Note on uniformly transient graphs

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Abstract. We study a special class of graphs with a strong transience feature called uniform transience. We characterize uniform transience via a Feller-type property and via validity of an isoperimetric inequality. We then give a further characterization via equality of the Royden boundary and the harmonic boundary and show that the Dirichlet problem has a unique solution for such graphs. The Markov semigroups and resolvents (with Dirichlet boundary conditions) on these graphs are shown to be ultracontractive. Moreover, if the underlying measure is finite, the semigroups and resolvents are trace class and their generators have ℓ^p independent pure point spectra (for $1 \le p \le \infty$).

Examples of uniformly transient graphs include Cayley graphs of hyperbolic groups as well as trees and Euclidean lattices of dimension at least three. As a surprising consequence, the Royden compactification of such lattices turns out to be the one-point compactification and the Laplacians of such lattices have pure point spectrum if the underlying measure is chosen to be finite.

Introduction

Spectral geometry is concerned with the interplay of spectral theory of Laplacians and the geometry of the underlying structure. The two basic paradigms are given by Riemannian manifolds and graphs. There are many similarities between the case of graphs and the case of manifolds. Indeed, a common framework is provided within the theory of Dirichlet forms. Still, there are also crucial differences. Structurally, a main difference is that manifolds give local Dirichlet forms whereas graphs give non-local Dirichlet forms.

As far as examples are concerned, there is also another difference: In the case of manifolds, the Riemannian structure provides both the Laplacian on smooth functions and a canonical measure. The situation on graphs is rather different.

One is given two pieces of data on a countably infinite set X, viz (in notation explained below in Section 1)

- a graph structure (b, c) and

- a measure m,

and these two pieces of data are completely *independent*. In this sense, there are more parameters available in the case of graphs.

One way to deal with the abundance of possible measures in the graph case is to restrict attention to certain special measures. In this context, two choices have attracted particular attention. One is the measure derived from b by taking m to be the vertex degree. The corresponding Laplacian is known as the normalized Laplacian. The other choice is the constant measure. Both of these choices have their merits. Indeed, it seems that, for a long time, the study of Laplacians on graphs was restricted to one of these two choices. In particular, the spectral geometry of normalized Laplacian has been quite a focus of attention, see e.g. [1], [3], [5], [8], [11], [34], [35], [39], [51], and references therein as well as [34] and [55] for higher order Laplacians.

Recently, however, there has been an outburst of all sorts of studies of Laplacians on graphs with general measures, see e.g. [2], [6], [7], [12], [17], [18], [19], [20], [23], [26], [29], [31], [32], [37], [36], [44], [43], [46], [52], [53], [54], [65], and references therein. In some sense, a comparable development can be seen in the study of manifolds. There, weighted manifolds have become a focus of attention in certain questions of spectral geometric nature, see e.g. [25], [24], [27].[1]

Given this situation, there is substantial interest in features of the graph which do NOT depend on the choice of the measure.

One such feature is transience/recurrence. Another feature is compactness of the underlying structure. Indeed, quite recently, the concept of a canonically compactifiable graph has been brought forward in [22]. Canonically compactifiable graphs have many claims to model a (relatively) compact situation.

Here, we present another property which is independent of the measure. This property is stronger than transience and weaker than canonical compactifiability. There are various ways to look at this property. Indeed, the main abstract result of this note (Theorem 2.1) shows that it can simultaneously be seen as a strong transience condition or as a strong Feller-type condition or as the validity of a strong isoperimetric inequality. We call it *uniform transience*. This property has already appeared in the literature in work of Barlow–Coulhon–Grigor'yan [4] and of Windisch [62] in, yet, other manifestations. Corresponding considerations are somewhat ad-hoc and a systematic treatment – as given below – was missing until now.

In this context, we emphasize that, unlike in other studies of graphs, our treatment below is not inspired by similar considerations in the manifold setting. In fact, it may be an interesting question to carry it over to the manifold case.

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[1]Indeed, one might argue that the normalized Laplacian on a graph corresponds to the Laplacian on a manifold with Riemannian volume and the choice of a general measure on a graph corresponds to the choice of a weight (i.e., a density with respect to the Riemannian volume) on the manifold.

As discussed below, the class of uniformly transient graphs contains all non-trivial trees and all Cayley graphs of hyperbolic groups (with standard weights) as well as all transient graphs with a quasitransitive automorphism group. In particular, all Euclidean lattices $\mathbb{Z}^d$ for $d \geq 3$ fall into this category.

Uniform transience has a certain compactness flavor to it. In fact, every canonically compactifiable graph is uniformly transient (Corollary 2.3). Thus, all models considered in [22] fall into our framework here. Moreover, it is possible to characterize uniform transience by a boundedness condition with respect to a certain metric (Theorem 3.2). Furthermore, it is possible to characterize uniform transience via the Royden boundary. As a consequence, we can show unique solvability of the Dirichlet problem for uniformly transient graphs. This is discussed in Section 4. The methods developed in Section 4 can be extended to reprove the (well-known) existence of solutions for the Dirichlet problem on general graphs. We include a discussion in Section 5.

As mentioned already, all canonically compactifiable graphs are uniformly transient. We can even characterize the canonically compactifiable graphs within the class of uniformly transient graphs as those for which all harmonic functions of finite energy are bounded (Theorem 6.1). In this context, we can also prove that for a transient graph the Royden compactification agrees with the one-point compactification if and only if the graph is uniformly transient and has the Liouville property (Corollary 6.2). As a particular class of examples for this we discuss Euclidean lattices.

Uniformly transient graphs yield ultracontractive semigroups independently of the underlying measure (Lemma 7.1). This can then be used to show that they yield ℓ^p independent pure point spectrum whenever the underlying measure is finite (Theorem 7.2).

Our abstract results give remarkable and somewhat surprising consequences for the Euclidean lattices $\mathbb{Z}^d$ for $d \geq 3$. These can easily be seen to be uniformly transient. From this, we then obtain that the Royden compactification of such a lattice is the one-point compactification. This is in sharp contrast to the case of smaller dimensions. In fact, the Royden compactification of the one-dimensional lattice is an enormous object (see [66]). Moreover, we infer that the Laplacian on such a Euclidean lattice has pure point spectrum whenever the lattice is equipped with a finite measure. Details are discussed in the last two sections.

Our considerations make use of a certain characterization of the domain of the Laplacian with Dirichlet boundary condition and of a certain characterization of transience. Both of these characterizations are probably well known. As we have not been able to find them in the literature, we have included corresponding discussions in one appendix each. We also include an appendix discussing the relation between harmonic functions and bounded harmonic functions.

Note added. After we received a referee report on this article we were made aware by a third person that uniformly transient graphs have already featured in the not so widely known papers [40], [41] under no particular name. In particular, Theorem 4.2 which provides one characterization of uniform transience is contained in Proposition 4.1 of [40]. However, we would like to stress that the contexts

of [40], [41] are quite different from ours. There, uniformly transient graphs are used as a mere tool to study the convergence of resistance forms and associated processes while we provide a systematic study of uniformly transient graphs and their properties. Furthermore, we were made aware that a related concept called uniform subcriticality was studied in the context of second order uniformly elliptic operators in a domain of $\mathbb{R}^d$ in [56].

1. Framework: graphs, forms and Laplacians

In this section we introduce the key objects of our study. These are forms on graphs and the associated semigroups and Laplacians. A convenient framework has recently been presented in [44], [43]. Here we follow these works and refer to them for further details and references.

Let X be a countably infinite set. The vector space of all real-valued functions on X is denoted by $C(X)$. The subspace of all real-valued functions vanishing outside of a finite set is denoted by $C_c(X)$ and the closure of $C_c(X)$ with respect to the *supremum norm*

$$\|f\|_\infty := \sup_{x \in X} |f(x)|$$

is denoted by $C_0(X)$. It is a complete normed space when equipped with the supremum norm.

A *graph* over X is a pair (b, c) such that $b : X \times X \to [0, \infty)$ is symmetric, has zero diagonal, and satisfies

$$\sum_{y \in X} b(x, y) < \infty$$

for all $x \in X$ and $c : X \to [0, \infty)$ is arbitrary. Then, X is called the *vertex set*, b the *edge weight* and c the *killing term* or *potential*. Elements $x, y \in X$ are said to be *neighbors* or *connected* by an edge of weight $b(x, y)$ if $b(x, y) > 0$. If the number of neighbors of each vertex is finite, then we call (b, c) or b *locally finite*. A finite sequence $(x_0, \ldots, x_n)$ of pairwise distinct vertices such that $b(x_{i-1}, x_i) > 0$ for $i = 1, \ldots, n$ is called a *path* from x_0 to x_n. We say that (b, c) or b is *connected* if, for every two distinct vertices $x, y \in X$, there is a path from x to y.

Given a weighted graph (b, c) over X, we define the *generalized form* $\widetilde{Q} : C(X) \to [0, \infty]$ by

$$\widetilde{Q}(f) := \frac{1}{2} \sum_{x,y \in X} b(x, y) \, |f(x) - f(y)|^2 + \sum_{x \in X} c(x) \, |f(x)|^2,$$

and define the *generalized form domain* by

$$\widetilde{D} := \{ f \in C(X) : \widetilde{Q}(f) < \infty \}.$$

Functions in $\widetilde{D}$ are said to have *finite energy*.

Clearly, $C_c(X) \subseteq \widetilde{D}$ holds as $b(x, \cdot)$ is summable for every $x \in X$. By Fatou's lemma, $\widetilde{Q}$ is lower semi-continuous with respect to pointwise convergence. The form $\widetilde{Q}$ gives rise to a semi-scalar product on $\widetilde{D}$ via

$$\widetilde{Q}(f,g) = \frac{1}{2} \sum_{x,y \in X} b(x,y) \left(f(x) - f(y)\right) \left(g(x) - g(y)\right) + \sum_{x \in X} c(x) \, f(x) \, g(x).$$

If $c \not\equiv 0$ and b is connected, the form $\widetilde{Q}$ defines a scalar product. Furthermore, the form $\widetilde{Q}$ is compatible with normal contractions. For each *normal contraction* $C : \mathbb{R} \to \mathbb{R}$ (i.e., C satisfies $|C(x) - C(y)| \leq |x - y|$ and $|C(z)| \leq |z|$ for arbitrary $x, y, z \in \mathbb{R}$) and each $f \in C(X)$ we have

$$\widetilde{Q}(C \circ f) \leq \widetilde{Q}(f).$$

We will need the following well-known lemma (see e.g. [22]).

Lemma 1.1. *Let (b,c) be a connected graph over X. Then, for any $x, y \in X$, there exists $d(x,y) \geq 0$ such that for any $f \in \widetilde{D}$*

$$|f(x) - f(y)|^2 \leq \widetilde{Q}(f) \, d(x,y).$$

We now choose a vertex $o \in X$ and define a semi-scalar product $\langle \cdot, \cdot \rangle_o$ on $\widetilde{D}$ by

$$\langle f, g \rangle_o = \widetilde{Q}(f,g) + f(o) \, g(o),$$

for $f, g \in \widetilde{D}$ and the corresponding semi-norm

$$\|f\|_o := \langle f, f \rangle_o^{1/2} = \left(\widetilde{Q}(f) + |f(o)|^2\right)^{1/2}.$$

If b is connected, then $\langle \cdot, \cdot \rangle_o$ defines a scalar product and $\|\cdot\|_o$ defines a norm on $\widetilde{D}$. In this case $(\widetilde{D}, \langle \cdot, \cdot \rangle_o)$ is a Hilbert space.

The space $\widetilde{D}$ has received a lot of interest since first being studied in [67]. A systematic investigation was given in the work of Soardi [59]. In this context, we also recall the following well-known lemma, which follows from Lemma 1.1 above.

Lemma 1.2. *If (b,c) is connected, then the point evaluation map*

$$\delta_x : (\widetilde{D}, \|\cdot\|_o) \to \mathbb{R}, \quad u \mapsto u(x),$$

is continuous for each $x \in X$.

As the choice of $o \in X$ in the previous lemma is arbitrary, we directly obtain the following consequence.

Lemma 1.3. *Let (b,c) be a connected graph over X and let $o_1, o_2 \in X$ be arbitrary. Then, the norms $\|\cdot\|_{o_1}$ and $\|\cdot\|_{o_2}$ are equivalent.*

Remark. In general $\|\cdot\|_o$ and $\widetilde{Q}^{1/2}$ are not equivalent norms on $C_c(X)$. In fact, they can be shown to be equivalent if and only if the underlying graph is transient (see Appendix B).

Our main focus of interest is a special subspace of $\widetilde{D}$. It is introduced next.

Definition 1.4 (The space $\widetilde{D}_0$). Let (b, c) be a connected graph over X and let $o \in X$ be fixed. Define $\widetilde{D}_0$ to be the closure of $C_c(X)$ in $\widetilde{D}$ with respect to $\|\cdot\|_o$.

Remark. We think of the elements of $\widetilde{D}_0$ as functions satisfying "Dirichlet boundary conditions at infinity". As is clear from Lemma 1.3, $\widetilde{D}_0$ does not depend on the choice of $o \in X$. In fact, $f \in \widetilde{D}$ belongs to $\widetilde{D}_0$ if and only if there exists a sequence (φ_n) in $C_c(X)$ with $\varphi_n \to f$ pointwise and $\widetilde{Q}(\varphi_n - f) \to 0, n \to \infty$.

Lemma 1.5. *Let (b, c) be connected and let $o \in X$ be fixed. Then $(\widetilde{D}_0, \langle \cdot, \cdot \rangle_o)$ is a Hilbert space. Furthermore, for each normal contraction $C : \mathbb{R} \to \mathbb{R}$ and each $f \in \widetilde{D}_0$ we have $C \circ f \in \widetilde{D}_0$.*

Proof. The fact that $\widetilde{D}$ is a Hilbert space is a consequence of the lower semi-continuity of $\widetilde{Q}$ with respect to pointwise convergence. It then follows that $\widetilde{D}_0$ is a Hilbert space as well. Let $f \in \widetilde{D}_0$ and a normal contraction C be given. Let (φ_n) be a sequence in $C_c(X)$ which approximates f with respect to $\|\cdot\|_o$. The compatibility of $\widetilde{Q}$ with normal contractions implies $\widetilde{Q}(C \circ \varphi_n) \leq \widetilde{Q}(\varphi_n)$. In particular, we obtain that $(C \circ \varphi_n)$ is a bounded sequence in the Hilbert space $(\widetilde{D}_0, \langle \cdot, \cdot \rangle_o)$. By the Banach–Saks theorem, it possesses a subsequence $(C \circ \varphi_{n_k})$ whose Cesàro means

$$\psi_L := \frac{1}{L} \sum_{k=1}^{L} C \circ \varphi_{n_k}$$

converge to a limit $g \in \widetilde{D}_0$ with respect to $\|\cdot\|_o$ as $L \to \infty$. By Lemma 1.2, convergence with respect to $\|\cdot\|_o$ implies pointwise convergence. From the choice of φ_n we obtain $\psi_L \to C \circ f$ pointwise while the construction of ψ_L yields $\psi_L \to g$ pointwise. We conclude $C \circ f = g \in \widetilde{D}_0$. This finishes the proof.   □

Finally, we will need the concept of capacity. In our context the *capacity* $\mathrm{cap}(x)$ of a point $x \in X$ is defined as

$$\mathrm{cap}(x) = \inf\{\widetilde{Q}(\varphi) : \varphi \in C_c(X), \varphi(x) = 1\}.$$

We now assume that we are additionally given a measure m on X of full support. Then $\ell^2(X, m)$ is the vector space of square summable (with respect to m) elements of $C(X)$. It is a Hilbert space with respect to the inner product

$$\langle f, g \rangle := \sum_{x \in X} f(x)\, g(x)\, m(x).$$

The associated norm is given by

$$\|f\| := \langle f, f \rangle^{1/2}.$$

Whenever (b, c) is a graph over X and a measure m of full support is given, we obtain the bilinear form $Q_m^{(D)} := Q^{(D)}$ by restricting $\widetilde{Q}$ to

$$D(Q^{(D)}) := \overline{C_c(X)}^{\|\cdot\|_{\widetilde{Q}}},$$

where the closure is taken with respect to the norm

$$\|u\|_{\widetilde{Q}} := \left(\widetilde{Q}(u) + \|u\|^2\right)^{1/2}.$$

By definition $Q^{(D)}(f,g) = \widetilde{Q}(f,g)$ holds for $f, g \in D(Q^{(D)})$. Thus, the key ingredient in the definition of $Q^{(D)}$ is the domain $D(Q^{(D)})$. Here, we have the following characterization. We have not been able to find it in the literature. Thus, we include proof in Appendix A. It may be of interest in other situations as well.

Lemma 1.6 (Characterization of $D(Q^{(D)})$). *Let (b,c) be a graph over X and m be a measure on X of full support. Then,*

$$D(Q_m^{(D)}) = \widetilde{D}_0 \cap \ell^2(X, m).$$

There then exists a unique self-adjoint operator $L := L_m^{(D)}$ with

$$\langle Lf, g \rangle = Q^{(D)}(f, g)$$

for all f in the domain of the operator $D(L)$ and $g \in D(Q^{(D)})$.

This operator is non-negative and gives rise to a semigroup $e^{-tL_m^{(D)}}, t \geq 0$, and resolvents $(L_m^{(D)} + \alpha)^{-1}$, $\alpha > 0$. The semigroup and the resolvents are bounded operators on $\ell^2(X, m)$. It turns out that their restrictions to $C_c(X)$ can be uniquely extended to give bounded operators on $\ell^p(X, m)$ for all $p \in [1, \infty)$, see [44].

2. Uniformly transient graphs

In this section we introduce the class of graphs under considerations. They can be characterized in three different ways viz via a transience property, via an isoperimetric inequality and via a Feller-type property.

Theorem 2.1 (The main characterization). *Let (b,c) be a connected graph over X. Then, the following assertions are equivalent:*

  (i) *The inclusion $\widetilde{D}_0 \subseteq C_0(X)$ holds ("Uniform transience").*

 (ii) *There exists $C \geq 0$ with $\|\varphi\|_\infty \leq C\widetilde{Q}^{1/2}(\varphi)$ for all $\varphi \in C_c(X)$ ("Supnorm isoperimetricity").*

(ii′) *For one (all) $o \in X$ there exists $C_o \geq 0$ with $\|\varphi\|_\infty \leq C_o \|\varphi\|_o$ for all $\varphi \in C_c(X)$.*

(iii) *The inclusion $D(Q_m^{(D)}) \subseteq C_0(X)$ holds for any measure m on X of full support. ("Uniform strong Feller property").*

(iii′) *The inclusion $D(Q_m^{(D)}) \subseteq C_0(X)$ holds for any measure m on X of full support with $m(X) < \infty$.*

 (iv) *The inequality $\inf_{x \in X} \operatorname{cap}(x) > 0$ holds ("Uniform positive capacity of points").*

Remark. Of course, one can replace the condition $\varphi \in C_c(X)$ by $\varphi \in \widetilde{D}_0$ in (ii) and (ii′).

Proof. We first show (i) $\implies$ (ii). Choose $o \in X$ arbitrary. By (i) and the closed graph theorem, the map

$$(\widetilde{D}_0, \|\cdot\|_o) \to (C_0(X), \|\cdot\|_\infty), \quad f \mapsto f,$$

is continuous. Thus, there exists $C_1 \geq 0$ with

$$\|f\|_\infty \leq C_1 \|f\|_o$$

for all $f \in \widetilde{D}_0$. Therefore, it suffices to show that there exists $C_2 \geq 0$ with

$$\|\varphi\|_o \leq C_2 \widetilde{Q}^{1/2}(\varphi)$$

for all $\varphi \in C_c(X)$.

Assume the contrary. Then, we can chose a sequence $(\varphi_n) \in C_c(X)$ with

$$\|\varphi_n\|_o > n \widetilde{Q}^{1/2}(\varphi_n)$$

for all n. Without loss of generality, we can assume that $\|\varphi_n\|_o = 1$ for all n. This yields $\widetilde{Q}(\varphi_n) \to 0, n \to \infty$ and then $|\varphi_n(o)| \to 1, n \to \infty$. By Lemma 1.1 and $\widetilde{Q}(|\varphi_n|) \leq \widetilde{Q}(\varphi_n) \to 0$ it follows that $|\varphi_n| \to 1$ pointwise as $n \to \infty$. By Fatou's lemma, $\widetilde{Q}(1) \leq \lim_{n \to \infty} \widetilde{Q}(|\varphi_n|) = 0$ so that $1 \in \widetilde{D}$ and $c \equiv 0$. Then the preceding considerations show, in fact, that the sequence $(|\varphi_n|)$ from $C_c(X)$ converges to 1 in the sense of $\|\cdot\|_o$. This in turn implies $1 \in \widetilde{D}_0$ which contradicts (i).

Due to the equivalence of all norms $\|\cdot\|_o$, $o \in X$, given in Lemma 1.3, the validity of (ii$'$) for one $o \in X$ is equivalent to the validity of (ii$'$) for all $o \in X$. The implications (ii) $\implies$ (ii$'$) $\implies$ (i) are then clear.

The equivalence between (i) and (iii) and (iii$'$) follows easily from the characterization

$$D(Q_m^{(D)}) = \widetilde{D}_0 \cap \ell^2(X, m)$$

given in Lemma 1.6. For (iii$'$) $\implies$ (i) note that for every $f \in \widetilde{D}_0$, there exists a measure m such that $f \in \ell^2(X, m)$.

Finally, the equivalence between (ii) and (iv) follows easily from the definition of the capacity of a point. $\qquad\square$

Remark. Note that in the above proof of (i) $\implies$ (ii) we have actually shown that if $\widetilde{Q}^{1/2}$ and $\|\cdot\|_o$ are not equivalent norms on $C_c(X)$, then $c \equiv 0$ and $1 \in \widetilde{D}_0$ which is equivalent to recurrence as discussed in Appendix B.

Definition 2.2 (Uniformly transient graphs). Let (b, c) be a connected graph over X. Then, (b, c) is called *uniformly transient* if it satisfies one of the equivalent conditions of the previous theorem.

Remark (Context of the definition). a) As is well known, see e.g. [59] or Appendix B, a connected graph (b, c) is *recurrent* if and only if the constant function 1 belongs to $\widetilde{D}_0$ and $\widetilde{Q}(1) = 0$ holds. It is *transient* if it is not recurrent. Obviously,

the function 1 cannot belong to $\widetilde{D}_0$ if $\widetilde{D}_0$ is contained in $C_0(X)$. Thus, condition (ii) of Theorem 2.1 gives that uniformly transient graphs do indeed satisfy a very uniform version of transience.

b) Condition (iv) is the definition of uniform transience given in [4].

c) The semigroup $e^{-tL_m^{(D)}}$, $t \geq 0$, associated to a graph (b,c) over X satisfies the *Feller property* if it maps $C_c(X)$ into $C_0(X)$. Now, the spectral calculus easily gives that the semigroup always maps $\ell^2(X,m)$ into $D(L_m^{(D)}) \subseteq D(Q_m^{(D)})$ for any $t > 0$. Thus, conditions (iii) and (iii′) of Theorem 2.1 give a strong form of the Feller property. For a recent study of the Feller property on graphs we refer the reader to [65].

d) Let us emphasize that uniform transience (like transience) does not depend on the measure but only on the form $\widetilde{Q}$, i.e., the graph structure (b,c).

e) As is well known, transience is stable under extending graphs, i.e., transience of a subgraph implies transience of the whole graph. This stability is not true for uniform transience. Indeed, gluing together a uniformly transient graph with a recurrent graph will result in a graph which is not uniformly transient. The same is true regarding the stability of the Feller property, see [65].

f) For a probabilistic approach to transience and various further aspects of random walks on graphs we refer the reader to the standard monograph [63].

We next present three classes of graphs which are uniformly transient.

Recall that a connected graph is *canonically compactifiable* in the sense of [22] if any function in $\widetilde{D}$ is bounded.

Corollary 2.3 (Canonically compactifiable graphs are uniformly transient). *Let (b,c) be a connected canonically compactifiable graph over X. Then, (b,c) is uniformly transient.*

Proof. If a graph is canonically compactifiable, it is not hard to infer from the closed graph theorem that the map $(\widetilde{D}, \|\cdot\|_o) \to \ell^\infty(X)$ is continuous so that such graphs satisfy property (ii′) of Theorem 2.1 (see [22] as well for details). □

Let us now turn to the another large class of uniformly transient graphs. For a measure m, we say the operator $L_m^{(D)}$ has a *spectral gap* if the bottom of the spectrum of $L_m^{(D)}$ is positive.

Corollary 2.4 (Spectral gap). *Let (b,c) a graph over X and suppose $L_m^{(D)}$ has a spectral gap for m satisfying $\delta := \inf_{x \in X} m(x) > 0$ (e.g. $m \equiv 1$). Then, (b,c) is uniformly transient.*

Proof. As $L_m^{(D)}$ has a spectral gap $\lambda > 0$, we have $Q_m^{(D)}(\varphi) \geq \lambda \|\varphi\|^2$ for all $\varphi \in C_c(X)$. Thus, we have for all $\varphi \in C_c(X)$

$$\|\varphi\|_\infty^2 \leq \delta^{-1} \|\varphi\|^2 \leq \delta^{-1} \lambda^{-1} \widetilde{Q}(\varphi)$$

which yields the statement by (ii) of Theorem 2.1. □

The corollary above implies that all graphs with standard weights, i.e., $b\colon X \times X \to \{0,1\}$, $c \equiv 0$ and $m \equiv 1$, which satisfy a strong isoperimetric inequality (see e.g. [15], [16], [9]) are uniformly transient. This includes, for example, trees with all vertex degrees at least three and Cayley graphs of hyperbolic groups.

In the example below we show that the reverse implication of Corollary 2.3 does not hold. Thus, there exist uniformly transient graph which are not canonically compactifiable.

Example 2.5. Consider a tree with standard weights with vertex degree larger than two. Then, the Laplacian on the tree has a spectral gap and is uniformly transient by Corollary 2.4. However, it can be seen to be not canonically compactifiable. Consider a path of vertices (x_n) in the tree and denote by T_n the subtrees emanating from x_n, $n \geq 0$, (i.e., the vertices of T_n are those vertices of X which are closer to x_n than to x_k for any $k \neq n$). Define a function φ by letting $\varphi(x) = \sum_{j=1}^{n} j^{-1}$ for $x \in T_n$. It is immediate that $\varphi \in \widetilde{D}$ but φ is not bounded.

Another, more abstract way, to see that the tree is non-compactifiable can be seen as follows (using the notation of [22]): by Theorem 4.3 in [22], a graph is canonically compactifiable if and only if the diameter with respect to the metric ρ, which is the square root of the free effective resistance, is finite. On trees ρ^2 equals the metric d ([22], Lemma 8.1) which is the path metric with weights $1/b(x, y)$. Clearly, d has infinite diameter in the standard weight case.

Finally, recall that a graph (b, c) over X is called *quasi-vertex-transitive* if there exists an $n \in \mathbb{N}$ and vertices $x_1, \ldots, x_n$ such that for any vertex y in X there exists an $j \in \{1, \ldots, n\}$ and a bijection $h\colon X \to X$ with $h(y) = x_j$ and $c(h(z)) = c(z)$ and $b(h(v), h(w)) = b(v, w)$ for all $z, v, w \in X$. If n can be chosen as 1, the graph is called *vertex-transitive*.

Corollary 2.6. *If a graph (b, c) over X is quasi-vertex-transitive and transient, then (b, c) is uniformly transient.*

Proof. By transience and (iii) of Theorem B.2 for any vertex o in the graph there exists a constant C_o with $|\varphi(o)|^2 \leq C_o \widetilde{Q}(\varphi)$ for all $\varphi \in C_c(X)$. By quasi-vertex-transitivity the constant C_o can be chosen independently of $o \in X$. Now, the desired statement follows directly from Theorem 2.1. $\qquad\square$

3. A metric criterion for uniform transience

In this section we present a characterization for uniform transience in terms of boundedness with respect to a certain metric.

Let (b, c) be a connected graph over X and let $o \in X$ be arbitrary. We define for $x, y \in X$,

$$\gamma_o(x, y) := \sup \{|\varphi(x) - \varphi(y)| : \varphi \in C_c(X), \|\varphi\|_o \leq 1\}$$
$$= \sup \{|f(x) - f(y)| : f \in \widetilde{D}_0, \|f\|_o \leq 1\}.$$

Here, the last equality follows by approximation and Lemma 1.2. Similarly, we define for $x, y \in X$,

$$\gamma(x, y) := \sup \{|\varphi(x) - \varphi(y)| : \varphi \in C_c(X), \widetilde{Q}(\varphi) \leq 1\}$$
$$= \sup \{|f(x) - f(y)| : f \in \widetilde{D}_0, \widetilde{Q}(f) \leq 1\}.$$

The crucial properties of γ_o and γ are given in the next proposition.

Proposition 3.1 (Properties of γ and γ_o). *The map $\gamma_o \colon X \times X \to [0, \infty)$ is a metric for any $o \in X$ and so is the map $\gamma \colon X \times X \to [0, \infty)$. Any $f \in \widetilde{D}_0$ is uniformly continuous with respect to γ. More specifically, $f \in \widetilde{D}_0$ satisfies*

$$|f(x) - f(y)| \leq \widetilde{Q}^{1/2}(f)\gamma(x, y)$$

for all $x, y \in X$. Moreover, $\gamma_o \leq \gamma$ and

$$\gamma = \sup_{o \in X} \gamma_o$$

holds.

Proof. We first show that γ_o is a metric. The values of γ_o are finite by Lemma 1.1. Symmetry and triangle inequality are clear. As the characteristic function of any $x \in X$ belongs to $C_c(X)$, the map γ_o is not degenerate. Similarly, it can be shown that γ is a metric.

The statement concerning uniform continuity is clear from the definition of γ.

From the definition of γ and γ_o is is clear that $\gamma_o \leq \gamma$ holds for any $o \in X$. To show the statement on the supremum, let $x, y \in X$ be given and choose for $\varepsilon > 0$ arbitrary $\varphi \in C_c(X)$ with $\widetilde{Q}(\varphi) \leq 1$ and

$$\gamma(x, y) \leq |\varphi(x) - \varphi(y)| + \varepsilon.$$

If we now choose $o \in X$ with $\varphi(o) = 0$, then we obtain $\|\varphi\|_o = \widetilde{Q}(\varphi)^{1/2} \leq 1$ and, hence, $\gamma_o(x, y) \geq |\varphi(x) - \varphi(y)|$. Altogether, we arrive at

$$\gamma(x, y) \leq \gamma_o(x, y) + \varepsilon.$$

As $\varepsilon > 0$ is arbitrary this gives the desired statement on the supremum. $\square$

Remark. a) It is not hard to see that the supremum over $f \in \widetilde{D}_0$ can be replaced by a maximum both for γ and γ_o (compare [22] for a similar reasoning).

b) If (b, c) is transient, then $\|\cdot\|_o$ and $\widetilde{Q}^{1/2}$ are equivalent norms on $C_c(X)$ for any $o \in X$ (see Appendix B). Thus, in this case, γ and γ_o are equivalent metrics (see Corollary B.3).

c) It seems that the metric in the definition is the square root of the metric denoted as *wired resistance metric* in [38] (in the transient case).

d) The analogous situation where the supremum is taken over $f \in \widetilde{D}$ instead of $f \in \widetilde{D}_0$ has received quite some attention (see e.g. [22] and references therein). The arising metric is the (square root of the) *resistance metric*. It has also played a role in considerations inspired by non-commutative geometry [14], [33].

Recall that a metric space is said to have *finite diameter* if there exists $C \geq 0$ such that the distance between any two points is bounded by C.

Theorem 3.2 (Metric criterion for uniform transience). *Let (b, c) a connected graph over X. Then, the following statements are equivalent:*

(i) *The graph (b, c) is uniformly transient.*

(ii) *The diameter of (X, γ_o) is finite for one (all) $o \in X$.*

(iii) *The diameter of (X, γ) is finite.*

Proof. (i) $\Longrightarrow$ (iii): By uniform transience and (i) of Theorem 2.1, there exists $C \geq 0$ with $\|\varphi\|_\infty \leq CQ(\varphi)^{1/2}$ for all $\varphi \in C_c(X)$. This directly shows

$$|\varphi(x) - \varphi(y)| \leq 2 \|\varphi\|_\infty \leq 2C$$

for all $x, y \in X$ and all $\varphi \in C_c(X)$ with $\widetilde{Q}(\varphi) \leq 1$. This implies (iii).

(iii) $\Longrightarrow$ (ii): By the previous proposition, we have $\gamma_o \leq \gamma$ for any $o \in X$. This gives (ii) (for all $o \in X$).

(ii) $\Longrightarrow$ (i): Let (ii) be valid for one $o \in X$. Note that by Lemma 1.3 it then follows that (ii) is valid for all $o \in X$. There then exists $C \geq 0$ such that $|f(x) - f(o)| \leq C$ for any $x \in X$ and any $f \in \widetilde{D}_o$ with $\|f\|_o \leq 1$. This gives

$$|f(x)| \leq |f(x) - f(o)| + |f(o)| \leq C + \|f\|_o \leq C + 1$$

for any $x \in X$ and any $f \in \widetilde{D}_o$ with $\|f\|_o \leq 1$. This then implies

$$\|f\|_\infty \leq (C + 1) \|f\|_o$$

for any $f \in \widetilde{D}_0$ and by part (ii′) of Theorem 2.1 the desired statement follows. □

4. Uniform transience, the Royden compactification and the Dirichlet problem on the boundary

In this section we first discuss a characterization of uniform transience in terms of the Royden boundary of a graph. This will then allow us to show unique solvability of the Dirichlet problem for uniformly transient graphs.

Recall that the Royden compactification of a graph (b, c) is the unique (up to homeomorphism) compact Hausdorff space R such that the following three conditions are satisfied:

- X is a dense open subset of R.
- Each function of the algebra $\widetilde{D} \cap \ell^\infty(X)$ can be uniquely extended to a continuous function on R.
- The algebra $\widetilde{D} \cap \ell^\infty(X)$ separates the points of R.

One can construct R by applying Gelfand theory to the algebra generated by the uniform closure of $\widetilde{D} \cap \ell^\infty(X)$ and the constant function 1 which is a commutative C^*-algebra. For more details of this construction for graphs, we refer the reader to Section 4 of [22] (see [57] for the original work of Royden on manifolds).

Definition 4.1 (Royden algebra $\mathcal{A}$). Let (b, c) be a connected graph over X. The uniform closure of $\widetilde{D} \cap \ell^\infty(X)$ in $\ell^\infty(X)$ is called the *Royden algebra* of (b, c) and is denoted by $\mathcal{A}$. The unique continuous extension of $f \in \mathcal{A}$ to a function on R will be denoted by $\hat{f}$.

Remark. a) Since $\widetilde{D} \cap \ell^\infty(X)$ separates the points of R, the algebra $\mathcal{A} + \mathrm{Lin}\{1\}$ is isomorphic to $C(R)$ by the Stone–Weierstrass theorem.

b) It was shown in [22] that $1 \in \mathcal{A}$ if and only if $c \in \ell^1(X)$.

c) For a different construction of R (when $c \equiv 0$) using a somewhat smaller Banach algebra and further discussion we refer the reader to Chapter 6 of [59].

The set $\partial_R X = R \setminus X$ is called the *Royden boundary* of (b, c). The importance of the Royden boundary is due to the fact that harmonic functions in $\widetilde{D} \cap \ell^\infty(X)$ are uniquely determined by their values on the closed subset

$$\partial_h X := \{x \in \partial_R X \,:\, \hat{f}(x) = 0 \text{ for all } f \in \widetilde{D}_0 \cap \ell^\infty(X)\},$$

see the discussion below. We call $\partial_h X$ the *harmonic boundary* of (b, c). In general it is strictly smaller than the Royden boundary. However, it turns out that the validity of $\partial_R X = \partial_h X$ is equivalent to uniform transience.

Theorem 4.2. *Let (b, c) be a connected graph over X. Then, the following assertions are equivalent:*

(i) *(b, c) is uniformly transient.*

(ii) *The equality $\partial_h X = \partial_R X$ holds.*

Proof. (i) $\implies$ (ii): Assume (b, c) is uniformly transient, i.e., $\widetilde{D}_0 \subseteq C_0(X)$ holds. Since X is dense in R, we can approximate any $x \in R$ by a net $(x_i) \subseteq X$. Any such net converging to a boundary point will eventually leave every finite subset of X. As functions in $C_0(X)$ eventually become arbitrarily small, we infer $\lim_i f(x_i) = 0$, for each $f \in \widetilde{D}_0 \subseteq C_0(X)$. Now, the statement follows from the continuity of $\hat{f}$ on R and the fact that $\hat{f}|_X = f$.

(ii) $\implies$ (i): Assume $\partial_h X = \partial_R X$ and suppose that (b, c) is not uniformly transient. Then there exists a function $f \in \widetilde{D}_0$, a constant $C > 0$ and a sequence $(x_n) \subseteq X$ leaving every finite subset of X such that

$$|f(x_n)| \geq C, \text{ for all } n \geq 1.$$

Without loss of generality we may assume that f is bounded. As R is compact, the sequence (x_n) has a convergent subnet with limit $x \in \partial_R X$. From the continuity of $\hat{f}$ we infer $|\hat{f}(x)| \geq C > 0$. But this implies $x \notin \partial_h X$, which is a contradiction. $\square$

We will now study the relation of $\partial_h X$ and harmonic functions in $\mathcal{A}$. Let us first recall the definition and some properties of harmonic functions. Given a weighted graph (b, c) over X we introduce the associated *formal Laplacian* $\mathcal{L}$ acting on

$$\widetilde{F} := \left\{ f \in C(X) : \sum_{y \in X} b(x, y)\,|f(y)| < \infty \text{ for all } x \in X \right\}$$

as

$$\mathcal{L}f(x) := \sum_{y \in X} b(x, y)\,(f(x) - f(y)) + c(x)f(x).$$

The operator $\mathcal{L}$ can be seen as a discrete analogue to the Laplace–Beltrami operator on a Riemannian manifold. We will be interested in *harmonic functions,* i.e., functions $f \in \widetilde{F}$ satisfying $\mathcal{L}f = 0$. The formal operator $\mathcal{L}$ is intimately linked to the form $\widetilde{Q}$ by the following lemma.

Lemma 4.3 (Green's formula). *The inclusion $\widetilde{D} \subseteq \widetilde{F}$ holds and for each $f \in \widetilde{D}$ and each $g \in C_c(X)$ the equality*

$$\widetilde{Q}(f, g) = \sum_{x \in X} (\mathcal{L}f)(x)\,g(x)$$

is satisfied. Furthermore, if $f \in \widetilde{D}$ is harmonic, the above equality extends to all $g \in \widetilde{D}_0$ and is equal to 0.

Proof. We first show the inclusion $\widetilde{D} \subseteq \widetilde{F}$ by using the argument of the proof of Proposition 3.8 of [31]. Letting $B_x := \sum_{y \in X} b(x, y)$ we estimate

$$\sum_{y \in X} b(x, y)|f(y)| \leq \sum_{y \in X} b(x, y)\,|f(x) - f(y)| + \sum_{y \in X} b(x, y)\,|f(x)|$$

$$\leq \left(\sum_{y \in X} b(x, y) \right)^{1/2} \left(\sum_{y \in X} b(x, y)\,|f(x) - f(y)|^2 \right)^{1/2} + B_x|f(x)|$$

$$\leq B_x^{1/2}\,\widetilde{Q}^{1/2}(f) + B_x|f(x)|$$

which shows the claim. Combining $\widetilde{D} \subseteq \widetilde{F}$ and Lemma 4.7 of [30] we obtain the equality

$$\sum_{x \in X} \mathcal{L}f(x)g(x) = \widetilde{Q}(f, g)$$

for $f \in \widetilde{D}$ and $g \in C_c(X)$. The furthermore statement follows from this equality and the denseness of $C_c(X)$ in $\widetilde{D}_0$ with respect to $\|\cdot\|_o$. □

The following lemmas are well known and easy to check, see Section 5 of [22].

Lemma 4.4. *Assume that (b, c) is connected. If $f \in \widetilde{F}$ is non-negative and not constant and satisfies $\mathcal{L}f \leq 0$ on X, then f does not attain a maximum on X.*

Lemma 4.5. *If $f \in \widetilde{F}$ satisfies $\mathcal{L}f = 0$ on X, then $|f|$ satisfies $\mathcal{L}|f| \leq 0$.*

Corollary 4.6 (Maximum principle for uniformly transient graphs). *Assume (b, c) is uniformly transient and let $f \in \mathcal{A}$ be harmonic. Then,*

$$\|f\|_\infty = \|\hat{f}|_{\partial_h X}\|_\infty.$$

Proof. Combining Lemma 4.4 and Lemma 4.5 we conclude that a non-constant harmonic function $f \in \mathcal{A}$ does not attain its maximum on X. As its continuation $\hat{f}$ is continuous on the compact set R, it attains its maximum at $\partial_R X$. Since uniform transience implies $\partial_h X = \partial_R X$, the statement follows. $\qquad\square$

Remark. The maximum principle shows that on uniformly transient graphs harmonic functions in the Royden algebra $\mathcal{A}$ are uniquely determined by their value on the harmonic boundary. In the next section we will prove that an analogous statement holds for transient graphs and harmonic functions in $\widetilde{D} \cap \ell^\infty(X)$ (instead of $\mathcal{A}$).

For our subsequent considerations we will need the following well-known statement, see e.g. Theorem 6.3 in [59].

Proposition 4.7 (Royden decomposition for uniformly transient graphs). *Let (b, c) be a connected uniformly transient graph over X. Then for any $f \in \widetilde{D}$ there exist unique $f_0 \in \widetilde{D}_0$ and $f_h \in \widetilde{D}$ harmonic with $f = f_0 + f_h$. Moreover, if $f \in \ell^\infty(X)$, then $f_h \in \ell^\infty(X)$.*

Proof. Uniqueness. Assume there exist $g_0, f_0 \in \widetilde{D}_0$ and harmonic $g_h, f_h \in \widetilde{D}$ such that $f = f_0 + f_h = g_0 + g_h$. Since $g_h - f_h$ is harmonic and $f_0 - g_0 \in \widetilde{D}_0$, by Lemma 4.3 we obtain

$$0 = \widetilde{Q}(g_h - f_h, f_0 - g_0) = \widetilde{Q}(f_0 - g_0).$$

Thus, the connectedness of (b, c) implies that $f_0 - g_0$ is constant. If $c \not\equiv 0$, then we obtain $f_0 - g_0 = 0$ immediately. If $c \equiv 0$, then the transience of $(b, 0)$ implies that the only constant function in $\widetilde{D}_0$ vanishes everywhere (see Theorem B.2). This shows uniqueness.

Existence. By standard Hilbert space theory, there exists a minimizer of the functional $u \mapsto \widetilde{Q}(u - f)$ on the set $\widetilde{D}_0$ which we denote by f_0. Now, let $\varphi \in C_c(X)$ and $\varepsilon > 0$ be arbitrary. Then, since $f_0 + \varepsilon\varphi \in \widetilde{D}_0$, we obtain

$$\widetilde{Q}(f_0 - f) \le \widetilde{Q}(f_0 + \varepsilon\varphi - f) = \widetilde{Q}(f_0 - f) + 2\varepsilon\,\widetilde{Q}(f_0 - f, \varphi) + \varepsilon^2\,\widetilde{Q}(\varphi).$$

As ε and φ were arbitrary this shows that $f_h := f - f_0$ is harmonic.

The "moreover" statement. Assume that f is bounded. Since (b, c) is uniformly transient, we have $f_0 \in C_0(X) \subseteq \ell^\infty(X)$ and the statement follows from $f_h = f - f_0$. This finishes the proof. $\qquad\square$

We will now show that for each function φ in

$$C_0(\partial_h X) := \{\hat{f}|_{\partial_h X} : f \in \mathcal{A}\}$$

the Dirichlet problem

$$\mathcal{L}f = 0 \text{ on } X, \quad \hat{f}|_{\partial_h X} = \varphi$$

has a unique solution provided (b, c) is uniformly transient. Let us first identify the space $C_0(\partial_h X)$.

Lemma 4.8. *Let (b, c) be transient. If we equip $\partial_h X$ with the subspace topology and denote by $C(\partial_h X)$ its continuous functions, the following is true:*

- *If $1 \in \mathcal{A}$, the equality $C_0(\partial_h X) = C(\partial_h X)$ holds.*
- *If $1 \notin \mathcal{A}$, there exists a point $\infty \in \partial_h X$ such that*

$$C_0(\partial_h X) = \{f \in C(\partial_h X) : f(\infty) = 0\}.$$

Proof. The inclusion $C_0(\partial_h X) \subseteq C(\partial_h X)$ is obviously satisfied. As $\partial_h X$ is compact each function in $C(\partial_h X)$ can be extended to a function in $C(R)$ by Tietze's extension theorem. If $1 \in \mathcal{A}$, the algebra $\mathcal{A}$ is isomorphic to $C(R)$ and the equality $C_0(\partial_h X) = C(\partial_h X)$ follows. If $1 \notin \mathcal{A}$, the functions in $\mathcal{A}$ vanish at exactly one point $\infty \in \partial_R X$ (as otherwise, since $\mathcal{A}$ separates points, the Stone–Weierstrass theorem would imply $1 \in \mathcal{A}$). Since $\widetilde{D}_0 \subseteq \mathcal{A}$, we obtain $\infty \in \partial_h X$. This finishes the proof. $\square$

Theorem 4.9 (The DP on uniformly transient graphs). *Assume (b, c) is connected and uniformly transient. For each $\varphi \in C_0(\partial_h X)$ the equation*

$$\mathcal{L}f = 0 \text{ on } X, \quad \hat{f}|_{\partial_h X} = \varphi$$

has a unique solution $f_\varphi \in \mathcal{A}$. Furthermore, the mapping $C_0(\partial_h X) \to \mathcal{A}$, $\varphi \mapsto f_\varphi$ is an isometry.

Proof. We will only treat the case where $1 \in \mathcal{A}$. The other case can be treated similarly.

Uniqueness and the isometry property. This is an immediate consequence of the maximum principle for uniformly transient graphs, Corollary 4.6.

Existence. Consider the set $G \subseteq C_0(\partial_h X)$ of functions φ for which there exists $f \in \widetilde{D} \cap \ell^\infty(X)$ such that $\hat{f} = \varphi$ on $\partial_h X$.

Step 1. For each $\varphi \in G$ there exists a solution to the boundary value problem.

Proof of Step 1. Assume that $\varphi = \hat{f}$ with $f \in \widetilde{D} \cap \ell^\infty(X)$. By the Royden decomposition, Proposition 4.7, the function f has a unique decomposition $f = f_0 + f_h$ with $f_0 \in \widetilde{D}_0$ and $f_h \in \widetilde{D} \cap \ell^\infty(X)$ harmonic. Since $\widetilde{D}_0$ vanishes on $\partial_h X$, we obtain $\varphi = \hat{f} = \hat{f}_h$ on $\partial_h X$ and the claim follows.

Step 2. Suppose that $f_n \in \widetilde{D} \cap \ell^\infty(X)$ solves $\mathcal{L}f_n = 0$ and $\hat{f}_n|_{\partial_h X} = \varphi_n$. Furthermore, suppose $\varphi_n \to \varphi$ uniformly as $n \to \infty$. Then, (f_n) converges uniformly to $f \in \mathcal{A}$ that solves $\mathcal{L}f = 0$ and $\hat{f}|_{\partial_h X} = \varphi$.

Proof of Step 2. By the maximum principle, Corollary 4.6, we have

$$\|f_n - f_m\|_\infty = \|\varphi_n - \varphi_m\|_\infty \to 0$$

which implies the uniform convergence of (f_n) to some $f \in \ell^\infty(X)$. We obtain $f \in \mathcal{A}$ by the definition of $\mathcal{A}$ and $\hat{f} = \varphi$ on $\partial_h X$ by the uniform convergence. It follows from Lebesgue's theorem of dominated convergence that $\mathcal{L} f_n \to \mathcal{L} f$. This finishes the proof of Step 2.

We can now conclude the existence part as follows. Since $\widetilde{D} \cap \ell^\infty(X)$ separates the points of R, the set G separates the points of $\partial_h X$. Furthermore, as $1 \in \mathcal{A}$, it vanishes nowhere. Thus, by the Stone–Weierstrass theorem, G is dense in $C(\partial_h X) = C_0(\partial_h X)$ and the statement follows by combining Step 1 and Step 2. $\square$

Remark. Let us put the above theorem into the perspective of the existing literature. Even though the existence of solutions to the Dirichlet problem is well known, see e.g. Theorem 6.47 in [59], uniqueness statements for the Dirichlet problem for large classes of graphs seem to be rather new, confer [22] for the corresponding result for canonically compactifiable graphs. We would also like to emphasize the simplicity of our arguments compared to the discussion in [59] which is based on harmonic measures on $\partial_h X$. Combining the Royden decomposition (based on Hilbert space arguments) and a maximum principle (based on the compactness of $X \cup \partial_h X$ in the uniformly transient setting) together with the Stone–Weierstrass theorem already yields existence. In the next section we will also demonstrate how to use this method for arbitrary transient graphs. However, one needs to exercise some more care in this case when proving a maximum principle as $\partial_h X$ might be strictly smaller than $\partial_R X$.

5. The Dirichlet problem on general graphs

In this section we show how to solve the Dirichlet problem for arbitrary connected graphs (b, c) over X by only using a maximum principle and the Royden decomposition.

Proposition 5.1 (Royden decomposition). *Let (b, c) be a connected transient graph over X. Then, to any $f \in \widetilde{D}$, there exist unique $f_0 \in \widetilde{D}_0$ and $f_h \in \widetilde{D}$ harmonic with $f = f_0 + f_h$. Moreover, if $-a \le f \le b$ for some $a, b \ge 0$ then $-a \le f_h \le b$. In particular, if $f \in \ell^\infty(X)$, then $f_h \in \ell^\infty(X)$.*

Proof. Existence and uniqueness of the decomposition can be proven as in the proof of the Royden decomposition, Proposition 4.7.

The "moreover" statement. Suppose $-a \le f \le b$ for some $a, b \ge 0$. Since $C_c(X)$ is dense in $\widetilde{D}_0$ with respect to $\widetilde{Q}$ and by the construction of f_h (see proof of the Royden decomposition, Proposition 4.7), there exists a sequence $(\varphi_n) \subseteq C_c(X)$ such that
$$\widetilde{Q}(f - \varphi_n) \to \widetilde{Q}(f_h), \quad \text{as } n \to \infty.$$
Since we assumed the bound $-a \le f \le b$, the equality $((f - \varphi_n) \vee (-a)) \wedge b = f - \psi_n$ holds for some compactly supported function ψ_n. By the compatibility of $\widetilde{Q}$ with normal contractions we obtain
$$\widetilde{Q}(f - \varphi_n) \ge \widetilde{Q}(\psi_n - f) = \widetilde{Q}(\psi_n - f_0) + \widetilde{Q}(f_h) \ge \widetilde{Q}(f_h),$$

showing the convergence $\psi_n \to f_0$ with respect to $\widetilde{Q}$. By transience, this implies pointwise convergence $\psi_n \to f_0$, see Theorem B.2. As by construction the inequalities $-a \le f - \psi_n \le b$ hold, we obtain the statement. □

Remark. In [59] the above proposition is stated without the positivity assumption on a, b. However, as we deal with possibly non vanishing potentials c, we need to assume $a, b \ge 0$ to ensure the inequality

$$\widetilde{Q}((f \vee (-a)) \wedge b) \le \widetilde{Q}(f).$$

The following proposition is a variant of Theorem 6.7 in [59].

Proposition 5.2. *Assume (b, c) is transient. Let $f \in \widetilde{D} \cap \ell^\infty(X)$ be harmonic and assume $\hat{f} \ge -C$ on $\partial_h X$ for some $C \ge 0$. Then, $f \ge -C$ on X.*

Proof. For $\varepsilon > 0$, let

$$F := \{x \in R : \hat{f}(x) + \varepsilon \le -C\}.$$

By our assumptions we have $F \cap \partial_h X = \emptyset$. Thus, for each $x \in F$, there exists a function $g_x \in \widetilde{D}_0 \cap \ell^\infty(X)$, such that $\hat{g}_x(x) \ne 0$. By Lemma 1.5 we have $|g_x| \in \widetilde{D}_0$. Thus, we may assume $\hat{g}_x \ge 0$ on R. Furthermore, let U_x be a neighborhood of x, such that $\hat{g}_x(y) > 0$ for each $y \in U_x$. Obviously, F is closed and, hence, compact. Thus, there exist finitely many points $x_i \in F$ such that the corresponding U_{x_i} cover F. With

$$\widetilde{g} = \sum_i \hat{g}_{x_i},$$

we let $\alpha = \inf\{\widetilde{g}(x) : x \in F\}$ and set $g = \min\{1, \alpha^{-1}\widetilde{g}\}$. Then, clearly, $g \ge 1$ on F, and by Lemma 1.5 the restriction of g to X belongs to $\widetilde{D}_0 \cap \ell^\infty(X)$. By choice of the set F we obtain

$$f + \|f\|_\infty\, g \ge -\varepsilon - C \quad \text{on } X.$$

Applying the Royden decomposition, Proposition 5.1 to the function $f + \|f\|_\infty g$ and noting that f is its harmonic part, we obtain $f \ge -\varepsilon - C$. Since ε was arbitrary, the claim follows. □

Corollary 5.3 (Maximum principle). *Assume (b, c) is transient and that the harmonic boundary is non-empty. Then, for each harmonic $f \in \widetilde{D} \cap \ell^\infty(X)$ the following equality holds:*

$$\|f\|_\infty = \|\hat{f}|_{\partial_h X}\|_\infty.$$

Proof. The statement is a direct consequence of Proposition 5.2. □

Remark. Note that, in general, the maximum principle holds for harmonic functions in $\widetilde{D} \cap \ell^\infty(X)$ only. Indeed, the failure of the maximum principle for functions in $\mathcal{A}$ may lead to non uniqueness of solutions to the Dirichlet problem.

It may happen that $\partial_h X = \emptyset$ even if (b, c) is transient. This is due to the fact that graphs (b, c) with non vanishing potential c are always transient, see Theorem B.2. Indeed, the following characterization holds.

Proposition 5.4. *Assume (b, c) is connected. Then, the following assertions are equivalent:*

(i) $\partial_h X = \emptyset$.

(ii) $1 \in \widetilde{D}_0$.

(iii) $c \in \ell^1(X)$ *and* $(b, 0)$ *is recurrent.*

Proof. The implication (ii) $\Longrightarrow$ (i) is immediate from the definitions.

To show the reverse implication (i) $\Longrightarrow$ (ii) we use the construction of g in the proof of Proposition 5.2 (with the set F being replaced by R) to obtain $1 \in \widetilde{D}_0$.

We let $\widetilde{Q}', \widetilde{D}'$ and $\widetilde{D}'_0$ be the form and the corresponding spaces associated to the graph $(b, 0)$.

(ii) $\Longrightarrow$ (iii): Since $\widetilde{Q}'(f) \leq \widetilde{Q}(f)$, we obviously have $\widetilde{D}_0 \subseteq \widetilde{D}'_0$. Thus, if $1 \in \widetilde{D}_0$, we obtain $1 \in \widetilde{D}'_0$ and $c \in \ell^1(X)$. As $\widetilde{Q}'(1) = 0$, the graph $(b, 0)$ is recurrent.

(iii) $\Longrightarrow$ (ii): Let $(b, 0)$ be recurrent and $c \in \ell^1(X)$. Then, by the definition of recurrence given in Appendix B, there exists a sequence of compactly supported functions (φ_n) converging to 1 pointwise such that $\widetilde{Q}'(\varphi_n) \to 0$ as $n \to \infty$. By the compatibility of $\widetilde{Q}'$ with normal contractions we may assume $0 \leq \varphi_n \leq 1$ for each n. Using Lebesgue's theorem and $c \in \ell^1(X)$ this shows that $\widetilde{Q}(1 - \varphi_n) \to 0$, as $n \to \infty$ and, hence, $1 \in \widetilde{D}_0$. $\qquad\square$

Theorem 5.5 (Existence of solutions to the DP). *Assume (b, c) is connected. For each $\varphi \in C_0(\partial_h X)$ the equation*

$$\mathcal{L}f = 0 \text{ on } X, \quad \hat{f}|_{\partial_h X} = \varphi$$

has a solution $f_\varphi \in \mathcal{A}$.

Proof. We only need to consider the case when $\partial_h X \neq \emptyset$. If (b, c) is recurrent, we then have $1 \in \widetilde{D}_0$ and hence $\partial_h X = \emptyset$. Thus, we may assume that (b, c) is transient. Now, the proof can be carried out as in the existence part of the proof of Theorem 4.9 using the Royden decomposition (Proposition 5.1) and the maximum principle (Corollary 5.3) for transient graphs with non-vanishing harmonic boundary. $\qquad\square$

Remark. a) The class of functions on the boundary for which we prove existence of solutions to the Dirichlet problem is somewhat smaller that the one in [59]. However, using further monotone convergence arguments we could recover these results. We refrain from giving details.

b) As mentioned above, uniqueness of solutions is not clear anymore as the maximum principle does not seem to hold for arbitrary harmonic functions in $\mathcal{A}$.

6. Canonically compactifiable graphs and the one-point compactification

In this section we will have a look at canonically compactifiable graphs in the context of uniformly transient graph. In particular, we present a characterization of canonical compactifiability in terms of uniform transience. Moreover, we will provide necessary and sufficient conditions for the Royden compactification to agree with the one-point compactification. Finally, we will show how Euclidean lattices in dimension at least three serve as examples for our results.

We can now characterize the canonically compactifiable graphs within the class of uniformly transient graphs. Recall that canonically compactifiable means that $\widetilde{D} \subseteq \ell^\infty(X)$ and that function in $\widetilde{D}$ are said to have finite energy.

Theorem 6.1. *Let (b, c) be a graph over X. Then, the following assertions are equivalent:*

(i) *The graph (b, c) is canonically compactifiable.*

(ii) *The graph (b, c) is uniformly transient and any harmonic function of finite energy is bounded.*

Proof. (i) $\implies$ (ii): By the very definition of canonical compactifiability any function of finite energy is bounded. Moreover, it has already been shown in Corollary 2.3 that any canonically compactifiable graph is uniformly transient. Thus, (i) implies (ii).

(ii) $\implies$ (i): As the graph is uniformly transient, we have $\widetilde{D}_0 \subseteq C_0(X)$ and any element of $\widetilde{D}_0$ is bounded. Moreover, by assumption, any harmonic function of finite energy is bounded. Thus, the Royden decomposition, Proposition 4.7, shows boundedness of all functions of finite energy and (i) follows. □

The next result shows that uniformly transient graphs as well as canonically compactifiable graphs appear naturally whenever the Royden compactification is the one-point compactification.

Theorem 6.2. *Let $(b, 0)$ be a connected graph. Then, the following assertions are equivalent:*

(i) *The graph $(b, 0)$ is uniformly transient and any harmonic function of finite energy is constant.*

(ii) *The graph $(b, 0)$ is canonically compactifiable and any harmonic function of finite energy is constant.*

(iii) *The graph $(b, 0)$ is transient and the Royden compactification of X is the one-point compactification.*

Proof. The equivalence between (i) and (ii) is immediate from the previous theorem.

(i) $\implies$ (iii): By the assumption on the harmonic functions and the Royden decomposition, Proposition 5.1, we obtain that the smallest algebra $\mathcal{A}_0$ containing $\widetilde{D}$ and the constant functions is given as $\mathcal{A}_0 = \widetilde{D}_0 + \mathrm{Lin}\{1\}$, where $\mathrm{Lin}\{1\}$ denotes the

the linear span of the constant functions. Moreover, uniform transience implies $\widetilde{D}_0 \subseteq C_0(X)$. Now, (iii) is immediate.

(iii) $\implies$ (i): This can be inferred from [59]. We include a proof for the convenience of the reader. As the graph is transient, the algebra $\mathcal{A}_0 := \widetilde{D}_0 \cap \ell^\infty(X)$ does not contain any non vanishing constant functions. As the Royden compactification has only one boundary point, any element of $\mathcal{A}_0$ must then actually vanish on the Royden boundary. (Assume the contrary: then $\mathcal{A}_0$ contains an element of the form $1 + f$ with f vanishing at the boundary point. By adding a suitable function with compact support we then obtain that $\mathcal{A}_0$ contains a uniformly positive function. By a suitable cut-off, $\mathcal{A}_0$ must then contain the constant functions.) Thus, the algebra $\mathcal{A}_0$ is contained in $C_0(X)$. By an easy cut-off argument, this yields that, in fact, $\widetilde{D}_0$ is contained in $C_0(X)$. Hence, the graph is uniformly transient. Now, let a harmonic function of finite energy be given. Then, by the unique solvability of the boundary value problem proven in the previous section, this function must be a multiple of the constant function. $\square$

The assumptions of the theorem turn out to be satisfied for a rather well-known class of examples.

Example: $\mathbb{Z}^d$ for $d \geq 3$. We consider the Euclidean integer lattice $\mathbb{Z}^d$ with $d \geq 3$ with standard weights, i.e., $c \equiv 0$ and $b(x, y) = 1$ if x and y have Euclidean distance one and $b(x, y) = 0$, otherwise. This is obviously a vertex transitive graph and (since $d \geq 3$) it is transient. Hence, it is uniformly transient by Corollary 2.6. It is folklore[2] that these lattices are weak Liouville, i.e., any bounded harmonic function is constant. Furthermore, it is well known (cf. Theorem C.1) that the absence of non-constant bounded harmonic functions implies the absence of non-constant harmonic functions of finite energy. Thus, $\mathbb{Z}^d$ does not support non-constant harmonic functions of finite energy. Therefore, the previous theorem applies and we find that the Royden compactification of $\mathbb{Z}^d$ is just the one-point compactification. This is remarkable as the Royden compactification of the one dimensional integer lattice is far from being the one-point compactification, but rather contains a 'huge' number of additional points, cf. [66].

Remark. The considerations of the previous example can easily be adapted to any transient vertex-(quasi)transitive graph with the Liouville type property that harmonic functions of finite energy are constant.

7. Spectral theory of uniformly transient graphs

In this section we consider some spectral features of uniformly transient graphs on ℓ^p. Our results here generalize the corresponding results for canonically compactifiable graphs in [22] as canonically compactifiable graphs are uniformly transient. In fact, in terms of proofs, we basically adapt the proofs given in [22].

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[2]There is even a webpage, http://artofproblemsolving.com/community/c6h5327, gathering favorite proofs on this.

Recall that the semigroup e^{-tL}, $t > 0$, and the resolvents $(L + \alpha)^{-1}$, $\alpha > 0$, arising from the forms $Q^{(D)}$ associated to a graph are called *ultracontractive* if they are bounded operators from $\ell^2(X, m)$ to $\ell^\infty(X)$.

Lemma 7.1 (Uniformly transient graphs are ultracontractive). *Let (b, c) be a uniformly transient graph over X. Let m be a measure on X of full support and $L = L_m^{(D)}$ the operator associated to $Q_m^{(D)}$. Then, the associated semigroup and the resolvent are ultracontractive.*

Proof. We only consider the semigroup operators e^{-tL}, $t > 0$. The statements on resolvents can then be derived by standard techniques. Let $t > 0$ be arbitrary. We have
$$e^{-tL}\ell^2(X, m) \subseteq D(L) \subseteq D(Q_m^{(D)}) \subseteq C_0(X) \subseteq \ell^\infty(X).$$
Since e^{-tL} is a continuous operator on $\ell^2(X, m)$ we now obtain, by a simple application of the closed graph theorem, that e^{-tL} can be seen as a continuous map from $\ell^2(X, m)$ to $\ell^\infty(X)$. This shows the desired statement. □

Theorem 7.2 (Spectral properties of uniformly transient graphs). *Let (b, c) be a uniformly transient graph over X. Let m be a measure on X of full support with $m(X) < \infty$ and let $L = L_m^{(D)}$ be the operator associated to $Q_m^{(D)}$. Then, the following statements hold:*

(a) *The operators e^{-tL}, $t > 0$, and $(L + \alpha)^{-1}$, $\alpha > 0$, are trace class.*

(b) *The spectrum of L is purely discrete.*

(c) *The infimum of the spectrum of L is bounded below by*
$$\alpha := \frac{1}{C^2 m(X)},$$
   where C is the constant appearing in (i) *of Theorem* 2.1.

(d) *The semigroups e^{-tL}, $t > 0$, and the resolvents $(L + \alpha)^{-1}$, $\alpha > 0$, are norm analytic and compact on all $\ell^p(X, m)$, $1 \leq p \leq \infty$, and the spectra of the generators of e^{-tL} on $\ell^p(X, m)$ agree for all $1 \leq p \leq \infty$.*

Remark. The proof of (a) of this theorem uses a technique sometimes known as the factorization principle having its roots in Grothendieck's work [28]. For questions of the type considered here, it has been introduced in [60] to which we refer for further discussion (see [10] as well for further application in a similar spirit).

Proof. (a) By $m(X) < \infty$, there is a canonical continuous embedding
$$j : \ell^\infty(X) \to \ell^2(X, m), \quad f \mapsto f.$$
Thus, by Lemma 7.1,
$$e^{-tL} = je^{-tL}$$
is a composition of a continuous maps from $\ell^2(X, m)$ to $\ell^\infty(X)$ with a continuous map from $\ell^\infty(X)$ to $\ell^2(X, m)$.

By the Grothendieck factorization principle (see preceding remark) it is then a Hilbert–Schmidt operator. Then, the operator

$$e^{-tL} = e^{-\frac{t}{2}L}e^{-\frac{t}{2}L}$$

is trace class as it is a product of two Hilbert-Schmidt operators.

(b) This follows directly from (a).

(c) By the definition of $D(Q^{(D)})$ and the closed graph theorem, the estimate (ii) of Theorem 2.1 holds for all $f \in D(Q^{(D)})$. Thus, we obtain directly

$$\|f\|_m^2 \leq m(X)\,\|f\|_\infty^2 \leq m(X)\,C^2\,\widetilde{Q}(f)$$

for all $f \in D(Q^{(D)})$. This easily gives (c).

(d) This follows directly from Theorem 2.1.4 and Theorem 2.1.5 of [13].   □

Remark. (a) Graphs with discrete spectrum have been investigated in, e.g., [23], [42], [45], [47], and [64]. A general discussion of characterizations and perturbation theory of self-adjoint operators with compact resolvent is recently given in [50]. The p-independence of the spectra of general graph Laplacians has recently been investigated in [6].

(b) The fact that L has compact resolvent under the given assumptions is indeed a special case of Theorem 2.4.1 of [48]. There, it is shown in the context of resistance forms that L has compact resolvent if the finite measure m has a second moment with respect to the metric γ, i.e.,

$$\sum_{x \in X} \gamma(o,x)^2\,m(x) < \infty$$

holds for some $o \in X$. Note that γ is the square root of the resistance metric used in [48]. In the uniformly transient setting every finite measure m has a second moment with respect to γ since it is bounded in this situation, cf. Theorem 3.2.

(c) Note that the theorem applies in particular to $\mathbb{Z}^d$ for $d \geq 3$ as this is a uniformly transient graph (as discussed previously).

We finish this section by giving a lower bound for the eigenvalues of L.

Theorem 7.3. *Let (b,c) be a uniformly transient graph over X. Let m be a measure on X of full support with $m(X) < \infty$ and let $L = L_m^{(D)}$ be the operator associated to $Q_m^{(D)}$. Let (x_n) be an enumeration of X. Then, the inequality*

$$\frac{1}{C^2\,m(X \setminus \{x_1,\ldots,x_n\})} \leq \lambda_{n+1}(L)$$

holds, where C is the constant appearing in (ii) *of Theorem 2.1 and $\lambda_n(L)$ is the n-th eigenvalue of L counted with multiplicity.*

Proof. To prove the lower bound we use the min-max principle (see e.g. the textbook [61]) and the fact that $C_c(X)$ is a form core for $Q_m^{(D)}$ to obtain

$$\lambda_{n+1}(L) = \sup_{\varphi_1,\ldots,\varphi_n \in \ell^2(V,m)} \inf_{0 \neq \varphi \in C_c(X) \cap \{\varphi_1,\ldots,\varphi_n\}^\perp} \frac{\widetilde{Q}(\varphi)}{\|\varphi\|^2}$$

$$\geq \inf_{0 \neq \varphi \in C_c(X) \,:\, \varphi(x_1) = \cdots = \varphi(x_n) = 0} \frac{\widetilde{Q}(\varphi)}{\|\varphi\|^2}$$

$$\geq \inf_{0 \neq \varphi \in C_c(X) \,:\, \varphi(x_1) = \cdots = \varphi(x_n) = 0} \frac{\|\varphi\|_\infty^2}{C^2 \|\varphi\|^2},$$

where we have used the uniform transience of (b,c) for the last inequality. Now, the statement on the lower bound follows from the elementary fact that bounded functions φ that vanish on the set $\{x_1, \ldots, x_n\}$ satisfy

$$\|\varphi\|^2 \leq m(X \setminus \{x_1, \ldots, x_n\}) \, \|\varphi\|_\infty^2.$$

This finishes the proof. $\square$

Remark. The best possible lower bound in the above theorem is achieved by choosing an enumeration $x_1, x_2, \ldots$ of X that satisfies $m(x_n) \geq m(x_{n+1})$ for each $n \geq 1$. In the case where $m(X) < \infty$ such an enumeration can always be chosen because m has to vanish at infinity.

A. A characterization of the domain of the form with Dirichlet boundary conditions

In this section we provide a proof for Lemma 1.6, i.e., we show that

$$\overline{C_c(X)}^{\|\cdot\|_{\widetilde{Q}}} = \widetilde{D}_0 \cap \ell^2(X, m)$$

whenever (b,c) is a graph over X and m an measure on X of full support. The statement is a special case of a theorem about general Dirichlet forms.

Proof. Let (Q, D) be a Dirichlet form on $L^2(Y, \mu)$ (where Y is a locally compact Hausdorff space and μ a Radon measure of full support). One can associate to (Q, D) the extended Dirichlet space D_e, where $u \in D_e$ if and only if there exists a Q-Cauchy sequence (u_n) with $u_n \to u$ μ-almost surely.

It is well known that the equality $D = D_e \cap L^2(Y, \mu)$ holds (see Theorem 1.5.2 in [21]). In the situation of graphs, i.e., $Q = Q^{(D)}, D = D(Q^{(D)}), Y = X, \mu = m$, it is easy to check that $D(Q^{(D)})_e = \widetilde{D}_0$ (see Proposition 3.8 in [58]). $\square$

B. A characterization of transience via equivalence of norms

In this section we present a characterization of transience via an equivalence of norms. This characterization is probably well known. As we have not been able to

find it in the literature, we include a proof. We also point out that it sheds some additional light on the corresponding equivalence in our main characterization of uniform transience.

There are various equivalent characterizations of transience. The following definition suits our purposes best. For further details and a discussion of the relationship to other characterizations we refer the reader to [21], [59], and [63].

Definition B.1. A connected graph (b, c) is *recurrent* if and only if 1 belongs to $\widetilde{D}_0$ and $\widetilde{Q}(1) = 0$ holds. The graph is called *transient* if it is not recurrent.

Remark. Obviously, $\widetilde{Q}(1) = 0$ holds if and only if $c \equiv 0$ holds.

Theorem B.2. *Let (b, c) be a connected graph over the countably infinite X. Then, the following assertions are equivalent:*

 (i) *The graph (b, c) is transient.*

 (ii) *The norms $\|\cdot\|_o$ and $\widetilde{Q}^{1/2}$ are equivalent on $C_c(X)$ for every $o \in X$.*

 (ii′) *The norms $\|\cdot\|_o$ and $\widetilde{Q}^{1/2}$ are equivalent on $C_c(X)$ for one $o \in X$.*

 (iii) *For every $o \in X$ there exists $C_o \geq 0$ with $|\varphi(o)| \leq C_o \widetilde{Q}^{1/2}(\varphi)$ for all $\varphi \in C_c(X)$.*

 (iii′) *For one $o \in X$ there exists $C_o \geq 0$ with $|\varphi(o)| \leq C_o \widetilde{Q}^{1/2}(\varphi)$ for all $\varphi \in C_c(X)$.*

 (iv) $\mathrm{cap}(o) > 0$ *for every $o \in X$.*

 (iv′) $\mathrm{cap}(o) > 0$ *for one $o \in X$.*

Remark. Note that properties (i), (ii), and (iv) of Theorem 2.1 directly strengthen properties (i), (iii), and (iv) of the previous theorem.

Proof. The equivalences between (ii), (iii) and (iv) and between (ii′), (iii′) and (iv′) are clear.

(i) $\Longrightarrow$ (ii): It was remarked after the proof of Theorem 2.1 that if $\|\cdot\|$ and $Q^{1/2}$ are not equivalent norms on $C_c(X)$, then $1 \in \widetilde{D}_0$ and $c \equiv 0$ so that the graph is recurrent.

(ii) $\Longrightarrow$ (ii′): This is clear.

(ii′) $\Longrightarrow$ (i): Let $o \in X$ be given such that $\|\cdot\|_o$ and $\widetilde{Q}^{1/2}$ are equivalent on $C_c(X)$. Therefore, there exists $C > 0$ such that $\|\varphi\|_o \leq C\widetilde{Q}^{1/2}(\varphi)$ for all $\varphi \in C_c(X)$. Assume that (b, c) is recurrent. Then, 1 belongs to $\widetilde{D}_0$ and $c \equiv 0$ holds. Hence, there exists a sequence (φ_n) in $C_c(X)$ converging to 1 with respect to $\|\cdot\|_o$. In particular, $\lim_{n \to \infty} \widetilde{Q}(\varphi_n) = 0$. As $c \equiv 0$, we then obtain

$$1 = \|1\|_o = \lim_{n \to \infty} \|\varphi_n\|_o \leq \lim_{n \to \infty} C\widetilde{Q}^{1/2}(\varphi_n) = 0$$

giving a contradiction. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary B.3. *Let (b, c) be a connected transient graph. Then, γ and γ_o are equivalent metrics for any $o \in X$.*

C. Harmonic functions

In this appendix we discuss the relation of bounded harmonic functions and harmonic functions of finite energy. The following theorem is certainly well known to experts. Due to the lack of a reference we include a proof for the convenience of the reader.

Theorem C.1. *Let* $(b, 0)$ *be a transient connected graph over* X *and suppose that there exists a non-constant harmonic function of finite energy on* X. *Then there exists a non-constant bounded harmonic function on* X.

Proof. Let $f \in \widetilde{D}$ be harmonic and non-constant and consider the functions $f_n := (f \wedge n) \vee (-n)$. We use the Royden decomposition (Proposition 5.1) to obtain bounded $(f_n)_0 \in \widetilde{D}_0$ and bounded harmonic functions $(f_n)_h$ such that $f_n = (f_n)_0 + (f_n)_h$. It suffices to show that $(f_n)_h$ is non-constant for some n.

By the compatibility of $\widetilde{Q}$ with normal contractions we obtain $\widetilde{Q}(f_n) \leq \widetilde{Q}(f)$ for each n. This implies that (f_n) is a bounded sequence in $(\widetilde{D}, \langle \cdot, \cdot \rangle_o)$ and hence has a weakly convergent subsequence. Without loss of generality we assume that (f_n) itself converges weakly. From this and the pointwise convergence of f_n to f we obtain

$$\widetilde{Q}(f - f_n) = \widetilde{Q}(f) + \widetilde{Q}(f_n) - 2\widetilde{Q}(f, f_n)$$
$$\leq 2\left(\widetilde{Q}(f) - \widetilde{Q}(f, f_n)\right) \to 0, \quad \text{as } n \to \infty.$$

Now assume that for each n the function $(f_n)_h$ is constant. Using Lemma 4.3 we obtain

$$\widetilde{Q}(f - f_n) = \widetilde{Q}(f - (f_n)_h) + \widetilde{Q}((f_n)_0) \geq \widetilde{Q}(f - (f_n)_h) = \widetilde{Q}(f),$$

where the last equality follows from the fact that $(f_n)_h$ is constant and $c \equiv 0$. Taking the limit $n \to \infty$ shows $\widetilde{Q}(f) = 0$ which contradicts the assumption that f was not constant. This finishes the proof. $\square$

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