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Tangent measures and absolute continuity of harmonic measure

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Abstract. We show that for uniform domains $\Omega \subseteq \mathbb{R}^{d+1}$ whose boundaries satisfy a certain nondegeneracy condition that harmonic measure cannot be mutually absolutely continuous with respect to α-dimensional Hausdorff measure unless $\alpha \leq d$. We employ a lemma that shows that, at almost every non-degenerate point, we may find a tangent measure of harmonic measure whose support is the boundary of yet another uniform domain whose harmonic measure resembles the tangent measure.

1. Introduction

In this paper we discuss when the harmonic measure ω_Ω for a domain $\Omega \subseteq \mathbb{R}^{d+1}$ can be mutually absolutely continuous with respect to some Hausdorff measure $\mathscr{H}^\alpha$. This is a popular problem in the case $\alpha = d$. For a simply connected planar domain $\Omega \subseteq \mathbb{C}$, $\omega_\Omega \ll \mathscr{H}^1|_{\partial\Omega} \ll \omega_\Omega$ if and only if $\partial\Omega$ is a rectifiable curve by the F. and M. Riesz theorem [31] (also see [17]). In higher dimensions, some extra geometric assumptions on the domain are necessary due to counterexamples by Wu and Ziemer [33], [34]. Building on work of Dahlberg [15], David and Jerison showed in [16] that harmonic measure is in fact A_∞-equivalent to $\mathscr{H}^d|_{\partial\Omega}$ if $\Omega \subseteq \mathbb{R}^{d+1}$ is a non-tangentially accessible domain with Ahlfors d-regular boundary.

Definition 1.1. We say $\Omega \subseteq \mathbb{R}^{d+1}$ is a C-*uniform domain* if, for every $x, y \in \overline{\Omega}$ there is a path $\gamma \subseteq \Omega$ connecting x and y such that

(1) the length of γ is at most $C|x-y|$ and

(2) for $t \in \gamma$, $\operatorname{dist}(t, \partial\Omega) \geq \operatorname{dist}(t, \{x,y\})/C$.

A curve satisfying the above conditions is called a *good curve* for x and y in Ω. We say Ω satisfies the C-*interior corkscrew condition* if for all $\xi \in \partial\Omega$ and $r \in (0, \operatorname{diam}\partial\Omega)$ there is a ball $B(x, r/C) \subseteq \Omega \cap B(\xi, r)$.

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If $B = B(\xi, r)$, we call $B' = B(x, r/C)$ the *corkscrew ball* of B and denote its center by x_B. We say Ω satisfies the *C-exterior corkscrew condition* if there is a ball $B(y, r/C) \subseteq B(\xi, r)\backslash\Omega$ for all $\xi \in \partial\Omega$ and $r \in (0, \operatorname{diam}\partial\Omega)$. A domain $\Omega \subseteq \mathbb{R}^{d+1}$ is *C-non-tangentially accessible* (or *C-NTA*) if it has the uniform, exterior and interior corkscrew properties with constants C.

Our definition of NTA domains is slightly different than that introduced by Jerison and Kenig in [19], but it is equivalent [8]. The appeal of these domains aside from their nice geometry are the convenient scale invariant properties of harmonic measure like being doubling. However, many of these properties have been generalized to other domains, see for example [1], [2], and [28].

One still has $\mathscr{H}^d|_{\partial\Omega} \ll \omega_\Omega$ if $\Omega \subseteq \mathbb{R}^{d+1}$ is NTA and we just assume $\mathscr{H}^d|_{\partial\Omega}$ is locally finite instead of Ahlfors d-regular [12] (or even when Ω is just uniform with rectifiable boundary [27]), but we do not get mutual absolute continuity [9]. For the most part, all these results require either assuming or establishing some rectifiability properties of the boundary of Ω. Recently it was shown that rectifiability is actually necessary to have $\omega_\Omega \ll \mathscr{H}^d$ even on a subset of $\partial\Omega$ [7]. See also [3], [18].

The focus for us, however, will be on the relationship between harmonic measure and $\mathscr{H}^\alpha$ for $\alpha \neq d$, Makarov showed that for simply connected planar domains we have $\omega_\Omega \perp \mathscr{H}^\alpha|_{\partial\Omega}$ for $\alpha > 1$ and $\omega_\Omega \ll \mathscr{H}^\alpha|_{\partial\Omega}$ if $\alpha < 1$ [24]. This is a uniquely planar property, though: for $d \geq 2$, there are NTA topological spheres in $\mathbb{R}^{d+1}$ called *Wolff snowflakes* for which either $\dim \omega_\Omega < d$ or $\dim \omega_\Omega > d$. In particular, we can have domains where $\omega_\Omega \ll \mathscr{H}^\alpha$ on a set of positive harmonic measure for some $\alpha > d$ and $\omega_\Omega \perp \mathscr{H}^\alpha$ for some $\alpha < d$. The $\mathbb{R}^3$ case is due to Wolff [32] and the result for higher dimensions is due to Lewis, Verchota, and Vogel [23]. A corollary of our main results, however, will show that, for NTA domains, mutual absolute continuity can only occur if $\alpha \leq d$.

Corollary I. *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be an* NTA *domain and $E \subseteq \mathbb{R}^{d+1}$ such that $\omega_\Omega(E) > 0$ and $\omega_\Omega|_E \ll \mathscr{H}^\alpha|_E \ll \omega_\Omega|_E$. Then $\alpha \leq d$.*

Our main result holds in more general circumstances. Firstly, the domain need not be NTA but just uniform, and the points in E need to satisfy a nondegeneracy condition.

Definition 1.2. For $\Omega \subseteq \mathbb{R}^{d+1}$ connected and $\beta, \delta \in (0, 1)$, we say $\xi \in \partial\Omega$ is *(β, δ)-non-degenerate* if

$$\eta_\delta(\xi) := \limsup_{r \to 0} \eta_\delta(\xi, r) \leq \beta, \tag{1.1}$$

where

$$\eta_\delta(\xi, r) := \sup_{\substack{x \in \Omega \\ |x-\xi| = \delta r}} \omega^x_{B(\xi,r)\cap\Omega}(\partial B(\xi, r) \cap \Omega).$$

We will say ξ is non-degenerate if it is (β, δ)-non-degenerate for some $\beta, \delta > 0$.

This may seem slightly messy, but it is satisfied at each point in $\partial\Omega$ when Ω satisfies the capacity density condition, which we will define later. In particular, this includes NTA domains.

Next, to establish the bound $\alpha \le d$, we do not need mutual absolute continuity but just some control on the upper densities of harmonic measure. Recall that we define the *upper and lower α-densities* for a measure μ as

$$\theta^{\alpha,*}(\mu,\xi) = \limsup_{r\to 0} \frac{\mu(B(\xi,r))}{r^s} \quad \text{and} \quad \theta^{\alpha}_{*}(\mu,\xi) = \liminf_{r\to 0} \frac{\mu(B(\xi,r))}{r^s}.$$

We can now state the main result.

Theorem I. *Let $d \ge 1$, and $\Omega \subseteq \mathbb{R}^{d+1}$ be a uniform domain. Suppose there is $\alpha > 0$, $x_0 \in \Omega$, and a set $E \subseteq \partial\Omega$ with $\omega_\Omega^{x_0}(E) > 0$ such that each $\xi \in E$ is non-degenerate and*

$$0 < \theta^{\alpha,*}(\omega_\Omega^{x_0},\xi) < \infty \quad \textit{for } \xi \in E. \tag{1.2}$$

Then $\alpha \le d$.

Observe that if $E \subseteq \partial\Omega$ is a set with $\mathscr{H}^\alpha(E) < \infty$ and $\omega_\Omega^{x_0} \ll \mathscr{H}^\alpha$ on E, then $\theta^{\alpha,*}(\omega_\Omega^{x_0},\xi) < \infty$ for $\omega_\Omega^{x_0}$-almost every $\xi \in E$, and so having finite densities is a weaker condition in this scenario. Indeed, for $\xi \in E$,

$$\begin{aligned}\theta^{\alpha,*}(\omega_\Omega^{x_0},\xi) &= \limsup_{r\to 0} \frac{\omega_\Omega^{x_0}(B(\xi,r))}{r^\alpha} \\ &= \limsup_{r\to 0} \frac{\omega_\Omega^{x_0}(B(\xi,r))}{\omega_\Omega^{x_0}(E\cap B(\xi,r))} \frac{\omega_\Omega^{x_0}(E\cap B(\xi,r))}{\mathscr{H}^\alpha(E\cap B(\xi,r))} \frac{\mathscr{H}^\alpha(E\cap B(\xi,r))}{r^\alpha}.\end{aligned}$$

The first quotient converges to 1 for ω_Ω-almost every $\xi \in E$ by Corollary 2.14 in [26]. The second quotient converges to a finite number for $\mathscr{H}^\alpha$-almost every (and hence ω_Ω-almost every) $\xi \in E$ by Theorem 2.12 in [26]. Finally, by Theorem 6.2 in [26], $\theta^{\alpha,*}(\mathscr{H}^d|_{\partial\Omega},\cdot) \in (0,\infty)$ and hence the limit of the third quotient has finite supremal limit for $\mathscr{H}^\alpha$-almost every (and hence ω_Ω-almost every) $\xi \in E$. Thus, the above equalities give

$$\theta^{\alpha,*}(\omega_\Omega^{x_0},\cdot) \le \frac{d\omega_\Omega|_E}{d\mathscr{H}^\alpha|_E} \cdot \theta^{\alpha,*}(\mathscr{H}^d|_{\partial\Omega},\cdot) \quad \omega_\Omega^{x_0}\text{-a.e. in } E \text{ if } \mathscr{H}^\alpha(E) < \infty. \tag{1.3}$$

In particular, this gives the following corollaries.

Corollary II. *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a uniform domain, $\alpha > d$, and $E \subseteq \partial\Omega$ be a set of non-degenerate points of finite $\mathscr{H}^\alpha$-measure such that $\omega_\Omega^{x_0}|_E \ll \mathscr{H}^\alpha|_E$. Then $\theta^{\alpha,*}(\omega_\Omega^{x_0},\xi) = 0$ for $\omega_\Omega^{x_0}$-almost every $\xi \in E$.*

Corollary III. *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a uniform domain, $E \subseteq \partial\Omega$ a set of non-degenerate points, and $\alpha > d$. It is impossible for $\omega_\Omega^{x_0}|_E \ll \mathscr{H}^\alpha|_E \ll \omega_\Omega^{x_0}|_E$ unless $\omega_\Omega^{x_0}(E) = 0$.*

Corollary III implies the conditions of Corollary II since $\mathscr{H}^\alpha|_E \ll \omega_\Omega|_E$ implies $\mathscr{H}^\alpha(E) < \infty$.

Mutual absolute continuity can in fact occur for $\alpha < d$. At the time of writing this manuscript, Alexander Volberg informed us that he constructed a uniform domain $\Omega \subseteq \mathbb{R}^{d+1}$ satisfying the capacity density condition (so every point is non-degenerate by Lemma 2.6 below) such that $\omega_\Omega \ll \mathscr{H}^\alpha|_{\partial\Omega} \ll \omega_\Omega$ for some $\alpha < d$. The construction is a modification of another example given by Bishop and Jones in the plane of a rectifiable set $E \subseteq \mathbb{R}^2$ that contains a set of positive harmonic measure but zero Hausdorff 1-measure [13].

We can also bound α from below in certain circumstances.

Theorem II. *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a uniform domain and let $E \subseteq \partial\Omega$ have positive harmonic measure such that*

$$0 < \theta_*^\alpha(\omega_\Omega^{x_0}, \xi) < \infty \quad \text{for all } \xi \in E. \tag{1.4}$$

Then $\alpha > d$. If for some $s > d-1$ we have, for each $\xi \in E$,

$$\liminf_{r\to 0} \frac{\mathscr{H}^s_\infty(B(\xi,r)\cap\partial\Omega)}{r^s} > 0, \tag{1.5}$$

then $\alpha \geq s$.

Below we present a few simple corollaries of the main results.

Corollary IV. *If Ω is an* NTA *domain satisfying*

$$0 < \theta_*^\alpha(\omega_\Omega^{x_0}, \xi) \leq \theta^{\alpha,*}(\omega_\Omega^{x_0}, \xi) < \infty$$

for ξ in a set of positive harmonic measure, then $\alpha = d$.

Indeed, it is not difficult to show that NTA domains satisfy (1.5) with $s = d$. If B is a ball centered on $\partial\Omega$ of radius r_B and B_0 is a ball of radius r_{B_0} comparable to r_B so that $2B_0 \subseteq \Omega\cap B$, then the existence of the exterior corkscrew ball implies that the radial projection of $\partial\Omega\cap B$ onto ∂B_0 has measure at least a constant times $r_{B_0}^d$ (and hence at least a constant times r_B^d). Since the radial projection onto ∂B_0 is Lipschitz on $(2B_0)^c$, this implies (1.5) with $s = d$. Also, any point satisfying (1.5) is non-degenerate by Lemma 2.7 below. Thus, the corollary follows from Theorem I and Theorem II .

This corollary is particularly interesting in the context of Wolff snowflakes. Recall that if μ is a Borel probability measure in $\mathbb{R}^{d+1}$, we define its *lower and upper pointwise dimensions* at the point $x \in \operatorname{supp}\mu$ to be

$$\underline{d}_\mu(x) = \liminf_{r\to 0} \frac{\log\mu(B(x,r))}{\log r} \quad \text{and} \quad \overline{d}_\mu(x) = \limsup_{r\to 0} \frac{\log\mu(B(x,r))}{\log r},$$

respectively. The common value $\underline{d}_\mu(x) = \overline{d}_\mu(x) = d_\mu(x)$, if it exists, we call it *pointwise dimension* of μ at $x \in \operatorname{supp}\mu$. Wolff in fact constructs domains $\Omega \subseteq \mathbb{R}^{d+1}$ where $d_{\omega_\Omega} < d$ ω_Ω-almost everywhere or $d_{\omega_\Omega} > d$ ω_Ω-almost everywhere. Note that if the upper and lower α-densities are finite and positive at a point, this implies the pointwise dimension at that point is α as well. In other words, ω_Ω having pointwise dimension α at ξ means that for all $\varepsilon > 0$, $r^{\alpha+\varepsilon} < \omega_\Omega(B(\xi, r)) < r^{\alpha-\varepsilon}$ for $r > 0$ small enough, while having positive lower density means that $cr^\alpha < \omega_\Omega(B(\xi, r)) < Cr^\alpha$ for r small and some constants $c, C > 0$. Thus, our results show that while the pointwise dimensions can be noninteger for these Wolff domains, the upper and lower α-densities cannot be finite and positive on a set of positive measure.

To prove Theorem I , we will rely heavily on the tangent measures of Preiss [29]. Recall that if μ is a Radon measure and $x \in \operatorname{supp} \mu$, then the *tangent measures of μ at x*, denoted $\operatorname{Tan}(\mu, x)$, is the set of measures ν that are weak limits of the form $\nu = \lim_{j\to\infty} c_j T_{x,r_j\#}\mu$, where $c_j \geq 0$, $r_j \downarrow 0$, and

$$T_{x,r}(y) = \frac{y - x}{r}.$$

Tangent measures have been employed to study the relationship between harmonic measure and the geometry of the boundary in several papers, see for example [11], [20], [21], and [22]. Other results which do not use tangent measures but employ more quantitative techniques modelled after tangent measure methods include [30].

The following is the main lemma we employ, whose proof takes up most of the paper and may be of independent interest.

Lemma I. *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a uniform domain, $d \geq 1$, and $x_0 \in \Omega$. Fix $\delta \in (0, 1)$, let $E \subseteq \partial\Omega$ be the set of (β, δ)-non-degenerate points, and suppose it has positive $\omega_\Omega^{x_0}$-measure. Then for $\omega_\Omega^{x_0}$-almost every $\xi_0 \in E$, $\operatorname{Tan}(\omega_\Omega^{x_0}, \xi_0) \neq \emptyset$. Moreover, if we have a tangent measure μ that is the weak limit of $T_{\xi_0,r_j\#}\omega_\Omega^{x_0}/\omega_\Omega^{x_0}(B(\xi_0, r_j))$, then we may pass to a subsequence such that the following hold.*

(1) *$\operatorname{supp} \mu$ is the boundary of a C'-uniform domain $\tilde\Omega$ that is Δ-regular (see Definition 2.4 below), where C' depends on C and d, and the Δ-regularity data depend additionally on δ and β.*

(2) *There is a uniform subdomain Ω^* dense in $\tilde\Omega$ such that, for all $x \in \Omega^*$, if $\Omega_j := T_{\xi_0,r_j}(\Omega)$, then $x \in \Omega_j$ for all sufficiently large j.*

(3) *Let $\omega_j := \omega_{\Omega_j}$. For $x \in \Omega^*$, ω_j^x converges weakly to $\omega_{\tilde\Omega}^x$.*

(4) *For continuous functions f vanishing at infinity, the harmonic functions $\int f d\omega_j$ converge to $\int f d\omega_{\tilde\Omega}$ uniformly on compact subsets of Ω^*.*

(5) *If (1.5) holds, then there exists $c' > 0$ depending on c and d so that $\mathscr{H}^s_\infty(B(\xi, r) \cap \partial\tilde\Omega) \geq c' r^s$ for all $\xi \in \partial\tilde\Omega$ and $r > 0$.*

(6) *Finally, there is C_0, depending on d and C, so that if $B' \subseteq B = B(\xi, r)$ are balls centered on $\partial\tilde\Omega$ and $B(x, r/C') \subseteq B \cap \Omega$, then*

$$C_0^{-1}\frac{\mu(B')}{\mu(B)} \leq \frac{\omega_{\tilde\Omega}^x(B')}{\omega_{\tilde\Omega}^x(B)} \leq C_0\frac{\mu(B')}{\mu(B)}. \tag{1.6}$$

Similar results were shown by Kenig and Toro in the case of NTA domains, [21], and in the case of two-sided NTA domains, [22]. For example, Lemma 3.8 in [21] shows the above for result for NTA domains, and the tangent measure μ is what they call the *tangent measure at* ∞ for $\tilde{\Omega}$. The inequality (1.6) is not stated there but follows from their work (see also Lemma 4.2 in [22]). What is special about the above lemma, however, is that it works for more general domains, and secondly, that we can fix a point in the limiting domain $\tilde{\Omega}$ and the scaled harmonic measures ω_j^x will converge to the corresponding harmonic measure in $\tilde{\Omega}$. In a recent paper with Tolsa, we also obtain slightly weaker versions of the blow up results of Kenig and Toro that held for Δ-regular domains without assuming uniformity (see [10]); however Δ-regularity is much stronger than the assumptions in Lemma I, and we did not obtain (1.6) in the purely Δ-regular (non-uniform) setting.

Note that $\tilde{\Omega}$ in Lemma I is a uniform Δ-regular domain, and thus a uniform domain satisfying the CDC (if $d \geq 2$). This latter set satisfies many useful properties (such as harmonic measure being doubling, see [2]) that the original domain Ω may not have enjoyed originally.

There are several possible venues for improvement and inquiry. Firstly, can we relax the uniformity and nondegeneracy conditions? These are used in quite crucial ways in the proof. Secondly, we note that in Volberg's example, $\theta_*^\alpha(\mathscr{H}^d|_{\partial\Omega}, \cdot) = 0$ $\mathscr{H}^\alpha$-almost everywhere, and hence $\theta_*^\alpha(\omega_\Omega|, \cdot)$ vanishes ω_Ω-almost everywhere, and we do not know if $\alpha = d$ otherwise.

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2. Preliminaries

2.1. Notation

We will work entirely in in $\mathbb{R}^{d+1}$ with $d \geq 1$. We write $a \lesssim b$ if there is $C > 0$ so that $a \leq Cb$ and $a \lesssim_t b$ if C depends on the parameter t. We write $a \sim b$ to mean $a \lesssim b \lesssim a$ and define $a \sim_t b$ similarly. To simplify notation, we will implicitly assume that all implied constants depend on d

The open ball centered at $\xi \in \mathbb{R}^{d+1}$ of radius $r > 0$ will be denoted $B(\xi, r)$, and in particular, we will write $\mathbb{B} := B(0,1)$. If $x \in \mathbb{R}^d$, we will denote the d-dimensional ball by $B(\xi, r) = B(\xi, r) \cap \mathbb{R}^d$. If B is a ball, its radius will be denoted r_B. If $\Omega \subseteq \mathbb{R}^{d+1}$ is a domain, we will write $\Omega^{\text{ext}} = (\overline{\Omega})^c$.

If Ω is uniform and B is centered on the boundary, we will write x_B for a point such that $B(x_B, r_B/C) \subseteq B \cap \Omega$ (or, to permit us some flexibility, any point x_B satisfying this with constant C comparable to the original constant in the definition of uniform domains).

For sets $A, B \subseteq \mathbb{R}^{d+1}$, we let

$$\operatorname{dist}(A,B) = \inf\{|x-y| : x \in A, y \in B\}, \quad \operatorname{dist}(x,A) = \operatorname{dist}(\{x\},A), \quad \text{and}$$
$$\operatorname{diam} A = \sup\{|x-y| : x, y \in A\}.$$

For $A \subseteq \mathbb{R}^{d+1}$, $\alpha > 0$, and $\delta \in (0, \infty]$, define

$$\mathscr{H}_\delta^\alpha(A) = \inf\left\{\sum r_i^\alpha : A \subseteq \bigcup B(x_i, r_i), x_i \in \mathbb{R}^{d+1}, \ \ r_i < \delta\right\}.$$

We define the *α-dimensional Hausdorff measure* as

$$\mathscr{H}^{\alpha}(A)=\lim_{\delta\downarrow 0}\mathscr{H}^{\alpha}_{\delta}(A),$$

the *d-dimensional Hausdorff content* as $\mathscr{H}^{\alpha}_{\infty}(A)$, and the *Hausdorff dimension of A* as $\dim A=\inf\{\alpha:\mathscr{H}^{\alpha}(A)=0\}$. See Chapter 4 in [26] for more information about Hausdorff measure.

If μ_j and ν are Borel measures, we will denote by $\mu_j \rightharpoonup \nu$ the weak convergence of μ_j to ν.

2.2. Regularity of harmonic functions

Here we collect some lemmas about harmonic measure.

Definition 2.1. A domain $\Omega\subseteq\mathbb{R}^{d+1}$ satisfies the *Harnack chain condition* if there is $C>0$ so that for all Λ there is $N(\Lambda)$ such that for all $\varepsilon>0$ and $x,y\in\Omega$ with $\operatorname{dist}(\{x,y\},\partial\Omega)\geq\varepsilon$ and $|x-y|\leq\Lambda\varepsilon$, there is a chain of balls $B_1,\dots,B_N\subseteq\Omega$ with

(1) $N\leq N(\Lambda)$,

(2) $r_{B_i}/C\leq\operatorname{dist}(B_i,\partial\Omega)\leq Cr_{B_i}$ for $i=1,\dots,N$,

(3) $B_i\cap B_{i+1}\neq\emptyset$ for $i=1,\dots,N-1$, and

(4) $x\in B_1$ and $y\in B_N$.

In particular, if u is a positive harmonic function on Ω, then by repeated use of Harnack's inequality on each B_i,

$$u(x)\sim_{\Lambda}u(y)\quad\text{if}\quad\frac{|x-y|}{\operatorname{dist}(\{x,y\},\partial\Omega)}\leq\Lambda. \tag{2.1}$$

Lemma 2.2 (Theorem 2.15 in [8]). *A domain is uniform if and only if it satisfies the interior corkscrew and Harnack chain conditions quantitatively.*

The way nondegeneracy will manifest in our proof is the following lemma.

Lemma 2.3. *Suppose $\Omega\subseteq\mathbb{R}^{d+1}$, $\delta\in(0,1)$, $\xi\in\partial\Omega$ and $\omega^{x}_{B\cap\Omega}(\partial B(\xi,r)\cap\Omega)\leq\beta<1$ for $x\in\partial B(\xi,\delta r)\cap\Omega$ and $r\in(0,R)$. Then there is $\alpha=\alpha(\beta,d)$ so that, for all $r\in(0,R)$,*

$$\omega^{x}_{\Omega}\big(\overline{B(\xi,r)}^{c}\big)\lesssim_{\beta,\delta}\left(\frac{|x-\xi|}{r}\right)^{\alpha}\quad\textit{for } x\in\Omega\cap B(\xi,r). \tag{2.2}$$

In particular, ξ is a regular point for $\partial\Omega$.

Proof. Let $B=B(\xi,r)$, $r<R$, and ϕ be a continuous function such that $\mathbb{1}_B\leq\phi\leq\mathbb{1}_{2B}$ and let $\psi=1-\phi$. Let $u_\psi=\int\psi d\omega_\Omega$. Then by the maximum principle, for $x\in\delta B$,

$$u_\psi(x)\leq\omega^{x}_{\Omega}(B^c)\leq\omega^{x}_{\Omega\cap B}(\partial B\cap\Omega)\leq\beta<1.$$

Thus, again by the maximum principle, for $x\in\delta^2B\cap\Omega$,

$$u_\psi(x)\leq\beta\omega^{x}_{\Omega\cap\delta B}(\partial(\delta B)\cap\Omega)\leq\beta^2,$$

and inductively, we have

$$u_\psi(x) \le \beta^j \quad \text{for } x \in \delta^j B \cap \Omega \text{ and } j \ge 0.$$

Thus there is $\alpha = \alpha(\beta,\delta) > 0$ such that

$$u_\psi(x) \lesssim_{\beta,\delta} \left(\frac{|x-\xi|}{r}\right)^\alpha \quad \text{for } x \in \delta B \cap \Omega.$$

Having ϕ decrease pointwise to $\mathbb{1}_{\overline{\mathbb{B}}}$, we have

$$\omega_\Omega^x(\overline{B}^c) \lesssim_{\beta,\delta} \left(\frac{|x-\xi|}{r}\right)^\alpha \quad \text{for } x \in \delta B \cap \Omega. \qquad \square$$

Definition 2.4 ([4]). For a domain $\Omega \subseteq \mathbb{R}^{d+1}$ and a ball B centered on $\partial\Omega$, and $x \in B \cap \Omega$. We say that Ω is *uniformly Δ-regular* if there are $\delta \in (0,1)$ and $R_\Delta \in (0,\infty]$ so that

$$\sup_{\xi \in \partial\Omega} \sup_{r < R_\Delta} \eta_\delta(\xi, r) < 1. \tag{2.3}$$

The original definition is given with $\delta = 1/2$, but it is not difficult (but still tedious) to show that for uniform domains if the definition holds for one value δ then it holds for any $\delta \in (0,1)$.

Definition 2.5 ([1]). Let $d \ge 2$ and let Cap denote the Newtonian capacity. A domain $\Omega \subseteq \mathbb{R}^{d+1}$ satisfies the *capacity density condition* (or CDC) if there is $R_\Omega > 0$ so that $\mathrm{Cap}(B\backslash\Omega) \gtrsim r_B^{d-1}$ for any ball B centered on $\partial\Omega$ of radius $r_B \in (0, R_\Omega)$.

The same definition works in the plane with the logarithmic capacity, but we will not use it here.

It was shown in [4] that the CDC is equivalent to Δ-regularity for $d \ge 2$.

Theorem 2.6 (Lemma 3 in [4]). *For $d \ge 2$, there is $c \ge 4$ so that if $\Omega \subseteq \mathbb{R}^{d+1}$ and B is centered on $\partial\Omega$, then* $\mathrm{Cap}(B\backslash\Omega) \gtrsim r_B^{d-1}$ *if and only if there is $\beta \in (0,1)$ so that $\omega_{cB\cap\Omega}^x(\partial(cB)\cap\Omega) \le \beta$ on $\partial(2B)\cap\Omega$. In particular, Ω is uniformly Δ-regular if and only if it satisfies the* CDC.

Lemma 2.7 (Lemma 1 in [14], Lemma 3.4 in [7]). *Let $d \ge 1$ and $\Omega \subseteq \mathbb{R}^{d+1}$ be a domain, $\xi \in \partial\Omega$, $r > 0$, $B := B(\xi, r)$, and suppose that $\rho := \mathscr{H}_\infty^s(\partial\Omega \cap \delta B)/(\delta r)^s$ for some $s > d-1$. Then*

$$\omega_{\Omega\cap B}^x(B) \gtrsim \rho \quad \textit{for all } x \in \delta B \cap \Omega. \tag{2.4}$$

In particular, if ξ satisfies (1.5), then ξ is a non-degenerate point. If $\mathscr{H}^s(\partial\Omega \cap \delta B)/(\delta r_B)^s \gtrsim 1$ for all balls B centered on $\partial\Omega$ of radius less than some r_0, then Ω is Δ-regular.

For the $d = 2$ case, see [14]; the general case is identical, but a proof is given in [7] as well.

Finally, we recall some lemmas from [28].

Lemma 2.8 (Theorem 1.3 in [28]). *Let $d \geq 1$, Ω be a C-uniform domain in $\mathbb{R}^{d+1}$ and let B be a ball centered at $\partial\Omega$. Let $p_1, p_2 \in \Omega$ be such that $\operatorname{dist}(p_i, B \cap \partial\Omega) \geq c_0^{-1} r_B$ for $i = 1, 2$. Then, for all $E \subset B \cap \partial\Omega$,*

$$\frac{\omega_\Omega^{p_1}(E)}{\omega_\Omega^{p_1}(B)} \sim_{c_0, C} \frac{\omega_\Omega^{p_2}(E)}{\omega_\Omega^{p_2}(B)}. \tag{2.5}$$

Lemma 2.9 (Lemma 10.1 in [28]). *Let $d \geq 1$ and let $\Omega \subsetneq \mathbb{R}^{d+1}$ be a C-uniform domain and B a ball centered at $\partial\Omega$ with radius r. Suppose that there exists a point $x_B \in \Omega$ so that the ball $B_0 := B(x_B, r/M)$ satisfies $4B_0 \subset \Omega \cap B$ for some $M > 1$. Then, for $0 < r \leq r_\Omega = r_\Omega(d, C)$, and $\tau > 0$,*

$$\omega_\Omega^x(B) \sim_{C,M,\tau,d} \omega_\Omega^{x_B}(B) G_\Omega(x, x_B)\, r^{d-1} \quad \text{for all } x \in \Omega \backslash (1+\tau)B. \tag{2.6}$$

Here, $r_\Omega = \infty$ if $\operatorname{diam}(\Omega) = \infty$.

2.3. Tangent measures

We recall some basic results.

Lemma 2.10 (Theorem 14.3 in [26]). *Let μ be a Radon measure on $\mathbb{R}^n$. If $a \in \mathbb{R}^n$ and*

$$\limsup_{r \to 0} \frac{\mu(B(a, 2r))}{\mu(B(a, r))} < \infty,$$

then every sequence $r_i \downarrow 0$ contains a subsequence such that $T_{a,r_j\#}\mu/\mu(B(a, r_j))$ converges to a measure $\nu \in \operatorname{Tan}(\mu, a)$.

Lemma 2.11 (Lemma 14.5 in [26]). *Let μ be a Radon measure on $\mathbb{R}^n$ and A a measurable set. Suppose $a \in \operatorname{supp}\mu$ is a point of density for A, meaning*

$$\lim_{r \to 0} \frac{\mu(B(a, r) \backslash A)}{\mu(B(a, r))} = 0.$$

If $c_i T_{a,r_i\#}\mu \to \nu \in \operatorname{Tan}(\mu, a)$, then so does $c_i T_{a,r_i\#}\mu|_A$. In particular, this holds for μ almost every $x \in A$.

The above lemma is not stated as such in [26], but it follows by an inspection of the proof (in particular the last two lines). The way we will use this lemma is the following.

Corollary 2.12. *With the assumptions of the previous lemma, if $c_i T_{a,r_i\#}\mu \to \nu \in \operatorname{Tan}(\mu, a)$, then for all $\xi \in \operatorname{supp}\nu$ and $\rho > 0$, $T_{a,r_i}(A) \cap B(\xi, \rho) \neq \emptyset$ for j sufficiently large. In particular, we may find $\xi_i \in T_{a,r_i}(A)$ so that $\xi_i \to \xi$.*

Lemma 2.13 (Lemma 14.7 in [26]). *Let μ be a Radon measure in $\mathbb{R}^n$, $s > 0$, and let A be the set of points ξ in $\mathbb{R}^n$ for which*

$$0 < a \leq \theta_*^s(\mu, \xi) \leq \theta^{s,*}(\mu, \xi) \leq b < \infty.$$

Then for almost every $\xi \in A$ and every $\nu \in \operatorname{Tan}(\mu, \xi)$,

$$a r^s \leq \nu(B(x, r)) \leq b r^s \quad \text{for } x \in \operatorname{supp}\nu, 0 < r < \infty. \tag{2.7}$$

A measure satisfying (2.7) for some $a, b > 0$ is called *Ahlfors s-regular*. We will need a slightly different version of this result suggested by Xavier Tolsa.

Lemma 2.14. *Let μ be a Radon measure in $\mathbb{R}^n$, $s > 0$. Then for μ almost every $\xi_0 \in S = \{\xi \in \mathbb{R}^n : 0 < \theta^{s,*}(\mu,\xi) < \infty\}$, there is $\nu \in \mathrm{Tan}(\mu,\xi_0)$ and r_j such that*

$$\mu_j = \frac{T_{\xi_0,r_j\#}\mu}{\mu(B(\xi_0,r_j))} \rightharpoonup \nu \tag{2.8}$$

and

$$\nu(B(x,r)) \leq r^s \quad \textit{for all } x \in \operatorname{supp}\nu, r > 0. \tag{2.9}$$

Proof. Note that the function $\theta^{*,s}(\mu,x)$ is a Borel function (see Exercise 3, Chapter 6 in [26]). By Egorov's theorem, for $k \in \mathbb{N}$, we may find a set $S_k \subseteq S \cap B(0,k)$ so that $\mu(S \cap B(0,k) \backslash S_k) < k^{-1}$ and $\theta^{*,s}(\mu,\cdot)$ is continuous on S_k.

For integers k, ℓ, m, let

$$S_{k,\ell,m} = \{\xi \in S_k : \mu(B(\xi,r))/r^s \leq (1+\ell^{-1})\theta^{*,s}(\mu,\xi) \text{ for } r \in (0,m^{-1})\}, \tag{2.10}$$

and let $S^*_{k,\ell,m}$ be the points of density for this set. Then for each $m \in \mathbb{N}$, almost all of S_k is in $\bigcup_m S^*_{k,\ell,m}$, and hence almost all of S_k is in

$$S^*_k := \bigcap_\ell \bigcup_m S^*_{k,\ell,m} \subseteq S_k.$$

Thus, almost all of S is in $S^* = \bigcup S^*_k$.

Let $\xi \in S^*$. Pick $r_j \downarrow 0$ such that

$$\frac{\mu(B(\xi,r_j))}{r_j^s} \to \theta^{*,s}(\mu,\xi) \in (0,\infty). \tag{2.11}$$

Let $\mu_j = T_{\xi,r_j\#}\mu/\mu(B(\xi,r_j))$. We first claim we can pick a subsequence so that this converges weakly to a nonzero measure $\nu \in \mathrm{Tan}(\mu,\xi)$. To see this, observe that since $\theta^{*,s}(\mu,\xi) < \infty$, for any R we have that

$$\begin{aligned}\limsup_j \mu_j(B(0,R)) &= \limsup_j \frac{\mu(B(\xi,Rr_j))}{\mu(B(\xi,r_j))} \\ &= \limsup_j \frac{\mu(B(\xi,Rr_j))}{(Rr_j)^s} \cdot \lim_{j\to\infty} \frac{r_j^s}{\mu(B(\xi,r_j))} R^s \overset{(2.11)}{\leq} R^s.\end{aligned}$$

Since this holds for all R, we can use a diagonalization argument to pick a subsequence so that μ_j converges weakly on all of $\mathbb{R}^n$ to a finite measure $\nu \in \mathrm{Tan}(\mu,\xi)$.

Let $x \in \operatorname{supp}\nu$ and $r > 0$. Let $\ell \in \mathbb{N}$, so $\xi \in S^*_{k,\ell,m}$ for some k, m. By Corollary 2.12, we may find $\xi_j \in T_{\xi,r_j}(S_{k,\ell,m})$ so that $\xi_j \to x$. Let $\zeta_j = T^{-1}_{\xi,r_j}(\xi_j)$. Then $\theta^{*,s}(\mu,\zeta_j) \to \theta^{*,s}(\mu,\xi)$ since $\zeta_j, \xi \in S_{k,\ell,m}$, $\zeta_j \to \xi$, and $\theta^{*,s}(\mu,\cdot)$ is continuous on $S_{k,\ell,m}$.

Hence,

$$\begin{aligned}\nu(B(x,r)) &\leq \liminf_{j\to\infty}\mu_j(B(x,r)) \leq \liminf_{j\to\infty}\mu_j(B(\xi_j,|\xi_j-x|+r))\\ &= \liminf_{j\to\infty}\frac{\mu(B(\zeta_j,r_j(|\xi_j-x|+r)))}{\mu(B(\xi,r_j))}\\ &\overset{(2.10)}{\leq} \liminf_{j\to\infty}\frac{(1+\ell^{-1})\theta^{*,s}(\mu,\zeta_j)r_j^s(|\xi_j-x|+r)^s}{r_j^s}\frac{r_j^s}{\mu(B(\xi,r_j))}\\ &\overset{(2.11)}{=} (1+\ell^{-1})\theta^{s,*}(\mu,\xi)(0+r)^s\cdot\theta^{s,*}(\mu,\xi)^{-1} = (1+\ell^{-1})r^s\end{aligned}$$

Since this holds for all $\ell\in\mathbb{N}$, the lemma follows. □

Lemma 2.15. *Let μ be a Radon measure in $\mathbb{R}^n$ and $s>0$. Let*

$$S=\{\xi\in\mathbb{R}^n : 0<\theta_*^s(\mu,\xi)<\infty, \limsup_{r\to 0}\mu(B(\xi,2r))/\mu(B(\xi,r))<\infty\}.$$

Then for almost every $\xi_0\in S$ there is $\nu\in\mathrm{Tan}(\mu,\xi_0)$ and r_j such that

$$\mu_j=\frac{T_{\xi_0,r_j\#}\mu}{\mu(B(\xi_0,r_j))}\rightharpoonup\nu \tag{2.12}$$

and

$$\nu(B(x,r))\geq r^s \text{ for all } x\in\operatorname{supp}\nu, r>0. \tag{2.13}$$

Proof. Again, $\theta_*^s(\mu,x)$ is a Borel function (see Exercise 3, Chapter 6 in [26]), so Egorov's theorem implies for $k\in\mathbb{N}$, we may find a set $S_k\subseteq S\cap B(0,k)$ so that $\mu(B(S\cap B(0,k))\backslash S')<k^{-1}$ and $\theta_*^s(\mu,\cdot)$ is continuous on S_k.

For integers k,ℓ,m, let

$$S_{k,\ell,m}=\{\xi\in S_k : \mu(B(\xi,r))/r^s\geq(1-\ell^{-1})\theta^{s_*}(\mu,\xi) \text{ for } r\in(0,m^{-1}),\} \tag{2.14}$$

and let $S_{k,\ell,m}^*$ be the points of density for this set. Then for each $m\in\mathbb{N}$, almost all of S_k is in $\bigcup_m S_{k,\ell,m}^*$, and hence almost all of S_k is in

$$S_k^* := \bigcap_\ell\bigcup_m S_{k,\ell,m}^*\subseteq S_k.$$

Thus, almost all of S is in $S^*=\bigcup S_k^*$.

Let $\xi\in S^*$. Pick $r_j\downarrow 0$ such that

$$\frac{\mu(B(\xi,r_j))}{r_j^\alpha}\to\theta_*^d(\mu,\xi)\in(0,\infty). \tag{2.15}$$

By Lemma 2.10 and the definition of S, we can pass to a subsequence so that $\mu_j = T_{\xi,r_j\#}\mu/\mu(B(\xi,r_j))$ converges weakly to a nonzero measure $\nu\in\mathrm{Tan}(\mu,\xi)$. Let $x\in\operatorname{supp}\nu$ and $r>0$. Let $\ell\in\mathbb{N}$, so $\xi\in S_{k,\ell,m}^*$ for some k,m. By Corollary 2.12, we may find $\xi_j\in T_{\xi,r_j}(S_{k,\ell,m})$ so that $\xi_j\to x$. Let $\zeta_j=T_{\xi,r_j}^{-1}(\xi_j)\in S_{k,\ell,m}$.

Then

$$\begin{aligned}
\limsup_{j\to\infty}\mu_j(B(x,r)) &\geq \limsup_{j\to\infty}\mu_j(B(\xi_j,r-|\xi_j-x|)) \\
&= \limsup_{j\to\infty}\frac{\mu(B(\zeta_j,r_j(r-|\xi_j-x|)))}{\mu(B(\xi,r_j))} \\
&\overset{(2.14)}{\geq} \limsup_{j\to\infty}\frac{(1-\ell^{-1})\theta_*^s(\mu,\zeta_j)r_j^s(r-|\xi_j-x|)^s}{r_j^s}\frac{r_j^s}{\mu(B(\xi,r_j))} \\
&\overset{(2.15)}{=} (1-\ell^{-1})\theta_*^s(\mu,\xi)(r-0)^s\cdot\theta_*^s(\mu,\xi)^{-1} = (1-\ell^{-1})r^s.
\end{aligned}$$

Since this holds for all $\ell\in\mathbb{N}$, for all $\varepsilon>0$, we have

$$\nu(B(x,r+\varepsilon)) \geq \nu(\overline{B(x,r)}) \geq \limsup_{j\to\infty}\mu_j(\overline{B(x,r)}) \geq \mu_j(B(x,r)) \geq r^s.$$

Thus $\nu(\overline{B(x,r)})\geq r^s$ for all $x\in\operatorname{supp}\nu$ and $r>0$. This easily implies $\nu(B(x,r))\geq r^s$ for all $x\in\operatorname{supp}\nu$ and $r>0$. □

Lemma 2.16. *Suppose $\Omega\subseteq\mathbb{R}^{d+1}$ is a uniform domain, $x_0\in\Omega$, $\xi_0\in\partial\Omega$ is non-degenerate, and $r_j\to 0$. Then there is a subsequence and a Borel measure ν such that*

$$T_{\xi_0,r_j\#}\omega_\Omega^{x_0}/\omega_\Omega^{x_0}(B(\xi_0,r_j)) =: \mu_j \rightharpoonup \nu\in\operatorname{Tan}(\omega_\Omega^{x_0},\xi_0).$$

Proof. Suppose ξ is (β,δ)-non-degenerate for some $\beta,\delta\in(0,1)$. By Theorem 14.3 in [26], we need only show that

$$\limsup_{r\to 0}\frac{\omega_\Omega^{x_0}(B(\xi_0,2r))}{\omega_\Omega^{x_0}(B(\xi_0,r))}<\infty. \tag{2.16}$$

Note that for $r>0$ small enough, $x_0\notin B(\xi_0,4r)$, and so we may apply Lemma 2.8 to get

$$\begin{aligned}
\frac{\omega_\Omega^{x_0}(B(\xi_0,2r))}{\omega_\Omega^{x_0}(B(\xi_0,r))} &\overset{(2.5)}{\sim}_C \omega_\Omega^{x_{B(\xi_0,2r)}}(B(\xi_0,r))^{-1} \overset{(2.1)}{\sim}_{C,\delta} \omega_\Omega^{x_{B(\xi_0,\delta r)}}(B(\xi_0,r))^{-1} \\
&\leq \omega_{\Omega\cap B(\xi_0,r)}^{x_{B(\xi_0,\delta r)}}(B(\xi_0,r)\cap\partial\Omega)^{-1}.
\end{aligned} \tag{2.17}$$

Thus, since ξ_0 is non-degenerate, we now have (2.16) since

$$\limsup_{r\to 0}\frac{\omega_\Omega^{x_0}(B(\xi_0,2r))}{\omega_\Omega^{x_0}(B(\xi_0,r))} \overset{(2.17)}{\lesssim} \limsup_{r\to 0}\omega_{\Omega\cap B(\xi_0,r)}^{x_{B(\xi_0,\delta r)}}(B(\xi_0,r)\cap\partial\Omega)^{-1}<\infty. \qquad \square$$

3. Proof of Lemma I

Let $\Omega\subseteq\mathbb{R}^{d+1}$ be a uniform domain and $\xi_0\in\partial\Omega$ and $r_j\to 0$. Let $T_j=T_{\xi_0,r_j}$ and suppose $\mu_j=T_{j\#}\omega_\Omega^{x_0}/\omega_\Omega^{x_0}(B(\xi_0,r_j))$ converges weakly to a measure μ. Let $\Omega_j=T_j(\Omega)$. Pass to a subsequence so that $\partial\Omega_j\cap\overline{B(0,n)}$ converges in the Hausdorff metric to a compact set Σ_n. Note that $\Sigma_n\subseteq\Sigma_{n+1}$, otherwise there is $x\in\Sigma_n$ of distance $r>0$ from Σ_{n+1}. For large j there is $\xi_j\in\partial\Omega_j\cap\overline{B(0,n)}\cap B(x,r/2)$. Since $\xi_j\in\partial\Omega_j\cap\overline{B(0,n+1)}$, $r/2\leq\operatorname{dist}(\xi_j,\Sigma_{n+1})\to 0$ as $j\to\infty$, a contradiction.

Let

$$\Sigma = \bigcup \Sigma_n.$$

Lemma 3.1. *We have* $\operatorname{supp}\mu \subseteq \Sigma$.

Proof. Suppose B is a ball centered on $\operatorname{supp}\mu$. Then $\mu_j(B) > 0$ for all j large, hence $B \cap \partial\Omega_j \neq 0$ for all j large since $\operatorname{supp}\mu_j = \partial\Omega_j$. If $\xi_j \in \partial\Omega_j \cap B$, then there is a subsequence converging to a point $\xi \in \Sigma \cap \overline{B}$. Thus, Σ intersects the closure of any ball centered on $\operatorname{supp}\mu$, which implies $\operatorname{supp}\mu \subseteq \Sigma$. □

Lemma 3.2. *By passing to a subsequence, we may assume that for all* $\xi \in \Sigma$ *and* $r > 0$ *there is a ball* B *of radius* $\frac{r}{8C}$ *so that* $2B \subseteq B(\xi, r) \cap \Omega_j$ *for all large* j.

Proof. Let A be a countably dense set in Σ. Let $\xi \in A$ and $r \in \mathbb{Q} \cap (0, \infty)$. Then there is $\xi_j \in \Omega_j \cap B(\xi, r/2)$ for j large enough. Since Ω_j satisfies the C-interior corkscrew condition, there is a ball $B(x_j, \frac{r}{2C}) \subseteq B(\xi_j, r/2) \cap \Omega_j$. By passing to a subsequence, we can find $x_{\xi,r}$ so that, for large j,

$$B\Big(x_{\xi,r}, \frac{2r}{5C}\Big) \subseteq \Omega_j \cap B(\xi_j, r/2) \subseteq \Omega_j \cap B(\xi, r).$$

By a diagonalization argument, we can assure that this holds for all $(\xi, r) \in A \times (0, r) \cap \mathbb{Q}$ for sufficiently large j. By the density of A and $(0, r) \cap \mathbb{Q}$, it follows that for all $\xi \in \Sigma$ and $r > 0$, there is $x_{\xi,r}$ so that $B(x_{\xi,r}, \frac{r}{4C}) \subseteq B(\xi, r) \cap \Omega_j$ for all j large. By taking $B = B(x_{\xi,r}, \frac{r}{8C})$, this proves the lemma. □

For all $x \in \mathbb{Q}^{d+1} \backslash \Sigma$, let $r_x = \operatorname{dist}(x, \Sigma)$ so that $B(x, r_x) \subsetneq \Sigma^c$ and $B_x := B(x, r_x/2) \subseteq (\partial\Omega_j)^c$ for all sufficiently large j. By a diagonalization argument, we may pass to a subsequence such that for each $x \in \mathbb{Q}^{d+1} \backslash \Sigma$, $B_x \subseteq \Omega_j$ for all but finitely many j or $B_x \subseteq \Omega_j^{\text{ext}}$ for all but finitely many j. Let

$$\Omega^* = \bigcup \{B_x : B_x \subseteq \Omega_j \text{ for all but finitely many } j\}. \tag{3.1}$$

Lemma 3.3. Ω^* *is a* C'*-uniform domain with constant depending on* C, $\partial\Omega^* = \Sigma$, *and* Ω^* *satisfies the* $8C$*-interior corkscrew property.*

Proof. By a covering argument, it follows that any ball B with $B \subsetneq \Omega_j$ for all j large satisfies $B \subseteq \Omega^*$. By Lemma 3.2, for every $\xi \in \Sigma$ and $r > 0$, there is a ball $B \subseteq B(\xi, r) \cap \Omega_j$ of radius $\frac{r}{8C}$ for j large, and hence $B \subseteq \Omega^*$. Since this holds for all $\xi \in \Sigma$ and $r > 0$, this implies $\Sigma \subseteq \overline{\Omega^*}$. By construction, however, $\Omega^* \subseteq \Sigma^c$, and so $\Sigma \subseteq \partial\Omega^*$. Hence, we have shown that Ω^* satisfies the interior corkscrew property with constant $8C$, and in particular, has nonempty interior.

Now we focus on uniformity, but to prove this, we will need the following theorem.

Theorem 3.4 (Theorem 5.1 in [25]). *Let* Ω *be a uniform domain. Then there is a constant* L, *depending only on the uniformity constant for* Ω, *such that for each pair of points* $x, y \in \Omega$ *there is an* L*-bi-Lipschitz embedding* $f\colon \overline{B(0, |x-y|)} \to \Omega$ *such that* $\{x, y\} \subseteq f(\overline{B(0, |x-y|)})$.

Let $x, y \in \mathbb{Q}^{d+1} \cap \Omega^*$ and $r = |x - y|$. Then $x, y \in \Omega_j$ for all j large, so there is an L-bi-Lipschitz $f_{x,y,j} : \overline{B(0,r)} \to \Omega_j$ such that $\{x, y\} \subseteq f_{x,y,j}(\overline{B(0,r)})$. By passing to a subsequence, we may assume $f_{x,y,j}$ converges uniformly to an L-bi-Lipschitz map $f_{x,y} : \overline{B(0,r)} \to \mathbb{R}^{d+1}$. If $\varepsilon > 0$ is small enough (depending on x and y), we may assume that for each j there is B_j of radius $\frac{\min\{r_x, r/2\}}{8L^2}$ so that $B_j \subseteq f_{x,y,j}(B(0,(1-\varepsilon)r)) \cap B_x$, see Figure 1.

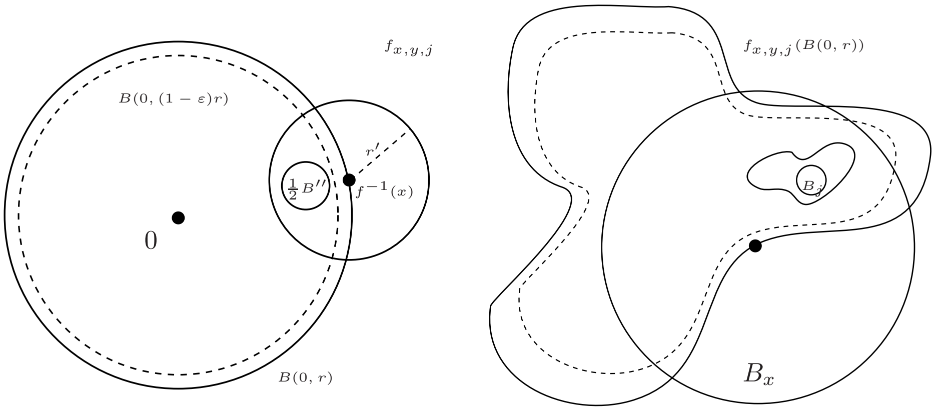

FIGURE 1. Here, the outer ball on the left is $B(0, r)$, the shaded smaller ball is $B(0,(1-\varepsilon)r)$, and the darker ball $\frac{1}{2}B'' \subseteq B(0,(1-\varepsilon)r)$ are on the left and their images are shown on the right. The image of $\frac{1}{2}B''$ is contained in B_x and contains a smaller ball B_j of desired radius.

To see this, let $r' = \min\{r_x, r/2\}$. Then there is $B_j'' \subseteq B(0,r) \cap B(f_{x,y,j}^{-1}(x), r'/L)$ of radius $\frac{r'}{4L}$. Then $f(B_j'') \subseteq B(x, r') \subseteq B_x$. Now, if $\varepsilon > 0$ is small enough, $\frac{1}{2}B_j'' \subseteq B(0,(1-\varepsilon)r)$ and $f(\frac{1}{2}B'')$ contains a ball B_j of radius $\frac{r'}{8L^2}$, which proves the claim.

By passing to a subsequence, we may assume there is a ball

$$B \subseteq f_{x,y,j}(B(0,(1-\varepsilon)r)) \cap B_x \subseteq \Omega_j \tag{3.2}$$

for all j large. Observe that

$$\operatorname{dist}(f_{x,y,j}(\overline{B(0,(1-\varepsilon/2)r)}), \partial\Omega_j) \geq \frac{\varepsilon}{2L} r$$

for all j large, and so

$$\operatorname{dist}(f_{x,y}(\overline{B(0,(1-\varepsilon/2)r)}), \Sigma) \geq \frac{\varepsilon}{2L} r.$$

Hence, $f_{x,y}(\overline{B(0,(1-\varepsilon/2)r)}) \subseteq \Sigma^c$, and by uniform convergence, we have

$$f_{x,y,j}(\overline{B(0,(1-\varepsilon)r)}) \subseteq \Sigma^c = \partial\Omega^*, \quad \text{for } j \text{ large}.$$

By (3.2), since $B \subseteq \Omega^*$ and since $f_{x,y,j}(\overline{B(0,(1-\varepsilon)r)})$ is connected,

$$f_{x,y,j}(\overline{B(0,(1-\varepsilon)r)}) \subseteq \Omega^*, \quad \text{for } j \text{ sufficiently large.}$$

Again, by uniform convergence, we also have $f_{x,y}(\overline{B(0,(1-\varepsilon)r)}) \subseteq \Omega^*$. Letting $\varepsilon \to 0$, we get $f_{x,y}(B(0,r)) \subseteq \Omega^*$. Note that $x, y \in \overline{f_{x,y}(B(0,r))}$.

Thus, for all $x, y \in \mathbb{Q}^{d+1} \cap \Omega^*$, we can find an L-bi-Lipschitz map

$$f_{x,y} : \overline{B(0,|x-y|)} \to \overline{\Omega^*}$$

containing x, y. By Arzelà–Ascoli, we can find such a map for every $x, y \in \Omega^*$. Since balls are uniform domains and bi-Lipschitz maps preserve uniformity, we have that $f_{x,y}(B(0,|x-y|))$ is a uniform domain (with constant depending on d and L, which in turn only depends on the uniformity constant of Ω). Thus, we can find a path γ satisfying the conditions of Definition 1.1 for the domain $f_{x,y}(B(0,|x-y|))$, and it will also satisfy Definition 1.1 for the domain Ω^*. This shows that Ω^* is uniform. □

Lemma 3.5. *For $\xi \in \partial\Omega^* \setminus \operatorname{supp}\mu$, let $B(\xi) = B(\xi, \operatorname{dist}(\xi, \operatorname{supp}\mu)/2)$. Let x_ξ be the center of the ball $B \subseteq B(\xi) \cap \Omega^*$ given by Lemma 3.2. Then $\omega_j^{x_\xi}(B(\xi)) \to 0$, and in particular, $\delta B(\xi) \cap \Omega^{*,\text{ext}} = \emptyset$, where δ is as in Lemma 2.7.*

Proof. Let $R \geq 1$ be so that $R\mathbb{B} \supseteq B(\xi)$. Let B' be the ball from Lemma 3.2 applied to $R\mathbb{B}$ and $x^{B'}$ its center. Since Ω_j is uniform, we may apply Lemma 2.8 and get for j large

$$\omega_j^{x_\xi}(B(\xi)) \overset{(2.1)}{\sim}_{r_{B'},R} \omega_j^{x^{B'}}(B(\xi)) \leq \frac{\omega_j^{x^{B'}}(B(\xi))}{\omega_j^{x^{B'}}(R\mathbb{B})} \overset{(2.5)}{\sim} \frac{\omega_j^{T_j(x_0)}(B(\xi))}{\omega_j^{T_j(x_0)}(R\mathbb{B})}$$

$$\leq \frac{\omega_j^{T_j(x_0)}(B(\xi))}{\omega_j^{T_j(x_0)}(\mathbb{B})} = \mu_j(B(\xi)). \tag{3.3}$$

Thus, since $2B(\xi) \cap \operatorname{supp}\mu = \emptyset$,

$$\limsup_{j\to\infty} \omega_j^{x_\xi}(B(\xi)) \overset{(3.3)}{\lesssim} \limsup_{j\to\infty} \mu_j(B(\xi)) \leq \mu(2B(\xi)) = 0.$$

Now, suppose $\delta B(\xi) \cap \Omega^{*,\text{ext}} \neq \emptyset$. Then there is a ball $B'' \subseteq \delta B(\xi) \cap \Omega_*^{\text{ext}}$ with rational center so that $B'' \subseteq \delta B(\xi) \cap \Omega_j^{\text{ext}}$ for all large j. Since we also have $B \subseteq \delta B(\xi) \cap \Omega_j$, this implies $\mathscr{H}_\infty^d(\delta B(\xi) \cap \partial\Omega_j) \gtrsim_{r_B, r_{B''}} r_{\delta B(\xi)}^d$ for all j, where the implied constant depends on r_B and $r_{B'}$ (the proof for Hausdorff measure is shown in Lemma 2.3 in [12], but the same proof works for Hausdorff content) and hence by Lemma 2.7,

$$\omega_j^x(B(\xi)) \gtrsim_{r_B, r_{B'}} 1 \text{ for all } x \in \delta B(\xi). \tag{3.4}$$

So in particular, this holds for $x = x_\xi$, but that would contradict the first half of this theorem. Thus, $\delta B(\xi) \cap \Omega^{*,\text{ext}} = \emptyset$. □

Lemma 3.6. *Let*

$$\tilde{\Omega} = \Omega^* \cup \bigcup \left\{ \frac{\delta}{2} B(\xi) : \xi \in \partial\Omega^* \backslash \operatorname{supp} \mu \right\}.$$

Then $\tilde{\Omega}$ *is also a uniform domain and* $\partial\tilde{\Omega} = \operatorname{supp} \mu$.

Proof. Since Ω^* is uniform, we know that for all $x, y \in \overline{\Omega^*}$ there is a good curve γ. But $\overline{\Omega^*} \cap \frac{\delta}{2}B(\xi) = \frac{\delta}{2}B(\xi)$ for any $\xi \in \partial\Omega^* \backslash \operatorname{supp}\mu$, and thus for any pair of points $x, y \in \tilde{\Omega}$ there is a good curve for x and y *with respect to* Ω^*, and this curve will also be good for $\tilde{\Omega}$. Since there are good curves for all pairs in $\tilde{\Omega}$, it is not hard to show that there are good curves between any pair of points in its closure.

Clearly $\operatorname{supp}\mu \subseteq \overline{\tilde{\Omega}}$. Moreover, $\operatorname{supp}\mu \subseteq \Sigma$ implies $\operatorname{supp}\mu \cap \Omega^* = \emptyset$, and by definition $\operatorname{supp}\mu \cap \frac{\delta}{2}B(\xi) = \emptyset$ for all $\xi \in \partial\Omega^* \backslash \operatorname{supp}\mu$, hence $\operatorname{supp}\mu \subseteq \partial\tilde{\Omega}$. Now suppose there is $\zeta \in \partial\tilde{\Omega} \backslash \operatorname{supp}\mu$. Since $\Omega^* \subseteq \tilde{\Omega}$, $\zeta \notin \Omega^*$. Since $\zeta \notin \frac{\delta}{2}B(\xi) \cup \operatorname{supp}\mu$ for any $\xi \in \partial\Omega^* \backslash \operatorname{supp}\mu$, we also know $\zeta \notin \partial\Omega^*$, and so $\zeta \in \Omega^{*\,\mathrm{ext}}$. In particular,

$$\zeta \in \overline{\bigcup_{\xi \in \partial\Omega^* \backslash \operatorname{supp}\mu} \tfrac{\delta}{2}B(\xi)}.$$

Thus there is $\xi \in \partial\Omega^* \backslash \operatorname{supp}\mu$ such that $\operatorname{dist}(\zeta, \frac{\delta}{2}B(\xi)) < \frac{\delta}{8}\operatorname{dist}(\zeta, \operatorname{supp}\mu)$, and so

$$\begin{aligned} |\zeta - \xi| &\leq r_{\frac{\delta}{2}B(\xi)} + \operatorname{dist}(\zeta, \tfrac{\delta}{2}B(\xi)) < \frac{\delta}{4}\operatorname{dist}(\xi, \operatorname{supp}\mu) + \frac{\delta}{8}\operatorname{dist}(\zeta, \operatorname{supp}\mu) \\ &< \frac{3\delta}{8}\operatorname{dist}(\xi, \operatorname{supp}\mu) + \frac{\delta}{8}|\zeta - \xi|, \end{aligned}$$

which implies

$$|\zeta - \xi| < (1 - \delta/8)^{-1}\frac{3\delta}{8}\operatorname{dist}(\xi, \operatorname{supp}\mu) < \delta \operatorname{dist}(\xi, \operatorname{supp}\mu)/2$$

for δ small enough. But then $\delta B \cap \Omega^{*,\mathrm{ext}} \neq \emptyset$, which contradicts Lemma 3.5. □

Lemma 3.7. *By passing to a subsequence, for any* $f \in C_0(\mathbb{R}^{d+1})$, *the function*

$$u_{f,j}(x) = \int f d\omega_j^x$$

converges to a harmonic function v_f *on* Ω^* *in the sense that for all compact subsets* $K \subseteq \Omega^*$, $K \subseteq \Omega_j$ *for* j *sufficiently large and* $u_{f,j}$ *converges uniformly to* v_f *on* K. *In particular,*

$$v_f = v_g \quad \text{for all } f, g \in C_0(\mathbb{R}^{d+1}) \text{ such that } f\mathbb{1}_{\partial\tilde{\Omega}} = g\mathbb{1}_{\partial\tilde{\Omega}}. \tag{3.5}$$

Proof. The set of continuous functions vanishing at infinity is separable in the L^∞-metric, so let A be a dense subset of $C_0(\mathbb{R}^{d+1})$ and $f \in A$. For each $x \in \mathbb{Q}^{d+1} \cap \Omega^*$, we can pass to a subsequence, so that $u_{f,j}$ converges uniformly on B_x (recall (3.1)), so by a diagonalization argument, we can guarantee $u_{f,j}$ converges uniformly on every B_x, and hence by a covering argument, on every compact subset of Ω^* to a harmonic function v_f.

Note that if B is a ball compactly contained in Σ^c, then $\omega_j(B) = 0$ for large j, and so

$$v_f = \lim_{j\to\infty} \int f d\omega_j = \lim_{j\to\infty} \int_{B^c} f d\omega_j.$$

Thus,

$$v_f = v_{f\mathbb{1}_{\partial\Omega^*}}. \tag{3.6}$$

Note that by (3.3), for $\xi \in \partial\Omega^* \backslash \operatorname{supp}\mu$

$$\int_{B(\xi)} |f| \, d\omega_j^{x_\xi} \leq ||f||_\infty \, \omega_j^{x_\xi}(B(\xi)) \lesssim \mu_j(B(\xi)) \to 0.$$

Since Ω^* is uniform, and because each $x \in \Omega^*$ is in an open ball contained in Ω_j for j large, we have

$$\Big| \int_{B(\xi)} f \, d\omega_j^x \Big| \leq \int_{B(\xi)} |f| \, d\omega_j^x \sim \int_{B(\xi)} |f| \, d\omega_j^{x_\xi} \to 0.$$

Since this holds for all $\xi \in \partial\Omega^* \backslash \operatorname{supp}\mu$ and (3.6) holds, we have

$$v_f = v_{f\mathbb{1}_{\operatorname{supp}\mu}}. \tag{3.7}$$

Thus, by a diagonalization argument and the density of A in $C_c^\infty(\mathbb{R}^{d+1})$, we can ensure that for all $f \in C_c^\infty$, there is a harmonic function $v_f : \Omega^* \to \mathbb{R}$ that is the uniform limit of $\int f d\omega_j$ on compact subsets of Ω^* and such that (3.5) holds. □

Combining all the previous lemmas, we have now shown the following.

Lemma 3.8. *Let Ω be a uniform domain, $\xi_0 \in \partial\Omega$ and $r_j \to 0$ such that*

$$\mu_j = T_{j\#}\omega_\Omega^{x_0}/\omega_\Omega^{x_0}(B(\xi_0, r_j)) \rightharpoonup \mu.$$

Then we may pass to a subsequence such that

(1) $\operatorname{supp}\mu$ *is the boundary of a C'-uniform domain $\tilde{\Omega}$, where C' depends on C and d.*

(2) *There is a uniform subdomain Ω^* dense in $\tilde{\Omega}$ such that for all $x \in \Omega^*$, if $\Omega_j := T_{\xi_0,r_j}(\Omega)$, then $x \in \Omega_j$ for all sufficiently large j.*

(3) *For $x \in \Omega^*$ and $\omega_j := \omega_{\Omega_j}$, and any continuous function f vanishing at infinity, $\int f d\omega_j$ converges to a harmonic function v_f uniformly on compact subsets of Ω^* such that (3.5) holds.*

Lemma 3.9. *Let $\Omega \subseteq \mathbb{R}^{d+1}$ be a uniform domain. For almost every non-degenerate point $\xi_0 \in \partial\Omega$, if $r_j \to 0$ and $\mu_j = T_{j\#}\omega_\Omega^{x_0}/\omega_\Omega^{x_0}(B(\xi_0, r_j))$, then there is a subsequence that converges weakly to a measure μ satisfying the conclusions of the previous lemma. In addition, we have $v_f = \int f d\tilde{\omega}$ for $f \in C_0(\mathbb{R}^{d+1})$, where $\tilde{\omega}$ is the harmonic measure for $\tilde{\Omega}$.*

Proof. First note that for $\delta < \tau$, by the maximum principle,

$$\sup_{|x-\xi|=\delta r} \omega^x_{B(\xi,r)\cap\Omega}(\partial B(\xi,r)\cap\Omega) \leq \sup_{|x-\xi|=\tau r} \omega^x_{B(\xi,r)\cap\Omega}(\partial B(\xi,r)\cap\Omega).$$

Thus, $E = \bigcup E_n$ where

$$E_n := \Big\{\xi \in \partial\Omega : \text{ for all } r \in (0, 1/n), \sup_{|x-\xi|=r/n} \omega^x_{B(\xi,r)\cap\Omega}(\partial B(\xi,r)\cap\Omega) \leq 1 - 1/n\Big\}.$$

Fix an n and let ξ_0 be a point of density for E_n with respect to the measure $\omega^{x_0}_\Omega$. By Lemma 2.16, we can pass to a subsequence so that μ_j converges weakly to a measure μ, and thus again to another subsequence so that the conclusions of Lemma 3.8 hold. Let $f \in C_0(\mathbb{R}^{d+1})$, $\varepsilon > 0$, and $\xi \in \partial\tilde{\Omega}$. Pick $r > 0$ small enough so that

$$|f(\zeta) - f(\xi)| < \varepsilon \quad \text{whenever } |\xi - \zeta| \leq r. \tag{3.8}$$

Consider the function

$$h(x) = f(\xi) + \varepsilon + 2||f||_\infty \omega^x_j(\overline{B(\xi,r)}^c) - \int f d\omega^x_j.$$

This is harmonic on Ω_j. We will show that h is nonnegative. By Theorem 5.2.6 in [5], it suffices to show that

$$\liminf_{x\to\infty} h(x) \geq 0 \quad \text{and} \quad \liminf_{x\to\zeta} h(x) \geq 0 \quad \text{for quasi-every } \zeta \in \partial\Omega. \tag{3.9}$$

Let $B(y,R)$ be a ball containing the support of f. Then $||f||_\infty R^{d-1}/|\cdot - y|^{d-1}$ is a subharmonic majorant of $|f|$, and thus

$$\left|\int f\, d\omega^x_j\right| \leq \int |f|\, d\omega^x_j \leq \frac{||f||_\infty R^{d-1}}{|x-y|^{d-1}} \to 0$$

as $x \to \infty$. Thus $\lim_{x\to\infty} h(x) \geq 0$, which proves the first part of (3.9).

To prove the second part, we recall that quasi-every point $\zeta \in \partial\Omega_j$ is regular (see Theorem 6.6.8 in [5]), and thus we only need to show $\lim_{x\to\zeta} h(x) \geq 0$ for $\zeta \in \partial\Omega_j$ regular.

1. If $\zeta \in \overline{B(\xi,r)}$, then

$$\liminf_{x\to\zeta} h(x) \geq f(\xi) + \varepsilon - \lim_{x\to\zeta}\int f d\omega^x_j = f(\xi) - f(\zeta) + \varepsilon \overset{(3.8)}{>} 0.$$

2. If $\zeta \notin \overline{B(\xi,r)}$, then the boundary data of $\omega^x_j(\overline{B(\xi,r)}^c)$ is continuous at ζ and thus

$$\liminf_{x\to\zeta} h(x) = f(\xi) + \varepsilon + 2||f||_\infty - f(\zeta) \geq \varepsilon > 0.$$

Thus, we have shown that $\liminf_{x\to\zeta} h(x) \geq 0$ for ζ regular, which proves the last part of (3.9), and hence $h \geq 0$.

We can similarly show that the function

$$f(\xi) - \varepsilon - 2\,||f||_\infty\,\omega_j^x(\overline{B(\xi,r)^c}) - \int f\,d\omega_j^x$$

is nonpositive. Combining our estimates, we obtain that

$$\Big|\int f\,d\omega_j^x - f(\xi)\Big| \le 2\,||f||_\infty\,\omega_j^x(\overline{B(\xi,r)}^c) + \varepsilon \quad \text{for } x \in \Omega_j. \tag{3.10}$$

Let $\rho \in (0, 1/10)$ and $\xi \in \operatorname{supp}\mu$. Let $R = 1 + |\xi| + \rho$.

Since ξ_0 is a point of density for E_n, by Corollary 2.12 we can ensure that for j large enough there is $\zeta_j \in T_j(E_n)$ with

$$|\xi - \zeta_j| < \rho r. \tag{3.11}$$

Setting $\xi_j = T_j^{-1}(\zeta_j)$, we have $\xi_j \in E_n \cap B(\xi_0, Rr_j)$ with $|T_j^{-1}(\xi) - \xi_j| < \rho r r_j$. Note that by the definition of E_n and by Lemma 2.3, for j large enough so that $\frac{1}{nr_j} > 100r$, we have

$$\omega_j^x(\overline{B(\zeta_j,(1-\rho)r)}^c) \lesssim_n \Big(\frac{|x-\zeta_j|}{(1-\rho)r}\Big)^\alpha \quad \text{for } x \in B(\zeta_j,(1-\rho)r) \cap \Omega_j.$$

Thus, we have for $x \in \Omega_j \cap B(\xi, r/4)\backslash B(\xi, 2\rho r) \subseteq B(\zeta_j,(1-\rho)r)$ that

$$|x - \zeta_j| \overset{(3.11)}{\le} |x - \xi| + \rho r \le \frac{3}{2}|x - \xi| \tag{3.12}$$

and

$$\omega_j^x(\overline{B(\xi,r)}^c) \le \omega_j^x(\overline{B(\zeta_j,(1-\rho)r)}^c) \lesssim \Big(\frac{|x-\zeta_j|}{(1-\rho)r}\Big)^\alpha \lesssim \Big(\frac{|x-\xi|}{r}\Big)^\alpha. \tag{3.13}$$

Combining (3.10) and (3.13), we get

$$\Big|\int f d\omega_j^x - f(\xi)\Big| \lesssim \Big(\frac{|x-\xi|}{r}\Big)^\alpha + \varepsilon \quad \text{if } x \in \Omega_j \cap B(\xi, r/4)\backslash B(\xi, 2\rho r).$$

Letting $j \to \infty$, and using the fact that $x \in \Omega^*$ implies $x \in \Omega_j$ for all large j, we have

$$|v_f(x) - f(\xi)| \lesssim \Big(\frac{|x-\xi|}{r}\Big)^\alpha + \varepsilon \quad \text{if } x \in \Omega^* \cap B(\xi, r/4)\backslash B(\xi, 2\rho r).$$

Now let $\rho \to 0$ and we get

$$|v_f(x) - f(\xi)| \lesssim \Big(\frac{|x-\xi|}{r}\Big)^\alpha + \varepsilon \quad \text{if } x \in B(\xi, r/4) \cap \Omega^*.$$

Hence,

$$f(\xi) - \varepsilon \le \liminf_{x\to\xi} v_f(x) \le \limsup_{x\to\xi} v_f(x) \le f(\xi) + \varepsilon.$$

Letting $\varepsilon \to 0$, we now have

$$\lim_{x\to\xi} v_f(x) = f(\xi).$$

Thus, v_f is a harmonic function on Ω^* whose limits at $\partial\tilde{\Omega} \subseteq \partial\Omega^*$ coincide with f. Let $\tilde{\omega} = \omega_{\tilde{\Omega}}$, $\tilde{u}_f = \int f d\tilde{\omega}$, and $F = \tilde{u}_f \mathbb{1}_{\partial\Omega^*\setminus \operatorname{supp}\mu} + f \operatorname{supp}\mu$. By (3.5), $v_f = v_F$. Moreover, v_F is harmonic in Ω^* and has boundary limit equal to F at every regular point in $\partial\Omega^*$. In particular, it equals f everywhere on $\operatorname{supp}\mu = \partial\tilde{\Omega}$ and equals $\tilde{u}_f$ at every regular point of $\partial\Omega^*\setminus\operatorname{supp}\mu$ since $\tilde{u}_f$ is continuous on $\partial\Omega^*\setminus\operatorname{supp}\mu$. Thus, $v_f = v_F = \int F d\omega_{\Omega^*}$. The function $\tilde{u}_f$ agrees with F at every boundary point of $\partial\Omega^*$ as well, hence $\tilde{u}_f = \int F d\omega_{\Omega^*} = v_f$. Therefore f extends harmonically to all of $\tilde{\Omega}$ and in fact $v_f = \int f d\tilde{\omega}$. □

Lemma 3.10. *With the assumptions of Lemma 3.9, if E is the set of (β,δ)-non-degenerate, then for almost every point $\xi_0 \in E$, Ω^* is Δ-uniform with constants depending on C, d, δ, and β.*

Proof. We will assume $\delta = 1/2$ for simplicity. Let $B = B(\xi, r)$ be a ball with $\xi \in \partial\tilde{\Omega}$, $r > 0$, and let $x \in \partial\frac{1}{100}B \cap \Omega^*$. By Lemma 4.1 in [6], there is a constant $C > 0$ depending only on the uniformity constant of Ω_j (which is the same constant for all j) so that for all j with $\partial\Omega_j \cap B \neq \emptyset$, there is a C-uniform domain $\Omega_j^B \subseteq \Omega_j \cap CB$ such that $B \cap \Omega_j \subseteq \Omega_j^B$, see Figure 2. By Lemma 3.9, we can pass to a subsequence and guarantee there are uniform domains $\Omega^{B,*} \subseteq \tilde{\Omega}^B$ (the former dense in the latter) so that $\omega^y_{\Omega_j^B}$ converges weakly to $\omega^y_{\tilde{\Omega}^B}$ for all $y \in \Omega^{B,*}$. By the definition of $\tilde{\Omega}^B$, we know $\tilde{\Omega}^B \subseteq CB \cap \tilde{\Omega}$ and $B \cap \tilde{\Omega} \subseteq \tilde{\Omega}^B$.

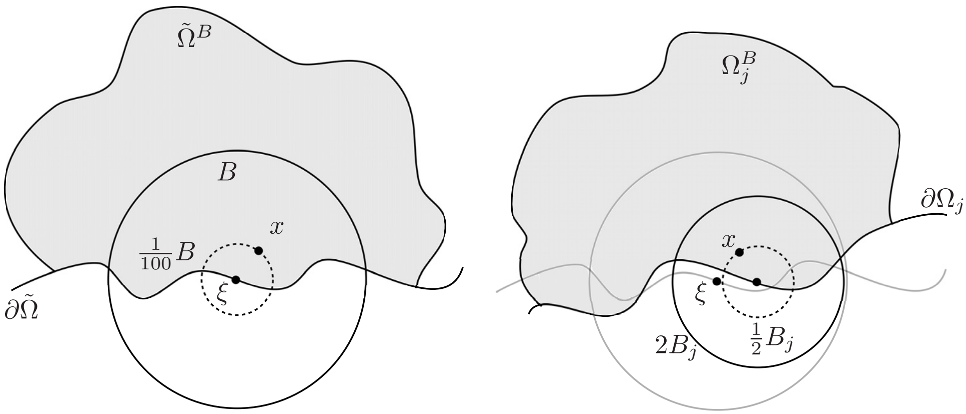

Figure 2. In the figure on the left, the shaded area depicts $\Omega^{B*} \subseteq CB \cap \Omega^*$ and on the right we have $\Omega_j^B \subseteq CB \cap \Omega_j$.

Let $E_n = \{\xi \in E : \eta_{1/2}(\xi, r) < \beta \text{ for } r < 1/n\}$. Then almost every $\xi_0 \in E$ is a point of density for some E_n, $n \in \mathbb{N}$. By Corollary 2.12, for each j sufficiently large we can pick $\xi_j \in T_j(E_n) \subseteq \partial\Omega_j$ converging to ξ and set $B_j = B(\xi_j, 2|\xi_j - x|)$ so that $B_j \subseteq \frac{1}{2}B$, $r_{B_j} \geq r_B/1000$, and $x \in \partial\frac{1}{2}B_j$. Then by the maximum principle, weak limits, the maximum principle again, and since $\xi_j \in T_j(E_n)$, we have

$$\begin{aligned}\omega^x_{CB\cap\tilde{\Omega}}(\overline{B}\cap\partial\tilde{\Omega}) &\geq \omega^x_{\tilde{\Omega}^B}(\overline{B}) \geq \limsup_{j\to\infty} \omega^x_{\Omega_j^B}\left(\tfrac{1}{2}\overline{B}\right)\\ &\geq \limsup_{j\to\infty} \omega^x_{\Omega_j\cap B_j}(\overline{B_j}\cap\partial\Omega_j) \geq 1-\beta > 0\end{aligned}$$

for some β depending only on δ and β. This implies $\omega^x_{CB\cap\tilde{\Omega}^B}(\partial B\cap\tilde{\Omega}) \leq \beta < 1$. Since $x \in \partial\frac{1}{100}B$ and our choice of ball B were arbitrary, we have thus shown Δ-uniformity. □

Lemma 3.11. *With the assumptions of Lemma 3.9, if $B' \subseteq B = B(\xi, r)$ are balls centered on $\partial\tilde{\Omega}$ and $B(x, \frac{r}{C'}) \subseteq B\cap\Omega$ is a C'-corkscrew ball (recall $\tilde{\Omega}$ is uniform), then*

$$\frac{\omega^x_{\tilde{\Omega}}(B')}{\omega^x_{\tilde{\Omega}}(B)} \sim_{C,d} \frac{\mu(B')}{\mu(B)}. \tag{3.14}$$

Proof. Let $T_j = T_{\xi_0, r_j}$ and $\mu_j := T_{j\#}\omega^{x_0}_\Omega/\omega^{x_0}_\Omega(B(\xi_0, r_j))$ be the subsequence obtained in Lemma I (note that $\mu_j(\mathbb{B}) = 1$). Let $\zeta \in \partial\tilde{\Omega}$, $B = B(\zeta, R)$, $\xi \in B\cap\partial\tilde{\Omega}$, and $r \in (0, R)$ so that

$$B' = B(\xi, r) \subseteq B(\xi, 2r) \subseteq B. \tag{3.15}$$

Fix $M \geq 1$ so that $2B \subseteq \frac{M}{4C}\mathbb{B}$.

Let $\xi_j \in \partial\Omega_j$ converge to 0 and $B_j = B(\xi_j, M - |\xi_j|)$, so for j large we have $\frac{M}{2}\mathbb{B} \subseteq B_j \subseteq M\mathbb{B}$. Let y_j be a corkscrew point for B_j in Ω_j, so $B(y_j, r_{B_j}/C) \subseteq B_j\cap\Omega_j$. By passing to a subsequence if necessary, and since $r_{B_j} \to M$, we can assume there is y so that $B(y, \frac{M}{2C}) \subseteq \Omega_j\cap B_j$ for all j large enough. Since $2B \subseteq \frac{M}{4C}\mathbb{B}$, we know $y \in \Omega_j\backslash 2B$, and for j large enough we know $x_j := T_j(x_0) \in \Omega_j\backslash M\mathbb{B} \subseteq \Omega_j\backslash MB_j$, so we can apply Lemma 2.8 twice to get, for $\varepsilon \in (0, 1)$,

$$\begin{aligned}\frac{\omega^x_{\tilde{\Omega}}(B')}{\omega^x_{\tilde{\Omega}}(B)} &\overset{(2.5)}{\sim} \frac{\omega^y_{\tilde{\Omega}}(B')}{\omega^y_{\tilde{\Omega}}(B)} \leq \liminf_{j\to\infty} \frac{\omega^y_{\Omega_j}(B')}{\omega^y_{\Omega_j}((1-\varepsilon)B)}\\ &\overset{(2.5)}{\sim} \liminf_{j\to\infty} \frac{\omega^{x_j}_{\Omega_j}(B')/\omega^{x_j}_{\Omega_j}(B_j)}{\omega^{x_j}_{\Omega_j}((1-\varepsilon)B))/\omega^{x_j}_{\Omega_j}(B_j)} = \liminf_{j\to\infty} \frac{T_{j\#}\omega^{x_0}_\Omega(B')}{T_{j\#}\omega^{x_0}_\Omega((1-\varepsilon)B))}\\ &= \liminf_{j\to\infty} \frac{\mu_j(B')}{\mu_j((1-\varepsilon)B))} \leq \frac{\mu(\overline{B'})}{\mu((1-\varepsilon)B)}\end{aligned}$$

Letting $\varepsilon \to 0$, we get

$$\frac{\omega^x_{\tilde{\Omega}}(B')}{\omega^x_{\tilde{\Omega}}(B)} \leq \frac{\mu(\overline{B'})}{\mu(B)}.$$

Now apply this to $\rho B'$ and take $\rho \uparrow 1$, we get

$$\frac{\omega_{\tilde{\Omega}}^{x}(B')}{\omega_{\tilde{\Omega}}^{x}(B)} = \lim_{\rho\uparrow 1} \frac{\omega_{\tilde{\Omega}}^{x}(\rho B')}{\omega_{\tilde{\Omega}}^{x}(B)} \lesssim \lim_{\rho\to 1} \frac{\mu(\rho \overline{B'})}{\mu(B)} = \frac{\mu(B')}{\mu(B)}.$$

Thus, we get one inequality in (3.14). The other inequality has a similar proof. □

Lemma 3.12. *With the assumptions of Lemma 3.9, suppose there is $E \subseteq \partial\Omega$ with $\omega_{\Omega}^{x_0}(E) > 0$ and $c > 0$ so that*

$$\liminf_{r\to 0} \frac{\mathscr{H}_{\infty}^{s}(B(\xi,r)\cap\partial\Omega)}{r^s} \geq c \quad \text{for all } \xi \in E. \tag{3.16}$$

Then for $\omega_{\Omega}^{x_0}$-almost every $\xi_0 \in E$, there is $c' > 0$ depending on s, d and c so that

$$\mathscr{H}_{\infty}^{s}(B(\xi,r)\cap\partial\tilde{\Omega}) \geq c' r^s \quad \text{for all } \xi \in \partial\tilde{\Omega} \text{ and } r > 0. \tag{3.17}$$

Proof. Let $\xi \in \partial\tilde{\Omega}$ and $r > 0$. Set

$$E_n = \{\xi \in \partial\Omega : \mathscr{H}_{\infty}^{s}(B(\xi,r)\cap\partial\Omega) > r^s \text{ for } r < n^{-1}\}.$$

Then $E = \bigcup E_n$. Let ξ_0 be a point of density in some E_n with respect to $\omega_{\Omega}^{x_0}$. Let $\xi \in \partial\tilde{\Omega} = \operatorname{supp}\mu$, $r > 0$. Then by Corollary 2.12 there is $\xi_j \in E_n \cap T_j^{-1}(B(\xi, r/2))$, and thus if j is large enough so that $rr_j/2 < 1/n$,

$$\begin{aligned}\mathscr{H}_{\infty}^{s}(B(\xi,r)\cap\partial\Omega_j) &= r_j^{-s}\mathscr{H}_{\infty}^{s}(T_j^{-1}(B(\xi,r))\cap\partial\Omega) \\ &\geq r_j^{-s}\mathscr{H}_{\infty}^{d}(B(\xi_j, rr_j/2)\cap\partial\Omega) \geq \frac{cr^s}{2^s}.\end{aligned}$$

Let ν_j be an s-Frostmann measure with support in $B(\xi,r)\cap\partial\Omega_j$ so that $\nu_j(B(\xi,r)) \gtrsim cr^s/2^s$. By passing to a subsequence, we can assume ν_j converges weakly to another s-Frostmann measure ν and $\nu(\overline{B(\xi,r)}) \gtrsim cr^s/2^s$. If $\zeta \in \operatorname{supp}\nu$, then for all $t > 0$, $\nu_j(B(\zeta,2t)) \geq \nu(B(\zeta,t))/2 > 0$ for j large enough, and so for j large

$$\mathscr{H}_{\infty}^{s}(B(\zeta,2t)\cap\partial\Omega_j) \gtrsim \nu_j(B(\zeta,2t)) \geq \nu(B(\zeta,t))/2 > 0.$$

Thus, there is $\zeta_j \in \partial\Omega_j \cap B(\zeta,2t)$, and so

$$\mathscr{H}_{\infty}^{s}(B(\zeta_j,4t)\cap\partial\Omega_j)/(4t)^s \geq \frac{\nu(B(\zeta,t))}{2(4t)^s} > 0.$$

Hence, by Lemma 2.7, for all j large,

$$\begin{aligned}\omega_j^{x_{B(\zeta_j,4t)}}(B((\zeta,4t(1+\delta^{-1}))) &\geq \omega_j^{x_{B(\zeta_j,4t)}}(B(\zeta_j,4t\delta^{-1})) \\ &\gtrsim \mathscr{H}_{\infty}^{s}(B(\zeta_j,4t)\cap\partial\Omega_j)/(4t)^s \gtrsim \frac{\nu(B(\zeta,t))}{2(4t)^s} > 0.\end{aligned}$$

and hence $\tilde{\omega}^{x_{B(\zeta_j,4t)}}(B((\zeta,4t(1+\delta^{-1}))) > 0$ for all $t > 0$, which implies $\zeta \in \operatorname{supp}\tilde{\omega} = \partial\tilde{\Omega}$. This implies, finally, that

$$\mathscr{H}_{\infty}^{s}(B(\xi,r)\cap\partial\tilde{\Omega}) \gtrsim \nu(B(\xi,r)) \gtrsim \frac{cr^s}{2^s}.$$

Since this holds for all $\xi \in \partial\tilde{\Omega}$ and $r > 0$, this finishes the proof. □

This finishes the proof of Lemma I.

4. Proof of Theorem I

In this section, all implied constants are assumed to depend on the uniformity constant and d. Let

$$F = \{\xi \in \partial\Omega : 0 < \theta^{\alpha,*}(\omega_\Omega^{x_0}, \xi) < \infty, \quad \xi \text{ non-degenerate}\}.$$

We fix $\xi_0 \in F$ such that the conclusions of Lemma 2.14 and Lemma I hold for $\mu_j = T_{\xi_0,r_j\#}\omega_\Omega^{x_0}/\omega_\Omega^{x_0}(B(\xi_0,r_j))$. Then for balls $B' \subseteq B$ centered on $\partial\tilde{\Omega}$,

$$\frac{\omega_{\tilde{\Omega}}^{x_B}(B')}{\omega_{\tilde{\Omega}}^{x_B}(B)} \sim \frac{\mu(B')}{\mu(B)} \lesssim \frac{r_{B'}^{\alpha}}{\mu(B)}. \tag{4.1}$$

Pick $\tilde{B} \subseteq \tilde{\Omega}$ so that there is $\zeta \in \partial\tilde{B} \cap \partial\tilde{\Omega}$. Let $\tilde{x}$ be the center of $\tilde{B}$. We claim that if $\alpha > d$, then the normal derivative of $G_{\tilde{B}}(\tilde{x},\cdot)$ at ζ is zero. Let $x \in [\zeta,\tilde{x}] \cap \partial B(\zeta, r_{\tilde{B}}/2)$. Let $B = B(\zeta, 2r_{\tilde{B}})$ and $B' = B(\zeta, |x-\zeta|)$, see Figure 3.

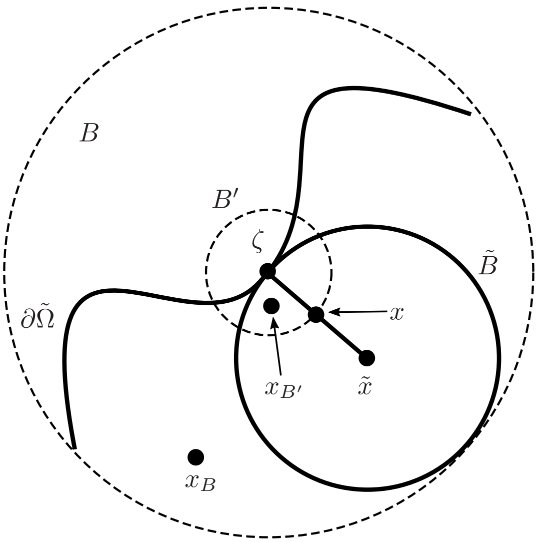

Figure 3. The balls $\tilde{B}$, B, and B'.

Since $G_{\tilde{B}}(\tilde{x},\zeta) = 0$ and because $G_{\tilde{B}} \leq G_{\tilde{\Omega}}$ by the maximum principle, we get

$$\frac{|G_{\tilde{B}}(\tilde{x},x) - G_{\tilde{B}}(\tilde{x},\zeta)|}{|x-\zeta|} = \frac{G_{\tilde{B}}(\tilde{x},x)}{|x-\zeta|} \leq \frac{G_{\tilde{\Omega}}(\tilde{x},x)}{|x-\zeta|}.$$

Now we apply the Harnack chain condition in each variable of the Green function and use Lemma 2.9 to get

$$\frac{G_{\tilde{\Omega}}(\tilde{x},x)}{|x-\zeta|} \sim \frac{G_{\tilde{\Omega}}(x_B,x_{B'})}{|x-\zeta|} \sim \frac{|x-\zeta|^{1-d}\,\omega_{\tilde{\Omega}}^{x_B}(B')}{|x-\zeta|\,\omega_{\tilde{\Omega}}^{x_B}(B)} = \frac{\omega_{\tilde{\Omega}}^{x_B}(B')}{|x-\zeta|^{d}\,\omega_{\tilde{\Omega}}^{x_B}(B)}.$$

Finally, by (4.1), we get

$$\frac{\omega_{\tilde{\Omega}}^{x_B}(B')}{|x-\zeta|^d\,\omega_{\tilde{\Omega}}^{x_B}(B)} \lesssim \frac{r_{B'}^{\alpha}}{|x-\zeta|^d\,\mu(B)} = \frac{|x-\zeta|^{\alpha-d}}{\mu(B)}.$$

Combining these estimates, we get

$$\frac{|G_{\tilde{B}}(\tilde{x},x)-G_{\tilde{B}}(\tilde{x},\zeta)|}{|x-\zeta|} \lesssim \frac{|x-\zeta|^{\alpha-d}}{\mu(B)}$$

so as $x\to\zeta$ along $[\zeta,\tilde{x}]$, this shows that the normal derivative at ζ must be zero, as wished. But $G_{\tilde{B}}(\tilde{x},\cdot)=|\tilde{x}-\cdot|^{1-d}-r_{\tilde{B}}^{1-d}$ on $\tilde{B}$, which clearly has nonzero normal derivative at ζ, and this gives a contradiction. Thus, $\alpha\le d$.

5. Proof of Theorem II

First assume $\theta_*^{\alpha}(\omega_{\Omega}^{x_0},\xi)\in(0,\infty)<\infty$ for each $\xi\in E$ and $\omega_{\Omega}^{x_0}(E)>0$. Then it is not hard to show that E has σ-finite $\mathscr{H}^{\alpha}$-measure. Indeed, note that if

$$E_{k,\ell}=\{\xi\in E:\omega_{\Omega}^{x_0}(B(\xi,r))>r^{\alpha}/\ell \text{ for } r\in(0,k^{-1}]\}$$

then $E=\bigcup_{k,\ell}E_k$. Fix $k\in\mathbb{N}$ and let $r<k^{-1}$ By the Besicovitch covering theorem, we may find a covering of $E_{k,\ell}$ by balls B_j of bounded overlap of radii r so that each B_j is centered on $E_{k,\ell}$. Then

$$\mathscr{H}_r^{\alpha}(E_{k,\ell})\le\sum r_{B_j}^{\alpha}\le k\sum\omega_{\Omega}^{x_0}(B_j)\lesssim_d k\omega_{\Omega}^{x_0}\Big(\bigcup B_j\Big)\le 1.$$

Letting $r\to 0$ shows $E_{k,\ell}$ has finite α measure. If $\alpha\le d-1$, then each $E_{k,\ell}$ has finite $(d-1)$-measure. This implies $E_{k,\ell}$ is polar and polar sets have harmonic measure zero (see e.g., Theorem 5.9.4 and Theorem 6.5.5 in [5]). Thus $\omega(E_{k,\ell})=0$ for each k,ℓ, and hence $\omega_{\Omega}^{x_0}=0$, which is a contradiction since $\omega_{\Omega}^{x_0}(E)>0$. Hence $\alpha>d-1$.

Now assume (1.5). Note that (1.5) and Lemma 2.7 imply each $\xi\in E$ is non-degenerate. Again, by Lemma I, we can find a tangent measure and domain $\tilde{\Omega}$ satisfying

$$\frac{\omega_{\tilde{\Omega}}^{x_B}(B')}{\omega_{\tilde{\Omega}}^{x_B}(B)}\sim\frac{\mu(B')}{\mu(B)}\gtrsim\frac{r_{B'}^{\alpha}}{r_B^{\alpha}}, \tag{5.1}$$

and so that condition (5) of Lemma I holds. This implies $\dim\partial\tilde{\Omega}\le\alpha$, but condition (5) implies $\dim\partial\tilde{\Omega}\ge s$, and so $\alpha\ge s$.

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