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A note on star-shaped compact hypersurfaces with prescribed scalar curvature in space forms

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Abstract. Guan, Ren and Wang obtained a C^2 a priori estimate for admissible 2-convex hypersurfaces satisfying the Weingarten curvature equation $\sigma_2(\kappa(X)) = f(X, \nu(X))$. In this note, we give a simpler proof of this result, and extend it to space forms.

1. Introduction

In [7], Guan, Ren and Wang solved the long standing problem of obtaining global C^2 estimates for a closed convex hypersurface $M \subset \mathbb{R}^{n+1}$ of prescribed kth elementary symmetric function of curvature in general form:

$$\sigma_k(\kappa(X)) = f(X, \nu(X)), \quad \forall X \in M. \tag{1.1}$$

In the case $k = 2$ of scalar curvature, they were able to prove the estimate for strictly starshaped 2-convex hypersurfaces. Their proof relies on new test curvature functions and elaborate analytic arguments to overcome the difficulties caused by allowing f to depend of ν.

In this note, we give a simpler proof for the scalar curvature case and we extend the result to space forms $N^{n+1}(K)$, with $K = -1, 0, 1$. Our main result is stated in Theorem 2.1 of section 2 and leads to the existence Theorem 3.3. For related results in the literature see [3], [6], [2] and [8].

2. Prescribed scalar curvature

Let $N^{n+1}(K)$ be a space form of sectional curvature $K = -1$, 0, and $+1$. Let $g^N := ds^2$ denote the Riemannian metric of $N^{n+1}(K)$. In Euclidean space $\mathbb{R}^{n+1}$, fix the origin O and let $\mathbb{S}^n$ denote the unit sphere centered at O. Suppose that (z, ρ)

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are spherical coordinates in $\mathbb{R}^{n+1}$, where $z \in \mathbb{S}^n$. The standard metric on S^n induced from $\mathbb{R}^{n+1}$ is denoted by dz^2. Let $\bar{b}$ be constant, $0 < \bar{b} \leq \infty$, $I = [0, \bar{b})$, and $\phi(\rho)$ a positive function on I. Then the new metric

$$g^N := ds^2 = d\rho^2 + \phi^2(\rho)dz^2. \tag{2.1}$$

on $\mathbb{R}^{n+1}$ is a model of N^{n+1}, which is Euclidean space $\mathbb{R}^{n+1}$ if $\phi(\rho) = \rho$, $\bar{b} = \infty$, a hemisphere of the unit sphere $\mathbb{S}^{n+1}$ if $\phi(\rho) = \sin(\rho)$, $\bar{b} = \pi/2$, and hyperbolic space $\mathbb{H}^{n+1}$ if $\phi(\rho) = \sinh(\rho)$, $\bar{b} = \infty$.

We recall some formulas for the induced metric, normal, and second fundamental form on $\mathcal{M}$ (see [2]). We will denote by ∇' the covariant derivatives with respect to the standard spherical metric e_{ij}, and by ∇ the covariant derivatives with respect to some local orthonormal frame on $\mathcal{M}$. Then we have

$$g_{ij} = \phi^2 e_{ij} + \rho_i \rho_j,\ g^{ij} = \frac{1}{\phi^2}\Big(e^{ij} - \frac{\rho^i \rho^j}{\phi^2 + |\nabla' \rho|^2}\Big), \tag{2.2}$$

$$\nu = \frac{(-\nabla'\rho, \phi^2)}{\sqrt{\phi^4 + \phi^2 |\nabla'\rho|^2}}, \tag{2.3}$$

and

$$h_{ij} = \frac{\phi}{\sqrt{\phi^2 + |\nabla'\rho|^2}}\Big(-\nabla'_{ij}\rho + \frac{2\phi'}{\phi}\rho_i\rho_j + \phi\phi' e_{ij}\Big). \tag{2.4}$$

Consider the vector field $V = \phi(\rho)\frac{\partial}{\partial \rho}$ in $N^{n+1}(K)$, and define $\Phi(\rho) = \int_0^\rho \phi(r)dr$. Then, $u := \langle V, \nu \rangle$ is the support function. By a straight forward calculation we have the following equations (see [5], lemmas 2.2 and 2.6):

$$\nabla_{ij}\Phi = \phi' g_{ij} - u h_{ij}, \tag{2.5}$$

$$\nabla_i u = g^{kl} h_{ik} \nabla_l \Phi, \quad \text{and} \tag{2.6}$$

$$\nabla_{ij} u = g^{kl} \nabla_k h_{ij} \nabla_l \Phi + \phi' h_{ij} - u g^{kl} h_{ik} h_{jl}. \tag{2.7}$$

Now let Γ_k be the connected component of $\{\lambda \in \mathbb{R}^n : \sigma_k(\lambda) > 0\}$, where

$$\sigma_k = \sum_{i_1 < i_2 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}$$

is the kth mean curvature. $\mathcal{M} := \{(z, \rho(z)) : z \in \mathbb{S}^n\}$ is an embedded hypersurface in N^{n+1}. We call ρ *k-admissible* if the principal curvatures $(\lambda_1(z), \ldots, \lambda_n(z))$ of $\mathcal{M}$ belong to Γ_k. Our problem is to study a smooth positive 2-admissible function ρ on $\mathbb{S}^n$ satisfying

$$\sigma_2(\lambda(b)) = \psi(V, \nu), \tag{2.8}$$

where $b = \{b_{ij}\} = \{\gamma^{ik} h_{kl} \gamma^{lj}\}$, $\{h_{ij}\}$ is the second fundamental form of $\mathcal{M}$, and γ^{ij} is $\sqrt{g^{-1}}$. Equivalently, we study the solution of the following equation:

$$F(b) = \begin{pmatrix} n \\ 2 \end{pmatrix}^{(-1/2)} \sigma_2(\lambda(b))^{1/2} = f(\lambda(b_{ij})) = \overline{\psi}(V, \nu). \tag{2.9}$$

Now we are ready to state and prove our main result.

Theorem 2.1. *Suppose* $\mathcal{M} = \{(z, \rho(z)) \mid z \in \mathbb{S}^n\} \subset N^{n+1}$ *is a closed* 2*-convex hypersurface which is strictly starshaped with respect to the origin and satisfies equation* (2.9) *for some positive function* $\overline{\psi}(V, \nu) \in C^2(\Gamma)$*, where* Γ *is an open neighborhood of the unit normal bundle of* $\mathcal{M}$ *in* $N^{n+1} \times \mathbb{S}^n$*. Suppose also we have uniform control* $0 < R_1 \le \rho(z) \le R_2 < \bar{b}$, $|\rho|_{C^1} \le R_3$*. Then there is a constant* C *depending only on* n, R_1, R_2, R_3 *and* $|\bar{\psi}|_{C^2}$*, such that*

$$\max_{z\in\mathbb{S}^n} |\kappa_i(z)| \le C. \tag{2.10}$$

Proof. Since $\sigma_1(\kappa) > 0$ on $\mathcal{M}$, it suffices to estimate from above the largest principal curvature of $\mathcal{M}$. Consider

$$M_0 = \max_{\mathbf{x}\in\mathcal{M}} e^{\beta\Phi} \frac{\kappa_{\max}}{u-a} ,$$

where $u \ge 2a$ and β is a large constant to be chosen (we will always assume $\beta\phi' + K > 0$). Then M_0 is achieved at $\mathbf{x}_0 = (z_0, \rho(z_0))$ and we may choose a local orthonormal frame $e_1, \dots, e_n$ around $\mathbf{x}_0$ such that $h_{ij}(\mathbf{x}_0) = \kappa_i\delta_{ij}$, where $\kappa_1, \dots, \kappa_n$ are the principal curvatures of Σ at $\mathbf{x}_0$. We may assume $\kappa_1 = \kappa_{\max}(\mathbf{x}_0)$. Thus at $\mathbf{x}_0$, $\log h_{11} - \log(u-a) + \beta\Phi$ has a local maximum. Therefore,

$$0 = \frac{\nabla_i h_{11}}{h_{11}} - \frac{\nabla_i u}{u-a} + \beta\Phi_i, \tag{2.11}$$

and

$$0 \ge \frac{\nabla_{ii} h_{11}}{h_{11}} - \left(\frac{\nabla_i h_{11}}{h_{11}}\right)^2 - \frac{\nabla_{ii} u}{u-a} + \left(\frac{\nabla_i u}{u-a}\right)^2 + \beta\Phi_{ii}. \tag{2.12}$$

By the Gauss and Codazzi equations, we have $\nabla_k h_{ij} = \nabla_j h_{ik}$ and

$$\nabla_{11} h_{ii} = \nabla_{ii} h_{11} + h_{11}h_{ii}^2 - h_{11}^2 h_{ii} + K(h_{11}\delta_{1i}\delta_{1i} - h_{11}\delta_{ii} + h_{ii} - h_{i1}\delta_{i1}). \tag{2.13}$$

Therefore,

$$\begin{aligned} F^{ii}\nabla_{11} h_{ii} &= F^{ii}\nabla_{ii} h_{11} + \kappa_1 \sum_i f_i\,\kappa_i^2 - \kappa_1^2 \sum f_i\,\kappa_i + K\Big(-\kappa_1 \sum_i f_i + \sum_i f_i\,\kappa_i\Big) \\ &= \sum_i f_i \nabla_{ii} h_{11} + \kappa_1 \sum_i f_i\,\kappa_i^2 - \overline{\psi}\kappa_1^2 + K\Big(-\kappa_1 \sum_i f_i + \overline{\psi}\Big) \end{aligned} \tag{2.14}$$

Covariantly differentiating equation (2.9) twice yields

$$F^{ii} h_{iik} = \bar{\psi}_V(\nabla_{e_k} V) + h_{ks}\bar{\psi}_\nu(e_s) \tag{2.15}$$

so that

$$\Big|\sum_i f_i h_{iis}\Phi_s\Big| \le C(1+\kappa_1) \tag{2.16}$$

and

$$\begin{aligned} F^{ii} h_{ii11} + F^{ij,kl} h_{ij1} h_{kl1} = \nabla_{11}(\overline{\psi}) &\ge -C(1+\kappa_1^2) + h_{11s}\bar{\psi}_\nu(e_s) \\ &\ge -C(1+\kappa_1^2 + \beta\kappa_1) \quad \text{(using (2.11))}. \end{aligned} \tag{2.17}$$

Combining (2.17) and (2.14) and using (2.5)–(2.7),(2.11)–(2.12), and (2.15)–(2.16) gives

$$\begin{aligned}
0 \geq \frac{1}{\kappa_1}\Big\{ &- C(1+\kappa_1^2+\beta\kappa_1) - F^{ij,kl}\nabla_1 h_{ij}\nabla_1 h_{kl} - \kappa_1 \sum f_i\,\kappa_i^2 + \kappa_1^2\overline{\psi} \\
&- K(-\kappa_1 \sum f_i + \overline{\psi})\Big\} - \frac{1}{\kappa_1^2}\sum f_i\,|\nabla_i h_{11}|^2 - \frac{1}{u-a}\sum f_i\{h_{iis}\Phi_s - u\kappa_i^2 + \phi'\kappa_i\} \\
&+ \sum f_i \frac{|\nabla_i u|^2}{(u-a)^2} - u\beta\overline{\psi} + \beta\phi'\sum f_i \\
\geq -C(\kappa_1+\beta) &- \frac{1}{\kappa_1}F^{ij,kl}\nabla_1 h_{ij}\nabla_1 h_{kl} + \frac{a}{u-a}\sum f_i\,\kappa_i^2 + (\beta\phi'+K)\sum f_i \\
&- \frac{1}{\kappa_1^2}\sum f_i\,|\nabla_i h_{11}|^2 + \sum f_i \frac{|\nabla_i u|^2}{(u-a)^2}
\end{aligned}$$

In other words,

$$\begin{aligned}
0 \geq &-C(\kappa_1+\beta) - \frac{1}{\kappa_1}F^{ij,kl}\nabla_1 h_{ij}\nabla_1 h_{kl} + \frac{a}{u-a}\sum f_i\,\kappa_i^2 \\
&+ (\beta\phi'+K)\sum f_i - \frac{1}{\kappa_1^2}\sum f_i\,|\nabla_i h_{11}|^2 + \sum f_i \frac{|\nabla_i u|^2}{(u-a)^2}.
\end{aligned} \tag{2.18}$$

By (2.11) we have, for any $\epsilon > 0$,

$$\frac{1}{\kappa_1^2}\sum f_i\,|\nabla_i h_{11}|^2 \leq (1+\epsilon^{-1})\beta^2 \sum f_i\,|\nabla_i \Phi|^2 + \frac{(1+\epsilon)}{(u-a)^2}\sum f_i\,|\nabla_i u|^2. \tag{2.19}$$

Using this in (2.18) we obtain

$$\begin{aligned}
0 \geq &-C(\kappa_1+\beta) - \frac{1}{\kappa_1}F^{ij,kl}\nabla_1 h_{ij}\nabla_1 h_{kl} + (\frac{a}{u-a} - C\epsilon)\sum f_i\,\kappa_i^2 \\
&+ [\beta\phi' + K - C\beta^2(1+\epsilon^{-1})]T\,,
\end{aligned} \tag{2.20}$$

where $T = \sum f_i$. Now we divide the remainder of the proof into two cases.

Case A. Assume $\kappa_n \leq -\kappa_1/n$. In this case, equation (2.20) implies (here ϵ is a small controlled multiple of a and we use $f_n \geq f_i$ which holds by concavity of f)

$$0 \geq -C(\kappa_1+\beta) + \frac{a}{C}\sum f_i\,\kappa_i^2 - C\beta^2 T \geq -C(\kappa_1+\beta) + \Big(\frac{1}{C}\kappa_1^2 - C\beta^2\Big)T\,. \tag{2.21}$$

Since $T \geq 1$ by the concavity of f, equation (2.21) implies $\kappa_1 \leq C\beta$ at $\mathbf{x}_0$.

Case B. Assume $\kappa_n > -\kappa_1/n$. Let us partition $\{1, \ldots, n\}$ into two parts,

$$I = \{j : f_j \leq n^2 f_1\} \quad \text{and} \quad J = \{j : f_j > n^2 f_1\}.$$

For $i \in I$, we have (by (2.11)), for any $\epsilon > 0$,

$$\frac{1}{\kappa_1^2}\, f_i\,|\nabla_i h_{11}|^2 \leq (1+\epsilon)\sum f_i\,\frac{|\nabla_i u|^2}{(u-a)^2} + C\,(1+\epsilon^{-1})\,\beta^2 f_1. \tag{2.22}$$

Inserting this into equation (2.18) gives (for ϵ a small controlled multiple of a^2)

$$
\begin{aligned}
0 \geq -C(\kappa_1+\beta) - \frac{1}{\kappa_1}F^{ij,kl}\,\nabla_1 h_{ij}\nabla_1 h_{kl} + \frac{a}{C}\sum f_i\,\kappa_i^2 + (\beta\phi' + K)\sum f_i \\
- \frac{1}{\kappa_1^2}\sum_{i\in J} f_i\,|\nabla_i h_{11}|^2 - C\beta^2 f_1.
\end{aligned} \tag{2.23}
$$

Now we use an inequality due to Andrews [1] and Gerhardt [4]:

$$
\begin{aligned}
-\frac{1}{\kappa_1}F^{ij,kl}\,\nabla_1 h_{ij}\nabla_1 h_{kl} &\geq \frac{1}{\kappa_1}\sum_{i\neq j}\frac{f_i - f_j}{\kappa_j - \kappa_i}\,|\nabla_1 h_{ij}|^2 \\
&\geq \frac{2}{\kappa_1}\sum_{j\geq 2}\frac{f_j - f_1}{\kappa_1 - \kappa_j}\,|\nabla_j h_{11}|^2 \geq \frac{2}{\kappa_1^2}\sum_{j\in J} f_j\,|\nabla_j h_{11}|^2.
\end{aligned} \tag{2.24}
$$

We now insert (2.24) into (2.23) to obtain

$$
0 \geq -C(\kappa_1+\beta) + \frac{a}{C}\sum f_i\,\kappa_i^2 + (\beta\phi' + K)\sum f_i - C\beta^2 f_1. \tag{2.25}
$$

Since $\kappa_n > -\kappa_1/n$ we have that

$$
\sum f_i = \frac{(n-1)\sigma_1}{2\binom{n}{2}\,\overline{\psi}} > \frac{\kappa_1 - \frac{n-1}{n}\kappa_1}{n\,\overline{\psi}} = \frac{\kappa_1}{n^2\,\overline{\psi}}\,.
$$

We also note that on $\mathcal{M}$, ϕ' is bounded below by a positive controlled constant, so we may assume $\beta\phi' + K$ is large. Therefore from (2.25) we obtain

$$
0 \geq \left(\frac{\beta\phi' + K}{n^2\overline{\psi}} - C\right)\kappa_1 - C\beta + \left(\frac{a}{C_2}\kappa_1^2 - C\beta^2\right) f_1. \tag{2.26}
$$

We now fix β large enough that $\frac{\beta\phi'+K}{n^2\overline{\psi}} > 2C$ which implies a uniform upper bound for κ_1 at $\mathbf{x}_0$. By the definition of M_0 we then obtain a uniform upper bound for κ_{max} on $\mathcal{M}$ which implies a uniform upper and lower bound for the principle curvatures. □

3. Lower order estimates

In this section, we obtain C^0 and C^1 estimates for the more general equation

$$
\sigma_k(\kappa) = \psi(V,\,\nu), \quad \text{where } k = 1, \ldots, n. \tag{3.1}
$$

3.1. C^0 estimates

The C^0-estimates were proved in [2] but for the reader's convenience we include the simple proof.

Lemma 3.1. *Let $1 \le k \le n$ and let $\psi \in C^2(N^{n+1} \times \mathbb{S}^n)$ be a positive function. Suppose there exist two numbers R_1 and R_2, $0 < R_1 < R_2 < \bar{b}$, such that*

$$\psi\Big(V, \frac{V}{|V|}\Big) \ge \sigma_k(1, \dots, 1)\, q^k(\rho), \quad \rho = R_1, \tag{3.2}$$

$$\psi\Big(V, \frac{V}{|V|}\Big) \le \sigma_k(1, \dots, 1)\, q^k(\rho), \quad \rho = R_2, \tag{3.3}$$

where $q(\rho) = \frac{1}{\phi}\frac{d\phi}{d\rho}$. Let $\rho \in C^2(\mathbb{S}^n)$ be a solution of equation (3.1). Then

$$R_1 \le \rho \le R_2.$$

Proof. Suppose that $\max_{z \in \mathbb{S}^n} \rho(z) = \rho(z_0) > R_2$. Then at z_0,

$$g^{ij} = \phi^{-2} e^{ij}, \quad h_{ij} = -\nabla'_{ij}\rho + \phi\phi' e_{ij} \ge \phi\phi' e_{ij}, \quad b_{ij} \ge q(\rho)\delta_{ij}\ .$$

Hence $\psi(V, \nu)(z_0) = \sigma_k(b_{ij})(z_0) > q^k(R_2)\sigma_k(1, \dots, 1)$, contradicting (3.3). The proof of (3.2) is similar. □

3.2. C^1 estimates

In this section, we follow the idea of [3] and [6] to derive C^1 estimates for the height function ρ. In other words, we are looking for a lower bound for the support function u. First, we need the following technical assumption: for any fixed unit vector ν,

$$\frac{\partial}{\partial \rho}(\phi(\rho)^k \psi(V, \nu)) \le 0, \quad \text{where } |V| = \phi(\rho). \tag{3.4}$$

Lemma 3.2. *Let M be a radial graph in N^{n+1} satisfying (3.1) and (3.4), and let ρ be the height function of M. If ρ has positive upper and lower bounds, then there is a constant C, depending on the minimum and maximum values of ρ, such that*

$$|\nabla \rho| \le C.$$

Proof. Consider $h = -\log u + \gamma(\Phi(\rho))$ and suppose h achieves its maximum at z_0. We will show that for a suitable choice of $\gamma(t)$, $u(z_0) = |V(z_0)|$, that is $V(z_0) = |V(z_0)|\nu(z_0)$, which implies a uniform lower bound for u on M. If not, we can choose a local orthonormal frame $\{e_1, \dots, e_n\}$ on M such that $\langle V, e_1\rangle \ne 0$, and $\langle V, e_i\rangle = 0$, $i \ge 2$. Then at z_0 we have

$$h_i = \frac{-u_i}{u} + \gamma'\, \nabla_i \Phi = 0, \tag{3.5}$$

and

$$\begin{aligned} 0 \ge h_{ii} &= \frac{-u_{ii}}{u} + \Big(\frac{u_i}{u}\Big)^2 + \gamma'\, \nabla_{ii}\Phi + \gamma''(\nabla_i \Phi)^2 \\ &= \frac{-1}{u}\left(h_{ii1}\nabla_1\Phi + \phi' h_{ii} - u h_{ii}^2\right) + [(\gamma')^2 + \gamma''](\nabla_i\Phi)^2 + \gamma'(\phi' g_{ii} - h_{ii} u). \end{aligned} \tag{3.6}$$

Equation (3.5) gives

$$h_{11} = u\gamma', \quad h_{i1} = 0, \quad i \geq 2, \tag{3.7}$$

so we may rotate $\{e_2, \ldots, e_n\}$ so that $h_{ij}(z_0, \rho(z_0)$ is diagonal. Hence,

$$\begin{aligned} 0 \geq & \frac{-1}{u}\left(\sigma_k^{ii}\, h_{ii1}\nabla_1\Phi + \phi' k\psi - u\,\sigma_k^{ii} h_{ii}^2\right) \\ & + [(\gamma')^2 + \gamma''](\nabla_1\Phi)^2\sigma^{11} + \gamma'\left(\phi'\sum\sigma_k^{ii} - k\psi u\right). \end{aligned} \tag{3.8}$$

Differentiating equation (3.1) with respect to e_1 we obtain

$$\sigma_k^{ii} h_{ii1} = d_V\psi(\nabla_{e_1}V) + h_{11}d_\nu\psi(e_1). \tag{3.9}$$

Substituting equation (3.9) and (3.7) into (3.8) yields

$$\begin{aligned} 0 \geq & \frac{-1}{u}[\langle V, e_1\rangle\, d_V\psi(\nabla_{e_1}V) + u\gamma'\,\langle V, e_1\rangle\, d_\nu\psi(e_1) + k\phi'\psi] \\ & + \sigma_k^{ii}\, h_{ii}^2 + [(\gamma')^2 + \gamma'']\,\langle V, e_1\rangle^2\,\sigma_k^{11} + \gamma'\phi'\sigma_k^{ii} - ku\gamma'\psi \\ = & \frac{-1}{u}\,[\langle V, e_1\rangle\, d_V\psi(\nabla_{e_1}V) + k\phi'\psi] + \sigma_k^{ii}\, h_{ii}^2 \\ & + [(\gamma')^2 + \gamma'']\,\langle V, e_1\rangle^2\,\sigma_k^{11} + \gamma'\phi'\sum\sigma_k^{ii} - u\gamma'\psi - \gamma'\,\langle V, e_1\rangle\, d_\nu\psi(e_1). \end{aligned} \tag{3.10}$$

Our assumption (3.4) is equivalent to

$$k\phi^{k-1}\phi'\psi + \phi^k\frac{\partial}{\partial\rho}\psi(V, \nu) \leq 0, \tag{3.11}$$

or

$$k\phi'\psi + d_V\psi(V, \nu) \leq 0. \tag{3.12}$$

Since at z_0, $V = \langle V, e_1\rangle\, e_1 + \langle V, \nu\rangle\, \nu$,

$$d_V\psi(V, \nu) = \langle V, e_1\rangle\, d_V\psi(\nabla_{e_1}V) + \langle V, \nu\rangle\, d_V\psi(\nabla_\nu V). \tag{3.13}$$

Therefore,

$$\begin{aligned} 0 \geq & \sigma_k^{ii} h_{ii}^2 + [(\gamma')^2 + \gamma'']\,\langle V, e_1\rangle^2\,\sigma_k^{11} + \gamma'\phi'\sum\sigma_k^{ii} \\ & - u\gamma'\psi - \gamma'\,\langle V, e_1\rangle\, d_\nu\psi(e_1) + d_V\psi(\nabla_\nu V). \end{aligned} \tag{3.14}$$

Now let $\gamma(t) = \alpha/t$, where $\alpha > 0$ is sufficiently large. Since $h_{11} \leq 0$ at z_0, and $\sum\sigma_k^{ii} = (n-k+1)\sigma_{k-1}$, we have that

$$\sigma_k^{11} = \sigma_{k-1}(\kappa|\kappa_1) \geq \sigma_{k-1} \geq \sigma_k^{(k-1)/k} = \psi^{(k-1)/k}. \tag{3.15}$$

Therefore,

$$[(\gamma')^2 + \gamma'']\,\langle V, e_1\rangle^2\,\sigma_k^{11} + \sigma_k^{ii} h_{ii}^2 + \gamma'\phi'\sum\sigma_k^{ii} \geq C\alpha^2\sigma_k^{11}, \tag{3.16}$$

for some C depending on $|\rho|_{C^0}$.

We conclude that

$$0 \geq C\alpha^2 \, \psi^{(k-1)/k} - \alpha \, |V| \, |d_\nu \psi(e_1)| - |d_V \psi(\nabla_\nu V)|, \tag{3.17}$$

which leads to a contradiction when α is large. Therefore at z_0 we have $u = |V|$, which completes the proof. □

By a standard continuity argument ([3]), we can prove the following theorem.

Theorem 3.3. *Suppose* $\psi \in C^2(\bar{B}_{r_2} \setminus B_{r_1} \times \mathbb{S}^n)$ *satisfies conditions* (3.2), (3.3), *and* (3.4). *Then there exists a unique* $C^{3,\alpha}$ *starshaped solution* $\mathcal{M}$ *satisfying equation* (2.8).

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