Rev. Mat. Iberoam. **33** (2017), no. 3, 1025–1036
DOI 10.4171/RMI/962

Measures with locally finite support and spectrum

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Abstract. This note answers an issue raised by Nir Lev and Alexander Olevskii in [7]. In the conclusion of [7] Lev and Olevskii mentioned a new and fascinating Poisson summation formula suggested by André Paul Guinand in [4]. Lev and Olevskii wanted to know to what class of functions this formula applies. This will be answered below (Theorem 4.1).

Another intriguing Poisson summation formula was elaborated in [10]. We show that it is coupled to the Epstein ζ function by a coupling discovered by Jean-Pierre Kahane and Szolem Mandelbrojt in [5].

1. Introduction

Let us begin with some definitions. The Fourier transform $\mathcal{F}(f) = \widehat{f}$ of a function f is defined by $\widehat{f}(y) = \int_{\mathbb{R}^n} \exp(-2\pi\, ix \cdot y) f(x)\, dx$.

Definition 1.1. A set $\Lambda \subset \mathbb{R}^n$ is locally finite if $\Lambda \cap B$ is finite for every compact set B.

A Dirac comb is a sum $\mu = \sum_{\gamma\in\Gamma} \delta_\gamma$ of Dirac masses on a lattice Γ. The Fourier transform of a Dirac comb on a lattice Γ is a Dirac comb on the dual lattice Γ^*. The simplest example is $\mu = \sum_{-\infty}^{\infty} \delta_k$ whose Fourier transform is μ. This is the usual Poisson summation formula. Some companions to the standard Poisson summation formula are described now.

Definition 1.2. Let σ_j be a Dirac comb supported by a coset $x_j + \Gamma_j$ of a lattice $\Gamma_j \subset \mathbb{R}^n$, $1 \le j \le N$. Let $F_j \subset \mathbb{R}^n$ be a finite set, and let $g_j(x) = \sum_{y\in F_j} c(y) \exp(2\pi i y \cdot x)$ be a trigonometric sum. Let $\mu_j = g_j\, \sigma_j$. Then $\mu = \mu_1 + \cdots + \mu_N$ will be called a generalized Dirac comb.

The Fourier transform of a generalized Dirac comb is a generalized Dirac comb.

Are there other atomic measures μ which together with their Fourier transform $\widehat{\mu}$ are supported by a locally finite set? This problem was investigated and

Mathematics Subject Classification (2010): Primary 42B10; Secondary 52C23.
Keywords: Quasicrystals, Poisson summation formula.

recently solved by Lev and Olevskii in a remarkable series of contributions [7], [8], and [9]. Two new solutions are proposed in this note. The first one is based on a theorem by Kahane and Mandelbrojt which is discussed in Section 2. This theorem will be applied to the Epstein ζ function in Section 3. This yields a new proof of the Poisson summation formula which was presented in [10]. In Section 4 we back Guinand's hint.

2. A theorem by Kahane and Mandelbrojt

Definition 2.1. A purely atomic measure μ on $\mathbb{R}^n$ is a crystalline measure if (a) the support Λ of μ is a locally finite set, (b) μ is a tempered distribution, and (c) the distributional Fourier transform $\widehat{\mu}$ of μ is also a purely atomic measure which is supported by a locally finite set.

Let μ be a crystalline measure. We then have

$$\mu = \sum_{\lambda \in \Lambda} a(\lambda)\, \delta_\lambda \quad \text{and} \quad \widehat{\mu} = \sum_{y \in S} b(y)\, \delta_y,$$

where $(a(\lambda))_{\lambda\in\Lambda}$ and $(b(y))_{y\in S}$ satisfy

$$a(\lambda) \neq 0,\ \lambda \in \Lambda, \quad b(y) \neq 0,\ y \in S\,. \tag{1}$$

The support Λ of μ is locally finite and the spectrum S of μ is also locally finite. Then for every test function $f \in \mathcal{S}(\mathbb{R}^n)$ the following generalized Poisson summation formula holds:

$$\sum_{\lambda \in \Lambda} a(\lambda) \widehat{f}(\lambda) = \sum_{y \in S} b(y) f(y)\,. \tag{2}$$

It is proved in [5] that such a Poisson summation formula exists if and only if the corresponding Dirichlet series satisfies a functional equation. Let us be more precise.

Let λ_k, $k \in \mathbb{Z}$, be a strictly increasing sequence of real numbers with

$$\lambda_{-k} = -\lambda_k \quad (\forall\, k \in \mathbb{Z})\,. \tag{3}$$

$$\lambda_k \to \pm\infty \quad (k \to \pm\infty)\,. \tag{4}$$

Let a_k, $k \in \mathbb{Z}$, be an even sequence of complex numbers. Four other conditions are assumed:

(i) the set $\Lambda = \{\lambda_k,\ k \in \mathbb{Z}\}$ is uniformly discrete,

(ii) $\mu_\Lambda = \sum_{-\infty}^{\infty} a_k \delta_{\lambda_k}$ is a tempered distribution,

(iii) the Dirichlet series $\phi_\Lambda(s) = \sum_1^\infty a_k \lambda_k^{-s}$ converges if $\Re s > 1$,

(iv) $(1-s)\phi_\Lambda(s)$ has an analytic extension to the complex plane.

Let us observe that (i) and (ii) imply $|a_k| \le C(1+|k|)^m$, $k \in \mathbb{Z}$, for some exponent m.

Let $\gamma(s) = \pi^{-s/2}\,\Gamma(s/2)$. Then Kahane and Mandelbrojt proved the following theorem:

Theorem 2.1. *If* (i), (ii), (iii) *and* (iv) *are satisfied, the following properties are equivalent:*

(a) $\gamma(s)\,\phi_\Lambda(s) = \gamma(1-s)\,\phi_\Lambda(1-s) \quad (\forall s \in \mathbb{C})$.

(b) *The function* $\theta_\Lambda(u) = \sum_{-\infty}^{\infty} a_k \exp(-\pi u \lambda_k^2)$, $u > 0$, *satisfies the functional equation* $\theta_\Lambda(u) = u^{-1/2}\theta_\Lambda(1/u)$.

(c) *We have* $\widehat{\mu}_\Lambda = \mu_\Lambda$.

The authors proved a similar theorem where $1-s$ is replaced by $3-s$ in the functional equation and the even measure $\sum_{-\infty}^{\infty} a_k\,\delta_{\lambda_k}$ is replaced by the odd measure $\sum_{\{k\in\mathbb{Z},\,k\neq 0\}} (a_k/\lambda_k)\,\delta_{\lambda_k}$. This will be the case in the Poisson summation formula of Theorem 3.1. Some pieces of the proof of Theorem 2.1 will be used in this essay.

We have (c) $\Rightarrow$ (b). Indeed the Fourier transform of $g_t(x) = \exp(-\pi t x^2)$ is $\widehat{g}_t(y) = t^{-1/2}\exp(-\pi y^2/t)$. Then $\langle \widehat{g}_t, \mu\rangle = \langle g_t, \widehat{\mu}\rangle$ yields (b). Conversely (b) $\Rightarrow$ (c). Indeed the collection of g_t, $t > 0$, is total in the set of even functions f in the Schwartz class $\mathcal{S}(\mathbb{R})$. Since μ is an even measure it suffices to check $\langle \widehat{f}, \mu\rangle = \langle f, \widehat{\mu}\rangle$ for even functions f, and since μ is a tempered distribution it suffices to do it for a dense collection of even test functions.

The coefficient a_0 does not appear in (a). Therefore the meaning of the implication (a) $\Rightarrow$ (b) shall be clarified.

The proof of (b) $\Rightarrow$ (a) is using Riemann's original approach to the functional equation satisfied by the ζ function. We define

$$\xi(s) = \pi^{-s/2}\,\Gamma(s/2)\,\phi_\Lambda(s)\,, \tag{5}$$

and we consider as above

$$\theta_\Lambda(u) = \sum_{-\infty}^{\infty} a_k \exp(-\pi u \lambda_k^2)\,. \tag{6}$$

We then have:

Lemma 2.1. *If* $a_0 = 0$, *there exists a positive* α *such that*

$$\theta_\Lambda(u) = O(\exp(-\alpha u)), \quad u \to \infty\,. \tag{7}$$

If $a_0 = 0$ *and* (b) *holds, then*

$$\theta_\Lambda(u) = O(u^{-1/2}\exp(-\alpha/u)), \quad u \to 0\,. \tag{8}$$

The proof of (7) is immediate since $a_0 = 0$, $0 < \lambda_1 < \lambda_2 < \dots$, Λ is uniformly discrete, and μ is a tempered distribution. Then (b) implies (8).

Lemma 2.2. *If $a_0 = 0$, the function $\xi(2s)$ is the Mellin transform of θ_Λ. More precisely we have*

$$2\,\xi(s) = \int_0^\infty \theta_\Lambda(t)\, t^{s/2}\, \frac{dt}{t}\,. \tag{9}$$

To prove Lemma 2.2 it suffices to exchange the summation and the integration in (9). If $\Re s > 1$ this is implied by Lemma 2.1. But (7) and (8) imply that the right hand side of (9) is an entire function of s. Then (9) is extended to the complex plane by analytic continuation. Once (9) is proved, the functional equation satisfied by θ_Λ implies (a). This simple calculation will be detailed in Section 3. The implication (a) ⇒ (b) amounts to inverting the Mellin transform in (9). This part of the proof of Theorem 2.1 will not be used below.

Does Theorem 2.1 apply to the Riemann zeta function $\zeta(s) = \sum_1^\infty k^{-s}$ in (a), the Jacobi θ function $\theta(z) = \sum_{-\infty}^\infty \exp(i\pi z k^2)$ in (b), and the Dirac comb $\mu = \sum_{k\in\mathbb{Z}} \delta_k$ in (c)? Unfortunately it is not the case since we do not have $a_0 = 0$ here which implies that the right hand side of (9) is always divergent. To fix this issue some renormalization is needed. The standard Dirac comb is replaced by the measure $\mu = \frac{1}{2}\sum_{-\infty}^\infty \delta_{k/2} - \frac{3}{2}\sum_{-\infty}^\infty \delta_k + \sum_{-\infty}^\infty \delta_{2k}$, which satisfies $\widehat{\mu} = \mu$. The corresponding ζ function is

$$\widetilde{\zeta}(s) = \frac{1}{2}\,(1 - 2^{-s})(1 - 2^{-s+1})\,\zeta(s),$$

where $\zeta(s)$ is the Riemann ζ function. This example will be seminal below.

In Theorem 2.1 the sequence λ_k, $k \in \mathbb{Z}$, is uniformly discrete (in fact a slightly weaker condition suffices). But this condition is not satisfied in Theorem 3.1. Theorem 3.1 belongs to the program launched by Kahane and Mandelbrojt but cannot be deduced from their work. Theorem 2.1 will only be used as a guideline.

3. The Epstein ζ function

The Epstein ζ function will play the role of $\phi_\Lambda(s)$ in Theorem 2.1. Then the Poisson summation formula which was studied in [10] can be coupled with the Epstein ζ function by the Kahane–Mandelbrojt scheme.

The Epstein ζ function is a holomorphic function of the complex variable s defined by the series

$$\zeta_E(s) = \sum_{\{k\in\mathbb{Z}^3,\, k\neq 0\}} |k|^{-s}\,, \tag{10}$$

which converges if $\Re s > 3$. It satisfies the functional equation

$$\pi^{-s/2}\,\Gamma(s/2)\,\zeta_E(s) = \pi^{-(3-s)/2}\,\Gamma((3-s)/2)\,\zeta_E(3-s)\,. \tag{11}$$

This does not yet make any sense since it simultaneously requires $\Re s > 3$ in the left hand side of (11) and $\Re s < 0$ in the right hand side which is not compatible. This functional equation is well known. A proof is given since some of its ingredients are needed in Theorem 3.1.

A simple modification of the Epstein function is

$$\widetilde{\zeta}_E(s) = \frac{1}{2}(1 - 2^{-s})(2^{3-s} - 1)\,\zeta_E(s)\,. \tag{12}$$

One defines $\chi\colon \mathbb{Z}^3 \mapsto \{-1/2, 0, 4\}$ by $\chi(k) = 0$ if $k \in 4\mathbb{Z}^3$, $\chi(k) = 4$ if $k \in 2\mathbb{Z}^3 \setminus 4\mathbb{Z}^3$ and $\chi(k) = -1/2$ if $k \in \mathbb{Z}^3 \setminus 2\mathbb{Z}^3$.

Then

$$\widetilde{\zeta}_E(s) = \sum_{k\in\mathbb{Z}^3} \chi(k)\,|k|^{-s}\,. \tag{13}$$

The sum runs over $k \in \mathbb{Z}^3$ since $\chi(0) = 0$. The series defined by (13) converges when $\Re s > 2$ which is not sufficient to give a meaning to the functional equation. But $\widetilde{\zeta}_E(s)$ extends to the complex plane as an entire function of $s \in \mathbb{C}$. This will be proved now and implies that $\zeta_E(s)$ is a meromorphic function with a simple pole at $s = 3$.

The Fourier transform of the three-dimensional measure $\mu = \sum_{k\in\mathbb{Z}^3} \chi(k)\delta_{k/2}$ is identical to μ. The corresponding theta function is defined by

$$\theta_E(t) = \sum_{k\in\mathbb{Z}^3} \chi(k)\exp(-\pi t|k|^2), \quad t > 0\,. \tag{14}$$

Since $\chi(0) = 0$ we obviously have $\theta_E(t) = O(\exp(-\pi t))$ as $t \to \infty$.

The Fourier transform of $g_t(x) = \exp(-\pi t|x|^2)$ is $\widehat{g}_t(y) = t^{-3/2}\exp(-\pi|y|^2/t)$. Then $\langle \widehat{g}_t, \mu\rangle = \langle g_t, \widehat{\mu}\rangle$ implies the functional equation

$$\theta_E(t) = \frac{1}{8}\,t^{-3/2}\,\theta_E\Big(\frac{1}{16t}\Big)\,. \tag{15}$$

Therefore

$$\theta_E(t) = O\Big(t^{-3/2}\exp\Big(-\frac{\pi}{16t}\Big)\Big), \quad t \to 0\,. \tag{16}$$

Following Riemann's proof of the functional equation we define

$$\widetilde{\xi}(s) = \pi^{-s/2}\,\Gamma(s/2)\,\widetilde{\zeta}_E(s)\,. \tag{17}$$

Lemma 3.1. *If* $\Re s > 3$ *we have*

$$\widetilde{\xi}(s) = \int_0^\infty \theta_E(t)\,t^{s/2}\,\frac{dt}{t}\,. \tag{18}$$

To prove Lemma 3.1 it suffices to exchange the summation and the integration, which is licit when $\Re s > 3$. Then the exponential decay of $\theta_E(t)$ at 0 and ∞ implies that the right hand side of (18) is an entire function of the complex variable s. This entire function is the analytic continuation of the modified Epstein function $\pi^{-s/2}\Gamma(s/2)\widetilde{\zeta}_E(s)$. Therefore $\zeta_E(s)$ is a meromorphic function of s with a single pole at $s = 3$.

The functional equation $\widetilde{\xi}(s) = 2^{(3-2s)}\widetilde{\xi}(3-s)$ is easily deduced from (15) and (18). Indeed

$$\widetilde{\xi}(s) = \int_0^\infty \theta_E(t)\, t^{s/2}\, \frac{dt}{t} = \int_0^{1/4} + \int_{1/4}^\infty = J(s) + K(s).$$

We perform the change of variable $t = 1/(16u)$ in $J(s)$ and use (15). Returning to the variable t we obtain

$$J(s) = 8 \cdot 4^{-s} \int_{1/4}^\infty \theta_E(t)\, t^{(3-s)/2} \frac{dt}{t}.$$

This implies

$$\widetilde{\xi}(s) = \int_{1/4}^\infty \left(t^{s/2} + \frac{8}{4^s}\, t^{(3-s)/2}\right) \theta_E(t)\, \frac{dt}{t}.$$

We then have

$$\widetilde{\xi}(3-s) = \frac{4^s}{8}\, \widetilde{\xi}(s)\,, \tag{19}$$

since the integrand satisfies this functional equation. Finally (19) implies (11) and the Epstein ζ function satisfies the well-known functional equation.

If condition (i) could be suppressed in the Kahane–Mandelbrojt theorem, the functional equation (11) would imply the following Poisson summation formula:

Theorem 3.1. *The Fourier transform of the one dimensional odd measure*

$$\tau = \sum_{k\in\mathbb{Z}^3} \chi(k)|k|^{-1}(\delta_{|k|/2} - \delta_{-|k|/2}) \tag{20}$$

is $-i\tau$.

Observe that the summation runs over $\mathbb{Z}^3 : k \neq 0$ is not imposed since $\chi(0) = 0$. Since τ is odd, proving Theorem 3.1 is equivalent to checking

$$\langle \tau, \widehat{f}\rangle = -i\langle \tau, f\rangle \tag{21}$$

for every odd function f in the Schwartz class $\mathcal{S}(\mathbb{R})$. But the family of odd functions $\psi_t(\cdot)$, $t > 0$, defined by

$$\psi_t(x) = x\exp(-\pi t x^2), \quad x\in\mathbb{R},\ t > 0, \tag{22}$$

is total in the subspace of odd functions in the Schwartz class $\mathcal{S}(\mathbb{R})$. The Fourier transform of $\psi_t(x)$ is

$$\widehat{\psi}_t(y) = -it^{-3/2}\, y\, \exp(-\pi y^2/t)\,. \tag{23}$$

We plug ψ_t into (21). Then (21) is identical to (15), which ends the proof.

We now give another description of the measure τ. By Legendre's theorem, an integer $n \geq 0$ can be written as a sum of three squares (0^2 being admitted) if and only if n is not of the form $4^j(8k+7)$. For instance 0, 1, 2, 3, 4, 5, 6 are sums of three squares but 7 is not. Let $r_3(n)$ be the number of decompositions of the integer $n \geq 1$ into a sum of three squares (with $r_3(n) = 0$ if n is not a sum of three squares). More precisely $r_3(n)$ is the number of points $k \in \mathbb{Z}^3$ such that $|k|^2 = n$. We have $r_3(4n) = r_3(n)$, $\forall n \in \mathbb{N}$, $r_3(0) = 1$, $r_3(1) = 6, r_3(2) = 12, \ldots$. Then $r_3(2^j) = 6$ if j is even and 12 if j is odd. The behavior of $r_3(n)$ as $n \to \infty$ is erratic. The mean behavior is more regular since ([2])

$$\sum_{0\leq n\leq x} r_3(n) = \frac{4}{3}\pi\, x^{3/2} + O(x^{3/4+\epsilon}) \tag{24}$$

holds for every positive ϵ.

Let $\chi(n) = -1$ if $n \in \mathbb{N}\setminus 4\mathbb{N}$, $\chi(n) = 4$ if $n \in 4\mathbb{N}\setminus 16\mathbb{N}$, and $\chi(n) = 0$ if $n \in 16\mathbb{N}$. We then have

$$\tau = \sum_1^\infty \chi(n)\, r_3(n)\, n^{-1/2}\,(\delta_{\sqrt{n}/2} - \delta_{-\sqrt{n}/2})\,. \tag{25}$$

4. Guinand's construction

The preceding construction yields an atomic measure μ enjoying the following properties (a) μ is supported by a locally finite set Λ and (b) $\widehat{\mu} = -i\mu$. A measure μ which satisfies $\widehat{\mu} = \pm i\mu$ is necessarily odd as it was the case in our construction. If $\widehat{\mu} = \mu$ the mesure μ is even.

Do there exist crystalline measures such $\widehat{\mu} = \mu$ besides the standard Poisson summation formula? The answer is yes as it was suggested by Guinand in [4].

Theorem 4.1. *There exists an even atomic measure μ supported by the set $\Lambda = \{\pm\sqrt{k+1/9},\, k \in \mathbb{N}\}$ such that $\widehat{\mu} = \mu$.*

Lev and Olevskii stressed that Guinand did not prove that the atomic measure μ defined in [4] is a tempered distribution. This point will be clarified and Guinand's proof will be completed.

The new ingredient of the proof of Theorem 4.1 is our Lemma 4.1. Let $\chi\colon \mathbb{Z} \mapsto \{-1, 0, 1\}$ be the Dirichlet character defined by $\chi(k+3) = \chi(k)$, $\chi(0) = 0$, $\chi(1) = 1$, and $\chi(2) = -1$. Let

$$F(q) = \sum_1^\infty k\, \chi(k)\, q^{k^2}, \tag{26}$$

with $|q| < 1$. As Philippe Michel kindly taught us $F(q)$ can be written as a product of Dedekind η functions [6], [11]. More precisely, let $q = \exp(2\pi i z)$ and

$$\eta(z) = q^{1/24} \prod_1^\infty (1 - q^n)\,. \tag{27}$$

Then we have

$$F(q) = \frac{\eta(3z)^2\,\eta(12z)^2}{\eta(6z)}\,. \tag{28}$$

The infinite product $\prod_1^\infty (1 - q^n)$ does not vanish in the open unit disc U. It implies that $F(q) \neq 0$ if $|q| < 1$, $q \neq 0$. Then the holomorphic function $G(q) = F(q)/q$ does not vanish on U. Therefore $H(q) = G(q)^{1/3}$ is uniquely defined as a holomorphic function in U such that $H(0) = 1$. We have

$$H(q) = \prod_1^\infty (1 - q^{3n})\,(1 + q^{6n})^{2/3}\,(1 + q^{3n})^{1/3}\,, \tag{29}$$

which implies

$$H(q) = \sum_0^\infty \gamma_k\, q^{3k}\,, \tag{30}$$

where $\gamma_0 = 1$, γ_k are real numbers, and

$$\sum_0^\infty \gamma_k\, q^k = \prod_1^\infty (1 - q^n)\,(1 + q^{2n})^{2/3}\,(1 + q^n)^{1/3}\,. \tag{31}$$

Lemma 4.1. *There exists a constant C such that*

$$|\gamma_k| \leq C\, k^{1/3}, \quad k \geq 1\,. \tag{32}$$

We first observe that $\sum_1^\infty k\, r^{k^2} \leq \frac{C}{1-r}$ for $0 < r < 1$. This together with (26) implies $|G(q)| \leq \frac{C}{1-|q|}$ and

$$|H(q)| \leq \frac{C}{(1 - |q|)^{1/3}}\,. \tag{33}$$

Since $H(q)$ is holomorphic in the open unit disc we have, for $0 < r < 1$,

$$2\pi\, \gamma_k = r^{-k} \int_0^{2\pi} H(r \exp(i\theta)) \exp(-ik\theta)\, d\theta\,. \tag{34}$$

To prove Lemma 4.1 it suffices to plug $r = 1 - 1/k$ into (34) and to use (33).

We now follow Guinand and complete the proof of Theorem 4.1. We set $\lambda_k = \sqrt{k + 1/9}$, $k \in \mathbb{N}$, and define a one dimensional atomic measure by

$$\mu = \sum_0^\infty \gamma_k\, (\delta_{\lambda_k} + \delta_{-\lambda_k})\,. \tag{35}$$

Lemma 4.1 implies that μ is a tempered distribution. Edmund Landau already knew the following identity:

Lemma 4.2. *For every $a > 0$ we have*

$$\sum_{1}^{\infty} k\,\chi(k)\,\exp(-\pi k^2 a^2/3) = a^{-3} \sum_{1}^{\infty} k\,\chi(k)\,\exp(-\pi k^2/3a^2)\,. \tag{36}$$

If $q = \exp(-\pi a^2/3)$ the left hand side of (36) is $F(q) = \sum_1^\infty k\,\chi(k) q^{k^2}$. Similarly, if $r = \exp(-\pi/3a^2)$ the right hand side of (36) is $a^{-3}F(r)$. Returning to the argument z as in (27) we set $z = ia^2/6$ and $\zeta = i(6a^2)^{-1}$. Then Lemma 4.2 follows from (28) and from the functional equation satisfied by the Dedeking η function.

But Lemma 4.2 also follows from the ordinary Poisson summation formula. The Fourier transform of the measure $\sigma = \sum_{-\infty}^{\infty} \chi(k)\delta_{k/\sqrt{3}}$ is $-i\sigma$, and the Fourier transform of $\psi_a(x) = x\exp(-\pi x^2 a^2)$ is $-ia^{-3}x\exp(-\pi x^2 a^{-2})$. Then (36) follows from $\langle \widehat{\sigma}, \psi_a \rangle = \langle \sigma, \widehat{\psi}_a \rangle$.

Let $q = \exp(-\pi a^2/3)$ and $r = \exp(-\pi/3a^2)$ as above. We have $F(q) = a^{-3}F(r)$, by (34). Extracting cubic roots on both sides and using the definition of H yields $q^{1/3}H(q) = a^{-1}r^{1/3}H(r)$. Since $H(q) = \sum_0^\infty \gamma_k\, q^{3k}$ we obtain

$$\sum_{0}^{\infty} \gamma_k \exp\Big(-\pi a^2\Big(k + \frac{1}{9}\Big)\Big) = a^{-1} \sum_{0}^{\infty} \gamma_k \exp\Big(-\pi\Big(k + \frac{1}{9}\Big)\, a^{-2}\Big)\,. \tag{37}$$

The Guinand θ function is defined by $\theta_G(u) = \sum_0^\infty \gamma_k \exp(-\pi u(k + \frac{1}{9}))$ and (37) can be written $\theta_G(u) = u^{-1/2}\theta_G(1/u)$. To end the proof of Theorem 4.1 it suffices to copy the argument of (b) $\Rightarrow$ (c) in the Kahane–Mandelbrojt theorem.

The Kahane–Mandelbrojt scheme paves the way to the following definition:

Definition 4.1. The Guinand ζ function is defined by the series

$$\zeta_G(s) = \sum_{0}^{\infty} \gamma_k\,(k + 1/9)^{-s/2},$$

which converges if $\Re s > 8/3$.

Then the same proof which was used for the Epstein ζ function yields the following.

Theorem 4.2. *The Guinand ζ function $\zeta_G(s)$ is an entire function of the complex variable $s \in \mathbb{C}$ and satisfies the functional equation* (a) *of Theorem 2.1.*

Instead of (28), let us use the identity

$$\sum_{1}^{\infty} \Big(\frac{-2}{k}\Big)\, k\, q^{k^2} = \frac{\eta(16z)^9}{\eta(8z)^3\,\eta(32z)^3} = F_1(z)\,, \tag{38}$$

where $\left(\frac{p}{q}\right)$ denotes the Jacobi symbol. Then the identity satisfied by the Dedekind η function implies

$$F_1(z) = 4^{-3}(-iz)^{-3/2} F_1\Big(-\frac{1}{256z}\Big). \tag{39}$$

However (39) can also be proved using the standard Poisson summation formula. Indeed we define c_k, $k \in \mathbb{Z}$, by the following properties (i) c_k is periodic of period 8, (ii) $c_k = 0$ if k is even, (iii) $c_k = 1$ if $k \equiv 1 \pmod 8$ or $k \equiv 3 \pmod 8$, and (iv) $c_k = -1$ if $k \equiv 5 \pmod 8$ or $k \equiv 7 \pmod 8$. We then have $c_k = \left(\frac{-2}{k}\right)$, $k \geq 1$, and the Fourier transform of the measure $\sigma = \sum_{-\infty}^{\infty} c_k \delta_{k/\sqrt{8}}$ is $-i\sigma$ as in the second proof of Lemma 4.2. We set $G_1(q) = F_1(q)/q$ where $F_1(q)$ is defined by (38). Then (38) implies that G_1 does not vanish on the unit open disc. We can define $H_1(q) = G_1(q)^{1/3}$. Using (38) again we have

$$H_1(q) = \sum_{1}^{\infty} \beta_k \, q^{8k}. \tag{40}$$

As above

$$|\beta_k| \leq C\, k^{1/3}, \quad k \geq 1. \tag{41}$$

We set $q = \exp(-\pi a^2/8)$, $r = \exp(-\pi a^{-2}/8)$, and $z = ia^2/16$. The argument used in proving Theorem 4.1 yields

$$q^{1/3}\, H_1(q) = a^{-1}\, r^{1/3}\, H_1(r). \tag{42}$$

This implies

$$\sum_{1}^{\infty} \beta_k \exp(-\pi a^2(k+1/24)) = a^{-1} \sum_{1}^{\infty} \beta_k \exp(-\pi a^{-2}(k+1/24)). \tag{43}$$

We set $\lambda_k = \sqrt{k+1/24}$, $k \in \mathbb{N}$, and define a one dimensional atomic measure by

$$\mu = \sum_{0}^{\infty} \beta_k \, (\delta_{\lambda_k} + \delta_{-\lambda_k}). \tag{44}$$

Let us observe that μ is even and is a tempered distribution.

Theorem 4.3. *There exists an atomic measure μ supported by the set $\Lambda = \{\pm\sqrt{k+1/24},\ k \in \mathbb{N}\}$ such that $\widehat{\mu} = \mu$.*

The corresponding ζ function is defined by the series

$$\zeta_M(s) = \sum_{0}^{\infty} \beta_k \, (k+1/24)^{-s/2},$$

which converges if $\Re s > 8/3$. This ζ function is an entire function of the complex variable s.

The same game can be played with the following identity.

$$\sum_{1}^{\infty}\left(\frac{-6}{k}\right) k\, q^{k^2} = \frac{\eta(48z)^{13}}{\eta(24z)^5\, \eta(96z)^5}\,, \tag{45}$$

and we end with $\Lambda = \{\pm\sqrt{k+1/72},\, k \in \mathbb{N}\}$.

5. Illustration of a theorem by N. Lev and A. Olevskii

We apply the Kahane–Mandelbrojt scheme to the functional equation satisfied by the Dedekind η function. We have

$$\eta(-1/z) = \sqrt{-iz}\,\eta(z)\,. \tag{46}$$

The Euler function is defined as

$$\phi(q) = \prod_{1}^{\infty}(1-q^n) = \sum_{-\infty}^{\infty}(-1)^n\, q^{(3n^2-n)/2}\,. \tag{47}$$

Let $q = \exp(-2\pi a^2)$. The functional equation (46) is applied to $z = ia^2$ and yields

$$\sum_{-\infty}^{\infty}(-1)^n \exp[-\pi a^2(3n^2 - n + 1/12)] = \theta(a) = a^{-1}\theta(1/a)\,. \tag{48}$$

We then denote by μ the even atomic measure which is the sum

$$\sum_{0}^{\infty}(-1)^n(\delta_{-\lambda_n^{\pm}} + \delta_{\lambda_n^{\pm}})\,, \tag{49}$$

where $\lambda_n^{\pm} = \sqrt{3n^2 \pm n + 1/12}$. We conclude as above to $\widehat{\mu} = \mu$. Does this construction provide a counterexample to the main theorem proved by Lev and Olevskii in [9]? This theorem asserts that a crystalline measure whose support and spectrum are uniformly discrete is a generalized Dirac comb. It is the case here since $\sqrt{3n^2 \pm n + 1/12} = n\sqrt{3} \pm \frac{1}{2\sqrt{3}}$. The measure μ which we constructed is a generalized Dirac comb.

6. Open problems

Let $\Lambda = \{\lambda_k,\ k \in \mathbb{Z}\}$ as in (3) and (4), and let $\mathcal{M}_\Lambda$ be the collection of all atomic measures supported by Λ and whose Fourier transform is also supported by Λ. We are interested in comparing the property $\mathcal{M}_\Lambda \neq \{0\}$ to the geometrical structures described in [1]. If it is the case, what is the dimension of $\mathcal{M}_\Lambda$? Given a $\theta \in (0,1)$ what happens if $\Lambda = \{\pm\sqrt{k+\theta},\ k \in \mathbb{N}\}$?

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Received May 11, 2016.

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