Rev. Mat. Iberoam. **33** (2017), no. 4, 1149–1171
DOI 10.4171/RMI/966

Geometry of spaces of real polynomials of degree at most n

Christopher Boyd and Anthony Brown

Abstract. We study the geometry of the unit ball of the space of integral polynomials of degree at most n on a real Banach space. We prove Šmul'yan type theorems for Gâteaux and Fréchet differentiability of the norm on preduals of spaces of polynomials of degree at most n. We show that the set of extreme points of the unit ball of the predual of the space of integral polynomials is $\big\{\pm\sum_{j=0}^{n}\phi^j : \phi\in E', \|\phi\|\leq 1\big\}$. This contrasts greatly with the situation for homogeneous polynomials where the set of extreme points of the unit ball is the set $\{\pm\phi^n : \phi\in E', \|\phi\|=1\}$.

1. Introduction

One of the most fundamental concepts in analysis is the idea of a continuously differentiable function on an open subset of a Banach space. Asking for a higher degree of smoothness leads us to consider spaces of k–times differentiable functions. While it is very important to understand such spaces, their scope and general structure is such that the possibilities for the use of many of the tools that have been developed over the past 60 years has been limited. In a recent paper [9], Choi, Hájek and Lee show that any k-times differentiable mapping on an open subset of a real Banach space E can be approximated by a polynomial of degree at most k. While the study of spaces of polynomials between Banach space has been an extremely active area of research over the past 45 years, most of this research has focused on spaces of homogeneous polynomials. In [4] the authors initiated a systematic study of Banach spaces of polynomials of degree at most n. There the spaces of continuous, approximable, integral and nuclear polynomials were defined and their duality investigated. In this paper we will further develop the geometric theory of spaces of polynomials of degree at most n. We will concentrate on obtaining a description of the sets of extreme points for spaces of integral polynomials of degree at most n over real Banach spaces. Our results show a significant difference between the geometry of spaces of homogeneous and that of non-homogeneous polynomials,

Mathematics Subject Classification (2010): Primary 46G25; Secondary 46B20.
Keywords: Polynomials on Banach spaces, Gâteaux and Fréchet differentiability, extreme points.

which will prove essential in [5], where we give a complete characterisation of the isometries between spaces of polynomials on real Banach spaces.

Let us recall some definition and results from [4] and [12]. Given Banach spaces E and F and a positive integer n, a continuous mapping $P\colon E \to F$ is said to be an n-homogeneous polynomial if there is a (continuous) n-linear mapping $L\colon E\times \overset{n \text{ times}}{\cdots} \times E \to F$ such that $P(x) = L(x, \dots, x)$ for all x in E. We denote the space of all n-homogeneous polynomials from E into F by $\mathscr{P}({}^nE, F)$. A mapping $P\colon E \to F$ is said to be a polynomial of degree at most n if $P = \sum_{j=0}^n P_j$ with $P_j \in \mathscr{P}({}^jE, F)$. The space of all polynomials of degree at most n is denoted by $\mathscr{P}({}^{\leqslant n}E, F)$. With pointwise addition and scalar multiplication $\mathscr{P}({}^{\leqslant n}E, F)$ becomes a vector space and, when endowed with the norm $\|P\| = \sup_{\|x\|\leqslant 1} \|P(x)\|$, a Banach space. Both $\mathscr{P}({}^nE, F)$ and $\mathscr{P}({}^{\leqslant n}E, F)$ possess certain natural subspaces. An n-homogeneous polynomial, P, is said to be of finite type if it can be written as $P(x) = \sum_{j=1}^k \phi_j(x)^n y_j$ with $(\phi_j)_{j=1}^k \subset E'$ and $(y_j)_{j=1}^k \subset F$. The space of finite type polynomials is denoted by $\mathscr{P}_f({}^nE, F)$. A polynomial P in $\mathscr{P}({}^{\leqslant n}E, F)$ is said to be of finite type if it can be written as $P = \sum_{j=0}^n P_j$ with P_j in $\mathscr{P}_f({}^jE, F)$. The closure of $\mathscr{P}_f({}^nE, F)$ (resp. $\mathscr{P}_f({}^{\leqslant n}E, F)$) with respect to the norm $\|\cdot\|$ is denoted by $\mathscr{P}_A({}^nE, F)$ (resp. $\mathscr{P}_A({}^{\leqslant n}E, F)$) and is called the space of n-homogeneous approximable polynomials (resp. approximable polynomials of degree at most n). When F is the field of scalars we write $\mathscr{P}({}^nE)$, $\mathscr{P}_f({}^nE)$, $\mathscr{P}_A({}^nE)$, $\mathscr{P}({}^{\leqslant n}E)$, $\mathscr{P}_f({}^{\leqslant n}E)$ and $\mathscr{P}_A({}^{\leqslant n}E)$.

To understand the duality theory of spaces of polynomials of degree at most n, we introduce the spaces of integral and nuclear polynomials of degree at most n. A polynomial P in $\mathscr{P}({}^{\leqslant n}E)$ is said to be an integral polynomial if there is a regular Borel measure, μ, on $(B_{E'}, \sigma(E', E))$ such that

$$(*) \qquad P(x) = \int_{\overline{B}_{E'}} \sum_{j=0}^{n} \phi(x)^j \, d\mu(\phi)$$

for all x in E. We denote the space of all integral polynomials of degree at most n by $\mathscr{P}_I({}^{\leqslant n}E)$. When endowed with the norm

$$\|P\|_I = \inf \left\{ |\mu| : \mu \text{ satisfies } (*) \right\}$$

the pair $(\mathscr{P}_I({}^{\leqslant n}E), \|\cdot\|_I)$ becomes a Banach space. A polynomial P in $\mathscr{P}({}^{\leqslant n}E)$ is said to be a nuclear polynomial if there are sequences $(\lambda_k)_k \subset \mathbb{K}$ and $(\phi_k)_k \subset \overline{B}_{E'}$ with $\sum_{k=1}^\infty |\lambda_k| < \infty$ such that

$$P(x) = \sum_{k=1}^{\infty} \lambda_k \sum_{j=0}^{n} \phi_k(x)^j$$

for all x in E. We denote the space of all nuclear polynomials of degree at most n by $\mathscr{P}_N({}^{\leqslant n}E)$. When endowed with the norm

$$\|P\|_N = \inf \left\{ \sum_{k=0}^{\infty} |\lambda_k| : P(x) = \sum_{k=1}^{\infty} \lambda_k \sum_{j=0}^{n} \phi_k(x)^j, \phi_k \in \overline{B}_{E'} \right\},$$

the pair $(\mathscr{P}_N({}^{\leqslant n}E), \|\cdot\|_N)$ becomes a Banach space.

Both $\mathscr{P}(^{\leqslant n}E)$ and $\mathscr{P}_I(^{\leqslant n}E)$ are dual spaces. We consider $\bigoplus_{j=0}^n \bigotimes_{j,s} E$, the direct sum of the spaces of j-fold symmetric tensors in E for j between 0 and n. We define the π-norm on $\bigoplus_{j=0}^n \bigotimes_{j,s} E$ by

$$\|\theta\|_\pi = \inf\Big\{\sum_{k=1}^m |\lambda_k| : \theta = \sum_{k=1}^m \lambda_k \sum_{j=0}^n \underbrace{x_k \otimes \cdots \otimes x_k}_{j \text{ terms}}, \|x_k\| \leqslant 1\Big\}$$

and the ϵ-norm on $\bigoplus_{j=0}^n \bigotimes_{j,s} E$ by

$$\Big\|\sum_{j=0}^n \sum_{k=1}^{m_j} \lambda_{k,j} \underbrace{x_{k,j} \otimes \cdots \otimes x_{k,j}}_{j \text{ terms}}\Big\|_\epsilon = \sup_{\|\phi\|\leq 1} \Big\{\Big|\sum_{j=0}^n \sum_{k=1}^{m_j} \lambda_{k,j} \phi(x_{k,j})^j\Big|\Big\}. \tag{1.1}$$

It is shown in [4] that $(\mathscr{P}(^{\leqslant n}E), \|\cdot\|)$ is isometrically isomorphic to

$$\Big(\bigoplus_{j=0}^n \bigotimes_{j,s} E, \|\cdot\|_\pi\Big)',$$

while $(\mathscr{P}_I(^{\leqslant n}E), \|\cdot\|_I)$ is isometrically isomorphic to

$$\Big(\bigoplus_{j=0}^n \bigotimes_{j,s} E, \|\cdot\|_\epsilon\Big)'.$$

For further information on spaces of homogeneous polynomials we refer the reader to [12].

Given a Banach space E, a point z in $\overline{B}_E$ is said to be an extreme point of $\overline{B}_E$ if z is not the midpoint of any line segment which is contained in $\overline{B}_E$. The extreme points of $\overline{B}_E$ are denoted by $\mathcal{E}\mathrm{xt}\,(\overline{B}_E)$. A point x in the closed unit ball of E is said to be an exposed point of $\overline{B}_E$ if we can find a $\phi \in E'$ with $\|\phi\| = 1$ such that

$$\phi(x) = 1 \quad \text{and} \quad \phi(y) < 1 \text{ for } y \in \overline{B}_E\backslash\{x\}.$$

If this is the case then we say that ϕ *exposes* x.

When $E = F'$ is a dual Banach space and the x is exposed by ϕ in F we say that x is a weak*-exposed point of E and that ϕ weak*-exposes the unit ball of E at x.

We say that x is said to be a strongly exposed point of $\overline{B}_E$ if we can find a $\phi \in E'$ such that

$$\phi(x) = 1$$

and whenever $(x_n)_n$ is a sequence in $\overline{B}_E$ with

$$\lim_{n\to\infty} \phi(x_n) = 1$$

then $(x_n)_n$ converges to x in norm. We will say that ϕ strongly exposes x.

If $E = F'$ is a dual Banach space and the x is strongly exposed by ϕ in F we say that x is a weak*-strongly exposed point of E and that ϕ weak*-strongly exposes the unit ball of E at x. For further details on the extremal structure of convex sets we refer to [13]. Further information on the geometry of Banach spaces including Šmul'yan's theorem can be found in [10].

2. Šmul'yan type theorems

In the first part of the paper we will prove various 'Šmul'yan type theorems' which will characterize Gâteaux and Fréchet differentiability of the norm of some of the spaces introduced in [4]. While our results are interesting in their own right, they will also be used in the second part of the paper where we will investigate the extremal structure of related spaces.

2.1. Gâteaux differentiability of the norm of $\big(\bigoplus_{j=0}^n \widehat{\bigotimes}_{j,s} E, \|\cdot\|_\epsilon\big)$

In [17], Šmul'yan shows that if E is a Banach space then the norm of E is Gâteaux differentiable at $x \in \overline{B}_E$ with derivative ϕ if and only if x weak*-exposes the closed unit ball of E' at ϕ. In Proposition 2.1 of [15], Ruess and Stegall characterise Gâteaux differentiability of the norms of certain operator spaces and in Theorem 7 of [7], the first author and Ryan proved a 'Šmul'yan type' theorem which relates the Gâteaux differentiability of the norm of $\widehat{\bigotimes}_{n,s,\epsilon} E$ to elements of $S_{E'}$ as well as to elements of $S_{\mathscr{P}_I({}^nE)}$. We now prove a non-homogeneous version of this result. Before we proceed, it is worth noting a couple of important differences. Firstly, when dealing with homogeneous polynomials, the cases n even and n odd have to be dealt with separately, This is not necessary for polynomials of degree n. Secondly, in our theorem, the elements in E' do not have to lie in the sphere, they only have to lie in the unit ball.

In the following we will find that the derivatives of points in $\big(\bigoplus_{j=0}^n \widehat{\bigotimes}_{j,s} E, \|\cdot\|_\epsilon\big)$ are of the form $\sum_{j=0}^n \phi^j$, where we interpret $\sum_{j=0}^n \phi^j$ to be 1 if $\phi \equiv 0$.

Theorem 2.1. *Let E be a real Banach space and let n be a positive integer. Let $T \in \big(\bigoplus_{j=0}^n \widehat{\bigotimes}_{j,s} E, \|\cdot\|_\epsilon\big)$ with $\|T\|_\epsilon = 1$ and $\phi \in \overline{B}_{E'}$. The following are equivalent.*

(a) *The norm of $\big(\bigoplus_{j=0}^n \widehat{\bigotimes}_{j,s} E, \|\cdot\|_\epsilon\big)$ is Gâteaux differentiable at T with derivative $\sum_{j=0}^n \phi^j$ $\big($respectively $-\sum_{j=0}^n \phi^j\big)$.*

(b) (i) *$T\big(\sum_{j=0}^n \phi^j\big) = 1$ $\big($respectively $T\big(-\sum_{j=0}^n \phi^j\big) = 1\big)$ and ϕ is unique in $\overline{B}_{E'}$ satisfying this condition.*

(ii) *There exists a real number α, with $-1<\alpha<1$, such that $T\big(\sum_{j=0}^n \psi^j\big) > \alpha$ $\big($respectively $T\big(-\sum_{j=0}^n \psi^j\big) > \alpha\big)$ for all $\psi \in \overline{B}_{E'}$.*

(c) (i) *$T\big(\sum_{j=0}^n \phi^j\big) = 1$ $\big($respectively $T\big(-\sum_{j=0}^n \phi^j\big) = 1\big)$.*

(ii) *There exists a real number α, with $-1<\alpha<1$, such that $T\big(\sum_{j=0}^n \psi^j\big) > \alpha$ $\big($respectively $T\big(-\sum_{j=0}^n \psi^j\big) > \alpha\big)$ for all $\psi \in \overline{B}_{E'}$.*

(iii) *If $(\phi_k)_k$ is a sequence in the closed unit ball of E' such that $\big(T\big(\sum_{j=0}^n \phi_k^j\big)\big)_k$ converges to 1 $\big($respectively $\big(T\big(-\sum_{j=0}^n \phi_k^j\big)\big)_k$ converges to 1$\big)$, then $(\phi_k)_k$ has a subnet $(\phi_{k_\alpha})_\alpha$ which converges to ϕ in the weak* topology on E'.*

Proof. We first note that by Proposition 5.1(a) of [4],

$$\mathcal{E}\mathrm{xt}\left(\overline{B}_{\mathscr{P}_I(\leqslant^n E)}\right) \subseteq \left\{\pm \sum_{j=0}^{n} \phi^j : \phi \in E', \|\phi\| \leqslant 1\right\}.$$

Since each weak*-exposed point is also an extreme point, it follows from a Theorem of Šmul'yan (see [17]) that the only possibilities for the derivatives are $\pm \sum_{j=0}^{n} \phi^j$. We will prove the case when the derivative is $\sum_{j=0}^{n} \phi^j$. The proof when the derivative is $-\sum_{j=0}^{n} \phi^j$ is analogous.

(a) ⇒ (b) (i) and (c) (i). Let us assume that the norm of $\big(\bigoplus_{j=0}^{n} \widehat{\bigotimes}_{j,s} E, \|\cdot\|_\epsilon\big)$ is Gâteaux differentiable at T with derivative $\sum_{j=0}^{n} \phi^j$. Then it follows from a theorem of Šmul'yan (see [17]) that T weak*-exposes the closed unit ball of $\mathscr{P}_I(^{\leqslant n}E)$ at $\sum_{j=0}^{n} \phi^j$. Thus $T\big(\sum_{j=0}^{n} \phi^j\big) = 1$ and $T(P) < 1$ for all other $P \in \overline{B}_{\mathscr{P}_I(\leqslant^n E)}$. In particular $T\big(\sum_{j=0}^{n} \psi^j\big) < 1$ for all $\psi \in \overline{B}_{E'}$ with $\psi \neq \phi$.

(a) ⇒ (b)(ii) and (c)(ii). We will assume that (b)(ii) or (c)(ii) does not hold and show that (a) does not hold. If (b)(ii) or (c)(ii) does not hold, then we can find a sequence $(\phi_k)_k \subset \overline{B}_{E'}$ such that $T\big(\sum_{j=0}^{n} \phi_k^j\big) \to -1$. However, $\overline{B}_{E'}$ is compact in the weak* topology, so we can find a subnet $(\phi_{k_\alpha})_\alpha$ of $(\phi_k)_k$ that converges weak* to some $\psi \in \overline{B}_{E'}$. But then $\big(\phi_{k_\alpha}(x)\big)_\alpha$ converges to $\psi(x)$ for all $x \in E$. Now consider a general element

$$\sum_{j=0}^{n}\sum_{m=1}^{p} \lambda_{j,m} \underbrace{x_{j,m} \otimes \cdots \otimes x_{j,m}}_{j \text{ terms}} \in \Big(\bigoplus_{j=0}^{n} \bigotimes_{j,s} E, \|\cdot\|_\epsilon\Big).$$

Since $\sum_{j=0}^{n} \phi^j$ is represented by a unit point mass at ϕ, we have, using Equation 11 of [4], that

$$\Big(\sum_{j=0}^{n} \phi_\alpha^j\Big)\Big(\sum_{j=0}^{n}\sum_{m=1}^{p} \lambda_{j,m} \underbrace{x_{j,m} \otimes \cdots \otimes x_{j,m}}_{j \text{ terms}}\Big) = \sum_{j=0}^{n}\sum_{m=1}^{p} \lambda_{j,m}\phi_\alpha(x_{j,m})^j$$

and

$$\Big(\sum_{j=0}^{n} \psi^j\Big)\Big(\sum_{j=0}^{n}\sum_{m=1}^{p} \lambda_{j,m} \underbrace{x_{j,m} \otimes \cdots \otimes x_{j,m}}_{j \text{ terms}}\Big) = \sum_{j=0}^{n}\sum_{m=1}^{p} \lambda_{j,m}\psi(x_{j,m})^j.$$

Thus $\big(\big(\sum_{j=0}^{n} \phi_{k_\alpha}^j\big)(\theta)\big)_\alpha$ converges to $\big(\sum_{j=0}^{n} \psi^j\big)(\theta)$ for all $\theta \in \big(\bigoplus_{j=0}^{n} \bigotimes_{j,s} E, \|\cdot\|_\epsilon\big)$. Since the space $\big(\bigoplus_{j=0}^{n} \bigotimes_{j,s} E, \|\cdot\|_\epsilon\big)$ is dense in $\big(\bigoplus_{j=0}^{n} \widehat{\bigotimes}_{j,s} E, \|\cdot\|_\epsilon\big)$, we have

$$\Big(\bigoplus_{j=0}^{n} \bigotimes_{j,s} E, \|\cdot\|_\epsilon\Big)' \overset{1}{\cong} \Big(\bigoplus_{j=0}^{n} \widehat{\bigotimes}_{j,s} E, \|\cdot\|_\epsilon\Big)' \overset{1}{\cong} \mathscr{P}_I(^{\leqslant n}E).$$

Since the unit ball of $\mathscr{P}_I(^{\leqslant n}E)$, $\overline{B}_{\mathscr{P}_I(\leqslant^n E)}$, is $\sigma\big(\mathscr{P}_I(^{\leqslant n}E), \big(\bigoplus_{j=0}^{n} \widehat{\bigotimes}_{j,s} E, \|\cdot\|_\epsilon\big)\big)$ compact and since the topology $\sigma\big(\mathscr{P}_I(^{\leqslant n}E), \big(\bigoplus_{j=0}^{n} \bigotimes_{j,s} E, \|\cdot\|_\epsilon\big)\big)$ is a weaker

Hausdorff topology on $\overline{B}_{\mathscr{P}_I(\leqslant^n E)}$, the relative $\sigma\big(\mathscr{P}_I(^{\leqslant n}E), \big(\bigoplus_{j=0}^n \bigotimes_{j,s} E, \|\cdot\|_\epsilon\big)\big)$ topology on $\overline{B}_{\mathscr{P}_I(\leqslant^n E)}$ is the same as the relative $\sigma\big(\mathscr{P}_I(^{\leqslant n}E), \big(\bigoplus_{j=0}^n \widehat{\bigotimes}_{j,s} E, \|\cdot\|_\epsilon\big)\big)$ topology. Thus we have that $\big(\sum_{j=0}^n \phi_{k_\alpha}^j\big)_\alpha$ $\sigma\big(\mathscr{P}_I(^{\leqslant n}E), \big(\bigoplus_{j=0}^n \widehat{\bigotimes}_{j,s} E, \|\cdot\|_\epsilon\big)\big)$ converges to $\sum_{j=0}^n \psi^j$. Hence

$$T\Big(-\sum_{j=0}^n \psi^j\Big) = \lim_\alpha T\Big(-\sum_{j=0}^n \phi_{k_\alpha}^j\Big) = 1.$$

Since $-\sum_{j=0}^n \psi^j \neq \sum_{j=0}^n \phi^j$, it follows that T does not weak*-expose $\overline{B}_{\mathscr{P}_I(\leqslant^n E)}$ at $\sum_{j=0}^n \phi^j$. This contradicts a Theorem of Šmul'yan (see [17]), so that (a) does not hold, as required.

(b) $\Rightarrow$ (c)(iii). We will assume that (c)(iii) does not hold and show that this implies that (b)(i) does not hold. Now, if (c)(iii) does not hold, then there exists a sequence $(\phi_k)_k \subset \overline{B}_{E'}$ such that $\lim_{k\to\infty} T\big(\sum_{j=0}^n \phi_k^j\big) = 1$ and such that no subnet of $(\phi_k)_k$ converges weak* to ϕ. In this case there must exist a weak* neighbourhood of $0 \in E'$, say V, such that $\{(\phi_k)_k \cap (\phi + V)\} \setminus \{\phi\} = \emptyset$. Since $\overline{B}_{E'}$ is weak* compact we can find a subnet $(\phi_{k_\alpha})_\alpha$ of $(\phi_k)_k$ which converges weak* to some $\psi \in \overline{B}_{E'}$, with $\psi \neq \phi$. Hence $T\big(\sum_{j=0}^n \psi^j\big) = \lim_\alpha T\big(\sum_{j=0}^n \phi_{k_\alpha}^j\big) = 1$ and this contradicts (b)(i), as required.

(c) $\Rightarrow$ (a). Suppose that (c) holds and (a) does not. If (a) does not hold then the norm of $\big(\bigoplus_{j=0}^n \widehat{\bigotimes}_{j,s} E, \|\cdot\|_\epsilon\big)$ is not Gâteaux differentiable at T with derivative $\sum_{i=0}^n \phi^i$. This means there exists $S \in \big(\bigoplus_{j=0}^n \widehat{\bigotimes}_{j,s} E, \|\cdot\|_\epsilon\big)$ such that

$$\lim_{t\to 0} \frac{\|T + tS\| - \|T\|}{t} \neq S\Big(\sum_{j=0}^n \phi^j\Big).$$

Hence, for each $\epsilon > 0$, we can find a null sequence $(t_k)_k$ of positive numbers (changing S to $-S$ if necessary) so that

$$\|T + t_k S\| - \|T\| - t_k\, S\Big(\sum_{j=0}^n \phi^j\Big) \geqslant \epsilon\, t_k \quad \text{for all } k \in \mathbb{N}.$$

Using [13], p. 640, we see that $\mathcal{E}\mathrm{xt}\,\big(\overline{B}_{\mathscr{P}_I(\leqslant^n E)}\big)$ is a boundary for $\big(\bigoplus_{j=0}^n \widehat{\bigotimes}_{j,s} E, \|\cdot\|_\epsilon\big)$. Combining this with Proposition 5.1(a) of [4], it follows that for each $k \in \mathbb{N}$, we can choose $\phi_k \in \overline{B}_{E'}$, $\alpha_k = \pm 1$ such that $\alpha_k (T + t_k S)\big(\sum_{j=0}^n \phi_k^j\big) = \|T + t_k S\|$. Then

$$\begin{aligned} 1 = \|T\| &\geqslant \alpha_k\, T\Big(\sum_{j=0}^n \phi_k^j\Big) = \alpha_k (T + t_k S)\Big(\sum_{j=0}^n \phi_k^j\Big) - \alpha_k\, t_k\, S\Big(\sum_{j=0}^n \phi_k^j\Big) \\ &\geqslant \|T + t_k S\| - t_k \|S\|. \end{aligned}$$

Now, as $k \to \infty$, $t_k \to 0$ so that $\|T + t_k S\| - t_k\|S\| \to \|T\| = 1$. Therefore $\alpha_k T\big(\sum_{j=0}^n \phi_k^j\big) \to 1$. But if (c)(ii) holds, then this implies that there must exist an $N \in \mathbb{N}$ such that $\alpha_k = 1$ for all $k \geqslant N$.

Next, as (c)(iii) holds, then $(\phi_k)_k$ must have a subnet $(\phi_{k_\alpha})_\alpha$ which converges weak* to ϕ. As we showed in proving that (a) implies (b)(ii) and (c)(ii), this implies that $\big(\sum_{j=0}^n \phi_{k_\alpha}^j\big)_\alpha$ converges weak* to $\sum_{j=0}^n \phi^j$. Hence, with the above ϵ,

$$\begin{aligned}\epsilon t_{k_\alpha} &\leqslant \|T + t_{k_\alpha} S\| - \|T\| - t_{k_\alpha} S\Big(\sum_{j=0}^n \phi^j\Big)\\ &= (T + t_{k_\alpha} S)\Big(\sum_{j=0}^n \phi_{k_\alpha}^j\Big) - \|T\| - t_{k_\alpha} S\Big(\sum_{j=0}^n \phi^j\Big)\\ &\leqslant t_{k_\alpha} S\Big(\sum_{j=0}^n \phi_{k_\alpha}^j\Big) - t_{k_\alpha} S\Big(\sum_{j=0}^n \phi^j\Big) = t_{k_\alpha}\Big(S\Big(\sum_{j=0}^n \phi_{k_\alpha}^j\Big) - S\Big(\sum_{j=0}^n \phi^j\Big)\Big)\end{aligned}$$

will eventually hold. However this is impossible since $\big(\sum_{j=0}^n \phi_{k_\alpha}^j\big)_\alpha$ converges weak* to $\sum_{j=0}^n \phi^j$, giving us our required contradiction. □

Remark 2.2. If E is separable then $\big(\overline{B}_{E'}, w*\big)$ is metrizable, so that its topology can be described using sequences rather than nets. Hence in this case, all the use of subnets in the proof of Theorem 2.1 can be replaced by use of subsequences.

Since $\mathscr{P}_A({}^{\leqslant n}E)$ is isometrically isomorphic to $\big(\bigoplus_{j=0}^n \widehat{\bigotimes}_{j,s} E', \|\cdot\|_\epsilon\big)$ (see Equation 8 of [4]), Theorem 2.1 can also be written in terms of approximable polynomials. To do this we will employ the Aron–Berner extension, [1], which gives us a canonical way to extend a polynomial, P, on E to a polynomial, $\overline{P}$, on E''. Note that if $P \in \mathscr{P}_f({}^{\leqslant n}E)$ is defined by $P = \sum_{j=0}^n \sum_{k=1}^{m_j} \phi_{j,k}^j$, then the Aron–Berner extension of P is given by

$$\overline{P}(x) = \sum_{j=0}^n \sum_{k=1}^{m_j} x(\phi_{j,k})^j. \tag{2.1}$$

We can now state our corollary.

Corollary 2.3. *Let E be a real Banach space, let n be a positive integer, let $P \in S_{\mathscr{P}_A({}^{\leqslant n}E)}$, with $\overline{P}$ its Aron–Berner extension, let $x \in \overline{B}_{E''}$ and let δ_x be the evaluation at x. Then the following are equivalent.*

(a) *The norm of $\mathscr{P}_A({}^{\leqslant n}E)$ is Gâteaux differentiable at P with derivative δ_x (respectively $-\delta_x$).*

(b) (i) *$\overline{P}(x) = 1$ $\big($respectively $\overline{P}(x) = -1\big)$ and x is the unique point in $\overline{B}_{E''}$ where this holds.*

(ii) *There exists a real number α, with $-1 < \alpha < 1$, such that $\overline{P}(y) > \alpha$ $\big($respectively $\overline{P}(y) < \alpha\big)$ for all $y \in \overline{B}_{E''}$.*

(c) (i) $\overline{P}(x) = 1$ (*respectively* $\overline{P}(x) = -1$).

(ii) *There exists a real number* α, *with* $-1 < \alpha < 1$, *such that* $\overline{P}(y) > \alpha$ (*respectively* $\overline{P}(y) < \alpha$) *for all* $y \in \overline{B}_{E''}$.

(iii) *If* $(x_k)_k$ *is a sequence in the closed unit ball of* E'' *such that the sequence* $\big(\overline{P}(x_k)\big)_k$ *converges to* 1 (*respectively* $\big(\overline{P}(x_k)\big)_k$ *converges to* -1) *then* $(x_k)_k$ *has a subnet* $(x_{k_\alpha})_\alpha$ *which converges to* x *in the weak* topology.*

Proof. Let $P \in \mathscr{P}_f({}^{\leqslant n}E)$ be given by $P = \sum_{j=0}^{n}\sum_{k=1}^{m_j} \phi_{j,k}^j$. Then, using (2.1) and the duality in Equation 12 of [4], we have

$$P\Big(\pm\sum_{j=0}^{n} x^j\Big) = \pm\sum_{j=0}^{n}\sum_{k=1}^{m_j} x(\phi_{j,k})^j = \pm\overline{P}(x) = \pm\delta_x(\overline{P}).$$

Extending by density to all $P \in \mathscr{P}_A({}^{\leqslant n}E)$, the result now follows immediately from Theorem 2.1. □

Remark 2.4. In Theorems 7 and 8 of [7], there are examples of points where the norm of $\widehat{\bigotimes}_{n,s,\epsilon}E$ is Gâteaux differentiable for each $n \geqslant 2$. Since $\widehat{\bigotimes}_{n,s,\epsilon}E' \overset{1}{\cong} \mathscr{P}_A({}^nE)$ these examples also give points where the norm of $\mathscr{P}_A({}^nE)$ is Gâteaux differentiable. However if we regard these polynomials as elements of $\mathscr{P}_A({}^{\leqslant n}E)$ then the situation is totally different. If $P \in \mathscr{P}_A({}^{\leqslant n}E)$ is j-homogeneous with j even, then $\overline{P}(x) = \overline{P}(-x)$ so that condition (b)(i) of Corollary 2.3 cannot hold. On the other hand, if $P \in \mathscr{P}_A({}^{\leqslant n}E)$ is j-homogeneous with j odd, then $\overline{P}(x) = -\overline{P}(-x)$ so that conditions (b)(i) and (b)(ii) cannot hold simultaneously. Thus the norm of $\mathscr{P}_A({}^{\leqslant n}E)$ is not Gâteaux differentiable at any j-homogeneous point for $j \geqslant 2$, and clearly the same holds for any point that is an even or odd polynomial.

Later on, in Corollaries 3.2 and 3.3, we will give results in the converse direction and list points where the norms of $\big(\bigoplus_{j=0}^n \widehat{\bigotimes}_{j,s}E, \|\cdot\|_\epsilon\big)$ and $\mathscr{P}_A({}^{\leqslant n}E)$ are Gâteaux differentiable.

2.2. Fréchet differentiability of the norm of $\big(\bigoplus_{j=0}^n \widehat{\bigotimes}_{j,s}E, \|\cdot\|_\epsilon\big)$

In [18], Šmul'yan shows that if E is a Banach space then the norm of E is Fréchet differentiable at $x \in \overline{B}_E\backslash\{0\}$ with derivative ϕ if and only if x weak*-strongly exposes the closed unit ball of E' at ϕ. In [15] and [16], Ruess and Stegall prove results about the Fréchet differentiability of the norm in various operator spaces and in Theorem 11 of [7] the first author and Ryan prove a Fréchet differentiability version of Theorem 7 of [7]. In this section we prove an analogue of Theorem 2.1 but this time characterising Fréchet differentiability instead of Gâteaux differentiability. Note that it may also be regarded as the non-homogeneous version of Theorem 11 of [7].

Theorem 2.5. *Let* E *be a real Banach space and let* n *be a positive integer. If* $T \in S_{(\bigoplus_{j=0}^n \widehat{\bigotimes}_{j,s}E, \|\cdot\|_\epsilon)}$ *and* $\phi \in \overline{B}_{E'}$, *then the following are equivalent.*

(a) *The norm of* $\big(\bigoplus_{j=0}^{n}\widehat{\bigotimes}_{j,s}E, \|\cdot\|_\epsilon\big)$ *is Fréchet differentiable at* T *with derivative* $\sum_{j=0}^{n}\phi^j$ $\big($*respectively* $-\sum_{j=0}^{n}\phi^j\big)$.

(b) (i) $T\big(\sum_{j=0}^{n}\phi^j\big)=1$ $\big($*respectively* $T\big(-\sum_{j=0}^{n}\phi^j\big)=1\big)$.

(ii) *There exists a real number* α, *with* $-1<\alpha<1$, *such that* $T\big(\sum_{j=0}^{n}\psi^j\big)>\alpha$ $\big($*respectively* $T\big(-\sum_{j=0}^{n}\psi^j\big)>\alpha\big)$ *for all* $\psi\in\overline{B}_{E'}$.

(iii) *If* $(\phi_k)_k$ *is a sequence in the closed unit ball of* E' *such that* $\big(T\big(\sum_{j=0}^{n}\phi_k^j\big)\big)_k$ *converges to* 1 $\big($*respectively* $\big(T\big(-\sum_{j=0}^{n}\phi_k^j\big)\big)_k$ *converges to* 1$\big)$, *then* $(\phi_k)_k$ *converges in norm to* ϕ.

(c) *The closed unit ball of* $\mathscr{P}_I(^{\leqslant n}E)$ *is weak*-strongly exposed by* T *at* $\sum_{j=0}^{n}\phi^j$ $\big($*respectively at* $-\sum_{j=0}^{n}\phi^j\big)$.

Proof. Since each weak*-strongly exposed point is also an extreme point, it follows, as in Theorem 2.1, that the only possibilities for the derivatives are $\pm\sum_{j=0}^{n}\phi^j$. Again, we will prove the case where the derivative is $\sum_{j=0}^{n}\phi^j$. The other case is similar.

(a) $\Leftrightarrow$ (c). This follows from a theorem of Šmul'yan (see [18]).

(a) $\Rightarrow$ (b)(i) and (b)(ii). If the norm of $\big(\bigoplus_{j=0}^{n}\widehat{\bigotimes}_{j,s}E, \|\cdot\|_\epsilon\big)$ is Fréchet differentiable at T with derivative $\sum_{i=0}^{n}\phi^i$ then it is also Gâteaux differentiable at T with derivative $\sum_{j=0}^{n}\phi^j$. Hence these implications follow from Theorem 2.1.

(c) $\Rightarrow$ (b)(iii). First note that if $x\in E$ has norm one, then using (1.1), we see that x is also an element of $\big(\bigoplus_{j=0}^{n}\widehat{\bigotimes}_{j,s}E, \|\cdot\|_\epsilon\big)$ of norm one. Moreover, using the duality between $\big(\bigoplus_{j=0}^{n}\widehat{\bigotimes}_{j,s}E, \|\cdot\|_\epsilon\big)$ and $\mathscr{P}_I(^{\leqslant n}E)$ (see Equation 11 of [4]), it follows that $\big(\sum_{j=0}^{n}\phi^j\big)(x)=\phi(x)$. Now let us assume that the closed unit ball of $\mathscr{P}_I(^{\leqslant n}E)$ is weak*-strongly exposed by T at $\sum_{j=0}^{n}\phi^j$. Since $\sum_{j=0}^{n}\phi^j$ is a weak*-strongly exposed point, we have by definition that if $\big(\sum_{j=0}^{n}\phi_k^j\big)_k$ is a sequence in the closed unit ball of $\mathscr{P}_I(^{\leqslant n}E)$ (so that $(\phi_k)_k$ lies in the closed unit ball of E') such that the sequence $\big(T\big(\sum_{j=0}^{n}\phi_k^j\big)\big)_k$ converges to 1 then $\big(\sum_{j=0}^{n}\phi_k^j\big)_k$ converges in the integral norm to $\sum_{j=0}^{n}\phi^j$. However

$$\|\phi_k-\phi\| = \sup_{x\in\overline{B}_E}|(\phi_k-\phi)(x)| \leqslant \sup_{S\in\overline{B}_{(\bigoplus_{j=0}^{n}\widehat{\bigotimes}_{j,s}E,\|\cdot\|_\epsilon)}}\Big|S\Big(\sum_{j=0}^{n}\phi_k^j-\sum_{j=0}^{n}\phi^j\Big)\Big|$$
$$= \Big\|\sum_{j=0}^{n}\phi_k^j-\sum_{j=0}^{n}\phi^j\Big\|_I.$$

Hence $(\phi_k)_k$ converges in norm to ϕ, as required.

(b) $\Rightarrow$ (a). We will assume that (a) does not hold and that (b) holds. Now, if the norm of $\big(\bigoplus_{j=0}^{n}\widehat{\bigotimes}_{j,s}E, \|\cdot\|_\epsilon\big)$ is not Fréchet differentiable at T with derivative

$\sum_{j=0}^{n} \phi^j$ then we can find an $\epsilon > 0$ and a sequence $(T_k)_k \in \big(\bigoplus_{j=0}^{n} \widehat{\bigotimes}_{j,s} E, \|\cdot\|_\epsilon\big)$ converging to 0 such that

$$\left| \|T + T_k\|_\epsilon - \|T\|_\epsilon - T_k\Big(\sum_{j=0}^{n} \phi^j\Big) \right| \geqslant \epsilon \|T_k\|_\epsilon \quad \text{for all } k \in \mathbb{N}. \tag{2.2}$$

Now, for each $k \in \mathbb{N}$, we choose $\phi_k \in \overline{B}_{E'}$ and $t_k = \pm 1$ such that

$$\|T + T_k\|_\epsilon < t_k (T + T_k)\Big(\sum_{j=0}^{n} \phi_k^j\Big) + \frac{1}{k}\, \|T_k\|_\epsilon. \tag{2.3}$$

Then

$$\begin{aligned} 1 = \|T\|_\epsilon \geqslant t_k T\Big(\sum_{j=0}^{n} \phi_k^j\Big) &= t_k (T + T_k)\Big(\sum_{j=0}^{n} \phi_k^j\Big) - t_k T_k\Big(\sum_{j=0}^{n} \phi_k^j\Big) \\ &> \|T + T_k\|_\epsilon - \frac{1}{k}\, \|T_k\|_\epsilon - \|T_k\|_\epsilon. \end{aligned}$$

However, as $k \to \infty$,

$$\|T + T_k\|_\epsilon - \frac{1}{k}\, \|T_k\|_\epsilon - \|T_k\|_\epsilon \to \|T\|_\epsilon = 1,$$

and so $t_k T\big(\sum_{j=0}^{n} \phi_k^j\big) \to 1$. Thus, by (b)(ii), we must have $t_k = 1$ for all sufficiently large k and this means that $T\big(\sum_{j=0}^{n} \phi_k^j\big) \to 1$. Hence, using (b)(iii), we see that $(\phi_k)_k$ converges in norm to ϕ. Now

$$\|T + T_k\|_\epsilon \geqslant (T + T_k)\Big(\sum_{j=0}^{n} \phi^j\Big) = \|T\|_\epsilon + T_k\Big(\sum_{j=0}^{n} \phi^j\Big),$$

so that $\|T+T_k\|_\epsilon - \|T\|_\epsilon - T_k\big(\sum_{j=0}^{n} \phi^j\big) \geqslant 0$. This allows us to remove the absolute value signs in (2.2), which together with (2.3) yields

$$\begin{aligned} \epsilon \|T_k\| &\leqslant \|T + T_k\|_\epsilon - \|T\|_\epsilon - T_k\Big(\sum_{j=0}^{n} \phi^j\Big) \\ &< (T + T_k)\Big(\sum_{j=0}^{n} \phi_k^j\Big) + \frac{1}{k}\|T_k\|_\epsilon - \|T\|_\epsilon - T_k\Big(\sum_{j=0}^{n} \phi^j\Big) \\ &\leqslant T_k\Big(\sum_{j=0}^{n} \phi_k^j - \sum_{j=0}^{n} \phi^j\Big) + \frac{1}{k}\, \|T_k\|_\epsilon \leqslant \|T_k\|_\epsilon \Big(\Big\| \sum_{j=0}^{n} \phi_k^j - \sum_{j=0}^{n} \phi^j \Big\|_I + \frac{1}{k} \Big) \end{aligned}$$

for sufficiently large k. Thus we will have our contradiction if we can show that $\sum_{j=0}^{n} \phi_k^j$ converges in the integral norm to $\sum_{j=0}^{n} \phi^j$. Since we have assumed that ϕ converges in norm to ϕ_k and since ϕ and all the ϕ_k have norm less than or equal to one, it will be sufficient to show that the map

$$\begin{aligned} i \colon \overline{B}_{E'} &\to \mathscr{P}_I({}^{\leqslant n}E) \\ \phi &\mapsto \sum\nolimits_{j=0}^{n} \phi^j \end{aligned}$$

is continuous. We will first show that for $j = 0, 1, \dots, n$, the map

$$\begin{aligned} i_j\colon \overline{B}_{E'} &\to \mathscr{P}_I({}^j E) \\ \phi &\mapsto \phi^j \end{aligned}$$

is continuous. Note that if $j = 0$ then we just have the constant mapping $\phi \mapsto 1$, so we may assume that $j \neq 0$. Also, if $j \neq 0$ and $\phi \neq 0$, we can represent ϕ^j using a mass of $\|\phi\|^j$ at $\phi/\|\phi\|$. Thus it follows that $\|\phi^j\|_I = \|\phi\|^j$ for all $\phi \in E'$. Now let $\phi, \psi \in \overline{B}_{E'}$. Then using Example 2.3(c) of [8] and the above comment, it follows that for $j = 1, 2, 3, \dots, n$, we have

$$\begin{aligned} \|\phi^j - \psi^j\|_I = \Big\|(\phi - \psi)\Big(\sum_{k=0}^{j-1} \phi^{j-k-1}\psi^k\Big)\Big\|_I &\leqslant e^j \,\|\phi - \psi\|_I \,\Big\|\sum_{k=0}^{j-1} \phi^{j-k-1}\psi^k\Big\|_I \\ &\leqslant e^j \,\|\phi - \psi\| \sum_{k=0}^{j-1} e^{j-1} \,\|\phi^{j-k-1}\|_I \,\|\psi^k\|_I \\ &= e^j \,\|\phi - \psi\| \sum_{k=0}^{j-1} e^{j-1} \,\|\phi\|^{j-k-1} \,\|\psi\|^k \leqslant j e^{2j-1} \,\|\phi - \psi\|. \end{aligned}$$

Hence i_j is continuous for $j = 0, 1, \dots, n$. However $\mathscr{P}_I({}^{\leqslant n}E)$ is a topological direct sum of $\mathscr{P}_I({}^j E)$, $j = 0, 1, 2, \dots, n$ (see Corollary 3.7 of [4]) so that the theorem now follows since i may be regarded as a sum of the i_j. □

As with Theorem 2.1, we can also express this result in terms of approximable polynomials.

Corollary 2.6. *Let E be a real Banach space, let n be a positive integer, let $P \in S_{\mathscr{P}_A({}^{\leqslant n}E)}$, with $\overline{P}$ its Aron–Berner extension and let $x \in \overline{B}_{E''}$. Then the following are equivalent.*

(a) *The norm of $\mathscr{P}_A({}^{\leqslant n}E)$ is Fréchet differentiable at P with derivative δ_x (respectively $-\delta_x$).*

(b) (i) $\overline{P}(x) = 1$ $\big($*respectively* $\overline{P}(x) = -1\big)$.

  (ii) *There exists a real number α, with $-1 < \alpha < 1$, such that $\overline{P}(x) > \alpha$ $\big($respectively $\overline{P}(x) < \alpha\big)$ for all $x \in \overline{B}_{E''}$.*

  (iii) *If $(x_k)_k$ is a sequence in the closed unit ball of E'' such that the sequence $\big(\overline{P}(x_k)\big)_k$ converges to 1 $\big($respectively $\big(\overline{P}(x_k)\big)_k$ converges to $-1\big)$ then $(x_k)_k$ converges in norm to x.*

(c) *The closed unit ball of $\mathscr{P}_I({}^{\leqslant n}E')$ is weak*-strongly exposed by P at δ_x (respectively $-\delta_x$).*

2.3. Fréchet differentiability of the norm of $\mathscr{P}(^{\leqslant n}E)$

In this section we use the isometric isomorphism between $\big(\bigoplus_{j=0}^{n}\widehat{\bigotimes}_{j,s}E, \|\cdot\|_\pi\big)'$ and $\mathscr{P}(^{\leqslant n}E)$ in Corollary 2.8 of [4] to give a characterisation of the Fréchet differentiability of the norm of $\mathscr{P}(^{\leqslant n}E)$. Before we proceed with the statement of the theorem we will fix some notation. If $\sum_{j=0}^{n} x\otimes \overset{j\text{ terms}}{\cdots} \otimes x \in \big(\bigoplus_{j=0}^{n}\widehat{\bigotimes}_{j,s}E, \|\cdot\|_\pi\big)$, $P=\sum_{j=0}^{n} P_j \in \mathscr{P}(^{\leqslant n}E)$ with each $P_j\in\mathscr{P}(^{j}E)$ and $\delta_x\in\mathscr{P}(^{\leqslant n}E)'$ denotes the evaluation functional at x, then using Equation 4 of [4],

$$P\Big(\sum_{j=0}^{n}\underbrace{x\otimes\cdots\otimes x}_{j\text{ terms}}\Big)=\sum_{j=0}^{n}P_j(x)=P(x)=\delta_x(P).$$

In the following theorem δ_x will stand for not only the element in $\mathscr{P}(^{\leqslant n}E)'$ but also the element $\sum_{j=0}^{n} x\otimes \overset{j\text{ terms}}{\cdots} \otimes x \in \big(\bigoplus_{j=0}^{n}\widehat{\bigotimes}_{j,s}E, \|\cdot\|_\pi\big)$. Thus, we will use the duality bracket $\langle P,\delta_x\rangle$ to stand for $P\big(\sum_{j=0}^{n} x\otimes \overset{j\text{ terms}}{\cdots} \otimes x\big)$ as well as for $\delta_x(P)$. Now for the theorem, which takes a somewhat similar form to Theorem 2.5.

Theorem 2.7. *Let E be a real Banach space and let n be a positive integer. If $P\in S_{\mathscr{P}(^{\leqslant n}E)}$ and $x\in\overline{B}_E$, then the following are equivalent.*

(a) *The norm of $\mathscr{P}(^{\leqslant n}E)$ is Fréchet differentiable at P with derivative δ_x (respectively $-\delta_x$).*

(b) (i) *$P(x)=1$ (respectively $P(x)=-1$).*

  (ii) *There exists a real number α with $-1<\alpha<1$ such that $P(y)>\alpha$ (respectively $P(y)<\alpha$) for all $y\in\overline{B}_E$.*

  (iii) *If $(x_k)_k$ is a sequence in the closed unit ball of E such that $(P(x_k))_k$ converges to 1 (respectively converges to -1) then the sequence $(x_k)_k$ converges in norm to x.*

(c) *The closed unit ball of $\big(\bigoplus_{j=0}^{n}\widehat{\bigotimes}_{j,s}E, \|\cdot\|_\pi\big)$ is strongly exposed by P at δ_x (respectively $-\delta_x$).*

Proof. As usual, we will prove the case where the derivative is δ_x, the other case being similar.

(a) ⇒ (c). If the norm of $\mathscr{P}(^{\leqslant n}E)$ is Fréchet differentiable at P with derivative δ_x, then it follows from a theorem of Šmul'yan (see [18]) that P weak*-strongly exposes δ_x regarded as an element of $\overline{B}_{\mathscr{P}(^{\leqslant n}E)'}$. Thus $\overline{B}_{(\bigoplus_{j=0}^{n}\widehat{\bigotimes}_{j,s}E,\|\cdot\|_\pi)}$ is strongly exposed by P at δ_x.

(c) ⇒ (b)(i). Since P has norm 1, this follows from the definition of strongly exposed point.

(c) ⇒ (b)(ii). We will assume that (b)(ii) does not hold and show that this implies that (c) does not hold. If (b)(ii) does not hold then we can find a sequence

$(x_k)_k \subset \overline{B}_E$ so that the sequence $(P(x_k))_k$ converges to -1. Since $P(x_k) = \langle P, \delta_{x_k}\rangle$ this means that the sequence $\big(\langle P, -\delta_{x_k}\rangle\big)_k$ converges to $\langle P, \delta_x\rangle = P(x) = 1$. Hence, since δ_x is a strongly exposed point of $\overline{B}_{(\bigoplus_{j=0}^n \widehat{\bigotimes}_{j,s} E, \|\cdot\|_\pi)}$, this means that the sequence $\big(-\delta_{x_k}\big)_k$ converges to δ_x in $\big(\bigoplus_{j=0}^n \widehat{\bigotimes}_{j,s} E, \|\cdot\|_\pi\big)$. That is,

$$\lim_{k\to\infty} \Big\| \sum_{j=0}^n \underbrace{x\otimes\cdots\otimes x}_{j \text{ terms}} - \Big(-\sum_{j=0}^n \underbrace{x_k\otimes\cdots\otimes x_k}_{j \text{ terms}}\Big)\Big\|_\pi = 0.$$

However the constant term in $\sum_{j=0}^n x\otimes \overset{j \text{ terms}}{\cdots} \otimes x - \big(-\sum_{j=0}^n x_k\otimes \overset{j \text{ terms}}{\cdots} \otimes x_k\big)$ is 2 for each x_k, and this is impossible, since it follows directly from the definition of the π-norm that if $\theta = \sum_{j=0}^n \theta_j \in \big(\bigoplus_{j=0}^n \widehat{\bigotimes}_{j,s} E, \|\cdot\|_\pi\big)$, then $|\theta_0| \geqslant \|\theta\|_\pi$. Hence (c) cannot hold.

(c) $\Rightarrow$ (b)(iii). If $(x_k)_k$ is a sequence in the closed unit ball of E such that the sequence $(P(x_k))_k$ converges to 1 then we have that the sequence $\big(\langle P, \delta_{x_k}\rangle\big)_k$ converges to $\langle P, \delta_x\rangle$. Since δ_x is a strongly exposed point of $\overline{B}_{(\bigoplus_{j=0}^n \widehat{\bigotimes}_{j,s} E, \|\cdot\|_\pi)}$, this means that the sequence $\big(\delta_{x_k}\big)_k$ converges to δ_x in $\big(\bigoplus_{j=0}^n \widehat{\bigotimes}_{j,s} E, \|\cdot\|_\pi\big)$. That is

$$\lim_{k\to\infty} \Big\| \sum_{j=0}^n \underbrace{x_k\otimes\cdots\otimes x_k}_{j \text{ terms}} - \sum_{j=0}^n \underbrace{x\otimes\cdots\otimes x}_{j \text{ terms}}\Big\|_\pi = 0.$$

Now let $\epsilon > 0$. By the definition of the π-norm, for each $k \in \mathbb{N}$ we can find elements $y_{k,1}, y_{k,2}, \ldots, y_{k,m_k}$ in $\overline{B}_E$ and $\lambda_{k,1}, \lambda_{k,2}, \ldots, \lambda_{k,m_k}$ in $\mathbb{R}$ such that

$$\sum_{j=0}^n \underbrace{x_k\otimes\cdots\otimes x_k}_{j \text{ terms}} - \sum_{j=0}^n \underbrace{x\otimes\cdots\otimes x}_{j \text{ terms}} = \sum_{l=1}^{m_k} \lambda_{k,l} \sum_{j=0}^n \underbrace{y_{k,l}\otimes\cdots\otimes y_{k,l}}_{j \text{ terms}} \tag{2.4}$$

and such that

$$\sum_{l=1}^{m_k} |\lambda_{k,l}| < \Big\| \sum_{j=0}^n \underbrace{x_k\otimes\cdots\otimes x_k}_{j \text{ terms}} - \sum_{j=0}^n \underbrace{x\otimes\cdots\otimes x}_{j \text{ terms}}\Big\|_\pi + \frac{\epsilon}{2}.$$

However we can also find a $K \in \mathbb{N}$ such that for all $k \geqslant K$

$$\Big\| \sum_{j=0}^n \underbrace{x_k\otimes\cdots\otimes x_k}_{j \text{ terms}} - \sum_{j=0}^n \underbrace{x\otimes\cdots\otimes x}_{j \text{ terms}}\Big\|_\pi < \frac{\epsilon}{2}.$$

Hence for all $k \geqslant K$, $\sum_{l=1}^{m_k} |\lambda_{k,l}| < \epsilon$. However, equating the terms in E (that is $\bigotimes_{1,s} E$) in (2.4), we see that $x_k - x = \sum_{l=1}^{m_k} \lambda_{k,l} y_{k,l}$. Thus, since all the $y_{k,l}$ have norm less than or equal to 1, we have

$$\|x_k - x\| = \Big\| \sum_{l=1}^{m_k} \lambda_{k,l} y_{k,l}\Big\| \leqslant \sum_{l=1}^{m_k} |\lambda_{k,l}|\cdot\|y_{k,l}\| \leqslant \sum_{l=1}^{m_k} |\lambda_{k,l}| < \epsilon \quad \text{for all } k \geqslant K.$$

Hence the sequence $(x_k)_k$ converges in norm to x.

(b) $\Rightarrow$ (a). We will assume that (b) holds and that (a) does not. If (a) does not hold then we can find an $\epsilon > 0$ and a sequence $(P_k)_k \subset \mathscr{P}(^{\leqslant n}E)$ converging to 0 such that for all $k \in \mathbb{N}$, $\big|\|P + P_k\| - \|P\| - \langle P_k, \delta_x\rangle\big| \geqslant \epsilon\|P_k\|$. That is

$$\big|\|P + P_k\| - \|P\| - P_k(x)\big| \geqslant \epsilon\|P_k\|. \tag{2.5}$$

Now for each $k \in \mathbb{N}$, we choose $x_k \in S_E$ and $t_k = \pm 1$ so that

$$t_k(P + P_k)(x_k) > \|P + P_k\| - \frac{1}{k}\|P_k\|. \tag{2.6}$$

Then

$$1 = \|P\| \geqslant t_k P(x_k) > \|P + P_k\| - \frac{1}{k}\|P_k\| - \|P_k\| \to \|P\| = 1 \quad \text{as } k \to \infty.$$

Thus $t_k P(x_k) \to 1$ and so using (b)(ii) we see that we must have $t_k = 1$ for all sufficiently large k. Then using (b)(iii), it follows that $(x_k)_k$ converges to x in norm. We now note that $\|P + P_k\| \geqslant (P + P_k)(x) = P(x) + P_k(x) = \|P\| + P_k(x)$ so that $\|P + P_k\| - \|P\| - P_k(x) \geqslant 0$. Hence using (2.5) and (2.6),

$$\epsilon\|P_k\| \leqslant \|P + P_k\| - \|P\| - P_k(x) < (P + P_k)(x_k) + \frac{1}{k}\|P_k\| - \|P\| - P_k(x)$$
$$\leqslant P_k(x_k) - P_k(x) + \frac{1}{k}\|P_k\| = \langle P_k, \delta_{x_k} - \delta_x\rangle + \frac{1}{k}\|P_k\| \leqslant \|P_k\| \left(\left\|\delta_{x_k} - \delta_x\right\|_\pi + \frac{1}{k} \right)$$

for sufficiently large k. Now if we denote by δ_x^j the map

$$\begin{aligned} \mathscr{P}(^jE) &\to \mathbb{K} \\ P &\mapsto P(x) \end{aligned}$$

then the map

$$\begin{aligned} E &\to \mathscr{P}(^jE)' \\ x &\mapsto \delta_x^j \end{aligned}$$

is continuous. Since we may regard δ_x as a sum of the δ_x^j, then we also have that the map

$$\begin{aligned} E &\to \mathscr{P}(^{\leqslant n}E)' \\ x &\mapsto \delta_x \end{aligned}$$

is continuous. Since $(x_k)_k$ converges in norm to x, it follows that $\left\|\delta_{x_k} - \delta_x\right\|_\pi \to 0$ as $k \to \infty$. Thus we have a contradiction and hence (b) implies (a). □

3. Extremal structure of spaces of real polynomials and their preduals

In this section we will prove results concerning the extremal structure of spaces of polynomials and their preduals and compare and contrast these results with the corresponding results for homogeneous spaces. It will become apparent below that there are substantial differences between them.

For our first theorem in this section we will employ Theorem 2.1. In Proposition 5.1(a) of [4], we obtained an upper bound on the set of extreme points of the closed unit ball of $\mathscr{P}_I({}^{\leqslant n}E)$. The following result is in the opposite direction: we obtain a lower bound on the set of weak*-exposed points of $\overline{B}_{\mathscr{P}_I({}^{\leqslant n}E)}$ for a certain class of Banach spaces. Note that it may be regarded as the non-homogeneous version of Theorem 8 of [7].

Theorem 3.1. *Let n be an integer greater than or equal to 2 and let E be a real separable Banach space. Then the set of weak*-exposed points of the closed unit ball of $\mathscr{P}_I({}^{\leqslant n}E)$ contains the set*

$$\Big\{\pm\sum_{j=0}^{n}\phi^j : \phi\in\overline{B}_{E'} \text{ and } \phi \text{ attains its norm}\Big\}$$

(where we include the case $\phi = 0$ in which case $\pm\sum_{j=0}^{n}\phi^j = \pm 1$).

Proof. We first note that if $T \in \big(\bigoplus_{j=0}^{n}\widehat{\bigotimes}_{j,s}E, \|\cdot\|_\epsilon\big)$ exposes $\sum_{j=0}^{n}\phi^j$, then $-T$ exposes $-\sum_{j=0}^{n}\phi^j$, so we only have to show that $\sum_{j=0}^{n}\phi^j$ is exposed by such a T. Suppose that ϕ is a norm attaining vector in $\overline{B}_{E'}$ and let $x \in S_E$ be such that $\phi(x) = \|\phi\|$. We will first construct a tensor $T_2 \in \widehat{\bigotimes}_{2,s,\epsilon}E$ with norm at most $1/2$ such that

$$0 < T_2\big(\psi^2\big) \leqslant \frac{1}{2} \quad \text{for all } \psi \in \overline{B}_{E'}\backslash \mathrm{sp}\{\phi\}$$

and such that $T_2\big(\phi^2\big) = 0$. Since E is separable, we can choose a sequence $(x_k)_k \subset S_E$ with the property that if $\phi \in E'$ then $\phi = 0$ if and only if $\phi(x_k) = 0$ for all k. When $\phi = 0$ the construction is slightly different so we will dispose of this case first. Let

$$T_2 = C\sum_{k=1}^{\infty}\frac{1}{k^2}\,x_k\otimes x_k,$$

where C is positive and is chosen so that T_2 has norm at most one half. Then, since $\phi = 0$,

$$T_2(\phi^2) = C\sum_{k=1}^{\infty}\frac{1}{k^2}\,\phi(x_k)^2 = 0.$$

Also, since T_2 has norm at most one half,

$$|T_2(\psi^2)| \leqslant \frac{1}{2} \quad \text{for all } \psi \in \overline{B}_{E'}.$$

Finally, if $\psi \in \overline{B}_{E'}\backslash\{0\}$ then there exists at least one k such that $\psi(x_k) \neq 0$. Hence

$$T_2(\psi^2) = C\sum_{k=1}^{\infty}\frac{1}{k^2}\,\psi(x_k)^2 > 0 \quad \text{for all } \psi \in \overline{B}_{E'}\backslash\{0\},$$

and we have constructed our T_2 if $\phi = 0$.

Let us now consider the case $\phi \neq 0$. If $y \in E$ then we can write y as

$$y = \frac{\phi(y)}{\|\phi\|}x + y - \frac{\phi(y)}{\|\phi\|}x, \quad \text{with } \frac{\phi(y)}{\|\phi\|}x \in \mathrm{sp}\{x\} \text{ and } y - \frac{\phi(y)}{\|\phi\|}x \in \ker\phi.$$

(Note $y - \frac{\phi(y)}{\|\phi\|}x \in \ker\phi$ since $\phi\big(y - \frac{\phi(y)}{\|\phi\|}x\big) = \phi(y) - \frac{\phi(y)}{\|\phi\|}\phi(x) = 0$). Thus E is the topological direct sum of $\mathrm{sp}\{x\}$ and $\ker\phi$. Hence

$$E' \cong (\mathrm{sp}\{x\} \oplus \ker\phi)' \cong \mathrm{sp}\{x\}^{\perp} \oplus (\ker\phi)^{\perp} \cong (\ker\phi)' \oplus \mathrm{sp}\{\phi\},$$

where we write $\psi \in E'$ as

$$\psi = \psi_1 + \psi_2 = \psi - \frac{\psi(x)}{\|\phi\|}\phi + \frac{\psi(x)}{\|\phi\|}\phi,$$

with $\psi_1 = \psi - \frac{\psi(x)}{\|\phi\|}\phi \in (\ker\phi)'$ and $\psi_2 = \frac{\psi(x)}{\|\phi\|}\phi \in \mathrm{sp}\{\phi\}$. Note that ψ_1 does lie in $(\ker\phi)' \cong \mathrm{sp}\{x\}^{\perp}$ since $\psi_1(\lambda x) = \psi(\lambda x) - \frac{\psi(x)}{\|\phi\|}\phi(\lambda x) = \lambda\big(\psi(x) - \frac{\psi(x)}{\|\phi\|}\phi(x)\big) = 0$ for all $\lambda \in \mathbb{R}$. Now consider the sequence $(y_k)_k$, where $y_k = x_k - \frac{\phi(x_k)}{\|\phi\|}x$. First note that since

$$\phi(y_k) = \phi\Big(x_k - \frac{\phi(x_k)}{\|\phi\|}x\Big) = \phi(x_k) - \frac{\phi(x_k)}{\|\phi\|}\phi(x) = 0,$$

we have that $(y_k)_k \subset \ker\phi$. Next, we show that if $\psi \in \overline{B}_{E'}\backslash\mathrm{sp}\{\phi\}$ then we must have $\psi(y_k) \neq 0$ for at least one k. To see this suppose that we do have $\psi(y_k) = 0$ for all $k \in \mathbb{N}$. Since $\psi_2 \in \mathrm{sp}\{\phi\}$ and $(y_k)_k \subset \ker\phi$ we have $\psi_2(y_k) = 0$ for all $k \in \mathbb{N}$. But $\psi = \psi_1 + \psi_2$ so it follows that we also have $\psi_1(y_k) = 0$ for all $k \in \mathbb{N}$. Now

$$\begin{aligned}
\psi_1(y_k) = 0 \quad \text{for all } k \quad &\Rightarrow \psi_1\Big(x_k - \frac{\phi(x_k)}{\|\phi\|}x\Big) = 0 \quad \text{for all } k \\
&\Rightarrow \psi_1(x_k) - \frac{\phi(x_k)}{\|\phi\|}\psi_1(x) = 0 \quad \text{for all } k \\
&\Rightarrow \psi_1(x_k) = 0 \quad \text{for all } k \text{ (since } \psi_1 \in \mathrm{sp}\{x\}^{\perp}) \\
&\Rightarrow \psi_1 = 0.
\end{aligned}$$

But this means that $\psi \in \mathrm{sp}\{\phi\}$ contrary to our assumption. Hence if $\psi \in \overline{B}_{E'}\backslash\mathrm{sp}\{\phi\}$ then we must have $\psi(y_k) \neq 0$ for at least one k. We now define our tensor T_2 by

$$T_2 = C\sum_{k=1}^{\infty} \frac{1}{k^2}\, y_k \otimes y_k,$$

where C is positive and is chosen so that T_2 has norm at most one half. Then, since $y_k \in \ker\phi$ for each k,

$$T_2(\phi^2) = C\sum_{k=1}^{\infty} \frac{1}{k^2}\,\phi(y_k)^2 = 0.$$

Also, since T_2 has norm at most one half,

$$|T_2(\psi^2)| \leqslant \frac{1}{2} \quad \text{for all } \psi \in \overline{B}_{E'}.$$

Finally, since $\psi(y_k) \neq 0$ for at least one k for all $\psi \in \overline{B}_{E'}\backslash \mathrm{sp}\{\phi\}$, we have that

$$T_2(\psi^2) = C \sum_{k=1}^{\infty} \frac{1}{k^2}\, \psi(y_k)^2 > 0 \quad \text{for all } \psi \in \overline{B}_{E'}\backslash \mathrm{sp}\{\phi\}$$

and we have constructed our T_2 in the case $\phi \neq 0$.

Now let $T \in \big(\bigoplus_{j=0}^{n} \widehat{\bigotimes}_{j,s} E, \|\cdot\|_\epsilon\big)$ be defined by

$$T = 1 - \frac{\|\phi\|^2}{8} + \frac{\|\phi\|}{4} x - \frac{1}{8} x \otimes x - T_2$$

for both $\phi = 0$ and $\phi \neq 0$. We will show that T has the following properties:

- $T\big(\sum_{j=0}^{n} \phi^j\big) = 1$,
- $T\big(\sum_{j=0}^{n} \psi^j\big) < 1$ for all $\psi \in \overline{B}_{E'}\backslash\{\phi\}$,
- $T\big(\sum_{j=0}^{n} \psi^j\big) > -1/2$ for all $\psi \in \overline{B}_{E'}$,
- $\|T\|_\epsilon = 1$.

Firstly,

$$\begin{aligned} T\Big(\sum_{j=0}^{n} \phi^j\Big) &= 1 - \frac{\|\phi\|^2}{8} + \frac{\|\phi\|}{4}\phi(x) - \frac{1}{8}\,\phi(x)^2 - T_2(\phi^2) \\ &= 1 - \frac{\|\phi\|^2}{8} + \frac{\|\phi\|^2}{4} - \frac{\|\phi\|^2}{8} - 0 = 1. \end{aligned}$$

Also, if $\psi \in \overline{B}_{E'}$ then

$$\begin{aligned} T\Big(\sum_{j=0}^{n} \psi^j\Big) &= 1 - \frac{\|\phi\|^2}{8} + \frac{\|\phi\|}{4}\psi(x) - \frac{1}{8}\psi(x)^2 - T_2(\psi^2) \\ &= 1 - \frac{1}{8}(\psi(x) - \|\phi\|)^2 - T_2(\psi^2). \end{aligned} \tag{3.1}$$

Now if $\psi \in \mathrm{sp}\{\phi\}\backslash\{\phi\}$ then $\big|\psi(x) - \|\phi\|\big| > 0$ and $T_2(\psi^2) = 0$, and if $\psi \in \overline{B}_{E'}\backslash \mathrm{sp}\{\phi\}$ then $T_2(\psi^2) > 0$ (if $\phi = 0$ then $T_2(\psi^2) > 0$ for all $\psi \in \overline{B}_{E'}\backslash\{0\}$). Thus

$$T\Big(\sum_{j=0}^{n} \psi^j\Big) < 1 \quad \text{for all } \psi \in \overline{B}_{E'}\backslash\{\phi\}.$$

Additionally, since $\big|\psi(x) - \|\phi\|\big| \leqslant 2$ for all $\psi \in \overline{B}_{E'}$ and since T_2 has norm less than or equal to one half it follows from (3.1) that

$$T\Big(\sum_{j=0}^{n} \psi^j\Big) > -\frac{1}{2} \quad \text{for all } \psi \in \overline{B}_{E'}.$$

Finally, using (1.1) and (3.1),

$$\|T\|_\epsilon = \sup_{\psi\in\overline{B}_{E'}} \Big|1 - \frac{\|\phi\|^2}{8} + \frac{\|\phi\|}{4}\psi(x) - \frac{1}{8}\psi(x)^2 - T_2(\psi^2)\Big| = \sup_{\psi\in\overline{B}_{E'}} \Big|T\Big(\sum_{j=0}^{n}\psi^j\Big)\Big|.$$

Using the three items that we have already proved, it follows that $\|T\|_\epsilon = 1$. We have now shown that conditions (b)(i) and (b)(ii) of Theorem 2.1 hold. Hence, using this theorem, it follows that the norm of $\big(\bigoplus_{j=0}^{n} \widehat{\bigotimes}_{j,s} E, \|\cdot\|_\epsilon\big)$ is Gâteaux differentiable at T with differential $\sum_{j=0}^{n}\phi^j$. Then, by a theorem of Šmul'yan (see [17]), it follows that T weak*-exposes the closed unit ball of $\mathscr{P}_I(^{\leqslant n}E)'$ at $\sum_{j=0}^{n}\phi^j$ and so $\sum_{j=0}^{n}\phi^j$ is a weak*-exposed point of $\mathscr{P}_I(^{\leqslant n}E)$. □

It is worth noting that in this theorem the condition $n \geqslant 2$ appears; this will be a necessary condition for the rest of this paper. Although Theorem 3.1 is trivially true if $n = 0$ (it just says that the weak*-exposed points of $\overline{B}_{\mathbb{R}}$ are ± 1), it is not in general true when $n = 1$, as we will now show by considering the case $E = \mathbb{R}$. Firstly, since $\mathscr{P}_I(^{\leqslant n}\mathbb{R})$ is finite dimensional, it is reflexive and so it follows from Corollary 4.6 and Theorem 5.2 of [4] that

$$\mathscr{P}(^{\leqslant n}\mathbb{R})' \overset{1}{\cong} \mathscr{P}(^{\leqslant n}\mathbb{R}')' \overset{1}{\cong} \mathscr{P}_N(^{\leqslant n}\mathbb{R})'' \overset{1}{\cong} \mathscr{P}_I(^{\leqslant n}\mathbb{R})'' \overset{1}{\cong} \mathscr{P}_I(^{\leqslant n}\mathbb{R}). \tag{3.2}$$

Now consider a polynomial $a_0 + a_1x \in \mathscr{P}(^{\leqslant 1}\mathbb{R})$. We have $\|a_0 + a_1x\| = |a_0| + |a_1|$, so that $\mathscr{P}(^{\leqslant 1}\mathbb{R}) \overset{1}{\cong} \mathbb{R} \oplus_1 \mathbb{R}$. Hence $\mathscr{P}_I(^{\leqslant 1}\mathbb{R}) \overset{1}{\cong} \mathscr{P}(^{\leqslant 1}\mathbb{R})' \overset{1}{\cong} (\mathbb{R} \oplus_1 \mathbb{R})' \overset{1}{\cong} \mathbb{R} \oplus_\infty \mathbb{R}$. Thus the only weak*-exposed points of $\overline{B}_{\mathscr{P}_I(^{\leqslant 1}\mathbb{R})}$ are the four polynomials $\pm 1 \pm x$. If Theorem 3.1 were true in this case, then we would have

$$\Big\{\pm\sum_{j=0}^{1}\phi^j : \phi \in \overline{B}_{\mathbb{R}'} \text{ and } \phi \text{ attains its norm}\Big\} \subseteq \mathcal{E}\mathrm{xp}_{w*}\big(\overline{B}_{\mathscr{P}(^{\leqslant 1}\mathbb{R})}\big). \tag{3.3}$$

Since all $\phi \in \overline{B}_{\mathbb{R}'} \overset{1}{\cong} \overline{B}_{\mathbb{R}}$ attain their norm, when we put (3.3) into the language of polynomials of degree 1 on $\mathbb{R}$ we obtain

$$\big\{\pm(1 + \lambda x) \colon \lambda \in [-1, 1]\big\} \subseteq \mathcal{E}\mathrm{xp}_{w*}\big(\overline{B}_{\mathscr{P}(^{\leqslant 1}\mathbb{R})}\big),$$

giving more weak*-exposed points than there actually are.

In our discussion in Remark 2.4, we indicated points where the norm of $\mathscr{P}_A(^nE)$ is Gâteaux differentiable and also that these are not points of Gâteaux smoothness if we regard them as elements of $\mathscr{P}_A(^{\leqslant n}E)$. Before we continue with our study of extremal structure, we note that in Theorem 3.1 we have constructed points where the norm of $\big(\bigoplus_{j=0}^{n} \widehat{\bigotimes}_{j,s} E, \|\cdot\|_\epsilon\big)$ (and hence also the norm of $\mathscr{P}_A(^{\leqslant n}E)$) is Gâteaux differentiable.

Corollary 3.2. *Let E be a real Banach space and let n be an integer greater than or equal to 2. If $x \in S_E$ and $(x_k)_k \subset S_E$ is a sequence with the property* *that if $\phi \in E'$ then $\phi = 0$ if and only if $\phi(x_k) = 0$ for all k, then the norm of $\left(\bigoplus_{j=0}^n \widehat{\bigotimes}_{j,s} E, \|\cdot\|_\epsilon\right)$ is Gâteaux differentiable at the points*

$$1 - \frac{1}{8} x \otimes x - C \sum_{k=1}^{\infty} \frac{1}{k^2} x_k \otimes x_k$$

and

$$1 - \frac{\|\psi\|^2}{8} + \frac{\|\psi\|}{4} x - \frac{1}{8} x \otimes x - D \sum_{k=1}^{\infty} \frac{1}{k^2} \left(x_k - \frac{\psi(x_k)}{\|\psi\|} x\right) \otimes \left(x_k - \frac{\psi(x_k)}{\|\psi\|} x\right),$$

where $\psi \in \overline{B}_{E'} \backslash \{0\}$ is any norm attaining vector which attains its norm at x, and C and D are any positive constants such that

$$\left\|C \sum_{k=1}^{\infty} \frac{1}{k^2} x_k \otimes x_k\right\| \leqslant \frac{1}{2} \quad \text{and} \quad \left\|D \sum_{k=1}^{\infty} \frac{1}{k^2} \left(x_k - \frac{\psi(x_k)}{\|\psi\|} x\right) \otimes \left(x_k - \frac{\psi(x_k)}{\|\psi\|} x\right)\right\| \leqslant \frac{1}{2}.$$

Using Equation 8 of [4] we can also express this in the language of approximable polynomials as follows.

Corollary 3.3. *Let E be a real Banach space and let n be an integer greater than or equal to 2. If $\phi \in S_{E'}$ and $(\phi_k)_k \subset S_{E'}$ is a sequence with the property that if $x \in E''$ then $x = 0$ if and only if $x(\phi_k) = 0$ for all k, then the norm of $\mathscr{P}_A(^{\leqslant n} E)$ is Gâteaux differentiable at the points*

$$1 - \frac{1}{8} \phi^2 - C \sum_{k=1}^{\infty} \frac{1}{k^2} \phi_k^2$$

and

$$1 - \frac{\|y\|^2}{8} + \frac{\|y\|}{4} x - \frac{1}{8} \phi^2 - D \sum_{k=1}^{\infty} \frac{1}{k^2} \left(\phi_k - \frac{y(\phi_k)}{\|y\|} \phi\right)^2,$$

where $y \in \overline{B}_{E''} \backslash \{0\}$ is any norm attaining vector which attains its norm at $\psi \in S_{E'}$, and C and D are any positive constants such that

$$\left\|C \sum_{k=1}^{\infty} \frac{1}{k^2} \phi_k^2\right\| \leqslant \frac{1}{2} \quad \text{and} \quad \left\|D \sum_{k=1}^{\infty} \frac{1}{k^2} \left(\phi_k - \frac{y(\psi_k)}{\|y\|} \psi\right)^2\right\| \leqslant \frac{1}{2}.$$

Remark 3.4. While Corollary 3.2 and Corollary 3.3 detail points of Gâteaux smoothness, we note that we have no characterisation of such points, apart from that given in Theorem 2.1 and Corollary 2.3.

After that slight diversion we will now resume our study of extremal structure. Since by James' theorem every continuous linear functional on a reflexive Banach space attains its norm, and since weak*-exposed points are also both exposed and extreme points, we can combine Proposition 5.1(a) of [4] and Theorem 3.1 to obtain the following result, which is the non-homogeneous version of Corollary 9 in [7].

Corollary 3.5. *Let n be an integer greater than or equal to 2 and let E be a real separable reflexive Banach space. Then the set $\{\pm\sum_{j=0}^n \phi^j : \phi \in \overline{B}_{E'}\}$ is equal to the set of weak*- exposed or exposed or extreme points of the closed unit ball of $\mathscr{P}_I(^{\leqslant n}E)$.*

Our next theorem gives a complete classification of the extreme points of the closed unit ball of $\mathscr{P}_I(^{\leqslant n}E)$. In the proof we will need the following lemma, which is Lemma 4 of [7].

Lemma 3.6. *Let E be a real normed space and let ϕ be a unit vector in E'. Suppose that for every finite dimensional subspace F of E there exists a subspace $\widetilde{F}$ of E containing F with the property that $\|\phi|_{\widetilde{F}}\| = 1$ and such that $\phi|_{\widetilde{F}}$ is an extreme point of the closed unit ball of $(\widetilde{F})'$. Then ϕ is an extreme point of the closed unit ball of E'.*

Now for our result.

Theorem 3.7. *If E is a real Banach space and n is an integer which is greater than or equal to 2 then the set of extreme points of the closed unit ball of $\mathscr{P}_I(^{\leqslant n}E)$ is the set $\{\pm\sum_{j=0}^n \phi^j : \phi \in \overline{B}_{E'}\}$.*

Proof. To prove this proposition we will consider $\mathscr{P}_I(^{\leqslant n}E)$ as

$$\mathscr{P}_I(^{\leqslant n}E) \overset{1}{\cong} \Big(\bigoplus_{j=0}^n \bigotimes_{j,s} E, \|\cdot\|_\epsilon\Big)'.$$

We will show that the conditions of Lemma 3.6 hold with $\big(\bigoplus_{j=0}^n \bigotimes_{j,s} E, \|\cdot\|_\epsilon\big)$ as the normed space and $\pm\sum_{j=0}^n \phi^j$ as the unit vector in $\big(\bigoplus_{j=0}^n \bigotimes_{j,s} E, \|\cdot\|_\epsilon\big)'$. We write $\pm\sum_{j=0}^n \phi^j$ to denote either $\sum_{j=0}^n \phi^j$ or $-\sum_{j=0}^n \phi^j$. Let X be a finite dimensional subspace of $\big(\bigoplus_{j=0}^n \bigotimes_{j,s} E, \|\cdot\|_\epsilon\big)$. Then we can find a finite dimensional subspace F of E such that X is a subspace of $\big(\bigoplus_{j=0}^n \bigotimes_{j,s} F, \|\cdot\|_\epsilon\big)$. Now if $\phi \in \overline{B}_{E'}$ then we clearly also have $\phi\big|_F \in \overline{B}_{F'}$. The space $\big(\bigoplus_{j=0}^n \bigotimes_{j,s} F, \|\cdot\|_\epsilon\big)$ now corresponds to the $\widetilde{F}$ in Lemma 3.6. Since $\big(\bigoplus_{j=0}^n \bigotimes_{j,s} F, \|\cdot\|_\epsilon\big)$ is finite dimensional, it follows from Corollary 3.5 that $\pm\sum_{j=0}^n \phi\big|_F^j$ is an extreme point of $\overline{B}_{\mathscr{P}_I(^{\leqslant n}F)}$. Since $\big\|\pm\sum_{j=0}^n \phi|_F^j\big\|_I = 1$, when regarded as an element of $\mathscr{P}_I(^{\leqslant n}F)$, we have now shown that the conditions of Lemma 3.6 hold. Thus $\pm\sum_{j=0}^n \phi^j$ is an extreme point of the closed unit ball of $\mathscr{P}_I(^{\leqslant n}E)$. Hence

$$\Big\{\pm\sum_{j=0}^n \phi^j : \phi \in \overline{B}_{E'}\Big\} \subseteq \mathcal{E}\mathrm{xt}\,\big(\overline{B}_{\mathscr{P}_I(^{\leqslant n}E)}\big).$$

However, by Proposition 5.1(a) of [4],

$$\mathcal{E}\mathrm{xt}\,\big(\overline{B}_{\mathscr{P}_I(^{\leqslant n}E)}\big) \subseteq \Big\{\pm\sum_{j=0}^n \phi^j : \phi \in \overline{B}_{E'}\Big\}. \qquad \square$$

It is worth noting that this theorem shows that there are more extreme points in the non-homogeneous case than in the homogeneous case. In Theorem 2.1 of [11] it is shown that

$$\mathcal{E}\text{xt}\left(\overline{B}_{\mathscr{P}_I({}^nE)}\right) = \left\{ \pm \phi^n \colon \phi \in S_{E'} \right\},$$

so that in the homogeneous case the ϕ must have norm one to yield an extreme point, while the ϕ in Theorem 3.7 only has to have norm less than or equal to one. Indeed, if we put $\phi = 0$ into this theorem, we see that the constant polynomial $P(x) = 1$ is an extreme point of the closed unit ball of $\mathscr{P}_I({}^{\leqslant n}E)$.

A bounded sequence $(P_k)_k$ in $\mathscr{P}({}^{\leqslant n}E)$ is said to converge Aron–Berner pointwise to P in $\mathscr{P}({}^{\leqslant n}E)$ if $\overline{P}_k(z)$ converges to $\overline{P}(z)$ for every z in E''. The following corollary shows that weak and Aron–Berner sequential convergence for $\mathscr{P}_A({}^{\leqslant n}E)$ coincide.

Corollary 3.8. *Let E be a Banach space and n be a positive integer.*

(a) *A bounded sequence $(P_k)_k$ in $\mathscr{P}_A({}^{\leqslant n}E)$ converges weakly to P in $\mathscr{P}_A({}^{\leqslant n}E)$ if and only if $(P_k)_k$ converges Aron–Berner pointwise to P.*

(b) *A bounded subset of $\mathscr{P}_A({}^{\leqslant n}E)$ is weakly relatively compact if and only if it is relatively countably compact for the Aron–Berner pointwise topology.*

Proof. We showed in Theorem 3.7 that the set of extreme points of the unit ball of $\mathscr{P}_A({}^{\leqslant n}E)' \overset{1}{\cong} \mathscr{P}_I({}^{\leqslant n}E')$ is $\{\pm\delta_z : z \in E'', \|z\| \leqslant 1\}$. Part (a) now follows from a result of Rainwater, [14], while part (b) is a consequence of a theorem of Bourgain and Talagrand, [3]. □

Example 3.9. If we specialize to the case $E = \mathbb{R}$ then, since the closed unit ball of the dual of $\mathbb{R}$ is the set of linear functions $\{x \mapsto \alpha x \colon \alpha \in \mathbb{R}, |\alpha| \leqslant 1\}$, we see that

$$\mathcal{E}\text{xt}\left(\overline{B}_{\mathscr{P}_I({}^{\leqslant n}\mathbb{R})}\right) = \left\{ \pm \sum_{j=0}^{n} \phi^j \colon \phi \in \overline{B}_{\mathbb{R}'} \right\} = \left\{ \pm \sum_{j=0}^{n} (\alpha x)^j \colon \alpha \in \mathbb{R}, |\alpha| \leqslant 1 \right\}.$$

If we further specialize to the case $n = 2$, we have

$$\mathcal{E}\text{xt}\left(\overline{B}_{\mathscr{P}_I({}^{\leqslant 2}\mathbb{R})}\right) = \left\{ \pm \left(1 + \alpha x + \alpha^2 x^2\right) \colon \alpha \in \mathbb{R}, |\alpha| \leqslant 1 \right\}.$$

Using Mathematica we can construct the convex hull of this set giving us a visual representation of the closed unit ball of $\mathscr{P}_I({}^{\leqslant 2}\mathbb{R})$. We have included this from two different viewpoints in Figures 1 and 2.

Note that using (3.2), we see that the figures also represent the unit ball of the dual of $\mathscr{P}({}^{\leqslant 2}\mathbb{R})$. We also note that the geometry of $\mathscr{P}({}^{\leqslant 2}\mathbb{R})$ was investigated in [2] and a plot of its unit ball can be found in Figure 2.2 of the aforementioned paper.

Combining Theorem 3.7 with Equation 16 of [4], we also have a lower bound on the set of extreme points of the closed unit ball of the space of nuclear polynomials of degree n.

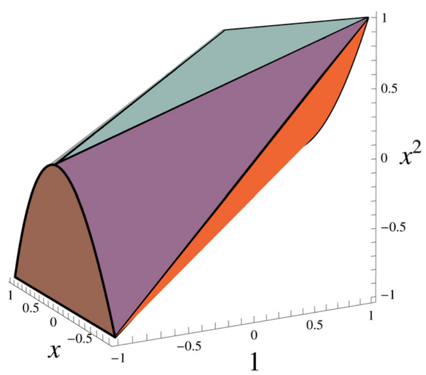

Figure 1. The unit ball of $\mathscr{P}_I({}^{\leqslant 2}\mathbb{R})$.

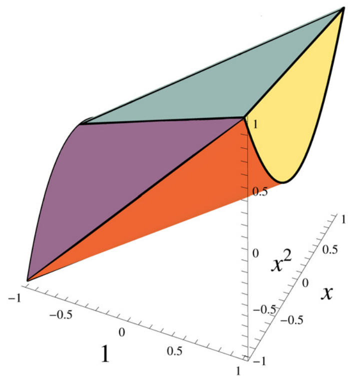

Figure 2. The unit ball of $\mathscr{P}_I({}^{\leqslant 2}\mathbb{R})$.

Corollary 3.10. *If E is a real Banach space and n is an integer which is greater than or equal to 2 then the set $\{\pm\sum_{j=0}^n \phi^j : \phi \in \overline{B}_{E'}\}$ is contained in the set of extreme points of the closed unit ball of $\mathscr{P}_N({}^{\leqslant n}E)$.*

We can also combine Theorem 3.7 with Theorem 5.2 of [4] to obtain the following result.

Corollary 3.11. *If E is a real Banach space such that l_1 does not embed into the space $\left(\bigoplus_{j=0}^n \widehat{\bigotimes}_{j,s} E, \|\cdot\|_\epsilon\right)$, and if n is an integer which is greater than or equal to 2, then the set $\{\pm\sum_{j=0}^n \phi^j : \phi \in \overline{B}_{E'}\}$ is equal to the set of extreme points of the closed unit ball of $\mathscr{P}_N({}^{\leqslant n}E)$.*

Example 3.12. Arguing as in Corollary 5.3 of [4], it follows that Corollary 3.11 applies to all Asplund spaces and also that Example 5.4 of [4], using [6], gives an example of a non-Asplund space where Corollary 3.11 does apply, so that we have

$$\mathcal{E}\mathrm{xt}\left(\overline{B}_{\mathscr{P}_N({}^{\leqslant n}JH)}\right) = \left\{\pm\sum_{j=0}^n \phi^j : \phi \in \overline{B}_{JH'}\right\}.$$

References

[1] Aron, R. M. and Berner, P. D.: A Hahn–Banach extension theorem for analytic mappings. *Bull. Soc. Math. France* **106** (1978), no.1, 3–24.

[2] Aron, R. M. and Klimek, M.: Supremum norms for quadratic polynomials. *Arch. Math. (Basel)* **76** (2001), no. 1, 73–80.

[3] Bourgain, J. and Talagrand, M.: Compacité extrémale. *Proc. Amer. Math. Soc.* **80** (1980), no. 1, 68–70.

[4] Boyd, C. and Brown, A.: Duality in spaces of polynomials of degree at most n. *J. Math. Anal. Appl.* **429** (2015), no. 2, 1271–1290.

[5] Boyd, C. and Brown, A.: Isometries between spaces of real polynomials of degree at most n. *Q. J. Math.* **67** (2016), no. 2, 183–200.

[6] Boyd, C., Costas, P. and Venkova, M.: Injective tensor products of tree spaces. Submitted.

[7] Boyd, C. and Ryan, R.: Geometric theory of spaces of integral polynomials and symmetric tensor products. *J. Funct. Anal.* **179** (2001), no. 1, 18–42.

[8] Carando, D., Dimant, V. and Muro, S.: Holomorphic functions and polynomial ideals on Banach spaces. *Collect. Math.* **63** (2012), no. 1, 71–91.

[9] Choi, Y. S., Hájek, P. and Lee, H. J.: Extensions of smooth mappings into biduals and weak continuity. *Adv. Math.* **234** (2013), 453–487.

[10] Deville, R., Godefroy, G. and Zizler, V.: *Smoothness and renorming in Banach spaces.* Pitman Monographs and Surveys in Pure and Applied Mathematics 64, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, New York, 1993.

[11] Dimant, V., Galicer, D. and García, R.: Geometry of integral polynomials, M-ideals and unique norm preserving extensions. *J. Funct. Anal.* **262** (2012), no. 5, 1987–2012.

[12] Dineen, S.: *Complex analysis on infinite dimensional spaces.* Springer Monographs in Mathematics, Springer-Verlag London, London, 1999.

[13] Fonf, V. P., Lindenstrauss, J. and Phelps, R. R.: Infinite dimensional convexity. In *Handbook of the geometry of Banach spaces, Vol. 1, 599–670.* North-Holland, Amsterdam, 2001.

[14] Rainwater, J.: Weak convergence of bounded sequences. *Proc. Amer. Math. Soc.* **14** (1963), 999.

[15] Ruess, W. M. and Stegall, C. P.: Exposed and denting points in duals of operator spaces. *Israel J. Math.* **53** (1986), no. 2, 163–190.

[16] Ruess, W. M. and Stegall, C. P.: Fréchet differentiability of the norm in operator spaces. *Math. Ann.* **280** (1988), no. 4, 527–536.

[17] Šmul'jan, V. L.: On some geometrical properties of the unit sphere in the space of type (B). *Rec. Math. N. S.* **6** (1939), 77–94,

[18] Šmul'jan, V.L.: Sur la dérivabilité de la norme dans l'espace de Banach. *C. R. (Doklady) Acad. Sci. URSS (N. S.)* **27** (1940), 643–648.

Received July 17, 2015.

Christopher Boyd: School of Mathematics and Statistics, University College Dublin, Belfield, Dublin 4, Ireland.
E-mail: Christopher.Boyd@ucd.ie

Anthony Brown: School of Mathematics and Statistics, University College Dublin, Belfield, Dublin 4, Ireland.
E-mail: Anthony.Brown@ucd.ie

The second author was supported by an Ad Astra Scholarship while this research was undertaken.