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# Global well-posedness of 3-D inhomogeneous Navier–Stokes system with ill-prepared initial data

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**Abstract.** In this paper, we investigate the global well-posedness of 3-D incompressible inhomogeneous Navier–Stokes system with ill-prepared initial data of the form

$$(1 + \varepsilon^\beta a_0(x_h, \varepsilon x_3), (\varepsilon^{1-\alpha} v_0^h, \varepsilon^{-\alpha} v_0^3)(x_h, \varepsilon x_3))$$

for any  $\alpha \in ]0, 1/3[$ ,  $\beta > 2\alpha$ , and  $\varepsilon$  being sufficiently small. This result improves the global well-posedness result for so-called well-prepared initial data, which corresponds to the case of  $\alpha = 0$ .

## 1. Introduction

In this paper, we consider the global well-posedness of the following incompressible inhomogeneous Navier–Stokes system in  $\mathbb{R}^3$ :

$$(1.1) \quad \begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \rho(\partial_t u + u \cdot \nabla u) - \Delta u + \nabla p = 0, \\ \operatorname{div} u = 0, \\ (\rho, u)|_{t=0} = (\rho_0, u_0), \end{cases}$$

where  $\rho, u = (u_1, u_2, u_3)$  represent the density and the velocity of the fluid, respectively, and  $p$  is a scalar pressure function.

Such a system describes a fluid which is obtained by mixing two immiscible incompressible fluids with different densities. It may also describe a fluid containing melted materials.

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*Mathematics Subject Classification* (2010): Primary 35Q30; Secondary 76D05.

*Keywords:* Inhomogeneous Navier–Stokes system, well-posedness.

When the initial density is away from zero, we denote by  $a \stackrel{\text{def}}{=} 1/\rho - 1$ . Then the system (1.1) can be equivalently reformulated as:

$$(1.2) \quad \begin{cases} \partial_t a + u \cdot \nabla a = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t u + u \cdot \nabla u + (1 + a)(\nabla p - \Delta u) = 0, \\ \operatorname{div} u = 0, \\ (a, u)|_{t=0} = (a_0, u_0). \end{cases}$$

As in the classical Navier–Stokes system (NS) (which corresponds to the case of  $a = 0$  in (1.2)), the inhomogeneous Navier–Stokes system (1.2) also has a scaling. Indeed, if  $(a, u)$  solves (1.2) with initial data  $(a_0, u_0)$ , then  $(a, u)_\ell$  is also a solution of (1.2) with initial data  $(a_0, u_0)_\ell$  for any  $\ell > 0$ , where

$$(1.3) \quad (a, u)_\ell \stackrel{\text{def}}{=} (a(\ell^2 \cdot, \ell \cdot), \ell u(\ell^2 \cdot, \ell \cdot)) \quad \text{and} \quad (a_0, u_0)_\ell \stackrel{\text{def}}{=} (a_0(\ell \cdot), \ell u_0(\ell \cdot)).$$

Ladyženskaja and Solonnikov [20] first established the unique solvability of (1.2) in a bounded domain  $\Omega$  with homogeneous Dirichlet boundary condition for  $u$ . Similar results were obtained by Danchin [17] in  $\mathbb{R}^d$  with initial data in the almost critical (corresponding to the scaling in (1.3)) Sobolev spaces. In [16], Danchin studied in general space dimension the unique solvability of the system (1.2) with initial data being small in the scaling invariant (or critical) homogeneous Besov spaces. This result was extended to more general Besov spaces by Abidi [1], and by Abidi, Paicu [2]. The smallness assumption on the initial density was removed in [3], [4].

Very recently, Danchin and Mucha [18] found that it was possible to establish the existence *and* uniqueness of solutions to (1.1) for discontinuous initial density with a small jump on a  $C^1$  interface. More precisely, they proved the global well-posedness of (1.1) with initial data,  $(\rho_0, u_0)$ , satisfying

$$(1.4) \quad \|\rho_0 - 1\|_{\mathcal{M}(B_{p,1}^{-1+d/p}(\mathbb{R}^d))} + \|u_0\|_{B_{p,1}^{-1+d/p}(\mathbb{R}^d)} \leq c$$

for some  $p \in [1, 2d[$  and small enough constant  $c$ . Here  $\mathcal{M}(B_{p,1}^{-1+d/p}(\mathbb{R}^d))$  denotes the multiplier space of the Besov space  $B_{p,1}^{-1+d/p}(\mathbb{R}^d)$ . One may check [18] for details. Let us remark that the classical Navier–Stokes system (NS) has a unique global solution provided that the initial data satisfies  $\|u_0\|_{B_{p,\infty}^{-1+d/p}(\mathbb{R}^d)} \leq c$  for any  $p \in ]1, \infty[$  (see [8]). The restriction of  $p \in [1, 2d[$  in [18] is due to the appearance of the free transport equation in (1.2), and it thus comes out the product of  $a$  with  $\nabla p$  in the momentum equation of (1.1).

Inspired by results concerning the global well-posedness of 3-D incompressible anisotropic Navier–Stokes equations with the third component of the initial velocity field being large (see for instance [22]), Paicu and the first author [23] relaxed the smallness condition in [2] so that (1.2) still has a unique global solution provided that

$$(1.5) \quad (\|a_0\|_{B_{p,1}^{3/p}} + \|u_0^h\|_{B_{p,1}^{-1+3/p}}) \exp(C_0 \|u_0^3\|_{B_{p,1}^{-1+3/p}}^2) \leq c_0$$

for some  $c_0$  sufficiently small and  $p \in ]1, 6[$ . This smallness condition (1.5) was improved by Huang, Paicu and the first author in [19] to

$$(1.6) \quad (\|a_0\|_{L^\infty} + \|u_0^h\|_{B_{p,r}^{-1+d/p}}) \exp(C_r \|u_0^d\|_{B_{p,r}^{-1+d/p}}^{2r}) \leq c_0$$

for some  $p \in ]1, d[, r \in ]1, \infty[$  and in general space dimension  $d$ . We emphasize that the proof in [19], [23] used in a fundamental way the algebraical structure of (1.2), namely,  $\operatorname{div} u = 0$ . The first step is to obtain energy estimates on the horizontal components of the velocity field, and then on the vertical component. Compared with [22], the additional difficulties with this strategy in [19, 23] are that: there appears a hyperbolic type equation in (1.2) and the pressure term is more difficult to be handled due to the appearance of  $a$  in the momentum equation of (1.2).

On the other hand, Chemin and Gallagher [11] initiated the global large solutions of 3-D classical Navier–Stokes equations (NS) with initial data, which is slowly varying in one direction,

$$(v_0^h + \varepsilon u_0^h, u_0^3)(x_h, \varepsilon x_3) \quad \text{with} \quad x_h = (x_1, x_2)$$

for smooth divergence free vector fields  $v_0^h$  and  $u_0 = (u_0^h, u_0^3)$ . The main idea behind the proof in [11] is that the solutions to 3-D Navier–Stokes system (NS) slowly varying in one space variable can be well approximated by solutions of 2-D Navier–Stokes equations. As in the classical 2-D Navier–Stokes system, 2-D inhomogeneous Navier–Stokes system is also globally well-posed with general initial data (see [17], [20] for instance). This motivates the authors in [13] to prove the global well-posedness of (1.2) with data of the form:

$$a_0^\varepsilon(x) = \varepsilon^\beta a_0(x_h, \varepsilon x_3), \quad u_0^\varepsilon(x) = (v_0^h(x_h, \varepsilon x_3), 0)$$

for any  $\beta > 1/4$ . Paicu and the first author [24] proved the global well-posedness of (1.2) with initial data of the form:

$$a_0^\varepsilon(x) = \varepsilon^\beta a_0(x_h, \varepsilon x_3), \quad u_0^\varepsilon(x) = (\varepsilon u_0^h, u_0^3)(x_h, \varepsilon x_3)$$

for any  $\beta > 0$ .

Furthermore, for the classical Navier–Stokes system (NS) with the so-called ill-prepared data

$$(1.7) \quad (\varepsilon^{1-\alpha} u_0^h, \varepsilon^{-\alpha} u_0^3)(x_h, \varepsilon x_3),$$

Chemin, Gallagher and Paicu [12] proved the global well-posedness of (NS) in  $\mathbb{R}^2 \times \mathbb{T}$  for  $\alpha = 0$ . Paicu and the second author [25] proved similar global well-posedness result in  $\mathbb{R}^3$  for  $\alpha = 1/2$ . This result was improved lately by the authors in [26] for any  $\alpha \in ]1/2, 1[$ . We remark that to prove such results as in [12], [25], [26], they need to use analytical type initial data and the tool developed by Chemin [9], which consists in making analytic-type estimates and controlling the size of the analyticity band simultaneously.

Motivated by [12], [25], [26], we shall consider the global solutions of (1.1) with ill-prepared initial data of the form

$$(1.8) \quad \rho_0(x) = \bar{\rho} + \varepsilon^\beta a_0(x_h, \varepsilon x_3), \quad u_0(x) = (\varepsilon^{1-\alpha} v_0^h, \varepsilon^{-\alpha} v_0^3)(x_h, \varepsilon x_3),$$

where  $\bar{\rho}$  is a positive constant,  $v_0^h = (v_0^1, v_0^2)$  and  $v_0 = (v_0^h, v_0^3)$  satisfies  $\operatorname{div} v_0 = 0$ . Obviously, this type data does not satisfy the smallness conditions (1.5) and (1.6) no matter how small  $\varepsilon$  is.

Our main result in this paper is stated as follows.

**Theorem 1.1.** *Let  $\delta > 0$ ,  $\alpha \in ]0, 1/3[$ ,  $\beta > 2\alpha$  and  $\gamma \in ]0, \gamma_0[$  with*

$$\gamma_0 \stackrel{\text{def}}{=} \min\left(\frac{\beta - 2\alpha}{5}, \frac{1 - 3\alpha}{5}\right).$$

*Let  $a_0$  and the solenoidal vector field  $v_0$  satisfy*

$$(1.9) \quad \|(a_0, v_0)\|_X \stackrel{\text{def}}{=} \|a_0\|_{X_1} + \|v_0\|_{X_2} + \|v_0\|_{X_3} < \infty,$$

*where*

$$(1.10) \quad \begin{aligned} \|a_0\|_{X_1} &\stackrel{\text{def}}{=} \|e^{\delta|D|} a_0\|_{B^{1-\gamma, 1/2+\gamma}} + \|e^{\delta|D|} a_0\|_{B^{1+\gamma, 1/2-\gamma}} + \|e^{\delta|D|} a_0\|_{B^{\gamma, 3/2-\gamma}}, \\ \|v_0\|_{X_2} &\stackrel{\text{def}}{=} \|e^{\delta|D|} v_0\|_{B^{-1/2+\gamma, -\gamma}} + \|e^{\delta|D|} v_0\|_{B^{0, -1/2}}, \\ \|v_0\|_{X_3} &\stackrel{\text{def}}{=} \|e^{\delta|D|} v_0\|_{B^{\gamma, 1/2-\gamma}} + \|e^{\delta|D|} v_0\|_{B^{-\gamma, 1/2+\gamma}}. \end{aligned}$$

*Then there exists a small positive constant  $\varepsilon_0$ , which depends on  $\|(a_0, v_0)\|_X$ , such that for  $\varepsilon \leq \varepsilon_0$ , the inhomogeneous Navier–Stokes system (1.1) with initial data given by (1.8) has a unique global smooth solution.*

**Remark 1.2.** The exact value of  $\varepsilon_0$  will be given by (2.15). In fact, we can also deduce from the proof of Theorem 1.1 that there exists a positive constant  $\eta$  such that for any  $a_0$  and divergence free vector field  $v_0$  satisfying

$$\|(a_0, v_0)\|_X \stackrel{\text{def}}{=} \|a_0\|_{X_1} + \|v_0\|_{X_2} + \|v_0\|_{X_3} \leq \eta,$$

the inhomogeneous Navier–Stokes system (1.1) with initial data given by (1.8) has a unique global smooth solution for any  $\varepsilon > 0$ .

Here the anisotropic Besov spaces,  $B^{\sigma, s}(\mathbb{R}^3)$ , and all the other functional framework will be presented in the next section.

Let us remark that besides the difficulties caused by proving global in time Cauchy–Kowalewskya type results in [12], [25], [26] for the classical Navier–Stokes system, here we shall encounter the following types of new difficulties:

- Note that after the scaling transformation, we shall obtain an inhomogeneous Navier–Stokes system (2.1) with anisotropic dissipation  $\Delta_h + \varepsilon^2 \partial_3^2$  and

anisotropic pressure gradient  $-\nabla^\varepsilon q$  for  $\nabla^\varepsilon = (\nabla_h, \varepsilon^2 \partial_3)$ . To capture the subtle dissipation in this new system, we shall use anisotropic Littlewood–Paley analysis, which has been used successfully for both homogeneous and inhomogeneous Navier–Stokes system [6], [13], [15], [25], [26] lately. However, due to the appearance of the free transport equation in (1.1), the assumption of analyticity only for the vertical variable in [12], [25], [26] will not be enough. Instead, we shall consider the initial data which is analytic in all the space variables. We emphasize once again that the algebraical structural of the system (2.1) and the tool developed by Chemin [9] will also play crucial roles in this paper.

- Since we can not use commutator’s argument to deal with the propagation of analytic regularity for transport equation, in order to control the inhomogeneity  $a_\Phi(t)$  in the critical anisotropic Besov space  $B^{1,1/2}(\mathbb{R}^3)$ , we require the global in time  $L^1$  estimate with values in Besov spaces, which are in the scalings of both the space  $B^{2,1/2}(\mathbb{R}^3)$  and in that of  $B_{2,1}^{1,1/2}(\mathbb{R}^3)$ , for the convection velocity field.
- However, in order to control  $\|v_\Phi\|_{L_t^1(B^{1,1/2})}$ , we would require the estimate of  $G(\varepsilon^\beta a)\nabla^\varepsilon q$  in the space  $L_t^1(B^{-1,1/2})$  for  $G(r) \stackrel{\text{def}}{=} r/(1+r)$ , which is impossible due to product laws in two space dimensions. The idea to overcome this difficulty is to use Lemma 4.2 so that we only need to handle the estimate of  $\| [G(\varepsilon^\beta a)\nabla^\varepsilon q]_\Phi \|_{L_t^1(B^{-1+\gamma,1/2-\gamma})}$  for some small positive constant  $\gamma$ . This in turn would require the estimates of  $a_\Phi$  in  $\tilde{L}_t^\infty(B^{1-\gamma,1/2+\gamma})$  and in  $\tilde{L}_t^\infty(B^{1+\gamma,1/2-\gamma})$ , and  $\|v_0\|_{X_2}$  to be finite. This explains the reason why the data in Theorem 1.1 is so much complicated.
- As in the proof of the global well-posedness of inhomogeneous Navier–Stokes system with initial data in the critical spaces, for instance in [3], [4], [18], [19], the pressure is always a big trouble. We point out that the assumption for  $\beta > 2\alpha$  in Theorem 1.1 will only be used to handle the estimates of  $q_{31}$  in (6.11) and of  $q_{41}$  in (6.16). Otherwise, the assumption for  $\beta > \alpha$  would be enough in Theorem 1.1.

Let us end this introduction by the notations we shall use in this context.

For  $a \lesssim b$ , we mean that there is a uniform constant  $C$ , which may be different on different lines but be independent of  $\varepsilon$ , such that  $a \leq Cb$ .

For  $X$  a Banach space and  $I$  an interval of  $\mathbb{R}$ , we denote by  $C(I; X)$  the set of continuous functions on  $I$  with values in  $X$ . For  $q$  in  $[1, +\infty]$ , the notation  $L^q(I; X)$  stands for the set of measurable functions on  $I$  with values in  $X$ , such that  $t \mapsto \|f(t)\|_X$  belongs to  $L^q(I)$ .

We denote by  $L_T^p(L_h^q(L_v^r))$  the space  $L^p([0, T]; L^q(\mathbb{R}_{x_h}; L^r(\mathbb{R}_{x_3})))$  with  $x_h = (x_1, x_2)$ , and  $\nabla_h = (\partial_{x_1}, \partial_{x_2})$ ,  $\Delta_h = \partial_{x_1}^2 + \partial_{x_2}^2$ .  $\nabla_\varepsilon = (\nabla_h, \varepsilon \partial_3)$ ,  $\Delta_\varepsilon = \Delta_h + \varepsilon^2 \partial_3^2$ , and  $\nabla^\varepsilon = (\nabla_h, \varepsilon^2 \partial_3)$ .

Finally,  $\{d_{k,\ell}\}_{k,\ell \in \mathbb{Z}}$  and  $\{d_{k,\ell}(t)\}_{k,\ell \in \mathbb{Z}}$  (resp.  $\{d_k\}_{k \in \mathbb{Z}}$  and  $\{d_k(t)\}_{k \in \mathbb{Z}}$ ) designate generic elements in the sphere of  $\ell^1(\mathbb{Z}^2)$  (resp.  $\ell^1(\mathbb{Z})$ ).

## 2. Structure of the proof

### 2.1. Reduction to a rescaled problem

For simplicity, we take  $\bar{\rho} = 1$  in (1.8) in what follows. As in [12], [24], [25], [26], we shall seek a solution of (1.1) with the form

$$\begin{aligned} \rho(t, x) &= 1 + \varepsilon^\beta a(t, x_h, \varepsilon x_3), \quad u(t, x) = (\varepsilon^{1-\alpha} v^h, \varepsilon^{-\alpha} v^3)(t, x_h, \varepsilon x_3), \\ q(t, x) &= p(t, x_h, \varepsilon x_3). \end{aligned}$$

This leads to the following rescaled inhomogeneous Navier–Stokes system:

$$(2.1) \quad \begin{cases} \partial_t a + \varepsilon^{1-\alpha} v \cdot \nabla a = 0, \\ (1 + \varepsilon^\beta a)(\partial_t v + \varepsilon^{1-\alpha} v \cdot \nabla v) - \Delta_\varepsilon v + \nabla^\varepsilon q = 0, \\ \operatorname{div} v = 0, \\ (a, v)|_{t=0} = (a_0, v_0). \end{cases}$$

Due to  $\operatorname{div} v = 0$ , the rescaled pressure  $q$  is determined by the following elliptic equation:

$$(2.2) \quad -\operatorname{div}\left(\frac{1}{1 + \varepsilon^\beta a} \nabla^\varepsilon q\right) = \varepsilon^{1-\alpha} \operatorname{div}(v \cdot \nabla v) - \operatorname{div}\left(\frac{1}{1 + \varepsilon^\beta a} \Delta_\varepsilon v\right),$$

which is degenerate in  $x_3$  direction when  $\varepsilon$  is small. Thus,  $\nabla q$  may not be uniformly bounded in the usual isentropic Besov spaces. In order to handle this problem and capture the subtle dissipative mechanism in (2.1), we need to use the anisotropic Littlewood–Paley theory.

As in [6], [10], [13], [14],[15], [21], [25], [26], the definitions of the spaces we are going to work with require anisotropic dyadic decomposition of the Fourier variables. Let us recall from [5] that

$$(2.3) \quad \begin{aligned} \Delta_k^h a &= \mathcal{F}^{-1}(\varphi(2^{-k}|\xi_h|)\widehat{a}), & \Delta_\ell^v a &= \mathcal{F}^{-1}(\varphi(2^{-\ell}|\xi_3|)\widehat{a}), \\ S_k^h a &= \mathcal{F}^{-1}(\chi(2^{-k}|\xi_h|)\widehat{a}), & S_\ell^v a &= \mathcal{F}^{-1}(\chi(2^{-\ell}|\xi_3|)\widehat{a}) \quad \text{and} \\ \Delta_j a &= \mathcal{F}^{-1}(\varphi(2^{-j}|\xi|)\widehat{a}), & S_j a &= \mathcal{F}^{-1}(\chi(2^{-j}|\xi|)\widehat{a}), \end{aligned}$$

where  $\xi_h = (\xi_1, \xi_2)$ ,  $\mathcal{F}a$  and  $\widehat{a}$  denote the Fourier transform of the distribution  $a$ , and  $\chi(\tau)$  and  $\varphi(\tau)$  are smooth functions such that

$$\begin{aligned} \operatorname{Supp} \varphi &\subset \left\{ \tau \in \mathbb{R} / \frac{3}{4} \leq |\tau| \leq \frac{8}{3} \right\} \quad \text{and} \quad \forall \tau > 0, \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\tau) = 1, \\ \operatorname{Supp} \chi &\subset \left\{ \tau \in \mathbb{R} / |\tau| \leq \frac{4}{3} \right\} \quad \text{and} \quad \chi(\tau) + \sum_{j \geq 0} \varphi(2^{-j}\tau) = 1. \end{aligned}$$

**Definition 2.1.** The anisotropic Besov space  $B^{s_1, s_2}(\mathbb{R}^3)$  is the space of distributions  $f$  in  $\mathcal{S}'_h(\mathbb{R}^3)$  (which means that  $f \in \mathcal{S}'(\mathbb{R}^3)$  and  $\lim_{j \rightarrow -\infty} \|S_j f\|_{L^\infty} = 0$ ) such that

$$\|f\|_{B^{s_1, s_2}} \stackrel{\text{def}}{=} \sum_{k, \ell \in \mathbb{Z}} 2^{ks_1 + \ell s_2} \|\Delta_k^h \Delta_\ell^v f\|_{L^2}$$

is finite.

We also need to use the norms of Chemin–Lerner type spaces,  $\tilde{L}_T^p(B^{s_1, s_2})$ ,

$$(2.4) \quad \|u\|_{\tilde{L}_T^p(B^{s_1, s_2})} \stackrel{\text{def}}{=} \sum_{k, \ell \in \mathbb{Z}} 2^{ks_1 + \ell s_2} \|\Delta_k^h \Delta_\ell^v u\|_{L_T^p(L^2)}.$$

It is easy to observe that  $\tilde{L}_T^1(B^{s_1, s_2}) = L_T^1(B^{s_1, s_2})$  and for any  $p > 1$ ,

$$(2.5) \quad \|u\|_{L_T^p(B^{s_1, s_2})} \leq \|u\|_{\tilde{L}_T^p(B^{s_1, s_2})}.$$

Theorem 1.1 can be deduced from the following theorem.

**Theorem 2.2.** *Under the same assumptions of Theorem 1.1, there exists a positive constant  $\varepsilon_0$ , which depends on  $\|(a_0, v_0)\|_X$ , such that the rescaled inhomogeneous Navier–Stokes system (2.1) has a unique global smooth solution for any  $\varepsilon \in ]0, \varepsilon_0[$ .*

**Remark 2.3.** More detailed information concerning the solution of (2.1) obtained in Theorem 2.2 will be presented in Subsection 2.3. As a matter of fact, we shall prove that for  $\theta(t), \psi(t)$  determined respectively by (2.7) and (2.9), there holds

$$\sup_{t \geq 0} \theta(t) \leq C\varepsilon^\gamma \|v_0\|_{X_2} \quad \text{and} \quad \sup_{t \geq 0} \Psi(t) \leq C(\|a_0\|_{X_1} + \|v_0\|_{X_3}).$$

### 2.2. The functional setting

The proof of Theorem 2.2 relies on the exponential decay estimate for the Fourier transform of the solution. For this end, we define

$$(2.6) \quad f_\Psi(t) \stackrel{\text{def}}{=} \mathcal{F}^{-1}(e^{\Psi(t, \cdot)} \widehat{f}(t, \cdot)).$$

We introduce the first key quantity  $\theta(t)$  describing the evolution of the analytic band of the solution, as follows

$$(2.7) \quad \begin{aligned} \dot{\theta}(t) = & \varepsilon^{1-\alpha} (\|v_\Phi^h(t)\|_{B^{1, 1/2}} + \|v_\Phi^h(t)\|_{B^{1-\gamma, 1/2+\gamma}} \\ & + \|v_\Phi^h(t)\|_{B^{1+\gamma, 1/2-\gamma}} + \varepsilon^{1+\gamma} \|v_\Phi^h(t)\|_{B^{-\gamma, 3/2+\gamma}}) \\ & + \varepsilon^\gamma (\|v_\Phi^3(t)\|_{B^{1, 1/2}} + \|v_\Phi^3(t)\|_{B^{1+\gamma, 1/2-\gamma}} + \varepsilon^{1+\gamma} \|v_\Phi^3(t)\|_{B^{-\gamma, 3/2+\gamma}}), \end{aligned}$$

with  $\theta(0) = 0$ , where the phase  $\Phi$  is given by

$$(2.8) \quad \Phi(t, \xi) \stackrel{\text{def}}{=} (\delta - \lambda\theta(t))|\xi|$$

for some  $\lambda > 0$  that will be chosen later on. To control the growth of  $\theta(t)$ , we need to introduce the second key quantity  $\Psi(t)$  via

$$(2.9) \quad \Psi(t) \stackrel{\text{def}}{=} \Psi_1(t) + \Psi_2(t) + \Psi_3(t) + \Psi_4(t),$$

where

$$\begin{aligned}
 \Psi_1(t) &\stackrel{\text{def}}{=} \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1,1/2})} + \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1+\gamma,1/2-\gamma})} + \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1-\gamma,1/2+\gamma})} \\
 &\quad + \varepsilon^{3\alpha+3\gamma} \|a_\Phi\|_{\tilde{L}_t^\infty(B^{\gamma,3/2-\gamma})}, \\
 \Psi_2(t) &\stackrel{\text{def}}{=} \|v_\Phi\|_{\tilde{L}_t^\infty(B^{0,1/2})} + \|v_\Phi\|_{\tilde{L}_t^\infty(B^{\gamma,1/2-\gamma})} + \|v_\Phi\|_{\tilde{L}_t^\infty(B^{-\gamma,1/2+\gamma})}, \\
 \Psi_3(t) &\stackrel{\text{def}}{=} \varepsilon^{2\alpha+2\gamma} (\|v_\Phi^h\|_{L_t^1(B^{2,1/2})} + \|v_\Phi^h\|_{L_t^1(B^{2+\gamma,1/2-\gamma})} + \|v_\Phi^h\|_{L_t^1(B^{2-\gamma,1/2+\gamma})} \\
 &\quad + \varepsilon^2 \|v_\Phi^h\|_{L_t^1(B^{0,5/2})} + \varepsilon^2 \|v_\Phi^h\|_{L_t^1(B^{\gamma,5/2-\gamma})} + \varepsilon^2 \|v_\Phi^h\|_{L_t^1(B^{-\gamma,5/2+\gamma})}) \\
 &\quad + \|v_\Phi^3\|_{L_t^1(B^{2,1/2})} + \|v_\Phi^3\|_{L_t^1(B^{2+\gamma,1/2-\gamma})} + \|v_\Phi^3\|_{L_t^1(B^{2-\gamma,1/2+\gamma})} \\
 &\quad + \varepsilon^2 \|v_\Phi^3\|_{L_t^1(B^{0,5/2})} + \varepsilon^2 \|v_\Phi^3\|_{L_t^1(B^{\gamma,5/2-\gamma})} + \varepsilon^2 \|v_\Phi^3\|_{L_t^1(B^{-\gamma,5/2+\gamma})}, \\
 \Psi_4(t) &\stackrel{\text{def}}{=} \varepsilon^{\alpha+\gamma} (\|v_\Phi^h\|_{\tilde{L}_t^2(B^{1,1/2})} + \|v_\Phi^h\|_{\tilde{L}_t^2(B^{1+\gamma,1/2-\gamma})} + \|v_\Phi^h\|_{\tilde{L}_t^2(B^{1-\gamma,1/2+\gamma})} \\
 &\quad + \varepsilon \|v_\Phi^h\|_{\tilde{L}_t^2(B^{0,3/2})} + \varepsilon \|v_\Phi^h\|_{\tilde{L}_t^2(B^{\gamma,3/2-\gamma})} + \varepsilon \|v_\Phi^h\|_{\tilde{L}_t^2(B^{-\gamma,3/2+\gamma})}) \\
 &\quad + \|v_\Phi^3\|_{\tilde{L}_t^2(B^{1,1/2})} + \|v_\Phi^3\|_{\tilde{L}_t^2(B^{1+\gamma,1/2-\gamma})} + \|v_\Phi^3\|_{\tilde{L}_t^2(B^{1-\gamma,1/2+\gamma})} \\
 &\quad + \varepsilon \|v_\Phi^3\|_{\tilde{L}_t^2(B^{0,3/2})} + \varepsilon \|v_\Phi^3\|_{\tilde{L}_t^2(B^{\gamma,3/2-\gamma})} + \varepsilon \|v_\Phi^3\|_{\tilde{L}_t^2(B^{-\gamma,3/2+\gamma})}.
 \end{aligned}$$

The proof of Theorem 2.2 will be based on the following two propositions, whose proofs will be presented in Section 7 and Section 8 respectively. Let us make the *a priori* assumption that

$$(2.10) \quad \Psi_1(T) \leq K,$$

which will be determined hereafter.

**Proposition 2.4.** *Under the assumption that  $\alpha \in ]0, 1/2[$ ,  $\beta > \alpha$  and  $0 < \gamma < \min((\beta - \alpha)/2, (1 - 2\alpha)/4)$ , there exists a positive constant  $C$  such that, for any positive  $\lambda$  and for any  $t$  satisfying  $\theta(t) \leq \delta/\lambda$ , and for  $\varepsilon$  given by (3.8),  $\varepsilon$  is so small that*

$$(2.11) \quad \varepsilon \leq \min \left( \left( \frac{\varepsilon}{K} \right)^{1/\beta}, \left( \frac{1}{2CK} \right)^{1/(\beta-\gamma)} \right).$$

Then we have

$$\theta(t) \leq C (\varepsilon^\gamma \|v_0\|_{X_2} + \max(\varepsilon^{\beta-\alpha-2\gamma}, \varepsilon^\alpha, \varepsilon^{1-2\alpha-2\gamma}) \Psi(t) \theta(t)).$$

**Proposition 2.5.** *Let  $\alpha \in ]0, 1/3[$ ,  $\beta > 2\alpha$ ,  $0 < \gamma \leq \min((\beta - 2\alpha)/2, (1 - 3\alpha)/4)$ , and  $\varepsilon$  satisfy (2.11). Then there exists a positive constant  $C$  such that, for any positive  $\lambda$  and for any  $t$  satisfying  $\theta(t) \leq \delta/\lambda$ , we have*

$$\begin{aligned}
 \Psi(t) &\leq C (\|a_0\|_{X_1} + \|v_0\|_{X_3}) \\
 &\quad + C \left( \frac{1}{\lambda} + \max \{ \varepsilon^\gamma, \varepsilon^{\beta-2\alpha-2\gamma}, \varepsilon^{1-3\alpha-4\gamma}, K \varepsilon^{\beta-2\alpha-\gamma} \} \Psi(t) \right) \Psi(t).
 \end{aligned}$$



**2.3. Proof of Theorem 2.2**

The proof of Theorem 2.2 essentially follows from the *a priori* estimates for sufficiently smooth solutions of (2.1) (see [12] for instance). For simplicity, here we only present the global *a priori* estimates for smooth solutions of (2.1). To this end, for  $\theta(t), \Psi(t)$  determined respectively by (2.7) and (2.9), we take  $K = K_0 \stackrel{\text{def}}{=} 4C(\|a_0\|_{X_1} + \|v_0\|_{X_3})$  in (2.10) and define

$$(2.12) \quad T^* \stackrel{\text{def}}{=} \sup \{ T > 0 : \theta(T) \leq 4C \varepsilon^\gamma \|v_0\|_{X_2} \quad \text{and} \quad \Psi(T) \leq K_0 \}.$$

Then it suffices to prove that  $T^* = +\infty$  provided that  $\varepsilon$  is sufficiently small. In order to use Propositions 2.4 and 2.5, we need to assume that  $\theta(T) \leq \delta/\lambda$ , which leads to the condition that

$$(2.13) \quad 4C \varepsilon^\gamma \|v_0\|_{X_2} \leq \frac{\delta}{\lambda}.$$

Then under the assumptions of Theorem 1.1, we infer from Proposition 2.4 and Proposition 2.5 that, for all  $T \in [0, T^*[,$

$$(2.14) \quad \begin{aligned} \theta(T) &\leq C(\varepsilon^\gamma + 4\varepsilon^{2\gamma}K_0)\|v_0\|_{X_2}, \quad \text{and} \\ \Psi(T) &\leq \frac{K_0}{4} + C\left(\frac{1}{\lambda} + \varepsilon^\gamma K_0 + \varepsilon^{2\gamma}K_0^2\right)K_0, \end{aligned}$$

provided that  $\varepsilon$  is so small that  $\gamma \leq \min\left(\frac{\beta-\alpha}{4}, \frac{\beta-2\alpha}{3}, \frac{1-3\alpha}{5}\right)$ . We then select  $\lambda$  so large that  $\lambda = 4C$ , and then choose  $\varepsilon$  to be so small that there holds (2.11), (2.13) and  $8C \varepsilon^\gamma K_0 \leq 1$ , that is

$$(2.15) \quad \varepsilon \leq \min \left\{ \left(\frac{\epsilon}{K_0}\right)^{1/\beta}, \left(\frac{1}{2CK_0}\right)^{1/(\beta-\gamma)}, \left(\frac{1}{8CK_0}\right)^{1/\gamma}, \left(\frac{\delta}{16C^2\|v_0\|_{X_2}}\right)^{1/\gamma} \right\}.$$

With this choice of  $\varepsilon$ , we deduce from (2.14) that for all  $T \in [0, T^*[,$

$$\theta(T) \leq 2C \varepsilon^\gamma \|v_0\|_{X_2} \quad \text{and} \quad \Psi(T) \leq \frac{3K_0}{4},$$

which contradicts with (2.12) if  $T^* < +\infty$ . This in turn shows that  $T^* = \infty$ , and whence we conclude the proof of Theorem 2.2. □

**3. The action of subadditive phases on products**

For any function  $f$ , we denote by  $f^+$  the inverse Fourier transform of  $[\widehat{f}]$ . Let us notice that the map  $f \mapsto f^+$  preserves the norm of the anisotropic Besov space  $B^{s_1, s_2}$  given by Definition 2.1. Throughout this section,  $\Phi$  denotes a locally bounded function on  $\mathbb{R}^+ \times \mathbb{R}^3$  which verifies

$$(3.1) \quad \Phi(t, \xi) \leq \Phi(t, \xi - \eta) + \Phi(t, \eta).$$

For the convenience of the readers, we recall the following anisotropic Bernstein type lemma from [14], [21]:

**Lemma 3.1.** *Let  $\mathcal{B}_h$  (resp.  $\mathcal{B}_v$ ) a ball of  $\mathbb{R}_h^2$  (resp.  $\mathbb{R}_v$ ), and  $\mathcal{C}_h$  (resp.  $\mathcal{C}_v$ ) a ring of  $\mathbb{R}_h^2$  (resp.  $\mathbb{R}_v$ ). Let  $1 \leq p_2 \leq p_1 \leq \infty$  and  $1 \leq q_2 \leq q_1 \leq \infty$ . Then,*

a) *if the support of  $\widehat{a}$  is included in  $2^k \mathcal{B}_h$ , then*

$$\|\partial_{x_h}^\alpha a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{k(|\alpha|+2(1/p_2-1/p_1))} \|a\|_{L_h^{p_2}(L_v^{q_1})};$$

b) *if the support of  $\widehat{a}$  is included in  $2^\ell \mathcal{B}_v$ , then*

$$\|\partial_{x_3}^\beta a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{\ell(\beta+(1/q_2-1/q_1))} \|a\|_{L_h^{p_1}(L_v^{q_2})};$$

c) *if the support of  $\widehat{a}$  is included in  $2^k \mathcal{C}_h$ , then*

$$\|a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{-kN} \sup_{|\alpha|=N} \|\partial_{x_h}^\alpha a\|_{L_h^{p_1}(L_v^{q_1})};$$

d) *if the support of  $\widehat{a}$  is included in  $2^\ell \mathcal{C}_v$ , then*

$$\|a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{-\ell N} \|\partial_{x_3}^N a\|_{L_h^{p_1}(L_v^{q_1})}.$$

**Lemma 3.2.** *Let  $\sigma_1 < \sigma < \sigma_2$  and  $s_2 < s < s_1$  with  $\sigma_1 + s_1 = \sigma + s = \sigma_2 + s_2$ . Then one has*

$$(3.2) \quad \|g_\Phi\|_{B^{\sigma,s}} \lesssim \|g_\Phi\|_{B^{\sigma_1,s_1}} + \|g_\Phi\|_{B^{\sigma_2,s_2}}.$$

*Proof.* According to Definition 2.1 and (2.6), one has

$$\|g_\Phi\|_{B^{\sigma,s}} = \sum_{k<\ell} 2^{k\sigma} 2^{\ell s} \|\Delta_k^h \Delta_\ell^v g_\Phi\|_{L^2} + \sum_{k\geq\ell} 2^{k\sigma} 2^{\ell s} \|\Delta_k^h \Delta_\ell^v g_\Phi\|_{L^2}.$$

However, it is easy to observe that

$$\begin{aligned} \sum_{k<\ell} 2^{k\sigma} 2^{\ell s} \|\Delta_k^h \Delta_\ell^v g_\Phi\|_{L^2} &\lesssim \sum_{k<\ell} d_{k,\ell} 2^{k(\sigma-\sigma_1)} 2^{\ell(s-s_1)} \|g_\Phi\|_{B^{\sigma_1,s_1}} \\ &\lesssim \sum_{k<\ell} d_{k,\ell} 2^{\ell(\sigma+s-\sigma_1-s_1)} \|g_\Phi\|_{B^{\sigma_1,s_1}} \lesssim \|g_\Phi\|_{B^{\sigma_1,s_1}} \end{aligned}$$

where we used the fact that  $\sigma + s = \sigma_1 + s_1$  and that  $\sigma > \sigma_1$  so that  $2^{k(\sigma-\sigma_1)} \leq 2^{\ell(\sigma-\sigma_1)}$ .

Along the same line, we find

$$\sum_{k\geq\ell} 2^{k\sigma} 2^{\ell s} \|\Delta_k^h \Delta_\ell^v g_\Phi\|_{L^2} \lesssim \|g_\Phi\|_{B^{\sigma_2,s_2}}.$$

This completes the proof of (3.2). □

To study the law of product in the anisotropic Besov spaces, we need to use Bony’s decomposition. We first recall the isotropic para-differential decomposition

from [7]: for  $a$  and  $b$  in  $S'_h(\mathbb{R}^3)$ ,

$$\begin{aligned}
 ab &= T(a, b) + R(a, b) + \bar{T}(a, b) \quad \text{and} \quad ab = T(a, b) + \mathcal{R}(a, b) \quad \text{with} \\
 T(a, b) &= \sum_{j \in \mathbb{Z}} S_{j-1} a \Delta_j b, \quad \bar{T}(a, b) = T(b, a), \quad \mathcal{R}(a, b) = \sum_{j \in \mathbb{Z}} \Delta_j a S_{j+2} b \quad \text{and} \\
 (3.3) \quad R(a, b) &= \sum_{j \in \mathbb{Z}} \Delta_j a \tilde{\Delta}_j b, \quad \text{with} \quad \tilde{\Delta}_j b = \sum_{j'=j-1}^{j+1} \Delta_{j'} b.
 \end{aligned}$$

Sometimes we shall use Bony’s decomposition for both horizontal and vertical variables simultaneously.

As an application of the above basic facts on Littlewood–Paley theory, we now present the following law of product in the anisotropic Besov spaces.

**Lemma 3.3.** *Let  $\sigma_1, \dots, \sigma_8$  and  $s_1, \dots, s_8$  be real numbers so that*

$$\begin{aligned}
 \sigma_1 + \sigma_2 = \sigma_3 + \sigma_4 = \sigma_5 + \sigma_6 = \sigma_7 + \sigma_8 > 0 \quad &\text{with} \quad \sigma_1, \sigma_4, \sigma_5, \sigma_8 \leq 1 \quad \text{and} \\
 s_1 + s_2 = s_3 + s_4 = s_5 + s_6 = s_7 + s_8 > 0 \quad &\text{with} \quad s_1, s_4, s_6, s_7 \leq 1/2.
 \end{aligned}$$

Then there holds

$$\begin{aligned}
 (3.4) \quad \|[ab]_\Phi\|_{B^{\sigma_1+\sigma_2-1, s_1+s_2-1/2}} &\lesssim \|a_\Phi\|_{B^{\sigma_1, s_1}} \|b_\Phi\|_{B^{\sigma_2, s_2}} + \|a_\Phi\|_{B^{\sigma_3, s_3}} \|b_\Phi\|_{B^{\sigma_4, s_4}} \\
 &\quad + \|a_\Phi\|_{B^{\sigma_5, s_5}} \|b_\Phi\|_{B^{\sigma_6, s_6}} + \|a_\Phi\|_{B^{\sigma_7, s_7}} \|b_\Phi\|_{B^{\sigma_8, s_8}}.
 \end{aligned}$$

*Proof.* We first observe from (3.1) that

$$(3.5) \quad |\mathcal{F}[\Delta_k^h \Delta_\ell^v(ab)]_\Phi(\xi)| \leq \mathcal{F}[\Delta_k^h \Delta_\ell^v(a_\Phi^+ b_\Phi^+)](\xi).$$

Hence, it suffices to prove (3.4) for  $\Phi = 0$ .

Indeed we get, by applying Bony’s decomposition (3.3) for both horizontal and vertical variables, that

$$ab = (T^h + R^h + \bar{T}^h)(T^v + R^v + \bar{T}^v)(a, b).$$

Considering the support to the Fourier transform of the terms in  $R^h R^v(a, b)$ , by applying Lemma 3.1, we obtain

$$\begin{aligned}
 \|\Delta_k^h \Delta_\ell^v R^h R^v(a, b)\|_{L^2} &\lesssim 2^k 2^{\ell/2} \sum_{\substack{k' \geq k-3 \\ \ell' \geq \ell-3}} \|\Delta_{k'}^h \Delta_{\ell'}^v a\|_{L^2} \|\tilde{\Delta}_{k'}^h \tilde{\Delta}_{\ell'}^v b\|_{L^2} \\
 &\lesssim 2^k 2^{\ell/2} \sum_{\substack{k' \geq k-3 \\ \ell' \geq \ell-3}} d_{k', \ell'} 2^{-k'(\sigma_1+\sigma_2)} 2^{-\ell'(s_1+s_2)} \|a\|_{B^{\sigma_1, s_1}} \|b\|_{B^{\sigma_2, s_2}} \\
 &\lesssim d_{k, \ell} 2^{-k(\sigma_1+\sigma_2-1)} 2^{-\ell(s_1+s_2-1/2)} \|a\|_{B^{\sigma_1, s_1}} \|b\|_{B^{\sigma_2, s_2}}.
 \end{aligned}$$

The same estimate holds for  $T^h T^v(a, b)$ ,  $T^h R^v(a, b)$ , and  $R^h T^v(a, b)$ .

While since  $\sigma_5 \leq 1$  and  $s_6 \leq 1/2$ , it follows from Lemma 3.1 that

$$\begin{aligned} \|S_{k'-1}^h \Delta_{\ell'}^v a\|_{L_h^\infty(L_v^2)} &\lesssim d_{\ell'} 2^{k'(1-\sigma_5)} 2^{-\ell' s_5} \|a\|_{B^{\sigma_5, s_5}} \quad \text{and} \\ \|\Delta_{k'}^h S_{\ell'-1}^v b\|_{L_h^2(L_v^\infty)} &\lesssim d_{k'} 2^{-k' \sigma_6} 2^{\ell'(1/2-s_6)} \|b\|_{B^{\sigma_6, s_6}}, \end{aligned}$$

from which we infer

$$\begin{aligned} \|\Delta_k^h \Delta_\ell^v T^h \bar{T}^v(a, b)\|_{L^2} &\lesssim \sum_{\substack{|k'-k| \leq 4 \\ |\ell'-\ell| \leq 4}} \|S_{k'-1}^h \Delta_{\ell'}^v a\|_{L_h^\infty(L_v^2)} \|\Delta_{k'}^h S_{\ell'-1}^v b\|_{L_h^2(L_v^\infty)} \\ &\lesssim d_{k, \ell} 2^{-k(\sigma_5+\sigma_6-1)} 2^{-\ell(s_5+s_6-1/2)} \|a\|_{B^{\sigma_5, s_5}} \|b\|_{B^{\sigma_6, s_6}}. \end{aligned}$$

The same estimate holds for  $R^h \bar{T}^v(a, b)$ .

In the same manner, we obtain

$$\begin{aligned} \|\Delta_k^h \Delta_\ell^v \bar{T}^h T^v(a, b)\|_{L^2} &\lesssim \sum_{\substack{|k'-k| \leq 4 \\ |\ell'-\ell| \leq 4}} \|\Delta_{k'}^h S_{\ell'-1}^v a\|_{L_h^2(L_v^\infty)} \|S_{k'-1}^h \Delta_{\ell'}^v b\|_{L_h^\infty(L_v^2)} \\ &\lesssim d_{k, \ell} 2^{-k(\sigma_7+\sigma_8-1)} 2^{-\ell(s_7+s_8-1/2)} \|a\|_{B^{\sigma_7, s_7}} \|b\|_{B^{\sigma_8, s_8}}. \end{aligned}$$

The same estimate holds for  $\bar{T}^h R^v(a, b)$ .

Finally, since  $\sigma_4 \leq 1, s_4 \leq 1/2$ , applying Lemma 3.1 yields

$$\|S_{k'-1}^h S_{\ell'-1}^v b\|_{L^\infty} \lesssim 2^{k'(1-\sigma_4)} 2^{\ell'(1/2-s_4)} \|b\|_{B^{\sigma_4, s_4}},$$

which ensures

$$\begin{aligned} \|\Delta_k^h \Delta_\ell^v \bar{T}^h \bar{T}^v(a, b)\|_{L^2} &\lesssim \sum_{\substack{|k'-k| \leq 4 \\ |\ell'-\ell| \leq 4}} \|\Delta_{k'}^h \Delta_{\ell'}^v a\|_{L^2} \|S_{k'-1}^h S_{\ell'-1}^v b\|_{L^\infty} \\ &\lesssim d_{k, \ell} 2^{-k(\sigma_3+\sigma_4-1)} 2^{-\ell(s_3+s_4-1/2)} \|a\|_{B^{\sigma_3, s_3}} \|b\|_{B^{\sigma_4, s_4}}. \end{aligned}$$

This completes the proof of (3.4). □

We remark that the law of product of Lemma 3.3 works also for Chemin–Lerner type spaces  $\tilde{L}_T^p(B^{s_1, s_2})$ . Indeed, the proof of Lemma 3.3 implies the following corollary.

**Corollary 3.4.** *Let  $p, p_1, p_2, p_3, p_4 \in [1, \infty]$  with  $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$ . Then under the assumptions that if  $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \leq 1$  and  $s_1, s_2, s_3, s_4$  satisfy  $0 < \sigma_1 + \sigma_2 = \sigma_3 + \sigma_4, s_1, s_4 \leq 1/2$  and  $0 < s_1 + s_2 = s_3 + s_4$ , or if  $\sigma_1, \sigma_2, \sigma_3, \sigma_4$  and  $s_1, s_2, s_3, s_4 \leq 1/2$  satisfy  $\sigma_1, \sigma_4 \leq 1, 0 < \sigma_1 + \sigma_2 = \sigma_3 + \sigma_4$ , and  $0 < s_1 + s_2 = s_3 + s_4$ , one has*

$$\begin{aligned} (3.6) \quad &\| [ab]_\Phi \|_{\tilde{L}_T^p(B^{\sigma_1+\sigma_2-1, s_1+s_2-1/2})} \\ &\lesssim \|a_\Phi\|_{\tilde{L}_T^{p_1}(B^{\sigma_1, s_1})} \|b_\Phi\|_{\tilde{L}_T^{p_2}(B^{\sigma_2, s_2})} + \|a_\Phi\|_{\tilde{L}_T^{p_3}(B^{\sigma_3, s_3})} \|b_\Phi\|_{\tilde{L}_T^{p_4}(B^{\sigma_4, s_4})}. \end{aligned}$$

In the particular case when  $\sigma_1, \sigma_2 \leq 1$  with  $\sigma_1 + \sigma_2 > 0$  and  $s_1, s_2 \leq 1/2$  with  $s_1 + s_2 > 0$ , one has

$$(3.7) \quad \|[ab]_{\Phi}\|_{\tilde{L}_T^p(B^{\sigma_1+\sigma_2-1, s_1+s_2-1/2})} \lesssim \|a_{\Phi}\|_{\tilde{L}_T^{p_1}(B^{\sigma_1, s_1})} \|b_{\Phi}\|_{\tilde{L}_T^{p_2}(B^{\sigma_2, s_2})}.$$

**Remark 3.5.** Let us remark that if  $\sigma_1, \sigma_2 \leq 1, \sigma_1 + \sigma_2 > 0$  and  $s_1 \leq 1/2$ , the proof of Lemma 3.3 also implies

$$\begin{aligned} & \|[\Delta_k^h \Delta_{\ell}^v T^v(a, b)]_{\Phi}(t)\|_{L^2} \\ & \leq C(d_k(t)d_{\ell} + d_{k, \ell}) 2^{k(1-\sigma_1-\sigma_2)} 2^{\ell(1/2-s_1-s_2)} \|a_{\Phi}(t)\|_{B^{\sigma_1, s_1}} \|b_{\Phi}\|_{\tilde{L}_t^{\infty}(B^{\sigma_2, s_2})}. \end{aligned}$$

**Lemma 3.6.** Let  $\sigma, s > 0$  and  $\sigma \leq 1$  or  $s \leq 1/2$ ; let  $G(r) = r/(1+r)$ . Then there exists  $\epsilon > 0$  such that, if

$$(3.8) \quad \|a_{\Phi}\|_{\tilde{L}_T^{\infty}(B^{1, 1/2})} \leq \epsilon,$$

there holds

$$\|[G(a)]_{\Phi}\|_{\tilde{L}_T^{\infty}(B^{\sigma, s})} \leq 2 \|a_{\Phi}\|_{\tilde{L}^{\infty}(B^{\sigma, s})}.$$

*Proof.* Indeed, under the assumption of Lemma 3.6, we deduce from Corollary 3.4 that

$$\|[ab]_{\Phi}\|_{\tilde{L}_T^{\infty}(B^{\sigma, s})} \leq C (\|a_{\Phi}\|_{\tilde{L}_T^{\infty}(B^{1, 1/2})} \|b_{\Phi}\|_{\tilde{L}_T^{\infty}(B^{\sigma, s})} + \|a_{\Phi}\|_{\tilde{L}_T^{\infty}(B^{\sigma, s})} \|b_{\Phi}\|_{\tilde{L}_T^{\infty}(B^{1, 1/2})}).$$

Then one can inductively prove that

$$\|[a^k]_{\Phi}\|_{\tilde{L}_T^{\infty}(B^{\sigma, s})} \leq k C^k \|a_{\Phi}\|_{\tilde{L}_T^{\infty}(B^{1, 1/2})}^{k-1} \|a_{\Phi}\|_{\tilde{L}_T^{\infty}(B^{\sigma, s})}.$$

On the other hand, Taylor’s expansion gives

$$G(r) = \sum_{k=1}^{\infty} (-1)^{k-1} r^k \quad \text{for } r \in ]-1, 1[,$$

from which and (3.8), we infer

$$\begin{aligned} \|[G(a)]_{\Phi}\|_{\tilde{L}_T^{\infty}(B^{\sigma, s})} & \leq \sum_{k=0}^{\infty} \|[a^{k+1}]_{\Phi}\|_{\tilde{L}_T^{\infty}(B^{\sigma, s})} \\ & \leq \|a_{\Phi}\|_{\tilde{L}_T^{\infty}(B^{\sigma, s})} \sum_{k=0}^{\infty} (k+1) (\epsilon C)^k \leq 2 \|a_{\Phi}\|_{\tilde{L}_T^{\infty}(B^{\sigma, s})} \end{aligned}$$

whenever  $\epsilon$  is so small that  $C\epsilon \leq \delta_0$  for some  $\delta_0$  sufficiently small. This yields the lemma. □

### 4. The action of the phase $\Phi$ on the heat semigroup

This section is devoted to studying the action of the Fourier multiplier  $e^{\Phi(t,D)}$  on the heat semigroup  $e^{t\Delta_\varepsilon}$  for the phase function  $\Phi(t, \xi)$  given by (2.8). Let us first present the classical parabolic smoothing estimates in the Chemin–Lerner type space.

**Lemma 4.1.** *Let  $\beta \in [0, 2], r \in [1, \infty]$  and  $\sigma, s \in \mathbb{R}$ . Let  $v_0 = (v_0^h, v_0^3)$  be a divergence free vector field. Then one has*

$$(4.1) \quad \begin{aligned} \varepsilon^{\beta/r} \left\| [e^{t\Delta_\varepsilon} v_0]_\Phi \right\|_{\tilde{L}_T^r(B^{\sigma,s})} &\lesssim \|e^{\delta|D|} v_0\|_{B^{\sigma-(2-\beta)/r, s-\beta/r}} \quad \text{and} \\ \varepsilon^{\beta/r} \left\| [e^{t\Delta_\varepsilon} v_0^3]_\Phi \right\|_{\tilde{L}_T^r(B^{\sigma,s})} &\lesssim \|e^{\delta|D|} v_0^h\|_{B^{\sigma+1-(2-\beta)/r, s-1-\beta/r}}. \end{aligned}$$

*Proof.* By virtue of (2.6) and (2.8), we get

$$\begin{aligned} \left\| \Delta_k^h \Delta_\ell^v [e^{t\Delta_\varepsilon} v_0]_\Phi \right\|_{L^2} &\lesssim e^{-ct(2^{2k} + \varepsilon^2 2^{2\ell})} \|e^{\delta|D|} \Delta_k^h \Delta_\ell^v v_0\|_{L^2} \\ &\lesssim d_{k,\ell} 2^{-k(\sigma-(2-\beta)/r)} 2^{-\ell(s-\beta/r)} e^{-ct(2^{2k} + \varepsilon^2 2^{2\ell})} \|e^{\delta|D|} v_0\|_{B^{\sigma-(2-\beta)/r, s-\beta/r}}, \end{aligned}$$

from which and

$$\left\| e^{-ct(2^{2k} + \varepsilon^2 2^{2\ell})} \right\|_{L_t^r} \leq C \min(2^{-2k}, \varepsilon^{-2} 2^{-2\ell})^{1/r},$$

we deduce

$$\varepsilon^{\beta/r} \left\| \Delta_k^h \Delta_\ell^v [e^{t\Delta_\varepsilon} v_0]_\Phi \right\|_{L_t^r(L^2)} \lesssim d_{k,\ell} 2^{-k\sigma} 2^{-\ell s} \|e^{\delta|D|} v_0\|_{B^{\sigma-(2-\beta)/r, s-\beta/r}},$$

which leads to the first inequality of (4.1).

Exactly by the same manner, we get by  $\operatorname{div} v_0 = 0$  and Lemma 3.1 that

$$\begin{aligned} \left\| \Delta_k^h \Delta_\ell^v [e^{t\Delta_\varepsilon} v_0^3]_\Phi \right\|_{L^2} &\lesssim 2^{-\ell} e^{-ct(2^{2k} + \varepsilon^2 2^{2\ell})} \|e^{\delta|D|} \Delta_k^h \Delta_\ell^v \operatorname{div}_h v_0^h\|_{L^2} \\ &\lesssim d_{k,\ell} 2^{-k(\sigma-(2-\beta)/r)} 2^{-\ell(s-\beta/r)} e^{-ct(2^{2k} + \varepsilon^2 2^{2\ell})} \|e^{\delta|D|} v_0\|_{B^{\sigma+1-(2-\beta)/r, s-1-\beta/r}}, \end{aligned}$$

and whence

$$\varepsilon^{\beta/r} \left\| \Delta_k^h \Delta_\ell^v [e^{t\Delta_\varepsilon} v_0^3]_\Phi \right\|_{L_t^r(L^2)} \lesssim d_{k,\ell} 2^{-k\sigma} 2^{-\ell s} \|e^{\delta|D|} v_0\|_{B^{\sigma+1-(2-\beta)/r, s-1-\beta/r}},$$

which implies the second inequality of (4.1). This completes the proof of the lemma.  $\square$

In what follows, we denote

$$(4.2) \quad E_\varepsilon f(t) \stackrel{\text{def}}{=} \int_0^t e^{(t-t')\Delta_\varepsilon} f(t') dt'.$$

**Lemma 4.2.** *Let  $\beta \in [0, 2], r_1, r_2 \in [1, \infty]$  with  $r_2 \leq r_1$ , and  $\sigma, s \in \mathbb{R}$ . Then there holds*

$$(4.3) \quad \varepsilon^{\beta/r} \left\| [E_\varepsilon f]_\Phi \right\|_{\tilde{L}_T^{r_1}(B^{\sigma,s})} \lesssim \|f_\Phi\|_{\tilde{L}_T^{r_2}(B^{\sigma-(2-\beta)/r, s-\beta/r})}$$

with  $1/r = 1 + 1/r_1 - 1/r_2$ .

*Proof.* Notice that

$$\|\Delta_k^h \Delta_\ell^v [E_\varepsilon f]_\Phi\|_{L^2} \lesssim \int_0^t e^{-c(t-t')(2^{2k} + \varepsilon^2 2^{2\ell})} \|\Delta_k^h \Delta_\ell^v f_\Phi(t')\|_{L^2} dt',$$

from which and Young's inequality, we infer

$$\begin{aligned} \|\Delta_k^h \Delta_\ell^v [E_\varepsilon f]_\Phi\|_{L_T^{r_1}(L^2)} &\lesssim \|e^{-t(2^{2k} + \varepsilon^2 2^{2\ell})}\|_{L_T^r} \|\Delta_k^h \Delta_\ell^v f_\Phi\|_{L_T^{r_2}(L^2)} \\ &\lesssim \min(2^{-2k}, \varepsilon^{-2} 2^{-2\ell})^{1/r} \|\Delta_k^h \Delta_\ell^v f_\Phi\|_{L_T^{r_2}(L^2)} \end{aligned}$$

with  $1/r = 1 + 1/r_1 - 1/r_2$ , which implies (4.3). □

The following lemma concerns the regularizing effect due to the analyticity.

**Lemma 4.3.** *Let  $\sigma, s \in \mathbb{R}$ , and  $p(D)$  be a Fourier multiplier with symbol  $p(\xi)$  satisfying  $|p(\xi)| \leq C|\xi_3|$ . Assume that  $f$  verifies*

$$(4.4) \quad \|\Delta_k^h \Delta_\ell^v f_\Phi(t)\|_{L^2} \lesssim (d_k(t)d_\ell + d_{k,\ell}) 2^{-k\sigma} 2^{-\ell s} \dot{\theta}(t) \|g_\Phi\|_{\tilde{L}_t^\infty(B^{\sigma,s})}$$

for  $\dot{\theta}(t)$  given by (2.7). Then there holds

$$(4.5) \quad \|[E_\varepsilon p(D)f]_\Phi\|_{\tilde{L}_T^\infty(B^{\sigma,s})} \leq \frac{C}{\lambda} \|g_\Phi\|_{\tilde{L}_T^\infty(B^{\sigma,s})}.$$

*Proof.* In view of (2.8), we write

$$(4.6) \quad \Phi(t, D) - \Phi(t', D) = -\lambda \int_{t'}^t \dot{\theta}(\tau) d\tau |D|,$$

from which and (4.4), we infer

$$\begin{aligned} \|\Delta_k^h \Delta_\ell^v [E_\varepsilon p(D)f]_\Phi\|_{L_t^\infty(L^2)} &\lesssim 2^\ell \int_0^t e^{-c\lambda \int_{t'}^t \dot{\theta}(\tau) d\tau 2^\ell} \|\Delta_k^h \Delta_\ell^v f_\Phi(t')\|_{L^2} dt' \\ &\lesssim 2^{-k\sigma} 2^{\ell(1-s)} \left( d_\ell \int_0^t e^{-c\lambda \int_{t'}^t \dot{\theta}(\tau) d\tau 2^\ell} d_k(t') \dot{\theta}(t') dt' \right. \\ &\quad \left. + d_{k,\ell} \int_0^t e^{-c\lambda \int_{t'}^t \dot{\theta}(\tau) d\tau 2^\ell} \dot{\theta}(t') dt' \right) \|g_\Phi\|_{\tilde{L}_t^\infty(B^{\sigma,s})}, \end{aligned}$$

which implies

$$\begin{aligned} \|[E_\varepsilon p(D)f]_\Phi\|_{\tilde{L}_T^\infty(B^{\sigma,s})} &= \sum_{k,\ell \in \mathbb{Z}} 2^{k\sigma} 2^{\ell s} \|\Delta_k^h \Delta_\ell^v [E_\varepsilon a(D)f]_\Phi\|_{L_t^\infty(L^2)} \\ &\leq C \sum_{\ell \in \mathbb{Z}} 2^\ell \left( d_\ell \int_0^t e^{-c\lambda \int_{t'}^t \dot{\theta}(\tau) d\tau 2^\ell} \dot{\theta}(t') dt' \right. \\ &\quad \left. + \sum_{k \in \mathbb{Z}} d_{k,\ell} \int_0^t e^{-c\lambda \int_{t'}^t \dot{\theta}(\tau) d\tau 2^\ell} \dot{\theta}(t') dt' \right) \|g_\Phi\|_{\tilde{L}_t^\infty(B^{\sigma,s})} \\ &\leq \frac{C}{\lambda} \|g_\Phi\|_{\tilde{L}_t^\infty(B^{\sigma,s})}. \end{aligned}$$

This proves (4.5). □

### 5. Propagation of analytic regularity for the transport equation

In this section, we investigate the propagation of analytic regularity for the following transport equation:

$$(5.1) \quad \partial_t a + \varepsilon^{1-\alpha} v \cdot \nabla a = f, \quad a(0, x) = a_0(x).$$

**Proposition 5.1.** *Let  $\sigma \in ]-1, 1]$ ,  $s \in ]-1/2, 1/2]$  and  $v$  be a solenoidal vector field. Let  $\theta(T) \leq \delta/\lambda$  and  $\Phi$  be the phase function given by (2.8). Assume that  $e^{\delta|D|}a_0 \in B^{\sigma,s}$ ,  $f_\Phi \in L^1_T(B^{\sigma,s})$ , and  $v_\Phi \in L^1_T(B^{1,1/2}) \cap L^1_T(B^{2,1/2})$ . Then (5.1) has a unique solution  $a$  on  $[0, T]$  so that for any  $t \in [0, T]$ , there holds*

$$(5.2) \quad \begin{aligned} \|a_\Phi\|_{\tilde{L}^\infty_t(B^{\sigma,s})} &\leq \|e^{\delta|D|}a_0\|_{B^{\sigma,s}} + \|f_\Phi\|_{L^1_t(B^{\sigma,s})} \\ &+ C\left(\frac{1}{\lambda} + \varepsilon^{1-\alpha} \|v_\Phi\|_{L^1_t(B^{2,1/2})}\right) \|a_\Phi\|_{\tilde{L}^\infty_t(B^{\sigma,s})}. \end{aligned}$$

*Proof.* Since both the existence and uniqueness of Proposition 5.1 basically follow from the Estimate (5.2). For simplicity, we just present the detailed derivation of the *a priori* estimate (5.2) for smooth solutions of (5.1). Indeed by virtue of (4.6), we first integrate (5.1) with respect to  $t$  and then apply the operator  $e^{\Phi(t,D)}$  to the resulting equation to get

$$(5.3) \quad \begin{aligned} a_\Phi(t) &= e^{\Phi(t,D)}a_0 - \varepsilon^{1-\alpha} \int_0^t e^{-\lambda \int_{t'}^t \dot{\theta}(\tau) d\tau |D|} [v \cdot \nabla a]_\Phi(t') dt' \\ &+ \int_0^t e^{-\lambda \int_{t'}^t \dot{\theta}(\tau) d\tau |D|} f_\Phi(t') dt'. \end{aligned}$$

We claim that

$$(5.4) \quad \begin{aligned} &\|\Delta_k^h \Delta_\ell^v \partial_3 [v^3 a]_\Phi(t)\|_{L^2} \\ &\leq C 2^{-k\sigma} 2^{-\ell s} \left( d_{k,\ell} 2^\ell \|v_\Phi^3(t)\|_{B^{1,1/2}} + d_k d_\ell(t) 2^k \|v_\Phi^h(t)\|_{B^{1,1/2}} \right. \\ &\left. + d_k(t) d_\ell 2^\ell \|v_\Phi^3(t)\|_{B^{1,1/2}} + d_{k,\ell}(t) \|v_\Phi^h(t)\|_{B^{2,1/2}} \right) \|a_\Phi\|_{\tilde{L}^\infty_t(B^{\sigma,s})}. \end{aligned}$$

Along the same line as the proof of Lemma 3.3, since the phase function  $\Psi$  given by (2.8) verifies (3.1) whenever  $\theta(T) \leq \delta/\lambda$ , it suffices to prove (5.4) for  $\Phi = 0$ . As a matter of fact, by using Bony’s decomposition for both horizontal and vertical variables to  $v^3 a$ , we write

$$v^3 a(t) = (T^h + R^h + \bar{T}^h)(T^v + R^v + \bar{T}^v)(v^3, a)(t).$$

Considering the support to the Fourier transform of the terms in  $R^h R^v(v^3, a)(t)$ ,



we get, by applying Lemma 3.1, that

$$\begin{aligned} \|\Delta_k^h \Delta_\ell^v R^h R^v(v^3, a)(t)\|_{L^2} &\lesssim 2^k 2^{\ell/2} \sum_{\substack{k' \geq k-3 \\ \ell' \geq \ell-3}} \|\Delta_{k'}^h \Delta_{\ell'}^v v^3(t)\|_{L^2} \|\widetilde{\Delta}_{k'}^h \widetilde{\Delta}_{\ell'}^v a\|_{L_t^\infty(L^2)} \\ &\lesssim 2^k 2^{\ell/2} \sum_{\substack{k' \geq k-3 \\ \ell' \geq \ell-3}} d_{k', \ell'} 2^{-k'(1+\sigma)} 2^{-\ell'(1/2+s)} \|v^3(t)\|_{B^{1,1/2}} \|a\|_{\widetilde{L}_t^\infty(B^{\sigma,s})} \\ &\lesssim d_{k,\ell} 2^{-k\sigma} 2^{-\ell s} \|v^3(t)\|_{B^{1,1/2}} \|a\|_{\widetilde{L}_t^\infty(B^{\sigma,s})}. \end{aligned}$$

The same estimate holds for  $T^h T^v(v^3, a)(t)$ ,  $T^h R^v(v^3, a)(t)$  and  $R^h T^v(v^3, a)(t)$ .

In the same manner, we have

$$\begin{aligned} \|\Delta_k^h \Delta_\ell^v \bar{T}^h R^v(v^3, a)(t)\|_{L^2} &\lesssim 2^{\ell/2} \sum_{\substack{|k'-k| \leq 4 \\ \ell' \geq \ell-3}} \|\Delta_{k'}^h \Delta_{\ell'}^v v^3(t)\|_{L^2} \|S_{k'-1}^h \widetilde{\Delta}_{\ell'}^v a\|_{L_t^\infty(L_h^\infty(L_v^2))} \\ &\lesssim d_k(t) d_\ell 2^{-k\sigma} 2^{-\ell s} \|v^3(t)\|_{B^{1,1/2}} \|a\|_{\widetilde{L}_t^\infty(B^{\sigma,s})}. \end{aligned}$$

The same estimate holds for  $\bar{T}^h T^v(v^3, a)(t)$ .

Whereas using the fact that  $\operatorname{div} v(t) = 0$  and Lemma 3.1, one has

$$\begin{aligned} \|S_{k'-1}^h \Delta_{\ell'}^v v^3(t)\|_{L_h^\infty(L_v^2)} &\lesssim 2^{-\ell'} \|S_{k'-1}^h \Delta_{\ell'}^v \partial_3 v^3(t)\|_{L_h^\infty(L_v^2)} \\ &\lesssim 2^{-\ell'} \|S_{k'-1}^h \Delta_{\ell'}^v \operatorname{div}_h v^h(t)\|_{L_h^\infty(L_v^2)} \lesssim d_{\ell'}(t) 2^{k'} 2^{-3\ell'/2} \|v^h(t)\|_{B^{1,1/2}}, \end{aligned}$$

from which we infer

$$\begin{aligned} \|\Delta_k^h \Delta_\ell^v T^h \bar{T}^v(v^3, a)(t)\|_{L^2} &\lesssim \sum_{\substack{|k'-k| \leq 4 \\ |\ell'-\ell| \leq 4}} \|S_{k'-1}^h \Delta_{\ell'}^v v^3(t)\|_{L_h^\infty(L_v^2)} \|\Delta_{k'}^h S_{\ell'-1}^v a\|_{L_t^\infty(L_h^2(L_v^\infty))} \\ &\lesssim d_k d_\ell(t) 2^{k(1-\sigma)} 2^{-\ell(1+s)} \|v^3(t)\|_{B^{1,1/2}} \|a\|_{\widetilde{L}_t^\infty(B^{\sigma,s})}. \end{aligned}$$

Finally using once again that  $\operatorname{div} v(t) = 0$  and Lemma 3.1, we obtain

$$\begin{aligned} \|\Delta_k^h \Delta_\ell^v R^h \bar{T}^v(v^3, a)(t)\|_{L^2} &\lesssim 2^k \sum_{\substack{k' \geq k-3 \\ |\ell'-\ell| \leq 4}} \|\Delta_{k'}^h \Delta_{\ell'}^v v^3(t)\|_{L^2} \|\widetilde{\Delta}_{k'}^h S_{\ell'-1}^v a\|_{L_t^\infty(L_h^2(L_v^\infty))} \\ &\lesssim 2^k \sum_{\substack{k' \geq k-3 \\ |\ell'-\ell| \leq 4}} 2^{-\ell'} \|\Delta_{k'}^h \Delta_{\ell'}^v \operatorname{div}_h v^h(t)\|_{L^2} \|\widetilde{\Delta}_{k'}^h S_{\ell'-1}^v a\|_{L_t^\infty(L_h^2(L_v^\infty))} \\ &\lesssim d_{k,\ell}(t) 2^{-k\sigma} 2^{-\ell(1+s)} \|v^h(t)\|_{B^{2,1/2}} \|a\|_{\widetilde{L}_t^\infty(B^{\sigma,s})}. \end{aligned}$$

The same estimate holds for  $\bar{T}^h \bar{T}^v(v^3, a)(t)$ . This completes the proof of (5.4) for  $\Phi = 0$ .

Exactly by the same manner as the proof of (5.4), we can also get

$$\begin{aligned} \|\Delta_k^h \Delta_\ell^v \operatorname{div}_h[v^h a]_\Phi(t)\|_{L^2} &\leq C 2^{-k\sigma} 2^{-\ell s} (d_{k,\ell} 2^k \|v_\Phi^h(t)\|_{B^{1,1/2}} \\ (5.5) \quad &+ d_k d_\ell(t) 2^k \|v_\Phi^h(t)\|_{B^{1,1/2}} + d_{k,\ell}(t) \|v_\Phi^h(t)\|_{B^{2,1/2}}) \|a_\Phi\|_{\widetilde{L}_t^\infty(B^{\sigma,s})}. \end{aligned}$$

By summing up (5.3), (5.4) and (5.5), we write

$$\begin{aligned} \|\Delta_k^h \Delta_\ell^v a_\Phi\|_{L_t^\infty(L^2)} &\leq \|e^{\delta|D|} \Delta_k^h \Delta_\ell^v a_0\|_{L^2} + \int_0^t \|\Delta_k^h \Delta_\ell^v f_\Phi(t')\|_{L^2} dt' \\ &+ C 2^{-k\sigma} 2^{-\ell s} \left( d_{k,\ell} (2^k + 2^\ell) \int_0^t e^{-c\lambda \int_{t'}^t \dot{\theta}(\tau) d\tau (2^k + 2^\ell)} \varepsilon^{1-\alpha} \|v_\Phi(t')\|_{B^{1,1/2}} dt' \right. \\ &+ d_\ell 2^\ell \int_0^t e^{-c\lambda \int_{t'}^t \dot{\theta}(\tau) d\tau 2^\ell} d_k(t') \varepsilon^{1-\alpha} \|v_\Phi^3(t')\|_{B^{1,1/2}} dt' \\ &+ d_k 2^k \int_0^t e^{-c\lambda \int_{t'}^t \dot{\theta}(\tau) d\tau 2^k} d_\ell(t') \varepsilon^{1-\alpha} \|v_\Phi^h(t')\|_{B^{1,1/2}} dt' \\ &\left. + \int_0^t d_{k,\ell}(t') \varepsilon^{1-\alpha} \|v_\Phi^h(t')\|_{B^{2,1/2}} dt' \right) \|a_\Phi\|_{\tilde{L}_t^\infty(B^{\sigma,s})}. \end{aligned}$$

Then (5.2) follows from Definition 2.1, (2.7) and

$$\begin{aligned} \int_0^t e^{-c\lambda \int_{t'}^t \dot{\theta}(\tau) d\tau (2^k + 2^\ell)} \varepsilon^{1-\alpha} \|v_\Phi(t')\|_{B^{1,1/2}} dt' \\ \leq \int_0^t e^{-c\lambda \int_{t'}^t \dot{\theta}(\tau) d\tau (2^k + 2^\ell)} \dot{\theta}(t') dt' \leq \frac{C}{\lambda} (2^k + 2^\ell)^{-1}, \end{aligned}$$

and

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \int_0^t e^{-c\lambda \int_{t'}^t \dot{\theta}(\tau) d\tau 2^\ell} d_k(t') \varepsilon^{1-\alpha} \|v_\Phi^3(t')\|_{B^{1,1/2}} dt' \\ \leq \int_0^t e^{-c\lambda \int_{t'}^t \dot{\theta}(\tau) d\tau 2^\ell} \dot{\theta}(t') dt' \leq \frac{C}{\lambda} 2^{-\ell}, \\ \sum_{\ell \in \mathbb{Z}} \int_0^t e^{-c\lambda \int_{t'}^t \dot{\theta}(\tau) d\tau 2^k} d_\ell(t') \varepsilon^{1-\alpha} \|v_\Phi^h(t')\|_{B^{1,1/2}} dt' \\ \leq \int_0^t e^{-c\lambda \int_{t'}^t \dot{\theta}(\tau) d\tau 2^k} \dot{\theta}(t') dt' \leq \frac{C}{\lambda} 2^{-k}. \end{aligned}$$

This completes the proof of Proposition 5.1. □

**Remark 5.2.** We mention here that we can not prove the uniform estimate of  $a_\Phi$  in the isotropic Besov space  $\tilde{L}_t^\infty(B_{2,1}^{3/2})$  as that in [13], [24]. The main reason is that we can not use commutator’s argument to prove the propagation of analytic regularity for the transport equation.

**Lemma 5.3.** *Let  $v(t)$  be a smooth solenoidal vector field and let  $\gamma \in ]0, 1[$ . Let  $\theta(T) \leq \delta/\lambda$  and  $\Phi$  be the phase function given by (2.8). Then one has*

$$\begin{aligned} \|\Delta_k^h \Delta_\ell^v [v \cdot \nabla a]_\Phi(t)\|_{L^2} \\ \leq C 2^{-k(1-\gamma)} 2^{-\ell(1/2+\gamma)} (d_{k,\ell} (2^k + 2^\ell) \|v_\Phi(t)\|_{B^{1,1/2}} \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1-\gamma,1/2+\gamma})} \\ (5.6) \quad + (d_k d_\ell(t) 2^k \|v_\Phi^h(t)\|_{B^{1-\gamma,1/2+\gamma}} + d_{k,\ell}(t) \|v_\Phi^h(t)\|_{B^{2-\gamma,1/2+\gamma}}) \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1,1/2})}), \end{aligned}$$

and

$$\begin{aligned}
 & \|\Delta_k^h \Delta_\ell^v [v \cdot \nabla a]_\Phi(t)\|_{L^2} \\
 & \leq C 2^{-k(1+\gamma)} 2^{-\ell(1/2-\gamma)} (d_{k,\ell}(2^k + 2^\ell) \|v_\Phi(t)\|_{B^{1,1/2}} \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1+\gamma,1/2-\gamma})} \\
 (5.7) \quad & + (d_k(t)d_\ell 2^\ell \|v_\Phi^3(t)\|_{B^{1+\gamma,1/2-\gamma}} + d_{k,\ell}(t)\|v_\Phi^h(t)\|_{B^{2+\gamma,1/2-\gamma}})\|a_\Phi\|_{\tilde{L}_t^\infty(B^{1,1/2})}),
 \end{aligned}$$

and

$$\begin{aligned}
 & \|\Delta_k^h \Delta_\ell^v [v \cdot \nabla a]_\Phi(t)\|_{L^2} \\
 & \leq C 2^{-k\gamma} 2^{-\ell(3/2-\gamma)} (d_{k,\ell}(2^k + 2^\ell) \|v_\Phi(t)\|_{B^{1,1/2}} \|a_\Phi\|_{\tilde{L}_t^\infty(B^{\gamma,3/2-\gamma})} + d_{k,\ell}(t) \\
 (5.8) \quad & \times (\|v_\Phi^h(t)\|_{B^{1,3/2}} \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1+\gamma,1/2-\gamma})} + \|v_\Phi^h(t)\|_{B^{1+\gamma,3/2-\gamma}} \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1,1/2})})).
 \end{aligned}$$

*Proof.* Once again similar to the proof of Lemma 3.3, it suffices to prove (5.6-5.8) for  $\Phi = 0$ . Indeed we first get, by using Bony’s decomposition for both horizontal and vertical variables, that

$$v^3 a = (T^h T^v + T^h \mathcal{R}^v + \mathcal{R}^h T^v + \mathcal{R}^h \mathcal{R}^v)(v^3, a)(t).$$

Note that

$$\|S_{k'+2}^h \Delta_{\ell'}^v a\|_{L_t^\infty(L_h^\infty(L_v^2))} \lesssim d_{k,\ell'} 2^{k'\gamma} 2^{-\ell'(1/2+\gamma)} \|a\|_{\tilde{L}_t^\infty(B^{1-\gamma,1/2+\gamma})},$$

from which, we deduce

$$\begin{aligned}
 \|\Delta_k^h \Delta_\ell^v \mathcal{R}^h T^v(v^3, a)(t)\|_{L^2} & \lesssim \sum_{\substack{k' \geq k - N_0 \\ |\ell' - \ell| \leq 4}} \|\Delta_{k'}^h S_{\ell'-1}^v v^3(t)\|_{L_h^2(L_v^\infty)} \|S_{k'+2}^h \Delta_{\ell'}^v a\|_{L_t^\infty(L_h^\infty(L_v^2))} \\
 & \lesssim d_{k,\ell} 2^{-k(1-\gamma)} 2^{-\ell(1/2+\gamma)} \|v^3(t)\|_{B^{1,1/2}} \|a\|_{\tilde{L}_t^\infty(B^{1-\gamma,1/2+\gamma})}.
 \end{aligned}$$

The same estimate holds for  $T^h T^v(v^3, a)(t)$ .

While due to  $\operatorname{div} v = 0$ , one has

$$\begin{aligned}
 \|S_{k'-1}^h \Delta_{\ell'}^v v^3(t)\|_{L_h^\infty(L_v^2)} & \lesssim 2^{-\ell'} \|S_{k'-1}^h \Delta_{\ell'}^v \operatorname{div}_h v^h(t)\|_{L_h^\infty(L_v^2)} \\
 & \lesssim d_{k,\ell}(t) 2^{k'\gamma} 2^{-\ell'(3/2+\gamma)} \|v^h(t)\|_{B^{2-\gamma,1/2+\gamma}},
 \end{aligned}$$

from which we infer

$$\begin{aligned}
 \|\Delta_k^h \Delta_\ell^v T^h \mathcal{R}^v(v^3, a)(t)\|_{L^2} & \lesssim \sum_{\substack{|k'-k| \leq 4 \\ \ell' \geq \ell - N_0}} \|S_{k'-1}^h \Delta_{\ell'}^v v^3(t)\|_{L_h^\infty(L_v^2)} \|S_{k'+2}^h \Delta_{\ell'+2}^v a\|_{L_t^\infty(L_h^2(L_v^\infty))} \\
 & \lesssim \sum_{\substack{|k'-k| \leq 4 \\ \ell' \geq \ell - N_0}} d_{k',\ell'}(t) 2^{-k'(1-\gamma)} 2^{-\ell'(3/2+\gamma)} \|v^h(t)\|_{B^{2-\gamma,1/2+\gamma}} \|a\|_{\tilde{L}_t^\infty(B^{1,1/2})} \\
 & \lesssim d_{k,\ell}(t) 2^{-k(1-\gamma)} 2^{-\ell(3/2+\gamma)} \|v^h(t)\|_{B^{2-\gamma,1/2+\gamma}} \|a\|_{\tilde{L}_t^\infty(B^{1,1/2})}.
 \end{aligned}$$

The same estimate holds for  $\mathcal{R}^h \mathcal{R}^v(v^3, a)(t)$ . Hence in view of Lemma 3.1, we obtain

$$\begin{aligned} \|\Delta_k^h \Delta_\ell^v \partial_3(v^3 a)(t)\|_{L^2} &\leq C 2^{-k(1-\gamma)} 2^{-\ell(1/2+\gamma)} (d_{k,\ell} 2^\ell \|v^3(t)\|_{B^{1,1/2}} \|a\|_{\tilde{L}_t^\infty(B^{1-\gamma,1/2+\gamma})} \\ &\quad + d_{k,\ell}(t) \|v^h(t)\|_{B^{2-\gamma,1/2+\gamma}} \|a\|_{\tilde{L}_t^\infty(B^{1,1/2})}). \end{aligned} \tag{5.9}$$

Following exactly the same strategy, we can also prove

$$\begin{aligned} \|\Delta_k^h \Delta_\ell^v \operatorname{div}_h(v^h a)(t)\|_{L^2} &\leq C 2^{-k(1-\gamma)} 2^{-\ell(1/2+\gamma)} (d_{k,\ell} 2^k \|v^h(t)\|_{B^{1,1/2}} \|a\|_{\tilde{L}_t^\infty(B^{1-\gamma,1/2+\gamma})} \\ &\quad + (2^k d_k d_\ell(t) \|v^h(t)\|_{B^{1-\gamma,1/2+\gamma}} + d_{k,\ell}(t) \|v^h(t)\|_{B^{2-\gamma,1/2+\gamma}}) \|a\|_{\tilde{L}_t^\infty(B^{1,1/2})}), \end{aligned} \tag{5.10}$$

and

$$\begin{aligned} \|\Delta_k^h \Delta_\ell^v \partial_3(v^3 a)(t)\|_{L^2} &\leq C 2^{-k(1+\gamma)} 2^{-\ell(1/2-\gamma)} (d_{k,\ell} 2^\ell \|v^3(t)\|_{B^{1,1/2}} \|a\|_{\tilde{L}_t^\infty(B^{1+\gamma,1/2-\gamma})} \\ &\quad + (2^\ell d_k(t) d_\ell \|v^3(t)\|_{B^{1+\gamma,1/2-\gamma}} + d_{k,\ell}(t) \|v^h(t)\|_{B^{2+\gamma,1/2-\gamma}}) \|a\|_{\tilde{L}_t^\infty(B^{1,1/2})}), \end{aligned} \tag{5.11}$$

and

$$\begin{aligned} \|\Delta_k^h \Delta_\ell^v \operatorname{div}_h(v^h a)(t)\|_{L^2} &\leq C 2^{-k(1+\gamma)} 2^{-\ell(1/2-\gamma)} (d_{k,\ell} 2^k \|v^h(t)\|_{B^{1,1/2}} \\ &\quad \times \|a\|_{\tilde{L}_t^\infty(B^{1+\gamma,1/2-\gamma})} + d_{k,\ell}(t) \|v^h(t)\|_{B^{2+\gamma,1/2-\gamma}} \|a\|_{\tilde{L}_t^\infty(B^{1,1/2})}). \end{aligned} \tag{5.12}$$

Combining (5.9) with (5.10), we conclude the proof of (5.6) for  $\Phi = 0$ . Whereas by summing up (5.11) and (5.12), we achieve (5.7).

On the other hand, since  $\gamma \in ]0, 1[$ , one has

$$\|S_{k'+2}^h \Delta_{\ell'}^v a\|_{L_t^\infty(L_h^\infty(L_v^2))} \lesssim d_{k,\ell} 2^{k'(1-\gamma)} 2^{-\ell'(3/2-\gamma)} \|a\|_{\tilde{L}_t^\infty(B^{\gamma,3/2-\gamma})},$$

which ensures

$$\begin{aligned} \|\Delta_k^h \Delta_\ell^v \mathcal{R}^h T^v(v^3, a)(t)\|_{L^2} &\lesssim \sum_{\substack{k' \geq k - N_0 \\ |\ell' - \ell| \leq 4}} \|\Delta_{k'}^h S_{\ell'-1}^v v^3(t)\|_{L_h^2(L_v^\infty)} \|S_{k'+2}^h \Delta_{\ell'}^v a\|_{L_t^\infty(L_h^\infty(L_v^2))} \\ &\lesssim d_{k,\ell} 2^{-k\gamma} 2^{-\ell(3/2-\gamma)} \|v^3(t)\|_{B^{1,1/2}} \|a\|_{\tilde{L}_t^\infty(B^{\gamma,3/2-\gamma})}. \end{aligned}$$

The same estimate holds for  $T^h T^v(v^3, a)(t)$ .

While, again due to  $\operatorname{div} v = 0$ , one has

$$\begin{aligned} \|S_{k'-1}^h \Delta_{\ell'}^v v^3(t)\|_{L_h^\infty(L_v^2)} &\lesssim 2^{-\ell'} \|S_{k'-1}^h \Delta_{\ell'}^v \operatorname{div}_h v^h(t)\|_{L_h^\infty(L_v^2)} \\ &\lesssim d_{k,\ell}(t) 2^{k'(1-\gamma)} 2^{-\ell'(5/2-\gamma)} \|v^h(t)\|_{B^{1+\gamma,3/2-\gamma}}, \end{aligned}$$

from which we deduce

$$\begin{aligned} \|\Delta_k^h \Delta_\ell^v T^h \mathcal{R}^v(v^3, a)(t)\|_{L^2} &\lesssim \sum_{\substack{|k'-k| \leq 4 \\ \ell' \geq \ell - N_0}} \|S_{k'-1}^h \Delta_{\ell'}^v v^3(t)\|_{L_h^\infty(L_v^2)} \|\Delta_{k'}^h S_{\ell'+2}^v a\|_{L_t^\infty(L_h^2(L_v^\infty))} \\ &\lesssim d_{k,\ell}(t) 2^{-k\gamma} 2^{-\ell(5/2-\gamma)} \|v^3\|_{B^{1+\gamma,3/2-\gamma}} \|a\|_{\tilde{L}_t^\infty(B^{1,1/2})}. \end{aligned}$$

The same estimate holds for  $\mathcal{R}^h \mathcal{R}^v(v^3, a)(t)$ . We thus obtain

$$(5.13) \quad \begin{aligned} \|\Delta_k^h \Delta_\ell^v \partial_3(v^3 a)(t)\|_{L^2} &\leq C 2^{-k\gamma} 2^{-\ell(3/2-\gamma)} (d_{k,\ell} 2^\ell \|v^3(t)\|_{B^{1,1/2}} \|a\|_{\tilde{L}_t^\infty(B^{\gamma,3/2-\gamma})} \\ &\quad + d_{k,\ell}(t) \|v^h(t)\|_{B^{1+\gamma,3/2-\gamma}} \|a\|_{\tilde{L}_t^\infty(B^{1,1/2})}). \end{aligned}$$

The same argument assures that

$$(5.14) \quad \begin{aligned} &\|\Delta_k^h \Delta_\ell^v \operatorname{div}_h(v^h a)(t)\|_{L^2} \\ &\leq C 2^{-k\gamma} 2^{-\ell(3/2-\gamma)} (d_{k,\ell} 2^k \|v^h\|_{B^{1,1/2}} \|a\|_{\tilde{L}_t^\infty(B^{\gamma,3/2-\gamma})} + d_{k,\ell}(t) \\ &\quad \times (\|v^h(t)\|_{B^{1,3/2}} \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1+\gamma,1/2-\gamma})} + \|v^h(t)\|_{B^{1+\gamma,3/2-\gamma}} \|a\|_{\tilde{L}_t^\infty(B^{1,1/2})})). \end{aligned}$$

By summing up (5.13) and (5.14), we complete the proof of (5.8), and also the lemma.  $\square$

With Lemma 5.3, we deduce from the proof of Proposition 5.1 that

**Proposition 5.4.** *Let  $a$  be a smooth enough solution of (5.1) on  $[0, T]$ . Then under the assumptions of Lemma 5.3, for any  $t \in ]0, T[$ , we have*

$$(5.15) \quad \begin{aligned} \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1-\gamma,1/2+\gamma})} &\leq \|e^{\delta|D|} a_0\|_{B^{1-\gamma,1/2+\gamma}} + \|f_\Phi\|_{L_t^1(B^{1-\gamma,1/2+\gamma})} \\ &\quad + \frac{C}{\lambda} (\|a_\Phi\|_{\tilde{L}_t^\infty(B^{1-\gamma,1/2+\gamma})} + \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1,1/2})}) \\ &\quad + C \varepsilon^{1-\alpha} \|v_\Phi^h\|_{L_t^1(B^{2-\gamma,1/2+\gamma})} \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1,1/2})}, \end{aligned}$$

and

$$(5.16) \quad \begin{aligned} \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1+\gamma,1/2-\gamma})} &\leq \|e^{\delta|D|} a_0\|_{B^{1+\gamma,1/2-\gamma}} + \|f_\Phi\|_{L_t^1(B^{1+\gamma,1/2-\gamma})} \\ &\quad + \frac{C}{\lambda} (\|a_\Phi\|_{\tilde{L}_t^\infty(B^{1+\gamma,1/2-\gamma})} + \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1,1/2})}) \\ &\quad + C \varepsilon^{1-\alpha} \|v_\Phi^h\|_{L_t^1(B^{2+\gamma,1/2-\gamma})} \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1,1/2})}, \end{aligned}$$

and

$$(5.17) \quad \begin{aligned} \|a_\Phi\|_{\tilde{L}_t^\infty(B^{\gamma,3/2-\gamma})} &\leq \|e^{\delta|D|} a_0\|_{B^{\gamma,3/2-\gamma}} + \|f_\Phi\|_{L_t^1(B^{\gamma,3/2-\gamma})} \\ &\quad + \frac{C}{\lambda} \|a_\Phi\|_{\tilde{L}_t^\infty(B^{\gamma,3/2-\gamma})} + C \varepsilon^{1-\alpha} (\|v_\Phi^h\|_{L_t^1(B^{1,3/2})} \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1+\gamma,1/2-\gamma})} \\ &\quad + \|v_\Phi\|_{L_t^1(B^{1+\gamma,3/2-\gamma})} \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1,1/2})}). \end{aligned}$$

### 6. Elliptic estimates in the analytical class

In this section, we present the estimates of the pressure function in the analytical class. Recall that the re-scaled pressure function  $q$  satisfies

$$(6.1) \quad -\operatorname{div}\left(\frac{1}{1+\varepsilon^\beta a} \nabla^\varepsilon q\right) = \varepsilon^{1-\alpha} \operatorname{div}(v \cdot \nabla v) - \operatorname{div}\left(\frac{1}{1+\varepsilon^\beta a} \Delta_\varepsilon v\right).$$

In the sequel, we always denote  $G(r) \stackrel{\text{def}}{=} r/(1+r)$ , and let  $\theta(t)$ ,  $\Phi(t)$  and  $\Psi(t)$  be given by (2.7), (2.8) and (2.9) respectively. Moreover, we always assume that  $\theta(T) \leq \delta/\lambda$ .

**Proposition 6.1.** *Let  $\alpha \in ]0, 1/2[$ ,  $\beta > \alpha$  and  $0 < \gamma \leq 1/2 \min(\beta - \alpha, 1 - 2\alpha)$ . Then there exists a positive constant  $C_0$  such that for  $\epsilon$  given by (3.8), if a satisfies*

$$(6.2) \quad \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1,1/2})} \leq K \quad \text{and} \quad \epsilon \leq \min\left(\left(\frac{1}{2C_0K}\right)^{1/(\beta-\gamma)}, \left(\frac{\epsilon}{K}\right)^{1/\beta}\right),$$

we have

$$(6.3) \quad \begin{aligned} \epsilon^{1-\alpha} \|q\|_{Y_t} &\leq C \max(\epsilon^{\beta-\alpha-2\gamma}, \epsilon^{1-2\alpha-2\gamma}) \theta(t) \Psi(t) \quad \text{with} \\ \|q\|_{Y_t} &\stackrel{\text{def}}{=} \|\nabla_h q_\Phi\|_{L_t^1(B^{-1,1/2})} + \|\nabla_\epsilon q_\Phi\|_{L_t^1(B^{-1+\gamma,1/2-\gamma})}. \end{aligned}$$

*Proof.* In view of (6.1) and  $\text{div } v = 0$ , we write

$$(6.4) \quad \begin{aligned} q &= -(-\Delta_\epsilon)^{-1} \nabla_\epsilon \cdot (G(\epsilon^\beta a) \nabla_\epsilon q) + \epsilon^{1-\alpha} (-\Delta_\epsilon)^{-1} \text{div}_h \text{div}_h (v^h \otimes v^h) \\ &\quad + 2\epsilon^{1-\alpha} (-\Delta_\epsilon)^{-1} \partial_3 \text{div}_h (v^3 v^h) - 2\epsilon^{1-\alpha} (-\Delta_\epsilon)^{-1} \partial_3 (v^3 \text{div}_h v^h) \\ &\quad + (-\Delta_\epsilon)^{-1} \text{div} (G(\epsilon^\beta a) \Delta_\epsilon v) \stackrel{\text{def}}{=} q_1 + \dots + q_5. \end{aligned}$$

To avoid the difficulty of product laws in the Besov space  $B_{2,1}^{-1}(\mathbb{R}^2)$ , we write

$$(6.5) \quad \begin{aligned} &\|\nabla_h [q_1]_\Phi\|_{L_t^1(B^{-1,1/2})} \\ &= \epsilon^{-\gamma} \| |D_h|^{-\gamma} |\epsilon D_3|^\gamma \nabla_h (-\Delta_\epsilon)^{-1} \nabla_\epsilon \cdot |D_h|^\gamma |D_3|^{-\gamma} [G(\epsilon^\beta a) \nabla_\epsilon q]_\Phi \|_{L_t^1(B^{-1,1/2})} \\ &\leq C \epsilon^{-\gamma} \| |D_h|^\gamma |D_3|^{-\gamma} [G(\epsilon^\beta a) \nabla_\epsilon q]_\Phi \|_{L_t^1(B^{-1,1/2})} \\ &\leq C \epsilon^{-\gamma} \| [G(\epsilon^\beta a) \nabla_\epsilon q]_\Phi \|_{L_t^1(B^{-1+\gamma,1/2-\gamma})}, \end{aligned}$$

where  $|D_h|$  and  $|D_3|$  denote the Fourier multipliers with symbols  $|\xi_h| = \sqrt{\xi_1^2 + \xi_2^2}$  and  $|\xi_3|$  respectively. In what follows, we shall frequently use this kind of tricks to deal with the estimate of the pressure function.

In view of (6.5), if  $\epsilon$  is so small that  $\epsilon^\beta K \leq \epsilon$ , we get, by applying Corollary 3.4 and Lemma 3.6, that

$$\begin{aligned} \|q_1\|_{Y_t} &\leq C \epsilon^{-\gamma} \| [G(\epsilon^\beta a) \nabla_\epsilon q]_\Phi \|_{L_t^1(B^{-1+\gamma,1/2-\gamma})} \\ &\leq C \epsilon^{\beta-\gamma} \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1,1/2})} \|\nabla_\epsilon q_\Phi\|_{L_t^1(B^{-1+\gamma,1/2-\gamma})}. \end{aligned}$$

Applying the law of product of Corollary 3.4 gives

$$\begin{aligned} \|q_2\|_{Y_t} &\leq C \epsilon^{1-\alpha} (\| [v^h v^h]_\Phi \|_{L_t^1(B^{0,1/2})} + \| [v^h v^h]_\Phi \|_{L_t^1(B^{\gamma,1/2-\gamma})}) \\ &\leq C \epsilon^{1-\alpha} \|v_\Phi^h\|_{L_t^1(B^{1,1/2})} (\|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{\gamma,1/2-\gamma})} + \|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{0,1/2})}), \end{aligned}$$

and

$$\begin{aligned} \|q_3\|_{Y_t} &\leq C \varepsilon^{-\alpha} (\| [v^3 v^h]_{\Phi} \|_{L_t^1(B^{0,1/2})} + \| [v^3 v^h]_{\Phi} \|_{L_t^1(B^{\gamma,1/2-\gamma})}) \\ &\leq C \varepsilon^{-\alpha} \|v_{\Phi}^3\|_{L_t^1(B^{1,1/2})} (\|v_{\Phi}^h\|_{\tilde{L}_t^\infty(B^{\gamma,1/2-\gamma})} + \|v_{\Phi}^h\|_{\tilde{L}_t^\infty(B^{0,1/2})}). \end{aligned}$$

Whereas we get, by applying first a similar trick as that in (6.5) and then Corollary 3.4, that

$$\begin{aligned} \|q_4\|_{Y_t} &\leq C \varepsilon^{-\alpha-\gamma} \| [v^3 \operatorname{div}_h v^h]_{\Phi} \|_{L_t^1(B^{-1+\gamma,1/2-\gamma})} \\ &\leq C \varepsilon^{-\alpha-\gamma} \|v_{\Phi}^3\|_{L_t^1(B^{1,1/2})} \|v_{\Phi}^h\|_{\tilde{L}_t^\infty(B^{\gamma,1/2-\gamma})}. \end{aligned}$$

To handle  $q_5$  in (6.4), we split it further as

$$\begin{aligned} (6.6) \quad q_5 &= (-\Delta_\varepsilon)^{-1} \operatorname{div}_h (G(\varepsilon^\beta a) \Delta_h v^h) + (-\Delta_\varepsilon)^{-1} \operatorname{div}_h (G(\varepsilon^\beta a) \varepsilon^2 \partial_3^2 v^h) \\ &\quad + (-\Delta_\varepsilon)^{-1} \partial_3 (G(\varepsilon^\beta a) \Delta_h v^3) + (-\Delta_\varepsilon)^{-1} \partial_3 (G(\varepsilon^\beta a) \varepsilon^2 \partial_3^2 v^3) \\ &\stackrel{\text{def}}{=} q_{5,1} + \dots + q_{5,4}. \end{aligned}$$

Similar to the estimate of  $q_1$ , one has

$$\begin{aligned} \|q_{5,1}\|_{Y_t} &\leq C \varepsilon^{\beta-\gamma} \|a_{\Phi}\|_{\tilde{L}_t^\infty(B^{1,1/2})} \|v_{\Phi}^h\|_{L_t^1(B^{1+\gamma,1/2-\gamma})}, \\ \|q_{5,3}\|_{Y_t} &\leq C \varepsilon^{-1+\beta-\gamma} \|a_{\Phi}\|_{\tilde{L}_t^\infty(B^{1,1/2})} \|v_{\Phi}^3\|_{L_t^1(B^{1+\gamma,1/2-\gamma})}. \end{aligned}$$

While noting that

$$\begin{aligned} &(-\Delta_\varepsilon)^{-1} \operatorname{div}_h (G(\varepsilon^\beta a) \varepsilon^2 \partial_3^2 v^h) \\ &= \varepsilon^{-1+\delta} |D_h|^{-1+\delta} |\varepsilon D_3|^{1-\delta} (-\Delta_\varepsilon)^{-1} \operatorname{div}_h |D_h|^{1-\delta} |D_3|^{-1+\delta} (G(\varepsilon^\beta a) \varepsilon^2 \partial_3^2 v^h), \end{aligned}$$

for  $\delta$  taking  $\gamma$  and  $2\gamma$ , we infer

$$\begin{aligned} \|[q_{5,2}]_{\Phi}\|_{Y_t} &\leq \varepsilon^{-1+\gamma} \| [G(\varepsilon^\beta a) \varepsilon^2 \partial_3^2 v^h]_{\Phi} \|_{L_t^1(B^{-\gamma,-1/2+\gamma})} \\ &\leq C \varepsilon^{1+\beta+\gamma} \|a_{\Phi}\|_{\tilde{L}_t^\infty(B^{1,1/2})} \|v_{\Phi}^h\|_{L_t^1(B^{-\gamma,3/2+\gamma})}. \end{aligned}$$

In a similar manner and using  $\operatorname{div} v = 0$ , one has

$$\begin{aligned} \|q_{54}\|_{Y_t} &\leq C \varepsilon^{1-\gamma} \| [G(\varepsilon^\beta a) \partial_3 \operatorname{div}_h v^h]_{\Phi} \|_{L_t^1(B^{-1+\gamma,1/2-\gamma})} \\ &\leq C \varepsilon^{1+\beta-\gamma} \|a_{\Phi}\|_{\tilde{L}_t^\infty(B^{1,1/2})} \|v_{\Phi}^h\|_{L_t^1(B^{\gamma,3/2-\gamma})}. \end{aligned}$$

By summing up the above estimates, we arrive at

$$\begin{aligned} \varepsilon^{1-\alpha} \|q\|_{Y_t} &\leq C \varepsilon^{\beta-\alpha} \|a_{\Phi}\|_{\tilde{L}_t^\infty(B^{1,1/2})} \left( \varepsilon^{1-\gamma} (\|q\|_{Y_t} + \|v_{\Phi}^h\|_{L_t^1(B^{1+\gamma,1/2-\gamma})}) \right. \\ &\quad \left. + \varepsilon^{-\gamma} \|v_{\Phi}^3\|_{L_t^1(B^{1+\gamma,1/2-\gamma})} + \varepsilon^{2-\gamma} \|v_{\Phi}^h\|_{L_t^1(B^{\gamma,3/2-\gamma})} + \varepsilon^2 \|v_{\Phi}^h\|_{L_t^1(B^{-\gamma,3/2+\gamma})} \right) \\ &\quad + C \varepsilon^{1-2\alpha} (\varepsilon \|v_{\Phi}^h\|_{L_t^1(B^{1,1/2})} + \varepsilon^{-\gamma} \|v_{\Phi}^3\|_{L_t^1(B^{1,1/2})}) \\ &\quad \times (\|v_{\Phi}^h\|_{\tilde{L}_t^\infty(B^{\gamma,1/2-\gamma})} + \|v_{\Phi}^h\|_{\tilde{L}_t^\infty(B^{0,1/2})}). \end{aligned}$$

While we get, by applying Lemma 3.2, that

$$\begin{aligned} \varepsilon^{2+\beta-\alpha-\gamma} \|v_\Phi^h\|_{L_t^1(B^{\gamma,3/2-\gamma})} \\ \leq C \varepsilon^{\beta-2\gamma} \varepsilon^{2-\alpha+\gamma} (\|v_\Phi^h\|_{L_t^1(B^{-\gamma,3/2+\gamma})} + \|v_\Phi^h\|_{L_t^1(B^{1+\gamma,1/2-\gamma})}). \end{aligned}$$

Then due to the assumptions of  $\alpha, \beta, \gamma$  in the proposition, (6.3) follows by choosing  $\varepsilon$  suitably small in (6.2).  $\square$

**Proposition 6.2.** *Let  $\alpha \in ]0, 1[$ ,  $\beta > \alpha$  and  $0 < \gamma \leq \min((\beta - \alpha)/4, (1 - \alpha)/3)$ . Then there exists some positive constant  $C_0$  such that for  $\varepsilon$  given by (3.8), if a satisfies*

$$(6.7) \quad \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1,1/2})} + \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1-\gamma,1/2+\gamma})} \leq K \quad \text{and} \quad \varepsilon^\beta \leq \min\left(\frac{1}{2C_0K}, \frac{\varepsilon}{K}\right),$$

there holds

$$(6.8) \quad \begin{aligned} \varepsilon^{2\alpha+\gamma} \|q\|_{Z_t} &\leq C \Psi^2(t), \quad \text{with} \\ \|q\|_{Z_t} &\stackrel{\text{def}}{=} \|\nabla_\varepsilon q_\Phi\|_{L_t^1(B^{\gamma,1/2-\gamma})} + \|\nabla_\varepsilon q_\Phi\|_{L_t^1(B^{-\gamma,1/2+\gamma})}. \end{aligned}$$

*Proof.* Following the same line as the proof of Proposition 6.1, we shall split the proof of (6.8) into the following steps:

- *Estimate of  $\nabla_\varepsilon q_1$ .*

By virtue of (6.4), we get, by applying Corollary 3.4, that

$$\begin{aligned} \|\nabla_\varepsilon [q_1]_\Phi\|_{L_t^1(B^{\gamma,1/2-\gamma})} &\lesssim \|[G(\varepsilon^\beta a)\nabla_\varepsilon q]_\Phi\|_{L_t^1(B^{\gamma,1/2-\gamma})} \\ &\lesssim \|[G(\varepsilon^\beta a)]_\Phi\|_{\tilde{L}_t^\infty(B^{1,1/2})} \|\nabla_\varepsilon q_\Phi\|_{L_t^1(B^{\gamma,1/2-\gamma})}, \end{aligned}$$

and

$$\begin{aligned} \|\nabla_\varepsilon [q_1]_\Phi\|_{L_t^1(B^{-\gamma,1/2+\gamma})} &\lesssim \|[G(\varepsilon^\beta a)\nabla_\varepsilon q]_\Phi\|_{L_t^1(B^{-\gamma,1/2+\gamma})} \\ &\lesssim \|[G(\varepsilon^\beta a)]_\Phi\|_{\tilde{L}_t^\infty(B^{1,1/2})} \|\nabla_\varepsilon q_\Phi\|_{L_t^1(B^{-\gamma,1/2+\gamma})} \\ &\quad + \|[G(\varepsilon^\beta a)]_\Phi\|_{\tilde{L}_t^\infty(B^{1-\gamma,1/2+\gamma})} \|\nabla_\varepsilon q_\Phi\|_{L_t^1(B^{0,1/2})}. \end{aligned}$$

While it follows from Lemma 3.2 that

$$\|\nabla_\varepsilon q_\Phi\|_{L_t^1(B^{0,1/2})} \lesssim \|\nabla_\varepsilon q_\Phi\|_{L_t^1(B^{\gamma,1/2-\gamma})} + \|\nabla_\varepsilon q_\Phi\|_{L_t^1(B^{-\gamma,1/2+\gamma})}.$$

Therefore, if  $\varepsilon$  is so small that  $\varepsilon^\beta K \leq \varepsilon$ , by applying Lemma 3.6, we obtain

$$(6.9) \quad \|q_1\|_{Z_t} \leq C \varepsilon^\beta (\|a_\Phi\|_{\tilde{L}_t^\infty(B^{1,1/2})} + \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1-\gamma,1/2+\gamma})}) \|q\|_{Z_t}.$$

- *Estimate of  $\nabla_\varepsilon q_2$ .*

Applying the law of product of Corollary 3.4 and Lemma 3.3 yields that

$$\begin{aligned} \|\nabla_\varepsilon [q_2]_\Phi\|_{L_t^1(B^{\gamma,1/2-\gamma})} &\lesssim \varepsilon^{1-\alpha} \|[v^h \otimes v^h]_\Phi\|_{L_t^1(B^{1+\gamma,1/2-\gamma})} \\ &\lesssim \varepsilon^{1-\alpha} \|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{0,1/2})} \|v_\Phi^h\|_{L_t^1(B^{2+\gamma,1/2-\gamma})}, \end{aligned}$$



and

$$\begin{aligned} \|\nabla_\varepsilon[q_2]_\Phi\|_{L^1_t(B^{-\gamma,1/2+\gamma})} &\lesssim \varepsilon^{1-\alpha} \|[v^h \otimes v^h]_\Phi\|_{L^1_t(B^{1-\gamma,1/2+\gamma})} \\ &\lesssim \varepsilon^{1-\alpha} \left( \|v^h_\Phi\|_{\tilde{L}^\infty(B^{0,1/2})} \|v^h_\Phi\|_{L^1_t(B^{2-\gamma,1/2+\gamma})} \right. \\ &\quad \left. + \|v^h_\Phi\|_{\tilde{L}^\infty(B^{-\gamma,1/2+\gamma})} \|v^h_\Phi\|_{L^1_t(B^{2,1/2})} \right). \end{aligned}$$

This gives rise to

$$(6.10) \quad \|q_2\|_{Z_t} \leq C \varepsilon^{1-\alpha} \left( \|v^h_\Phi\|_{\tilde{L}^\infty(B^{0,1/2})} \left( \|v^h_\Phi\|_{L^1_t(B^{2+\gamma,1/2-\gamma})} + \|v^h_\Phi\|_{L^1_t(B^{2-\gamma,1/2+\gamma})} \right) \right. \\ \left. + \|v^h_\Phi\|_{\tilde{L}^\infty(B^{-\gamma,1/2+\gamma})} \|v^h_\Phi\|_{L^1_t(B^{2,1/2})} \right).$$

• *Estimate of  $\nabla_\varepsilon q_3$ .*

To deal with  $\nabla_\varepsilon q_3$  given by (6.4), we first use Bony’s decomposition (3.3) for the vertical variable to split it as

$$(6.11) \quad q_3 = \varepsilon^{1-\alpha} (-\Delta_\varepsilon)^{-1} \partial_3 \operatorname{div}_h (T^v(v^3, v^h)) + \varepsilon^{1-\alpha} (-\Delta_\varepsilon)^{-1} \partial_3 \operatorname{div}_h (\mathcal{R}^v(v^3, v^h)) \\ \stackrel{\text{def}}{=} q_{31} + q_{32}.$$

Applying Lemma 3.1 and  $\operatorname{div} v = 0$  yields

$$\begin{aligned} \|\Delta_k^h \Delta_\ell^v \mathcal{R}^h \mathcal{R}^v(v^3, v^h)\|_{L^1_t(L^2)} &\lesssim \sum_{\substack{k' \geq k - N_0 \\ \ell' \geq \ell - N_0}} \|\Delta_{k'}^h \Delta_{\ell'}^v v^3\|_{L^1_t(L^2)} \|S_{k'+2}^h S_{\ell'+2}^v v^h\|_{L^\infty(L^\infty)} \\ &\lesssim \sum_{\substack{k' \geq k - N_0 \\ \ell' \geq \ell - N_0}} 2^{-\ell'} \|\Delta_{k'}^h \Delta_{\ell'}^v \operatorname{div}_h v^h\|_{L^1_t(L^2)} \|S_{k'+2}^h S_{\ell'+2}^v v^h\|_{L^\infty(L^\infty)} \\ &\lesssim d_{k,\ell} 2^{-k\gamma} 2^{-\ell(3/2-\gamma)} \|v^h\|_{L^1_t(B^{2,1/2})} \|v^h\|_{\tilde{L}^\infty(B^{\gamma,1/2-\gamma})}. \end{aligned}$$

The same estimate holds for  $T^h \mathcal{R}^v(v^3, v^h)$ . This gives

$$\|\mathcal{R}^v(v^3, v^h)\|_{L^1_t(B^{\gamma,3/2-\gamma})} \lesssim \|v^h\|_{L^1_t(B^{2,1/2})} \|v^h\|_{\tilde{L}^\infty(B^{\gamma,1/2-\gamma})}.$$

In view of (3.5), similar estimate holds for  $[\mathcal{R}^v(v^3, v^h)]_\Phi$ , which ensures

$$(6.12) \quad \|\nabla_\varepsilon[q_{32}]_\Phi\|_{L^1_t(B^{\gamma,1/2-\gamma})} \lesssim \varepsilon^{1-\alpha} \|[\mathcal{R}^v(v^3, v^h)]_\Phi\|_{L^1_t(B^{\gamma,3/2-\gamma})} \\ \lesssim \varepsilon^{1-\alpha} \|v^h_\Phi\|_{L^1_t(B^{2,1/2})} \|v^h_\Phi\|_{\tilde{L}^\infty(B^{\gamma,1/2-\gamma})}.$$

Again due to  $\operatorname{div} v = 0$ , we have

$$\begin{aligned} \|S_{k'-1}^h \Delta_{\ell'}^v v^3\|_{L^1_t(L^\infty(L^2_\eta))} &\lesssim 2^{-\ell'} \|S_{k'-1}^h \Delta_{\ell'}^v \operatorname{div}_h v^h\|_{L^1_t(L^\infty(L^2_\eta))} \\ &\lesssim d_{k',\ell'} 2^{k'\gamma} 2^{-\ell'(3/2+\gamma)} \|v^h\|_{L^1_t(B^{2-\gamma,1/2+\gamma})}, \end{aligned}$$

which implies

$$\begin{aligned} & \|\Delta_k^h \Delta_\ell^v T^h \mathcal{R}^v(v^3, v^h)\|_{L_t^1(L^2)} \\ & \lesssim \sum_{\substack{|k'-k|\leq 4 \\ \ell' \geq \ell - N_0}} \|S_{k'}^h \Delta_{\ell'}^v v^3\|_{L_t^1(L_h^\infty(L_t^2))} \|\Delta_{k'}^h S_{\ell'+2}^v v^h\|_{L_t^\infty(L_h^2(L_t^\infty))} \\ & \lesssim d_{k,\ell} 2^{k\gamma} 2^{-\ell(3/2+\gamma)} \|v^h\|_{L_t^1(B^{2-\gamma, 1/2+\gamma})} \|v^h\|_{\tilde{L}_t^\infty(B^{0, 1/2})}. \end{aligned}$$

The same estimate holds for  $R^h \mathcal{R}^v(v^3, v^h)$  and  $\bar{T}^h \mathcal{R}^v(v^3, v^h)$ . This leads to

$$\|\mathcal{R}^v(v^3, v^h)\|_{L_t^1(B^{-\gamma, 3/2+\gamma})} \lesssim \|v^h\|_{L_t^1(B^{2-\gamma, 1/2+\gamma})} \|v^h\|_{\tilde{L}_t^\infty(B^{0, 1/2})}.$$

A similar estimate holds for  $[\mathcal{R}^v(v^3, v^h)]_\Phi$ , which implies

$$\begin{aligned} (6.13) \quad \|\nabla_\varepsilon [q_{32}]_\Phi\|_{L_t^1(B^{-\gamma, 1/2+\gamma})} & \lesssim \varepsilon^{1-\alpha} \|[\mathcal{R}^v(v^3, v^h)]_\Phi\|_{L_t^1(B^{-\gamma, 3/2+\gamma})} \\ & \lesssim \varepsilon^{1-\alpha} \|v_\Phi^h\|_{L_t^1(B^{2-\gamma, 1/2+\gamma})} \|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{0, 1/2})}. \end{aligned}$$

Combining (6.12) with (6.13), we obtain

$$\begin{aligned} (6.14) \quad \|q_{32}\|_{Z_t} & \lesssim \varepsilon^{1-\alpha} (\|v_\Phi^h\|_{L_t^1(B^{2, 1/2})} \|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{\gamma, 1/2-\gamma})} \\ & \quad + \|v_\Phi^h\|_{L_t^1(B^{2-\gamma, 1/2+\gamma})} \|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{0, 1/2})}). \end{aligned}$$

While using Bony's decomposition (3.3) to  $T^v(v^3, v^h)$  for the horizontal variables, one has

$$T^v(v^3, v^h) = (T^h + R^h + \bar{T}^h)T^v(v^3, v^h),$$

from which, and similarly as in proof of Lemma 3.3, we deduce that

$$\begin{aligned} & \| [T^v(v^3, v^h)]_\Phi \|_{L_t^1(B^{1+\gamma, 1/2-\gamma})} \\ & \lesssim \|v_\Phi^3\|_{\tilde{L}_t^2(B^{1, 1/2})} \|v_\Phi^h\|_{\tilde{L}_t^2(B^{1+\gamma, 1/2-\gamma})} + \|v_\Phi^3\|_{L_t^1(B^{2, 1/2})} \|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{\gamma, 1/2-\gamma})}, \end{aligned}$$

and

$$\| [T^v(v^3, v^h)]_\Phi \|_{L_t^1(B^{1-\gamma, 1/2+\gamma})} \lesssim \|v_\Phi^3\|_{\tilde{L}_t^2(B^{1, 1/2})} \|v_\Phi^h\|_{\tilde{L}_t^2(B^{1-\gamma, 1/2+\gamma})},$$

so that there holds

$$\begin{aligned} (6.15) \quad \|q_{31}\|_{Z_t} & \lesssim \varepsilon^{-\alpha} (\| [T^v(v^3, v^h)]_\Phi \|_{L_t^1(B^{1+\gamma, 1/2-\gamma})} + \| [T^v(v^3, v^h)]_\Phi \|_{L_t^1(B^{1-\gamma, 1/2+\gamma})}) \\ & \lesssim \varepsilon^{-\alpha} (\|v_\Phi^3\|_{\tilde{L}_t^2(B^{1, 1/2})} (\|v_\Phi^h\|_{\tilde{L}_t^2(B^{1+\gamma, 1/2-\gamma})} + \|v_\Phi^h\|_{\tilde{L}_t^2(B^{1-\gamma, 1/2+\gamma})}) \\ & \quad + \|v_\Phi^3\|_{L_t^1(B^{2, 1/2})} \|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{\gamma, 1/2-\gamma})}). \end{aligned}$$

- *Estimate of  $\nabla_\varepsilon q_4$ .*

Along the same line as the manipulation of  $\nabla_\varepsilon q_3$ , we first split  $q_4$  as

$$\begin{aligned} (6.16) \quad q_4 & = \varepsilon^{1-\alpha} (-\Delta_\varepsilon)^{-1} \partial_3 T^v(v^3, \operatorname{div}_h v^h) + \varepsilon^{1-\alpha} (-\Delta_\varepsilon)^{-1} \partial_3 (\mathcal{R}^v(v^3, \operatorname{div}_h v^h)) \\ & \stackrel{\text{def}}{=} q_{41} + q_{42}. \end{aligned}$$

Similar to (6.5), we have

$$\|q_{42}\|_{Z_t} \lesssim \varepsilon^{1-\alpha-2\gamma} \|[\mathcal{R}^v(v^3, \operatorname{div}_h v^h)]_\Phi\|_{L_t^1(B^{-1+\gamma, 3/2-\gamma})},$$

from which and with a similar proof as that of (6.14), we infer

$$(6.17) \quad \|q_{42}\|_{Z_t} \lesssim \varepsilon^{1-\alpha-2\gamma} \|v_\Phi^h\|_{L_t^1(B^{2, 1/2})} \|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{\gamma, 1/2-\gamma})}.$$

A similar proof to that of Lemma 3.3 gives rise to

$$\|[T^v(v^3, \operatorname{div}_h v^h)]_\Phi\|_{L_t^1(B^{\gamma, 1/2-\gamma})} \lesssim \|v_\Phi^3\|_{\tilde{L}_t^2(B^{1, 1/2})} \|v_\Phi^h\|_{\tilde{L}_t^2(B^{1+\gamma, 1/2-\gamma})},$$

and

$$\|[T^v(v^3, \operatorname{div}_h v^h)]_\Phi\|_{L_t^1(B^{-\gamma, 1/2+\gamma})} \lesssim \|v_\Phi^3\|_{\tilde{L}_t^2(B^{1, 1/2})} \|v_\Phi^h\|_{\tilde{L}_t^2(B^{1-\gamma, 1/2+\gamma})}.$$

We thus obtain

$$(6.18) \quad \begin{aligned} \|q_{41}\|_{Z_t} &\lesssim \varepsilon^{-\alpha} (\|[T^v(v^3, \operatorname{div}_h v^h)]_\Phi\|_{L_t^1(B^{\gamma, 1/2-\gamma})} + \|[T^v(v^3, \operatorname{div}_h v^h)]_\Phi\|_{L_t^1(B^{-\gamma, 1/2+\gamma})}) \\ &\lesssim \varepsilon^{-\alpha} \|v_\Phi^3\|_{\tilde{L}_t^2(B^{1, 1/2})} (\|v_\Phi^h\|_{\tilde{L}_t^2(B^{1+\gamma, 1/2-\gamma})} + \|v_\Phi^h\|_{\tilde{L}_t^2(B^{1-\gamma, 1/2+\gamma})}). \end{aligned}$$

- *Estimate of  $\nabla_\varepsilon q_5$ .*

We shall use the decomposition (6.6) to deal with  $q_5$ . Applying Corollary 3.4 gives

$$\begin{aligned} \|q_{51}\|_{Z_t} &\lesssim \|[G(\varepsilon^\beta a)\Delta_h v^h]_\Phi\|_{L_t^1(B^{\gamma, 1/2-\gamma})} + \|[G(\varepsilon^\beta a)\Delta_h v^h]_\Phi\|_{L_t^1(B^{-\gamma, 1/2+\gamma})} \\ &\lesssim \|[G(\varepsilon^\beta a)]_\Phi\|_{\tilde{L}_t^\infty(B^{1, 1/2})} (\|\Delta_h v_\Phi^h\|_{L_t^1(B^{\gamma, 1/2-\gamma})} + \|\Delta_h v_\Phi^h\|_{L_t^1(B^{-\gamma, 1/2+\gamma})}) \\ &\quad + \|[G(\varepsilon^\beta a)]_\Phi\|_{\tilde{L}_t^\infty(B^{1-\gamma, 1/2+\gamma})} \|\Delta_h v_\Phi^h\|_{L_t^1(B^{0, 1/2})}. \end{aligned}$$

From this,  $\varepsilon^\beta K \leq \epsilon$ , and Lemma 3.6, we conclude

$$(6.19) \quad \begin{aligned} \|q_{51}\|_{Z_t} &\lesssim \varepsilon^\beta \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1, 1/2})} (\|v_\Phi^h\|_{L_t^1(B^{2+\gamma, 1/2-\gamma})} + \|v_\Phi^h\|_{L_t^1(B^{2-\gamma, 1/2+\gamma})}) \\ &\quad + \varepsilon^\beta \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1-\gamma, 1/2+\gamma})} \|v_\Phi^h\|_{L_t^1(B^{2, 1/2})}. \end{aligned}$$

The same argument yields

$$(6.20) \quad \begin{aligned} \|q_{52}\|_{Z_t} &\lesssim \varepsilon^{2+\beta} \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1, 1/2})} (\|v_\Phi^h\|_{L_t^1(B^{\gamma, 5/2-\gamma})} + \|v_\Phi^h\|_{L_t^1(B^{-\gamma, 5/2+\gamma})}) \\ &\quad + \varepsilon^{2+\beta} \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1-\gamma, 1/2+\gamma})} \|v_\Phi^h\|_{L_t^1(B^{0, 5/2})}. \end{aligned}$$

Note that

$$\begin{aligned} \|q_{53}\|_{Z_t} &\lesssim \varepsilon^{-2\gamma} \|[G(\varepsilon^\beta a)\Delta_h v^3]_\Phi\|_{L_t^1(B^{-1+\gamma, 3/2-\gamma})} \\ &\lesssim \varepsilon^{-2\gamma} (\|[G(\varepsilon^\beta a)]_\Phi\|_{\tilde{L}_t^\infty(B^{1, 1/2})} \|\Delta_h v_\Phi^3\|_{L_t^1(B^{-1+\gamma, 3/2-\gamma})} \\ &\quad + \|[G(\varepsilon^\beta a)]_\Phi\|_{\tilde{L}_t^\infty(B^{\gamma, 3/2-\gamma})} \|\Delta_h v_\Phi^3\|_{L_t^1(B^{0, 1/2})}), \end{aligned}$$

which together with Lemma 3.6 and  $\operatorname{div} v = 0$  ensures that

$$(6.21) \quad \begin{aligned} \|q_{53}\|_{Z_t} &\lesssim \varepsilon^{\beta-2\gamma} (\|a_\Phi\|_{\tilde{L}_t^\infty(B^{1,1/2})} \|v_\Phi^h\|_{L_t^1(B^{2+\gamma,1/2-\gamma})} \\ &\quad + \|a_\Phi\|_{\tilde{L}_t^\infty(B^{\gamma,3/2-\gamma})} \|v_\Phi^3\|_{L_t^1(B^{2,1/2})}). \end{aligned}$$

Similarly, due to  $\operatorname{div} v = 0$ , we have

$$\|\nabla_\varepsilon[q_{54}]_\Phi\|_{L_t^1(B^{\gamma,1/2-\gamma})} \lesssim \varepsilon^\beta \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1,1/2})} \varepsilon \|v_\Phi^h\|_{L_t^1(B^{1+\gamma,3/2-\gamma})},$$

and

$$\begin{aligned} &\|\nabla_\varepsilon[q_{54}]_\Phi\|_{L_t^1(B^{-\gamma,1/2+\gamma})} \\ &\lesssim \varepsilon^{1+\beta} (\|a_\Phi\|_{\tilde{L}_t^\infty(B^{1,1/2})} \|v_\Phi^h\|_{L_t^1(B^{1-\gamma,3/2+\gamma})} + \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1-\gamma,1/2+\gamma})} \|v_\Phi^h\|_{L_t^1(B^{1,3/2})}). \end{aligned}$$

This gives rise to

$$(6.22) \quad \begin{aligned} \|q_{54}\|_{Z_t} &\lesssim \varepsilon^{1+\beta} \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1,1/2})} (\|v_\Phi^h\|_{L_t^1(B^{1+\gamma,3/2-\gamma})} + \|v_\Phi^h\|_{L_t^1(B^{1-\gamma,3/2+\gamma})}) \\ &\quad + \varepsilon^{1+\beta} \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1-\gamma,1/2+\gamma})} \|v_\Phi^h\|_{L_t^1(B^{1,3/2})}. \end{aligned}$$

By summing up the above estimates, we conclude that

$$(6.23) \quad \begin{aligned} \|q\|_{Z_t} &\lesssim \varepsilon^\beta (\|a_\Phi\|_{\tilde{L}_t^\infty(B^{1,1/2})} + \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1-\gamma,1/2+\gamma})}) \|q\|_{Z_t} \\ &\quad + \varepsilon^\beta \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1,1/2})} (\|\Delta_\varepsilon v_\Phi^h\|_{L_t^1(B^{\gamma,1/2-\gamma})} + \|\Delta_\varepsilon v_\Phi^h\|_{L_t^1(B^{-\gamma,1/2+\gamma})} \\ &\quad + \varepsilon^{-2\gamma} \|v_\Phi^h\|_{L_t^1(B^{2+\gamma,1/2-\gamma})} + \varepsilon \|v_\Phi^h\|_{L_t^1(B^{1+\gamma,3/2-\gamma})} + \varepsilon \|v_\Phi^h\|_{L_t^1(B^{1-\gamma,3/2+\gamma})}) \\ &\quad + \varepsilon^\beta \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1-\gamma,1/2+\gamma})} (\|v_\Phi^h\|_{L_t^1(B^{2,1/2})} + \varepsilon^2 \|v_\Phi^h\|_{L_t^1(B^{0,5/2})} \\ &\quad \quad + \varepsilon \|v_\Phi^h\|_{L_t^1(B^{1,3/2})}) \\ &\quad + \varepsilon^{1-\alpha} (\|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{0,1/2})} (\|v_\Phi^h\|_{L_t^1(B^{2+\gamma,1/2-\gamma})} + \|v_\Phi^h\|_{L_t^1(B^{2-\gamma,1/2+\gamma})}) \\ &\quad \quad + \|v_\Phi^h\|_{L_t^1(B^{2,1/2})} (\|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{-\gamma,1/2+\gamma})} + \|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{\gamma,1/2-\gamma})})) \\ &\quad + \varepsilon^{1-\alpha-2\gamma} \|v_\Phi^h\|_{L_t^1(B^{2,1/2})} \|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{\gamma,1/2-\gamma})} \\ &\quad + \varepsilon^{\beta-2\gamma} \|a_\Phi\|_{\tilde{L}_t^\infty(B^{\gamma,3/2-\gamma})} \|v_\Phi^3\|_{L_t^1(B^{2,1/2})} \\ &\quad + \varepsilon^{-\alpha} (\|v_\Phi^3\|_{L_t^1(B^{2,1/2})} \|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{\gamma,1/2-\gamma})} \\ &\quad \quad + \|v_\Phi^3\|_{\tilde{L}_t^2(B^{1,1/2})} (\|v_\Phi^h\|_{\tilde{L}_t^2(B^{1+\gamma,1/2-\gamma})} + \|v_\Phi^h\|_{\tilde{L}_t^2(B^{1-\gamma,1/2+\gamma})})), \end{aligned}$$

While it follows from Definition 2.1 that

$$(6.24) \quad \begin{aligned} \varepsilon \|v_\Phi^h\|_{L_t^1(B^{1+\gamma,3/2-\gamma})} &= \varepsilon \sum_{k,\ell \in \mathbb{Z}} 2^{k(1+\gamma)} 2^{\ell(3/2-\gamma)} \|\Delta_k^h \Delta_\ell^v v_\Phi^h\|_{L_t^1(L^2)} \\ &\leq 1/2 \sum_{k,\ell \in \mathbb{Z}} (2^{k(2+\gamma)} 2^{\ell(1/2-\gamma)} + \varepsilon^2 2^{k\gamma} 2^{\ell(5/2-\gamma)}) \|\Delta_k^h \Delta_\ell^v v_\Phi^h\|_{L_t^1(L^2)} \\ &= \frac{1}{2} (\|v_\Phi^h\|_{L_t^1(B^{2+\gamma,1/2-\gamma})} + \varepsilon^2 \|v_\Phi^h\|_{L_t^1(B^{\gamma,5/2-\gamma})}). \end{aligned}$$

The same argument gives

$$(6.25) \quad \begin{aligned} \varepsilon \|v_\Phi\|_{L_t^1(B^{1,3/2})} &\leq \frac{1}{2} (\|v_\Phi^h\|_{L_t^1(B^{2,1/2})} + \varepsilon^2 \|v_\Phi^h\|_{L_t^1(B^{0,5/2})}), \\ \varepsilon \|v_\Phi\|_{L_t^1(B^{1-\gamma,3/2+\gamma})} &\leq \frac{1}{2} (\|v_\Phi^h\|_{L_t^1(B^{2-\gamma,1/2+\gamma})} + \varepsilon^2 \|v_\Phi^h\|_{L_t^1(B^{-\gamma,5/2+\gamma})}). \end{aligned}$$

Therefore, in view of (2.9), we deduce from (6.7) and (6.23) that

$$\begin{aligned} \|q\|_{Z_t} &\leq C (K\varepsilon^\beta \|q\|_{Z_t} + \varepsilon^{\beta-3\alpha-5\gamma} \Psi_1(t) \Psi_3(t) \\ &\quad + (\varepsilon^{-\alpha} + \varepsilon^{1-3\alpha-4\gamma}) \Psi_2(t) \Psi_3(t) + \varepsilon^{-2\alpha-\gamma} \Psi_4^2(t)), \end{aligned}$$

which together with the assumptions on  $\alpha, \beta$  and  $\gamma$  leads to (6.8), and we complete the proof of the proposition.  $\square$

**Remark 6.3.** It is easy to observe from the proof of Proposition 6.2 that if  $\beta > 2\alpha$ ,  $\gamma \leq \min((1 - 3\alpha)/4, (\beta - 2\alpha)/2)$  and  $\varepsilon$  is so small that  $\varepsilon^\beta \leq \min(1/(2C_0K), \varepsilon/K)$ , then there holds

$$\begin{aligned} \|q_1\|_{Z_t} + \|q_2\|_{Z_t} + \|q_{32}\|_{Z_t} + \|q_{42}\|_{Z_t} + \|q_{51}\|_{Z_t} + \|q_{52}\|_{Z_t} + \|q_{54}\|_{Z_t} \\ \leq C \max(\varepsilon^{\beta-2\alpha-2\gamma}, \varepsilon^{1-3\alpha-4\gamma}, K\varepsilon^{\beta-2\alpha-\gamma}) \Psi^2(t). \end{aligned}$$

### 7. Classical parabolic type estimates

This section is devoted to the estimate of the analytic band  $\theta$ , i.e, the proof of Proposition 2.4. To achieve this, we first rewrite the momentum equation of (2.1) as follows:

$$(7.1) \quad \begin{aligned} \partial_t v - \Delta_\varepsilon v &= F_1 + F_2 + F_3 \quad \text{with} \\ F_1 &\stackrel{\text{def}}{=} -\varepsilon^{1-\alpha} v \cdot \nabla v, \quad F_2 \stackrel{\text{def}}{=} -\frac{\varepsilon^\beta a}{1 + \varepsilon^\beta a} \Delta_\varepsilon v, \quad F_3 \stackrel{\text{def}}{=} -\frac{1}{1 + \varepsilon^\beta a} \nabla^\varepsilon q. \end{aligned}$$

For  $E_\varepsilon$  given by (4.2), applying the Duhamel formula to (7.1) gives

$$(7.2) \quad v(t) = e^{t\Delta_\varepsilon} v_0 + E_\varepsilon (F_1 + F_2 + F_3).$$

In what follows, we denote

$$(7.3) \quad \|f\|_{H_t} \stackrel{\text{def}}{=} \|f_\Phi\|_{L_t^1(B^{1,1/2})} + \|f_\Phi\|_{L_t^1(B^{1+\gamma,1/2-\gamma})} + \varepsilon^{1+\gamma} \|f_\Phi\|_{L_t^1(B^{-\gamma,3/2+\gamma})}.$$

First of all, it follows from Lemma 4.1 that

$$\varepsilon^{1-\alpha} \|[e^{t\Delta_\varepsilon} v_0^h]_\Phi\|_{L_t^1(B^{1,1/2})} \lesssim \varepsilon^\gamma \|e^{\delta|D|} v_0^h\|_{B^{-\alpha-\gamma,-1/2+\alpha+\gamma}}.$$

However, since  $0 < \gamma < (1 - 2\alpha)/4$ , we have  $-1/2 + \gamma < -\alpha - \gamma < 0$  and  $-1/2 < -1/2 + \alpha + \gamma < -\gamma$ , so that applying Lemma 3.2 yields

$$\varepsilon^{1-\alpha} \|[e^{t\Delta_\varepsilon} v_0^h]_\Phi\|_{L_t^1(B^{1,1/2})} \lesssim \varepsilon^\gamma (\|e^{\delta|D|} v_0^h\|_{B^{-1/2+\gamma,-\gamma}} + \|e^{\delta|D|} v_0^h\|_{B^{0,-1/2}}) \leq \varepsilon^\gamma \|v_0^h\|_{X_2}$$

for the norm  $\|\cdot\|_{X_2}$  given by (1.10).

Along the same lines, one has

$$\begin{aligned} & \varepsilon^{1-\alpha} \left\| [e^{t\Delta_\varepsilon} v_0^h]_\Phi \right\|_{L_t^1(B^{1+\gamma, 1/2-\gamma})} + \varepsilon^{2-\alpha+\gamma} \left\| [e^{t\Delta_\varepsilon} v_0^h]_\Phi \right\|_{L_t^1(B^{-\gamma, 3/2+\gamma})} \\ & \lesssim \varepsilon^\gamma \left( \left\| e^{\delta|D|} v_0^h \right\|_{B^{-\alpha, -1/2+\alpha}} + \left\| e^{\delta|D|} v_0^h \right\|_{B^{-\alpha-\gamma, -1/2+\alpha+\gamma}} \right) \lesssim \varepsilon^\gamma \|v_0^h\|_{X_2}. \end{aligned}$$

While it follows from the second inequality of (4.1) that

$$\left\| [e^{t\Delta_\varepsilon} v_0^3]_\Phi \right\|_{L_t^1(B^{1, 1/2})} + \varepsilon^{1+\gamma} \left\| [e^{t\Delta_\varepsilon} v_0^3]_\Phi \right\|_{L_t^1(B^{-\gamma, 3/2+\gamma})} \lesssim \left\| e^{\delta|D|} v_0^h \right\|_{B^{0, -1/2}}.$$

While it follows from the proof of Lemma 4.1 and  $\operatorname{div} v_0 = 0$  that

$$\begin{aligned} & \left\| \Delta_k^h \Delta_\ell^v [e^{t\Delta_\varepsilon} v_0^3]_\Phi \right\|_{L_t^1(L^2)} \\ & \lesssim 2^{-2k} \left\| e^{\delta|D|} \Delta_k^h \Delta_\ell^v v_0^3 \right\|_{L^2}^{2\gamma/(1+\gamma)} \left( 2^{-\ell} \left\| e^{\delta|D|} \Delta_k^h \Delta_\ell^v \operatorname{div}_h v_0^h \right\|_{L^2} \right)^{(1-\gamma)/(1+\gamma)} \\ & \lesssim d_{k,\ell} 2^{-k(1+\gamma)} 2^{-\ell(1/2-\gamma)} \left\| e^{\delta|D|} v_0^3 \right\|_{B^{0, -1/2}}^{2\gamma/(1+\gamma)} \left\| e^{\delta|D|} v_0^h \right\|_{B^{-\gamma, -1/2+\gamma}}^{(1-\gamma)/(1+\gamma)}, \end{aligned}$$

which gives

$$\left\| [e^{t\Delta_\varepsilon} v_0^3]_\Phi \right\|_{L_t^1(B^{1+\gamma, 1/2-\gamma})} \lesssim \|v_0\|_{X_2}.$$

As a consequence, we obtain

$$(7.4) \quad \varepsilon^{1-\alpha} \|e^{t\Delta_\varepsilon} v_0^h\|_{H_t} + \varepsilon^\gamma \|e^{t\Delta_\varepsilon} v_0^3\|_{H_t} \leq C \varepsilon^\gamma \|v_0\|_{X_2}.$$

**Step 1.** Estimate of the horizontal velocity

- *Estimate of  $E_\varepsilon(F_1^h)$ .*

Since  $\operatorname{div} v = 0$ ,  $v \cdot \nabla v^h = \nabla_h \cdot (v^h \otimes v^h) + \partial_3(v^3 v^h)$ , we get, by applying Lemma 4.2 and the law of product of Corollary 3.4, that

$$\begin{aligned} & \varepsilon^{1-\alpha} \left\| [E_\varepsilon(F_1^h)]_\Phi \right\|_{L_t^1(B^{1, 1/2})} \\ & \lesssim \varepsilon^{2(1-\alpha)} \left\| \operatorname{div}_h [v^h \otimes v^h]_\Phi \right\|_{L_t^1(B^{-1, 1/2})} + \varepsilon^{1-2\alpha} \left\| \partial_3 [v^3 v^h]_\Phi \right\|_{L_t^1(B^{0, -1/2})} \\ & \lesssim \varepsilon^{2(1-\alpha)} \left\| [v^h v^h]_\Phi \right\|_{L_t^1(B^{0, 1/2})} + \varepsilon^{1-2\alpha} \left\| [v^3 v^h]_\Phi \right\|_{L_t^1(B^{0, 1/2})} \\ & \lesssim \varepsilon^{1-2\alpha} \left( \varepsilon \left\| v_\Phi^h \right\|_{L_t^1(B^{1, 1/2})} + \left\| v_\Phi^3 \right\|_{L_t^1(B^{1, 1/2})} \right) \left\| v_\Phi^h \right\|_{\tilde{L}_t^\infty(B^{0, 1/2})}, \end{aligned}$$

Along the same lines, we have

$$\begin{aligned} & \varepsilon^{1-\alpha} \left\| [E_\varepsilon(F_1^h)]_\Phi \right\|_{L_t^1(B^{1+\gamma, 1/2-\gamma})} \\ & \lesssim \varepsilon^{2(1-\alpha)} \left\| [v^h \otimes v^h]_\Phi \right\|_{L_t^1(B^{\gamma, 1/2-\gamma})} + \varepsilon^{1-2\alpha} \left\| [v^3 v^h]_\Phi \right\|_{L_t^1(B^{\gamma, 1/2-\gamma})} \\ & \lesssim \varepsilon^{1-2\alpha} \left( \varepsilon \left\| v_\Phi^h \right\|_{L_t^1(B^{1, 1/2})} + \left\| v_\Phi^3 \right\|_{L_t^1(B^{1, 1/2})} \right) \left\| v_\Phi^h \right\|_{\tilde{L}_t^\infty(B^{\gamma, 1/2-\gamma})}. \end{aligned}$$

and if  $\alpha \leq 1/2$ ,

$$\begin{aligned} & \varepsilon^{2-\alpha+\gamma} \left\| [E_\varepsilon(F_1^h)]_\Phi \right\|_{L_t^1(B^{-\gamma, 3/2+\gamma})} \\ & \lesssim \varepsilon^{2(1-\alpha)} \left\| [v^h \otimes v^h]_\Phi \right\|_{L_t^1(B^{0, 1/2})} + \varepsilon^{1-2\alpha+\gamma} \left\| \partial_3 [v^3 v^h]_\Phi \right\|_{L_t^1(B^{-\gamma, -1/2+\gamma})} \\ & \lesssim \varepsilon^{1-2\alpha} \left( \varepsilon \left\| v_\Phi^h \right\|_{L_t^1(B^{1, 1/2})} \left\| v_\Phi^h \right\|_{\tilde{L}_t^\infty(B^{0, 1/2})} + \left\| v_\Phi^3 \right\|_{L_t^1(B^{1, 1/2})} \left\| v_\Phi^h \right\|_{\tilde{L}_t^\infty(B^{-\gamma, 1/2+\gamma})} \right. \\ & \quad \left. + \varepsilon^\gamma \left\| v_\Phi^3 \right\|_{L_t^1(B^{1-\gamma, 1/2+\gamma})} \left\| v_\Phi^h \right\|_{\tilde{L}_t^\infty(B^{0, 1/2})} \right). \end{aligned}$$

We thus obtain

$$\begin{aligned} &\varepsilon^{1-\alpha} \|E_\varepsilon(F_1^h)\|_{H_t} \\ &\lesssim \varepsilon^{1-2\alpha} \left( (\varepsilon \|v_\Phi^h\|_{L_t^1(B^{1,1/2})} + \|v_\Phi^3\|_{L_t^1(B^{1,1/2})}) (\|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{0,1/2})} + \|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{\gamma,1/2-\gamma})}) \right. \\ &\quad \left. + \|v_\Phi^3\|_{L_t^1(B^{1,1/2})} \|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{-\gamma,1/2+\gamma})} + \varepsilon^\gamma \|v_\Phi^3\|_{L_t^1(B^{1-\gamma,1/2+\gamma})} \|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{0,1/2})} \right). \end{aligned}$$

However, noticing that

$$\begin{aligned} &\varepsilon^\gamma \|v_\Phi^3\|_{L_t^1(B^{1-\gamma,1/2+\gamma})} \\ &= \sum_{k,\ell \in \mathbb{Z}} 2^{k(1-\gamma)} 2^{\ell(1/2+\gamma)} (\varepsilon^{1+\gamma} \|\Delta_k^h \Delta_\ell^v v_\Phi^3\|_{L_t^1(L^2)})^{\gamma/(1+\gamma)} \|\Delta_k^h \Delta_\ell^v v_\Phi^3\|_{L_t^1(L^2)}^{1/(1+\gamma)} \\ &\lesssim (\varepsilon^{1+\gamma} \|v_\Phi^3\|_{L_t^1(B^{-\gamma,3/2+\gamma})})^{\gamma/(1+\gamma)} \|v_\Phi^3\|_{L_t^1(B^{1,1/2})}^{1/(1+\gamma)}, \end{aligned}$$

we infer

$$(7.5) \quad \varepsilon^{1-\alpha} \|E_\varepsilon(F_1^h)\|_{H_t} \lesssim \varepsilon^{1-2\alpha-\gamma} \theta(t) \Psi_2(t).$$

- *Estimate of  $E_\varepsilon(F_2^h)$ .*

Similar to the estimate of  $E_\varepsilon(F_1^h)$ , since  $\varepsilon^\beta K \leq \varepsilon$ , we get, by applying Lemma 4.2, the law of product of Corollary 3.4 and Lemma 3.6, that

$$\begin{aligned} \|E_\varepsilon(G(\varepsilon^\beta a)\Delta_h v^h)\|_{H_t} &\lesssim \varepsilon^{-\gamma} \|[G(\varepsilon^\beta a)\Delta_h v^h]_\Phi\|_{L_t^1(B^{-1+\gamma,1/2-\gamma})} \\ &\lesssim \varepsilon^{\beta-\gamma} \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1,1/2})} \|\Delta_h v_\Phi^h\|_{L_t^1(B^{-1+\gamma,1/2-\gamma})}, \end{aligned}$$

and

$$\begin{aligned} \varepsilon^2 \|E_\varepsilon(G(\varepsilon^\beta a)\partial_3^2 v^h)\|_{H_t} &\lesssim \varepsilon^{1+\gamma} \|[G(\varepsilon^\beta a)\partial_3^2 v^h]_\Phi\|_{L_t^1(B^{-\gamma,-1/2+\gamma})} \\ &\lesssim \varepsilon^{1+\beta-\gamma} \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1,1/2})} \|\partial_3^2 v_\Phi^h\|_{L_t^1(B^{-\gamma,-1/2+\gamma})}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} &\varepsilon^{1-\alpha} \|E_\varepsilon(F_2^h)\|_{H_t} \\ &\lesssim \varepsilon^{1-\alpha+\beta-\gamma} \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1,1/2})} (\|v_\Phi^h\|_{L_t^1(B^{1+\gamma,1/2-\gamma})} + \varepsilon \|v_\Phi^h\|_{L_t^1(B^{-\gamma,3/2+\gamma})}) \\ (7.6) \quad &\lesssim \varepsilon^{\beta-\gamma} \theta(t) \Psi_1(t). \end{aligned}$$

- *Estimate of  $E_\varepsilon(F_3^h)$ .*

Due to (2.11), it follows from Lemma 4.2 and Lemma 3.6 that

$$\begin{aligned} \|E_\varepsilon(F_3^h)\|_{H_t} &\lesssim \|q\|_{Y_t} + \varepsilon^{-\gamma} \|[G(\varepsilon^\beta a)\nabla_h q]_\Phi\|_{L_t^1(B^{-1+\gamma,1/2-\gamma})} \\ &\lesssim \|q\|_{Y_t} + \varepsilon^{\beta-\gamma} \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1,1/2})} \|\nabla_h q_\Phi\|_{L_t^1(B^{-1+\gamma,1/2-\gamma})}, \end{aligned}$$

from which, the assumption that  $\varepsilon^{\beta-\gamma} K \leq 1$  and Proposition 6.1, we infer

$$(7.7) \quad \varepsilon^{1-\alpha} \|E_\varepsilon(F_3^h)\|_{H_t} \leq C \varepsilon^{1-\alpha} \|q\|_{Y_t} \leq C \min(\varepsilon^{\beta-\alpha-2\gamma}, \varepsilon^{1-2\alpha-2\gamma}) \theta(t) \Psi(t).$$

By summing up (7.4)–(7.7), we conclude that

$$(7.8) \quad \varepsilon^{1-\alpha} \|v^h\|_{H_t} \leq C(\varepsilon^\gamma \|v_0^h\|_{X_2} + \max(\varepsilon^{\beta-\alpha-2\gamma}, \varepsilon^{1-2\alpha-2\gamma}) \theta(t) \Psi(t)).$$

**Step 2.** Estimate of the vertical velocity

- *Estimate of  $E_\varepsilon(F_1^3)$ .*

Again since  $\operatorname{div} v = 0$ , we write

$$v \cdot \nabla v^3 = \nabla_h \cdot (v^h v^3) - 2(v^3 \operatorname{div}_h v^h),$$

from which, Lemma 4.2 and the law of product of Corollary 3.4, we deduce that

$$\begin{aligned} \|[E_\varepsilon(F_1^3)]_\Phi\|_{L_t^1(B^{1,1/2})} &\lesssim \varepsilon^{1-\alpha} (\|[v^h v^3]_\Phi\|_{L_t^1(B^{0,1/2})} + \varepsilon^{-\gamma} \|[v^3 \operatorname{div}_h v^h]_\Phi\|_{L_t^1(B^{-1+\gamma,1/2-\gamma})}) \\ &\lesssim \varepsilon^{1-\alpha} \|v_\Phi^3\|_{L_t^1(B^{1,1/2})} (\|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{0,1/2})} + \varepsilon^{-\gamma} \|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{\gamma,1/2-\gamma})}), \end{aligned}$$

and

$$\begin{aligned} \|[E_\varepsilon(F_1^3)]_\Phi\|_{L_t^1(B^{1+\gamma,1/2-\gamma})} &\lesssim \varepsilon^{1-\alpha} (\|[v^h v^3]_\Phi\|_{L_t^1(B^{\gamma,1/2-\gamma})} + \|[v^3 \operatorname{div}_h v^h]_\Phi\|_{L_t^1(B^{-1+\gamma,1/2-\gamma})}) \\ &\lesssim \varepsilon^{1-\alpha} \|v_\Phi^3\|_{L_t^1(B^{1,1/2})} \|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{\gamma,1/2-\gamma})}, \end{aligned}$$

and

$$\begin{aligned} \varepsilon^{1+\gamma} \|[E_\varepsilon(F_1^3)]_\Phi\|_{L_t^1(B^{-\gamma,3/2+\gamma})} &\lesssim \varepsilon^{1-\alpha} (\|[v^h v^3]_\Phi\|_{L_t^1(B^{0,1/2})} + \varepsilon^{-\gamma} \|[v^3 \operatorname{div}_h v^h]_\Phi\|_{L_t^1(B^{-1+\gamma,1/2-\gamma})}) \\ &\lesssim \varepsilon^{1-\alpha} \|v_\Phi^3\|_{L_t^1(B^{1,1/2})} (\|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{0,1/2})} + \varepsilon^{-\gamma} \|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{\gamma,1/2-\gamma})}). \end{aligned}$$

Therefore, if  $\gamma \leq 1 - \alpha$ , we obtain

$$(7.9) \quad \begin{aligned} \|E_\varepsilon(F_1^3)\|_{H_t} &\leq C \varepsilon^{1-\alpha} \|v_\Phi^3\|_{L_t^1(B^{1,1/2})} (\|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{0,1/2})} + \varepsilon^{-\gamma} \|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{\gamma,1/2-\gamma})}) \\ &\leq C \varepsilon^{1-\alpha-2\gamma} \theta(t) \Psi_2(t). \end{aligned}$$

- *Estimate of  $E_\varepsilon(F_2^3)$ .*

Similar to the estimate of (7.6), we have

$$\begin{aligned} \|E_\varepsilon(F_2^3)\|_{H_t} &\lesssim \varepsilon^{-\gamma} \|[G(\varepsilon^\beta a) \Delta_h v^3]_\Phi\|_{L_t^1(B^{-1+\gamma,1/2-\gamma})} + \varepsilon^{1+\gamma} \|[G(\varepsilon^\beta a) \partial_3^2 v^3]_\Phi\|_{L_t^1(B^{-\gamma,-1/2+\gamma})} \\ &\lesssim \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1,1/2})} (\varepsilon^{\beta-\gamma} \|v_\Phi^3\|_{L_t^1(B^{1+\gamma,1/2-\gamma})} + \varepsilon^{1+\beta+\gamma} \|v_\Phi^3\|_{L_t^1(B^{-\gamma,3/2+\gamma})}), \end{aligned}$$

so that we get

$$(7.10) \quad \|E_\varepsilon(F_2^3)\|_{H_t} \leq C \varepsilon^{\beta-2\gamma} \Psi_1(t) \theta(t).$$



• *Estimate of  $E_\varepsilon(F_3^3)$*

It follows from a similar derivation of (7.7) that for  $\gamma \leq \alpha$ ,

$$(7.11) \quad \|E_\varepsilon(F_3^3)\|_{H_t} \leq C \varepsilon^{1-\gamma} \|(1 - G(\varepsilon^\beta a))\varepsilon \partial_3 q\|_\Phi \|L_t^1(B^{-1+\gamma, 1/2-\gamma})\|$$

$$(7.12) \quad \leq C \varepsilon^{\alpha-\gamma} \varepsilon^{1-\alpha} \|q\|_{Y_t} \leq C \varepsilon^{\alpha-\gamma} \theta(t) \Psi(t).$$

Since  $\gamma < (\beta - \alpha)/2$ , we have  $\beta - 2\gamma > \alpha - \gamma$ . Then by summing up (7.4) and (7.9)–(7.12), we arrive at

$$(7.13) \quad \|v^3\|_{H_t} \leq C (\|v_0\|_{X_2} + \max(\varepsilon^{1-\alpha-2\gamma}, \varepsilon^{\alpha-\gamma}) \theta(t) \Psi(t)).$$

Proposition 2.4 follows by combining (7.8) with (7.13).

### 8. Regularizing effect of the analyticity

The goal of this section is to present the proof of Proposition 2.5. Here we need to use the regularizing effect of the heat semigroup. As a convention throughout this section, we always assume that there holds (2.10).

**Step 1.** Estimate of the density

In view of (2.9), we get, by applying (5.2) and (5.15)–(5.17), that

$$\begin{aligned} \Psi_1(t) \leq & \|e^{\delta|D|} a_0\|_{B^{1,1/2}} + \|e^{\delta|D|} a_0\|_{B^{1+\gamma, 1/2-\gamma}} + \|e^{\delta|D|} a_0\|_{B^{1-\gamma, 1/2+\gamma}} \\ & + \varepsilon^{3\alpha+3\gamma} \|e^{\delta|D|} a_0\|_{B^{\gamma, 3/2-\gamma}} + C \left( \frac{1}{\lambda} \Psi_1(t) + \varepsilon^{1-\alpha} (\|v_\Phi\|_{L_t^1(B^{2-\gamma, 1/2+\gamma})} \right. \\ & + \|v_\Phi\|_{L_t^1(B^{2,1/2})} + \|v_\Phi\|_{L_t^1(B^{2+\gamma, 1/2-\gamma})}) \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1,1/2})} + \varepsilon^{1+2\alpha+3\gamma} \\ & \left. \times (\|v_\Phi^h\|_{L_t^1(B^{1,3/2})} \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1+\gamma, 1/2-\gamma})} + \|v_\Phi\|_{L_t^1(B^{1+\gamma, 3/2-\gamma})} \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1,1/2})}) \right). \end{aligned}$$

However, it is easy to observe from Lemma 3.2 that

$$\|e^{\delta|D|} a_0\|_{B^{1,1/2}} \lesssim \|e^{\delta|D|} a_0\|_{B^{1+\gamma, 1/2-\gamma}} + \|e^{\delta|D|} a_0\|_{B^{1-\gamma, 1/2+\gamma}},$$

and it follows from (6.24) and (6.25) that

$$\begin{aligned} \varepsilon^{1+2\alpha+2\gamma} (\|v_\Phi^h\|_{L_t^1(B^{1,3/2})} + \|v_\Phi^h\|_{L_t^1(B^{1+\gamma, 3/2-\gamma})}) & \lesssim \varepsilon^{2\alpha+2\gamma} (\|v_\Phi^h\|_{L_t^1(B^{2,1/2})} \\ & + \varepsilon^2 \|v_\Phi^h\|_{L_t^1(B^{0,5/2})} + \|v_\Phi\|_{L_t^1(B^{2+\gamma, 1/2-\gamma})} + \varepsilon^2 \|v_\Phi^h\|_{L_t^1(B^{\gamma, 5/2-\gamma})}). \end{aligned}$$

Therefore, due to  $0 < \gamma \leq (1 - 3\alpha)/3$ , we obtain

$$(8.1) \quad \Psi_1(t) \leq C \|a_0\|_{X_1} + C \left( \frac{1}{\lambda} + \varepsilon^\gamma \Psi_3(t) \right) \Psi_1(t).$$

**Step 2.** Estimate of  $\Psi_2(t)$

In the remaining of this section, we denote

$$\|f\|_{K_t} \stackrel{\text{def}}{=} \|f_\Phi\|_{\tilde{L}_t^\infty(B^{\gamma, 1/2-\gamma})} + \|f_\Phi\|_{\tilde{L}_t^\infty(B^{-\gamma, 1/2+\gamma})}.$$

Then it follows from Lemma 4.1 that

$$(8.2) \quad \|e^{t\Delta_\varepsilon} v_0\|_{K_t} \leq C \|v_0\|_{X_3}.$$

**Step 2.1** The estimate of the horizontal velocity.

In order to estimate  $\|v^h\|_{K_t}$ , we still need to deal with the source term in (7.2).

• *Estimate of  $E_\varepsilon(F_1^h)$*

In view of (7.2), by using Bony’s decomposition (3.3) in the horizontal variable for  $v^3 v^h$ , we write  $F_1^h$  as

$$F_1^h = -\varepsilon^{1-\alpha} \nabla_h \cdot (v^h \otimes v^h) - \varepsilon^{1-\alpha} \partial_3 \mathcal{R}^v(v^3, v^h) - \varepsilon^{1-\alpha} \partial_3 T^v(v^3, v^h) \stackrel{\text{def}}{=} F_{11}^h + F_{12}^h + F_{13}^h.$$

Applying Lemma 4.2 and the law of product of Corollary 3.4 yields

$$\begin{aligned} \|E_\varepsilon(F_{11}^h)\|_{K_t} &\lesssim \varepsilon^{1-\alpha} (\| [v^h \nabla_h v^h]_\Phi \|_{L_t^1(B^{\gamma, 1/2-\gamma})} + \| [v^h \nabla_h v^h]_\Phi \|_{L_t^1(B^{-\gamma, 1/2+\gamma})}) \\ &\lesssim \varepsilon^{1-\alpha} \left( \|v_\Phi^h\|_{L_t^1(B^{2, 1/2})} (\|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{\gamma, 1/2-\gamma})} + \|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{-\gamma, 1/2+\gamma})}) \right. \\ &\quad \left. + \|v_\Phi^h\|_{L_t^1(B^{2-\gamma, 1/2+\gamma})} \|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{0, 1/2})} \right). \end{aligned}$$

For  $\tilde{\varphi}$  in  $C_c^\infty(\mathbb{R}^+ \setminus \{0\})$  which equals 1 on the support of  $\varphi$  in (2.3), let

$$\tilde{\varphi}(\xi_3) \stackrel{\text{def}}{=} \frac{\tilde{\varphi}(|\xi_3|)}{i\xi_3}.$$

Then we may write

$$\Delta_\ell^v v^3 = 2^{-\ell} \tilde{\varphi}(2^{-\ell} |D_3|) \Delta_\ell^v \partial_3 v^3,$$

and due to  $\partial_3 v^3 = -\text{div}_h v^h$ , we have

$$\mathcal{R}^v(v^3, v^h) = - \sum_{\ell \in \mathbb{Z}} 2^{-\ell} \tilde{\varphi}(2^{-\ell} |D_3|) \Delta_\ell^v \text{div}_h v^h S_{\ell+2}^v v^h,$$

from which, using Bony’s decomposition in the horizontal variables for  $\mathcal{R}^v(v^3, v^h)$ , one may deduce, by a similar derivation to that of Lemma 3.3, that  $E_\varepsilon(F_{12}^h)$  shares the same estimate as  $E_\varepsilon(F_{11}^h)$ .

Whereas it follows from Remark 3.5 that

$$\|\Delta_k^h \Delta_\ell^v [F_{13}^h]_\Phi(t)\|_{L^2} \lesssim (d_k(t) d_\ell + d_{k,\ell}) 2^{-k\sigma} 2^{-\ell s} \|v_\Phi^3(t)\|_{B^{1, 1/2}} \|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{\sigma, s})}$$

for any  $\sigma \in ]-1, 1]$ ,  $s \in \mathbb{R}$ , from which, and Lemma 4.3, we infer

$$\|E_\varepsilon(F_{13}^h)\|_{K_t} \leq \frac{C}{\lambda} (\|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{\gamma, 1/2-\gamma})} + \|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{-\gamma, 1/2+\gamma})}).$$

Hence, we obtain

$$(8.3) \quad \|E_\varepsilon(F_1^h)\|_{K_t} \leq C \left( \frac{1}{\lambda} + \varepsilon^{1-3\alpha-2\gamma} \Psi_3(t) \right) \Psi_2(t).$$

• *Estimate of  $E_\varepsilon(F_2^h)$*

Again due to Lemma 4.2, one has

$$\|E_\varepsilon(F_2^h)\|_{K_t} \lesssim \| [G(\varepsilon^\beta a)\Delta_\varepsilon v^h]_\Phi \|_{L_t^1(B^{\gamma,1/2-\gamma})} + \| [G(\varepsilon^\beta a)\Delta_\varepsilon v^h]_\Phi \|_{L_t^1(B^{-\gamma,1/2+\gamma})},$$

which together with Corollary 3.4 and Lemma 3.6 ensures that

$$\begin{aligned} \|E_\varepsilon(F_2^h)\|_{K_t} &\lesssim \varepsilon^\beta \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1,1/2})} (\|v_\Phi^h\|_{L_t^1(B^{2+\gamma,1/2-\gamma})} + \varepsilon^2 \|v_\Phi^h\|_{L_t^1(B^{\gamma,5/2-\gamma})} \\ &\quad + \|v_\Phi^h\|_{L_t^1(B^{2-\gamma,1/2+\gamma})} + \varepsilon^2 \|v_\Phi^h\|_{L_t^1(B^{-\gamma,5/2+\gamma})}) \\ &\quad + \varepsilon^\beta \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1-\gamma,1/2+\gamma})} (\|v_\Phi^h\|_{L_t^1(B^{2,1/2})} + \varepsilon^2 \|v_\Phi^h\|_{L_t^1(B^{0,5/2})}). \end{aligned}$$

Whenever  $\varepsilon$  is so small that  $\varepsilon^\beta K \leq \epsilon$  for  $\epsilon$  determined by (3.8). This gives rise to

$$(8.4) \quad \|E_\varepsilon(F_2^h)\|_{K_t} \leq C \varepsilon^{\beta-2\alpha-2\gamma} \Psi_1(t) \Psi_3(t).$$

• *Estimate of  $E_\varepsilon(F_3^h)$* .

In view of (6.11), we get, by a similar proof of (6.15), that

$$\begin{aligned} \|\Delta_k^h \Delta_\ell^v \nabla_h [q_{31}]_\Phi(t)\|_{L^2} &\lesssim \varepsilon^{1-\alpha} 2^\ell \|\Delta_k^h \Delta_\ell^v [T^v(v^3, v^h)]_\Phi(t)\|_{L^2} \\ &\lesssim d_{k\ell} 2^\ell 2^{-k\gamma} 2^{-\ell(1/2-\gamma)} \varepsilon^{1-\alpha} \|v_\Phi^3(t)\|_{B^{1,1/2}} \|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{\gamma,1/2-\gamma})}, \end{aligned}$$

and

$$\begin{aligned} \|\Delta_k^h \Delta_\ell^v \nabla_h [q_{31}]_\Phi(t)\|_{L^2} &\lesssim d_{k\ell} 2^\ell 2^{k\gamma} 2^{-\ell(1/2+\gamma)} \varepsilon^{1-\alpha} \|v_\Phi^3(t)\|_{B^{1,1/2}} \|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{-\gamma,1/2+\gamma})}, \end{aligned}$$

so that applying Lemma 4.3 yields

$$(8.5) \quad \|E_\varepsilon(\nabla_h q_{31})\|_{K_t} \leq \frac{C}{\lambda} \Psi_2(t).$$

Similarly, according to (6.16), one gets, by using a similar derivation of (6.18), that

$$\begin{aligned} \|\Delta_k^h \Delta_\ell^v \nabla_h [q_{41}]_\Phi(t)\|_{L^2} &\lesssim \varepsilon^{1-\alpha} 2^{-k} 2^\ell \|\Delta_k^h \Delta_\ell^v [T^v(v^3, \operatorname{div}_h v^h)]_\Phi(t)\|_{L^2} \\ &\lesssim d_{k\ell} 2^\ell 2^{-k\gamma} 2^{-\ell(1/2-\gamma)} \varepsilon^{1-\alpha} \|v_\Phi^3(t)\|_{B^{1,1/2}} \|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{\gamma,1/2-\gamma})}, \end{aligned}$$

and

$$\begin{aligned} \|\Delta_k^h \Delta_\ell^v \nabla_h [q_{41}]_\Phi(t)\|_{L^2} &\lesssim \varepsilon^{1-\alpha-2\gamma} 2^{-k(1-2\gamma)} 2^{\ell(1-2\gamma)} \|\Delta_k^h \Delta_\ell^v [T^v(v^3, \operatorname{div}_h v^h)]_\Phi(t)\|_{L^2} \\ &\lesssim d_{k\ell} 2^\ell 2^{k\gamma} 2^{-\ell(1/2+\gamma)} \varepsilon^{1-\alpha-2\gamma} \|v_\Phi^3(t)\|_{B^{1,1/2}} \|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{\gamma,1/2-\gamma})}, \end{aligned}$$

so that applying Lemma 4.3 and using  $1 - \alpha \geq 3\gamma$ , we get

$$(8.6) \quad \|E_\varepsilon(\nabla_h q_{41})\|_{K_t} \leq \frac{C}{\lambda} \Psi_2(t).$$

Let us examine  $q_{53}$ . In order to do it, by using Bony’s decomposition (3.3) for  $\Delta_h v^3 G(\varepsilon^\beta a)$  in the vertical variable, we write

$$q_{53} = (-\Delta_\varepsilon)^{-1} \partial_3 T^v (\Delta_h v^3, G(\varepsilon^\beta a)) + (-\Delta_\varepsilon)^{-1} \partial_3 \mathcal{R}^v (\Delta_h v^3, G(\varepsilon^\beta a)).$$

Note that Remark 3.5 and Lemma 3.6 ensure

$$\begin{aligned} & \|\Delta_k^h \Delta_\ell^v [T^v (\Delta_h v^3, G(\varepsilon^\beta a))]_\Phi(t)\|_{L^2} \\ & \lesssim \varepsilon^\beta (d_k(t) d_\ell + d_{k\ell}) 2^{k(1-\gamma)} 2^{-\ell(1/2-\gamma)} \|v_\Phi^3(t)\|_{B^{1+\gamma, 1/2-\gamma}} \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1, 1/2})}, \end{aligned}$$

from which, with a similar derivation of (8.5) and (8.6), we infer

$$\|E_\varepsilon (\nabla_h (-\Delta_\varepsilon)^{-1} \partial_3 T^v (\Delta_h v^3, G(\varepsilon^\beta a)))\|_{K_t} \leq \frac{C \varepsilon^{\beta-\gamma}}{\lambda} \Psi_1(t).$$

Whereas by using Bony’s decomposition (3.3) for  $\mathcal{R}^v (\Delta_h v^3, G(\varepsilon^\beta a))$  for the horizontal variables and using  $\operatorname{div} v = 0$ , one has

$$\|\partial_3 [\mathcal{R}^v (\Delta_h v^3, G(\varepsilon^\beta a))]_\Phi\|_{L_t^1(B^{-1+\gamma, 1/2-\gamma})} \lesssim \varepsilon^\beta \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1, 1/2})} \|v_\Phi^h\|_{L_t^1(B^{2+\gamma, 1/2-\gamma})}.$$

Then applying Lemma 4.2 gives

$$\begin{aligned} & \|E_\varepsilon (\nabla_h (-\Delta_\varepsilon)^{-1} \partial_3 \mathcal{R}^v (\Delta_h v^3 G(\varepsilon^\beta a)))\|_{K_t} \\ & \lesssim \varepsilon^{-2\gamma} \|\partial_3 [\mathcal{R}^v (\Delta_h v^3, G(\varepsilon^\beta a))]_\Phi\|_{L_t^1(B^{-1+\gamma, 1/2-\gamma})} \\ & \lesssim \varepsilon^{\beta-2\gamma} \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1, 1/2})} \|v_\Phi^h\|_{L_t^1(B^{2+\gamma, 1/2-\gamma})}. \end{aligned}$$

Hence, thanks to Remark 6.3, for  $\gamma \leq \min((1 - 3\alpha)/4, (\beta - 2\alpha)/2)$  and under the assumption of (2.11), we deduce that

$$(8.7) \quad \|E_\varepsilon (F_3^h)\|_{K_t} \leq C \left( \frac{1}{\lambda} + \max(\varepsilon^{\beta-2\alpha-2\gamma}, \varepsilon^{1-3\alpha-4\gamma}, K \varepsilon^{\beta-2\alpha-\gamma}) \Psi(t) \right) \Psi(t).$$

In view of (7.2), by summing up (8.2)–(8.7), we arrive at

$$(8.8) \quad \begin{aligned} & \|v^h\|_{K_t} \leq C \|v_0\|_{X_3} \\ & + C \left( \frac{1}{\lambda} + \max(\varepsilon^{\beta-2\alpha-2\gamma}, \varepsilon^{1-3\alpha-4\gamma}, K \varepsilon^{\beta-2\alpha-\gamma}) \Psi(t) \right) \Psi(t). \end{aligned}$$

**Step 2.2** The estimate of the vertical velocity.

Since  $\varepsilon$  satisfies  $\varepsilon^\beta K \leq \epsilon$ , applying Lemma 4.2 gives

$$\begin{aligned} & \|E_\varepsilon (F_2^3)\|_{K_t} \lesssim \|[F_2^3]_\Phi\|_{L_t^1(B^{\gamma, 1/2-\gamma})} + \|[F_2^3]_\Phi\|_{L_t^1(B^{-\gamma, 1/2+\gamma})} \\ & \lesssim \varepsilon^\beta \left( \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1, 1/2})} \left( \|\Delta_\varepsilon v_\Phi^3\|_{L_t^1(B^{\gamma, 1/2-\gamma})} + \|\Delta_\varepsilon v_\Phi^3\|_{L_t^1(B^{-\gamma, 1/2+\gamma})} \right) \right. \\ & \quad \left. + \|a_\Phi\|_{\tilde{L}_t^\infty(B^{1-\gamma, 1/2+\gamma})} \|\Delta_\varepsilon v_\Phi^3\|_{L_t^1(B^{0, 1/2})} \right), \end{aligned}$$

which gives

$$(8.9) \quad \|E_\varepsilon(F_2^3)\|_{K_t} \leq C \varepsilon^\beta \Psi_1(t) \Psi_3(t).$$

While again as  $\varepsilon^\beta K \leq \varepsilon$ ,  $2\alpha + 2\gamma < 1$ , it follows from Lemma 4.2 and Proposition 6.2 that

$$(8.10) \quad \|E_\varepsilon(F_3^3)\|_{K_t} \leq C\varepsilon\|q\|_{Z_t} \leq C \varepsilon^{1-2\alpha-\gamma} \Psi(t)^2.$$

Finally note that

$$F_1^3 = -\varepsilon^{1-\alpha}(v^h \cdot \nabla_h v^3) + \varepsilon^{1-\alpha}(v^3 \operatorname{div}_h v^h).$$

Then we get, by using Lemma 4.2 and the law of product Corollary 3.4, that

$$\begin{aligned} \varepsilon^{1-\alpha} \|E_\varepsilon(v^h \cdot \nabla_h v^3)\|_{K_t} &\lesssim \varepsilon^{1-\alpha} \left( \|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{0,1/2})} \left( \|v_\Phi^3\|_{L_t^1(B^{2+\gamma,1/2-\gamma})} + \|v_\Phi^3\|_{L_t^1(B^{2-\gamma,1/2+\gamma})} \right) \right. \\ &\quad \left. + \left( \|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{\gamma,1/2-\gamma})} + \|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{-\gamma,1/2+\gamma})} \right) \|v_\Phi^3\|_{L_t^1(B^{2,1/2})} \right), \end{aligned}$$

and

$$\begin{aligned} \varepsilon^{1-\alpha} \|E_\varepsilon(v^3 \operatorname{div}_h v^h)\|_{K_t} &\lesssim \varepsilon^{1-\alpha} \left( \|v_\Phi^3\|_{\tilde{L}_t^\infty(B^{0,1/2})} \left( \|v_\Phi^h\|_{L_t^1(B^{2+\gamma,1/2-\gamma})} + \|v_\Phi^h\|_{L_t^1(B^{2-\gamma,1/2+\gamma})} \right) \right. \\ &\quad \left. + \left( \|v_\Phi^3\|_{\tilde{L}_t^\infty(B^{\gamma,1/2-\gamma})} + \|v_\Phi^3\|_{\tilde{L}_t^\infty(B^{-\gamma,1/2+\gamma})} \right) \|v_\Phi^h\|_{L_t^1(B^{2,1/2})} \right), \end{aligned}$$

which ensure

$$\|E_\varepsilon(F_1^3)\|_{K_t} \leq C \varepsilon^{1-3\alpha-2\gamma} \Psi_2(t) \Psi_3(t).$$

From this, and (8.9) and (8.10), we achieve

$$(8.11) \quad \|v^3\|_{K_t} \leq C(\|v_0\|_{X_3} + \max(\varepsilon^\beta, \varepsilon^{1-3\alpha-2\gamma}) \Psi^2(t)).$$

Since Lemma 3.2 implies

$$\|f_\Phi\|_{\tilde{L}_t^\infty(B^{0,1/2})} \leq \|f\|_{K_t}$$

by combining (8.8) with (8.11), we conclude that

$$(8.12) \quad \begin{aligned} \Psi_2(t) &\leq C \left( \|v_0\|_{X_3} + \frac{1}{\lambda} \Psi(t) \right) \\ &\quad + \max(\varepsilon^{\beta-2\alpha-2\gamma}, \varepsilon^{1-3\alpha-4\gamma}, K\varepsilon^{\beta-2\alpha-\gamma}) \Psi^2(t). \end{aligned}$$

**Step 3.** Estimate of  $\Psi_3(t)$ .

Let

$$\begin{aligned} \|f\|_{L_t} &\stackrel{\text{def}}{=} \|f_\Phi\|_{L_t^1(B^{2+\gamma,1/2-\gamma})} + \|f_\Phi\|_{L_t^1(B^{2-\gamma,1/2+\gamma})} \\ &\quad + \varepsilon^2 \|f_\Phi\|_{L_t^1(B^{\gamma,5/2-\gamma})} + \varepsilon^2 \|f_\Phi\|_{L_t^1(B^{-\gamma,5/2+\gamma})}. \end{aligned}$$

Then we deduce from Lemma 4.1 that

$$(8.13) \quad \|e^{t\Delta_\varepsilon} v_0\|_{L_t} \leq C \|v_0\|_{X_3}.$$

Whereas applying Lemma 4.2 gives

$$\begin{aligned} & \varepsilon^{2\alpha+2\gamma} \|E_\varepsilon(\varepsilon^{1-\alpha} \nabla_h \cdot (v^h \otimes v^h))\|_{L_t} \\ & \lesssim \varepsilon^{1+\alpha+2\gamma} (\|[v^h \otimes v^h]_\Phi\|_{L_t^1(B^{1+\gamma,1/2-\gamma})} + \|[v^h \times v^h]_\Phi\|_{L_t^1(B^{1-\gamma,1/2+\gamma})}), \end{aligned}$$

and

$$\begin{aligned} & \varepsilon^{2\alpha+2\gamma} \|E_\varepsilon(\varepsilon^{1-\alpha} \partial_3(v^h v^3))\|_{L_t} \\ & \lesssim \varepsilon^{\alpha+2\gamma} (\|[v^h v^3]_\Phi\|_{L_t^1(B^{1+\gamma,1/2-\gamma})} + \|[v^h v^3]_\Phi\|_{L_t^1(B^{1-\gamma,1/2+\gamma})}) \\ & \quad + \varepsilon^{1+\alpha+2\gamma} (\|[v^h v^3]_\Phi\|_{L_t^1(B^{\gamma,3/2-\gamma})} + \|[v^h v^3]_\Phi\|_{L_t^1(B^{-\gamma,3/2+\gamma})}), \end{aligned}$$

so that we get, by applying the law of product of Corollary 3.4, that

$$\begin{aligned} & \varepsilon^{2\alpha+2\gamma} \|E_\varepsilon(F_1^h)\|_{L_t} \\ & \lesssim \varepsilon^{1+\alpha+2\gamma} \left( (\|v_\Phi^h\|_{L_t^1(B^{2+\gamma,1/2-\gamma})} + \|v_\Phi^h\|_{L_t^1(B^{2-\gamma,1/2+\gamma})}) \|v_\Phi^h\|_{\tilde{L}_t^\infty(B^{0,1/2})} \right. \\ & \quad + \|v_\Phi^3\|_{\tilde{L}_t^2(B^{1,1/2})} (\|v_\Phi^h\|_{\tilde{L}_t^2(B^{\gamma,3/2-\gamma})} + \|v_\Phi^h\|_{\tilde{L}_t^2(B^{-\gamma,3/2+\gamma})}) \\ & \quad \left. + \|v_\Phi^h\|_{\tilde{L}_t^2(B^{1,1/2})} (\|v_\Phi^3\|_{\tilde{L}_t^2(B^{\gamma,3/2-\gamma})} + \|v_\Phi^3\|_{\tilde{L}_t^2(B^{-\gamma,3/2+\gamma})}) \right) \\ & + \varepsilon^{\alpha+2\gamma} \left( \|v_\Phi^3\|_{\tilde{L}_t^2(B^{1,1/2})} (\|v_\Phi^h\|_{\tilde{L}_t^2(B^{1+\gamma,1/2-\gamma})} + \|v_\Phi^h\|_{\tilde{L}_t^2(B^{1-\gamma,1/2+\gamma})}) \right. \\ & \quad \left. + \|v_\Phi^h\|_{\tilde{L}_t^2(B^{1,1/2})} (\|v_\Phi^3\|_{\tilde{L}_t^2(B^{1+\gamma,1/2-\gamma})} + \|v_\Phi^3\|_{\tilde{L}_t^2(B^{1-\gamma,1/2+\gamma})}) \right). \end{aligned}$$

Due to (2.9), we arrive at

$$(8.14) \quad \varepsilon^{2\alpha+2\gamma} \|E_\varepsilon(F_1^h)\|_{L_t} \leq C(\varepsilon^{1-\alpha} \Psi_2(t) \Psi_3(t) + \varepsilon^\gamma \Psi_4^2(t)).$$

In the same manner, we have

$$\begin{aligned} \|E_\varepsilon(F_1^3)\|_{L_t} & \lesssim \varepsilon^{1-\alpha} (\|[v^h v^3]_\Phi\|_{L_t^1(B^{1+\gamma,1/2-\gamma})} + \|[v^h v^3]_\Phi\|_{L_t^1(B^{1-\gamma,1/2+\gamma})}) \\ & \quad + \|[v^3 \operatorname{div}_h v^h]_\Phi\|_{L_t^1(B^{\gamma,1/2-\gamma})} + \|[v^3 \operatorname{div}_h v^h]_\Phi\|_{L_t^1(B^{-\gamma,1/2+\gamma})}). \end{aligned}$$

Then applying the law of product of Corollary 3.4 yields

$$\begin{aligned} \|E_\varepsilon(F_1^3)\|_{L_t} & \lesssim \varepsilon^{1-\alpha} (\|v_\Phi^h\|_{\tilde{L}_t^2(B^{1,1/2})} (\|v_\Phi^3\|_{\tilde{L}_t^2(B^{1+\gamma,1/2-\gamma})} + \|v_\Phi^3\|_{\tilde{L}_t^2(B^{1-\gamma,1/2+\gamma})}) \\ & \quad + \|v_\Phi^3\|_{\tilde{L}_t^2(B^{1,1/2})} (\|v_\Phi^h\|_{\tilde{L}_t^2(B^{1+\gamma,1/2-\gamma})} + \|v_\Phi^h\|_{\tilde{L}_t^2(B^{1-\gamma,1/2+\gamma})})), \end{aligned}$$

from which we deduce that

$$(8.15) \quad \|E_\varepsilon(F_1^3)\|_{L_t} \leq C \varepsilon^{1-2\alpha-\gamma} \Psi_4^2(t).$$

Similarly, due to  $\varepsilon^\beta K \leq \varepsilon$ , it follows from Lemma 4.2, Lemma 3.6 and Corollary 3.4 that

$$\begin{aligned} \varepsilon^{2\alpha+2\gamma} \|E_\varepsilon(F_2^h)\|_{L_t} &\lesssim \varepsilon^{2\alpha+2\gamma} (\| [G(\varepsilon^\beta a)\Delta_\varepsilon v^h]_\Phi \|_{L_t^1(B^{\gamma,1/2-\gamma})} + \| [G(\varepsilon^\beta a)\Delta_\varepsilon v^h]_\Phi \|_{L_t^1(B^{-\gamma,1/2+\gamma})}) \\ &\lesssim \varepsilon^{\beta+2\alpha+2\gamma} (\| a_\Phi \|_{\tilde{L}_t^\infty(B^{1,1/2})} (\| \Delta_\varepsilon v_\Phi^h \|_{L_t^1(B^{\gamma,1/2-\gamma})} + \| \Delta_\varepsilon v_\Phi^h \|_{L_t^1(B^{-\gamma,1/2+\gamma})}) \\ &\quad + \| a_\Phi \|_{\tilde{L}_t^\infty(B^{1-\gamma,1/2+\gamma})} \| \Delta_\varepsilon v_\Phi^h \|_{L_t^1(B^{0,1/2})}), \end{aligned}$$

which gives

$$(8.16) \quad \varepsilon^{2\alpha+2\gamma} \|E_\varepsilon(F_2^h)\|_{L_t} \leq C \varepsilon^\beta \Psi_1(t) \Psi_3(t).$$

Along the same lines, we have

$$\begin{aligned} \|E_\varepsilon(F_2^3)\|_{L_t} &\lesssim \varepsilon^\beta (\| a_\Phi \|_{\tilde{L}_t^\infty(B^{1,1/2})} (\| \Delta_\varepsilon v_\Phi^3 \|_{L_t^1(B^{\gamma,1/2-\gamma})} + \| \Delta_\varepsilon v_\Phi^3 \|_{L_t^1(B^{-\gamma,1/2+\gamma})}) \\ &\quad + \| a_\Phi \|_{\tilde{L}_t^\infty(B^{1-\gamma,1/2+\gamma})} \| \Delta_\varepsilon v_\Phi^3 \|_{L_t^1(B^{0,1/2})}), \end{aligned}$$

which implies

$$(8.17) \quad \|E_\varepsilon(F_2^3)\|_{L_t} \leq C \varepsilon^\beta \Psi_1(t) \Psi_3(t).$$

Finally, since  $2\alpha + 2\beta \leq 1$ , by applying Lemma 4.2 and Proposition 6.2, one has

$$(8.18) \quad \varepsilon^{2\alpha+2\gamma} \|E_\varepsilon(F_3^h)\|_{L_t} + \|E_\varepsilon(F_3^3)\|_{L_t} \leq C \varepsilon^{2\alpha+2\gamma} \|q\|_{Z_t} \leq C \varepsilon^\gamma \Psi(t)^2.$$

Summing up (8.13)–(8.18), we conclude that

$$(8.19) \quad \begin{aligned} \Psi_3(t) &\leq C (\varepsilon^{2\alpha+2\gamma} \|v^h\|_{L_t} + \|v^3\|_{L_t}) \\ &\leq C (\|v_0\|_{X_3} + \max(\varepsilon^\gamma, \varepsilon^{1-2\alpha-\gamma}) \Psi^2(t)). \end{aligned}$$

Here we used Lemma 3.2 so that

$$\|f_\Phi\|_{L_t^1(B^{2,1/2})} + \varepsilon^2 \|f_\Phi\|_{L_t^1(B^{0,5/2})} \leq C \|f\|_{L_t}.$$

**Step 4.** Estimate of  $\Psi_4(t)$ .

Finally, it is easy to observe from (2.4) and (2.9) that

$$\Psi_4(t) \leq \Psi_2^{1/2}(t) \Psi_3^{1/2}(t) \leq 1/2 (\Psi_2(t) + \Psi_3(t)),$$

which together with (8.1), (8.12) and (8.19) leads to Proposition 2.5.

**Acknowledgments.** Ping Zhang would like to thank Professor Jean-Yves Chemin for profitable discussion on this topic. Part of this work was done when we were visiting Morningside Center of the Academy of Mathematics and Systems Sciences, CAS. We appreciate the hospitality and the financial support from MCM.

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Received November 8, 2015.

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P. Zhang is partially supported by NSF of China under Grant 11371347, the fellowship from Chinese Academy of Sciences and innovation grant from National Center for Mathematics and Interdisciplinary Sciences. Z. Zhang is partially supported by NSF of China under Grant 11371039 and 11425103.