



On the parabolic Lipschitz approximation of parabolic uniform rectifiable sets

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Abstract. We prove the existence of big pieces of regular parabolic Lipschitz graphs for a class of parabolic uniform rectifiable sets satisfying what we call a synchronized two cube condition. An application to the fine properties of parabolic measure is given.

1. Introduction and statement of main results

A by now classical result due to G. David and D. Jerison [2] states that if $\Sigma \subset \mathbb{R}^n$ is a closed set which is Ahlfors–David regular with respect to the restriction of the $(n - 1)$ -dimensional Hausdorff measure to Σ , σ , and if Σ satisfies what they call a two disc condition, then Σ contains big pieces of Lipschitz graphs in a sense which has to be made precise, see [2]. This result has many applications and implications and we here briefly discuss two. First, if Ω is one component of $\mathbb{R}^n \setminus \Sigma$, and if, in addition, Ω is an NTA-domain in the sense of [11], then the harmonic measure on $\partial\Omega$ is in the Muckenhoupt class A_∞ with respect to σ . Second, combining [2] with the monumental work of G. David and S. Semmes [4], see also [3], one can, if Σ is as above, conclude the equivalences between the statements

- (1.1) (i) Σ is uniformly rectifiable,
 (ii) Σ is good for (elliptic) singular integrals,
 (iii) Σ has a Corona decomposition.

We refer to [4] for definitions and precise statements of (1.1) and it is fair to say that [4] remains a source of continuous inspiration for anyone interested in the analysis of and on uniformly rectifiable sets.

This paper is the first in a sequence of two papers, motivated by applications to the analysis of parabolic partial differential equations on domains not locally given by graphs, devoted to parabolic versions of the above results. In this paper

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we prove that if $\Sigma \subset \mathbb{R}^{n+1}$ is a closed set which is Ahlfors–David regular in the parabolic sense, see Definition 1.1, and if Σ satisfies what we call a synchronized two cube condition, see Definition 1.9, then Σ contains big pieces of $\text{Lip}(1, 1/2)$ graphs in a sense made precise below. Furthermore, if in addition Σ is uniformly rectifiable in the parabolic sense, see Definition 1.3, then we prove that Σ contains big pieces of regular parabolic Lipschitz graphs, graphs which are slightly more regular compared to $\text{Lip}(1, 1/2)$ graphs and which have turned out to be the appropriate graphs from the point of view of parabolic singular integrals and parabolic measure. Building on our results concerning big pieces we in this paper also give an implication for parabolic measure, defined in a component of $\mathbb{R}^n \setminus \Sigma$. In the subsequent paper, see [16], the goal is to establish, if Σ is as above, the equivalences between the statements

- (i') Σ is uniformly rectifiable in the parabolic sense,
- (ii') Σ is good for (parabolic) singular integrals,
- (iii') Σ has a parabolic Corona decomposition.

In general the theory of parabolic boundary value problems, and the analysis of and on parabolic uniformly rectifiable sets, is less developed compared to the elliptic counterparts. The main results in the field are due to Hofmann, Lewis, Murray, Silver, see [6], [7], [8], [12], [13] and Hofmann, Lewis, Nyström, see [9], [10]. In particular, in [6], [7], [12], [13], the authors establish the correct notion of (time-dependent) regular parabolic Lipschitz graphs from the point of view of parabolic singular integrals and parabolic measure. In [9], [10] the first author, together with Hofmann and Lewis, introduced the correct notion of parabolic uniformly rectifiable sets and proved the existence of big pieces of regular parabolic Lipschitz graphs under the additional assumption that Σ is Reifenberg flat in the parabolic sense. Apart from these results the field is essentially open and as such we claim that this paper and our subsequent paper [16] advance the field substantially. In general we believe that our results will pave the way for further developments of boundary and free boundary problems for parabolic partial differential equations on domains not locally given by graphs. The ambition is to pursue this line of research in subsequent papers.

1.1. Notation

Points in Euclidean $(n + 1)$ -space \mathbb{R}^{n+1} are denoted by $(X, t) = (x_1, \dots, x_n, t)$, where $X = (x_1, \dots, x_n) \in \mathbb{R}^n$ and t represents the time-coordinate. We will always assume that $n \geq 1$. We let $\bar{E}, \partial E$, be the closure and boundary of the set $E \subset \mathbb{R}^{n+1}$. $\langle \cdot, \cdot \rangle$ denotes the standard inner product on \mathbb{R}^n and we let $|X| = \langle X, X \rangle^{1/2}$ be the Euclidean norm of X . We let $\|(X, t)\| := |X| + |t|^{1/2}$. Given $(X, t), (Y, s) \in \mathbb{R}^{n+1}$ we let $d_p(X, t, Y, s) = |X - Y| + |t - s|^{1/2}$ and we define $d(X, t, E)$ to equal the parabolic distance, defined with respect to $d_p(\cdot, \cdot)$, from $(X, t) \in \mathbb{R}^{n+1}$ to E . We let

$$Q_r(X, t) = \{(Y, s) \in \mathbb{R}^{n+1} : |y_i - x_i| < r, |t - s| < r^2\},$$

whenever $(X, t) \in \mathbb{R}^{n+1}$, $r > 0$, and we call $Q_r(X, t)$ a parabolic cube of size r . We let dx denote Lebesgue n -measure on \mathbb{R}^n and, given an integer $k \geq 1$, we let \mathcal{H}^k

denote the k -dimensional Hausdorff measure. Given a Borel set $E \subset \mathbb{R}^{n+1}$ we let $\sigma(E) = \int_E d\sigma_t dt$ where $d\sigma_t$ denotes the restriction of the $(n - 1)$ -dimensional Hausdorff measure to the time slice $E \cap (\mathbb{R}^n \times \{t\})$. $\Sigma \subset \mathbb{R}^{n+1}$ will denote a closed set. Below we will, for $(X, t) \in \Sigma$ and $r > 0$, use the notation $\Delta(X, t, r) = \Sigma \cap Q_r(X, t)$. The extremal time coordinates of Σ will be denoted by $T_0 = \inf\{t : \exists(X, t) \in \Sigma\}$ and $T_1 = \sup\{t : \exists(X, t) \in \Sigma\}$. When we consider a domain Ω defined as a connected component of $\mathbb{R}^{n+1} \setminus \Sigma$, T_0 and T_1 will be defined relative to $\partial\Omega$.

1.2. Regular parabolic graphs

A function $\psi : \mathbb{R}^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}$ is called $\text{Lip}(1, 1/2)$, or $\text{Lip}(1, 1/2)$ regular, with constant M , if

$$(1.3) \quad |\psi(x, t) - \psi(y, s)| \leq M(|x - y| + |t - s|^{1/2})$$

whenever $(x, t) \in \mathbb{R}^n$, $(y, s) \in \mathbb{R}^n$. We call $\Omega \subset \mathbb{R}^{n+1}$ a (unbounded) $\text{Lip}(1, 1/2)$ graph domain, with constant M , if

$$(1.4) \quad \Omega = \Omega_\psi = \{(x, x_n, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} : x_n > \psi(x, t)\}$$

for some $\text{Lip}(1, 1/2)$ function ψ having $\text{Lip}(1, 1/2)$ constant bounded by M . We say that $\psi = \psi(x, t) : \mathbb{R}^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}$ is a regular parabolic $\text{Lip}(1, 1/2)$ function with parameters b_1 and b_2 , if ψ has compact support and satisfies

$$(1.5) \quad \begin{aligned} & \text{(i)} \quad |\psi(x, t) - \psi(y, t)| \leq b_1|x - y|, x, y \in \mathbb{R}^{n-1}, t \in \mathbb{R}, \\ & \text{(ii)} \quad D_{1/2}^t \psi \in \text{BMO}(\mathbb{R}^n), \quad \|D_{1/2}^t \psi\|_* \leq b_2 < \infty. \end{aligned}$$

Here $D_{1/2}^t \psi(x, t)$ denotes the $1/2$ derivative in t of $\psi(x, \cdot)$, x fixed. This half derivative in time can be defined by way of the Fourier transform or by

$$(1.6) \quad D_{1/2}^t \psi(x, t) \equiv \hat{c} \int_{\mathbb{R}} \frac{\psi(x, s) - \psi(x, t)}{|s - t|^{3/2}} ds$$

for properly chosen \hat{c} . $\|\cdot\|_*$ denotes the norm in parabolic $\text{BMO}(\mathbb{R}^n)$. For a definition of the space $\text{BMO}(\mathbb{R}^n)$ we refer to [9]. It is well known, see [7], that if ψ is a regular parabolic $\text{Lip}(1, 1/2)$ function with parameters b_1 and b_2 , then ψ is $\text{Lip}(1, 1/2)$ regular with constant $M = M(b_1, b_2)$. However, there are examples of functions ψ which are $\text{Lip}(1, 1/2)$ regular but not regular parabolic $\text{Lip}(1, 1/2)$, see [13]. We call $\Omega \subset \mathbb{R}^{n+1}$ a (unbounded) regular parabolic $\text{Lip}(1, 1/2)$ graph domain, with constants (b_1, b_2) , if (1.4) holds for some regular parabolic $\text{Lip}(1, 1/2)$ function ψ having constants (b_1, b_2) .

1.3. Parabolic uniform rectifiability

Definition 1.1. Let $\Sigma \subset \mathbb{R}^{n+1}$ be a closed set. We say that Σ is parabolic Ahlfors–David regular, parabolic ADR for short, with constant M , $M \geq 1$, if

$$(1.7) \quad M^{-1} r^{n+1} \leq \sigma(\Delta(X, t, r)) \leq M r^{n+1},$$

whenever $0 < r < \text{diam } \Sigma$, $(X, t) \in \Sigma$, $T_0 < t < T_1$ and where $\text{diam } \Sigma$ is the (parabolic) diameter of Σ (which may be infinite).

Definition 1.2. Assume that $\Sigma \subset \mathbb{R}^{n+1}$ is parabolic ADR in the sense of Definition 1.1. Let

$$\gamma(Z, \tau, r) = \inf_P \left(r^{-n-1} \int_{\Delta(Z, \tau, r)} \left(\frac{d(Y, s, P)}{r} \right)^2 d\sigma(Y, s) \right)^{1/2},$$

whenever $(Z, \tau) \in \Sigma$, $r > 0$, and where the infimum is taken with respect to all n dimensional planes P containing a line parallel to the t axis. Let

$$(1.8) \quad d\nu(Z, \tau, r) = \gamma(Z, \tau, r) d\sigma(Z, \tau) r^{-1} dr.$$

We say that ν is a Carleson measure on $\Delta(Y, s, R) \times (0, R)$ if there exists $\tilde{M} < \infty$ such that

$$(1.9) \quad \nu(\Delta(X, t, \rho) \times (0, \rho)) \leq \tilde{M} \rho^{n+1},$$

whenever $(X, t) \in \Sigma$ and $Q_\rho(X, t) \subset Q_R(Y, s)$. The least such \tilde{M} in (1.9) is called the Carleson norm of $\Delta(Y, s, R) \times (0, R)$.

Definition 1.3. Assume that $\Sigma \subset \mathbb{R}^{n+1}$ is parabolic ADR in the sense of Definition 1.1 with constant M . Let ν be defined as in (4.1). Then Σ is parabolic uniformly rectifiable, parabolic UR for short, with UR constants (M, \tilde{M}) if

$$(1.10) \quad \|\nu\|_+ := \sup_{(X, t) \in \Sigma, \rho > 0} \rho^{-n-1} \nu(\Delta(X, t, \rho) \times (0, \rho)) \leq \tilde{M}.$$

1.4. Uniform big pieces

Assume that $\Sigma \subset \mathbb{R}^{n+1}$ is parabolic ADR in the sense of Definition 1.1. Let in the following π denote the orthogonal projection onto the plane $\{(x, x_n, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} : x_n = 0\}$. At instances we identify \mathbb{R}^n with $\mathbb{R}^{n-1} \times \{0\} \times \mathbb{R}$, and put

$$I_r(z, \tau) = \{(y, s) \in \mathbb{R}^n : |y_i - z_i| < r, i = 1, \dots, n - 1, |s - \tau| < r^2\}$$

for $(z, \tau) \in \mathbb{R}^n$, $r > 0$.

Definition 1.4. We say that Σ contains uniform big pieces of $\text{Lip}(1, 1/2)$ graphs with constants (ϵ, \hat{M}) if the following hold. Given $(X, t) \in \Sigma$, $T_0 < t < T_1$ and $0 < R < \text{diam } \Sigma$, there exists, after a possible rotation in the space variable, a $\text{Lip}(1, 1/2)$ function ψ with constant \hat{M} , and $\epsilon > 0$, such that

$$(1.11) \quad \mathcal{H}^n(\pi(\Sigma_\psi \cap \Delta(X, t, R))) \geq \epsilon R^{n+1},$$

where

$$(1.12) \quad \Sigma_\psi = \{(x, x_n, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} : x_n = \psi(x, t)\}.$$

Definition 1.5. We say that Σ contains uniform big pieces of regular parabolic $\text{Lip}(1, 1/2)$ graphs with constants (ϵ, b_1, b_2) if, whenever $(X, t) \in \Sigma$, $T_0 < t < T_1$ and $0 < R < \text{diam } \Sigma$, (1.11) and (1.12) hold with a regular parabolic $\text{Lip}(1, 1/2)$ function ψ , satisfying (1.5) with constants b_1, b_2 , and for $\epsilon > 0$.

Definition 1.6. Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain, i.e., a connected open set, defined as a connected component of $\mathbb{R}^{n+1} \setminus \Sigma$. We then say that $\partial\Omega$ satisfies a uniform interior big pieces of $\text{Lip}(1, 1/2)$ graphs condition with constants (ϵ, \hat{M}, C) if the following hold. Given $(\hat{X}, \hat{t}) \in \Omega$, we can find a domain $\tilde{\Omega}$ and $C \geq 1$ such that the following holds with $d := d(\hat{X}, \hat{t}, \partial\Omega)$.

- (i) $\tilde{\Omega} \subset \Omega \cap Q_{Cd}(\hat{X}, \hat{t})$.
- (ii) After a possible rotation in the space variables we have

$$\tilde{\Omega} = \{(y, y_n, s) : (y, s) \in I_{\hat{M}^{-1}d}(\hat{x}, \hat{t}), \psi(y, s) < y_n < \hat{x}_n + d/2\},$$
 where $(\hat{X}, \hat{t}) = (\hat{x}, \hat{x}_n, \hat{t})$ and for some $\text{Lip}(1, 1/2)$ function ψ with constant \hat{M} .
- (iii) $\mathcal{H}^n(\pi(\partial\Omega \cap \partial\tilde{\Omega}) \cap I_{\hat{M}^{-1}d}(\hat{x}, \hat{t})) \geq \epsilon d^{n+1}$.

Definition 1.7. Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain, i.e., a connected open set, defined as a connected component of $\mathbb{R}^{n+1} \setminus \Sigma$. We then say that $\partial\Omega$ satisfies a uniform interior big pieces of regular parabolic $\text{Lip}(1, 1/2)$ graphs condition with constants (ϵ, b_1, b_2, C) if the following hold. Given $(\hat{X}, \hat{t}) \in \Omega$, we can find a domain $\tilde{\Omega}$ such that (i)–(iii) of Definition 1.6 hold for some regular parabolic $\text{Lip}(1, 1/2)$ function ψ with constants (b_1, b_2) and for some constant $\hat{M} = \hat{M}(b_1, b_2)$.

Remark 1.8. Note that (1.11) implies, as Hausdorff measure does not increase under projections, that

$$(1.14) \quad \sigma(\Sigma_\psi \cap \Delta(X, t, R)) \geq \epsilon R^{n+1}.$$

1.5. Statement of main results – existence of big pieces

Definition 1.9. Assume that $\Sigma \subset \mathbb{R}^{n+1}$ is parabolic ADR in the sense of Definition 1.1 with constant M . Let $\gamma_0 \geq 1$ be given. We say that Σ satisfies a synchronized two cube condition with constant γ_0 if there exist, for all $(X, t) \in \Sigma$, $T_0 < t < T_1$ and $0 < r < \text{diam } \Sigma$, two parabolic cubes $Q_\rho(X_1, t_1)$ and $Q_\rho(X_2, t_2)$ such that $Q_\rho(X_1, t_1) \cap (\mathbb{R}^n \times (T_0, T_1))$ and $Q_\rho(X_2, t_2) \cap (\mathbb{R}^n \times (T_0, T_1))$ belong to different connected components of $\mathbb{R}^{n+1} \setminus \Sigma$, with

$$\gamma_0^{-1}r < \rho < r, \quad t_1 = t = t_2, \quad \|(X, t) - (X_1, t)\| \leq r, \quad \|(X, t) - (X_2, t)\| \leq r.$$

Definition 1.10. Assume that $\Sigma \subset \mathbb{R}^{n+1}$ is parabolic ADR in the sense of Definition 1.1 with constant M . Let $\gamma_0 \geq 1$ be given. Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain defined as a connected component of $\mathbb{R}^{n+1} \setminus \Sigma$. We say that $\partial\Omega$ satisfies a synchronized two cube condition with constant γ_0 , if there exist, for all $(X, t) \in \partial\Omega$, $T_0 < t < T_1$ and $0 < r < \text{diam } \Sigma$, two parabolic cubes $Q_\rho(X_1, t_1)$, $Q_\rho(X_2, t_2)$, such that $Q_\rho(X_1, t_1) \cap (\mathbb{R}^n \times (T_0, T_1)) \subset \Omega$, $Q_\rho(X_2, t_2) \cap (\mathbb{R}^n \times (T_0, T_1)) \subset \mathbb{R}^{n+1} \setminus \tilde{\Omega}$, satisfying

$$\gamma_0^{-1}r < \rho < r, \quad t_1 = t = t_2, \quad \|(X, t) - (X_1, t)\| \leq r, \quad \|(X, t) - (X_2, t)\| \leq r.$$

Remark 1.11. We use the term a synchronized two cube condition to emphasize that if $(X, t) \in \Sigma$, then the t -coordinates of the centers of the cylinders $Q_\rho(X_1, t_1)$, $Q_\rho(X_2, t_2)$ coincide with t .

Concerning big pieces, the following two theorems are our main results.

Theorem 1.12. *Let Σ be a closed subset of \mathbb{R}^{n+1} which is parabolic ADR with constant M . Assume that Σ satisfies a synchronized two cube condition with constant $\gamma_0 \geq 1$, in the sense of Definition 1.9. Then Σ contains uniform big pieces of $\text{Lip}(1, 1/2)$ graphs with constants (ϵ, \hat{M}) for some $\hat{M} = \hat{M}(n, M, \gamma_0)$, and $\epsilon = \epsilon(n, M, \gamma_0)$. Let $\Omega \subset \mathbb{R}^{n+1}$ be a connected component of $\mathbb{R}^{n+1} \setminus \Sigma$. Assume that $\partial\Omega$ satisfies a synchronized two cube condition with constant $\gamma_0 \geq 1$ in the sense of Definition 1.10. Then $\partial\Omega$ satisfies a uniform interior big pieces of $\text{Lip}(1, 1/2)$ graphs conditions with constants (ϵ, \hat{M}, C) for some $\epsilon = \epsilon(n, M, \gamma_0)$, $\hat{M} = \hat{M}(n, M, \gamma_0)$, and $C = C(n, M, \gamma_0)$.*

Theorem 1.13. *Let Σ be a closed subset of \mathbb{R}^{n+1} which is parabolic ADR with constant M . Assume that Σ satisfies a synchronized two cube condition with constant $\gamma_0 \geq 1$ in the sense of Definition 1.9. Assume in addition that Σ is UR with constants (M, \tilde{M}) . Then Σ contains uniform big pieces of regular parabolic $\text{Lip}(1, 1/2)$ graphs with constants (ϵ, b_1, b_2) for some $b_1 = b_1(n, M, \tilde{M}, \gamma_0)$, $b_2 = b_2(n, M, \tilde{M}, \gamma_0)$ and $\epsilon = \epsilon(n, M, \tilde{M}, \gamma_0)$. Let $\Omega \subset \mathbb{R}^{n+1}$ be a connected component of $\mathbb{R}^{n+1} \setminus \Sigma$. Assume that $\partial\Omega$ satisfies a synchronized two cube condition with constant $\gamma_0 \geq 1$ in the sense of Definition 1.10. Then $\partial\Omega$ satisfies a uniform interior big pieces of regular parabolic $\text{Lip}(1, 1/2)$ graphs conditions with constants (ϵ, b_1, b_2, C) for some $\epsilon = \epsilon(n, M, \gamma_0)$, $b_1 = b_1(n, M, \tilde{M}, \gamma_0)$, $b_2 = b_2(n, M, \tilde{M}, \gamma_0)$ and $C = C(n, M, \gamma_0)$.*

Remark 1.14. Note that Theorem 1.12 is a parabolic version of the main result in [2]. In Theorem 1 in [9] the conclusion of Theorem 1.13 was established assuming that Σ is parabolic ADR with constant M , parabolic UR and, in addition, that Σ has the separation property and is δ -Reifenberg flat for a sufficiently small δ . If Σ has the separation property and is δ -Reifenberg flat, then Σ satisfies a synchronized two cube condition. This implication can not be reversed. Hence Theorem 1.13 generalizes Theorem 1 in [9] beyond the hypothesis of Σ being Reifenberg flat.

Remark 1.15. In [17]–[19], the author claims several of the results that we establish in this paper and in [16]. However, in [17]–[19] the author, unfortunately, in general either gives no proofs of his claims or he supplies proofs which have obvious gaps. Here and in [16] we give complete proofs of our results.

1.6. Proofs and organization of the paper

The proof of Theorem 1.12 is given in Section 2. We prove Theorem 1.12 by adapting to the parabolic setting the original and quite ingenious argument of G. David and D. Jerison [2]. In Section 3 we prove Theorem 1.13 and to complete the proof we can, after some preliminaries, reuse important results from [9]. In Section 4 we

prove further results devoted to connectivity, big pieces and parabolic measure. In particular, in Theorem 4.3 we prove that connectivity of $\Omega \subset \mathbb{R}^{n+1}$, formulated in terms of a uniform Carleson chain condition, implies the synchronized two cube condition, and hence the existence of big pieces. In the elliptic setting, see [15], and [1], [14], this type of results have been proved to be useful. In Theorem 4.7, assuming the synchronized two cube condition and the Harnack chain property, see Definition 4.4 below, we formulate a consequence of our main results to the fine properties of parabolic measure.

2. The proof of Theorem 1.12

We here prove Theorem 1.12. We will only give the proof in the case when $\text{diam } \Sigma = \infty$, $T_0 = -\infty$ and $T_1 = \infty$. If T_0 or T_1 is finite, the proof will be completely analogous but all sets occurring have to be intersected with $\mathbb{R}^n \times (T_0, T_1)$ and the notation will be more cumbersome. Let Σ be a closed subset of \mathbb{R}^{n+1} which is parabolic ADR with constant M . Assume that Σ satisfies a synchronized two cube condition with constant γ_0 . We redefine

$$(2.1) \quad M \text{ to equal } \max\{M, \gamma_0, 4n\}.$$

Based on this we can in the following, without loss of generality, assume that Σ is parabolic ADR with constant M and that there exist, for all $(X, t) \in \Sigma$ and $r > 0$, two parabolic cubes $Q_\rho(X_1, t_1), Q_\rho(X_2, t_2)$, belonging to different connected components of $\mathbb{R}^{n+1} \setminus \Sigma$, with $t_1 = t = t_2, \|(X, t) - (X_1, t)\| \leq r, \|(X, t) - (X_2, t)\| \leq r$, and

$$(2.2) \quad \rho = M^{-1}r.$$

Consider $(X, t) \in \Sigma$ and $r > 0$. We first note, using (2.2) and as the statements and conclusions of Theorem 1.12 are invariant under parabolic scalings and Euclidean translations, that to prove Theorem 1.12 we can without loss of generality assume that

$$r = 2M \quad \text{and} \quad (X, t) = (0, 0).$$

Then, by construction there is a point $(Y, 0)$, with $\|(Y, 0)\| \leq 2M$, and a cube $Q_2(Y, 0)$ contained in a component \mathcal{U} of $\mathbb{R}^{n+1} \setminus \Sigma$. Furthermore, if we let $\rho_0 = (2(1+M))^{-1}$, then there is a point $(Z, 0)$, with $\|(Z, 0)\| \leq M\rho_0$, such that the cube $Q_{\rho_0}(Z, 0)$ is contained in a component of $\mathbb{R}^{n+1} \setminus \Sigma$ different from \mathcal{U} . Indeed, this follows if we apply the synchronized two cube condition at $(0, 0)$ and on scale $M\rho_0$. Consider the time-independent hyperplane which passes through $(0, 0)$ and which has (spatial) normal defined by Y . In the following we can assume, after a possible rotation in the spatial coordinates, if necessary, that any point $(X, t) \in \mathbb{R}^{n+1}$ is given as $(X, t) = (x, x_n, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$, and that $Y = (0, \bar{M})$, where $2 \leq \bar{M} \leq 2M$. We identify the hyperplane with $\mathbb{R}^{n-1} \times \{0\} \times \mathbb{R}$, we let π denote the orthogonal projection onto this plane and we let π^\perp denote the orthogonal projection onto the normal to this plane. We let

$$I_r(z, \tau) = \{(y, s) \in \mathbb{R}^n : |y_i - z_i| < r, i = 1, \dots, n - 1, |s - \tau| < r^2\},$$

whenever $(z, \tau) \in \mathbb{R}^n, r > 0$, we set $I^0 = I_1(0, 0)$ and introduce

$$(2.3) \quad I^{\bar{M}} := \{(x, x_n, t) : (x, t) \in I^0, x_n = \bar{M}\}.$$

By construction $I^{\bar{M}} \subset \mathcal{U}, d(I^{\bar{M}}, \Sigma) \geq 1$ and by the choice $\rho_0 = (2(1 + M))^{-1}$ we see that

$$D := \pi(Q_{\rho_0}(Z, 0)) \subset \frac{1}{2} I^0,$$

and that

$$(2.4) \quad \text{there exists } \gamma > 0, \text{ depending only on } n \text{ and } M,$$

such that

$$(2.5) \quad \sigma(D) \geq 2\gamma.$$

Note that any line in the x_n -direction connecting $D \times \{x_n = -M\}$ with $I^{\bar{M}}$ has to intersect $Q_{\rho_0}(Z, 0)$ and $Q_2(Y, 0)$, thus it also intersects Σ . Given a degree of freedom $h > 0$ we let

$$\Gamma = \Gamma_h = \{(X, t) \in \mathbb{R}^{n+1} : x_n \geq h\|(x, t)\|\},$$

i.e., Γ is a parabolic cone with aperture defined by h , and we let

$$S = \{(X, t) = (x, x_n, t) \in \Sigma : -M \leq x_n \leq \bar{M} \text{ and all } (Y, s) \in (X, t) + \Gamma \text{ with } y_1 = \bar{M} \text{ belong to } I^{\bar{M}}\}.$$

Let \bar{M} and M be as above. Below we will construct h subject to

$$h \geq 3M \geq \bar{M} + M.$$

Then, by construction we see that if $(X, t) = (x, x_n, t) \in \Sigma$, and if $(x, t) \in D$, then any $(Y, s) \in (X, t) + \Gamma$ such that $y_n = \bar{M}$, belongs to $I^{\bar{M}}$. In particular, $D \subset \pi(S)$ and thus by (2.5),

$$(2.6) \quad \mathcal{H}^n(\pi(S)) \geq 2\gamma.$$

To prove the first statement in Theorem 1.12 it therefore suffices to prove the following lemma.

Lemma 2.1. *Let γ be as in (2.4), (2.5). Then there exists $h > 0$, depending only on n and M , such that if we let $\Gamma = \Gamma_h$, and if we define*

$$W := \{(x, t) \in I^0 : \exists (X, t) = (x, x_n, t) \in S, ((X, t) + \Gamma) \cap S = \{(X, t)\}\},$$

then $\mathcal{H}^n(\pi(S) \setminus W) \leq \gamma$.

Indeed, using Lemma 2.1 we conclude that there is a $\text{Lip}(1, 1/2)$ graph G with constant h such that $W = \pi(\Sigma \cap G)$ satisfies $\mathcal{H}^n(\pi(S) \setminus W) \leq \gamma$, implying $\mathcal{H}^n(W) \geq \gamma$. Thus $\sigma(\Sigma \cap G \cap Q_{2M}(0, 0)) \geq \gamma$ and the proof of the first statement in Theorem 1.12 is complete. Note that by this we have inferred the existence of a $\text{Lip}(1, 1/2)$ function ψ such that

$$G \cap \{(X, t) \in G : \pi(X, t) \in I^0\} = \{(x, \psi(x, t), t) : (x, t) \in I^0\}.$$

Define $\tilde{\Omega}$ to be the set

$$(2.7) \quad \tilde{\Omega} = \{(X, t) : (x, t) = \pi(X, t) \in \text{int } I^0, \psi(x, t) < x_n < \bar{M}\},$$

where $\text{int } I^0$ is the interior of I^0 . Then

$$(2.8) \quad \tilde{\Omega} \subset \Omega.$$

The reason for this is that if $(X, t) \in S$ and $\pi(X, t) \in W$, then

$$((X, t) + \Gamma) \cap \{x_n \leq \bar{M}\} \setminus \{(X, t)\} \subset \Omega,$$

and that $((X, t) + \Gamma) \cap \{x_n = \bar{M}\} \subset I^{\bar{M}}$. Furthermore, $\partial\tilde{\Omega}$ is $\text{Lip}(1, 1/2)$ regular and $\sigma(\partial\tilde{\Omega} \cap \partial\Omega) \geq \gamma$.

To prove the second statement in Theorem 1.12, we have to complement the argument above leading up to (2.7), (2.8). Indeed, consider $(\hat{X}, \hat{t}) \in \Omega$ and let $d(\hat{X}, \hat{t}, \partial\Omega) = d$. We first claim that there exists $C = C(n, M) \geq 1$, and

$$(2.9) \quad (X, \hat{t}) \in \partial\Omega \text{ such that } \|(X, \hat{t}) - (\hat{X}, \hat{t})\| \leq Cd.$$

To prove (2.9), we use the synchronized two cube condition. Indeed, let $(Z, t) \in \partial\Omega$ be a point that satisfies $\|(Z, t) - (\hat{X}, \hat{t})\| = d$. Then there exist two cubes $Q_d(X_1, t_1) \subset \mathbb{R}^{n+1} \setminus \Omega$ and $Q_d(X_2, t_2) \subset \Omega$ such that $t_1 = t = t_2$ and $\|(Z, t) - (X_i, t_i)\| \leq Md, i = 1, 2$. By construction $Q_d(X_1, t_1)$ contains a point (\hat{X}_1, \hat{t}) that belongs to $\mathbb{R}^{n+1} \setminus \Omega$. Thus the line connecting (\hat{X}, \hat{t}) and (\hat{X}_1, \hat{t}) intersects $\partial\Omega$ at a point (X, \hat{t}) . It is easy to see that $\|(X, \hat{t}) - (\hat{X}, \hat{t})\| \leq (M + \sqrt{n} + 2)d$. This proves (2.9) with $C = M + \sqrt{n} + 2$. We will now use (2.9) to prove that $\partial\Omega$ satisfies a uniform interior big pieces of $\text{Lip}(1, 1/2)$ graphs condition in the sense of Definition 1.6. To start the argument, we note that it follows, as $d(\hat{X}, \hat{t}, \partial\Omega) = d$ and $M \geq 1$, that the cube $Q_{c(n)d/M}(\hat{X}, \hat{t})$ is contained in Ω if $c(n) \leq (\sqrt{n} + 1)^{-1}$. By a parabolic scaling and an Euclidean translation we can in the following, without loss of generality, assume that $(X, \hat{t}) = (0, 0)$ and $Cd = 2M$, where (X, \hat{t}) and C are as in (2.9). By this change of coordinates we see that (\hat{X}, \hat{t}) and $Q_{c(n)d/M}(\hat{X}, \hat{t})$ are transformed into $(\hat{Y}, 0)$ and $Q_{2c(n)/C}(\hat{Y}, 0)$, respectively. From this point the argument proceeding Lemma 2.1 can be carried out with $Q_{2c(n)/C}(\hat{Y}, 0)$ in place of $Q_2(Y, 0)$. Furthermore, we must replace Q_{ρ_0} and I^0 by the corresponding sets rescaled by the factor $c(n)/C$. Doing this we obtain, after a rotation of the spacial variables if necessary, and as in (2.7), (2.8), the interior domain

$$\tilde{\Omega} = \{(X, t) : (x, t) \in \text{int } I_{c(n)/C}(0, 0), \psi(x, t) < x_n < \bar{M}\}.$$

By translating and scaling back, but not rotating, to the original coordinates we can conclude that

$$\tilde{\Omega} = \{(y, y_n, s) : (y, s) \in \text{int } I_{c(n)d/(2M)}(\hat{x}, \hat{t}), \psi(y, s) < y_n < \hat{x}_n\}.$$

Since $d(\hat{X}, \hat{t}, \partial\Omega) = d(0, \hat{x}, \hat{t}, \partial\Omega) = d$, we may redefine $\tilde{\Omega}$ as

$$\tilde{\Omega} = \{(y, y_n, s) : (y, s) \in \text{int } I_{c(n)d/(2M)}(\hat{x}, \hat{t}), \psi(y, s) < y_n < \hat{x}_n + d/2\}.$$

By construction we have $\tilde{\Omega} \subset \Omega \cap Q_{2Cd}(\hat{X}, \hat{t})$. This proves that Ω satisfies a uniform interior big pieces of $\text{Lip}(1, 1/2)$ graphs with constants (ϵ, \hat{M}, C) where $\epsilon = \gamma$, $\hat{M} = 2M/c(n)$ and $C = C(n, M)$. Based on (2.1) we see that in the end all of these constants depend on n, M and γ_0 . This completes the proof of the second statement in Theorem 1.12.

2.1. Proof of Lemma 2.1

For any

$$(X, t) \in \{(x, x_n, t) : (x, t) \in I^0, x_n \in [-M, \bar{M}]\},$$

we let $L(X, t)$ be the open line segment in the x_n direction which connects (X, t) to (x, \bar{M}, t) . If $(X, t) \in \Sigma$, then the length of $L(X, t)$ is at least 1 as $d(I^{\bar{M}}, \Sigma) \geq 1$. Define $\tilde{\mathcal{G}}$ to be the closure of the set of all such points $(X, t) \in \Sigma$ which satisfy $L(X, t) \cap \Sigma = \emptyset$, and let

$$\mathcal{G} := \tilde{\mathcal{G}} \cap S \subset \Sigma.$$

Consider $(x, t) \in I^0$ and define

$$M(x, t) = \sup\left\{\frac{1}{|I|}\sigma(\pi^{-1}(I) \cap Q_{2M}(0, 0)) : I \text{ contains } (x, t)\right\}.$$

Then, by weak estimates for the Hardy–Littlewood maximal function,

$$(2.10) \quad \mathcal{B} := \{(x, t) \in \mathbb{R}^n : M(x, t) \geq N\} \text{ satisfies } \sigma(\mathcal{B}) \leq c/N,$$

for some constant $c = c(n, M) \geq 1$. If N is large then \mathcal{B} contains those points (x, t) which are such that there is a cube I , containing (x, t) , onto which a large set on Σ is projected. Obviously

$$(2.11) \quad \text{there exists } N \geq 1, \text{ depending only on } n, M \text{ and } \gamma,$$

such that

$$(2.12) \quad \mathcal{H}^n(\mathcal{B}) \leq \gamma/2.$$

We fix N with respect to (2.12). In particular, both γ and N are from now on fixed constants depending only on n, M .

Having fixed γ and N , there will appear, in the construction to be outlined, four important constants: $\Lambda_0, \Lambda_1, \Lambda_2$, and Λ_3 , with $1 \leq \Lambda_i < \infty$ for $i \in \{0, 1, 2, 3\}$. In

general all constants appearing will depend at most on $n, M, \Lambda_0, \Lambda_1, \Lambda_2,$ and Λ_3 . We will choose the degrees of freedom $\Lambda_0, \Lambda_1, \Lambda_2,$ and Λ_3 to depend only on n, M, γ and N , and hence to depend only on n, M . Furthermore, Λ_i for $i \in \{0, 1, 2, 3\}$, will be chosen to be of the form 2^{N_i} for some integer $N_i \geq 1$.

We let, for Λ_1 fixed and as above, A be defined by

$$(2.13) \quad \log_2 A = \lceil \log_2(2M\Lambda_1) \rceil + 1.$$

Then A is a power of 2 such that $A \geq 2M\Lambda_1$. Based on A we define, for $j \in \{0, 1, \dots\}$,

$$\Sigma_j = \{(x, t) \in I^0 : \text{there exist } (X, t) \in \mathcal{G} \text{ and } (Y, s) \in S \text{ such that} \\ (X, t) = (x, x_n, t), (Y, s) \in (X, t) + \Gamma \text{ and } A^{-j} \leq y_n - x_n \leq A^{-j+1}\}.$$

If $(x, x_n, t) \in S$, there exists a maximal \hat{x}_n such that $(x, \hat{x}_n, t) \in S$. This follows since $x_n \leq \bar{M}$ if $(x, x_n, t) \in S, I^M \subset \mathcal{U}$ and S is closed. Thus $(x, \hat{x}_n, t) \in \mathcal{G}$, which shows that $\pi(\mathcal{G}) = \pi(S)$. When $(x, t) \in \pi(S) \setminus W$ we have $((X, t) + \Gamma) \cap S \neq \emptyset$ whenever $(X, t) = (x, x_n, t) \in S$. We conclude that for $(X, t) = (x, \hat{x}_n, t) \in \mathcal{G}, ((X, t) + \Gamma) \cap S \neq \emptyset$, so there exists $(Y, s) \in S$ such that $(Y, s) \in (X, t) + \Gamma$. By our restriction on A we have

$$\pi(S) \setminus W \subset \cup_j \Sigma_j.$$

Furthermore, as by construction $\mathcal{H}^n(\mathcal{B}) \leq \gamma/2$, the proof of Lemma 2.1 is reduced to proving that

$$(2.14) \quad \mathcal{H}^n(\cup_j \Sigma_j \cap (\mathbb{R}^n \setminus \mathcal{B})) \leq \gamma/2.$$

To continue the proof we will need the following lemma, Lemma 2.2, the proof of which is postponed until subsection 2.2.

Lemma 2.2. *Let $\varepsilon > 0$ be given. Let Λ_1 be as above and define A as in (2.13). Then there exist $\Lambda_2,$ and Λ_3 as above and as in the statement of Lemma 2.6, and an integer $N_0 = N_0(\varepsilon, \Lambda_2) \geq 1,$ such that if we let $\Lambda_0 = \Lambda_2^{\tilde{N}_0},$ for some $\tilde{N}_0 \geq N_0,$ and if we restrict h to satisfy $h \geq 2A\Lambda_0\Lambda_1\Lambda_3,$ then the following is true. Let $j \geq 0$ and $I \subset I^0$ be a dyadic cube of size $l(I) = A^{-j}.$ Then the number of dyadic cubes J of size $l(J) = \Lambda_0^{-1}A^{-j}$ that are contained in I and satisfy $J \cap (\Sigma_j \cap (\mathbb{R}^n \setminus \mathcal{B})) \neq \emptyset,$ is less than $\varepsilon\Lambda_0^{n+1}.$*

Let in the following $\varepsilon > 0$ be a degree of freedom to be fixed in (2.23) below. To proceed with the proof of (2.14), given $j \geq 0$ we dyadically subdivide I^0 into (disjoint) dyadic cubes $\{J_{j,l}\}$ of size $l(J_{j,l}) = A^{-j}.$ Note that there are $A^{(n+1)j}$ such cubes. We then apply Lemma 2.2 to each cube $J_{j,l}$ and, as a result, for Λ_0 as in Lemma 2.2 we can produce a set of dyadic cube $\{J_{j,l,k}\},$ each of size $l(J_{j,l,k}) = \Lambda_0^{-1}A^{-j},$ such $J_{j,l,k} \subseteq J_{j,l}$ and such that $J_{j,l,k} \cap (\Sigma_j \cap (\mathbb{R}^n \setminus \mathcal{B})) \neq \emptyset.$ By Lemma 2.2 we have, for each $J_{j,l},$ that the number of such cubes is at most $\varepsilon\Lambda_0^{n+1}.$ Using this

we see that

$$\begin{aligned}
 \mathcal{H}^n(\cup_j \Sigma_j \cap (\mathbb{R}^n \setminus \mathcal{B})) &\leq \sum_j \sum_l \mathcal{H}^n(J_{j,l} \cap (\Sigma_j \cap (\mathbb{R}^n \setminus \mathcal{B}))) \\
 (2.15) \qquad \qquad \qquad &\leq \sum_j \sum_l \sum_{k=1}^{k_l} \mathcal{H}^n(J_{j,l,k}).
 \end{aligned}$$

Hence, to prove (2.14) it suffices to prove that

$$(2.16) \qquad \sum_j \sum_l \sum_{k=1}^{k_l} \mathcal{H}^n(J_{j,l,k}) \leq \gamma/2.$$

In the following we will, to simplify the notation, simply denote the set of all cubes $\{J_{j,l,k}\}$ by $\{I_{j,k}\}$. To each $I_{j,k}$ we will associate a surface $S(I_{j,k})$, see Definition 2.3 below, and we intend to estimate the measure of $I_{j,k}$ in terms of the measure of $S(I_{j,k})$. The surfaces $\{S(I_{j,k})\}$ will not be uniquely defined, but if $j \neq j'$, then we will have $S(I_{j,k}) \cap S(I_{j',k'}) = \emptyset$ for all k and k' , see Lemma 2.4 below. For the sake of brevity we will in the following denote points $(X, t) = (x, x_n, t)$ by boldface letters, i.e., $\mathbf{X} := (X, t) = (x, x_n, t)$. To proceed with the construction, consider $I = I_{j,k}$ and choose any $\mathbf{X} \in \mathcal{G}$ and $\mathbf{Y} \in S$ such that $\pi(\mathbf{X}) \in I$, $\mathbf{Y} \in \mathbf{X} + \Gamma$ and $A^{-j} \leq y_n - x_n < A^{-j+1}$. Applying the synchronized two-cube condition we see that there exists a cube $Q \subset \mathbb{R}^{n+1}$ of size $R_j = M^{-1}\Lambda_1^{-1}A^{-j}$ and with center \mathbf{U} , such that $\|\mathbf{U} - \mathbf{Y}\| \leq \Lambda_1^{-1}A^{-j}$ and such that Q belongs to a component of $\mathbb{R}^{n+1} \setminus \Sigma$ different from \mathcal{U} . We recall that \mathcal{U} is the component that contains I^M . Furthermore, the t -coordinate of \mathbf{U} and \mathbf{Y} , coincide. Given I we let $J = J(I) = I_{\Lambda_1^{-1}R_j}(\pi(\mathbf{U}))$ and $\mathcal{S} = (\Sigma \cap Q_{2M}(0, 0)) \cup (I^0 \times \{x_n = -A\})$.

Definition 2.3. Given $I = I_{j,k}$ we define $S(I)$ to be the set of all $\mathbf{V} \in \mathcal{S}$ such that $\pi(\mathbf{V}) \in J = J(I) = I_{\Lambda_1^{-1}R_j}(\pi(\mathbf{U}))$, $v_n < u_n - R_j$ and such that the line segment joining \mathbf{V} to $\pi(\mathbf{V}) + (0, u_n - R_j, 0)$ does not meet Σ .

Note that $\pi(S(I)) = J$ as $\pi(\mathcal{S}) \supset J$. Moreover, it is easy to check that $J \subset 2I$. To complete the proof we will need the following lemma, Lemma 2.4, the proof of which is also postponed until subsection 2.2.

Lemma 2.4. *Let $\varepsilon > 0$ be given. Let $\Lambda_2, \Lambda_3, N_0$, be as in the statement of Lemma 2.2. Then there exists an integer $\tilde{N}_0 \geq N_0$, depending only on n, M, Λ_2 , and Λ_3 , such that if we let $\Lambda_0 = \Lambda_2^{\tilde{N}_0}$, $\Lambda_1 = 2^{\tilde{N}_0}$, define A as in (2.13), and if we restrict h to satisfy $h \geq 2A\Lambda_0\Lambda_1\Lambda_3$, then*

$$S(I_{j,k}) \cap S(I_{j',k'}) = \emptyset \quad \text{for all } k \text{ and } k' \text{ whenever } j \neq j'.$$

We can now use Lemma 2.2 and Lemma 2.4 to complete the proof of (2.16) and hence the proof of Lemma 2.1. Let K_j be the number of cubes $J_{j,l}$ that contain at

least one of the $J_{j,l,k}$. Then, given $\epsilon, \tilde{N}_0 \geq N_0$ and h as stated in Lemma 2.4, we see, using Lemma 2.2, that

$$(2.17) \quad \sum_l \sum_{k=1}^{k_l} \mathcal{H}^n(J_{j,l,k}) \leq K_j \epsilon \Lambda_0^{n+1} (2\Lambda_0^{-1}A^{-j})^{n+1} = K_j \epsilon 2^{n+1} A^{-j(n+1)}.$$

Fix l and assume that $J_{j,l} \supset J_{j,l,k}$ for some k . Then for all $k \in \{1, \dots, k_l\}$ one has $\pi(S(J_{j,l,k})) \subset 2J_{j,l}$. By construction

$$(2.18) \quad \mathcal{H}^n(\pi(\cup_{k=1}^{k_l} S(J_{j,l,k}))) \cap 2J_{j,l} \geq (2\Lambda_1^{-1}R_j)^{n+1}.$$

Hence, summing over l we see that

$$(2.19) \quad \begin{aligned} \sum_l \mathcal{H}^n(\pi(\cup_{k=1}^{k_l} S(J_{j,l,k}))) \cap 2J_{j,l} &\geq K_j (2\Lambda_1^{-1}R_j)^{n+1} \\ &= K_j (M^{-1}\Lambda_1^{-2})^{n+1} 2^{n+1} A^{-j(n+1)}. \end{aligned}$$

Combining (2.17) and (2.19) we see that

$$(2.20) \quad \sum_l \sum_{k=1}^{k_l} \mathcal{H}^n(J_{j,l,k}) \leq c\epsilon (M\Lambda_1^2)^{n+1} \sum_l \mathcal{H}^n(\pi(\cup_{k=1}^{k_l} S(J_{j,l,k})))$$

for all $j \geq 0$, where the constant $c = c(n)$ comes from the possible overlap between the sets $2J_{j,l}$ in (2.19). Next, using Lemma 2.4 we know that the $S(J_{j,l,k}) \cap S(J_{j',l,k}) = \emptyset$ whenever $j \neq j'$. Furthermore, by definition,

$$S(J_{j,l,k}) \subset \mathcal{S} = (\Sigma \cap Q_{2M}(0,0)) \cup (I^0 \times \{x_n = -A\}).$$

This implies that

$$(2.21) \quad \begin{aligned} \sum_j \sum_l \mathcal{H}^n(\pi(\cup_{k=1}^{k_l} S(J_{j,l,k}))) &\leq \mathcal{H}^n(\pi(\Sigma) \cap I^0) + \mathcal{H}^n(\pi(I^0 \times \{x_n = -A\})) \\ &= 2\mathcal{H}^n(I^0). \end{aligned}$$

Together (2.20) and (2.21) imply the bound

$$(2.22) \quad \sum_l \sum_{k=1}^{k_l} \mathcal{H}^n(J_{j,l,k}) \leq C\epsilon (M\Lambda_1^2)^{n+1},$$

where $C = C(n)$, $1 \leq C < \infty$. Let now ϵ be defined through the relation

$$(2.23) \quad C\epsilon (M\Lambda_1^2)^{n+1} = \gamma/2.$$

Then $\varepsilon = \varepsilon(n, M, \Lambda_1, \gamma) = \varepsilon(n, M, \gamma) = \varepsilon(n, M)$ and we see that Lemma 2.1 holds with $h = 2A\Lambda_0\Lambda_1\Lambda_3$ and, by construction, $h = h(n, M)$. In particular, the proof of Lemma 2.1 is now, modulo Lemma 2.2 and Lemma 2.4, complete.

2.2. Proof of Lemma 2.2 and Lemma 2.4

Lemma 2.5. *Let $I \subset I^0$ be a cube of size $l(I)$. Suppose there exist two line segments L and L' in different components of $\mathbb{R}^{n+1} \setminus \Sigma$ such that $\pi(L) \in I$, $\pi(L') \in I$ and such that the length of the line segment $\pi^\perp(L) \cap \pi^\perp(L') \cap (-3M/2, 3M/2)$ is greater than $4^{n+2}MNl(I)$. Then $2I \subset \mathcal{B}$.*

Proof. Any line segment orthogonal to x_n that connects L and L' intersects Σ , and by construction there exist $4^{n+1}MN$ such line segments with mutual distance greater than $2l(I)$. Thus there exist $4^{n+1}MN$ points $(X^i, t^i) \in \Sigma$ such that $\pi(X^i, t^i) \in I$ and $\|\pi^\perp(X^i, t^i) - \pi^\perp(X^j, t^j)\| \geq 2l(I)$ for $i \neq j$. Hence the cubes $Q_{l(I)}(X^i, t^i)$ are disjoint and satisfy $\pi(Q_{l(I)}(X^i, t^i)) \subset 2I$. This gives the estimate

$$\begin{aligned} \sigma(\pi^{-1}(2I) \cap Q_{2M}(0, 0)) &\geq \sigma\left(\bigcup_i Q_{l(I)}(X^i, t^i)\right) \\ &\geq 4^{n+1}MN M^{-1}l(I)^{n+1} \geq N\mathcal{H}^n(2I). \end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.6. *Let $h \geq 1$ be given. There exist constants $\Lambda_2 = 2^{N_2}$ and $\Lambda_3 = 2^{N_3}$, where N_2 and N_3 are positive integers, depending only on n, M and N , such that the following holds. If $I \subset I^0$ is a dyadic cube of size $l(I)$, $\Lambda_3 h^{-1} A^{-j+1} \leq l(I) \leq A^{-j}$, then there is a dyadic cube J of size $\Lambda_2^{-1}l(I)$ such that $J \subset \mathcal{B} \cup (\mathbb{R}^n \setminus \Sigma_j)$.*

Proof. We first note that it suffices to prove the existence of cube J , as in the statement of the lemma, but which is not dyadic, as J then contains a dyadic cube of size $2^{-1}\Lambda_2^{-1}l(I)$ and we may replace Λ_2 by $2\Lambda_2$. Furthermore, we in the following just establish the existence of Λ_2 and Λ_3 . That these constants can be chosen on the form $\Lambda_2 = 2^{N_2}$ and $\Lambda_3 = 2^{N_3}$ is then obvious. Given I we let $K \geq 4$ be a degree of freedom, which we fix in (2.25) and which only depends on n and M , and in the following we let J_k , for $k \geq 1$, denote the cube which has the same center as I but which has size $2^{-k}l(I)$. By construction,

$$I \supset J_1 \supset J_2 \supset \dots \supset J_k \supset \dots$$

Assume that $J_k \cap \pi(\mathcal{G}) = \emptyset$ for some $k \in \{1, \dots, K+1\}$. Then $J_k \subset (\mathbb{R}^n \setminus \Sigma_j)$ and we can choose $J = J_{K+1}$, $\Lambda_2 = 2^{(K+1)}$, $\Lambda_3 = 1$, to complete the proof of the lemma. Hence we in the following assume that $J_k \cap \pi(\mathcal{G}) \neq \emptyset$ for all $1 \leq k \leq K+1$. In this case we introduce

$$\lambda_k = \sup\{x_n : \mathbf{X} \in \mathcal{G} \text{ and } \pi(\mathbf{X}) \in J_k\}.$$

Obviously $\{\lambda_k\}$ is a non-increasing sequence. Assume first that $\lambda_k > \lambda_{k+1} + A^{-j}/10$ for all $k \in \{2, \dots, K\}$. Then there exist K points $\mathbf{X}^k \in \mathcal{G}$ such that $\pi(\mathbf{X}^k) \in J_k$ and

$$\|\mathbf{X}^k - \mathbf{X}^{k'}\| \geq |x_n^k - x_n^{k'}| > A^{-j}/10, \quad k \neq k'.$$

Using that Σ is parabolic ADR we see that

$$\begin{aligned}
 \sigma(\pi^{-1}(I)) &\geq \sum_{k=1}^K \sigma(Q_{A^{-j}/20}(\mathbf{X}^k) \cap \Sigma) \\
 (2.24) \qquad \qquad &\geq KM^{-1}A^{-j(n+1)}20^{-(n+1)} \geq c^{-1}KM^{-1}\mathcal{H}^n(I).
 \end{aligned}$$

Hence, if we choose

$$(2.25) \qquad \qquad K = 2cMN,$$

then $I \subset \mathcal{B}$ by the definition of the set \mathcal{B} . In this case we can again choose $J = J_{K+1}$, $\Lambda_2 = 2^{(K+1)}$, $\Lambda_3 = 1$, to complete the proof of the lemma. Next, assume that $\lambda_k < \lambda_{k+1} + A^{-j}/10$ for some $k \in \{2, \dots, K\}$ and choose $\mathbf{X} \in \mathcal{G}$ satisfying $\pi(\mathbf{X}) \in J_{k+1}$, $x_n = \lambda_{k+1}$. As $\mathbf{X} \in \Sigma$ we can use the synchronized two-cube condition to conclude that there is a cube Q with size

$$l(Q) = (10M)^{-1}2^{-k}l(I)$$

and center \mathbf{Y} , having the same t -coordinate as \mathbf{X} , such that $\|\mathbf{X} - \mathbf{Y}\| \leq 2^{-k}l(I)/10$ and such that Q belongs to a component \mathcal{V} , different from \mathcal{U} , of $\mathbb{R}^{n+1} \setminus \Sigma$. In this case we let $J \subset I^0$ be the cube with center $\pi(\mathbf{Y})$ and size

$$l(J) = (4^{n+1}MN)^{-1}l(Q).$$

If $J \cap \Sigma_j = \emptyset$, then the conclusion of the lemma holds with this choice of J , with Λ_2 defined through the relation

$$(2.26) \qquad \qquad \Lambda_2^{-1} = (4^{n+1}MN)^{-1}(10M)^{-1}2^{-K},$$

and with $\Lambda_3 = 1$. If $J \cap \Sigma_j \neq \emptyset$, then there exist $\mathbf{U} \in \mathcal{G}$, $\mathbf{V} \in S$ such that $\pi(\mathbf{U}) \in J$, $\mathbf{V} \in \mathbf{U} + \Gamma$ and $A^{-j} \leq v_n - u_n \leq A^{-j+1}$. From the definition of Γ we get

$$\|\pi(\mathbf{U}) - \pi(\mathbf{V})\| \leq h^{-1}(v_n - u_n) \leq h^{-1}A^{-j+1} \leq \Lambda_3^{-1}l(I),$$

provided $l(I) \geq \Lambda_3 h^{-1}A^{-j+1}$. Hence, if we choose

$$(2.27) \qquad \qquad \Lambda_3 = 100M^2 4^{n+1} N 2^K,$$

then

$$\|\pi(\mathbf{U}) - \pi(\mathbf{V})\| \leq l(J)/10.$$

We claim that $\pi(\mathbf{V}) \in J_k$. Indeed, recall that $\pi(\mathbf{X}) \in J_{k+1}$ and

$$(2.28) \quad \|\pi(\mathbf{V}) - \pi(\mathbf{X})\| \leq \|\pi(\mathbf{X}) - \pi(\mathbf{Y})\| + \|\pi(\mathbf{Y}) - \pi(\mathbf{U})\| + \|\pi(\mathbf{U}) - \pi(\mathbf{V})\|$$

where

$$\begin{aligned}
 \|\pi(\mathbf{X}) - \pi(\mathbf{Y})\| &\leq \|\mathbf{X} - \mathbf{Y}\| = 2^{-k}l(Q)/10, \\
 \|\pi(\mathbf{Y}) - \pi(\mathbf{U})\| &\leq (n-1)^{1/2}l(J) + l(J) \leq nl(J),
 \end{aligned}$$

and

$$\|\pi(\mathbf{U}) - \pi(\mathbf{V})\| \leq l(J)/10.$$

Hence, using this, (2.28) and the definitions of $l(Q)$ and $l(J)$ we see that

$$\|\pi(\mathbf{V}) - \pi(\mathbf{X})\| < \frac{1}{2} 2^{-(k+1)} r.$$

Therefore $\pi(\mathbf{V}) \in J_k$ as $\pi(\mathbf{X}) \in J_{k+1}$. Thus

$$(2.29) \quad v_n \leq \lambda_k \leq \lambda_{k+1} + A^{-j}/10 = x_n + A^{-j}/10.$$

Using this, and that $u_n \leq v_n - A^{-j}$ and $\|\mathbf{Y} - \mathbf{X}\| < A^{-j}/10$, we conclude that

$$u_n \leq x_n - 9A^{-j}/10 \leq y_n - 8A^{-j}/10.$$

Furthermore, as $\mathbf{U} \in \mathcal{G}$ there exists a point \mathbf{U}' , arbitrarily close to \mathbf{U} , such that the vertical line segment L' connecting \mathbf{U}' to I^M belongs to \mathcal{U} . Choose \mathbf{U}' such that $\|\mathbf{U} - \mathbf{U}'\| < l(J)/10$. Then $\pi(\mathbf{U}') \in 2J$. Let L be the line segment of length $2l(Q)$ which is the intersection between Q and the line parallel to L' through \mathbf{Y} . By construction L and L' belong to different components of $\mathbb{R}^{n+1} \setminus \Sigma$ and the length of L' is greater than $1/2$ since $d(I^M, \Sigma) \geq 1$. Moreover,

$$u'_n \leq y_n - 8A^{-j}/10 + l(J)/10 \leq y_n - 7A^{-j}/10 < y_n - l(Q),$$

which shows that the length of $\pi^\perp(L) \cap \pi^\perp(L')$ is $2l(Q)$. As $l(Q) \geq 4^{n+1}MNl(J) > 2^{n+1}MN(2l(J))$ we conclude, using Lemma 2.5, that $4J \subset \mathcal{B}$. Hence in this case we can complete the proof by choosing this J , Λ_2 as in (2.26) and Λ_3 as in (2.27). In particular, the conclusion holds in all cases with

$$(2.30) \quad \Lambda_2 = (4^{n+1}MN)(10M)2^K, \quad \Lambda_3 = 100M^2 4^{n+1} N 2^K.$$

This completes the proof of the lemma. □

Proof of Lemma 2.2. Let Λ_2 be the constant from Lemma 2.6. Lemma 2.6 guarantees, if $\Lambda_3 h^{-1} A^{-j+1} \leq l(I) \leq A^{-j}$, that at least one of the $\Lambda_2^{(n+1)}$ dyadic cubes of size $\Lambda_2^{-1} A^{-j}$ covering I does not intersect $\Sigma_j \cap (\mathbb{R}^n \setminus \mathcal{B})$. If we apply Lemma 2.6 again to the remaining cubes it follows, if $\Lambda_3 h^{-1} A^{-j+1} \leq \Lambda_2^{-1} l(I) \leq A^{-j}$, that the number of dyadic cubes of size $\Lambda_2^{-2} A^{-j}$ that intersect $\Sigma_j \cap (\mathbb{R}^n \setminus \mathcal{B})$ is bounded by $(\Lambda_2^{n+1} - 1)^2$. Repeating this procedure \tilde{N}_0 times, we see, if $\Lambda_3 h^{-1} A^{-j+1} \leq \Lambda_2^{-\tilde{N}_0+1} l(I) \leq A^{-j}$, that the number of dyadic cubes of size $\Lambda_2^{-\tilde{N}_0} A^{-j}$ that intersect $\Sigma_j \cap (\mathbb{R}^n \setminus \mathcal{B})$ is bounded by

$$(\Lambda_2^{n+1} - 1)^{\tilde{N}_0} = \Lambda_2^{\tilde{N}_0(n+1)} (1 - \Lambda_2^{-n-1})^{\tilde{N}_0}.$$

Let $N_0 = N_0(\varepsilon, \Lambda_2)$ be the smallest positive integer such that

$$(\Lambda_2^{n+1} - 1)^{N_0} \leq \varepsilon \Lambda_0^{n+1}.$$

Consider $\tilde{N}_0 \geq N_0$ and let $\Lambda_0 = \Lambda_2^{\tilde{N}_0}$. Then

$$(\Lambda_2^{n+1} - 1)^{\tilde{N}_0} \leq \epsilon \Lambda_0^{(n+1)}.$$

Furthermore, we see that our choice of h implies that

$$\Lambda_3 h^{-1} A \leq \Lambda_2^{-\tilde{N}_0} / 2 \leq \Lambda_2^{-\tilde{N}_0+1},$$

and hence, as $l(I) = A^{-j}$, we can ensure that $\Lambda_3 h^{-1} A^{-j+1} \leq \Lambda_2^{-\tilde{N}_0+1} l(I) \leq A^{-j}$. This completes the proof of the lemma. \square

Proof of Lemma 2.4. To prove the lemma we argue by contradiction. Indeed, assume that there exist $I = I_{j,k}, I' = I_{j',k'}, j > j'$, such that $S(I) \cap S(I') \neq \emptyset$. Let $\mathbf{V} \in S(I) \cap S(I')$. Let $\mathbf{X}, \mathbf{Y}, \mathbf{U}$ and Q be associated to the construction of $S(I)$ and let $\mathbf{X}', \mathbf{Y}', \mathbf{U}'$ and Q' be associated to the construction of $S(I')$, see Definition 2.3 and the notation introduced above that definition. We have

$$\begin{aligned} \|\pi(\mathbf{X}) - \pi(\mathbf{V})\| &\leq \|\pi(\mathbf{X}) - \pi(\mathbf{Y})\| + \|\pi(\mathbf{Y}) - \pi(\mathbf{V})\| \\ &\leq h^{-1} A^{-j+1} + \|\pi(\mathbf{Y}) - \pi(\mathbf{V})\|, \end{aligned}$$

and

$$\|\pi(\mathbf{Y}) - \pi(\mathbf{V})\| \leq \|\mathbf{Y} - \mathbf{U}\| + \|\pi(\mathbf{U}) - \pi(\mathbf{V})\| \leq \Lambda_1^{-1} A^{-j} + n \Lambda_1^{-1} R_j.$$

Thus

$$\begin{aligned} \|\pi(\mathbf{X}) - \pi(\mathbf{V})\| &\leq h^{-1} A^{-j+1} + \Lambda_1^{-1} A^{-j} + n M^{-1} \Lambda_1^{-2} A^{-j} \\ &\leq (h^{-1} A + \Lambda_1^{-1} + n M^{-1} \Lambda_1^{-2}) A^{-j} \leq 2 \Lambda_1^{-1} A^{-j} \end{aligned}$$

by our choices for M and h . The distance between $\pi(\mathbf{X})$ and $\pi(\mathbf{U}')$ can be estimated as

$$\|\pi(\mathbf{X}) - \pi(\mathbf{U}')\| \leq \|\pi(\mathbf{X}) - \pi(\mathbf{V})\| + \|\pi(\mathbf{V}) - \pi(\mathbf{U}')\| \leq 2 \Lambda_1^{-1} A^{-j} + n \Lambda_1^{-1} R_{j'}.$$

By construction, $A > 2M\Lambda_1$, which implies that $2\Lambda_1^{-1} A^{-j} \leq 2\Lambda_1^{-1} A^{-1} A^{-j'} \leq \Lambda_1^{-1} R_{j'}$ and that

$$\|\pi(\mathbf{X}) - \pi(\mathbf{U}')\| \leq (n + 1) \Lambda_1^{-1} R_{j'}.$$

Hence we can conclude that $\pi(\mathbf{X}) \in (n + 1) I_{\Lambda_1^{-1} R_{j'}}(\pi(\mathbf{U}'))$. As $\mathbf{X} \in \mathcal{G}$, there is a point \mathbf{W} such that $\|\mathbf{W} - \mathbf{X}\| \leq \Lambda_1^{-1} R_j$ and such that the vertical line L connecting \mathbf{W} to $I^{\bar{M}}$ belongs to \mathcal{U} . Note that $\|\pi(\mathbf{W}) - \pi(\mathbf{U}')\| \leq (n + 2) \Lambda_1^{-1} R_{j'}$. We will now consider the two possibilities $x_n < u'_n - R_{j'}$ and $x_n \geq u'_n - R_{j'}$ separately, and will derive a contradiction to the assumption that there exists a point $\mathbf{V} \in S(I) \cap S(I')$.

Assume first that $x_n < u'_n - R_{j'}$, let L' be the vertical line segment in Q' through \mathbf{U}' of length $2R_{j'}$. Then the projections of L and L' belong to $(n + 2) I_{\Lambda_1^{-1} R_{j'}}(\pi(\mathbf{U}'))$. Furthermore, as $\|\mathbf{W} - \mathbf{X}\| < R_{j'}$ it follows that the length of the line segment $\pi^\perp(L') \cap \pi^\perp(L)$ is greater than $R_{j'}$. We now want to apply

Lemma 2.5. Indeed, if $R_{j'} > 4^{n+2}MN(n+2)\Lambda_1^{-1}R_{j'}$ then Lemma 2.5 implies that $(n+2)I_{\Lambda_1^{-1}R_{j'}}(\pi(\mathbf{U}')) \subset \mathcal{B}$ if we choose \tilde{N}_0 , and hence Λ_1 , is large enough. Furthermore, $\pi(\mathbf{X}) \in I \cap (n+1)I_{\Lambda_1^{-1}R_{j'}}(\pi(\mathbf{U}'))$, and as the size of I is $\Lambda_0^{-1}A^{-j}$, we have $I \subset (n+2)I_{\Lambda_1^{-1}R_{j'}}(\pi(\mathbf{U}')) \subset \mathcal{B}$ if \tilde{N}_0 , and hence Λ_0 , is large enough. This contradicts the assumption that I meets $\Sigma_j \cap (\mathbb{R}^n \setminus \mathcal{B})$.

Assume next that $x_n \geq u'_n - R_{j'}$. Let

$$\mathbf{Z} = \pi(\mathbf{V}) + (0, u_n - \Lambda_1^{-1}R_j, 0)$$

and let L'' be the line segment joining \mathbf{V} and \mathbf{Z} . Then L'' does not meet Σ since $\mathbf{V} \in S(I)$, and L'' belongs to a component of $\mathbb{R}^{n+1} \setminus \Sigma$ other than \mathcal{U} since it intersects Q . We have

$$\begin{aligned} z_n &= u_n - \Lambda_1^{-1}R_j \geq y_n - \Lambda_1^{-1}A^{-j} - \Lambda_1^{-1}R_j \geq y_n - 2\Lambda_1^{-1}A^{-j} \\ (2.31) \quad &\geq x_n + A^{-j} - 2\Lambda_1^{-1}A^{-j} \geq x_n + A^{-j}/2 \geq w_n + A^{-j}/4. \end{aligned}$$

Since $\mathbf{V} \in S(I')$, we also have

$$(2.32) \quad v_n \leq u'_n - R_{j'} \leq x_n.$$

By (2.31) and (2.32), we have $z_n - v_n \geq A^{-j}/2$. Thus the length of L'' is at least $A^{-j}/2$, and the length of $\pi^\perp(L'') \cap \pi^\perp(L)$ is at least $A^{-j}/4$, again by (2.31). Since $\|\pi(\mathbf{V}) - \pi(\mathbf{W})\| \leq 3\Lambda_1^{-1}A^{-j}$, we conclude, using Lemma 2.5, that the cube $I^\mathbf{W}$ of size $3\Lambda_1^{-1}A^{-j}$ and center $\pi(\mathbf{W})$ belongs to \mathcal{B} if \tilde{N}_0 , and hence Λ_1 , is large enough. If \tilde{N}_0 , and hence Λ_0 , is large enough we again arrive at the contradiction $I \subset I^\mathbf{W} \subset \mathcal{B}$. \square

3. The proof of Theorem 1.13

We here prove Theorem 1.13. To prove the first statement in Theorem 1.13 we note that we can conclude, by the first statement in Theorem 1.12, that there exists, given $(X, t) \in \Sigma$, $R > 0$, and after a possible rotation in the space variables, a Lip(1, 1/2) function $\hat{\psi}$ with constant $\hat{M} = \hat{M}(n, M)$, and $\epsilon = \epsilon(n, M)$, such that if we let π denote the orthogonal projection onto the plane $\{(x, x_n, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} : x_n = 0\}$, then

$$(3.1) \quad \mathcal{H}^n(\pi(F)) \geq \epsilon R^{n+1}, \quad F := \Sigma_{\hat{\psi}} \cap \Delta(X, t, R),$$

where

$$(3.2) \quad \Sigma_{\hat{\psi}} = \{(x, x_n, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} : x_n = \hat{\psi}(x, t)\}.$$

To prove Theorem 1.13 we need to invoke the Carleson measure condition used in the very definition of parabolic uniform rectifiability. Let

$$f(Z, \tau) = \int_0^{100R} \gamma(Z, \tau, r) r^{-1} dr, \quad (Z, \tau) \in \Delta(X, t, R).$$

Then, using (1.9) we see that

$$\int_{\Delta(X,t,100R)} f(Z, \tau) d\sigma(Z, \tau) \leq \|\nu\| (100R)^{n+1},$$

where $\|\nu\|$ is the Carleson norm of ν on $\Delta(X, t, 800R) \times (0, 800R)$. Using this and weak estimates we see that if $A = 1000\epsilon^{-1}$, then

$$(3.3) \quad \begin{aligned} \sigma(\{(Z, \tau) \in \Delta(X, t, 100R) : f(Z, \tau) \geq A^{n+1} \|\nu\|\}) &\leq (100R/A)^{n+1} \\ &\leq (\epsilon R/10)^{n+1}. \end{aligned}$$

Using this inequality, (3.1) and the fact that Hausdorff measure does not increase under a projection, we deduce the existence of a closed set $F_1 = F_1(A)$ with $F_1 \subset F$, such that

$$(3.4) \quad f(Z, \tau) \leq A^{n+1} \|\nu\|, \quad (Z, \tau) \in F_1,$$

and

$$(3.5) \quad \mathcal{H}^n(\pi(F_1)) \geq \frac{\epsilon}{2} R^{n+1}.$$

We will now construct the approximating graph by extending $\hat{\psi}$ off $\pi(F_1)$. To do this we again identify \mathbb{R}^n with $\mathbb{R}^n \times \{0\}$, and put

$$I_r(z, \tau) = \{(y, s) \in \mathbb{R}^n : |y_i - z_i| < r, i = 1, \dots, n - 1, |s - \tau| < r^2\},$$

whenever $(z, \tau) \in \mathbb{R}^n, r > 0$. Let $\{\bar{I}_i = \overline{I_{r_i}(\hat{x}_i, \hat{t}_i)}\}$ be a Whitney decomposition of $\mathbb{R}^n \setminus \pi(F_1)$ into cubes, such that $I_i \cap I_j = \emptyset, i \neq j$, and

$$(3.6) \quad 10^{-10n} d(I_i, \pi(F_1)) \leq r_i \leq 10^{-8n} d(I_i, \pi(F_1)).$$

Let $\{v_i\}$ be a partition of unity adapted to $\{I_i\}$, i.e.,

$$(3.7) \quad \begin{aligned} (a) \quad &\sum v_i \equiv 1 \text{ on } \mathbb{R}^n \setminus \pi(F_1), \\ (b) \quad &v_i \equiv 1 \text{ on } I_i \text{ and } v_i \equiv 0 \text{ in } \mathbb{R}^n \setminus \overline{I_{2r_i}(\hat{x}_i, \hat{t}_i)} \text{ for all } i, \\ (c) \quad &v_i \text{ is infinitely differentiable on } \mathbb{R}^n \text{ with} \\ &r_i^{-l} \left| \frac{\partial^l}{\partial x^l} v_i \right| + r_i^{-2l} \left| \frac{\partial^l}{\partial t^l} v_i \right| \leq c(l, n) \quad \text{for } l = 1, 2, \dots \end{aligned}$$

In (c), $\partial^l/\partial x^l$ denotes an arbitrary partial derivative with respect to the space variable x and of order l . Next for each i we choose $(x'_i, t'_i) \in \pi(F_1)$ with

$$\rho_i = d(\{(x'_i, t'_i)\}, I_i) = d(\pi(F_1), I_i),$$

and we set $\Lambda = \{i : \bar{I}_i \cap \overline{I_{2R}(x, t)} \neq \emptyset\}$, where (x, t) is the projection of (X, t) onto \mathbb{R}^n . We now let

$$(3.8) \quad \psi(y, s) = \begin{cases} \hat{\psi}(y, s), & (y, s) \in \pi(F_1), \\ \sum_{i \in \Lambda} (\hat{\psi}(x'_i, t'_i) + \hat{M} \rho_i) v_i(y, s), & (y, s) \in \mathbb{R}^n \setminus \pi(F_1). \end{cases}$$

Then, $\psi \equiv 0$ on $\mathbb{R}^n \setminus Q_{4R}(X, t)$, and

$$(3.9) \quad \mathcal{H}^n(\pi(F_1)) \geq \frac{\epsilon}{2} R^{n+1}, \quad F_1 := \Sigma_\psi \cap \Delta(X, t, R),$$

where

$$(3.10) \quad \Sigma_\psi = \{(x, x_n, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} : x_n = \psi(x, t)\}.$$

We intend to prove that the function ψ is a regular parabolic $\text{Lip}(1, 1/2)$ function with constants $b_1 = b_1(n, M, \tilde{M})$, $b_2 = b_2(n, M, \tilde{M})$. By a standard Whitney type argument one can use the fact that ψ is a $\text{Lip}(1, 1/2)$ function with constant $\tilde{M} = \tilde{M}(n, M)$, and (3.6)–(3.8), to conclude that (1.3) holds with M replaced by $c\tilde{M}$. To verify this the more delicate case occurs when (y, s) is in the closure of two cubes say I_i, I_j with $i \in \Lambda, j \notin \Lambda$. However this case follows easily from the fact that $|\hat{\psi}| \leq c\tilde{M}R$ and $|\partial v_k / \partial y_l|(y, s) \leq c/R$ for $1 \leq l \leq n - 1, k = i, j$. Hence it only remains to prove that

$$(3.11) \quad \|D_{1/2}^t \psi\|_* \leq b_2 \quad \text{for some } b_2 = b_2(n, M, \tilde{M}).$$

The proof of (3.11) consists of two steps. The first step is to prove that the relevant Carleson norm with respect to Σ_ψ can be controlled using the local Carleson norm, $\|\nu\|$, defined relative to $\Delta(X, t, 800R)$. Let γ_ψ, ν_ψ , be as in the statement Definition 1.2 but with Σ replaced by Σ_ψ as the underlying closed set. Using this notation the first step is to prove that

$$(3.12) \quad \|\nu_\psi\|_+ \leq c(\hat{M})(1 + \|\nu\|).$$

The second step is to prove that the $\text{Lip}(1, 1/2)$ character of ψ , and $\|\nu_\psi\|_+$, can be used to control $\|D_{1/2}^t \psi\|_*$ and that, in particular,

$$(3.13) \quad \|D_{1/2}^t \psi\|_* \leq c(\hat{M})(1 + \|\nu\|).$$

To start the proof of (3.12) we note that if $(Y, s) \in \Sigma_\psi \cap \overline{Q_{100R}(X, t)}$, then

$$(3.14) \quad d(Y, s, \Sigma) \leq \tilde{c} \hat{M}^2 d(y, s, \pi(F_1)).$$

This inequality is trivial when $(Y, s) \in F_1$. Assume $(Y, s) = (y, s, \hat{\psi}(y, s))$ with $(y, s) \in \bar{I}_i$ for some i . If $\bar{I}_i \cap \bar{I}_j \neq \emptyset$ for some $j \notin \Lambda$, then $c\rho_i \geq R$ by (3.6), and again (3.14) is trivially satisfied as $(Y, s) \in \Sigma_\psi \cap \overline{Q_{100R}(X, t)}$. Hence, assume that $\bar{I}_i \cap \bar{I}_j = \emptyset$ for all $j \notin \Lambda$. Then

$$(3.15) \quad \begin{aligned} d((Y, s), \Sigma) &\leq d((y, s, \hat{\psi}(y, s)), (x'_i, t'_i, \hat{\psi}(x'_i, t'_i))) \\ &\leq c\rho_i + \left| \sum_{i' \in \Lambda} (\hat{\psi}(x'_{i'}, t'_{i'}) - \hat{\psi}(x'_i, t'_i) + \hat{M} \rho_{i'}) v_{i'}(y, s) \right| \leq c\hat{M} \rho_i, \end{aligned}$$

by the construction and the $\text{Lip}(1, 1/2)$ character of $\hat{\psi}$. The proof of (3.12) now follows exactly as in the proof of the corresponding statement in display (2.26)

in [9] but with $\delta_1 = 1$. In particular, at this stage, having constructed ψ as above and having verified (3.14), the corresponding argument in [9] does not use Reifenberg flatness in any decisive step. We omit further details. Finally, for the proof of (3.13), assuming (3.12), we refer to p. 368–373 in [9]. This completes our proof of Theorem 1.13.

4. Connectivity, big pieces, and an application to parabolic measure

Let $\Sigma \subset \mathbb{R}^{n+1}$ be a closed set. Let $\tilde{\gamma} > 0$ be given. In the following we let $\Omega \subset \mathbb{R}^{n+1}$ be a domain, i.e., a connected open set, defined as a connected component of $\mathbb{R}^{n+1} \setminus \Sigma$. Assume that $\Sigma \subset \mathbb{R}^{n+1}$ is parabolic ADR in the sense of Definition 1.1 with constant M .

4.1. Connectivity and big pieces

Definition 4.1. Let Σ and Ω be as above. Let $\epsilon \in (0, 1)$ be given. Let $(X, t) \in \partial\Omega$ and $r > 0$. We say that Ω satisfies a local interior chain condition at (X, t) , on scale r , with constant $\tilde{\gamma}$, if the following is true. Suppose $(U_i, s_i) \in \Omega \cap Q_r(X, t)$, $i = 1, 2$, and that $d((U_1, s_1), \partial\Omega) \geq r/100$, $d((U_2, s_2), \partial\Omega) \geq r/100$. Then there exists a finite family of cubes $\{Q_{r_i}(X_i, t_i)\}_1^l$, from (U_1, s_1) to (U_2, s_2) , such that

- (i) $(U_1, s_1) \in Q_{r_1}(X_1, t_1)$, $(U_2, s_2) \in Q_{r_l}(X_l, t_l)$,
- (ii) $Q_{r_{i+1}}(X_{i+1}, t_{i+1}) \cap Q_{r_i}(X_i, t_i) \neq \emptyset$ for $i = 1, 2, \dots, l - 1$,
- (iii) $r_i = d(X_i, t_i, \partial\Omega)/2$, for $i = 1, 2, \dots, l$, and $l \leq \tilde{\gamma}$.

Definition 4.2. Let Σ and Ω be as in Definition 4.1. Let $\tilde{\gamma} > 0$ be given. Let $E \subset \partial\Omega \times (0, \infty)$ be the set of all pairs $((X, t), r)$, $(X, t) \in \partial\Omega$ and $r > 0$, for which Ω does not satisfy a local interior chain condition at (X, t) , on scale r , with constant $\tilde{\gamma}$. Let

$$(4.1) \quad d\tilde{\nu}(Z, \tau, r) = \chi_E(Z, \tau, r) d\sigma(Z, \tau) r^{-1} dr,$$

where χ_E is the indicator function for the set E . Given $(Y, s) \in \partial\Omega$, $R > 0$, we say that $\Omega \cap Q_R(Y, s)$ satisfies a Carleson chain condition if there exists $\hat{\gamma}$ such that

$$(4.2) \quad \tilde{\nu}([\partial\Omega \cap Q_\rho(X, t)] \times (0, \rho)) \leq \hat{\gamma} \rho^{n+1},$$

whenever $(X, t) \in \partial\Omega$ and $Q_\rho(X, t) \subset Q_R(Y, s)$. The least such $\hat{\gamma}$ in (4.2) is called the Carleson norm of $[\partial\Omega \cap Q_R(Y, s)] \times (0, R)$. We write $\|\tilde{\nu}\|_+$ for the Carleson norm of $\tilde{\nu}$ when (4.2) holds for all $(X, t) \in \partial\Omega$, $R, \rho > 0$.

Concerning connectivity we prove the following theorem.

Theorem 4.3. *Let Σ be a closed subset of \mathbb{R}^{n+1} which is parabolic ADR with constant M . Assume that Σ is parabolic UR with constants (M, \tilde{M}) . Let $\Omega \subset \mathbb{R}^{n+1}$*

be a domain defined as a connected component of $\mathbb{R}^{n+1} \setminus \Sigma$. Assume that Ω satisfies a uniform Carleson chain condition in the sense of Definition 4.2 and with constants $\tilde{\gamma}$ and $\hat{\gamma}$. Then $\partial\Omega$ satisfies a uniform interior big pieces of regular parabolic $\text{Lip}(1, 1/2)$ graphs conditions with constants (ϵ, b_1, b_2, C) for some $\epsilon = \epsilon(n, M, \tilde{M}, \tilde{\gamma}, \hat{\gamma})$, $b_1 = b_1(n, M, \tilde{M}, \tilde{\gamma}, \hat{\gamma})$, $b_2 = b_2(n, M, \tilde{M}, \tilde{\gamma}, \hat{\gamma})$ and $C = C(n, M, \tilde{M}, \tilde{\gamma}, \hat{\gamma})$.

Proof. To prove Theorem 4.3 we only have to prove that if Ω is as in the statement of Theorem 4.3, then the additional information that Ω satisfies a uniform Carleson chain condition, with constants $\tilde{\gamma}$ and $\hat{\gamma}$, in the sense of Definition 4.2, implies that Ω satisfies a synchronized two cube condition in the sense of Definition 1.10. To start the proof of this we introduce, for $(Z, \tau) \in \Sigma, r > 0$,

$$(4.3) \quad \gamma_\infty(Z, \tau, r) = \inf_P \sup_{(Y,s) \in \Delta(Z,\tau,r)} \frac{d(Y, s, P)}{r},$$

where the infimum is taken over all n planes P containing a line parallel to the t axis. Given $(Z, \tau), r$ as above it follows, see display (2.2) in [9] that

$$(4.4) \quad \gamma_\infty(Z, \tau, r)^{n+3} \leq 16^{n+3} \gamma(Z, \tau, 2r).$$

In particular, introducing for $\epsilon > 0$ the set

$$(4.5) \quad F_\epsilon = \{(Z, \tau, r) \in \Sigma \times (0, R) : \gamma_\infty(Z, \tau, r) > \epsilon\},$$

we see, by elementary estimates, that

$$(4.6) \quad \int_0^\infty \int_\Sigma 1_{F_\epsilon \cap (Q_\rho(X,t) \times (0,\rho))}(Z, \tau, r) d\sigma \frac{dr}{r} \leq c(\epsilon) \tilde{M} \rho^{n+1}$$

whenever $(X, t) \in \Sigma$ and $0 < \rho \leq R$, and where we consider R large and fixed. Consider Definition 4.2 with constants $\tilde{\gamma}, \hat{\gamma}$. Let $E_\epsilon \subset \partial\Omega \times (0, R)$ be the set of all pairs $((X, t), r)$, $(X, t) \in \partial\Omega$ and $0 < r \leq R$, for which Ω does not satisfy a local interior chain condition at (X, t) , on scale ϵr . Then, by assumption, and by the scale invariance of the Carleson measure condition,

$$(4.7) \quad \int_0^\infty \int_\Sigma 1_{E_\epsilon \cap (Q_\rho(X,t) \times (0,\rho))}(Z, \tau, r) d\sigma \frac{dr}{r} \leq \hat{\gamma} \rho^{n+1},$$

whenever $(X, t) \in \Sigma$ and $0 < \rho \leq R$. To proceed, consider $(\tilde{Z}, \tilde{\tau}) \in \Sigma$ and $0 < \varrho \leq R$. Let $\beta, 0 < \beta \ll 1$, be a degree of freedom to be chosen and let $A_\beta(\tilde{Z}, \tilde{\tau}, \varrho)$ be the set of all pairs $((Y, s), \rho)$ such that $(Y, s) \in Q_{\varrho/2}(\tilde{Z}, \tilde{\tau}) \cap \Sigma$, $\beta\varrho \leq \rho \leq \varrho/4$, and such that $((Y, s), \rho)$ is in the intersection of the complements of the sets F_ϵ and E_ϵ . Assume that $A_\beta(\tilde{Z}, \tilde{\tau}, \varrho)$ is empty. This implies that if $(Y, s) \in Q_{\varrho/2}(\tilde{Z}, \tilde{\tau}) \cap \Sigma$, $\beta\varrho \leq \rho \leq \varrho/4$, then $((Y, s), \rho) \in F_\epsilon \cup E_\epsilon$. Hence

$$(4.8) \quad \begin{aligned} (c(\epsilon)\tilde{M} + \hat{\gamma})\varrho^{n+1} &\geq \int_0^\infty \int_\Sigma 1_{F_\epsilon \cup E_\epsilon \cap (Q_{\varrho/2}(\tilde{Z}, \tilde{\tau}) \times (0,\varrho))}(Z, \tau, r) d\sigma \frac{dr}{r} \\ &\geq c^{-1} \varrho^{n+1} \int_{\beta\varrho}^{\varrho/4} \frac{dr}{r} = c^{-1} \varrho^{n+1} \log(1/(2\beta)). \end{aligned}$$

We can conclude that if in this construction we choose $\beta = \beta(n, M, \tilde{M}, \epsilon, \tilde{\gamma}, \hat{\gamma})$ small enough, then the set $A_\beta(\tilde{Z}, \tilde{\tau}, \varrho)$ is non-empty whenever $(\tilde{Z}, \tilde{\tau}) \in \Sigma$ and $0 < \varrho \leq R$. Hence, consider $((Y, s), \rho) \in A_\beta(\tilde{Z}, \tilde{\tau}, \varrho)$. Then by construction, and the definition of the set F_ϵ , there exists a time-independent plane P such that

$$(4.9) \quad d((\tilde{Y}, \tilde{s}), P) \leq \epsilon\rho \quad \text{whenever } (\tilde{Y}, \tilde{s}) \in \Delta(Y, s, \rho).$$

Let $(n_P, 0)$ denote the normal to the plane P and set $(Y_1, s_1) = (Y + 10\epsilon\rho n_P, s)$, $(Y_2, s_2) = (Y - 10\epsilon\rho n_P, s)$. Using (4.9) we see that $(Y_1, s_1), (Y_2, s_2) \in Q_{1000\epsilon}(Y, s)$. Furthermore, to connect (Y_1, s_1) and (Y_2, s_2) by a sequence of cubes as in Definition 4.2 we see that the number of cubes needed is at least on the magnitude of $\log(\epsilon^{-1})$. However, by construction Ω satisfies a local interior chain condition at (Y, s) , on scale $\epsilon\rho$, with constant $\tilde{\gamma}$. Hence, if $(Y_1, s_1), (Y_2, s_2)$ would be in the same component these two statements would contradict each other. Hence $(Y_1, s_1), (Y_2, s_2)$ must be in different components and we can conclude that Ω satisfies a synchronized two cube condition for some constant $\gamma_0(n, M, \tilde{M}, \tilde{\gamma}, \hat{\gamma})$. This completes the proof of Theorem 4.3. \square

4.2. An application to parabolic measure

Let Σ and Ω be as above. We are interested in the bounded continuous Dirichlet problem and the parabolic measure in Ω . To start our outline, we recall the notion of thermal capacity introduced in [5]. Let K be a closed subset of \mathbb{R}^{n+1} and let $\mathcal{M}^+(K)$ be the collection of all non-negative Radon measures on \mathbb{R}^{n+1} with support contained in K . We define, following [5], the thermal capacity of K as

$$\text{Cap}(K) = \sup\{\mu(\mathbb{R}^{n+1}) : \mu \in \mathcal{M}^+(K), P_\mu \leq 1 \text{ on } \mathbb{R}^{n+1}\},$$

where

$$P_\mu(X, t) = \int_{\mathbb{R}^{n+1}} W(X - Y, t - s) d\mu(Y, s),$$

and where W is the standard heat kernel on \mathbb{R}^{n+1} . Let $(X, t) \in \partial\Omega$ and consider, for $k \in \mathbb{Z}_+$, the sets

$$A(X, t, 2^{-k}) = \{(Y, s) \in \mathbb{R}^{n+1} : (4\pi 2^{-k})^{-n/2} \leq W(X - Y, t - s) \leq (2\pi 2^{-k})^{-n/2}\}.$$

Consider the set $\partial\Omega \cap A(X, t, 2^{-k})$ and let

$$\mu_k(E) = \sigma(E \cap \partial\Omega) / (2^{-kn/2} \sigma(\partial\Omega \cap A(X, t, 2^{-k}))),$$

whenever $E \subset \mathbb{R}^{n+1}$ is a Borel set. It is then easy to see, using that Σ is parabolic ADR, that $P_{\mu_k} \leq c$ on \mathbb{R}^{n+1} , for some c independent of k . Hence,

$$\text{Cap}(\partial\Omega \cap A(X, t, 2^{-k})) \geq c^{-1} 2^{-kn/2},$$

and using the Wiener criterium in [5] we can conclude that any point $(X, t) \in \partial\Omega$ is regular for the bounded continuous Dirichlet problem for the heat equation in Ω . By analogy, the same also holds for the adjoint heat equation in Ω . Using this,

exhausting Ω by sets of the form $\Omega \cap Q_R(0, 0)$, letting $R \rightarrow \infty$, and applying Perron–Wiener–Brelot type arguments we can conclude that the bounded continuous Dirichlet problems for the heat equation, adjoint heat equation, in Ω always have unique solutions. Given $(\hat{X}, \hat{t}) \in \Omega$ we let $\omega(\hat{X}, \hat{t}, \cdot)$ be the caloric measure, at $(\hat{X}, \hat{t}) \in \Omega$, associated to the heat equation in Ω . To proceed we will now assume that Ω has additional connectivity properties.

Definition 4.4. Let Σ and Ω be as in Definition 4.1. We say that Ω has the Harnack chain property if the following condition is satisfied for some $\gamma \geq 100$. Suppose $(U_i, s_i) \in \Omega$, $i = 1, 2$, with $(s_2 - s_1)^{1/2} > \gamma^{-1} d((U_1, s_1), (U_2, s_2))$. Then there exists a Harnack chain $\{Q_{r_i}(X_i, t_i)\}_1^l$ from (U_1, s_1) to (U_2, s_2) such that the following hold for some $c(\gamma) \geq 1$.

- (i) $(U_1, s_1) \in Q_{r_1}(X_1, t_1)$, $(U_2, s_2) \in Q_{r_l}(X_l, t_l)$,
- (ii) $Q_{r_{i+1}}(X_{i+1}, t_{i+1}) \cap Q_{r_i}(X_i, t_i) \neq \emptyset$ for $i = 1, 2, \dots, l - 1$,
- (iii) $c(\gamma)^{-1} d((X_i, t_i), \partial\Omega) \leq r_i \leq c(\gamma) d((X_i, t_i), \partial\Omega)$, for $i = 1, 2, \dots, l$,
- (iv) $t_{i+1} - t_i \geq c(\gamma)^{-1} r_i^2$, for $i = 1, 2, \dots, l$,
- (v) $l \leq c(\gamma) \log \left(2 + \frac{d((U_1, s_1), (U_2, s_2))}{\min[d((U_1, s_1), \partial\Omega), d((U_2, s_2), \partial\Omega)]} \right)$.

Remark 4.5. Note that the connectivity conditions in Definition 4.2 and Definition 4.4 are different. The condition in Definition 4.2 is a condition concerning the (local) connectivity of Ω formulated in terms of a Carleson measure condition. The condition in Definition 4.4 is constructed in order to be able to use the Harnack inequality for the heat equation in an efficient way to compare the values of non-negative solutions respecting the time-lag in the parabolic Harnack inequality. Definition 4.2 makes no reference to Harnack chains.

Assume in the following that $\partial\Omega$ satisfies a synchronized two cube condition in the sense of Definition 1.9. For (X, t) , $r > 0$, and $A \geq 100$, we define

$$(4.10) \quad \Gamma_A^+(X, t, r) = \{(Y, s) : |Y - X|^2 \leq A|s - t|, |s - t| \geq 5r^2, s > t\}.$$

Definition 4.6. Let $(X, t) \in \partial\Omega$, $r > 0$, and consider $(\hat{X}, \hat{t}) \in \Omega \cap \Gamma_A^+(X, t, 4r)$. We say that $\omega(\cdot) = \omega(\hat{X}, \hat{t}, \cdot)$ satisfies the A_∞ condition on $\partial\Omega \cap Q_r(X, t)$, with constants L and $\lambda > 0$ if the following is true. $d\omega/d\sigma = h$ exists on $\partial\Omega \cap Q_r(X, t)$ and

$$(4.11) \quad \int_{\partial\Omega \cap Q_\rho(\tilde{X}, \tilde{t})} h^{1+\lambda} d\sigma \leq L \sigma(Q_\rho(\tilde{X}, \tilde{t}))^{-\lambda} (\omega(Q_\rho(\tilde{X}, \tilde{t})))^{1+\lambda}$$

whenever $(\tilde{X}, \tilde{t}) \in \partial\Omega$, $Q_{2\rho}(\tilde{X}, \tilde{t}) \subset Q_r(X, t)$.

Theorem 4.7. Let Σ be a closed subset of \mathbb{R}^{n+1} which is parabolic ADR with constant M . Assume that Σ is parabolic UR with constants (M, \tilde{M}) . Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain defined as a connected component of $\mathbb{R}^{n+1} \setminus \Sigma$. Assume that Ω satisfies a synchronized two cube condition with constant γ_0 in the sense of Definition 1.10.

Assume that Ω has the Harnack chain property in the sense of Definition 4.4 and with constant γ . Let $(X, t) \in \partial\Omega$, $r > 0$, $A \geq 100$, and consider $(\hat{X}, \hat{t}) \in \Omega \cap \Gamma_A^+(X, t, 4r)$. Then $\omega(\hat{X}, \hat{t}, \cdot)$ is a doubling measure in the sense that there exists a constant $c = c(n, M, \gamma_0, \gamma, A)$ such that

$$(4.12) \quad \omega(\hat{X}, \hat{t}, \partial\Omega \cap Q_{2\rho}(\tilde{X}, \tilde{t})) \leq c\omega(\hat{X}, \hat{t}, \partial\Omega \cap Q_\rho(\tilde{X}, \tilde{t})),$$

whenever $(\tilde{X}, \tilde{t}) \in \partial\Omega$, $Q_\rho(\tilde{X}, \tilde{t}) \subset Q_{2r}(X, t)$. Furthermore, $\omega(\hat{X}, \hat{t}, \cdot)$ satisfies the A_∞ condition on $\partial\Omega \cap Q_r(X, t)$ in the sense of Definition 4.6 with constants $L = L(n, M, \tilde{M}, \gamma_0, \gamma, A)$ and $\lambda(n, M, \tilde{M}, \gamma_0, \gamma, A) > 0$.

Proof. If Ω satisfies a synchronized two cube condition in the sense of Definition 1.10, and if Ω has the Harnack chain property in the sense of Definition 4.4 with constant γ , then Ω is a parabolic NTA-domain in the sense of [12] and [10]. Therefore, we see that (4.12) follows from [12] and [10]. To prove the statement that $\omega(\hat{X}, \hat{t}, \cdot)$ satisfies the A_∞ condition on $\partial\Omega \cap Q_r(X, t)$ in the sense of Definition 4.6, we first observe that the assumptions stated in the theorem imply, see Theorem 1.13, that Ω contains uniform interior big pieces of regular parabolic $\text{Lip}(1, 1/2)$ graphs. Using this, and that Ω is a parabolic NTA-domain, we can follow the proof of Theorem 1 in [10] essentially verbatim to conclude that $\omega(\hat{X}, \hat{t}, \cdot)$ satisfies the A_∞ condition on $\partial\Omega \cap Q_r(X, t)$. At this stage in [9], having established uniform interior big pieces of regular parabolic $\text{Lip}(1, 1/2)$ graphs, the assumption concerning Reifenberg flatness imposed in [9] is only used to ensure that Ω is a parabolic NTA-domain, something we in our setting already have concluded. We therefore omit further details of the proof and simply refer to the proof of Theorem 1 in [10]. \square

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References

- [1] BENNEWITZ, B. AND LEWIS, J. L.: On weak reverse Hölder inequalities for nondoubling harmonic measures. *Complex Var. Theory Appl.* **49** (2004), no. 7-9, 571–582.
- [2] DAVID, G. AND JERISON, D.: Lipschitz approximation to hypersurfaces, harmonic measure, and singular integrals. *Indiana Univ. Math. J.* **39** (1990), no. 3, 831–845.
- [3] DAVID, G. AND SEMMES, S.: Singular integrals and rectifiable sets in \mathbb{R}^n : Beyond Lipschitz graphs. *Astérisque* **193** (1991), 152 pp.
- [4] DAVID, G. AND SEMMES, S.: *Analysis of and on uniformly rectifiable sets*. Mathematical Surveys and Monographs 38, American Mathematical Society, Providence, RI, 1993.
- [5] EVANS, L. C. AND GARIEPY, R. F.: Wiener's criterion for the heat equation. *Arch. Rational Mech. Anal.* **78** (1982), no. 4, 293–314.
- [6] HOFMANN, S: Parabolic singular integrals of Calderón type, rough operators, and caloric layer potentials. *Duke Math. J.* **90** (1997), no. 2, 209–259.

- [7] HOFMANN, S. AND LEWIS, J.: L^2 solvability and representation by caloric layer potentials in time-varying domains. *Ann. of Math. (2)* **144** (1996), no. 2, 349–420.
- [8] HOFMANN, S. AND LEWIS, J.: The Dirichlet problem for parabolic operators with singular drift terms. *Mem. Amer. Math. Soc.* **151** (2001), no. 719, viii+113 pp.
- [9] HOFMANN, S., LEWIS, J. AND NYSTRÖM, K.: Existence of big pieces of graphs for parabolic problems. *Ann. Acad. Sci. Fenn. Math.* **28** (2003), no. 2, 355–384.
- [10] HOFMANN, S., LEWIS, J. AND NYSTRÖM, K.: Caloric measure in parabolic flat domains. *Duke Math. J.* **122** (2004), no. 2, 281–346.
- [11] JERISON, D. AND KENIG, C.: Boundary behavior of harmonic functions in nontangentially accessible domains. *Adv. in Math.* **46** (1982), no. 1, 80–147.
- [12] LEWIS, J. L. AND MURRAY, M.: The method of layer potentials for the heat equation in time-varying domains. *Mem. Amer. Math. Soc.* **114** (1995), no. 545, viii+157 pp.
- [13] LEWIS, J. L. AND SILVER, J.: Parabolic measure and the Dirichlet problem for the heat equation in two dimensions. *Indiana Univ. Math. J.* **37** (1988), no. 4, 801–839.
- [14] LEWIS, J. L. AND VOGEL, A. L.: Uniqueness in a free boundary problem. *Comm. Partial Differential Equations* **31** (2006), no. 10-12, 1591–1614.
- [15] LEWIS, J. L. AND VOGEL, A. L.: Symmetry theorems and uniform rectifiability. *Bound. Value Probl.* **2007** (2007), article ID 030190, 59 pages.
- [16] NYSTRÖM, K. AND STRÖMQVIST, M.: Analysis of and on parabolic uniformly rectifiable sets satisfying a synchronized two cube condition. In preparation.
- [17] RIVERA-NORIEGA, J.: A parabolic version of Corona decompositions. *Illinois J. Math.* **53** (2009), no. 2, 533–559.
- [18] RIVERA-NORIEGA, J.: Two results over sets with big pieces of parabolic Lipschitz graphs. *Houston J. Math.* **36** (2010), no. 2, 619–635.
- [19] RIVERA-NORIEGA, J.: Parabolic singular integrals and uniformly rectifiable sets in the parabolic sense. *J. Geom. Anal.* **23** (2013), no. 3, 1140–1157.

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