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Higher order rectifiability of measures via averaged discrete curvatures

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Abstract. We provide a sufficient geometric condition for $\mathbb{R}^n$ to be countably (μ, m) rectifiable of class $\mathscr{C}^{1,\alpha}$ (using the terminology of Federer), where μ is a Radon measure having positive lower density and finite upper density μ almost everywhere. Our condition involves integrals of certain many-point interaction functions (discrete curvatures) which measure flatness of simplexes spanned by the parameters.

1. Introduction

Let $\mathcal{H}^m$ denote the m dimensional Hausdorff measure over $\mathbb{R}^n$. A measure μ is said to be AD(m) *regular* if there exists $A \in [1,\infty)$ such that $A^{-1} \le r^{-m}\mu(\mathbf{B}(x,r)) \le A$ for $x \in \operatorname{spt}\mu$ and $r \in (0,1)$. We say that μ is *uniformly rectifiable* if it has the *big pieces of Lipschitz images* property, which means that it is AD(m) regular and there exist $\theta, M \in (0,\infty)$ such that for $x \in \operatorname{spt}\mu$ and $r \in (0,1)$ there exists a Lipschitz map $f\colon \mathbf{B}(0,r)\cap\mathbb{R}^m \to \mathbb{R}^n$ such that $\operatorname{Lip} f < M$ and $\mu(\mathbf{B}(x,r)\cap f[\mathbf{B}(0,r)]) \ge \theta r^m$. David and Semmes [9], [10] were studying uniformly rectifiable sets $\Sigma \subseteq \mathbb{R}^n$ in the context of harmonic analysis and in the search for a geometric criterion yielding boundedness of certain singular integral operators on $L^2(\mathcal{H}^m \llcorner \Sigma)$. To characterize these sets, they introduced the *beta-numbers* defined for $p \in [1,\infty)$, $x \in \mathbb{R}^n$, $r \in (0,\infty)$ as follows:

$$\beta^m_{\mu,p}(x,r) = r^{-1} \inf_L \Big(r^{-m} \int_{\mathbf{B}(x,r)} \operatorname{dist}(y,L)^p \,\mathrm{d}\mu(y) \Big)^{1/p},$$

where the infimum is taken with respect to all affine m dimensional planes L in $\mathbb{R}^n$. For $p, q \in [1,\infty)$ and $B \subseteq \mathbb{R}^n$ Borel we set

$$J_{\mu,p,q}(B) = \int_B \int_0^{\operatorname{diam} B} \beta^m_{\mu,q}(x,r)^p \,\frac{\mathrm{d}r}{r}\,\mathrm{d}\mu(x)\,.$$

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Following Definition 1.2 on p. 313 of [10], we say that μ satisfies the (p,q) *geometric lemma* if μ is AD(m) regular and there exists some $C \in (0,\infty)$ such that $J_{\mu,p,q}(B) \le C\mu(B)$ for all balls $B \subseteq \mathbb{R}^n$. Measures which satisfy the $(2,2)$ geometric lemma are uniformly rectifiable; see p. 22 in [10].

If $T = (a_0,\dots,a_{m+1}) \in (\mathbb{R}^n)^{m+2}$, let $\triangle T$ be the convex hull of the set $\{a_0,\dots,a_{m+1}\}$. For $B \subseteq \mathbb{R}^n$ Borel we define the *integral Menger curvature* of a measure μ on B by

$$\mathcal{M}_\mu(B) = \int_{B^{m+2}} \frac{\mathcal{H}^{m+1}(\triangle(a_0,\dots,a_{m+1}))^2}{\operatorname{diam}(\{a_0,\dots,a_{m+1}\})^{(m+2)(m+1)}} \,\mathrm{d}\mu^{m+2}(a_0,\dots,a_{m+1})\,.$$

This was (up to a constant) one of the functionals considered by Lerman and Whitehouse in [23], [24], were they showed that if μ is AD(m) regular, then $J_{\mu,2,2}$ is comparable to $\mathcal{M}_\mu$ on balls establishing a new characterization of uniform rectifiability and providing a connection between the beta-numbers and Menger-type curvatures. In Corollary 6.1 of [23] a characterization of measures satisfying the (p,p) geometric lemma for $1 \le p < \infty$ is also given.

Following §3.2.14 of Federer [12] and Anzellotti and Serapioni [4], we say that a set $E \subseteq \mathbb{R}^n$ is *countably* (μ,m) *rectifiable of class* $\mathscr{C}^{1,\alpha}$ if there exists a countably family $\mathcal{A}$ of m dimensional submanifolds of $\mathbb{R}^n$ of class $\mathscr{C}^{1,\alpha}$ such that $\mu(E \sim \bigcup\mathcal{A}) = 0$. Let $\boldsymbol{\Theta}^m_*(\mu,a)$ and $\boldsymbol{\Theta}^{m*}(\mu,a)$ denote the lower an upper densities of μ at $a \in \mathbb{R}^n$ as defined in §2.10.19 of [12].

Quite recently Meurer [33] proved, assuming a priori that $\Sigma \subseteq \mathbb{R}^n$ is merely a *Borel* set and $\mu = \mathcal{H}^m \llcorner \Sigma$, that if $\mathcal{M}_\mu(\mathbb{R}^n)$ is finite, then Σ is countably $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^1$. His work can be seen as a higher dimensional counterpart of the result of David [8] and Léger [22] established in connection with the famous Vitushkin conjecture on removable sets for bounded analytic functions. Azzam and Tolsa [5] and Tolsa [38] proved that if μ is Radon and satisfies $0 < \boldsymbol{\Theta}^{m*}(\mu,a) < \infty$ for μ almost all a, then $\mathbb{R}^n$ is countably (μ,m) rectifiable of class $\mathscr{C}^1$ *if and only if* $\int_0^1 \beta^m_{\mu,2}(x,r)^2 \frac{\mathrm{d}r}{r} < \infty$ for μ almost all a. However, in this generality the quantities considered in [33] and in [5, 38] are not known to be directly comparable – in contrast to the AD(m) regular case due to [23], [24]. A partial result concerning comparability of the two notions for non AD(m) regular measures was proven by the author in [17].

Higher order rectifiability, mostly of functions (see Definition 2.5 in [4]), has also been studied for a long time. In the context of functions it is rather called a *Lusin-type approximation*. Calderón and Zygmund (cf. Theorems 9 and 13 in [7]), Rešetnjak [34] and, more recently, Liu and Tai [27], Lin and Liu [26] gave conditions for higher order rectifiability of functions in terms of existence of approximating polynomials at almost all points. The classical Alexandrov's theorem (see [2] or [13]) and its generalization by Alberti [1] can be seen as a $\mathscr{C}^2$ rectifiability result for convex functions.

Higher order rectifiability is an important feature of sets in geometric analysis. It was observed by Schätzle in §3 of [35] that it can be used for proving regularity of sets governed by a PDE. This philosophy was employed later by Menne [28], [29], [30], [31], and Menne and the author [18] for showing certain regularity results for varifolds.

In the present article we provide a sufficient condition for rectifiability of class $\mathscr{C}^{1,\alpha}$ in terms of functionals similar to $\mathcal{M}_\mu$. Whenever μ is a measure over $\mathbb{R}^n$, and $l \in \{1, 2, \dots, m+2\}$, and $\alpha \in [0,1]$, and $p \in [1, \infty)$, and $a, a_0, \dots, a_{m+1} \in \mathbb{R}^n$, and $T = (a_0, \dots, a_{m+1})$, and $r \in (0, \infty]$ we set

$$\kappa(T) = \frac{\mathcal{H}^{m+1}(\triangle T)}{\operatorname{diam}(\triangle T)^{m+1}} \quad \text{if } \operatorname{diam}(\triangle T) > 0\,, \quad \kappa(T) = 0 \quad \text{otherwise}\,,$$

$$\kappa^{l,p,\alpha}_{\mu,a,r}(a_1, \dots, a_{l-1}) = \left(\mu^{\mathrm{m+2\text{-}l}}\right) \operatorname*{ess\,sup}_{a_l, \dots, a_{m+1} \in \mathbf{B}(a,r)} \frac{\kappa(a, a_1, \dots, a_{m+1})^p}{\operatorname{diam}(\{a, a_1, \dots, a_{m+1}\})^{m(l-1)+\alpha p}}\,,$$

$$\text{(1.1)} \quad \mathcal{K}^{l,p,\alpha}_\mu(a,r) = \int_{\mathbf{B}(a,r)} \cdots \int_{\mathbf{B}(a,r)} \kappa^{l,p,\alpha}_{\mu,a,r}(b_1, \dots, b_{l-1}) \,\mathrm{d}\mu(b_1) \cdots \,\mathrm{d}\mu(b_{l-1})$$

with the understanding that there is no essential supremum in case $l = m+2$ and there is no integral in case $l = 1$.

If $p > m(l-1)$, and $\alpha = 1 - m(l-1)/p$, and $\mu = \mathcal{H}^m \llcorner \Sigma$ for some compact set $\Sigma \subseteq \mathbb{R}^n$, then $\mathcal{E}(\mu) = \int \mathcal{K}^{l,p,\alpha}_\mu(a, \infty) \,\mathrm{d}\mu(a)$ coincides with one of the functionals analyzed by Strzelecki and von der Mosel [37], Blatt and the author [6], Szumańska and the author [21], Strzelecki and von der Mosel and the author [19], [20], and by the author [16]. Analogues of the Morrey–Sobolev embedding theorem for sets, where κ plays roughly the role of the second weak derivative, were studied in [37], [16], and [21]. In [20] the authors use $\mathcal{E}$ to solve geometric variational problems with topological constraints. In [6] and [19], a full geometric characterization of graphs of some (fractional) Sobolev maps is established.

If $l = m+2$, and $p = 2$, and $\alpha = 0$, then $\mathcal{E}(\mu)$ equals $\mathcal{M}_\mu(\mathbb{R}^n)$ which was considered in [23], [24], and [33]. Formally, if one sets $l = m+2$, and $p = 2$, and $\alpha = m - (m+2)(m+1)/2$, then $\mathcal{E}$ yields also one of the quantities used in §4 of [25] to approximate the least square error of a measure.

Our main result reads as follows.

Theorem 1.1. *Let μ be a Radon measure over $\mathbb{R}^n$ such that*

$$\text{(1.2)} \qquad 0 < \boldsymbol{\Theta}^m_*(\mu, x) \le \boldsymbol{\Theta}^{m*}(\mu, x) < \infty \quad \text{for } \mu \text{ almost all } x\,,$$

and $l \in \{1, 2, \dots, m+2\}$, and $\alpha \in (0,1]$, and $p \in [1, \infty)$. Assume $\mathcal{K}^{l,p,\alpha}_\mu(a,1) < \infty$ for μ almost all a. Then $\mathbb{R}^n$ is countably (μ, m) rectifiable of class $\mathscr{C}^{1,\alpha}$ and μ is absolutely continuous with respect to $\mathcal{H}^m$.

Moreover, if $\alpha < 1$, then for any $\varepsilon \in (0, 1-\alpha)$ there exists a measure μ satisfying (1.2) and $\mathcal{K}^{l,p,\alpha}_\mu(a) < \infty$ for μ almost all a and such that $\mathbb{R}^n$ is not countably (μ, m) rectifiable of class $\mathscr{C}^{1,\alpha+\varepsilon}$.

The converse of Theorem 1.1 does not hold due to the example constructed in [21]. In view of the characterization of graphs of functions of Sobolev–Slobodeckij class $W^{1+\alpha,p}$ obtained in [6] one could expect that finiteness μ almost everywhere of $\mathcal{K}^{l,p,\alpha}_\mu$ should rather characterize "rectifiability of class $W^{1+\alpha,p}$" – a notion not yet defined.

We prove the first part of Theorem 1.1 in Section 7. First we use standard methods of geometric measure theory to reduce the problem to the case when μ is roughly AD(m) regular, which is possible because we assume $0 < \boldsymbol{\Theta}^m_*(\mu, x)$ for μ almost all x. Then, for μ almost all points x, we find m dimensional planes that approximate the measure in smaller and smaller scales around x and, due to the condition $\alpha > 0$, we prove that they converge in the Grassmannian to some plane which must contain the approximate tangent cone of μ at x – this is the heart of the proof; see Lemma 7.4. From there we conclude using §2.8(5) of Allard [3] that $\mathbb{R}^n$ is countably (μ, m) rectifiable of class $\mathscr{C}^1$. This allows to reduce the problem further to the case when $\mu = \mathcal{H}^m \llcorner \Sigma$, where Σ is a subset of a graph of some $\mathscr{C}^1$ function. Next, we use the decay rates obtained in Lemma 7.4 together with Corollary 5.5, which is a consequence of Lemma A.1 in [35], to get the conclusion.

The proof of the second part of Theorem 1.1 is contained in Section 8. It bases on the fact that for any $0 \le \alpha < \beta \le 1$ there exists a set which is a graph of some $\mathscr{C}^{1,\alpha}$ map but is not countably $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{1,\beta}$ which can be deduced from the Appendix of [4]. In Section 8 we also give examples of other functions that could be used in Theorem 1.1 in place of κ.

Our result can be seen as an extension of [33] but is *not* stronger. A glance at the outline of the proof given above reveals why our problem is extremely simpler than that considered in [33] or [5]. The first reason is that we assume $\alpha > 0$ which immediately gives convergence of the approximating planes in Lemma 7.4. The second, but actually the crucial one, is that we assume $\boldsymbol{\Theta}^m_*(\mu, a) > 0$ for μ almost all a which allows to reduce, roughly, to the AD(m) regular case.

2. Notation

In principle we shall use the book of Federer [12] as our main reference and source of definitions, and we shall adopt some, but not all, of its notation. In particular we shall write $\{x \in X : P(x)\}$, in contrast to $X \cap \{x : P(x)\}$, for the set of those $x \in X$ which satisfy some predicate P. We also prefer to say that a function is "injective" rather than "univalent". The symbols $\mathbb{R}$ and $\mathbb{N}$ shall be used for the set of real and natural numbers including zero. Moreover, whenever $s, t \in \mathbb{R} \cup \{-\infty, \infty\}$ and $s < t$, we shall write (s, t), $[s, t]$ for the open and closed intervals in $\mathbb{R}$ and also $(s, t]$ and $[s, t)$ with the usual meaning. If A and B are sets, we write $A \sim B$ for the set theoretic difference. If X is a vector space, $A, B \subseteq X$, $c \in X$ and $r \in (0, \infty)$ we adopt the notation $c + A = \{c + a : a \in A\}$, $rA = \{ra : a \in A\}$ and $A + B = \{a + b : a \in A,\ b \in B\}$. When we write $\mathbb{R}^n$ we always mean the n dimensional Euclidean space with the standard scalar product denoted $u \bullet v$ for $u, v \in \mathbb{R}^n$. We shall write $\mathbf{U}(a, r)$ and $\mathbf{B}(a, r)$ for the open and closed ball centered at a and of radius r in the metric space to which a belongs to. We adopt the definition of a *measure* from §2.1.2 of [12], which is sometimes called an *outer measure* in the literature. The symbols $\mathcal{H}^m$ and $\mathcal{L}^m$ stand for the m dimensional Hausdorff and Lebesgue (outer) measures as defined in §2.10.2(1) and §2.6.5 of [12]. Whenever $m \in \mathbb{N} \sim \{0\}$ we use the symbol $\boldsymbol{\alpha}(m)$ for the Lebesgue measure of the unit ball in $\mathbb{R}^m$. If X and Y are normed vector spaces, $U \subseteq X$ is open, $k \in \mathbb{N}$, and $\alpha \in [0,1]$, then a function $f\colon U \to Y$ is said to be of class $\mathscr{C}^{k,\alpha}$ if f is continuous, has continuous derivatives up to order k (cf. §3.1.1 and §3.1.11 of [12]), and the k^{th} order derivative $D^k f$ satisfies the Hölder condition with exponent α (cf. §5.2.1 of [12]); in this case we write $f \in \mathscr{C}^{k,\alpha}(U,Y)$. The image of a set $A \subseteq X$ under a mapping $f\colon X \to Y$ is denoted $f[A]$ and similarly $f^{-1}[B]$ denotes the preimage of a set $B \subseteq Y$. We write id_X for the identity function on X. Whenever X is a metric space, $A \subseteq X$, and $x \in X$, we use the notation $\operatorname{dist}(x,A)$ for the distance of x from A. We write A^l to denote the Cartesian product of $l \in \mathbb{N} \sim \{0\}$ copies of a set A and if $f\colon A \to B$, then $f^l\colon A^l \to B^l$ is the Cartesian product of l copies of f, i.e., $f^l(a_1,\dots,a_l) = (f(a_1),\dots,f(a_l))$. Similarly, μ^l shall denote the product of l copies of a measure μ (cf. §2.6.1 of [12]). For the Grassmannian of m dimensional planes in $\mathbb{R}^n$ we write $\mathbf{G}(n,m)$ (cf. §1.6.2 of [12]). With each $P \in \mathbf{G}(n,m)$ we associate the orthogonal projection $P_\natural : \mathbb{R}^n \to P \subseteq \mathbb{R}^n$ onto P and the orthogonal complement $P^\perp = \ker P_\natural$. Whenever $v_1,\dots,v_k$ are vectors in some vector space X, we write $\operatorname{span}\{v_1,\dots,v_l\}$ for the linear span of these vectors. If μ measures some set X, $f\colon X \to \mathbb{R}$ is μ-measurable and $A \subseteq X$ is μ-measurable, we write $⨍_A f \,\mathrm{d}\mu = \mu(A)^{-1} \int_A f\,\mathrm{d}\mu$ for the *mean value* of f on A. By $X \ni x \mapsto f(x)$ we mean an *unnamed function* with domain X mapping $x \in X$ to $f(x)$. For the *essential supremum* of a function $f\colon X \to \mathbb{R}$ with respect to a measure μ over X we write $(\mu)\operatorname{ess\,sup} f$, which is defined to be equal to $(\mu)_{(\infty)}(f)$ in the notation of Federer (see §2.4.12 of [12]). To optimize space we shall sometimes write $(\mu)\operatorname{ess\,sup}_{x\in X} f(x)$ instead of $(\mu)\operatorname{ess\,sup}(X \ni x \mapsto f(x))$.

The reader might also want to recall the definitions of: the space of orthogonal projections $\mathbf{O}^*(n,m)$ (cf. §1.7.4 of [12]), the exterior algebra $\bigwedge_* X$ of a vector space X with its associated wedge product $\wedge$ (cf. §1.3 of [12]), tangent cone $\operatorname{Tan}(S,x)$ of a set $S \subseteq \mathbb{R}^n$ (cf. §3.1.21 of [12]) at $x \in \mathbb{R}^n$, approximate m-tangent cone $\operatorname{Tan}^m(\mu,x)$ of a measure μ (cf. §3.2.16 of [12]) at $x \in \mathbb{R}^n$.

We often write "$\Gamma_{x.y}$" to denote the number (the "constant") that appeared under the name "Γ" in the formulation of item x.y. Throughout the paper n and m shall denote two integers satisfying $1 \le m < n$.

3. Approximate tangent cones

Remark 3.1. For $a, v \in \mathbb{R}^n$, $\varepsilon \in (0,\infty)$ define the cone

$$\mathbf{E}(a,v,\varepsilon) = \left\{ b \in \mathbb{R}^n : \exists\, t \in (0,\infty)\ |t(b-a) - v| < \varepsilon \right\}.$$

Then (cf. §3.2.16 of [12]) $v \in \operatorname{Tan}^m(\mu,a)$ if and only if

$$\mathbf{\Theta}^{*m}(\mu \llcorner \mathbf{E}(a,v,\varepsilon)) > 0 \quad \text{for all } \varepsilon \in (0,\infty).$$

Note that, if $0 < \varepsilon < |v|$, then $b \in \mathbf{E}(a,v,\varepsilon)$ if and only if

$$b \neq a \quad \text{and} \quad \frac{b-a}{|b-a|} \bullet \frac{v}{|v|} > \left(1 - \frac{\varepsilon^2}{|v|^2}\right)^{1/2}.$$

The notion of an approximate tangent cone $\operatorname{Tan}^m(\mu, a)$ (cf. §3.2.16 of [12]) that we use is different from the notion of the tangent space defined by blow-ups in §11.2 of [36]. However, we have the following.

Proposition 3.2. *Suppose μ is a Radon measure over $\mathbb{R}^n$, and $T \in \mathbf{G}(n,m)$, and $a \in \mathbb{R}^n$, and*

$$\lim_{r \downarrow 0} r^{-m} \int_{\mathbf{B}(a,r)} \frac{|T_\natural^\perp (b-a)|}{|b-a|} \, \mathrm{d}\mu(b) = 0 \,. \tag{3.1}$$

Then $\operatorname{Tan}^m(\mu,a) \subseteq T$.

Proof. If $\operatorname{Tan}^m(\mu,a) \sim \{0\} = \varnothing$, the conclusion is evident. In all other cases we shall prove the proposition by contradiction. If there existed $v \in \operatorname{Tan}^m(\mu,a) \sim T$; then $|T_\natural^\perp v| > 0$. Recalling Remark 3.1, if $\varepsilon \in (0,\infty)$ satisfied $\varepsilon < \frac{1}{4}|T_\natural^\perp v|$, then for each $b \in \mathbf{E}(a,v,\varepsilon)$ setting $t = (b-a) \bullet v |b-a|^{-2}$, we would have

$$\begin{aligned} \left| \frac{b-a}{|b-a|} \bullet \frac{v}{|v|} \right| \cdot \left| T_\natural^\perp \frac{b-a}{|b-a|} \right| = \frac{|T_\natural^\perp t(b-a)|}{|v|} &\geq \frac{|T_\natural^\perp v| - |T_\natural^\perp (v - t(b-a))|}{|v|} \\ &\geq \frac{|T_\natural^\perp v| - \varepsilon}{|v|} \geq \frac{3}{4} \frac{|T_\natural^\perp v|}{|v|} > 0 \,. \end{aligned}$$

Hence, for any $r > 0$ and $\varepsilon \in \left(0, \frac{1}{4}|T_\natural^\perp v|\right)$, we would obtain

$$\begin{aligned} r^{-m} \int_{\mathbf{B}(a,r)} \frac{|T_\natural^\perp (b-a)|}{|b-a|} \, \mathrm{d}\mu(b) &\geq r^{-m} \int_{\mathbf{E}(a,v,\varepsilon) \cap \mathbf{B}(a,r)} \frac{|T_\natural^\perp (b-a)|}{|b-a|} \, \mathrm{d}\mu(b) \\ &\geq \frac{3}{4} \frac{|T_\natural^\perp v|}{|v|} \left(1 - \frac{\varepsilon^2}{|v|^2} \right)^{-1/2} \frac{\mu(\mathbf{E}(a,v,\varepsilon) \cap \mathbf{B}(a,r))}{r^m} \,. \end{aligned} \tag{3.2}$$

Since we assumed $v \in \operatorname{Tan}^m(\mu,a)$, we could argue that $\mathbf{\Theta}^{*m}(\mu \llcorner \mathbf{E}(a,v,\varepsilon), a) > 0$ for all $\varepsilon \in (0,\infty)$. Then, for $\varepsilon \in (0, \frac{1}{4}|T_\natural^\perp v|)$, taking $\limsup_{r \downarrow 0}$ on both sides of (3.2), we would get

$$\limsup_{r \downarrow 0} r^{-m} \int_{\mathbf{B}(a,r)} \frac{|T_\natural^\perp (b-a)|}{|b-a|} \, \mathrm{d}\mu(b) > 0 \,,$$

which is impossible due to the assumption (3.1). Thereby, we conclude that it was not possible to choose $v \in \operatorname{Tan}^m(\mu,a) \sim T$; thus $\operatorname{Tan}^m(\mu,a) \subseteq T$. □

Remark 3.3. Observe that

$$\lim_{r \downarrow 0} r^{-m-1} \int_{\mathbf{B}(a,r)} |T_\natural^\perp (b-a)| \, \mathrm{d}\mu(b) = 0 \tag{3.3}$$

implies (3.1) which can be verified by representing the integral over $\mathbf{B}(a,r)$ by a series of integrals over "annuli" $\mathbf{B}(a, 2^{-k}r) \sim \mathbf{U}(a, 2^{-k-1}r)$ for $k \in \mathbb{N}$. Hence, the conclusion of Proposition 3.2 holds also with assumption (3.1) replaced by (3.3).

4. Graphs of functions and the slope of the tangent plane to a graph

Remark 4.1. A convenient way to work with graphs of functions defined on some $T \in \mathbf{G}(n,m)$ and with values in $T^\perp$ is to express the function using orthonormal bases for T and $T^\perp$. To do that we choose orthogonal projections $\mathfrak{p} \in \mathbf{O}^*(n,m)$ and $\mathfrak{q} \in \mathbf{O}^*(n, n-m)$ (cf. §1.7.4 of [12]) such that $\operatorname{im} \mathfrak{p}^* = T$ and $\operatorname{im} \mathfrak{q}^* = T^\perp$. Then if $A \subseteq \mathbb{R}^m$ and $f \colon A \to \mathbb{R}^{n-m}$ we define $F = \mathfrak{p}^* + \mathfrak{q}^* \circ f$ and then $\operatorname{im} F$ is the graph of f with $\mathbb{R}^m$ identified with T.

The following remark, made in the spirit of §8.9(5) of [3], allows to express the "slope" of the tangent plane to a graph by the norm of the derivative of the function; see Corollary 4.3.

Remark 4.2. Assume $T \in \mathbf{G}(n,m)$ and $\eta \in \operatorname{Hom}(T, T^\perp)$. Set $S = \{v + \eta(v) : v \in T\} \in \mathbf{G}(n,m)$. Observe that the function $[0,\infty) \ni t \mapsto t^2(1+t^2)^{-1}$ is increasing; hence, using §8.9(3) of [3],

$$\begin{aligned}
\|S_\natural - T_\natural\|^2 &= \|T_\natural^\perp \circ S_\natural\|^2 \\
&= \sup\left\{ |T_\natural^\perp u|^2 |u|^{-2} : u \in S \sim \{0\} \right\} \\
&= \sup\left\{ |\eta(w)|^2 |w + \eta(w)|^{-2} : w \in T \sim \{0\} \right\} \\
&= \sup\left\{ |\eta(w)|^2 (1 + |\eta(w)|^2)^{-1} : w \in T\,,\ |w| = 1 \right\} = \frac{\|\eta\|^2}{1 + \|\eta\|^2}\,.
\end{aligned}$$

Corollary 4.3. *Let $\mathfrak{p}$, $\mathfrak{q}$, T, A, f, and F be as in Remark 4.1. Assume that $\Sigma \subseteq \operatorname{im} F$, $a \in \Sigma$ is such that $\operatorname{Tan}(\Sigma, a) \in \mathbf{G}(n,m)$ and that f is differentiable at $x = \mathfrak{p}(a)$. Then, employing Remark 4.2 with $Df(x)$ in place of η, we obtain*

$$\|\operatorname{Tan}(\Sigma,a)_\natural - T_\natural\|^2 = \frac{\|Df(x)\|^2}{1 + \|Df(x)\|^2}\,. \tag{4.1}$$

Let $b \in \Sigma$ and set $y = \mathfrak{p}(b)$. Then $b = F(y)$ and $DF(x)(y-x) \in \operatorname{Tan}(\Sigma, a)$. Define

$$\begin{gathered}
u = \mathfrak{q}^*(f(y) - f(x) - Df(x)(y-x)) = F(y) - F(x) - DF(x)(y-x) \in T^\perp \\
\textit{and} \quad v = \operatorname{Tan}(\Sigma,a)_\natural^\perp (b-a) = \operatorname{Tan}(\Sigma,a)_\natural^\perp u\,.
\end{gathered}$$

Then, by (4.1), we get

$$\begin{aligned}
|u - v| = |\operatorname{Tan}(\Sigma,a)_\natural u| = |\operatorname{Tan}(\Sigma,a)_\natural T_\natural^\perp u| &\le \|\operatorname{Tan}(\Sigma,a)_\natural - T_\natural\| |u| \\
&= \frac{\|Df(x)\|\, |u|}{(1 + \|Df(x)\|^2)^{1/2}}\,.
\end{aligned}$$

In consequence,

$$\begin{aligned}
|\operatorname{Tan}(\Sigma,a)_\natural^\perp (b-a)| &\le |f(y) - f(x) - Df(x)(y-x)| \\
&\le \left(1 - \frac{\|Df(x)\|}{(1 + \|Df(x)\|^2)^{1/2}}\right)^{-1} |\operatorname{Tan}(\Sigma,a)_\natural^\perp (b-a)|\,.
\end{aligned} \tag{4.2}$$

5. Higher order rectifiability criterion for graphs

To talk about approximate features of functions (e.g. limits, continuity, differentiability; cf. §2.9.2, §3.1.2, §3.2.16 of [12]) one needs to provide two parameters: a measure and a Vitali relation (cf. §2.8.16 of [12]). It will be convenient to define a standard family of Vitali relations.

Definition 5.1. For $k \in \mathbb{N} \sim \{0\}$, we set

$$\mathcal{V}_k = \left\{ (x, \mathbf{B}(x,r)) : x \in \mathbb{R}^k \,,\ r \in (0,\infty) \right\}.$$

Remark 5.2. If $k \in \mathbb{N} \sim \{0\}$ and ϕ is a measure over $\mathbb{R}^k$ such that all open sets are ϕ measurable and $\phi(A) < \infty$ for all bounded sets $A \subseteq \mathbb{R}^k$, then due to §2.8.18 of [12] the family $\mathcal{V}_k$ is a ϕ Vitali relation.

In the following proposition, whenever we write "$\operatorname{ap} Df$" we mean the approximate differential with respect to $(\mathcal{L}^m, \mathcal{V}_m)$.

Proposition 5.3. *Suppose $\alpha \in (0,1]$, and $A \subseteq \mathbb{R}^m$ is $\mathcal{L}^m$-measurable and such that $\mathbf{\Theta}^m(\mathcal{L}^m \llcorner A, a) = 1$ for all $a \in A$. Let $f \colon A \to \mathbb{R}^{n-m}$ be $(\mathcal{L}^m, \mathcal{V}_m)$ approximately differentiable on A and satisfy one of the following conditions:*

$$\limsup_{r \downarrow 0} r^{-m} \int_{A \cap \mathbf{B}(y,r)} \frac{P_f(y,z)}{|z-y|^{1+\alpha}} \, \mathrm{d}\mathcal{L}^m(z) < \infty \quad \textit{for all } y \in A\,, \tag{5.1}$$

or

$$(\mathcal{L}^m, \mathcal{V}_m) \operatorname{ap} \limsup_{z \to y} \frac{P_f(y,z)}{|z-y|^{1+\alpha}} < \infty \quad \textit{for all } y \in A\,, \tag{5.2}$$

where $P_f(y,z) = |f(z) - f(y) - \operatorname{ap} Df(y)(z-y)|$ for $y, z \in A$.

Then there exist functions $f_k \in \mathscr{C}^{1,\alpha}(\mathbb{R}^m, \mathbb{R}^{n-m})$ for $k = 1, 2, \ldots$, such that

$$\mathcal{L}^m \left(A \sim \textstyle\bigcup_{k=1}^{\infty} \{ x \in A : f(x) = f_k(x) \text{ and } \operatorname{ap} Df(x) = Df_k(x) \} \right) = 0.$$

In particular, the graph of f is countably $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{1,\alpha}$.

Proof. In case $\alpha = 1$ this was proven in Lemma A.1 of [35]. If $0 < \alpha < 1$, exactly the same proof, with relevant occurrences of 2 replaced by $1 + \alpha$, establishes the assertion. □

Remark 5.4. In case condition (5.2) is satisfied in Proposition 5.3, the conclusion of Proposition 5.3 follows also from Theorem 1.5 of Lin and Liu [26]. However, one should note that at a single point $y \in A$ condition (5.1) does *not* imply (5.2) at y. It is rather a consequence of the proposition that condition (5.1) at all points of A implies condition (5.2) for $\mathcal{L}^m$ almost all $y \in A$. Inspecting the proof of Lemma A.1 in [35] one can extract a sufficient condition, which is implied by (5.1) as well as by (5.2), for the proposition to hold; namely, it is enough to assume that for all $y \in A$ there exists some $K \in (0,\infty)$ such that

$$\limsup_{r\downarrow 0} \frac{\mathcal{L}^m\big(\mathbf{B}(y,r)\sim\{z\in A : |f(z)-f(y)-\operatorname{ap} Df(y)(z-y)| < Kr^{1+\alpha}\}\big)}{\boldsymbol{\alpha}(m)r^m} < \varepsilon_0(m)\,,$$

where $\varepsilon_0(m) \in (0,1)$ is a small constant depending only on m.

Corollary 5.5. *Let $\mathfrak{p}$, $\mathfrak{q}$, T, A, f, F be as in Remark 4.1. Suppose $\alpha \in (0,1]$, and A is $\mathcal{L}^m$ measurable, and f is $(\mathcal{L}^m, \mathcal{V}_m)$ approximately differentiable on A, and $\Sigma = F[A]$ satisfies $\mathcal{H}^m(\Sigma) < \infty$. Assume that one of the following conditions is satisfied for $\mathcal{H}^m$ almost all $a \in \Sigma$:*

$$\limsup_{r\downarrow 0} r^{-m} \int_{\Sigma\cap\mathbf{B}(a,r)} \frac{|\operatorname{Tan}^m(\mathcal{H}^m \llcorner \Sigma, a)_\natural^\perp(b-a)|}{|b-a|^{1+\alpha}}\, \mathrm{d}\mathcal{H}^m(b) < \infty \tag{5.3}$$

or

$$(\mathcal{H}^m \llcorner \Sigma, \mathcal{V}_n) \operatorname{ap}\limsup_{b\to a} \frac{|\operatorname{Tan}^m(\mathcal{H}^m \llcorner \Sigma, a)_\natural^\perp(b-a)|}{|b-a|^{1+\alpha}} < \infty\,. \tag{5.4}$$

Then Σ is $\mathcal{H}^m$ measurable and countably $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{1,\alpha}$.

Proof. Employing §3.1.8 of [12] we can divide A into a countable family of $\mathcal{L}^m$ measurable sets $\{A_i : i \in \mathbb{N}\}$ such that f restricted to each of A_i is Lipschitz and $\bigcup_{i\in\mathbb{N}} A_i = A$. Then $F|_{A_i}$ is bilipschitz and, since $\mathcal{H}^m$ and $\mathcal{L}^m$ are Borel regular, $\Sigma_i = F[A_i]$ is $\mathcal{H}^m$ measurable for each $i \in \mathbb{N}$. Hence, $\Sigma = \bigcup_{i\in\mathbb{N}} \Sigma_i$ is also $\mathcal{H}^m$ measurable. Moreover, if one of the conditions (5.3) or (5.4) is satisfied for $\mathcal{H}^m$ almost all $a \in \Sigma$, then the same condition holds for $\mathcal{H}^m$ almost all $a \in \Sigma_i$ for each $i \in \mathbb{N}$. Hence, it suffices to prove the corollary separately for each A_i and Σ_i in place of A and Σ. In the sequel we will assume this replacement has been done and that f has been extended to the whole of $\mathbb{R}^m$ by means of the Kirszbraun theorem (cf. §2.10.43 of [12]), so that we have

$$f : \mathbb{R}^m \to \mathbb{R}^{n-m} \quad \text{satisfies} \quad L = \operatorname{Lip}(f) < \infty$$
$$\text{and} \quad \operatorname{Tan}^m(\mathcal{H}^m \llcorner \Sigma, a) = \operatorname{Tan}(\Sigma, a) \quad \text{for } \mathcal{H}^m \text{ almost all } a \in \Sigma.$$

If (5.3) holds for $\mathcal{H}^m$ almost all $a \in \Sigma$, then let $\Sigma' \subseteq \Sigma$ be the set of $a \in \Sigma$ for which (5.3) holds. If (5.4) holds for $\mathcal{H}^m$ almost all $a \in \Sigma$, let $\Sigma' \subseteq \Sigma$ be the set of $a \in \Sigma$ for which (5.4) holds. Since $\mathcal{H}^m(\Sigma \sim \Sigma') = 0$, we know Σ' is $\mathcal{H}^m$ measurable. Recall the definitions of $\mathfrak{p}$, $\mathfrak{q}$, and F from Remark 4.1. Set $B' = \mathfrak{p}[\Sigma']$ and note that $B' = F^{-1}[\Sigma']$ so it is $\mathcal{L}^m$ measurable. Next, set $\tilde{B} = \{x \in B' : Df(x) \text{ exists}\}$. Then $\mathcal{L}^m(B' \sim \tilde{B}) = 0$ due to the Rademacher theorem (cf. §3.1.6 of [12]); hence, $\tilde{B}$ is also $\mathcal{L}^m$ measurable. Define $B = \{x \in \tilde{B} : \boldsymbol{\Theta}^m(\mathcal{L}^m \llcorner \tilde{B}, x) = 1\}$. Then, by §2.9.11 of [12], B is $\mathcal{L}^m$ measurable, $\mathcal{L}^m(\tilde{B} \sim B) = 0$ and $\boldsymbol{\Theta}^m(\mathcal{L}^m \llcorner B, x) = 1$ for *all* $x \in B$. Observe that, because F is Lipschitz,

$$\mathcal{H}^m(\Sigma \sim F[B]) = \mathcal{H}^m(\Sigma \sim \Sigma') + \mathcal{H}^m(F[B' \sim B]) = 0. \tag{5.5}$$

Hence, it suffices to check that Proposition 5.3 applies to $f|_B$.

Set $\lambda = (1+L^2)^{-1/2} \in (0,1]$ and note that $\operatorname{Lip}(F) \le \lambda^{-1}$; hence,

$$F[\mathbf{B}(\mathfrak{p}(a), \lambda r) \cap B] \subseteq \mathbf{B}(a,r) \cap F[B] \quad \text{for each } a \in \Sigma \text{ and } r \in (0,\infty). \tag{5.6}$$

For $x, y \in B$ define $P_f(x,y) = |f(y) - f(x) - Df(x)(y-x)|$. Employing (5.5) combined with (4.2) and then applying the area formula (cf. §3.2.3 of [12]) together with (5.6), we obtain

$$\begin{aligned} r^{-m} \int_{\mathbf{B}(a,r)\cap\Sigma} & \frac{|\operatorname{Tan}(\Sigma,a)_\natural^\perp(b-a)|}{|b-a|^{1+\alpha}} \, \mathrm{d}\mathcal{H}^m(b) \\ &\ge \lambda^{1+\alpha}(1-\lambda L) r^{-m} \int_{\mathbf{B}(a,r)\cap F[B]} \frac{P_f(\mathfrak{p}(a),\mathfrak{p}(b))}{|\mathfrak{p}(b)-\mathfrak{p}(a)|^{1+\alpha}} \, \mathrm{d}\mathcal{H}^m(b) \\ &\ge \lambda^{1+\alpha+m}(1-\lambda L)(\lambda r)^{-m} \int_{\mathbf{B}(x,\lambda r)\cap B} \frac{P_f(x,y)}{|y-x|^{1+\alpha}} \, \mathrm{d}\mathcal{L}^m(y) \end{aligned}$$

for $r \in (0,\infty)$, $x \in B$, and $a = F(x)$. Hence, if (5.3) holds, then one can employ Proposition 5.3 to see that $F[B]$ is countably $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{1,\alpha}$ and, due to (5.5), so is Σ.

Fix $a \in F[B]$ and set $x = \mathfrak{p}(a)$. For $y \in B$ and $b \in F[B]$ define

$$g(b) = \frac{|\operatorname{Tan}(\Sigma,a)_\natural^\perp(b-a)|}{|b-a|^{1+\alpha}}, \quad h(y) = \frac{|f(y)-f(x)-Df(x)(y-x)|}{|y-x|^{1+\alpha}},$$
$$\text{and} \quad \phi = \mathcal{H}^m \llcorner F[B] = \mathcal{H}^m \llcorner \Sigma,$$

Setting $\Delta = \lambda^{1+\alpha}(1-\lambda L)$ we obtain, by (4.2) and the area formula (cf. §3.2.3 of [12]),

$$\Delta h(\mathfrak{p}(b)) \le g(b) \quad \text{and} \quad \mathcal{L}^m(S) \le \phi(F[S]) \le \lambda^{-m}\mathcal{L}^m(S)$$

whenever $b \in F[B]$ and $S \subseteq B$ is $\mathcal{L}^m$ measurable. Hence, for each $r, t \in (0,\infty)$

$$\{y \in B : h(y) > t\} \subseteq \mathfrak{p}\big[\{b \in F[B] : g(b) > \Delta t\}\big],$$
$$\frac{\mathcal{L}^m(\mathbf{B}(x,\lambda r) \cap \{y \in B : h(y) > t\})}{\mathcal{L}^m(\mathbf{B}(x,r)\cap B)} \le \frac{\phi(\mathbf{B}(a,r) \cap \{b \in F[B] : g(b) > \Delta t\})}{\lambda^m \phi(\mathbf{B}(a,r))}.$$

Therefore,

$$\begin{aligned} \inf\Big\{t \in \mathbb{R} : \lim_{r\downarrow 0} & \frac{\mathcal{L}^m(\mathbf{B}(x,\lambda r)\cap\{y \in B : h(y)>t\})}{\mathcal{L}^m(\mathbf{B}(x,r)\cap B)} = 0\Big\} \\ &\le \inf\Big\{t \in \mathbb{R} : \lim_{r\downarrow 0} \frac{\phi(\mathbf{B}(a,r)\cap\{b \in F[B] : g(b) > \Delta t\})}{\lambda^m \phi(\mathbf{B}(a,r))} = 0\Big\}. \end{aligned} \tag{5.7}$$

For any $x \in B$ we have $\boldsymbol{\Theta}^m(\mathcal{L}^m \llcorner B, x) = 1$ so it follows that

$$\lim_{r\downarrow 0} \frac{\mathcal{L}^m(\mathbf{B}(x,\lambda r))}{\mathcal{L}^m(\mathbf{B}(x,r)\cap B)} = \lambda^m < \infty. \tag{5.8}$$

Recalling $x = \mathfrak{p}(a) \in B$ was chosen arbitrarily and combining (5.7) with (5.8) yields

$$(\mathcal{L}^m, \mathcal{V}_m) \operatorname{ap}\limsup_{y\to x} h(y) \le (\phi, \mathcal{V}_n) \operatorname{ap}\limsup_{b\to a} g(b)$$

for all $x \in B$ and $a = F(x)$. Consequently, if (5.4) holds, then one can employ Proposition 5.3 to see that $F[B]$ is countably $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{1,\alpha}$ and, because of (5.5), so is Σ. □

6. Existence of balanced balls

Definition 6.1. For $\delta \in [0,1]$, and $a \in \mathbb{R}^n$, and $r \in (0,\infty]$ we set

$$X_\delta(a,r) = \left\{(b_1,\dots,b_m) \in \mathbf{B}(a,r)^m : |(b_1-a)\wedge\cdots\wedge(b_m-a)| \geq \delta r^m\right\}.$$

Remark 6.2. The following Lemma 6.3 is a variant of Lemma 3.1 from [5]. Similar results can also be found in Proposition 3.1 of [23], and in Lemma 5.8 of [9], and in Lemma 4.2 of [33].

Lemma 6.3. *Suppose*

$$\mu \text{ is a Radon measure over } \mathbb{R}^n, \quad a,b\in\mathbb{R}^n, \quad r\in(0,\infty), \quad \mu(\mathbf{B}(a,r))>0,$$
$$t,\gamma\in(0,1), \quad k\in\mathbb{N}, \quad k<m, \quad L_k\in\mathbf{G}(n,k).$$

Then one of the following alternatives holds:

(a) There exist $x_{k+1},\dots,x_m \in \mathbf{B}(a,r)$ such that if $L_j = L_k + \operatorname{span}\{x_{k+1}-b,\dots,x_j-b\}$ for $j=k+1,\dots,m$, then

$$\operatorname{dist}(x_j-b, L_{j-1}) > \gamma r \quad \text{for } j=k+1,\dots,m,$$
$$\mu(\mathbf{B}(x_j,tr)\cap\mathbf{B}(a,r)) \geq \Gamma^{-1}t^n\mu(\mathbf{B}(a,r)) \quad \text{for } j=k+1,\dots,m,$$

where $\Gamma=\Gamma(n)\in[1,\infty)$.

(b) There exist $\lambda, N\in\mathbb{N}$ and $L_\lambda\in\mathbf{G}(n,\lambda)$ and $y_1,\dots,y_N\in\mathbf{B}(a,r)\cap(b+L_\lambda)$ satisfying

$$k\leq\lambda<m, \quad N\leq\Gamma\gamma^{-\lambda}, \quad L_k\subseteq L_\lambda,$$
$$\left\{\mathbf{B}(y_i,40\gamma r) : i=1,\dots,N\right\} \text{ is disjointed},$$
$$\sum_{i=1}^N \mu(\mathbf{B}(y_i,4\gamma r)) \geq \Gamma^{-1}\mu(\mathbf{B}(a,r)),$$
$$\frac{\mu(\mathbf{B}(y_i,4\gamma r))}{(4\gamma r)^m} \geq \Gamma^{-1}\gamma^{-(m-\lambda)}\frac{\mu(\mathbf{B}(a,r))}{r^m} \quad \text{for } i=1,\dots,N,$$

where $\Gamma=\Gamma(m)\in[1,\infty)$.

Proof. We mimic the proof of Lemma 3.1 in [5].

Set $\varepsilon = 2^{-(n+1)}t^n$. Choose $x_{k+1},\dots,x_m$ inductively so that for $j=k+1,\dots,m$

$$L_j = L_k + \operatorname{span}\{x_{k+1}-b,\dots,x_j-b\},$$
$$x_j\in\mathbf{B}(a,r)\sim\left(L_{j-1}+\mathbf{B}(0,\gamma r)\right),$$
$$s_j = \sup\left\{\mu(\mathbf{B}(x,tr)\cap\mathbf{B}(a,r)) : x\in\mathbf{B}(a,r)\sim\left(L_{j-1}+\mathbf{B}(0,\gamma r)\right)\right\}.$$
$$\mu(\mathbf{B}(x_j,tr)\cap\mathbf{B}(a,r)) \geq \tfrac{1}{2}s_j.$$

If $s_j \geq \varepsilon\mu(\mathbf{B}(a,r))$ for $j = k+1, \dots, m$, then alternative (a) holds with $\Gamma = 2^{n+1}$.

Assume there exists $\lambda \in \{k, \dots, m-1\}$ such that $s_{\lambda+1} \leq \varepsilon\mu(\mathbf{B}(a,r))$. Then, since $\mathbf{B}(a,r) \sim \big(L_\lambda + \mathbf{B}(0,\gamma r)\big)$ can be covered by $2^n t^{-n}$ balls with centers in $\mathbf{B}(a,r) \sim \big(L_\lambda + \mathbf{B}(0,\gamma r)\big)$ and radii tr, we get

$$\mu\big(\mathbf{B}(a,r) \sim \big(L_\lambda + \mathbf{B}(0,\gamma r)\big)\big) \leq 2^n\, t^{-n}\, \varepsilon\, \mu(\mathbf{B}(a,r))\,.$$

Recalling $\varepsilon = \frac{1}{2}\, 2^{-n}\, t^n$, this implies that

$$\mu\big(\mathbf{B}(a,r) \cap \big(L_\lambda + \mathbf{B}(0,\gamma r)\big)\big) \geq \tfrac{1}{2}\, \mu(\mathbf{B}(a,r))\,. \tag{6.1}$$

Since $\dim(L_\lambda) = \lambda$ there exists a finite set $I \subseteq \mathbf{B}(a,r) \cap L_\lambda$ such that

$$\mathbf{B}(a,r) \cap \big(L_\lambda + \mathbf{B}(0,\gamma r)\big) \subseteq \bigcup\big\{\mathbf{B}(z, 4\gamma r) : z \in I\big\}\,,$$
$$\big\{\mathbf{B}(z, \tfrac{1}{2}\gamma r) : z \in I\big\} \text{ is disjointed}\,, \quad \mathcal{H}^0(I) \leq K = (4\gamma)^{-\lambda}\,. \tag{6.2}$$

Next, we define $J = \big\{z \in I : \mu(\mathbf{B}(z, 4\gamma r)) \geq (4K)^{-1}\mu(\mathbf{B}(a,r))\big\}$ and note that

$$\mu\big(\bigcup\big\{\mathbf{B}(z, 4\gamma r) : z \in I \sim J\big\}\big) \leq \tfrac{1}{4}\, \mu(\mathbf{B}(a,r))\,;$$

hence, employing (6.1),

$$\mu\big(\bigcup\big\{\mathbf{B}(z, 4\gamma r) : z \in J\big\}\big) \geq \tfrac{1}{4}\, \mu(\mathbf{B}(a,r))\,. \tag{6.3}$$

Now we construct $Y = \{y_1, \dots, y_N\} \subseteq J$ inductively so that for $i = 1, \dots, N$

$$J_1 = J\,,$$
$$J_i = \big\{z \in J : \mathbf{B}(z, 40\gamma r) \cap \mathbf{B}(y_j, 40\gamma r) = \varnothing \text{ for } j = 1, 2, \dots, i-1\big\} \quad \text{if } i \geq 2\,,$$
$$y_i \in J_i\,, \quad \mu(\mathbf{B}(y_i, 4\gamma r)) \geq \mu(\mathbf{B}(z, 4\gamma r)) \quad \text{for } z \in J_i\,, \quad J_{N+1} = \varnothing\,. \tag{6.4}$$

For $i = 1, 2, \dots, N$ we see from (6.2) that $J_i \sim J_{i+1}$ contains at most 20^λ points. Thus, using (6.3) and (6.4), we obtain

$$\mu(\mathbf{B}(a,r)) \leq 4 \sum_{i=1}^{N} \mu\big(\bigcup\big\{\mathbf{B}(z, 4\gamma r) : z \in J_i \sim J_{i+1}\big\}\big) \leq 4 \cdot 20^\lambda \sum_{i=1}^{N} \mu\big(\mathbf{B}(y_i, 4\gamma r)\big)\,.$$

Finally, since $\mu(\mathbf{B}(z, 4\gamma r)) \geq (4K)^{-1}\mu(\mathbf{B}(a,r))$ for $z \in J$ and $K = (4\gamma)^{-\lambda}$, we conclude that

$$\frac{\mu(\mathbf{B}(z, 4\gamma r))}{(4\gamma r)^m} \geq \frac{\mu(\mathbf{B}(a,r))}{4K(4\gamma r)^m} \geq 4^{-2m}\, \gamma^{-(m-j+1)}\, \frac{\mu(\mathbf{B}(a,r))}{r^m} \quad \text{for } z \in J\,.$$

Hence, alternative (b) holds with $\Gamma = 4 \cdot 20^m$. □

Corollary 6.4. *Suppose $a \in \mathbb{R}^n$, and $r_0 \in (0, \infty)$, and $A \in [1, \infty)$,*

$$\text{and} \quad A^{-1}\boldsymbol{\alpha}(m) r^m \leq \mu(\mathbf{B}(a,r)) \quad \text{for } r \in (0, r_0]\,,$$
$$\text{and} \quad \mu(\mathbf{B}(z,r)) \leq A\, \boldsymbol{\alpha}(m)\, r^m \quad \text{for } z \in \mathbb{R}^n \text{ and } r \in (0, r_0]\,.$$

Then there exist $\delta = \delta(A, m) \in (0,1]$ and $\sigma = \sigma(A, n, m) \in (0,1]$ such that

$$\mu^m(X_\delta(a,r)) \geq \sigma\, \mu(\mathbf{B}(a,r))^m \quad \text{for } r \in (0, 4r_0]\,.$$

Proof. Let $r \in (0, r_0]$ and set $\gamma = \frac{1}{2}(\Gamma_{6.3(b)} A^2)^{1/(m-1)}$ and $t = \left(1 + \frac{1}{2}\gamma^m\right)^{1/m} - 1$ and $k = 0$ and $L_k = \{0\}$ and $b = a$ and $\delta = \frac{1}{2}\gamma^m$ and apply Lemma 6.3 with this choice of a, b, γ, r, t, k, L_k. Observe that for this particular γ alternative (b) of Lemma 6.3 cannot hold due to the assumed lower bound on $\mu(\mathbf{B}(a,r))$ and upper bound on $\mu(\mathbf{B}(z, 4\gamma r))$ for $z \in \mathbb{R}^n$. Therefore, there exist $x_1, \ldots, x_m \in \mathbf{B}(a,r)$ such that alternative (a) of Lemma 6.3 holds. Then clearly

$$|(x_1 - a) \wedge \cdots \wedge (x_m - a)| \geq \gamma^m r^m .$$

A simple computation, as in Proposition 1.7 of [16], shows that if we choose arbitrary points $y_i \in \mathbf{B}(x_i, tr)$ for $i = 1, \ldots, m$, then

$$|(y_1 - a) \wedge \cdots \wedge (y_m - a)| \geq \tfrac{1}{2}\gamma^m r^m = \delta\, r^m .$$

This shows that

$$\big(\mathbf{B}(x_1, tr) \cap \mathbf{B}(a,r)\big) \times \cdots \times \big(\mathbf{B}(x_m, tr) \cap \mathbf{B}(a,r)\big) \subseteq X_\delta(a,r) ;$$

hence, item (a) of Lemma 6.3 yields

$$\mu^m(X_\delta(a,r)) \geq \Gamma_{6.3(a)}^{-m}\, t^{nm}\, \mu(\mathbf{B}(a,r))^m . \qquad \square$$

7. Proof of higher order rectifiability

Remark 7.1. If $\delta \in [0,1]$, and $a, c \in \mathbb{R}^n$, and $r \in (0, \infty]$, and $(b_1, \ldots, b_m) \in X_\delta(a,r)$ (see Definition 6.1), and $P = \operatorname{span}\{b_1 - a, \ldots, b_m - a\}$, then

$$\kappa(a, b_1, \ldots, b_m, c) \geq \frac{\delta \operatorname{dist}(c - a, P)}{2^m (m+1)!\, 2r} .$$

Lemma 7.2. *Let $r, \delta \in (0, \infty)$, $\varepsilon \in (0,1)$, $P, Q \in \mathbf{G}(n,m)$, $v_1, \ldots, v_m \in \mathbb{R}^n$ satisfy*

$$Q = \operatorname{span}\{v_1, \ldots, v_m\} , \quad |v_1 \wedge \cdots \wedge v_m| \geq \delta r^m , \quad |v_i| \leq r , \quad |P_\natural^\perp v_i| \leq \varepsilon r$$

for $i = 1, \ldots, m$. Then $\|P_\natural - Q_\natural\| \leq m\delta^{-1}\varepsilon$.

Proof. By §8.9(3) of [3], there exists $u \in Q$ such that

$$|u| = 1 \quad \text{and} \quad \|P_\natural - Q_\natural\| = \|P_\natural^\perp \circ Q_\natural\| = |P_\natural^\perp u| .$$

Choose $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ such that $u = \sum_{i=1}^m \alpha_i v_i$. For each $i = 1, \ldots, m$ we have

$$\alpha_i = \big(v_1 \wedge \cdots \wedge v_{i-1} \wedge u \wedge v_{i+1} \wedge \cdots \wedge v_m\big) \bullet \frac{v_1 \wedge \cdots \wedge v_m}{|v_1 \wedge \cdots \wedge v_m|^2}$$
$$\text{and} \quad |\alpha_i| = \frac{|v_1 \wedge \cdots \wedge v_{i-1} \wedge u \wedge v_{i+1} \wedge \cdots \wedge v_m|}{|v_1 \wedge \cdots \wedge v_m|} \leq \frac{1}{\delta r} ;$$

hence,

$$\|P_\natural - Q_\natural\| = |P_\natural^\perp u| \leq \sum_{i=1}^m |\alpha_i|\, |P_\natural^\perp v_i| \leq m\delta^{-1}\varepsilon . \qquad \square$$

Remark 7.3. We shall frequently use the Chebyshev inequality in the following form. Whenever ν measures some set X, $f\colon X \to \mathbb{R}$ is a ν measurable function, $t \in (0,\infty)$, and $A \subseteq X$ is ν measurable, then for any $K \in (0,\infty)$, we have

$$\nu\left(\left\{x \in A : |f(x)| > K \textstyle\fint_A |f| \,\mathrm{d}\nu\right\}\right) \le K^{-1}\nu(A)\,.$$

Lemma 7.4. *Assume $l \in \{1,2,\ldots,m+2\}$, and $\alpha \in (0,1]$, and $p \in [1,\infty)$, and μ is a Radon measure, and $\delta,\sigma \in (0,1)$, and $A \in [1,\infty)$, and $r_0 \in (0,\infty)$. Define S to be the set of those $a \in \mathbb{R}^n$ for which*

$$\mathcal{K}^{l,p,\alpha}_{\mu}(a,4r_0) < \infty\,,$$

$$A^{-1}\boldsymbol{\alpha}(m)\, r^m \le \mu(\mathbf{B}(a,r)) \le A\,\boldsymbol{\alpha}(m)\, r^m \quad \textit{for } r \in (0,r_0]\,, \tag{7.1}$$

$$\mu^m(X_\delta(a,r)) \ge \sigma\, \mu(\mathbf{B}(a,r))^m \quad r \in (0,r_0]\,. \tag{7.2}$$

Then there exists a constant $C = C(m,l,p,\sigma,\alpha,\delta,A)$ and for each $a \in S$ there exists $T(a) \in \mathbf{G}\,(n,m)$ such that

(a) *in case $l < m+2$: for μ almost all $b \in \mathbf{B}(a,r_0)$,*

$$|T(a)^{\perp}_{\natural}(b-a)| \le C\,\mathcal{K}^{l,p,\alpha}_{\mu}(a,|b-a|)^{1/p}\,|b-a|^{1+\alpha}$$

and, whenever $b \in S \cap \mathbf{B}(a,\tfrac{1}{2}r_0)$,

$$\|T(a)_\natural - T(b)_\natural\| \le C\,\mathcal{K}^{l,p,\alpha}_{\mu}(a,|b-a|)^{1/p}\,|b-a|^{\alpha}\,;$$

(b) *in case $l = m+2$: for any $r \in (0,r_0]$,*

$$\left(\fint_{\mathbf{B}(a,r)} \operatorname{dist}(c-a,T(a))^p \,\mathrm{d}\mu(c)\right)^{1/p} \le C\,\mathcal{K}^{l,p,\alpha}_{\mu}(a,4r)^{1/p} r^{1+\alpha}\,.$$

In particular, $\operatorname{Tan}^m(\mu,a) \subseteq T(a)$ *for all $a \in S$, by Proposition 3.2 and Remark 3.3.*

Proof. Obviously we can assume S is not empty −otherwise there is nothing to prove. Set $M = (2^{m+m^2}A^{2m}+2)\sigma^{-1}$. If $2 \le l \le m+1$, define

$$Y(a,r) = \left\{(b_1,\ldots,b_m) \in \mathbf{B}(a,r)^m : \kappa^{l,p,\alpha}_{\mu,a,r}(b_1,\ldots,b_{l-1}) > \frac{M\,\mathcal{K}^{l,p,\alpha}_{\mu}(a,r)}{\mu(\mathbf{B}(a,r))^{l-1}}\right\},$$

if $l = m+2$, set

$$Y(a,r) = \Bigg\{(b_1,\ldots,b_m) \in \mathbf{B}(a,r)^m : \int_{\mathbf{B}(a,r)} \kappa^{l,p,\alpha}_{\mu,a,r}(b_1,\ldots,b_m,c)\,\mathrm{d}\mu^1(c) > \frac{M\,\mathcal{K}^{l,p,\alpha}_{\mu}(a,r)}{\mu(\mathbf{B}(a,r))^m}\Bigg\},$$

and if $l = 1$, set $Y(a,r) = \varnothing$. Employing Chebyshev's inequality (cf. Remark 7.3) we obtain

$$\mu^m(Y(a,r)) \le M^{-1}\,\mu(\mathbf{B}(a,r))^m \tag{7.3}$$

for $l \in \{1,\ldots,m+2\}$, and $a \in \mathbb{R}^n$, and $r \in (0,\infty]$.

Since $M > \sigma^{-1}$, using (7.2), we get

$$\mu^m(X_\delta(a,r) \sim Y(a,r)) \geq \big(\sigma - \tfrac{1}{M}\big)\, \mu(\mathbf{B}(a,r))^m > 0 \tag{7.4}$$

for $a \in S$ and $0 < r \leq r_0$. For $a \in S$, and $0 < r \leq r_0$, and $(g_1, \ldots, g_m) \in X_\delta(a,r) \sim Y(a,r)$ if $P = \operatorname{span}\{g_1 - a, \ldots, g_m - a\}$ and $1 \leq l \leq m+1$, then using Remark 7.1 together with (7.1) and (7.2) we get

$$\begin{aligned}
\frac{M\,\mathcal{K}^{l,p,\alpha}_{\mu}(a,r)}{\big(A^{-1}\boldsymbol{\alpha}(m)\, r^m\big)^{l-1}} &\geq \frac{M\,\mathcal{K}^{l,p,\alpha}_{\mu}(a,r)}{\mu(\mathbf{B}(a,r))^{l-1}} \geq \kappa^{l,p,\alpha}_{\mu,a,r}(g_1, \ldots, g_{l-1}) \\
&\geq (\mu) \operatorname*{ess\,sup}_{b \in \mathbf{B}(a,r)} \frac{\kappa(a, g_1, \ldots, g_m, b)^p}{\operatorname{diam}(\{a, g_1, \ldots, g_m, b\})^{m(l-1)+\alpha p}} \\
&\geq \frac{(\mu) \operatorname{ess\,sup}_{b \in \mathbf{B}(a,r)} \big(\delta \operatorname{dist}(b-a, P)\big)^p}{(2^m (m+1)!)^p\, (2r)^{m(l-1)+(1+\alpha)p}}
\end{aligned}$$

which implies

$$(\mu) \operatorname*{ess\,sup}_{b \in \mathbf{B}(a,r)} \big(\operatorname{dist}(b-a, P)\big) \leq C_1\, \mathcal{K}^{l,p,\alpha}_{\mu}(a,r)^{1/p}\, r^{1+\alpha}\,, \tag{7.5}$$

$$\text{where} \quad C_1 = A^{(l-1)/p} M^{1/p} (m+1)!\, \boldsymbol{\alpha}(m)^{(1-l)/p}\, 2^{m+m(l-1)/p+1+\alpha}\, \delta^{-1}\,.$$

An analogous computation shows that in case $l = m+2$, for $a \in S$, and $0 < r \leq r_0$, and $(g_1, \ldots, g_m) \in X_\delta(a,r) \sim Y(a,r)$ if $P = \operatorname{span}\{g_1 - a, \ldots, g_m - a\}$, then

$$\Big(⨍_{\mathbf{B}(a,r)} \operatorname{dist}(c-a, P)^p \,\mathrm{d}\mu(c) \Big)^{1/p} \leq C_1\, \mathcal{K}^{l,p,\alpha}_{\mu}(a,r)^{1/p}\, r^{1+\alpha}\,. \tag{7.6}$$

Now, we shall prove the Lemma in case $1 \leq l \leq m+1$. Set $E = \{x \in \mathbb{R}^n : \boldsymbol{\Theta}^{*m}(\mu, x) > 0\}$. Due to (7.4), for each $a \in S$ and $0 < r \leq r_0$ there exists an m-tuple

$$\big(g_1(a,r), \ldots, g_m(a,r)\big) \in E^m \cap X_\delta(a,r) \sim Y(a,r)$$

and we can define

$$P(a,r) = \operatorname{span}\big\{(g_1(a,r) - a), \ldots, (g_m(a,r) - a)\big\} \in \mathbf{G}\,(n,m)\,.$$

Whenever $a \in S$ and $0 \leq s \leq r \leq r_0$, noting $g_i(a,s) \in E \cap \mathbf{B}(a,r)$ for $i = 1, \ldots, m$, we may employ (7.5) together with Lemma 7.2 to obtain

$$\|P(a,r)_\natural - P(a,s)_\natural\| \leq m\, \delta^{-1}\, C_1\, \mathcal{K}^{l,p,\alpha}_{\mu}(a,r)^{1/p}\, r^{\alpha}\,.$$

Therefore, for each $a \in S$, the spaces $P(a,r)$ converge as $r \to 0$ to some $T(a) \in \mathbf{G}\,(n,m)$ and

$$\|P(a,r)_\natural - T(a)_\natural\| \leq C_2\, \mathcal{K}^{l,p,\alpha}_{\mu}(a,r)^{1/p}\, r^{\alpha}\,, \quad \text{where } C_2 = m\, \delta^{-1}\, C_1\,.$$

Moreover, by (7.5) and the triangle inequality, for any $a \in S$ and $b \in E \cap \mathbf{B}(a, r_0)$

$$|T(a)_\natural^\perp (b-a)| \leq (C_1 + C_2)\, \mathcal{K}^{l,p,\alpha}_{\mu}(a, |b-a|)^{1/p}\, |b-a|^{1+\alpha}\,.$$

Assume $a \in S$, and $r \in (0, r_0]$, and $b \in S \sim \{a\}$ are such that $|b - a| = \frac{1}{2}r$. Then for each $i = 1, \ldots, m$ there holds $|g_i(b, \frac{1}{2}r) - a| \le r$ and it follows from (7.5) that

$$\left|P(a,r)_\natural^\perp\big(g_i(b,\tfrac{1}{2}r) - a\big)\right| \le 2\,C_1\,\mathcal{K}_\mu^{l,p,\alpha}(a,r)^{1/p}\,r^{1+\alpha}\,;$$

hence, employing again Lemma 7.2, we get

$$\left\|P(a,r)_\natural - P\big(b,\tfrac{1}{2}r\big)_\natural\right\| \le 2^{1+\alpha}\,C_2\,\mathcal{K}_\mu^{l,p,\alpha}(a,r)^{1/p}\,|b-a|^\alpha\,.$$

In consequence, for all $a, b \in S$, $r \in (0,\infty)$ with $|a - b| = r/2 \le r_0/2$,

$$\begin{aligned}\|T(a)_\natural - T(b)_\natural\| &\le \|T(a)_\natural - P(a,r)_\natural\| + \|P(a,r)_\natural - P\left(b,\tfrac{r}{2}\right)_\natural\| + \|P\left(b,\tfrac{r}{2}\right)_\natural - T(b)_\natural\| \\ &\le C_3\,\mathcal{K}_\mu^{l,p,\alpha}(a,r)^{1/p}\,|b-a|^\alpha\,, \quad \text{where } C_3 = C_2(2 + 2^{1+\alpha}).\end{aligned}$$

This finishes the proof in case $1 \le l \le m+1$.

Next, we shall consider the case $l = m+2$. For $a \in S$ and $i = \mathbb{N}$ define inductively

$$\rho_i = 2^{-i} r_0\,, \quad Q_0(a) = P(a,\rho_0)\,,$$

$$\begin{aligned} Z_i(a) = \bigg\{ c \in & \Sigma(a,\rho_i) : \\ & \operatorname{dist}(c-a, Q_i(a))^p > M \fint_{\mathbf{B}(a,\rho_i)} \operatorname{dist}(z-a, Q_i(a))^p \,\mathrm{d}\mu(z) \bigg\}\,,\end{aligned}$$

$$W_i(a) = \big\{ (c_1, \ldots, c_m) \in \mathbf{B}(a,\rho_i)^m : c_j \in Z_i(a) \text{ for some } j \in \{1,\ldots,m\} \big\}\,,$$

and, whenever $i \ge 1$,

$$\begin{gathered}\big(h_{i,1}(a), \ldots, h_{i,m}(a)\big) \in X(a,\rho_i) \sim (Y(a,\rho_i) \cup W_{i-1}(a))\,, \\ Q_i(a) = \operatorname{span}\big\{h_{i,1}(a) - a, \ldots, h_{i,m}(a) - a\big\}\,.\end{gathered}$$

Note that $(h_{i,1}(a), \ldots, h_{i,m}(a))$ exists for all $i \in \mathbb{N}$ and $a \in S$. Indeed, for $i \in \mathbb{N}$ and $a \in S$ Chebyshev's inequality (cf. Remark 7.3) yields

$$\begin{gathered}\mu(Z_i(a)) \le M^{-1}\mu(\mathbf{B}(a,\rho_i))\,; \\ \text{hence,} \quad \mu^m(W_i(a)) \le \big(\big(1 + M^{-1}\big)^m - 1\big)\mu(\mathbf{B}(a,\rho_i))^m\,,\end{gathered}$$

which implies, combining (7.1) with (7.3) and noting $((1 + M^{-1})^m - 1) \le 2^m M^{-1}$ and $M > (2^{m+m^2} A^{2m} + 1)\sigma^{-1}$, the following: for $i \in \mathbb{N} \sim \{0\}$,

$$\begin{aligned}&\mu^m\big(X(a,\rho_i) \sim (Y(a,\rho_i) \cup W_{i-1}(a))\big) \\ &\qquad \ge \big(\boldsymbol{\alpha}(m)\,\rho_i^m\big)^m \big(A^m(\sigma - M^{-1}) - M^{-1}\,2^{m+m^2} A^m\big) > 0\,.\end{aligned}$$

Observe that for $a \in S$, and $i = \mathbb{N} \sim \{0\}$, and $j = 1, 2, \ldots, m$, employing (7.6),

$$\begin{aligned}\operatorname{dist}\big(h_{i,j}(a) - a, Q_{i-1}(a)\big) &\le \Big(M \fint_{\mathbf{B}(a,\rho_{i-1})} \operatorname{dist}(z-a, Q_{i-1}(a))^p \,\mathrm{d}\mu(z)\Big)^{1/p} \\ &\le 2^{1+\alpha}\,M^{1/p}\,C_1\,\mathcal{K}_\mu^{l,p,\alpha}(a,\rho_{i-1})^{1/p}\,\rho_i^{1+\alpha}\,.\end{aligned}$$

Therefore, Lemma 7.2 yields for $a \in S$ and $i \in \mathbb{N} \sim \{0\}$

$$\|Q_i(a)_\natural - Q_{i-1}(a)_\natural\| \le C_4\, \mathcal{K}_\mu^{l,p,\alpha}(a,\rho_{i-1})^{1/p} \rho_i^\alpha\,, \quad \text{where } C_4 = m\,\delta^{-1}\,2^{1+\alpha} M^{1/p}\, C_1\,.$$

Summing up a geometric series we see that for $a \in S$ the spaces $Q_i(a)$ converge as $i \to \infty$ to some $T(a) \in \mathbf{G}\,(n,m)$ satisfying

$$\|Q_i(a)_\natural - T(a)_\natural\| \le C_5\, \mathcal{K}_\mu^{l,p,\alpha}(a,2\rho_i)^{1/p}\,\rho_i^\alpha\,, \quad \text{where } C_5 = (1-2^{-\alpha})^{-1}\,C_4\,.$$

Let $a \in S$, and $\rho \in (0,\infty)$, and $i \in \mathbb{N}$ be such that $\rho_{i+1} < \rho \le \rho_i \le r_0$. Then

$$\begin{aligned}\Big(\fint_{\mathbf{B}(a,\rho)} \operatorname{dist}(c-a,T(a))^p \,\mathrm{d}\mu^1(c)\Big)^{1/p} &\le \Big(\fint_{\mathbf{B}(a,\rho)} \operatorname{dist}(c-a,Q_i(a))^p\,\mathrm{d}\mu^1(c)\Big)^{1/p} \\ &\quad + \Big(\fint_{\mathbf{B}(a,\rho)} \|Q_i(a)-T(a)\|^p|c-a|^p\,\mathrm{d}\mu^1(c)\Big)^{1/p} \\ &\le (C_1+C_5)\,\mathcal{K}_\mu^{l,p,\alpha}(a,2\rho_i)^{1/p}\rho_i^{1+\alpha} \le C_6\,\mathcal{K}_\mu^{l,p,\alpha}(a,4\rho)^{1/p}\rho^{1+\alpha}\,,\end{aligned}$$

where $C_6 = 2^{1+\alpha}(C_1+C_5)$. □

The following Theorem 7.5, cited from [3], will allow us to reduce our main Theorem 1.1 roughly to the case when $\mu = \mathcal{H}^m \llcorner \Sigma$, where Σ is a subset of a graph of a $\mathscr{C}^1$ function.

Theorem 7.5 (cf. §2.8(5) of [3]). *Suppose μ is a Radon measure over $\mathbb{R}^n$ such that for μ almost all a the following two conditions hold: $0 < \boldsymbol{\Theta}^{m*}(\mu,a) < \infty$ and there exists $T \in \mathbf{G}\,(n,m)$ such that $\operatorname{Tan}^m(\mu,a) \subseteq T$. Then $\mu = \boldsymbol{\Theta}^m(\mu,\cdot)\mathcal{H}^m$ and $\mathbb{R}^n$ is countably (μ,m) rectifiable of class $\mathscr{C}^1$.*

Now we are ready to prove the first part of the main Theorem 1.1.

Theorem 7.6. *Suppose μ is a Radon measure satisfying the density bounds (1.2) and $\mathcal{K}_\mu^{l,p,\alpha}(a,1) < \infty$ for μ almost all a. Then $\mathbb{R}^n$ is countably (μ,m) rectifiable of class $\mathscr{C}^{1,\alpha}$.*

Proof. For $j \in \mathbb{N} \sim \{0\}$ set

$$A_j = \big\{a \in \mathbb{R}^n : j^{-1}\boldsymbol{\alpha}(m)r^m < \mu(\mathbf{B}(a,r)) \le j\boldsymbol{\alpha}(m)r^m \text{ for } 0 < r < j^{-1}\big\}\,,$$
$$A_0 = \varnothing\,, \quad B_j = A_j \sim A_{j-1}\,, \quad \mu_j = \mu \llcorner B_j\,.$$

Since $(0,\infty) \ni r \mapsto \mu(\mathbf{B}(a,r))$ is right-continuous for each $a \in \mathbb{R}^n$ and $\mathbb{R}^n \ni a \mapsto \mu(\mathbf{B}(a,r))$ is upper semi-continuous for each $r \in (0,\infty)$ we deduce that A_j are Borel sets. Clearly $A_j \subseteq A_{j+1}$ for $j \in \mathbb{N}$ so $\{B_j : j \in \mathbb{N}\}$ is disjointed and, because μ satisfies (1.2), we have

$$\mu\big(\mathbb{R}^n \sim \textstyle\bigcup\{A_j : j \in \mathbb{N}\}\big) = 0\,;$$

hence, it suffices to show that $\mathbb{R}^n$ is countably (μ_j,m) rectifiable of class $\mathscr{C}^{1,\alpha}$ for each $j \in \mathbb{N}$. Fix $j \in \mathbb{N} \sim \{0\}$. Define Borel sets $A_{j,0} = \varnothing$ and, for $k \in \mathbb{N} \sim \{0\}$,

$$A_{j,k} = \big\{a \in A_j : (2j)^{-1}\boldsymbol{\alpha}(m)r^m < \mu_j(\mathbf{B}(a,r)) \le j\boldsymbol{\alpha}(m)r^m \;\text{ for } 0 < r < k^{-1}\big\}.$$

Observe that

$$\mu_j(\mathbf{B}(a,r)) \le j\,\boldsymbol{\alpha}(m)\,r^m \quad \text{for } a \in A_j \text{ and } 0 < r \le j^{-1}\,,$$

$$\boldsymbol{\Theta}^m_*(\mu_j, a) \ge j^{-1} \quad \text{for } \mu_j \text{ almost all } a \text{ by §2.9.11 of [12] since } B_j \text{ is Borel;}$$

$$\text{thus,} \quad \mu_j\big(\mathbb{R}^n \sim \textstyle\bigcup\{A_{j,k} : k \in \mathbb{N}\}\big) = 0\,. \tag{7.7}$$

For each $k \in \mathbb{N}$ and $a \in A_{j,k}$ we apply Corollary 6.4 with μ_j, k^{-1}, a, $2^m j$ in place of μ, r_0, a, A to find out that there exists $\delta = \delta(n,m,j,k) \in (0,1]$ and $\sigma = \sigma(n,m,j,k) \in (0,1]$ such that

$$\mu_j^m(X_\delta(a,r)) \ge \sigma\mu_j(\mathbf{B}(a,r)) \quad \text{for } 0 < r < k^{-1}\,.$$

Next, for each $k \in \mathbb{N}$ we apply Lemma 7.4 with μ_j, $\delta(n,m,k,j)$, $\sigma(n,m,k,j)$, $2j$, k^{-1}, $A_{j,k}$ in place of μ, δ, σ, A, r_0, S to see that for each $k \in \mathbb{N}$ and $a \in A_{j,k}$ there exists some $T(a) \in \mathbf{G}\,(n,m)$ such that

$$\operatorname{Tan}^m(\mu_j, a) \subseteq T(a)\,, \tag{7.8}$$

and, if $l < m+2$, then

$$\frac{|T(a)^{\perp}_{\natural}(b-a)|}{|b-a|^{1+\alpha}} \le C\,\mathcal{K}^{l,p,\alpha}_{\mu}(a, |b-a|)^{1/p} \quad \text{for } \mu_j \text{ almost all } b \in \mathbf{B}(a, k^{-1})\,, \tag{7.9}$$

and, if $l = m+2$, then, using Hölder's inequality,

$$⨍_{\mathbf{B}(a,r)} \frac{|T(a)^{\perp}_{\natural}(c-a)|}{r^{1+\alpha}}\,\mathrm{d}\mu_j(c) \le C\,\mathcal{K}^{l,p,\alpha}_{\mu}(a, 4r)^{1/p} \quad \text{for } 0 < r < k^{-1}\,. \tag{7.10}$$

Employing Theorem 7.5 together with (7.8) and (7.7), we see that A_j is countably (μ_j, m) rectifiable of class $\mathscr{C}^1$. Hence, there exists a countable family $\mathcal{A}$ of $\mathscr{C}^1$ submanifolds of $\mathbb{R}^n$ such that

$$\mu_j = \boldsymbol{\Theta}^m(\mu_j, \cdot)\mathcal{H}^m \llcorner \textstyle\bigcup\mathcal{A}\,. \tag{7.11}$$

Thus, to finish the proof it suffices to show that for each $k \in \mathbb{N}$ and $M \in \mathcal{A}$ the set $M \cap A_{j,k}$ is countably $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{1,\alpha}$.

Fix $M \in \mathcal{A}$ and $k \in \mathbb{N}$ such that $\mu_j(M \cap A_{j,k}) > 0$. Employing the definition of a submanifold (cf. §3.1.19(5) of [12]), we can represent M locally, around any $a \in M$, as a graph over the tangent plane $\operatorname{Tan}(M,a)$ of some $\mathscr{C}^1$ function, i.e., we can find a neighborhood U_a of a in $\mathbb{R}^n$ and projections $\mathfrak{p}_a \in \mathbf{O}^*(n,m)$, $\mathfrak{q}_a \in \mathbf{O}^*(n, n-m)$ such that

$$\operatorname{im}\mathfrak{p}_a^* = \operatorname{Tan}(M,a)\,, \quad \operatorname{im}\mathfrak{q}_a^* = \operatorname{Tan}(M,a)^{\perp}\,, \quad \mathfrak{p}_a|_{M\cap U_a} \text{ is injective}\,,$$

$$(\mathfrak{p}_a|_{M\cap U_a})^{-1} : \mathfrak{p}_a[U_a] \to \mathbb{R}^n \text{ is of class } \mathscr{C}^1\,, \quad D\big((\mathfrak{p}_a|_{M\cap U_a})^{-1}\big)(\mathfrak{p}_a(a)) = \mathfrak{p}_a^*\,.$$

Set $F_a = (\mathfrak{p}_a|_{M\cap U_a})^{-1}$ and $f_a = \mathfrak{q}_a \circ F_a$; then

$$F_a = \mathfrak{p}_a^* + \mathfrak{q}_a^* \circ f_a\,, \quad f_a(\mathfrak{p}_a(a)) = 0\,, \quad Df_a(\mathfrak{p}_a(a)) = 0\,.$$

Define an open "cuboid" adjusted to $\operatorname{Tan}(M,a)$ of radius $r \in (0,\infty)$ by the formula

$$\mathbf{C}(a,r) = \big\{ y \in \mathbb{R}^n : |\mathfrak{p}_a(y-a)| < r \text{ and } |\mathfrak{q}_a(y-a)| < r \big\}\,.$$

Recall U_a is a neighborhood of a in $\mathbb{R}^n$ so $a \in \operatorname{Int} U_a$. Thus, for all $a \in M$ there exists a radius $r_a > 0$ such that

$$M \cap \mathbf{C}(a, r_a) = F_a[\mathbf{U}(\mathfrak{p}_a(a), r_a)] \quad \text{and} \quad \|Df_a(x)\| \le \tfrac{1}{2} \tag{7.12}$$

for $x \in \mathbf{B}(\mathfrak{p}_a(a), r_a)$.

Next, observe that $M \cap A_{j,k}$ is a second-countable space as a subspace of a second-countable space $\mathbb{R}^n$; hence, it has the Lindelöf property (cf. §3.8 of [11]). Thus, from the open covering $\{\mathbf{C}(a, r_a) : a \in M \cap A_{j,k}\}$ of $M \cap A_{j,k}$, one can choose a countable subcovering $\{\mathbf{C}(a_i, r_{a_i}) : i \in \mathbb{N}\}$ of $M \cap A_{j,k}$. Now it suffices to prove that $M \cap A_{j,k} \cap \mathbf{C}(a_i, r_{a_i})$ is countably $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{1,\alpha}$ for each $i \in \mathbb{N}$.

Fix an $i \in \mathbb{N}$. Since $A_{j,k}$ is Borel it follows that $E = \mathfrak{p}_{a_i}[M \cap A_{j,k} \cap \mathbf{C}(a_i, r_{a_i})]$ is $\mathcal{L}^m$ measurable by §2.2.13 of [12]. By (7.12) we see that $\mathcal{H}^m(F_{a_i}[E]) < \infty$. Clearly we also have $T(a_i) = \operatorname{Tan}(M, a_i)$ because of (7.8) and $\dim T(a_i) = m$. Hence, we can apply Corollary 5.5 with $\mathfrak{p}_{a_i}$, $\mathfrak{q}_{a_i}$, $T(a_i)$, f_{a_i}, F_{a_i}, E, $F_{a_i}[E]$ in place of $\mathfrak{p}$, $\mathfrak{q}$, T, f, F, A, Σ. With these substitutions, recalling Remark 3.3 and (7.11) and $j^{-1} \le \mathbf{\Theta}^m(\mu_j, x) \le j$ for μ_j almost all $x \in A_j$, we see that (7.9) implies (5.4) in case $l < m+2$ and (7.10) implies (5.3) in case $l = m+2$. Consequently, we deduce that $M \cap A_{j,k} \cap \mathbf{C}(a_i, r_{a_i})$ is countably $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{1,\alpha}$. □

Remark 7.7. After finishing this work Ulrich Menne pointed out to the author that if $l < m+2$ and μ satisfies (1.2) and is of the form $\mu = \mathcal{H}^m \llcorner \Sigma$ for some $\Sigma \subseteq \mathbb{R}^n$ and $\mathcal{K}_\mu^{l,p,\alpha}(a, 1) < \infty$ for μ almost all a, then it follows from Lemma 7.4 that for μ almost all a Σ is *pointwise differentiable of order* $(1, \alpha)$ *at* a in the sense of Definition 3.3 of [32], which is stronger than just being rectifiable of class $\mathscr{C}^{1,\alpha}$. To prove this assertion one needs only to note that conditions (7.1) and (7.2) of Lemma 7.4 are satisfied at μ almost all points with A, δ, σ varying from point to point.

8. Sharpness of the exponent α and other admissible integrands

Here we prove the second part of our main Theorem 1.1. We also consider a few different functions which can replace κ in the definition of $\mathcal{K}_\mu^{l,p,\alpha}$.

Definition 8.1. For $T = (a_0, \ldots, a_{m+1}) \in (\mathbb{R}^n)^{m+2}$ we set

$$h_{\min}(T) = \min\bigl\{\operatorname{dist}\bigl(a_j, \operatorname{aff}(\{a_0, \ldots, a_{m+1}\} \sim \{a_j\})\bigr) : j = 0, 1, \ldots, m+1\bigr\},$$

where $\operatorname{aff}(A)$ denotes the smallest affine plane containing the set $A \subseteq \mathbb{R}^n$; and

$$\kappa_{\mathrm{h}}(T) = \frac{h_{\min}(T)}{\operatorname{diam}(\triangle T)} \quad \text{if } \operatorname{diam}(\triangle T) > 0\,, \quad \kappa_{\mathrm{h}}(T) = 0 \quad \text{otherwise}.$$

(A suggestion to use κ_{h} instead of κ appeared in §10 of [24].)

For any $T \in (\mathbb{R}^n)^{m+2}$ to estimate $h_{\min}(T)$ it suffices to consider T inside the $(m+1)$ dimensional vector subspace of $\mathbb{R}^n$ containing $\triangle T$. Hence, the following estimate follows immediately from Lemma 3 of Gritzmann and Lassak [14]

Lemma 8.2. *Suppose* $h \in [0, \infty)$, *and* $T \in (\mathbb{R}^n)^{m+2}$, *and* $a \in \mathbb{R}^n$, *and* $S \in \mathbf{G}(n,m)$, *and* $\triangle T \subseteq \{b \in \mathbb{R}^n : |S_\natural^\perp(b-a)| \le h\}$. *Then* $h_{\min}(T) \le (m+2)h$.

Corollary 8.3. *Let* $\mathfrak{p}$, $\mathfrak{q}$, T, A, f, F *be as in Remark 4.1. Suppose* $\alpha \in [0,1]$, *and* $A = \mathbb{R}^m$, *and* f *is of class* $\mathscr{C}^{1,\alpha}$, *and* $\Sigma = \operatorname{im} F$, *and* $T = (a_0, \dots, a_{m+1}) \in \Sigma^{m+2}$ *satisfy* $h_{\min}(T) > 0$, *and* $d = \operatorname{diam}(\triangle T)$, *and*

$$M = \sup\left\{ \|Df(\mathfrak{p}(b) - Df(\mathfrak{p}(a_0))\| \cdot |\mathfrak{p}(b) - \mathfrak{p}(a_0)|^{-\alpha} : b \in \Sigma\,,\ 0 < |b - a_0| \le d \right\}.$$

Then $h_{\min}(T) \le M(m+2)d^{1+\alpha}$.

Proof. Note $|f(\mathfrak{p}(b) - f(\mathfrak{p}(a_0)) - Df(\mathfrak{p}(a_0))(\mathfrak{p}(b) - \mathfrak{p}(a_0))| \le M|\mathfrak{p}(b) - \mathfrak{p}(a_0)|^{1+\alpha} \le M|b - a_0|^{1+\alpha}$ for all $b \in \Sigma$ with $0 < |b - a_0| \le d$. Hence, employing (4.2), we obtain $\triangle T \subseteq \{b \in \mathbb{R}^n : |\operatorname{Tan}(\Sigma, a_0)_\natural^\perp(b - a_0)| \le Md^{1+\alpha}\}$. Thus, Lemma 8.2 yields $h_{\min}(T) \le M(m+2)d^{1+\alpha}$. □

Corollary 8.4. *Let* $\mathfrak{p}$, $\mathfrak{q}$, T, A, f, F *be as in Remark 4.1. Suppose* $\alpha, \beta \in [0,1]$, *and* $\alpha < \beta$, *and* $A = \mathbb{R}^m$, *and* f *is of class* $\mathscr{C}^{1,\beta}$, *and* $\Sigma = F[\mathbf{B}(0,1)]$, *and* $p \in [1, \infty)$, *and* $l \in \{1, \dots, m+2\}$, *and* $\mu = \mathcal{H}^m \llcorner \Sigma$, *and* $\mathcal{K}^{l,p,\alpha}_{h;\mu}$ *is defined as* $\mathcal{K}^{l,p,\alpha}_{\mu}$ *in (1.1) but with* κ_{h} *in place of* κ. *Then there exists* $\Gamma = \Gamma(m, l, f, p, \alpha, \beta) \in (0, \infty)$ *such that*

$$\mathcal{K}^{l,p,\alpha}_{h;\mu}(a,1) \le \Gamma \quad \textit{for all } a \in \mathbb{R}^n\,.$$

Proof. Set

$$\begin{aligned} M &= \sup\left\{ \|Df(x) - Df(y))\| \cdot |x-y|^{-\beta} : x, y \in \mathbf{B}(0,1) \right\}, \\ L &= \sup\{ \|DF(x)\| : x \in \mathbf{B}(0,1) \}. \end{aligned}$$

If $p(\beta - \alpha) \ge m(l-1)$ (in particular if $l = 1$), then by Corollary 8.3

$$\mathcal{K}^{l,p,\alpha}_{h;\mu}(a,1) \le M^p\,(m+2)^p\,2^{p(\beta-\alpha)-m(l-1)} \quad \text{for } a \in \mathbb{R}^n\,.$$

Assume now $p(\beta - \alpha) < m(l-1)$ (in particular $l > 1$). For $a \in \mathbb{R}^n$ and $i \in \mathbb{N}$ define

$$A_i(a) = \left\{ (a_1, \dots, a_{l-1}) \in (\mathbb{R}^n)^{l-1} : 2^{-i} < \operatorname{diam}(\triangle(a, a_1, \dots, a_{l-1})) \le 2^{-i+1} \right\}.$$

Then $A_i(a) \subseteq \mathbf{B}(a, 2^{-i+1})^{l-1}$. Employing Corollary 8.3 and the area formula (cf. §3.2.3 of [12]) we get

$$\begin{aligned} \mathcal{K}^{l,p,\alpha}_{h;\mu}(a,1) &\le \sum_{i=0}^{\infty} \int_{A_i(a)} \frac{(M(m+2))^p \,\mathrm{d}\mu^{l-1}(a_1, \dots, a_{l-1})}{\operatorname{diam}(\{a, a_1, \dots, a_{l-1}\})^{m(l-1)-p(\beta-\alpha)}} \\ &\le (M(m+2))^p \sum_{i=0}^{\infty} \mu(A_i(a))\, 2^{i(m(l-1)-p(\beta-\alpha))} \\ &\le (M(m+2))^p\,(2L)^{m(l-1)}\,\boldsymbol{\alpha}(m)^{l-1} \sum_{i=0}^{\infty} 2^{-ip(\beta-\alpha)} < \infty \end{aligned}$$ □

Corollary 8.5. *Let $l \in \{1, 2, \ldots, m+2\}$, and $\alpha \in (0,1)$, and $p \in [1,\infty)$. Then for any $\varepsilon \in (0, 1-\alpha)$ there exists a Radon measure μ satisfying (1.2) and $\mathcal{K}^{l,p,\alpha}_{h;\mu}(a) < \infty$ for μ almost all a and such that $\mathbb{R}^n$ is not countably (μ, m) rectifiable of class $\mathscr{C}^{1,\alpha+\varepsilon}$.*

Proof. Let $f \colon \mathbb{R}^m \cap \mathbf{U}(0,2) \to \mathbb{R}^{n-m}$ be of class $\mathscr{C}^{1,\alpha+\varepsilon/2}$ such that the graph of f is not $(\mathcal{H}^m, m)$ rectifiable of class $\mathscr{C}^{1,\alpha+\varepsilon}$ in the sense of Definition 3.1 of [4] – constructions of such functions can be found in the Appendix of [4] or in [15]. Let $\mathfrak{p}$, $\mathfrak{q}$, T, F be related to f as in Remark 4.1. Set $\Sigma = F[\mathbf{B}(0,1)]$ and $\mu = \mathcal{H}^m \llcorner \Sigma$. Clearly $\mathbb{R}^n$ is not countably (μ, m) rectifiable of class $\mathscr{C}^{1,\alpha+\varepsilon}$. However, since f is of class $\mathscr{C}^{1,\alpha+\varepsilon/2}$, we see that $\mathcal{K}^{l,p,\alpha}_{h;\mu}(a) < \infty$ by Corollary 8.4. □

Definition 8.6. For $T = (a_0, a_1, \ldots, a_{m+1}) \in (\mathbb{R}^n)^{m+2}$ satisfying $h_{\min}(T) > 0$ and $i \in I = \{0, 1, \ldots, m+1\}$ define

$$p_m \sin_i(T) = \frac{|(a_1 - a_0) \wedge \cdots \wedge (a_{m+1} - a_0)|}{\prod_{j=0\,, j \neq i}^{m+1} |a_j - a_i|}\,,$$

$$\kappa_{\min}(T) = \min\{p_m \sin_i(T) : i \in I\}\,, \quad \kappa_{\max}(T) = \max\{p_m \sin_i(T) : i \in I\}\,,$$

$$\kappa_{\mathrm{dls}}(T) = \inf\Big\{ \big(\textstyle\sum_{i=0}^{m+1} \operatorname{dist}(a_i, L)^2\big)^{1/2} \operatorname{diam}(\triangle T)^{-1} : L \text{ an affine } m\text{-plane in } \mathbb{R}^n \Big\}\,.$$

If $h_{\min}(T) = 0$, then set $\kappa_{\min}(T) = \kappa_{\max}(T) = \kappa_{\mathrm{dls}}(T) = 0$.

Remark 8.7. The definitions of $\kappa_{\min}$, $\kappa_{\max}$ are motivated by §6.1.1 of [23] and the definition of κ_{dls} by §4 of [25].

Lemma 8.8. *There exists $\Gamma = \Gamma(m) \in [1, \infty)$ such that for $T \in (\mathbb{R}^n)^{m+2}$,*

$$\kappa(T) \le \Gamma \min\big\{\kappa_{\min}(T)\,, \kappa_{\max}(T)\,, \kappa_{\mathrm{dls}}(T)\,, \kappa_{\mathrm{h}}(T)\big\}$$
$$\text{and} \quad \max\big\{\kappa_{\min}(T)\,, \kappa_{\mathrm{dls}}(T)\,, \kappa(T)\big\} \le \Gamma \kappa_{\mathrm{h}}(T)\,.$$

Proof. Let $T = (a_0, \ldots, a_{m+1}) \in (\mathbb{R}^n)^{m+2}$. If $h_{\min}(T) = 0$, then we get zero on both sides of both inequalities; thus, assume $h_{\min}(T) > 0$. Permuting the tuple T we can assume $h_{\min}(T) = |P_\natural^\perp(a_{m+1} - a_0)|$, where $P = \operatorname{span}\{a_1 - a_0, \ldots, a_m - a_0\}$. Using the triangle inequality we can find $i \in \{0, 1, \ldots, m\}$ such that $2|a_{m+1} - a_i| \ge \operatorname{diam}(\triangle T)$; thus, permuting the tuple $(a_0, \ldots, a_m)$, we can also assume $i = 0$. Then

$$\kappa_{\min}(T) \le p_m \sin_i(T) = \frac{|(a_1 - a_0) \wedge \cdots \wedge (a_m - a_0)| \cdot |P_\natural^\perp(a_{m+1} - a_0)|}{|a_1 - a_0| \cdots |a_m - a_0| \cdot |a_{m+1} - a_0|} \le 2\kappa_{\mathrm{h}}(T)\,,$$

$$\text{and} \quad \kappa(T) = \frac{|(a_1 - a_0) \wedge \cdots \wedge (a_m - a_0)| \cdot |P_\natural^\perp(a_{m+1} - a_0)|}{(m+1)!\, \operatorname{diam}(\triangle T)^m \cdot \operatorname{diam}(\triangle T)} \le \frac{\kappa_{\mathrm{h}}(T)}{(m+1)!}\,,$$

$$\text{and} \quad \kappa_{\mathrm{h}}(T)^2 = \textstyle\sum_{i=0}^{m+1} \operatorname{dist}(a_i, a_0 + P)^2\,, \quad \text{so} \quad \kappa_{\mathrm{dls}}(T) \le \kappa_{\mathrm{h}}(T)\,.$$

Hence, $\max\{\kappa_{\min}(T), \kappa_{\mathrm{dls}}(T), \kappa(T)\} \le 2\kappa_{\mathrm{h}}(T)$.

Clearly $\kappa_{\min}(T) \le \kappa_{\max}(T)$. Since $|a_i - a_j| \le \operatorname{diam}(\triangle T)$ for $i, j \in \{0, 1, \ldots, m+1\}$ it is also clear that $\kappa_{\min}(T) \ge (m+1)!\,\kappa(T)$. From equation (A.2) in [24] we further deduce $\kappa_{\mathrm{h}}(T) \le (m+2)\kappa_{\mathrm{dls}}(T)$. Therefore,

$$\kappa(T) \le (m+2) \min\{\kappa_{\min}(T), \kappa_{\max}(T), \kappa_{\mathrm{dls}}(T), \kappa_{\mathrm{h}}(T)\}\,. \qquad \square$$

Corollary 8.9. *Let μ be a Radon measure over $\mathbb{R}^n$ satisfying (1.2), and $l \in \{1, 2, \ldots, m+2\}$, and $\alpha \in (0, 1]$, and $p \in [1, \infty)$, and $\mathcal{K}^{l,p,\alpha}_{*;\mu}$ be defined as $\mathcal{K}^{l,p,\alpha}_{\mu}$ in (1.1) but with κ replaced by one of κ, $\kappa_{\min}$, $\kappa_{\max}$, κ_{dls}, κ_{h}. Assume $\mathcal{K}^{l,p,\alpha}_{*;\mu}(a, 1) < \infty$ for μ almost all a. Then $\mathbb{R}^n$ is countably (μ, m) rectifiable of class $\mathscr{C}^{1,\alpha}$ and μ is absolutely continuous with respect to $\mathcal{H}^m$.*

Moreover, if κ is replaced by one of κ, $\kappa_{\min}$, κ_{dls}, κ_{h} and $\alpha < 1$, then for any $\varepsilon \in (0, 1-\alpha)$ there exists a measure μ satisfying (1.2) and $\mathcal{K}^{l,p,\alpha}_{*;\mu}(a) < \infty$ for μ almost all a and such that $\mathbb{R}^n$ is not countably (μ, m) rectifiable of class $\mathscr{C}^{1,\alpha+\varepsilon}$.

Proof. The claim readily follows from Theorem 7.6 and Corollary 8.5 combined with Lemma 8.8. $\square$

Remark 8.10. The author does not know whether the second part of Corollary 8.9 holds if one uses $\kappa_{\max}$ in place of κ.

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