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A new result on averaging theory for a class of discontinuous planar differential systems with applications

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Abstract. We develop the averaging theory at any order for computing the periodic solutions of periodic discontinuous piecewise differential system of the form

$$\frac{dr}{d\theta} = r' = \begin{cases} F^+(\theta, r, \varepsilon) & \text{if} \quad 0 \le \theta \le \alpha, \\ F^-(\theta, r, \varepsilon) & \text{if} \quad \alpha \le \theta \le 2\pi, \end{cases}$$

where $F^{\pm}(\theta, r, \varepsilon) = \sum_{i=1}^{k} \varepsilon^i F_i^{\pm}(\theta, r) + \varepsilon^{k+1} R^{\pm}(\theta, r, \varepsilon)$ with $\theta \in \mathbb{S}^1$ and $r \in D$, where D is an open interval of $\mathbb{R}^+$, and ϵ is a small real parameter.

Applying this theory, we provide lower bounds for the maximum number of limit cycles that bifurcate from the origin of quartic polynomial differential systems of the form $\dot{x} = -y + xp(x, y)$, $\quad \dot{y} = x + yp(x, y)$, with $p(x, y)$ a polynomial of degree 3 without constant term, when they are perturbed, either inside the class of all continuous quartic polynomial differential systems, or inside the class of all discontinuous piecewise quartic polynomial differential systems with two zones separated by the straight line $y = 0$.

1. Introduction and statement of the main results

The determination and distribution of limit cycles of the planar differential equations is one of the main open problems in the qualitative theory of such differential systems, see for instance [13]. Thus, in recent years the bifurcation of limit cycles using averaging theory in continuous planar differential systems is being largely studied, see for instance [6], [10], [11], [21], [22], [27]. But in the real world many phenomena are described using discontinuous differential equations, see for example [5], [28] and the references therein. These last years a big interest

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appeared for studying the limit cycles of discontinuous piecewise differential systems. Thus in [23] the averaging theory is extended up to order 1 for studying the periodic solutions of systems of the form $x' = \varepsilon\big(F(t,x,\varepsilon) + \operatorname{sign}(h(x))G(t,x,\varepsilon)\big)$ using techniques of regularization. In [19] the averaging theory has been extended to order 2 for a big class of discontinuous piecewise differential equations of the form $x' = \varepsilon F_1(t,x,\varepsilon)$. And in [20] this theory has been improved for analyzing the periodic solutions of discontinuous piecewise differential equations of the form $x' = F_0(t,x) + \varepsilon F_1(t,x,\varepsilon)$.

Our main goal in this paper is to develop the averaging theory at any order for a particular class of discontinuous piecewise differential systems, see subsection 1.1.

This averaging theory provides a straightforward calculation method to determine the number of limit cycles which can bifurcate from the periodic orbits of the regarded particular family of differential systems. The relevance of this lies in the fact that one of the main issues regarding the estimation of the number of limit cycles of differential systems is the computational constraints involved. In general the calculations require powerful computerized resources and in the case of the averaging theory, the higher the averaging order is, the more complex are the computational operations to calculate it. Then, as an application of this theory, we present in subsection 1.2 the bifurcation of limit cycles of quartic polynomial differential systems of the form $\dot{x} = -y + xp(x,y), \quad \dot{y} = x + yp(x,y)$ when they are perturbed, either inside the class of all continuous quartic polynomial differential systems, or inside the class of all discontinuous piecewise quartic polynomial differential systems with two zones separated by the straight line $y = 0$. Here $p(x,y)$ is a polynomial of degree 3 without constant term.

We remark that, using the averaging theory developed in this paper, we were able to improve previous results in [14], where the bifurcation of limit cycles in cubic polynomial differential systems is studied. The improvement is due not only because in the present work we study polynomial differential systems one degree higher than in that previous one, but also because the efficiency of the averaging method developed in Theorem 1 of this paper allowed us to obtain results up to the fourth order.

1.1. Results in averaging theory

Usually the discontinuous differential systems in the plane are studied for a straight line of discontinuity. Here we want to study the periodic solutions of discontinuous differential systems having the line of discontinuity composed by two half-straight lines starting at the origin and forming an angle α. Thus we develop the averaging theory at any order for computing the periodic solutions of periodic discontinuous piecewise differential systems of the form

$$\frac{dr}{d\theta} = r' = \begin{cases} F^+(\theta,r,\varepsilon) & \text{if} \quad 0 \le \theta \le \alpha, \\ F^-(\theta,r,\varepsilon) & \text{if} \quad \alpha \le \theta \le 2\pi, \end{cases} \tag{1.1}$$

where

$$F^\pm(\theta,r,\varepsilon) = \sum_{i=1}^{k} \varepsilon^i F_i^\pm(\theta,r) + \varepsilon^{k+1} R^\pm(\theta,r,\varepsilon),$$

and ε is a real small parameter. The set of discontinuity of system (1.1) is $\Sigma = \{\theta = 0\} \cup \{\theta = \alpha\}$ if $0 < \alpha < 2\pi$. Here $F_i^{\pm}\colon \mathbb{S}^1 \times D \to \mathbb{R}$ for $i = 0, 1, \dots, n$, and $R^{\pm} : \mathbb{S}^1 \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ are $\mathcal{C}^{k+1}$ functions, where D is an open and bounded interval of $(0, \infty)$, and $\mathbb{S}^1 \equiv \mathbb{R}/(2\pi)$. This last convention is equivalent to say that the functions involved in system (1.1) are 2π-periodic in the first variable.

We point out that taking $\alpha = 2\pi$ system (1.1) becomes continuous. So the averaging theory developed in this section also applies to continuous differential systems.

For $i = 1, 2, \dots, k$, we define the *averaged function* $f_i\colon D \to \mathbb{R}$ of order i as

$$f_i(\rho) = \frac{y_i^+(\alpha, \rho) - y_i^-(\alpha - 2\pi, \rho)}{i!}. \tag{1.2}$$

The functions $y_i^{\pm} : \mathbb{S}^1 \times D \to \mathbb{R}$, for $i = 1, 2, \dots, k$, are defined recurrently as

$$y_1^{\pm}(\theta, \rho) = \int_0^{\theta} F_1^{\pm}(\phi, \rho)\, d\phi,$$

$$y_i^{\pm}(\theta, \rho) = i! \int_0^{\theta} \Bigg(F_i^{\pm}(\phi, \rho) + \sum_{l=1}^{i} \sum_{S_l} \frac{1}{b_1!\, b_2!\, 2!^{b_2} \cdots b_l!\, l!^{b_l}} \cdot \partial^L F_{i-l}^{\pm}(\phi, \rho) \prod_{j=1}^{l} y_j^{\pm}(\phi, \rho)^{b_j} \Bigg) d\phi. \tag{1.3}$$

Here $\partial^L G(\phi, \rho)$ denotes the derivative of order L of a function G with respect to the variable ρ, and S_l is the set of all l-tuples of non-negative integers $(b_1, b_2, \dots, b_l)$ satisfying $b_1 + 2b_2 + \cdots + lb_l = l$, and $L = b_1 + b_2 + \cdots + b_l$. For sake of simplicity, in expression (1.3), we are assuming that $F_0 = 0$.

As we shall see the averaged functions f_i's control the existence of isolated periodic solutions of system (1.1). Since these functions are obtained directly from $y_i^{\pm}$ using (1.3), we give in the Appendix the explicit formulae of $y_i^{\pm}$'s up to $i = 7$. Recently in [24] the Bell polynomials were used to provide an alternative formula for the recurrence (1.3). This new formula can make easier the computational implementation of the averaged functions (1.2).

Our main result on the periodic solutions of system (1.1) is the following.

Theorem 1. *Assume that, for some $\ell \in \{1, 2, \dots, k\}$, $f_i = 0$ for $i = 1, 2, \dots, \ell - 1$ and $f_\ell \neq 0$. If there exists $\rho^* \in D$ such that $f_\ell(\rho^*) = 0$ and $f_\ell'(\rho^*) \neq 0$, then for $|\varepsilon| \neq 0$ sufficiently small there exists a 2π-periodic solution $r(\theta, \varepsilon)$ of (1.1) such that $r(0, \varepsilon) \to \rho^*$ when $\varepsilon \to 0$.*

Theorem 1 is proved in section 2.

The assumption $D \subset \mathbb{R}$ is not restrictive. In fact, if one consider D as being an open subset of $\mathbb{R}^n$ the conclusion of Theorem 1 still holds by assuming that the Jacobian matrix $Jf_l(\rho^*)$ is nonsingular, that is $\det(Jf_l(\rho^*)) \neq 0$. In this case the derivative $\partial^L G(\phi, \rho)$ is a symmetric L-multilinear map which is applied to a "product" of L vectors of $\mathbb{R}^n$, denoted as $\prod_{j=1}^{L} y_j \in \mathbb{R}^{nL}$ (see [21]).

1.2. Periodic solutions in planar differential quartic systems with a uniform isochronous center-focus type singular point

Assume that a differential polynomial system in $\mathbb{R}^2$ has a center at the point $q \in \mathbb{R}^2$, then without loss of generality we can suppose doing a translation that q is at the origin of coordinates. A center q is an *isochronous center* if it has a neighborhood such that in this neighborhood all the periodic orbits have the same period. An isochronous center is *uniform* if in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, it can be written as $\dot{r} = G(\theta, r)$, $\dot{\theta} = k$, $k \in \mathbb{R} \setminus \{0\}$, for more details see Conti [9].

The study of the analytic uniform isochronous centers has increased in the last decades. These systems have a unique singular point, which is the uniform isochronous center, and up to a linear change of coordinates they can be written as

$$\dot{x} = -y + x\,p(x,y), \quad \dot{y} = x + y\,p(x,y), \tag{1.4}$$

where p is an analytic function and $p(0,0) = 0$.

The present big interest in studying the uniform isochronous centers is due, on one hand to their importance in the general problem of isochronicity. On the other hand, system (1.4) in polar coordinates $x = r\cos\theta, y = r\sin\theta$ becomes

$$\dot{r} = \sum_{k\geq 1} p_k(\cos\theta, \sin\theta) r^{k+1}, \quad \dot{\theta} = 1,$$

where p_k is a homogeneous polynomial of degree k. These differential systems can be transformed into generalized Abel differential equations of the form

$$\frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} = \sum_{k\geq 1} p_k(\cos\theta, \sin\theta) r^{k+1}. \tag{1.5}$$

Then equation (1.5) gives information about system (1.4), and vice versa. For more details see [3], [1].

The bifurcation of limit cycles in planar differential polynomial systems of the form (1.4) of degree n has been intensively studied, see for instance [7], [9], [12] and the bibliography therein.

Consider a planar differential polynomial system and $q \in \mathbb{R}^2$ a singular point of this system. We say that q is a *weak focus* if it is a center for the linearized system at q.

The next result is well known, and a proof for it can be found in [15].

Proposition 2. *Assume that a planar differential polynomial system $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$ of degree n has a center at the origin of coordinates. Then, this center is uniform isochronous if and only if by doing a linear change of variables and a rescaling of time it can be written under the form (1.4).*

Algaba, Reyes, Ortega and Bravo [2], in 1999, and Chavarriga, García and Giné [8], in 2001, independently provided the following characterization of non-homogeneous quartic polynomial systems with an isolated uniform isochronous center at the origin.

Theorem 3. *Consider $p(x,y) = \sum_{i=1}^{3} g_i(x,y)$ where $g_i(x,y)$ for $i = 1,2,3$ are homogeneous polynomials of degree i, $g_1^2+g_2^2 \neq 0$ and $g_3 \neq 0$ such that (1.4) is a quartic polynomial differential system. Then system (1.4) has a uniform isochronous center at the origin if and only if it is reversible. In this case, modulo a rotation and a rescaling of the independent variable, system (1.4) can be written as*

$$\begin{aligned} \dot{x} &= -y + x(A_1 x + B_2 xy + C_1 x^3 + C_3 xy^2), \\ \dot{y} &= \quad x + y(A_1 x + B_2 xy + C_1 x^3 + C_3 xy^2). \end{aligned} \tag{1.6}$$

where $A_1, B_2, C_1, C_3 \in \mathbb{R}$.

In the case of homogeneous uniform isochronous centers, Conti [9] proved the following theorem in 1994.

Theorem 4. *Let $p(x,y) = \sum_{i+j=n-1} g_{i,j} x^i y^j$ be a homogeneous polynomial of degree $n-1$. Then system (1.4) has a uniform isochronous center at the origin if either n is even, or if n is odd and*

$$\sum_{\nu=0}^{n-1} \left[g_{n-1-\nu,\nu} \int_0^{2\pi} \cos^{n-1-\nu}\theta \sin^{\nu}\theta \; d\theta \right] = 0.$$

By Theorem 4 the homogeneous quartic polynomial systems of the form (1.4) always have a uniform isochronous center at the origin.

A classification of the global phase portraits of the planar quartic polynomial differential systems of the form (1.6) is provided in [15].

We study the limit cycles that bifurcate from the origin of the planar differential quartic polynomial systems of the form (1.4).

More precisely, let $H_c(n)$ denote the maximum number of limit cycles that bifurcate from the origin of system (1.4), when it is perturbed inside the class of all continuous polynomial differential systems of degree n, and $H_d(n)$ denotes the maximum number of limit cycles that bifurcate from the origin of system (1.4), when it is perturbed inside the class of all discontinuous piecewise polynomial differential systems of degree n with two zones separated by the straight line $y = 0$. We provide lower bounds for $H_c(4)$ and $H_d(4)$ in both cases when the origin is either a uniform isochronous center, or a weak focus. The method used for obtaining these lower bounds is based on the averaging theory.

To the best of our knowledge, this is the first work that provides an estimation of $H_d(4)$. Estimations for $H_c(4)$ have been studied for particular cases. For instance, in [1], the authors prove, using a different method than ours, that a system (1.4) with $p(x,y) = P_1(x,y)+P_m(x,y)$, where P_i is a homogeneous polynomial of degree i, $i \in \{1, m\}$, $P_1 P_m \neq 0$ might have $[m/2] + 1$ limit cycles. By setting $m = 3$ we have a quartic polynomial differential system, and the number of limit cycles to this particular case is 2.

This work extends previous results in [14], where we studied the bifurcation of limit cycles in cubic polynomial differential systems of the form (1.4).

When the averaged functions are polynomial a good tool to estimate the number of their simple zeros is the *Descartes theorem* (see [4]), which in this case provides, for $|\varepsilon| \neq 0$ small, lower bounds for the number of limit cycles of the system.

Theorem 5 (Descartes theorem). *Consider the real polynomial $r(x) = a_{i_1}x^{i_1} + a_{i_2}x^{i_2} + \cdots + a_{i_r}x^{i_r}$ with $0 = i_1 < i_2 < \cdots < i_r$ and $a_{i_j} \neq 0$ real constants for $j \in \{1, 2, \ldots, r\}$. When $a_{i_j}a_{i_{j+1}} < 0$, we say that a_{i_j} and $a_{i_{j+1}}$ have a variation of sign. If the number of variations of signs is m, then $r(x)$ has at most m positive real roots. Moreover, it is always possible to choose the coefficients of $r(x)$ in such a way that $r(x)$ has exactly $r-1$ positive real roots.*

When the averaged functions are more general functions there are other techniques to deal with the problem of estimating the number of their zeros. See for instance the theory of Chebyshev systems [18], [25].

We consider the following family of continuous systems

$$\begin{aligned} \dot{x} &= -y + x\,p(x,y) + \sum_{i=1}^{4} \varepsilon^i p_i(x,y), \\ \dot{y} &= \quad x + y\,p(x,y) + \sum_{i=1}^{4} \varepsilon^i q_i(x,y), \end{aligned} \tag{1.7}$$

where

$$\begin{aligned} p_j &= \alpha_0^j + \alpha_1^j x + \alpha_2^j y + \alpha_3^j x^2 + \alpha_4^j xy + \alpha_5^j y^2 + \alpha_6^j x^3 + \alpha_7^j x^2y + \alpha_8^j xy^2 + \alpha_9^j y^3 \\ &\quad + \alpha_{10}^j x^4 + \alpha_{11}^j x^3y + \alpha_{12}^j x^2y^2 + \alpha_{13}^j xy^3 + \alpha_{14}^j y^4, \\ q_j &= \beta_0^j + \beta_1^j x + \beta_2^j y + \beta_3^j x^2 + \beta_4^j xy + \beta_5^j y^2 + \beta_6^j x^3 + \beta_7^j x^2y + \beta_8^j xy^2 + \beta_9^j y^3 \\ &\quad + \beta_{10}^j x^4 + \beta_{11}^j x^3y + \beta_{12}^j x^2y^2 + \beta_{13}^j xy^3 + \beta_{14}^j y^4, \end{aligned}$$

being α_i^j and β_i^j, for $i = 0, \ldots, 14$ and $j = 1, \ldots, 4$, real constants. We also consider the discontinuous systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathcal{X}(x,y) = \begin{cases} X_1(x,y) & \text{if} \quad y > 0, \\ X_2(x,y) & \text{if} \quad y < 0, \end{cases} \tag{1.8}$$

where

$$X_1(x,y) = \begin{pmatrix} -y + xp(x,y) + \sum_{i=1}^{k} \varepsilon^i p_i(x,y) \\ x + yp(x,y) + \sum_{i=1}^{k} \varepsilon^i q_i(x,y) \end{pmatrix},$$

$$X_2(x,y) = \begin{pmatrix} -y + xp(x,y) + \sum_{i=1}^{k} \varepsilon^i u_i(x,y) \\ x + yp(x,y) + \sum_{i=1}^{k} \varepsilon^i v_i(x,y) \end{pmatrix},$$

$$\begin{aligned} u_j &= \gamma_0^j + \gamma_1^j x + \gamma_2^j y + \gamma_3^j x^2 + \gamma_4^j xy + \gamma_5^j y^2 + \gamma_6^j x^3 + \gamma_7^j x^2y + \gamma_8^j xy^2 + \gamma_9^j y^3 \\ &\quad + \gamma_{10}^j x^4 + \gamma_{11}^j x^3y + \gamma_{12}^j x^2y^2 + \gamma_{13}^j xy^3 + \gamma_{14}^j y^4, \\ v_j &= \delta_0^j + \delta_1^j x + \delta_2^j y + \delta_3^j x^2 + \delta_4^j xy + \delta_5^j y^2 + \delta_6^j x^3 + \delta_7^j x^2y + \delta_8^j xy^2 + \delta_9^j y^3 \\ &\quad + \alpha_{10}^j x^4 + \delta_{11}^j x^3y + \delta_{12}^j x^2y^2 + \delta_{13}^j xy^3 + \delta_{14}^j y^4, \end{aligned}$$

being γ_i^j and δ_i^j, for $i = 0, \dots, 14$ and $j = 1, \dots, k$, real constants, and $k \in \{4, 7\}$. In both cases, the continuous and the discontinuous one we have to consider either

$$p(x,y) = t_{10}x + t_{01}y + t_{20}x^2 + t_{11}xy + t_{02}y^2 + t_{30}x^3 + t_{21}x^2y + t_{12}xy^2 + t_{03}y^3, \tag{1.9}$$

with $t_{ij} \in \mathbb{R}$, $i + j = 1, 2, 3$, $t_{30}^2 + t_{21}^2 + t_{12}^2 + t_{03}^2 \neq 0$, or

$$p(x,y) = t_{10}x + t_{11}xy + t_{30}x^3 + t_{12}xy^2, \tag{1.10}$$

with $t_{30}^2 + t_{12}^2 \neq 0$, or

$$p(x,y) = t_{30}x^3 + t_{21}x^2y + t_{12}xy^2 + t_{03}y^3. \tag{1.11}$$

We remark that the polynomials $p(x, y)$ in (1.10) and (1.11) are used to study the cases of quartic polynomial differential systems with a uniform isochronous center at the origin, either having a non-homogeneous nonlinear part (using (1.6) of Theorem 3), or a homogeneous nonlinear part (see Theorem 4), respectively. On the other hand, since (1.9) is a general cubic polynomial in x and y without constant term, we can use it for studying the bifurcation of limit cycles in both cases when the origin is either a uniform isochronous center or a weak focus.

In what follows we state our results. We remark that in their statements we are referring the order of averaging we are using. Since we are providing lower bounds for the numbers H_c and H_d, the results could be improved using higher orders of the averaging Theorem 1.

Theorem 6. *Using averaging theory of order 4 we obtain, for $|\varepsilon| \neq 0$ sufficiently small, $H_d(4) \geq 6$ for the differential system (1.8) with $p(x, y)$ of the form (1.9) (i.e., system (1.8) has a weak focus or a uniform isochronous center at the origin).*

Theorem 6 is proved in section 3.

Theorem 7. *Using averaging theory of order 4 we obtain, for $|\varepsilon| \neq 0$ sufficiently small, $H_d(4) \geq 5$ for the differential system (1.8) with $p(x, y)$ either of the form (1.10) or (1.11) (i.e., system (1.8) has a uniform isochronous center at the origin).*

Theorem 7 is proved in section 4.

Theorem 8. *Using the averaging theory of order 7 we obtain, for $|\varepsilon| \neq 0$ sufficiently small, $H_d(4) \geq 6$ for the differential system (1.8) with $p(x, y)$ of the form (1.10) and $\alpha_0^j = \beta_0^j = \gamma_0^j = \delta_0^j = 0$, $j = 1, \dots, 7$.*

Theorem 8 is proved in section 5.

Theorem 9. *Using the averaging theory of order 4 we obtain, for $|\varepsilon| \neq 0$ sufficiently small, $H_c(4) \geq 2$ for the differential system (1.7) with $p(x, y)$ of the form (1.9).*

Theorem 10. *Using the averaging theory of order 4 we obtain, for* $|\varepsilon| \neq 0$ *sufficiently small,* $H_c(4) \geq 1$ *for the differential system* (1.7) *with* $p(x, y)$ *either of the form* (1.10) *or* (1.11).

Theorems 9 and 10 are proved in section 6.

To prove Theorems 6 and 7 (respectively Theorems 9 and 10) we shall use the averaging theory of order 4 for discontinuous (respectively continuous) differential systems, together with a rescaling of the variables. In these proofs we can see, using Descartes Theorem, that the lower bounds which appear in the theorems are actually upper bounds for the averaging theory of order 4, too.

In Theorem 6, due to Descartes theorem, an upper bound for the maximum number of the limit cycles is 6 for system (1.8) with $p(x, y)$ of the form (1.9), which can be obtained applying the averaging theory of order 4. We remark that in this case we can have either a weak focus or a uniform isochronous center at the origin. In Theorem 7 we study system (1.8) with $p(x, y)$ of the form (1.10) and (1.11), and hence we have a uniform isochronous center at the origin. In this case an upper bound for the maximum number of limit cycles which can be obtained using the averaging theory of order 4 is 5, which is one less than in the general case studied in Theorem 6. Hence in Theorem 6 if a differential system presents the maximum number of limit cycles bifurcating from its singular point obtained using the averaging theory of order 4, which is 6, this singular point is a weak focus, since by Theorem 7 if it was a uniform isochronous center this number would not exceed 5. Similar reasoning holds for the continuous case, discussed in Theorems 9 and 10.

In the case of limit cycles bifurcating from ovals of the period annulus of a uniform isochronous center, there are examples of quartic polynomial systems which has at least 8 limit cycles, see [16].

All calculations were performed with the assistance of the software `Mathematica`.

2. Proof of Theorem 1

The next lemma is a key result to prove Theorem 1. It has been proved, for a more general context, in [21], [22]. For sake of completeness we shall give a abridged version of its proof.

Lemma 11 ([21], [22]). *Let* $r^{\pm}(\cdot, \rho, \varepsilon)\colon [0, \theta_\rho) \to \mathbb{R}$ *be the solution of* $r' = F^{\pm}(\theta, r, \varepsilon)$ *with* $r^{\pm}(0, \rho, \varepsilon) = \rho$, *and being* $[0, \theta_\rho)$ *its interval of definition. If* $\theta_\rho > 2\pi$, *then*

$$r^{\pm}(\theta, \rho, \varepsilon) = \rho + \sum_{i=1}^{k} \varepsilon^i \, \frac{y_i^{\pm}(\theta, \rho)}{i!} + \mathcal{O}(\varepsilon^{k+1}),$$

where $y_i^{\pm}(\theta, \rho)$ *for* $i = 1, 2, \ldots, k$ *are defined in* (1.3).

Proof. First we see that the solution $r^{\pm}(\theta,\rho,\varepsilon)$ reads

$$r^{\pm}(\theta,\rho,\varepsilon)=\rho+\sum_{i=0}^{k}\varepsilon^i\int_0^t F_i(s,r^{\pm}(s,\rho,\varepsilon))\,ds+\mathcal{O}(\varepsilon^{k+1}). \tag{2.1}$$

The above remainder $\mathcal{O}(\varepsilon^{k+1})$ is easily obtained from the continuity of the solution and of the system, and from the compactness of the domain $[0,2\pi]\times\overline{D}\times[-\varepsilon_1,\varepsilon_1]$.

For $|\varepsilon|\neq 0$ sufficiently small the function $F_i^{\pm}(t,r^{\pm}(\theta,\rho,\varepsilon))$, for $i=0,1,\dots,k-1$, can be written in power series of ε as

$$\begin{aligned} F_i^{\pm}(t,r^{\pm}(\theta,\rho,\varepsilon))=\; & F_i^{\pm}(\theta,\rho)\\ & +\sum_{l=1}^{k-i}\frac{\varepsilon^l}{l!}\Big(\frac{\partial^l}{\partial\varepsilon^l}F_i^{\pm}(t,r^{\pm}(\theta,\rho,\varepsilon))\Big)\Big|_{\varepsilon=0}+\mathcal{O}(\varepsilon^{k-i+1}). \end{aligned} \tag{2.2}$$

Using the Faà di Bruno's formula (see [17]) to compute the l-derivative of $F_i^{\pm}(t,r^{\pm}(\theta,\rho,\varepsilon))$ in ε, for $i=0,1,\dots,k-1$, we have

$$\begin{aligned} \frac{\partial^l}{\partial\varepsilon^l}F_i^{\pm}(t,r^{\pm}(\theta,\rho,\varepsilon))\Big|_{\varepsilon=0} &=\sum_{S_l}\frac{l!}{b_1!\,b_2!\,2!^{b_2}\cdots b_l!\,l!^{b_l}}\\ &\quad\cdot\big(\partial^L F_i^{\pm}(t,r^{\pm}(\theta,\rho,\varepsilon))\big)\Big|_{\varepsilon=0}\prod_{j=1}^{l}z_j^{\pm}(\theta,\rho)^{b_j}. \end{aligned} \tag{2.3}$$

Here S_l is the set of all l-tuples of non-negative integers $(b_1,b_2,\dots,b_l)$ which are solutions of the equation $b_1+2b_2+\cdots+lb_l=l$, $L=b_1+b_2+\cdots+b_l$, and

$$z_j^{\pm}(\theta,\rho)=\Big(\frac{\partial^j}{\partial\varepsilon^j}r^{\pm}(\theta,\rho,\varepsilon)\Big)\Big|_{\varepsilon=0}. \tag{2.4}$$

Using the expression (2.3), (2.2) becomes

$$\begin{aligned} F_i^{\pm}(s,r^{\pm}(s,\rho,\varepsilon))=\; & F_i^{\pm}(s,\rho)+\sum_{l=1}^{k-i}\sum_{S_l}\frac{\varepsilon^l}{b_1!\,b_2!\,2!^{b_2}\cdots b_l!\,l!^{b_l}}\\ & \cdot\partial^L F_i^{\pm}(s,\rho)\prod_{j=1}^{l}z_j^{\pm}(s,\rho)^{b_j}+\mathcal{O}(\varepsilon^{k-i+1}), \end{aligned} \tag{2.5}$$

for $i=0,1,\dots,k-1$. Moreover, for $i=k$ we have that

$$F_k^{\pm}(s,r^{\pm}(s,\rho,\varepsilon))=F_k^{\pm}(s,\rho)+\mathcal{O}(\varepsilon). \tag{2.6}$$

Now, substituting (2.5) and (2.6) into (2.1) and doing a change of index we get

$$\begin{aligned} r^{\pm}(\theta,\rho,\varepsilon)=\rho+\sum_{i=1}^{k}\varepsilon^i\Big(\int_0^t\Big[F_i^{\pm}(s,\rho)+\sum_{l=1}^{i}\sum_{S_l}\frac{1}{b_1!\,b_2!\,2!^{b_2}\cdots b_l!\,l!^{b_l}}\\ \cdot\,\partial^L F_{i-l}^{\pm}(s,\rho)\prod_{j=1}^{l}z_j^{\pm}(s,\rho)^{b_j}\,ds\Big]\Big)+\mathcal{O}(\varepsilon^{k+1}). \end{aligned} \tag{2.7}$$

From (2.4) and (2.7) we conclude that the functions $z_i^{\pm}(\theta,\rho)$, for $i=1,2,\dots,k$, can be computed recurrently as

$$(2.8)\qquad \begin{aligned} z_1^{\pm}(\theta,\rho) &= \int_0^{\theta} F_1^{\pm}(\phi,\rho)\,d\phi,\\ z_i^{\pm}(\theta,\rho) &= i!\int_0^{\theta}\Big(F_i^{\pm}(\phi,\rho)+\sum_{l=1}^{i}\sum_{S_l}\frac{1}{b_1!\,b_2!\,2!^{b_2}\cdots b_l!\,l!^{b_l}}\\ &\qquad\qquad \cdot\,\partial^L F_{i-l}^{\pm}(\phi,\rho)\prod_{j=1}^{l} z_j^{\pm}(\phi,\rho)^{b_j}\Big)\,d\phi. \end{aligned}$$

Moreover, the recurrences (1.3) and (2.8) are the same. Since $z_1^{\pm}(\theta,\rho)=y_1^{\pm}(\theta,\rho)$ we conclude that $z_i^{\pm}(\theta,\rho)=y_i^{\pm}(\theta,\rho)$, for $i=1,2,\dots,k$, and therefore

$$r^{\pm}(\theta,\rho,\varepsilon)=\rho+\sum_{i=1}^{k}\varepsilon^i\frac{y_i(\theta,\rho)}{i!}+\mathcal{O}(\varepsilon^{k+1}).$$

This completes the proof of Lemma 11. □

Proof of Theorem 1. First of all we have to show that there exists ε_0 sufficiently small such that for each $\rho\in\overline{D}$ and for every $\varepsilon\in[-\varepsilon_0,\varepsilon_0]$ the solutions $r^{\pm}(\theta,\rho,\varepsilon)$ are defined for every $\theta\in[0,T]$. Indeed, by the *existence and uniqueness theorem* of solutions (see, for example, Theorem 1.2.4 of [26]), $r^{\pm}(\theta,\rho,\varepsilon)$ is defined for all $0\le\theta\le\inf\left(T,d/M^{\pm}(\varepsilon)\right)$, for each x with $|r-\rho|<d$ and for every $\rho\in\overline{D}$, where

$$M^{\pm}(\varepsilon)\ge\Big|\sum_{i=1}^{k}\varepsilon^iF_i^{\pm}(\theta,\rho)+\varepsilon^{k+1}R^{\pm}(\theta,\rho,\varepsilon)\Big|.$$

Clearly ε can be taken sufficiently small in order that $\inf(T,d/M^{\pm}(\varepsilon))=T$ for all $\rho\in\overline{D}$. Moreover, since the vector fields $F^{\pm}(\theta,r,\varepsilon)$ are T-periodic, the solutions $r^{\pm}(\theta,\rho,\varepsilon)$ can be extended for $\theta\in\mathbb{R}$.

We denote

$$f(\rho,\varepsilon)=r^{+}(\alpha,\rho,\varepsilon)-r^{-}(\alpha-T,\rho,\varepsilon).$$

It is easy to see that system (1.1) for $\varepsilon=\bar\varepsilon\in(-\varepsilon_0,\varepsilon_0)$ has a periodic solution passing through $\bar\rho\in D$ if and only if $f(\bar\rho,\bar\varepsilon)=0$.

From Lemma 11 we have that

$$f(\rho,\varepsilon)=\sum_{i=1}^{k}\varepsilon^i\,\frac{y_i(\theta,\rho)-y_i(\theta,\rho)}{i!}+\mathcal{O}(\varepsilon^{k+1})=\sum_{i=1}^{k}\varepsilon^if_i(\rho)+\mathcal{O}(\varepsilon^{k+1})$$

where the function f_i is the one defined in (1.2) for $i=1,2,\dots,k$. From the hypothesis of the statement of Theorem 1 we have

$$f(\rho,\varepsilon)=\varepsilon^{\ell}f_{\ell}(\rho)+\cdots+\varepsilon^kf_k(\rho)+\mathcal{O}(\varepsilon^{k+1}).$$

Since $f_\ell(\rho^*)=0$ and $f_\ell'(\rho^*)\ne0$, the *implicit function theorem* applied to the function $\mathcal{F}(\rho,\varepsilon)=f(\rho,\varepsilon)/\varepsilon^{\ell}$ guarantees the existence of a differentiable function $\rho(\varepsilon)$ such that $\rho(0)=\rho^*$ and $f(\rho(\varepsilon),\varepsilon)=\varepsilon^{\ell}\mathcal{F}(\rho(\varepsilon\ ),\varepsilon)=0$ for every $|\varepsilon|\ne0$ sufficiently small. Then the proof of the theorem follows. □

3. Proof of Theorem 6

Consider system (1.8) with $p(x,y)$ of the general form (1.9). In order to analyze the Hopf bifurcation for this system, applying Theorem 1, we set $\alpha = \pi$ and we introduce a small parameter ε doing the change of coordinates $x = \varepsilon X,\ y = \varepsilon Y$. After that we perform the polar change of coordinates $X = r\cos\theta, Y = r\sin\theta$, and by doing a Taylor expansion truncated at the 4^{th} order in ε we obtain an expression for $dr/d\theta$ of the form (1.1), with $\alpha = \pi$. The explicit expression is quite large so we omit it.

System (1.8) is a polynomial system, so the functions $F_i^{\pm}(\theta,r)$ and $R_i^{\pm}(\theta,r,\varepsilon)$, $i = 1,\dots,4$ are analytic, and consequently, locally Lipschitz. Moreover, since the variable θ appears through sinus and cosinus, system (1.8) in the form $dr/d\theta$ is 2π-periodic. It suffices to take $D = \{r : 0 < r < r_0\}$, where the unperturbed system has periodic solutions passing through the points $(0,r)$ with $0 < r < r_0$.

We obtain each y_i^+ and $y_i^-, i = 1.\dots,4$ applying expression (1.3) respectively for X_1 and X_2 of system (1.8), after the changes described in the first paragraph of this section. Then we calculate the averaged functions $f_i, i = 1.\dots,4$ using equation (1.2). Hence, by Theorem 1 we have the averaged function of first order

$$f_1(r) = A_1 r + A_0,$$

where

$$\begin{aligned}
A_1 &= \frac{1}{2}\pi(3t_{01}(\alpha_0^1 + \gamma_0^1) + \alpha_1^1 + \beta_2^1 + \gamma_1^1 + \delta_2^1 - 3t_{10}(\beta_0^1 + \delta_0^1)),\\
A_0 &= 2(\beta_2^1\alpha_0^1 + (\alpha_0^1)^2 t_{01} - \beta_0^1(\alpha_0^1 t_{10} + \beta_1^1) - \gamma_0^1\delta_2^1 - (\gamma_0^1)^2 t_{01} + \delta_0^1(\gamma_0^1 t_{10} + \delta_1^1)\\
&\qquad + \beta_0^2 - \delta_0^2).
\end{aligned}$$

The rank of the Jacobian matrix of the function $\mathcal{A} = (A_0, A_1)$ with respect to the variables $t_{01}, t_{10}, \alpha_0^1, \alpha_1^1, \beta_0^1, \beta_1^1, \beta_2^1, \gamma_0^1, \gamma_1^1, \delta_0^1, \delta_1^1, \delta_2^1$ is maximal. Then the coefficients A_0 and A_1 are linearly independent in their variables.

Clearly $f_1(r) = 0$ has at most one solution in D. Thus applying Theorem 1 it is proved that at most 1 limit cycle can bifurcate from the origin of system (1.8) with $p(x,y)$ of the form (1.9), using the averaging theory of first order. Solving A_1 for α_1^1 and A_0 for δ_0^2 we have $f_1(r) = 0$, and we can apply the averaging theory of order 2. Its corresponding averaged function is

$$f_2(r) = B_3 r^3 + B_2 r^2 + B_1 r + B_0,$$

where

$$\begin{aligned}
B_3 &= 2\pi(t_{02} + t_{20}),\\
B_2 &= \frac{1}{3}(-4)(3t_{01}(2\alpha_0^1 t_{10} - \alpha_2^1 + 4\gamma_0^1 t_{10} + \gamma_2^1) - 8t_{02}(\alpha_0^1 - \gamma_0^1) - \alpha_0^1 t_{20} - \alpha_4^1 - \beta_3^1\\
&\quad - 2\beta_5^1 + 6\gamma_1^1 t_{10} + \gamma_0^1 t_{20} + \gamma_4^1 + \delta_3^1 + 2\delta_5^1 + 3t_{01}^2(\beta_0^1 - \delta_0^1) - 3\beta_0^1 t_{10}^2\\
&\quad - 15\delta_0^1 t_{10}^2 + 3\beta_2^1 t_{10} + 3\delta_2^1 t_{10} + 4\beta_0^1 t_{11} - 4\delta_0^1 t_{11}),
\end{aligned}$$

$$
\begin{aligned}
B_1 = \frac{1}{4}\pi(&-8\alpha_3^1\beta_0^1 + 8\alpha_0^1\beta_5^1 - 3t_{01}(t_{10}(-\alpha_0^1\gamma_0^1 + 15\beta_0^1\delta_0^1 + 8(\alpha_0^1)^2 + 8(\beta_0^1)^2 \\
&+ (\gamma_0^1)^2 - 7(\delta_0^1)^2) + 3\alpha_2^1\gamma_0^1 - 5\beta_0^1\delta_2^1 - 4\alpha_0^1\alpha_2^1 - 4\beta_0^1\beta_2^1 - 5\beta_0^1\gamma_1^1 \\
&- \beta_1^1\gamma_0^1 + 5\gamma_1^1\delta_0^1 + \gamma_0^1\delta_1^1 - 7\gamma_0^1\gamma_2^1 + \delta_0^1\delta_2^1 - 4\alpha_0^2 - 4\gamma_0^2) \\
&+ 3t_{01}^2(8\alpha_0^1\beta_0^1 + \gamma_0^1(15\beta_0^1 - 7\delta_0^1)) + 16t_{02}((\alpha_0^1)^2 + (\gamma_0^1)^2) + \alpha_0^1\gamma_1^1 t_{10} \\
&- 3\alpha_2^1\gamma_1^1 - 3\alpha_2^1\delta_2^1 + \beta_1^1\delta_2^1 + 24\alpha_0^1\beta_0^1 t_{10}^2 - 3\alpha_0^1\delta_0^1 t_{10}^2 \\
&- 24\alpha_0^1\beta_2^1 t_{10} + 9\alpha_2^1\delta_0^1 t_{10} + \alpha_0^1\delta_2^1 t_{10} - 16\alpha_0^1\beta_0^1 t_{11} + 4\alpha_0^1\alpha_4^1 - 4\beta_0^1\beta_4^1 \\
&+ \beta_1^1\gamma_1^1 - 8\gamma_3^1\delta_0^1 - \gamma_1^1\delta_1^1 + 3\gamma_2^1\delta_2^1 + 8\gamma_0^1\delta_5^1 + 3\gamma_0^1\delta_0^1 t_{10}^2 \\
&- 9\gamma_2^1\delta_0^1 t_{10} - \gamma_0^1\delta_2^1 t_{10} - \gamma_0^1\gamma_1^1 t_{10} - 16\gamma_0^1\delta_0^1 t_{11} + 3\gamma_1^1\gamma_2^1 + 4\gamma_0^1\gamma_4^1 \\
&- \delta_1^1\delta_2^1 - 4\delta_0^1\delta_4^1 + 4\alpha_1^2 + 4\beta_2^2 + 4\gamma_1^2 + 4\delta_2^2 - 3\beta_1^1\delta_0^1 t_{10} \\
&+ 24\beta_0^1\beta_1^1 t_{10} + 3\delta_0^1\delta_1^1 t_{10} - 24\beta_0^2 t_{10} + 16(\beta_0^1)^2 t_{20} + 16(\delta_0^1)^2 t_{20}), \\
B_0 = \;&-4(-\alpha_0^1\beta_2^1\delta_1^1 + \alpha_0^1\beta_1^1\beta_2^1 + \alpha_0^1\beta_0^1\beta_4^1 - (\alpha_0^1)^2\beta_5^1 + t_{01}((\alpha_0^1)^2(\beta_1^1 - \delta_1^1) \\
&+ \alpha_0^1(3\alpha_2^1\gamma_0^1 + 2\beta_0^1(\gamma_1^1 + \delta_2^1)) + t_{10}(-6\beta_0^1\alpha_0^1\delta_0^1 - (\alpha_0^1)^2\gamma_0^1 + (\alpha_0^1)^3 \\
&- 6\gamma_0^1(3\beta_0^1\delta_0^1 + (\beta_0^1)^2 - (\delta_0^1)^2)) + \gamma_0^1(6\beta_0^1\delta_2^1 + 3\beta_0^1\beta_2^1 + 6\beta_0^1\gamma_1^1 \\
&- 4\gamma_1^1\delta_0^1 + 3\gamma_0^1\gamma_2^1 - \delta_0^1\delta_2^1 + 3\alpha_0^2 + 3\gamma_0^2)) + 3\gamma_0^1 t_{01}^2(2\alpha_0^1\beta_0^1 \\
&+ \gamma_0^1(3\beta_0^1 - \delta_0^1)) + t_{02}((\gamma_0^1)^3 - (\alpha_0^1)^3) + \alpha_0^1\beta_0^1\gamma_0^1 t_{10}^2 - \alpha_0^1\beta_2^1\gamma_0^1 t_{10} \\
&+ \beta_0^1(\delta_2^1)^2 + \alpha_0^1\alpha_2^1\gamma_1^1 + \beta_0^1\beta_1^1\delta_1^1 + \alpha_0^1\alpha_2^1\delta_2^1 + \beta_0^1\beta_2^1\delta_2^1 \\
&+ \beta_1^1\beta_0^2 + \beta_0^1\beta_1^2 - \alpha_0^1\beta_2^2 - (\alpha_0^1)^2\beta_0^1 t_{10}^2 + \alpha_0^1\beta_0^1\delta_1^1 t_{10} - 2\alpha_0^1\beta_0^1\beta_1^1 t_{10} \\
&+ (\alpha_0^1)^2\beta_2^1 t_{10} - 3\alpha_0^1\alpha_2^1\delta_0^1 t_{10} + \alpha_0^1\beta_0^2 t_{10} + (\alpha_0^1)^2\beta_0^1 t_{11} - \alpha_0^1(\beta_0^1)^2 t_{20} \\
&- \beta_0^1(\beta_1^1)^2 - (\beta_0^1)^2\beta_3^1 + 2\beta_0^1\gamma_1^1\delta_2^1 + \beta_0^1(\gamma_1^1)^2 + \beta_0^1\beta_2^1\gamma_1^1 - (\gamma_1^1)^2\delta_0^1 \\
&- \gamma_1^1\delta_2^1\delta_0^1 - \gamma_0^1\delta_4^1\delta_0^1 + \gamma_0^1\gamma_2^1\delta_2^1 + (\gamma_0^1)^2\delta_5^1 + \gamma_1^1\alpha_0^2 - \delta_1^1\beta_0^2 \\
&- \delta_0^1\delta_1^2 + \gamma_1^1\gamma_0^2 + \gamma_0^1\delta_2^2 - 6\beta_0^1\gamma_1^1\delta_0^1 t_{10} - 2(\beta_0^1)^2\gamma_1^1 t_{10} \\
&+ \beta_0^1\beta_1^1\gamma_0^1 t_{10} + 4\gamma_1^1(\delta_0^1)^2 t_{10} - 3\gamma_0^1\gamma_2^1\delta_0^1 t_{10} - \gamma_0^1\beta_0^2 t_{10} - (\gamma_0^1)^2\delta_0^1 t_{11} \\
&+ \gamma_0^1(\delta_0^1)^2 t_{20} + \delta_3^1(\delta_0^1)^2 + \gamma_0^1\gamma_1^1\gamma_2^1 + \delta_2^1\alpha_0^2 - 3\delta_0^1\alpha_0^2 t_{10} \\
&+ \delta_2^1\gamma_0^2 - 3\delta_0^1\gamma_0^2 t_{10} - \beta_0^3 + \delta_0^3 + 9\beta_0^1(\delta_0^1)^2 t_{10}^2 + 6(\beta_0^1)^2\delta_0^1 t_{10}^2 \\
&- 3(\delta_0^1)^3 t_{10}^2 - 3\beta_0^1\beta_2^1\delta_0^1 t_{10} - 6\beta_0^1\delta_2^1\delta_0^1 t_{10} - 2(\beta_0^1)^2\delta_2^1 t_{10} + \delta_2^1(\delta_0^1)^2 t_{10}),
\end{aligned}
$$

and since the rank of the Jacobian matrix of the function $\mathcal{B} = (B_0, B_1, B_2, B_3)$ with respect to its variables is maximal, B_i, $i = 0, \ldots, 3$ are linearly independent in their variables.

Hence $f_2(r) = 0$ has at most 3 solutions in D, see Theorem 5. Applying Theorem 1 it is proved that at most 3 limit cycles can bifurcate from the origin of system (1.8) with $p(x, y)$ of the form (1.9), using the averaging theory of order 2. Solving B_3 for t_{02}, B_2 for α_4^1, B_1 for β_2^2 and B_0 for δ_0^3 we obtain $f_2(r) = 0$, and we can apply the averaging theory of order 3, which corresponding averaged function is of the form

$$
rf_3(r) = C_4r^4 + C_3r^3 + C_2r^2 + C_1r + C_0,
$$

and C_i for $i = 0, \dots, 4$ are linearly independent in their variables, because the rank of the Jacobian matrix of the function $\mathcal{C} = (C_0, \dots, C_4)$ with respect to its variables is maximal. We do not explicitly provide their expressions, since they are very long. Therefore $f_3(r) = 0$ has at most 4 solutions in D, by Theorem 5. Applying Theorem 1 it is proved that at most 4 limit cycles can bifurcate from the origin of system (1.8) with $p(x, y)$ of the form (1.9) using the averaging theory of order 3. By conveniently choosing variables to cancel the coefficients $C_i, i = 0, \dots, 4$ we have $f_3(r) = 0$. Hence we apply the averaging theory of order 4 to obtain the averaged function of order 4

$$rf_4(r) = D_6r^6 + D_5r^5 + D_4r^4 + D_3r^3 + D_2r^2 + D_1r + D_0.$$

Since the rank of the Jacobian matrix of the function $\mathcal{D} = (D_0, \dots, D_6)$ with respect to its variables is maximal, the coefficients D_i, $i = 0, \dots, 6$ are linearly independent in their variables. Their expressions are very long so we do not provide them here. As a result of these calculations, it follows that $f_4(r) = 0$ has at most 6 solutions in D by Theorem 5. Applying Theorem 1 we conclude that at most 6 limit cycles can bifurcate from the origin of system (1.8) with $p(x, y)$ of the form (1.9), using the averaging theory of order 4. This result is a lower bound for $H_d(4)$, hence Theorem 6 is proved.

4. Proof of Theorem 7

First we consider the systems of the form (1.8) with $p(x, y)$ of the form (1.10). According to Theorem 3, the corresponding unperturbed system has a uniform isochronous center at the origin. In order to study the Hopf bifurcation for this case, we apply the results obtained in the proof of Theorem 6, by conveniently vanishing the coefficients of (1.9), used in that proof. More precisely, we take $t_{01} = t_{20} = t_{02} = t_{21} = t_{03} = 0$.

We also consider the systems of the form (1.8), with $p(x, y)$ of the form (1.11), whose corresponding unperturbed system also has a uniform isochronous center at the origin, see Theorem 4. Again, we use the results obtained in the proof of Theorem 6, vanishing the appropriate coefficients of (1.9), that is, we take $t_{01} = t_{10} = t_{20} = t_{11} = t_{02} = 0$.

Considering the above restrictions to the coefficients of $p(x, y)$ we obtain the averaged functions $f_i, i = 1, \dots, 4$ and since they are similar to those calculated in the proof of Theorem 6 we do not explicitly present them here. It is interesting to observe that the same number of limit cycles in each averaging order was obtained with $p(x, y)$ of the form (1.10) and (1.11).

The following Table 1 summarizes the results obtained in this proof and in the proof of Theorem 6.

It follows that if system (1.8) has 6 limit cycles up to the averaging theory of order 4, then it must have a weak focus at the origin.

Averaging order	# limit cycles	
	Theorem 6	Theorem 7 with $p(x,y)$ given by (1.10) or (1.11)
1	1	1
2	3	2
3	4	4
4	6	5

TABLE 1. Number of limit cycles for discontinuous differential systems (1.8).

5. Proof of Theorem 8

Consider system (1.8) with $p(x,y)$ of the form (1.10) and take $\alpha_0^j = \beta_0^j = \gamma_0^j = \delta_0^j = 0$, for $j = 1,\dots,7$. In this case the corresponding unperturbed system has a uniform isochronous center at the origin, see Theorem 3. In order to analyze the Hopf bifurcation for this case, applying Theorem 1, we set $\alpha = \pi$ and we introduce a small parameter ε doing the rescaling $x = \varepsilon X,\ y = \varepsilon Y$. After that doing the polar change of coordinates $X = r\cos\theta, Y = r\sin\theta$ and a Taylor expansion truncated at the 7^{th} order in ε we obtain an expression for $dr/d\theta$ of the form (1.1), with $\alpha = \pi$. The explicit expression is quite large so we omit it. All hypotheses for applying Theorem 1 to this case are satisfied using similar arguments to those presented for the proof of Theorem 6.

We obtain each y_i^+ and $y_i^-, i = 1\dots,7$ applying expression (1.3) respectively for X_1 and X_2 of system (1.8), after the changes previously described. Then we calculate the averaged functions $f_i, i = 1\dots,7$ using equation (1.2). We remark that, up to the averaging theory of order 4, the results in this case can be easily obtained from those already calculated in the proof of Theorem 7, taking into account the condition $\alpha_0^j = \beta_0^j = \gamma_0^j = \delta_0^j = 0$, $j = 1,\dots,7$, so we do not explicitly present the averaging functions from order 1 to 3 here. Starting from the averaged function of order 4 we have

$$f_4(r) = R_4r^4 + R_3r^3 + R_2r^2 + R_1r,$$

and R_i for $i = 1,\dots,4$ are linearly independent in their variables, since the rank of the Jacobian matrix of the function $\mathcal{R} = (R_1,\dots,R_4)$ with respect to its variables is maximal. We do not explicitly provide their expressions, because they are very long. Therefore $f_4(r) = 0$ has at most 3 solutions in D, by Theorem 5. Applying Theorem 1 it is proved that at most 3 limit cycles can bifurcate from the origin of system (1.8) with $p(x,y)$ of the form (1.10), and $\alpha_0^j = \beta_0^j = \gamma_0^j = \delta_0^j = 0$, $j = 1,\dots,7$ using the averaging theory of order 4.

The next averaging functions are calculated in a similar way, so we obtain

$$f_5(r) = S_5r^5 + S_4r^4 + S_3r^3 + S_2r^2 + S_1r,$$

and S_i for $i = 1,\dots,5$ are linearly independent in their variables,

$$f_6(r) = T_6r^6 + T_5r^5 + T_4r^4 + T_3r^3 + T_2r^2 + T_1r,$$

and T_j for $j = 1, \ldots, 6$ are linearly independent in their variables,

$$f_7(r) = U_7 r^7 + U_6 r^6 + U_5 r^5 + U_4 r^4 + U_3 r^3 + U_2 r^2 + U_1 r,$$

and U_k for $k = 1, \ldots, 7$ are linearly independent in their variables. The expressions of S_i, $i = 1, \ldots, 5$, T_j, $j = 1, \ldots, 6$ and U_k, $k = 1, \ldots, 7$ are very long so we do not provide them here.

Thus $f_5(r) = 0$, $f_6(r) = 0$ and $f_7(r) = 0$ has at most 4, 5 and 6 solutions in D, respectively, see Theorem 5. Applying Theorem 1 we conclude that at most 4, 5, and 6 limit cycles can bifurcate from the origin of system (1.8) with $p(x, y)$ of the form (1.10), and $\alpha_0^j = \beta_0^j = \gamma_0^j = \delta_0^j = 0$, $j = 1, \ldots, 7$ using the averaging theory of order 5, 6 and 7, respectively. Therefore Theorem 8 is proved.

The following Table 2 summarizes our results for this case:

Averaging order	# limit cycles
1	0
2	1
3	2
4	3
5	4
6	5
7	6

TABLE 2. Limit cycles for quartic discontinuous differential systems with a uniform isochronous center at the origin.

6. Proof of Theorems 9 and 10

System (1.1) becomes continuous by taking $\alpha = 2\pi$ and therefore the averaging theory developed in subsection 1.1 also applies to continuous differential systems.

First, consider the continuous differential system (1.7) with $p(x, y)$ of the form (1.9). In order to study the limit cycles for this system we only need the expressions of $y_i^+, i = 1. \ldots, 4$, which were already calculated for studying the previous cases. Hence, the averaged functions $f_i, i = 1. \ldots, 4$ can be obtained by the same algorithm used for the discontinuous differential systems, by taking $\alpha = 2\pi$.

The unperturbed continuous differential system corresponding to the perturbed system (1.7), with either $p(x, y)$ of the form (1.10) or (1.11) has a uniform isochronous center at the origin, according to Theorems 3 and 4, respectively. We apply the same arguments as in the previous paragraph, by taking $\alpha = 2\pi$ and using the expressions of $y_i^+, i = 1. \ldots, 4$ calculated in the proof of Theorem 7 to obtain the averaged functions $f_i, i = 1. \ldots, 4$ for this case. We remark that the same number of limit cycles was obtained in both cases where $p(x, y)$ is either of the form (1.10) or (1.11), in each averaging order studied.

Since the calculations and arguments are quite similar to those used in the previous proofs, we omit the explicit expressions of the averaged functions. We summarize our results in the following Table 3.

Averaging order	# limit cycles	
	general case	Uniform center
1	0	0
2	1	0
3	1	1
4	2	1

TABLE 3. Number of limit cycles for continuous differential systems (1.7).

We remark that from this proof, it follows that system (1.7) with $p(x, y)$ of the form (1.9) has a weak focus at the origin provided that it has 2 limit cycles up to the averaging theory of order 4.

7. Appendix

$$
\begin{aligned}
y_1^{\pm}(\theta,\rho) &= \int_0^{\theta} F_1^{\pm}(\phi,\rho)\,d\phi,\\
y_2^{\pm}(\theta,\rho) &= \int_0^{\theta} \big(2F_2^{\pm}(\phi,\rho) + 2\partial F_1^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)\big)\,d\phi,\\
y_3^{\pm}(\theta,\rho) &= \int_0^{\theta} \big(6F_3^{\pm}(\phi,\rho) + 6\partial F_2^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho) + 3\partial^2 F_1^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^2\\
&\qquad + 3\partial F_1^{\pm}(\phi,\rho)\,y_2^{\pm}(\phi,\rho)\big)\,d\phi,\\
y_4^{\pm}(\theta,\rho) &= \int_0^{\theta} \big(24F_4^{\pm}(\phi,\rho) + 24\partial F_3^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho) + 12\partial^2 F_2^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^2\\
&\qquad + 12\partial F_2^{\pm}(\phi,\rho)y_2^{\pm}(\phi,\rho) + 12\partial^2 F_1^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)y_2^{\pm}(\phi,\rho)\\
&\qquad + 4\partial^3 F_1^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^3 + 4\partial F_1^{\pm}(\phi,\rho)y_3^{\pm}(\phi,\rho)\big)\,d\phi,\\
y_5^{\pm}(\theta,\rho) &= \int_0^{\theta} \big(120F_5^{\pm}(\phi,\rho) + 120\partial F_4^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)\\
&\qquad + 60\partial^2 F_3^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^2 + 60\partial F_3^{\pm}(\phi,\rho)y_2^{\pm}(\phi,\rho)\\
&\qquad + 60\partial^2 F_2^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)y_2^{\pm}(\phi,\rho) + 20\partial^3 F_2^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^3\\
&\qquad + 20\partial F_2^{\pm}(\phi,\rho)y_3^{\pm}(\phi,\rho) + 20\partial^2 F_1^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)y_3^{\pm}(\phi,\rho)\\
&\qquad + 15\partial^2 F_1^{\pm}(\phi,\rho)y_2^{\pm}(\phi,\rho)^2 + 30\partial^3 F_1^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^2 y_2^{\pm}(\phi,\rho)\\
&\qquad + 5\partial^4 F_1^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^4 + 5\partial F_1^{\pm}(\phi,\rho)y_4^{\pm}(\phi,\rho)\big)\,d\phi,\\
y_6^{\pm}(\theta,\rho) &= \int_0^{\theta} \big(720F_6^{\pm}(\phi,\rho) + 720\partial F_5^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)\\
&\qquad + 360\partial^2 F_4^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^2 + 360\partial F_4^{\pm}(\phi,\rho)y_2^{\pm}(\phi,\rho)\\
&\qquad + 120\partial^3 F_3^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^3 + 360\partial^2 F_3^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)y_2^{\pm}(\phi,\rho)
\end{aligned}
$$

$$
\begin{aligned}
&+ 120\partial F_3^{\pm}(\phi,\rho)y_3^{\pm}(\phi,\rho) + 30\partial^4 F_2^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^4\\
&+ 180\partial^3 F_2^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^2 y_2^{\pm}(\phi,\rho) + 120\partial^2 F_2^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)y_3^{\pm}(\phi,\rho)\\
&+ 90\partial^2 F_2^{\pm}(\phi,\rho)y_2^{\pm}(\phi,\rho)^2 + 30\partial F_2^{\pm}(\phi,\rho)y_4^{\pm}(\phi,\rho)\\
&+ 60\partial^4 F_1^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^3 y_2^{\pm}(\phi,\rho) + 60\partial^3 F_1^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^2 y_3^{\pm}(\phi,\rho)\\
&+ 90\partial^3 F_1^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)y_2^{\pm}(\phi,\rho)^2 + 30\partial^2 F_1^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)y_4^{\pm}(\phi,\rho)\\
&+ 60\partial^2 F_1^{\pm}(\phi,\rho)y_2^{\pm}(\phi,\rho)y_3^{\pm}(\phi,\rho) + 6\partial^5 F_1^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^5\\
&+ 6\partial F_1^{\pm}(\phi,\rho)y_5^{\pm}(\phi,\rho)\big)\, d\phi,
\end{aligned}
$$

$$
\begin{aligned}
y_7^{\pm}(t,\rho) = \int_0^t &\big(5040 F_7^{\pm}(\phi,\rho) + 5040\partial F_6^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)\\
&+ 2520\partial^2 F_5^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^2 + 2520\partial F_5^{\pm}(\phi,\rho)y_2^{\pm}(\phi,\rho)\\
&+ 2520\partial^2 F_4^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)y_2^{\pm}(\phi,\rho) + 840\partial^3 F_4^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^3\\
&+ 840\partial F_4^{\pm}(\phi,\rho)y_3^{\pm}(\phi,\rho) + 840\partial^2 F_3^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)y_3^{\pm}(\phi,\rho)\\
&+ 630\partial^2 F_3^{\pm}(\phi,\rho)y_2^{\pm}(\phi,\rho)^2 + 1260\partial^3 F_3^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^2 y_2^{\pm}(\phi,\rho)\\
&+ 210\partial^4 F_3^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^4 + 210\partial F_3^{\pm}(\phi,\rho)y_4^{\pm}(\phi,\rho)\\
&+ 210\partial^2 F_2^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)y_4^{\pm}(\phi,\rho) + 420\partial^3 F_2^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^2 y_3^{\pm}(\phi,\rho)\\
&+ 420\partial^4 F_2^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^3 y_2^{\pm}(\phi,\rho) + 630\partial^3 F_2^{\pm}(\phi,\rho)y_2^{\pm}(\phi,\rho)^2 y_1^{\pm}(\phi,\rho)\\
&+ 42\partial^5 F_2^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^5 + 420\partial^2 F_2^{\pm}(\phi,\rho)y_2^{\pm}(\phi,\rho)y_3^{\pm}(\phi,\rho)\\
&+ 42\partial F_2^{\pm}(\phi,\rho)y_5^{\pm}(\phi,\rho) + 630\partial^3 F_2^{\pm}(\phi,\rho)y_2^{\pm}(\phi,\rho)^2 y_1^{\pm}(\phi,\rho)\\
&+ 7\partial^6 F_1^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^6 + 105\partial^5 F_1^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^4 y_2^{\pm}(\phi,\rho)\\
&+ 140\partial^4 F_1^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^3 y_3^{\pm}(\phi,\rho) + 630\partial^4 F_1^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^2 y_2^{\pm}(\phi,\rho)^2\\
&+ 105\partial^3 F_1^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)^2 y_4^{\pm}(\phi,\rho) + 42\partial^2 F_1^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)y_5^{\pm}(\phi,\rho)\\
&+ 420\partial^3 F_1^{\pm}(\phi,\rho)y_1^{\pm}(\phi,\rho)y_2^{\pm}(\phi,\rho)y_3^{\pm}(\phi,\rho)\\
&+ 105\partial^3 F_1^{\pm}(\phi,\rho)y_2^{\pm}(\phi,\rho)^3 + 105\partial^2 F_1^{\pm}(\phi,\rho)y_2^{\pm}(\phi,\rho)y_4^{\pm}(\phi,\rho)\\
&+70\partial^2 F_1^{\pm}(\phi,\rho)y_3^{\pm}(\phi,\rho)^2 + 7\partial F_1^{\pm}(\phi,\rho)y_6^{\pm}(\phi,\rho)\big)\, d\phi.
\end{aligned}
$$

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