  such that

$$\lim_{\ell \rightarrow \infty} \varepsilon(r_\ell, R_\ell) = 0$$

and

$$\lim_{\ell \rightarrow \infty} \varepsilon(r_\ell, R_\ell)^p r_\ell^{p/(p-1)} \frac{1}{M(r_\ell)^{1/(p-1)}} = 0.$$

Hence the result. □

Assume now that  $(M, g)$  is non parabolic and that the Riesz transform is bounded on  $L^p$  and on  $L^{p/(p-1)}$ . The Hodge projector

$$\Pi = d\Delta^{-1}d^* = (d\Delta^{-1/2})(\Delta^{-1/2}d^*) = (d\Delta^{-1/2})(d\Delta^{-1/2})^*$$

extends from  $L^2 \cap L^p$  to a bounded operator on  $L^p(T^*M)$ . Hence  $\Pi(\overline{d\mathcal{C}_0^\infty(T^*M)})$  is dense in  $\Pi(L^p(T^*M))$ .

Also, by definition,  $\Pi$  is the identity on  $d\mathcal{C}_0^\infty(M)$ , hence

$$\overline{d\mathcal{C}_0^\infty(M)}^{L^p} \subset \Pi(L^p(T^*M)).$$

If  $\alpha \in \mathcal{C}_0^\infty(T^*M)$  then we have  $\Pi(\alpha) = df$ , where

$$f(x) = \int_M G(x, y) d^* \alpha(y) dy$$

and where  $G(x, y)$  is the Green kernel of  $(M, g)$ . Because  $d^* \alpha$  has compact support, the growth of  $r \mapsto \int_{B(o,r)} f^p$  is controlled by

$$\mathcal{G}_p(r) := \int_{B(o,r)} G(x, o)^p dx.$$

Hence a direct consequence of the last lemma is the following proposition.

**Proposition 6.2.** *If  $(M, g)$  is a complete Riemannian manifold such that*

- *$(M, g)$  is non parabolic and its Green kernel satisfies*

$$\int_1^\infty \left(\frac{r}{\mathcal{G}_p(r)}\right)^{1/(p-1)} dr = \infty,$$

- *the Riesz transform is bounded on  $L^p$  and on  $L^{p/(p-1)}$ ,*

then

$$\Pi(L^p(T^*M)) = \overline{d\mathcal{C}_0^\infty(M)}^{L^p}.$$

Recall that  $(M, g)$  is said to be  $p$ -parabolic if we can find a sequence of smooth function with compact support  $(\chi_k)$  such that

$$0 \leq \chi_k \leq 1, \quad \lim_{k \rightarrow \infty} \|d\chi_k\|_{L^p} = 0 \quad \text{and} \quad \chi_k \rightarrow 1 \text{ uniformly on compact sets.}$$

A consequence of the definition is that on a  $p$ -parabolic manifold, any bounded function with  $L^p$  gradient has its gradient in the  $L^p$ -closure of  $d\mathcal{C}_0^\infty(M)$ .

**Remark 6.3.** If  $(M, g)$  is non-parabolic (i.e., 2-hyperbolic) then its Green kernel  $G(x, y)$  is bounded outside its pole: that is to say if  $r > 0$  then  $x \in M \setminus B(y, r) \mapsto G(x, y)$  is positive and it is bounded by  $\max_{x \in \partial B(y,r)} G(x, y)$ .

In particular we get:

**Proposition 6.4.** *Assume that  $(M, g)$  is a complete Riemannian manifold such that*

- *$(M, g)$  is non 2-parabolic and  $p$ -parabolic for some  $p > 2$ ,*
- *the Riesz transform is bounded on  $L^p$  and on  $L^{p/(p-1)}$ .*

Then

$$\Pi(L^p(T^*M)) = \overline{d\mathcal{C}_0^\infty(M)}^{L^p}.$$

These two results should be compared with the one of Lemma 7.1 in [8], where a Sobolev inequality was assumed. Then we can show the following adaptation of Corollary 7.5 in [8].

**Theorem 6.5.** *Under the hypotheses of Proposition 6.4,  $(M, g)$  has only one end.*

*Proof.* If  $M$  has at least two ends, we can find a compact set  $K \subset M$  such that  $M \setminus K = \mathcal{O}_- \cup \mathcal{O}_+$ , with  $\mathcal{O}_- \cap \mathcal{O}_+ = \emptyset$  and such that both  $\mathcal{O}_\pm$  are unbounded. And we can build a smooth function  $\varphi$  such that  $\varphi = \pm 1$  on  $\mathcal{O}_\pm$ . Then  $\Delta\varphi \in \mathcal{C}_0^\infty(M)$ ,  $d\varphi \in \mathcal{C}_0^\infty(T^*M)$ . We can define  $h: M \rightarrow [-1, 1]$  by

$$h(x) = \varphi(x) - \int_M G(x, y) \Delta\varphi(y) dy.$$

The function  $h$  is harmonic; by Remark 6.3, we know that  $h$  is bounded, and by construction,

$$dh = d\varphi - \Pi(d\varphi).$$

As  $(M, g)$  is assumed to be  $p$ -parabolic, we get that

$$dh \in \overline{d\mathcal{C}_0^\infty(M)}^{L^p}.$$

So that  $\Pi(dh) = dh$ ; but on  $L^2$ , we have by construction  $\Pi(dh) = 0$ , hence the contradiction.  $\square$

**Corollary 6.6.** *Let  $p > 2$ . On a non-parabolic and  $p$ -parabolic manifold with at least two ends, the Riesz transform can not be bounded simultaneously on  $L^p$  and on  $L^{p/(p-1)}$ .*

### 6.2. Sublinear harmonic function and the Riesz transform

Our next result shows that the existence of non constant sublinear harmonic function implies some  $L^p$  unboundedness of the Riesz transform.

**Proposition 6.7.** *Let  $(M, g)$  be a complete Riemannian manifold whose Ricci curvature satisfies*

$$\text{Ricci}_g \geq -\frac{\kappa^2}{r^2(x)} g.$$

*Assume moreover that  $(M, g)$  is doubling and satisfies the (RCE) condition, and that anchored balls satisfy, for some  $\mu \geq \nu > 2$  and positive constant  $c$ ,*

$$1 < r < R \implies c \left(\frac{R}{r}\right)^\nu \text{vol} B(o, r) \leq \text{vol} B(o, R) \leq c R^\mu.$$

*Let  $\alpha \in [0, 1)$ . If  $(M, g)$  carries a non constant harmonic function  $h$  with  $\alpha$ -growth:*

$$h(x) = \mathcal{O}(r^\alpha(x)),$$

*then the Riesz transform is not bounded on  $L^p$  for any  $p > 2$  and  $p \geq \mu/(1 - \alpha)$ .*

A quick inspection of the proof below shows that we only need  $\nu > p/(p - 1)$ .

*Proof.* By contraposition, we assume that the Riesz transform is bounded on  $L^p$ , with  $p(1 - \alpha) \geq \mu$ , and consider a harmonic function  $h$  such that

$$h(x) = \mathcal{O}(r^\alpha(x)).$$

We are going to show that necessary  $h$  is constant. We remark that our conditions imply a relative Faber–Krahn inequality, hence by Theorem 1.2 or [14], the Riesz transform is bounded on  $L^{p/(p-1)}$ . The quadratic decay of the negative part of the Ricci curvature, together with the Cheng–Yau gradient estimate, implies that

$$dh(x) = \mathcal{O}(r^{\alpha-1}(x)).$$

The volume growth condition

$$\text{vol } B(o, r) = \mathcal{O}(r^\mu)$$

implies

$$dh \in L^p.$$

Moreover, we also have

$$M(r) = \int_{B(o,r)} h^p \leq C r^{\mu+p\alpha} \leq C r^p,$$

so that

$$\left( \int_1^\infty \frac{r}{M(r)} \right)^{1/(p-1)} dr = \infty,$$

and with Lemma 6.1, we get that

$$dh \in \overline{d\mathcal{C}_0^\infty(M)}^{L^p}.$$

The volume growth assumption also implies that  $(M, g)$  is  $p$ -parabolic, and Proposition 6.2 yields

$$\Pi(dh) = dh.$$

Let  $\alpha \in \mathcal{C}_0^\infty(T^*M)$ . As the Hodge projector is bounded on  $L^p$  and on  $L^{p/(p-1)}$ , we get

$$\langle dh, \alpha \rangle = \langle \Pi(dh), \alpha \rangle = \langle dh, \Pi(\alpha) \rangle.$$

But

$$\Pi(\alpha) = df,$$

where  $f$  is given by

$$f(x) = \int_M G(x, y) d^* \alpha(y) dy.$$

There is a  $R > 0$  so that  $f$  is harmonic outside  $B(o, R)$ , and using the Green kernel estimate, we know that  $f$  tends to zero at infinity. We are going to estimate the decay of  $f$ . Let  $q = p/(p - 1)$ . Because  $\Pi$  is bounded on  $L^q$ , we have  $df \in L^q$ .

Let  $x \in M \setminus B(o, 2R)$ . The function  $f$  is harmonic on the remote ball  $B(x, r(x)/2)$ , and  $u = |df|$  satisfies the elliptic inequality

$$\Delta u \leq \frac{C}{r(x)^2} u \quad \text{on } B(x, r(x)/2).$$

The lower bound on the Ricci curvature implies that

$$u^q(x) \leq \frac{C}{\text{vol } B(x, r(x)/2)} \int_{B(x, r(x)/2)} u^q = \frac{o(1)}{\text{vol } B(o, r(x))}.$$

Hence,

$$|df(x)| \leq \frac{o(1)}{(\text{vol } B(o, r(x)))^{1/q}}.$$

Using the (RCE) condition, we can integrate this inequality along a path starting from  $x$  and escaping to infinity to get

$$|f(x)| \leq \frac{o(r(x))}{(\text{vol } B(o, r(x)))^{1/q}} + o(1) \int_{r(x)}^\infty \frac{1}{(\text{vol } B(o, s))^{1/q}} ds.$$

Using the reverse doubling condition  $(RD_\nu)$  for the anchored balls, we get that

$$|f(x)| \leq C \frac{o(r(x))}{(\text{vol } B(o, r(x)))^{1/q}}.$$

Now we define a function  $\chi_k$  by

$$\chi_k(x) = \begin{cases} 1 & \text{on } B(o, k), \\ 2 - r(x)/k & \text{on } B(o, 2k) \setminus B(o, k), \\ 0 & \text{on } M \setminus B(o, 2k). \end{cases}$$

Because  $dh \in L^p$  and  $df \in L^q$ , we have

$$\langle dh, \alpha \rangle = \langle dh, df \rangle = \lim_{k \rightarrow \infty} \langle dh, \chi_k df \rangle = \lim_{k \rightarrow \infty} \langle dh, d(\chi_k f) \rangle - \langle f dh, d\chi_k \rangle.$$

However,  $h$  is harmonic, hence

$$\langle dh, d(\chi_k f) \rangle = 0.$$

Moreover,

$$\begin{aligned} |\langle f dh, d\chi_k \rangle| &\leq C \frac{\text{vol } B(o, 2k) o(k)}{(\text{vol } B(o, k))^{1/q}} k^{(\alpha-1)} k^{-1} \\ &\leq (\text{vol } B(o, k))^{1/p} k^{\alpha-1} o(1) \leq C k^{\mu/p} k^{\alpha-1} o(1). \end{aligned}$$

Our hypotheses imply that this quantity tends to zero when  $k$  tends to infinity. Eventually we obtain that, for all  $\alpha \in \mathcal{C}_0^\infty(T^*M)$ ,  $\langle dh, \alpha \rangle = 0$ , hence  $dh = 0$ .  $\square$



In fact, the  $\alpha$ -Hölder elliptic estimates imply that a sublinear harmonic function with  $\beta$ -growth and  $\beta < \alpha$  is necessary constant.

**Lemma 6.8.** *Assume that  $(M, g)$  carries a non constant sublinear harmonic function  $h$  with*

$$h(x) = \mathcal{O}(r^\beta(x)) .$$

*If  $\alpha > \beta$ , then  $(M, g)$  can not satisfy the  $\alpha$ -Hölder elliptic estimates .*

*Proof.* Indeed, assume that  $(M, g)$  satisfies  $(EH_\alpha)$  and consider a harmonic function  $h: M \rightarrow \mathbb{R}$  such that for some positive constants  $C$  and  $\beta < \alpha$ ,

$$\forall x \in M, \quad |h(x)| \leq C(1 + d(x, o)^\beta) .$$

Using  $(EH_\alpha)$ , we get that for any  $x \in B(o, R)$ ,

$$|h(x) - h(o)| \leq \Lambda \left( \frac{d(x, o)}{R} \right)^\alpha \sup_{y \in B(o, 2R)} |h(y)| .$$

Hence we get that for any  $x \in M$  and any  $R \geq r(x)$ ,

$$|h(x) - h(o)| \leq C \Lambda \left( \frac{d(x, o)}{R} \right)^\alpha (1 + (2R)^\beta) .$$

Letting  $R \rightarrow +\infty$ , we get that  $h(x) = h(o)$  for any  $x \in M$ . □

## 7. Examples

In this section, we describe three series of applications of our results.

### 7.1. Manifolds with conical ends

These manifolds  $(M^n, g)$  are isometric outside a compact set to a truncated cone

$$\mathcal{C}_R(\Sigma) := ((R, \infty) \times \Sigma, (dr)^2 + r^2 h) ,$$

where  $(\Sigma, h)$  is compact Riemannian manifold. We are going to explain how our results can be used to recover the Theorem 1.3 of H-Q. Li, C. Guillarmou and A. Hassell and of A. Hassell and P. Lin ([35], [27], [28], [32]). From the explicit form of the metric, it is easy to remark that the conditions (QD), (VC) and (RCE) are satisfied. Concerning the volume growth of geodesic balls, there is a positive constant  $C$  such that for all  $x \in M$  and  $R > 0$ ,

$$C^{-1} R^n \leq \text{vol } B(x, R) \leq C R^n .$$

**\* A general positive result.** The hypotheses of Theorem A are satisfied for  $\nu = n$ , hence the Riesz transform is bounded on  $L^p$  for all  $p \in (1, n) \cup \{2\}$ .

★ **Negative results if  $M^n$  has several ends (i.e., if  $\Sigma$  is not connected).** In the case where  $n > 2$ , Corollary 7.5 in [8] already told us that the Riesz transform is unbounded on  $L^p$  when  $p \geq n$ . In the case where  $n = 2$ , it is well known that  $(M, g)$  carry a non constant harmonic function with logarithmic growth. Indeed,  $(M^2, g)$  is conformally equivalent to a compact surface with a finite number of points removed:

$$(M^2, g) = (\Sigma \setminus \{p_1, \dots, p_r\}, \varphi^{-2}\bar{g}),$$

where  $(\Sigma, \bar{g})$  is compact Riemann surface. Moreover, around each  $p_i$  we have

$$d_{\bar{g}}(x, p_i) \simeq d_g(x, o)^{-1}.$$

Consider the Green kernel  $\bar{G}$  on  $(\Sigma, \bar{g})$ . Then the function  $f(x) = G(x, p_1) - G(x, p_2)$  is harmonic on  $(\Sigma \setminus \{p_1, p_2\}, \bar{g})$  and satisfies, for  $i = 1, 2$ ,

$$f(x) \simeq (-1)^i \log((d_{\bar{g}}(x, p_i)).$$

Recall  $\Delta_g = \varphi^2 \Delta_{\bar{g}}$ , hence  $f$  is a harmonic function on  $(M^2, g)$  and it has logarithmic growth:  $f(x) = \mathcal{O}(\log r(x))$ . Proposition E implies that if  $p > 2$  then the Riesz transform can not be bounded in  $L^p$ .

Assume now that  $(\Sigma, h)$  is connected. Let

$$0 < \lambda_1 \leq \lambda_2 \leq \dots$$

be the spectrum of the Laplace operator of  $(\Sigma, h)$ , where the eigenvalues are repeated according to their multiplicity. And let  $\varphi_0, \varphi_1 \dots$  be an associated set of normalized eigenfunctions:  $\Delta_h \varphi_i = \lambda_i \varphi_i$ . We define  $0 < \alpha_1 \leq \alpha_2 \dots$  by

$$\alpha_i(n - 2 - \alpha_i) = \lambda_i.$$

★ **Negative results if  $M^n$  has only one end.** According to Lemma 6 in [12],  $(M, g)$  carries a harmonic function  $f$  such that

$$f(r, \theta) = r^{\alpha_1} \varphi_1(\theta) + o(r^{\alpha_1}).$$

Define now

$$\alpha = \min(\alpha_1, 1).$$

Proposition E implies that if  $p(1 - \alpha) \geq n$ , then the Riesz transform can not be bounded in  $L^p$ .

★ **Positive results if  $M^n$  has only one end.** Assume that  $(\Sigma, h)$  is connected. Let  $\mathcal{B}(r) = K \cup ([1, r] \times \Sigma)$ . Following the analysis of Proposition 4.1 in [1], it can be shown that there is a constant  $C$  such that for all harmonic function  $f$  defined over  $\mathcal{B}(R)$  and all  $1 < r < \rho \leq R$ ,

$$\frac{r^{2-2\alpha}}{r^n} \int_{\mathcal{B}(r)} |df|^2 \leq C \frac{\rho^{2-2\alpha}}{\rho^n} \int_{\mathcal{B}(\rho)} |df|^2.$$

From this result and Proposition 5.5, we obtain that  $(M, g)$  satisfies the  $\alpha$ -Hölder estimates  $(EH_\alpha)$ . Hence if  $p(1 - \alpha) < n$ , the Riesz transform is bounded on  $L^p$ .

### 7.2. Model manifolds

We consider a Riemannian manifold  $(M^n, g)$  that is isometric outside a compact set to the warped product

$$\mathcal{C}_R(\Sigma) := ((R, \infty) \times \Sigma, (dr)^2 + f^2(r)h),$$

where  $f(r) = e^{u(\ln(r))}$ , for some function  $u: [1, \infty) \rightarrow \mathbb{R}$  with bounded second derivative and

$$\alpha \leq u' \leq 1.$$

We assume that  $(\Sigma, h)$  is a compact manifold with non negative Ricci curvature so that  $(M, g)$  satisfies the quadratic decay on the negative part of the Ricci curvature and the conditions (VC) and (RCE). Moreover, anchored balls satisfy the reverse doubling condition  $(RD_\nu)$  for the exponent  $\nu = (n - 1)\alpha + 1$ .

If  $u$  satisfies the asymptotic condition

$$(7.1) \quad u'(\theta) \geq \bar{\alpha} - \psi(\theta) \quad \text{for some nonnegative } \psi \in L^1,$$

then the exponent in the reverse doubling is improved to  $\bar{\nu} = (n - 1)\bar{\alpha} + 1$ . Hence Theorem A implies that the Riesz transform is bounded on  $L^p$  for  $p < (n - 1)\bar{\alpha} + 1$ .

If

$$\int_1^\infty \frac{1}{f^{n-1}(r)} dr < \infty,$$

then  $(M, g)$  is non-parabolic, and if

$$\int_1^\infty \frac{1}{f^{(n-1)/(p-1)}(r)} dr = \infty,$$

then  $(M, g)$  is  $p$ -parabolic. Hence, by Theorem C, if  $p > 2$  and if  $\Sigma$  is non connected then the Riesz transform is not bounded on  $L^p$ .

If  $\Sigma$  is connected and if  $\int_1^\infty (1 - u'(r)) dr = \infty$ , then the diameter of geodesic sphere grows slowly:

$$\text{diam } \partial B(o, R) = \sup_{x, y \in \partial B(o, R)} d(x, y) = o(R),$$

and we can apply Corollary F and we know that the Riesz transform is bounded on  $L^p$  for all  $p \in (1, +\infty)$ . In conclusion, we get:

**Proposition 7.1.**

- If the function  $u$  satisfies the condition (7.1) and if  $p < (n - 1)\bar{\alpha} + 1$ , then the Riesz transform is bounded on  $L^p$ .
- Assume that  $1 < (n - 1)\bar{\alpha}$  and that  $\Sigma$  is non connected. If

$$\int_1^\infty \frac{1}{f^{(n-1)/(p-1)}(r)} dr = \infty,$$

then the Riesz transform not bounded on  $L^p$ .

- If  $\Sigma$  is connected and if  $\int_1^\infty (1 - u'(r)) dr = \infty$ , then the Riesz transform is bounded on  $L^p$  for all  $p \in (1, +\infty)$ .

### 7.3. Examples with infinite topological type

Our last example is inspired by a construction of J. Lott and Z. Shen, [37]. Assume that  $(P^n, g)$  is a compact Riemannian manifold with boundary  $\partial P = \Sigma_- \cup \Sigma_+$ , and that  $\Sigma_{\pm}$  have collared neighborhoods  $\mathcal{U}_{\pm}$ . Assume moreover that there is a diffeomorphism  $f: \mathcal{U}_- \rightarrow \mathcal{U}_+$  with

$$f^*g = 4g.$$

For  $\ell \in \mathbb{N}$ , we define  $2^\ell P$  to be the rescaled Riemannian manifold

$$(P, 4^\ell g).$$

Using the map  $f$  we can glue all the  $2^\ell P$  and get a Riemannian manifold  $(X, g)$  with boundary  $\partial X = \Sigma_-$ . If there is a compact manifold  $K$  with boundary diffeomorphic to  $\Sigma_-$  then we can form  $X_0 = X \cup K$ . We can also form  $X_1 = X \#_{\Sigma_-} (-X)$  the double of  $X$ . These manifolds have quadratic curvature decay and Euclidean growth. Hence our results yield the following.

**Proposition 7.2.**

- On  $X_0$  and  $X_1$ , the Riesz transform is bounded on  $L^p$  for every  $p \in (1, n)$ .
- On  $X_1$ , the Riesz transform is not bounded on  $L^p$  if  $p \geq n > 2$ .
- Assume that  $P$  is connected, then there is some  $\varepsilon \in (0, 1]$  such that on  $X_0$ , the Riesz transform is  $L^p$ -bounded for every  $p \in (1, n/(1 - \varepsilon))$ .

## 8. Some perspectives

In this last section, we conclude by some remarks and perspectives.

### 8.1. Perturbations

In [13], T. Coulhon and N. Dungey obtained a nice property of stability under perturbation for  $L^p$  boundedness of the Riesz transform. This result implies that if  $(M, g)$  is a complete Riemannian manifold satisfying the hypotheses of Theorem A (resp., Theorem D) and a non collapsing hypothesis

$$\inf_{x \in M} \text{vol } B(x, 1) > 0,$$

then the same conclusions hold for any other metric  $\tilde{g}$  that satisfies for some  $\varepsilon > 0$ ,

$$\tilde{g} - g = \mathcal{O}(r^{-\varepsilon}(x)).$$

### 8.2. Riesz transform associated to Schrödinger operator

Let  $(M, g)$  be a complete Riemannian manifold satisfying the geometric condition (QD), (VC) and (RCE), and let  $V: M \rightarrow \mathbb{R}$  be a locally bounded function such that

$$V(x) = \mathcal{O}(r^{-2}(x)).$$

A challenging question is to study the boundedness of the Riesz transform associated to Schrödinger operator  $\Delta + V$ :

$$R_V = d(\Delta + V)^{-1/2}.$$

In this case, the situation can be very complicated even in the case of manifolds with conical ends (see [27], [28], [32]). Some part of our analysis extends easily to this case but some crucial points are missing; we leave as an open question of finding the appropriate spectral condition for the operator  $\Delta + V$  that leads to results similar to Theorem A or Theorem D. Note that in the case where the potential is small in some Kato class, then the  $L^p$ -boundedness of the Riesz transform  $R_0$  implies the  $L^p$ -boundedness of the Riesz transform  $R_V$  ([21]).

### 8.3. Gaussian estimates on the heat kernel on 1-forms

In the setting of Theorem D, it will be interesting to understand under which geometric assumptions, one get a Gaussian estimate on the heat kernel of the Hodge Laplacian on 1-forms. Recall that such an estimate implies a boundedness result for the Riesz transform ([15], [16], [45]). According to [18], [19], [21], a sub-critical assumption on the Ricci curvature implies such an upper bound. It has recently been shown that this sub-critical assumption yields results for the Riesz transform on 1 and 2-forms ([38]).

### 8.4. Riesz transform associated to differential forms

Let  $\Delta_k = dd^* + d^*d: \mathcal{C}_0^\infty(\Lambda^k T^*M) \rightarrow \mathcal{C}_0^\infty(\Lambda^k T^*M)$  be the Hodge Laplacian acting on  $k$ -differential forms. Following the questions asked in [8], one would like to understand the  $L^p$  boundedness of the Riesz transforms

$$d(\Delta_k)^{-1/2}, \quad d^*(\Delta_k)^{-1/2}, \quad \nabla(\Delta_k)^{-1/2}.$$

On manifolds with conical ends, these questions have been recently investigated by C. Guillarmou and D. Sher ([29]), and it appears that it is a very difficult problem. It is tempting to analyze what can be done on a manifold whose curvature tensor decays quadratically; however, in order to use our analysis, we will need Gaussian upper bound on the heat kernel on forms, and such an estimate already provided some boundedness result for the Riesz transform according to a general principle ([15], [45]).

### 8.5. The Riesz transform of second order

Another interesting question is about the  $L^p$  boundedness of the Riesz transform of second order

$$\nabla d\Delta^{-1}.$$

The  $L^p$  boundedness of the operator  $\nabla d(\Delta + 1)^{-1}$  has been recently investigated by B. Güneysu and S. Pigola ([30]). Motivated by this paper, in a future work, we

intend to consider applications of these ideas for the operator  $\nabla d\Delta^{-1}$  on manifold where the full curvature tensor decays quadratically:

$$\|Rm(x)\| = \mathcal{O}\left(\frac{1}{r^2(x)}\right).$$

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